



รายงานวิจัยฉบับสมบูรณ์

โครงการ การพัฒนาระเบียงวีเชิงตัวเลขที่มีประสิทธิภาพสำหรับจำลอง
ตัวกลางที่ทำมาจากวัสดุหลายสนาม

โดย รศ.ดร. จรูญ รุ่งอมรรัตน์ และคณะ

มิถุนายน 2562

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ที่ทำมาจากวัสดุหลายสนาม

โดย รศ.ดร. จุฑา รุ่งอมรรัตน์
หน่วยปฏิบัติการวิจัยกลศาสตร์ประยุกต์และโครงสร้าง
ภาควิชาวิศวกรรมโยธา
คณะวิศวกรรมศาสตร์
จุฬาลงกรณ์มหาวิทยาลัย

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และจุฬาลงกรณ์มหาวิทยาลัย

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. และจุฬาลงกรณ์มหาวิทยาลัย
ไม่จำเป็นต้องเห็นด้วยเสมอไป)

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ABSTRACT

Project Code: RSA5980032

Project Title: Development of efficient numerical techniques for modeling multi-field material media

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Project Period: 3 years

Abstract: This research offers the computationally efficient and accurate numerical techniques, based on a scaled boundary finite element method (SBFEM) and a boundary integral equation method (BIEM), capable of solving various boundary value problems involving two-dimensional and three-dimensional semi-infinite and infinite domains. Basic governing equations are formulated in a general, unified framework allowing the treatment of various classes of linear problems such as steady-state heat conduction and flow in porous media, linear elasticity problems, problems involving fully-coupled multi-field materials, etc. Both SBFEM and BIEM are implemented within a general context such that embedded discontinuities, general boundary conditions, and excitations can be treated. After the proposed SBFEM and BIEM are fully tested with available benchmark solutions, they are applied to solve various problems under different scenarios to demonstrate their accuracy, convergence, efficiency and vast capability.

Keywords: SBFEM, BIEM, Multi-field materials, Discontinuities, Cracks

บทคัดย่อ

รหัสโครงการ: RSA5980032

ชื่อโครงการ: การพัฒนาระเบียบวิธีเชิงตัวเลขที่มีประสิทธิภาพสำหรับจำลองตัวกลางที่ทำมาจากวัสดุหลายสนาม

ชื่อนักวิจัย: รศ.ดร.จรูญ รุ่งอมรรัตน์ และ ศ.ดร. อธิพงศ์ แสนจันทร์ชัย ภาควิชาวิศวกรรมโยธา คณะวิศวกรรมศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

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ระยะเวลาโครงการ: 3 ปี

บทคัดย่อ: งานวิจัยนี้นำเสนอระเบียบวิธีเชิงตัวเลขที่มีประสิทธิภาพเชิงคำนวณและมีความถูกต้องสูง จากระเบียบวิธีสเกลบาวดาร์ไฟไนต์เอลิเมนต์ และระเบียบวิธีสมการปริพันธ์ขอบ ที่มีความสามารถในการแก้ปัญหาค่าขอบที่หลากหลายและเกี่ยวข้องกับโดเมนไร้ขอบเขตและโดเมนที่มีขอบเขตจำกัดทั้งกรณีสองมิติและสามมิติ สมการกำกับพื้นฐานถูกสร้างขึ้นในกรอบทั่วไปทำให้สามารถจัดการกับปัญหาเชิงเส้นประเภทต่างๆ ได้ อาทิเช่น ปัญหาการเหนี่ยวนำความร้อนแบบสถานะคงที่ การไหลในวัสดุพูนแบบสถานะคงที่ ปัญหาความยืดหยุ่นเชิงเส้น และปัญหาที่เกี่ยวข้องกับวัสดุร่วมหลายสนาม เป็นต้น ระเบียบวิธีสเกลบาวดาร์ไฟไนต์เอลิเมนต์และระเบียบวิธีสมการปริพันธ์ขอบถูกพัฒนาขึ้นในรูปแบบทั่วไปทำให้สามารถจัดการกับความไม่ต่อเนื่องภายในรวมทั้งเงื่อนไขขอบและการกระตุ้นแบบทั่วไปได้ หลังจากสอบเทียบกับผลเฉลยอ้างอิงที่เชื่อถือได้แล้ว ระเบียบวิธีที่พัฒนาขึ้นถูกนำไปใช้ในการวิเคราะห์ปัญหาประเภทต่างๆ ภายใต้ชุดของข้อมูลที่หลากหลาย เพื่อแสดงให้เห็นถึงความถูกต้อง การลู่เข้าของผลเฉลย ประสิทธิภาพ และความสามารถเชิงคำนวณ

คำสำคัญ: SBFEM, BIEM, วัสดุหลายสนาม, ความไม่ต่อเนื่อง, รอยร้าว

EXECUTIVE SUMMARY

1. Motivation and Significance

For the past several decades, multi-functional, high-performance, coupled-field materials including advanced composites and smart media (e.g., piezoelectric, piezomagnetic, and piezoelectromagnetic materials) have been considerably employed in various disciplines such as the aircraft and aerospace industries, medical applications, acoustic areas, nautical and civil engineering due to their remarkable properties and attractive fully-coupled effects. The key reason for using those advanced materials is to exploit their desirable inherent characteristics to further improve and enhance the performance of designed components and parts such as the increase in strength, stiffness and strength to weight ratio, the improvement of durability and environmentally friendly characteristics, the satisfaction of multi-purpose usages, and the insurance of integrity and safety under severe service conditions. Besides their positive features, the behavior of this class of materials is relatively complex and generally involves fully-coupled, multiple fields such as mechanical, electrical, and magnetic fields. This, as a direct consequence, renders responses of constituted components and parts under various scenarios increasingly complicated relative to those made from conventional materials and the ability to fully understand, model and predict such physical phenomena is quite challenging and poses various interesting research problems in recent years. It can be remarked that fundamental knowledge and basic information attained from those investigations should provide an essential basis for the design procedure of related components. Theoretical modeling and simulations are ones among existing methodologies commonly employed to gain fundamental understanding of basic behaviors and responses of various physical problems. However, this particular methodology, when applied to investigate complex problems, requires not only physically admissible and very sophisticated mathematical models but also powerful and efficient solution procedures to capture the physical phenomena to the level of complexity involved.

Modeling of a physical body by an idealized and simplified mathematical domain occupying a half or full plane and a half or full space is standard and widely employed when excitations and responses of interest are only localized in a region with its length scale much smaller than the characteristic dimensions of the body. The influence of the remote boundary of a domain on the local responses is generally insignificant for this particular case and can, therefore, be discarded in the modeling without loss of accuracy of the predicted solutions. Such situations arise in many applications and certain cases are described here only to emphasize the common use of such idealization. In petroleum and reservoir applications, simulations of crack growth in hydraulic fracturing process are often achieved by assuming that fractures are embedded either in an infinite medium if the fracturing process is carried out deeply beneath the ground or in a semi-infinite medium if the process is carried out very near the ground surface. In the damage/fatigue/failure assessment of components and parts of structures and various devices based on the linear fracture mechanics concept, the modeling and analysis of pre-existing flaws to gain essential fracture data are essential steps. There are many situations encountered in practices where damages and flaws are induced relatively near the surface and their size is much smaller than the characteristic dimensions of the components and parts. A half plane and a half

pace containing cracks supplied by a proper set of governing physics are ones of mathematical models commonly utilized and found adequate for approximating those situations. In the modeling of materials with micro-structures, a dilute theory is one of fundamental theories often applied to study the effective properties of the materials containing a collection of defects, voids, inclusions or localized impurities. In such theory, fundamental problems associated with an isolated inhomogeneity embedded in an idealized infinite medium must be solved first and results can be further employed along with micromechanics-based concepts to post-process for quantities of interest. In the subsurface and near-surface modeling of flows in porous media, an idealized flow domain in terms of a semi-infinite and an infinite body is also often employed in the simulations of flow in a local region surrounding the wellbores, the influence of impermeable natural seals on the production rate, and flow in naturally fractured media.

Various analytical techniques such as integral transform methods, methods based on representation theories, and potential-function-based approaches have been proposed and used extensively to solve various types of boundary value problems with semi-infinite and infinite domains. However, their applications are very limited to either two-dimensional problems with a simple set of data or three-dimensional cases with extremely idealized settings. Such limitation becomes more apparent when the complexity of involved physical phenomena increases (e.g., complexity introduced by the complex material constitutive laws, singularities induced by the embedded discontinuities, general prescribed information such as boundary and loading conditions, etc.). To enhance the modeling capability, a variety of numerical procedures have continuously been developed and employed to solve full-plane, half-plane, full-space, and half-space problems. A group of domain-based techniques such as the finite element and finite difference methods has been well-established in the past several decades, successfully applied to solve numerous problems in various disciplines, and also well recognized as general, efficient, and powerful computational tools for the numerical analysis. However, when they are applied to solve unbounded-domain problems, the computational efficiency can be significantly degraded and they also exhibit some major shortcomings and require nontrivial treatments. For instance, a standard discretization procedure cannot directly be applied to infinite and semi-infinite domains. A domain truncation supplied by a set of proper remote boundary conditions is commonly employed to establish an approximate domain of finite dimensions prior to the discretization. It should be remarked that defining such suitable truncated surface and the corresponding boundary conditions remains the key issue and it can significantly influence the quality of approximate solutions. Another limited capability of those methods to attain sufficiently accurate results with reasonably cheap computational cost is apparent when they are applied to solve fracture-related problems. In the analysis, they generally require substantially fine meshes in a region surrounding the crack front in order to accurately capture the complex (singular) field and extract essential local fracture information such as the intensity factors and non-singular terms along the crack front.

Another group of numerical techniques based mainly on the boundary discretization such as boundary element integral equations methods (BIEMs) and scaled boundary finite element method (SBFEM) has been continuously developed and found an attractive alternative for the analysis of boundary value problems. The BIEMs have been proven efficient for modeling infinite media, semi-infinite media, and bodies with a large volume to surface ratio which contain embedded discontinuities, since their underlying formulation involves merely unknowns on the domain boundary and the discontinuity surfaces. This, as a result, not only significantly reduces the number of degrees of freedom resulting from the discretization but also renders the

flexibility in the mesh generation and adaptation. Similarly, the SBFEM possesses many advantages and salient features including that the discretization of the boundary is only required in the approximation; the key governing equation does not involve the fundamental solutions; the technique is convenient for the treatment of unbounded domains, material interfaces, and embedded discontinuities; and the offered solutions possess a semi-analytical characteristic. While the development of both BIEMs and SBFEM has been established in various contexts, their applications to the modeling of semi-infinite and infinite media made of multi-field materials have not well recognized and deserve further rigorous investigations.

2. Objectives

The key objectives of the proposed research are

- 2.1 to develop an efficient numerical procedure based on the scaled boundary element method to solve two-dimensional, semi-infinite and infinite bodies made of multi-field materials,
- 2.2 to develop an efficient numerical procedure based on the weakly singular boundary integral equation method to solve a three-dimensional, semi-infinite and infinite bodies made of multi-field materials, and
- 2.3 to fully investigate the computational performance including the accuracy, convergence, and the computational efficiency of the developed techniques.

3. Methodology and Procedure

Procedure and methodology for the proposed research project can be briefly summarized below.

- 3.1 An extensive literature survey is conducted to identify the most recent advances and state of the art in the numerical analysis by scaled boundary finite element methods and boundary integral equation methods, and to properly refine the research problem and scope of the present study.
- 3.2 Fundamental laws of conservations, linear constitutive laws and the laws of kinematics are used as the basis in the formulation of general basic field equations governing field quantities of interest for both two-dimensional and three-dimensional settings. The formulation will be established in a unified manner allowing the treatment of various classes of boundary value problems such as steady-state heat conduction problems, steady-state Darcy's flow problems, membrane problems, Laplace's equation, linear elasticity problems, and problems involving fully coupled materials such as linear piezoelectric, linear piezomagnetic and linear piezoelectromagnetic solids.
- 3.3 For the two-dimensional case, a numerical technique based on the scaled boundary finite element method will be developed following procedures and methodology indicated below.
 - 3.3.1 The standard weighted residual technique along with the integration by parts via Gauss-divergence theorem is employed to derive the weak-form equation.
 - 3.3.2 A scaled boundary coordinate transformation is utilized to describe the geometry of the two-dimensional domain and then the formulation is transformed to that involving the scaled boundary coordinates.
 - 3.3.3 Scaled boundary finite element approximation is introduced to approximate the weak-form equation. The geometry of the domain, the field quantities, and the weight function are interpolated from values or functions along the scaling coordinate direction using standard basis functions defined in the boundary coordinate direction.

- 3.3.4 Scaled boundary finite element equations are obtained from the weak-form via the use of scaled boundary finite element approximation along with the integration by parts with respect to the scaling coordinate. The original system of linear partial differential equations is transformed into a system of linear, non-homogeneous, second-order, ordinary differential equations and two sets of boundary conditions on the inner and outer boundaries.
- 3.3.5 A standard technique for solving the linear differential equations is utilized to determine the homogeneous solution and this process results in an equivalent linear eigenvalue problem. All eigenvalues and the corresponding eigenvectors are solved using a selected efficient numerical procedure such as a polynomial root search method.
- 3.3.6 The particular solution of the system of linear differential equations associated with the prescribed distributed body source and the prescribed state variable and the prescribed surface flux on the side faces is constructed via the well-known method of undetermined coefficients.
- 3.3.7 The final general solution is obtained by enforcing the boundary conditions on both inner and outer boundaries of the body. All field quantities of interest within the body can be post-processed using the nodal basis functions along with the approximate solution obtained along the scaling coordinate direction.
- 3.3.8 The proposed algorithm is to be implemented in terms of an in-house computer code and it will be tested by comparing computed results with existing analytical and reference solutions for various scenarios.
- 3.4 For the three-dimensional case, a numerical technique based on the boundary integral equation method is to be developed. A set of singularity-reduced boundary integral relations/equations for both the full space and the half space containing a surface discontinuity is established by taking the general, unified constitutive law into account. For the half-space case, a systematic regularization procedure is adopted to handle various types of boundary conditions on the free surface and to derive the completely regularized boundary integral relations/equations. A well-known, weakly singular, symmetric Galerkin boundary element method (SGBEM) is implemented to solve both the full-space and half-space problems. The procedure and methodology can be summarized below.
 - 3.4.1 Fundamental solutions for both the state variable and body flux are established for the full space using the Radon integral transform technique.
 - 3.4.2 For the full space, standard boundary integral relations for the state variable and body flux are developed by generalizing Somigliana's identity along with exploiting constitutive laws. For the half space under symmetrical and anti-symmetrical conditions on the free surface, standard boundary integral relations for both the state variable and body flux can be established by using existing fundamental solutions for the full space and properly exploiting the symmetrical and anti-symmetrical conditions.
 - 3.4.3 Singularity-reduced boundary integral relations and weakly singular boundary integral equations for the full space containing discontinuities are obtained by applying a systematic regularization procedure to general multi-field materials.
 - 3.4.4 A set of singularity-reduced boundary integral relations and weakly singular boundary integral equations for the half space containing discontinuities and subjected to symmetrical and anti-symmetrical conditions on the free surface is

- obtained using a similar regularization procedure together with symmetry and anti-symmetry conditions.
- 3.4.5 Results for the half-space subjected to other types of boundary conditions on the free surface can be constructed for certain cases using the method of superposition. The boundary value problems associated with the half-space without the discontinuities but subjected to prescribed, non-zero state variable or surface flux on its boundary can be solved using the method of potential theory and representations.
 - 3.4.6 A weakly singular SGBEM is employed to construct numerical solutions of the weakly singular, weak-form boundary integral equation for the surface flux for both full-space and half-space problems. In the discretization, special elements are used along the boundary of discontinuities to enhance the approximation of the near-front data. Special numerical quadrature based on the integrand regularization via a series of variable transformation is utilized to numerically evaluate both weakly singular and nearly singular double surface integrals. An interpolation technique is employed to compute weakly singular kernels for general multi-field materials.
 - 3.4.7 The sum of the state variables on the discontinuity surface is solved from the weakly singular, weak-form boundary integral equation for the state variable using standard Galerkin method.
 - 3.4.8 Explicit formulae are established to compute the intensity factors and non-singular term along the boundary of the discontinuity surface.
 - 3.4.9 The proposed algorithm is to be implemented in terms of an in-house computer code and it will be tested by comparing computed results with existing analytical and reference solutions for various scenarios.
- 3.5 The accuracy, convergence and computational efficiency of the developed techniques (both the SBFEM and weakly singular SGBEM) will be fully investigated. Comparisons of obtained results and performance of the proposed schemes with existing techniques such as standard finite element methods will also be considered.

4. Contribution

The proposed research offer alternative, accurate and computationally efficient numerical techniques (in terms of both fully tested procedures and in-house computer packages) capable of solving a variety of linear boundary value problems involving both two-dimensional and three-dimensional infinite and semi-infinite domains commonly encountered in the physical modeling. The key contribution and the novel aspect of the proposed work stem directly from the key formulation and implementations of the scaled boundary finite element method (SBFEM) and the weakly singular boundary integral equation method (BIEM), which will be established in a very general framework allowing many governing physics (e.g., phenomena governed by Laplace and generalized Laplace equation, linear elasticity problems with and without displacement discontinuities, problems associated with fully coupled, multi-field materials, etc.) and various scenarios (e.g., semi-infinite and infinite bodies, general external excitations, prescribed boundary conditions, embedded surface discontinuities, etc.) to be treated in a single, unified fashion. The full investigation of the computational performance of the proposed procedures should also offer their pros and cons in comparison with other existing domain-based numerical techniques (e.g., finite element and finite difference methods) and also provide a general guideline for selecting the SBFEM and BIEM as an efficient tool in the solution search

of a particular situation. In addition, the framework of the SBFEM and BIEM established in the present study provides a sufficient and essential basis for their generalization to treat more complex, more practical, and larger scale problems.

5. Publications

5.1 Published/accepted articles in international Journals

1. Nguyen C.V., **Rungamornrat J.**, Pheinsusom P. Scaled boundary finite element method for two-dimensional linear multi-field media. *Engineering Journal*, 2017, **21(7)**, pp.333-360. (ISI/SCOPUS, IF = NA, SJR = 0.194)
2. **Rungamornrat J.**, Nguyen T.B., Pich V.C., Phongtinnaboot W., Wijeyewickrema A.C. Generalized T-stress solutions for penny-shaped cracks in transversely isotropic piezoelectric media. *Engineering Fracture Mechanics*, 2018, **192**, pp. 225-241. (ISI/SCOPUS, IF = 2.908, SJR = 1.331)
3. Pham T.N., **Rungamornrat J.**, Pansuk W., Sato Y. Weakly singular BIEM for analysis of cracks embedded in symmetric elastic whole space. *Lowland Technology International*, 2018, **20(2)**, pp. 169-186. (SCOPUS, SJR = 0.192)
4. **Rungamornrat J.**, Nguyen V.C. Scaled boundary finite element method with exact defining curves for two-dimensional linear multi-field media. *Frontiers of Structural and Civil Engineering*, 2019, **13(1)**, pp. 201-214. (ISI/SCOPUS, IF = 1.272, SJR = 0.401)
5. **Rungamornrat J.**, Sukulthanasorn N., Mear M.E. Analysis for T-stress of cracks in 3D anisotropic elastic media by weakly singular integral equation method. *Computer Methods in Applied Mechanics and Engineering*, 2019, **347**, pp. 1004-1029. (ISI/SCOPUS, IF = 4.821, SJR = 2.996)
6. **Rungamornrat J.**, Chansavang B, Phongtinnaboot W, Nguyen VC. Investigation of generalized SIFs of cracks in 3D piezoelectric media under various crack-face conditions. *Frontiers of Structural and Civil Engineering*, **Accepted for publication**. (ISI/SCOPUS, IF = 1.272, SJR = 0.401)

5.2 Published articles in international conference proceedings

1. Limwibul V, Phongtinnaboot W, **Rungamornrat J.** Analysis for generalized intensity factors and T-stresses of permeable cracks in 3D linear piezoelectric media. Proceedings of the Fifteenth East Asia-Pacific Conference on Structural Engineering & Construction (EASEC-15), Xi'an, China, October 11-13, 2017
2. Sukulthanasorn N, **Rungamornrat J.**, Furukawa A. Efficient BIEM for analysis of SIFs and T-stresses for cracks in 3D linear elastic whole space. Proceedings of the Thirtieth KKHTCNN Symposium on Civil Engineering, Taiwan, November 2-4, 2017
3. Le TM, **Rungamornrat J.**, Bui TQ. Influence of surface and couple stresses on response of surface-loaded elastic half-plane. Proceedings of the 10th International Conference on Computational Methods (ICCM2019), Singapore, July 9-13, 2019
4. Pham BV, Nguyen TB, **Rungamornrat J.** Analysis of cracked body strengthened by adhesively bonded patches by BEM-FEM coupling. Proceedings of the 10th International Conference on Computational Methods (ICCM2019), Singapore, July 9-13, 2019

5.3 Published articles in national conference proceedings

1. Watanavit P, **Rungamornrat J**, Samittakorn W. Analysis of near interface cracks by weakly singular boundary integral equation method. Proceedings of the 22nd National Convention on Civil Engineering, Khao Yai, Nakhon Ratchasima, Thailand, July 18-20, 2017
2. Sengcompan C, **Rungamornrat J**. Calculations of SIFs for sub-interface cracks in bi-material. Proceedings of the 24th National Convention on Civil Engineering, Udonthani, Thailand, July 10-12, 2019
3. Wongviboonsin W, **Rungamornrat J**. Size-dependence elastic responses of layered media under surface loading. Proceedings of the 24th National Convention on Civil Engineering, Udonthani, Thailand, July 10-12, 2019

6. Budget

หมวดงบประมาณ	ปีที่ 1	ปีที่ 2	ปีที่ 2	รวม
1. หมวดค่าตอบแทน	240,000.00-	240,000.00-	240,000.00-	720,000.00-
- ค่าตอบแทนหัวหน้าโครงการ	240,000.00-	240,000.00-	240,000.00-	720,000.00-
2. หมวดค่าจ้าง	150,000.00-	170,000.00-	170,000.00-	490,000.00-
- ค่าจ้างผู้ช่วยวิจัย (2 คน)	120,000.00-	120,000.00-	120,000.00-	360,000.00-
- ค่าจ้างเหมาสืบทัน รวบรวม และ สรุปข้อมูล	30,000.00-	0.00-	0.00-	30,000.00-
- ค่าจ้างเหมาเตรียมข้อมูลและ ประมวลผลโปรแกรม	0.00-	50,000.00-	50,000.00-	100,000.00-
3. หมวดค่าวัสดุ	40,000.00-	40,000.00-	40,000.00-	120,000.00-
- ค่าอุปกรณ์เพิ่มความจำและ อุปกรณ์เก็บข้อมูล	10,000.00-	10,000.00-	10,000.00-	30,000.00-
- ค่าอุปกรณ์คอมพิวเตอร์	10,000.00-	10,000.00-	10,000.00-	30,000.00-
- ค่ากระดาษและหมึกพิมพ์	10,000.00-	10,000.00-	10,000.00-	30,000.00-
- ค่าอุปกรณ์สำนักงาน	10,000.00-	10,000.00-	10,000.00-	30,000.00-
4. หมวดค่าใช้สอย	30,000.00-	30,000.00-	30,000.00-	90,000.00-
- ค่าเดินทางและการประชุม วิชาการในประเทศ	10,000.00-	10,000.00-	10,000.00-	30,000.00-
- ค่าส่งไปรษณีย์และพัสดุภัณฑ์	5,000.00-	5,000.00-	5,000.00-	15,000.00-
- ค่าหนังสือ เอกสารอ้างอิง ค่า ถ่ายสำเนาและค่าส่งโทรสาร	10,000.00-	5,000.00-	5,000.00-	20,000.00-
- ค่าจัดทำรายงานและสิ่งตีพิมพ์	0.00-	5,000.00-	5,000.00-	10,000.00-
- ค่าใช้สอยเบ็ดเตล็ด	5,000.00-	5,000.00-	5,000.00-	15,000.00-
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CHAPTER 1

INTRODUCTION

This chapter briefly summarizes the key motivation and significance of the current investigation. The objectives, scope of work, and the methodology and research procedures are then clearly addressed. Finally, the contribution of the present study is summarized.

1.1 Motivation and Significance

For the past several decades, multi-functional, high-performance, coupled-field materials including advanced composites and smart materials (e.g., piezoelectric, piezomagnetic, and piezoelectromagnetic materials) have been considerably employed in various disciplines such as the aircraft and aerospace industries, medical applications, acoustic areas, nautical and civil engineering due to their remarkable properties and attractive fully-coupled effects. The key reason for using those advanced materials is to exploit their desirable inherent characteristics to further improve or enhance the performance of designed components and parts such as the increase in strength, stiffness, and strength to weight ratio, the improvement of durability and environmentally friendly characteristics, the satisfaction of multi-purpose usages, and the insurance of integrity and safety under severe service conditions. Besides their positive features, the behavior of this class of materials is complex and generally involves fully-coupled, multiple fields such as mechanical, electrical, and magnetic fields. This, as a direct consequence, renders responses of constituted components and parts under various scenarios increasingly complicated relative to those made from conventional materials and the ability to fully understand, model and predict such physical phenomena is quite challenging and poses various interesting research problems in recent years. It can be remarked that fundamental knowledge and basic information attained from those investigations should provide an essential basis for the design procedure of related components. Theoretical modeling and simulations are ones among existing methodologies commonly employed to gain fundamental understanding of basic behaviors and responses of various physical problems. However, this particular methodology, when applied to investigate complex problems, requires not only physically admissible and sophisticated mathematical models but also powerful and efficient solution procedures to capture the physical phenomena to the level of complexity involved.

Modeling of a physical body by an idealized and simplified mathematical domain occupying a half plane, a full plane, a half space, and a full space is standard and widely employed when excitations and responses of interest are only localized in a region with its length scale much smaller than the characteristic dimensions of the body. The influence of the remote boundary of a domain on the responses is generally insignificant for this particular case and can, therefore, be discarded in the modeling without loss of accuracy of the predicted solutions. Such situations arise in many applications and certain cases are described here only to emphasize the common use of such idealization. In petroleum and reservoir applications, simulations of crack growth in hydraulic fracturing process are often achieved by assuming that fractures are embedded either in an infinite medium if the fracturing process is carried out deeply beneath the ground or in a semi-infinite medium if the process is carried out very near the ground surface (e.g., Mendelsohn, 1984a,b; Yew, 1997). In the damage, fatigue and failure assessments of

components and parts of structures and various devices based on the linear fracture mechanics concept, the modeling and analysis of pre-existing flaws to gain essential fracture data are essential steps. There are many situations encountered in practices where damages and flaws are induced relatively near the surface and their size is much smaller than the characteristic dimensions of the components and parts. A half plane and a half space containing cracks supplied by a proper set of governing physics are ones of mathematical models commonly utilized and found adequate for approximating those situations (e.g., Murakami, 1985; Noguchi and Smith, 1995; Movchan and Willis, 2000; Kit et al., 2000). In the modeling of materials with microstructures, a dilute theory is one of fundamental theories often applied to study the effective properties of the materials containing a collection of defects, voids, inclusions or localized impurities. In such theory, fundamental problems associated with an isolated inhomogeneity embedded in an idealized infinite medium must be solved first and results can be further employed along with micromechanics-based concepts to post-process for quantities of interest (e.g., Mura, 1991; Nemat-Nasser and Hori, 1993). In the subsurface and near-surface modeling of flows in porous media, an idealized flow domain in terms of a semi-infinite and an infinite body is also often employed in the simulations of flow in a local region surrounding the wellbores, the influence of impermeable natural seals on the production rate, and flow in naturally fractured media (e.g., Lough et al. 1998; Rungtornnrat and Wheeler, 2006).

Various analytical techniques such as integral transform methods, methods based on representation theories, and potential-function-based approaches have been proposed and used extensively to solve various types of boundary value problems with semi-infinite and infinite domains (e.g., Srivastava and Singh, 1969; Mayrhofer and Fischer, 1989; Chen et al., 2000; Chen and Shioya, 2000). However, their applications are very limited to either two-dimensional problems with a simple set of data or three-dimensional cases with extremely idealized settings. Such limitation becomes more apparent when the complexity of involved physical phenomena increases (e.g., complexity introduced by the complex material constitutive laws, singularities induced by the embedded discontinuities, general prescribed information such as boundary and loading conditions, etc.).

To enhance the modeling capability, a variety of numerical procedures have continuously been developed and employed to solve full-plane, half-plane, full-space, and half-space problems. A group of domain-based techniques such as the finite element and finite difference methods has been well-established in the past several decades (e.g., Oden and Carey, 1984; Hughes, 2000; Zienkiewicz and Taylor, 2000), successfully applied to solve numerous problems in various disciplines, and also well recognized as general, efficient, and powerful computational tools for the numerical analysis. However, when they are applied to solve unbounded-domain problems, the computational efficiency can be significantly degraded and they also exhibit some major shortcomings and require nontrivial treatments. For instance, a standard discretization procedure cannot directly be applied to infinite and semi-infinite domains. A domain truncation supplied by a set of proper remote boundary conditions is commonly employed to establish an approximate domain of finite dimensions prior to the discretization. It should be remarked that defining such suitable truncated surface and the corresponding boundary conditions remains the key issue and it can significantly influence the quality of approximate solutions. Another limited capability of those methods to attain sufficiently accurate results with reasonably cheap computational cost is apparent when they are applied to solve fracture-related problems. In the analysis, they generally require substantially fine meshes in a region surrounding the crack front in order to accurately capture the complex (singular) field and extract essential local fracture

information such as the intensity factors and non-singular terms along the crack front (e.g., Swenson and Ingrassia, 1988; Martha et al., 1993; Ayhan et al., 2003).

Another group of numerical techniques based mainly on the boundary discretization such as boundary element integral equations methods (BIEMs) and scaled boundary finite element method (SFBEM) has been continuously developed and found an attractive alternative for the analysis of boundary value problems. The BIEMs have been proven efficient for modeling infinite media, semi-infinite media, and bodies with a large volume to surface ratio which contain embedded discontinuities, since their underlying formulation involves merely unknowns on the domain boundary and the discontinuity surfaces. This, as a result, not only significantly reduces the number of degrees of freedom resulting from the discretization but also renders the flexibility in the mesh generation and adaptation. Similarly, the SFBEM, which was originally developed by Wolf and Song (1995a, 1995b), possesses many advantages and salient features including that the discretization of the boundary is only required in the approximation; the key governing equation does not involve the fundamental solutions; the technique is convenient for the treatment of unbounded domains, material interfaces, and embedded discontinuities; and the offered solutions possess a semi-analytical characteristic. While the development of both BIEMs and SFBEM has been established in various contexts, their applications to the modeling of semi-infinite and infinite media made of multi-field materials have not well recognized and deserve further rigorous investigations.

1.2 Objectives

The key objectives of the proposed research are

1. to develop an efficient numerical procedure based on the scaled boundary element method to solve two-dimensional, semi-infinite and infinite bodies made of multi-field materials,
2. to develop an efficient numerical procedure based on the weakly singular boundary integral equation method to solve a three-dimensional, semi-infinite and infinite bodies made of multi-field materials, and
3. to fully investigate the computational performance including the accuracy, convergence, and the computational efficiency of the developed techniques.

1.3 Scope of Work

Scope of the present study and assumptions relevant to the development are summarized as follows:

1. all involved governing field equations and corresponding boundary value problems are linear;
2. only semi-infinite domains including the half plane and half space and infinite domains including the full plane and the full space are considered;
3. only static boundary value problems (i.e., all quantities and governing equations are time independent) are treated;
4. for a two-dimensional case, only domains completely described by a single defining curve and a single scaling center are considered;
5. for a three-dimensional case, only domains free of distributed body source and made of a homogeneous material are treated; and

6. for half-space problems, only four types of boundary conditions on the free surface including symmetrical, anti-symmetrical, zero state variable, and zero surface flux conditions are treated.

1.4 Methodology and Procedure

Some of fundamental theories, key methodology, research procedure and results obtained from the previous two TRF projects (Grant No. MRG5080309 and No. MRG5380159) are employed as the basis for the present investigation. Additional theories and solution procedures (e.g., mechanics of composites and smart media, representation theories based on potential functions, a standard weighted residual technique, Stokes' theorem, integration by parts, superposition method, scaled boundary coordinate transformation, eigenvalue analysis, method of undetermined coefficient, etc.) are also integrated to establish the proposed numerical techniques and also enhance their modeling capability. The procedure and methodology can be briefly summarized below.

1. An extensive literature survey is conducted to identify the most recent advances and state of the art in the numerical analysis by the scaled boundary finite element method and the boundary integral equation method and to properly refine the scope of the present study.
2. Fundamental laws of conservations (e.g., conservation of linear and angular momentum, conservation of mass, conservation of heat flow, etc.), linear constitutive laws (e.g., Darcy's law, Fourier's law, Hookes' law, generalized Hookes' law, etc) and the laws of kinematics (e.g., strain-displacement relations, electric potential-field relations, etc.) are used as the basis in the formulation of general basic field equations governing field quantities of interest for both two-dimensional and three-dimensional settings. The formulation is to be established in a unified manner allowing the treatment of various classes of boundary value problems such as steady-state heat conduction problems, steady-state Darcy's flow problems, membrane problems, Laplace's equation, linear elasticity problems, and problems involving fully coupled materials such as linear piezoelectric, linear piezomagnetic and linear piezoelectromagnetic solids.
3. For the two-dimensional case, a numerical technique based on the scaled boundary finite element method is to be developed following procedures and methodology indicated below.
 - 3.1 The standard weighted residual technique along with the integration by parts via Gauss-divergence theorem is employed to derive the weak-form equation.
 - 3.2 A scaled boundary coordinate transformation is utilized to describe the geometry of the two-dimensional domain and then the formulation is transformed to that involving the scaled boundary coordinates.
 - 3.3 Scaled boundary finite element approximation is introduced to approximate the weak-form equation. The geometry of the domain, the field quantities, and the weight function are interpolated from values or functions along the scaling coordinate direction using standard basis functions defined in the boundary coordinate direction.
 - 3.4 Scaled boundary finite element equations are obtained from the weak-form via the use of scaled boundary finite element approximation along with the integration by parts with respect to the scaling coordinate. The original system of linear partial differential equations is transformed into a system of linear, non-homogeneous, second-order, ordinary differential equations and two sets of boundary conditions on the inner and outer boundaries.

- 3.5 A standard technique for solving the linear differential equations is utilized to determine the homogeneous solution and this process results in an equivalent linear eigenvalue problem. All eigenvalues and the corresponding eigenvectors are solved using a selected efficient numerical procedure such as a polynomial root search method.
- 3.6 The particular solution of the system of linear differential equations associated with the prescribed distributed body source and the prescribed state variable and the prescribed surface flux on the side faces is constructed via the well-known method of undetermined coefficients.
- 3.7 The final general solution is obtained by enforcing the boundary conditions on both inner and outer boundaries of the body. All field quantities of interest within the body can be post-processed using the nodal basis functions along with the approximate solution obtained along the scaling coordinate direction.
- 3.8 The proposed algorithm is to be implemented in terms of an in-house computer code and it will be tested by comparing computed results with existing analytical and reference solutions for various scenarios.
4. For the three-dimensional case, a numerical technique based on the boundary integral equation method is to be developed. A set of singularity-reduced boundary integral relations/equations for both the full space and the half space containing a surface discontinuity to be established in the present study is the non-trivial generalization of results proposed by Li (1996) by taking the general, unified constitutive law into account. For the half-space case, a systematic regularization procedure proposed by Rungamornrat and Mear (2008a) and Rungamornrat and Senjuntichai (2009) is adopted along with techniques used by Li (1996) in handling various types of boundary conditions on the free surface to derive the completely regularized boundary integral relations/equations. A well-known, weakly singular, symmetric Galerkin boundary element method (SGBEM) as employed by Li et al. (1998) and Rungamornrat and Mear (2008b) is implemented to solve both full-space and half-space problems. The procedure and methodology can be summarized below.
 - 4.1 Fundamental solutions for both the state variable and body flux are established for the full space using the Radon integral transform technique.
 - 4.2 For the full space, standard boundary integral relations for the state variable and body flux can be developed by generalizing Somigliana's identity along with exploiting constitutive laws. For the half space under symmetrical and anti-symmetrical conditions on the free surface, standard boundary integral relations for both the state variable and body flux can be established by using existing fundamental solutions for the full space and properly exploiting the symmetrical and anti-symmetrical conditions.
 - 4.3 Singularity-reduced boundary integral relations and weakly singular boundary integral equations for the full space containing discontinuities are obtained by generalizing a systematic regularization procedure proposed by Rungamornrat and Mear (2008a) and Rungamornrat and Senjuntichai (2009) to general multi-field materials.
 - 4.4 By following a systematic regularization procedure suggested by Rungamornrat and Mear (2008a) and Rungamornrat and Senjuntichai (2009), a set of singularity-reduced boundary integral relations and weakly singular boundary integral equations for the half space containing discontinuities and subjected to symmetrical and anti-symmetrical conditions on the free surface can also be obtained. Specifically, special decompositions of both strongly and hyper-singular kernels are established first using existing results for the full space and symmetry and anti-symmetry conditions. The regularization procedure

is based mainly on the integration by parts to transfer the derivative of involved kernels appearing in the decompositions to the boundary data via Stokes' theorem.

- 4.5 Results for the half-space subjected to other types of boundary conditions on the free surface can be constructed for certain cases using the method of superposition similar to that employed by Li (1996) for the isotropic case. The boundary value problems associated with the half-space without the discontinuities but subjected to prescribed, non-zero state variable or surface flux on its boundary can be solved using the method of potential theory and representations. Again, a similar regularization procedure as that employed for symmetrical and anti-symmetrical cases is utilized to obtain the final singularity-reduced integral relations/equations.
- 4.6 a weakly singular SGBEM is employed to construct numerical solutions of the weakly singular, weak-form boundary integral equation for the surface flux for both full-space and half-space problems. In the discretization, special elements proposed by Li et al. (1998) and Rungamornrat and Mear (2008b) are used along the boundary of the discontinuities to enhance the approximation of the near-front data. Special numerical quadrature based on the integrand regularization via a series of variable transformation is utilized to numerically evaluate both weakly singular and nearly singular double surface integrals. An interpolation technique is employed to compute weakly singular kernels for general multi-field materials to reduce the computational cost associated with the direct evaluation of the closed contour integral for every pair of points.
- 4.7 the sum of the state variables on the discontinuity surface is solved from the weakly singular, weak-form boundary integral equation for the state variable using standard Galerkin method.
- 4.8 Explicit formula proposed by Rungamornrat and Mear (2008b) and Subsathaphol (2014) are generalized and used to compute the intensity factors and non-singular term along the boundary of the discontinuity surface.
- 4.9 The proposed algorithm is to be implemented in terms of an in-house computer code and it will be tested by comparing computed results with existing analytical and reference solutions for various scenarios
5. The accuracy, convergence and computational efficiency of the developed techniques (both the SBFEM and weakly singular SGBEM) will be fully investigated. Comparisons of obtained results and performance of the proposed schemes with existing techniques such as standard finite element methods will also be considered.

1.5 Contribution

The proposed research offer alternative, accurate and computationally efficient numerical techniques (in terms of both fully tested procedures and in-house computer packages) capable of solving a variety of linear boundary value problems involving both two-dimensional and three-dimensional infinite and semi-infinite domains commonly encountered in the physical modeling. The key contribution and the novel aspect of the proposed work stem directly from the key formulation and implementations of the scaled boundary finite element method (SBFEM) and the weakly singular boundary integral equation method (BIEM), which will be established in a very general framework allowing many governing physics (e.g., phenomena governed by Laplace and generalized Laplace equation, linear elasticity problems with and without displacement discontinuities, problems associated with fully coupled, multi-field materials, etc.)

and various scenarios (e.g., semi-infinite and infinite bodies, general external excitations, prescribed boundary conditions, embedded surface discontinuities, etc.) to be treated in a single, unified fashion. The full investigation of the computational performance of the proposed procedures should also offer their pros and cons in comparison with other existing domain-based numerical techniques (e.g., finite element and finite difference methods) and also provide a general guideline for selecting the SBFEM and BIEM as an efficient tool in the solution search of a particular situation. In addition, the framework of the SBFEM and BIEM established in the present study provides a sufficient and essential basis for their generalization to treat more complex, more practical, and larger scale problems.

1.6 Organization of Report

The remaining part of this report is organized into five chapters. Chapter 2 presents the development of a scaled boundary finite element method for second-order, linear, boundary value problems for multi-field media. Next, Chapter 3 summarizes the derivation a set of generalized T-stress solutions and the integral formula for calculations of generalized T-stress for a penny-shaped crack in linear piezoelectric whole space. The development of the weakly singular boundary integral equation method for analysis of cracks in a linear multi-field whole space is reported in Chapter 4. Next, the extension of the BIEM to handle cracks in a linear elastic half-space under various boundary conditions on the free surface is presented in Chapter 5. Finally, all significant findings and concluding remarks are addressed in Chapter 6.

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CHAPTER 2

SCALED BOUNDARY FINITE ELEMENT METHOD FOR MULTI-FIELD PROBLEMS

This chapter presents an efficient and accurate numerical technique, based on a scaled boundary finite element method (SBFEM) that is capable of solving two-dimensional, second-order, linear, multi-field boundary value problems. Basic governing equations are established in a general, unified context allowing the treatment of various classes of linear problems such as steady-state heat conduction problems, steady-state flow in porous media, linear elasticity, linear piezoelectricity, and linear piezomagnetic and piezoelectromagnetic problems. A scaled boundary finite element approximation is also formulated within a general framework integrating the influence of the distributed body source, general boundary conditions, contributions of the general side-face data, and the flexibility of scaled boundary approximations. Standard procedures for numerical integration, search of eigenvalues and eigenvectors, determination of particular solutions, and solving a system of linear algebraic equations are adopted. After fully tested with available benchmark solutions, the proposed SBFEM is applied to solve various classes of linear problems under different scenarios to demonstrate its vast capability, computational efficiency and robustness.

2.1 Background and Review

A scaled boundary finite element method (SBFEM) has been found an attractive alternative analysis tool in the modelling of various problems in applied mechanics for the past two decades [1-12]. The method is recognized as a semi-analytical technique combining features of both analytical schemes and the finite element approximation and, due to the reduction of a spatial dimension in the discretization, it can be also categorized as a particular type of boundary element methods (BEMs). However, the key difference between the SBFEM and conventional BEMs is that the former is generally free of fundamental solutions; as a direct consequence, the non-trivial numerical treatment of singular integrals is not required. The concept of the SBFEM was originally introduced by Wolf and Song [13-15] using the mechanical-based approach to properly and efficiently model linear elastic unbounded media in the dynamic analysis of soil-structure interactions. To further reduce the complexity of the formulation presented in the original work, other techniques such as the standard weighted residual procedure and the principle of virtual work were also utilized to obtain the scaled boundary finite element equations [16-18]. As a result of the computational efficiency of the SBFEM in modelling unbounded media and the reduction of the discretization cost, various researches have been continuously and increasingly conducted, since its first emergence, to introduce other novel aspects and further enhance its capability.

In recent years, the SBFEM has been also applied to solve various boundary value problems. One salient feature of the SBFEM is that the whole domain can be generated by scaling its boundary with respect to a single point termed the scaling center. The SBFEM only requires meshing on a representative boundary of the domain and does not involve fundamental

solutions. For a domain with the complex geometries, its interior can be discretized into sub-domains to satisfy the scaling requirement. For fracture problems, the scaling center is commonly located at the crack tip and, as a result, the stress field can be expressed analytically along the direction radiating from the crack tip. As a result, the strength of singularity and associated information can be directly and accurately calculated from the obtained solution [19, 20]. Based on this positive feature, the method has been extensively utilized in the investigation and simulations of fracture problems under various scenarios such as crack formation, static and dynamic crack propagation, and transient responses of bodies containing interfacial cracks, stress singularities and bi-material interfaces [21, 22]. In the modelling of wave propagations, the SBFEM has been also widely employed to study elastic guided waves in an unbounded domain and the radiation and diffraction of linear water waves in shallow water with sidewalls [23-25]. Results from those investigations demonstrated that the SBFEM generally yields more accurate numerical solutions and is more computationally efficient, in terms of the number of degrees of freedom involved, than other existing approaches. Li et al. [26] applied the SBFEM to investigate the structural behavior of offshore monopoles with the influence of the ocean wave loads. The basic concept of the SBFEM was adopted to formulate the governing monopole's equation and the analytical wave equation. Later, Li et al. [6, 7] extended the SBFEM to study three-dimensional wave-pile problems with the emphasis on the wave behavior and pile-group responses. In their formulation, the scaled boundary finite element approximation is applied to Helmholtz equation by separating the vertical component from the velocity potential. The two-dimensional SBFEM was adopted to examine the wave field near the free surface level. Results from these studies indicated that the SBFEM is an accurate and efficient computational tool for the analysis of wave problems and wave-structure interactions. However, their formulation and implementations were still limited to particular settings.

To further enhance the capability of the technique, Deeks and Wolf [27] also integrated an h -hierarchical adaptive procedure to the standard SBFEM. This technique exploited the ability of the SBFEM to model the stress singularities at the scaling center and to avoid the discretization of certain adjacent segments of the boundary. Later, Vu and Deeks [28] investigated the performance of high-order elements in the SBFEM. In their study, both the spectral element and hierarchical approaches were employed and they found that the spectral element approach was better than the hierarchical approach in terms of the computational efficiency. Deeks and Augarde [29] also incorporated the meshless scheme into the standard SBFEM to model the far-field problems. Although, the SBFEM has been successfully implemented in various applications, it also possesses certain drawbacks when the number of degrees of freedom resulting from the discretization becomes large, rendering the computational expenses substantial. To overcome such disadvantage, Vu and Deeks [30] integrated a p -adaptive scheme in the SBFEM for solving two dimensional boundary value problems. In this study, an alternative set of refinement criteria was considered to maximize the solution accuracy while minimizing the computational cost. Furthermore, He et al. [31] presented a new element-free Galerkin scaled boundary finite element method for the approximation in the circumferential direction. This particular technique was then applied to solve a number of standard linear elasticity problems and the technique was found to offer higher and better rate of convergence than the original SBFEM. Additionally, He et al. [32] investigated the possibility of using the Fourier shape functions in the SBFEM. The developed technique was then used to solve elastostatics and steady-state heat transfer problems. It was found that the accuracy and

convergence of numerical solutions were better than those of results obtained from using the polynomial shape functions and an element-free Galerkin scheme.

Applications of the SBFEM to the analysis of linear problems involving piezoelectric materials have also been recognized in the literature. Liu and Lin [4] used the SBFEM to investigate two-dimensional electrostatic problems. Li et al. [5] developed the SBFEM to solve two-dimensional, linear piezoelectric fracture problems. The stress and electric displacement intensity factors for both static and dynamic cases were calculated directly from results of the SBFEM. It was indicated that the technique requires no asymptotic solution, no local mesh refinement, and no special treatment around the crack tip. Recently, Dieringer and Becker [33] employed the SBFEM to investigate linear problems within the framework of a classical laminated plate theory. In their study, the scaled boundary finite element equations for composites were formulated in terms of the displacement, and the stress singularity at a notch was fully examined. They also demonstrated that the enhanced SBFEM can evaluate the singular stress field as an explicit function of the notch opening displacement. Li et al. [34] and Li et al. [11] also employed the SBFEM to perform two-dimensional simulations of dynamic cracks and interfacial cracks in piezoelectric composites and also study the influence of thermal loads on the fracture response. While the SBFEM has been applied successfully to solve linear piezoelectric problems, the underlying formulation and existing implemented procedures are still limited to certain scenarios and, in particular, the extension of the technique to treat other general coupled-field media has not been found in the literature.

Other recent applications of the SBFEM have been also recognized. For instance, Ooi et al. [9,10] developed an efficient procedure based on the SBFEM together with polygon elements for simulating dynamic crack propagation in elastic media; He et al. [35] presented the SBFEM for the numerical analysis of two-dimensional elastic bodies with the rotationally periodic symmetry and subjected to arbitrary loading conditions; and, most recently, Vu and Deeks [8] integrated the information of the fundamental solutions into the scaled boundary finite element method and then used the technique to investigate problems in linear elasticity with concentrated loads. As become evident from the vast amount of related publications, researches on both the novel development and the enhancement of existing SBFEM have continuously and increasingly grown. However, most of existing SBFEMs were developed specifically for problems considered and this, as a consequence, limits its generality and flexibility to treat general field problems under various scenarios when compared with standard finite element methods. A systematic generalization of the SBFEMs to treat a broader class of boundary value problems obviously requires further rigorous investigations.

The present study aims to offer the SBFEM capable of solving two-dimensional, linear, second-order, multi-field boundary value problems. The key and novel feature is that all basic field equations governing responses of interest are formulated in a general framework allowing various types of linear problems such as steady-state heat conduction and flow in porous media, Laplace's equation, linear elasticity, linear piezoelectricity and other coupled-field problems to be treated in a unified manner. In addition, the treatment of distributed body source, general boundary conditions, prescribed conditions on the side faces, and the scaled boundary approximations are integrated in the implementation.

2.2 Problem Formulation

This section summarizes the clear description of the problem, basic governing field equations for two-dimensional, linear multi-field problems, and the corresponding weak formulation. Then, the scale boundary coordinates transformations and the scale boundary finite element approximations are introduced. Finally, the scale boundary finite element equations are presented.

2.2.1 Problem description

Let us consider a two-dimensional body occupying a region Ω in $\mathbb{R}^2$ as shown schematically in Figure 2.1. The region is assumed *smooth* in the sense that all involved mathematical operators (e.g., integrations and differentiations) can be performed over this region. In addition, the boundary of the body Ω , denoted by $\partial\Omega$, is assumed piecewise smooth and an outward unit normal vector at any smooth point on $\partial\Omega$ is denoted by $\mathbf{n} = \{n_1 \ n_2\}^T$. The interior of the body is denoted by $\text{int } \Omega$. More restrictions about the geometry of the body Ω will be placed later in the following development of the key formulation.

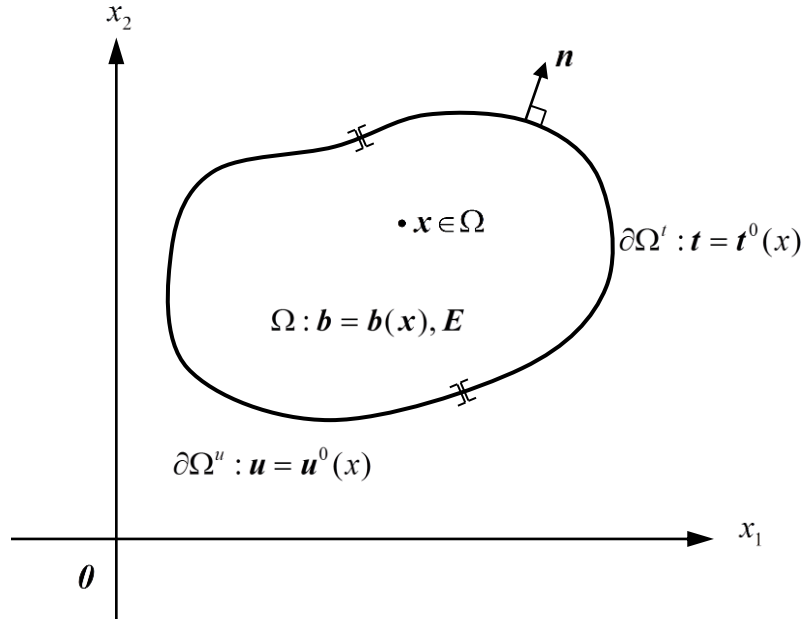


Figure 2.1 Schematic of two-dimensional, multi-field body subjected to external excitations

For convenience in further reference and presentations, a two-dimensional, reference Cartesian coordinate system $\{\theta; x_1, x_2\}$ is introduced where θ denotes the origin and x_1, x_2 denote the orthogonal coordinate axes (see Figure 2.1). A symbol $f_{,\alpha}$ is used throughout to denote a partial derivative of a function f with respect to the spatial coordinate x_α , $\alpha \in \{1, 2\}$ (i.e., $f_{,\alpha} = \partial f / \partial x_\alpha$) and, here and in what follows, standard indicial notations apply for subscripts

with lower case Greek and upper case indices. In particular, lower Greek subscripts range from 1 to 2 whereas upper case subscripts range from 1 to $\Lambda \in \{1, 2, 3, \dots\}$ and repeated subscripts imply the summation over their range unless stated otherwise.

The body is made of a homogeneous material with its behavior completely characterized by $4\Lambda^2$ constants denoted by a set $\{E_{\alpha I J \beta}\}$ and subjected to a prescribed distributed body source field at any point $\mathbf{x} \in \Omega$, denoted by an Λ -component vector $\mathbf{b} = \mathbf{b}(\mathbf{x}) = \{b_1(\mathbf{x}) \ b_2(\mathbf{x}) \ \dots \ b_\Lambda(\mathbf{x})\}^T$. In the present study, the constants $E_{\alpha I J \beta}$ are assumed to satisfy the following symmetry condition $E_{\alpha I J \beta} = E_{\beta J I \alpha}$. Responses of the body due to the applied distributed body source $\mathbf{b} = \mathbf{b}(\mathbf{x})$ are assumed to be completely described by the following three field quantities: the state variable $\mathbf{u} = \mathbf{u}(\mathbf{x})$, the state-variable gradient $\bar{\mathbf{e}} = \bar{\mathbf{e}}(\mathbf{x})$, and the body flux $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$. The state variable $\mathbf{u} = \mathbf{u}(\mathbf{x})$ contains Λ components denoted by $u_j(\mathbf{x})$ and is represented in a vector form by

$$\mathbf{u}(\mathbf{x}) = \{u_1(\mathbf{x}) \ u_1(\mathbf{x}) \ \dots \ u_\Lambda(\mathbf{x})\}^T \quad (2.1)$$

The state-variable gradient $\bar{\mathbf{e}} = \bar{\mathbf{e}}(\mathbf{x})$ and the body flux $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$ contain $2 \times \Lambda$ components denoted by $\bar{e}_{\alpha J}$ and $\sigma_{\alpha J}$, respectively and they can also be represented in a vector form by

$$\bar{\mathbf{e}}(\mathbf{x}) = \{\bar{e}_{11}(\mathbf{x}) \ \bar{e}_{12}(\mathbf{x}) \ \dots \ \bar{e}_{1\Lambda}(\mathbf{x}) \ \bar{e}_{21}(\mathbf{x}) \ \bar{e}_{22}(\mathbf{x}) \ \dots \ \bar{e}_{2\Lambda}(\mathbf{x})\}^T \quad (2.2)$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \{\sigma_{11}(\mathbf{x}) \ \sigma_{12}(\mathbf{x}) \ \dots \ \sigma_{1\Lambda}(\mathbf{x}) \ \sigma_{21}(\mathbf{x}) \ \sigma_{22}(\mathbf{x}) \ \dots \ \sigma_{2\Lambda}(\mathbf{x})\}^T \quad (2.3)$$

In addition, the surface flux at any smooth point on the boundary is denoted by an Λ -component vector $\mathbf{t} = \mathbf{t}(\mathbf{x}) = \{t_1(\mathbf{x}) \ t_2(\mathbf{x}) \ \dots \ t_\Lambda(\mathbf{x})\}^T$. The boundary of the given body $\partial\Omega$ can be decomposed into two disjoint portions; one is denoted by $\partial\Omega^u$ where the state variable $\mathbf{u}$ is fully prescribed ($\mathbf{u} = \mathbf{u}^0(\mathbf{x}) \ \forall \mathbf{x} \in \partial\Omega^u$ where $\mathbf{u}^0(\mathbf{x})$ is a prescribed vector) and the other is denoted by $\partial\Omega^t$ where the surface flux $\mathbf{t}$ is fully prescribed (i.e., $\mathbf{t} = \mathbf{t}^0(\mathbf{x}) \ \forall \mathbf{x} \in \partial\Omega^t$ where $\mathbf{t}^0(\mathbf{x})$ is a prescribed vector). In the present study, the prescribed vector value functions $\mathbf{b}(\mathbf{x})$, $\mathbf{u}^0(\mathbf{x})$ and $\mathbf{t}^0(\mathbf{x})$ are assumed sufficiently smooth to ensure the existence of the responses $\mathbf{u} = \mathbf{u}(\mathbf{x})$, $\bar{\mathbf{e}} = \bar{\mathbf{e}}(\mathbf{x})$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$.

The statement of the problem is to find the state variable $\mathbf{u} = \mathbf{u}(\mathbf{x})$, the state-variable gradient $\bar{\mathbf{e}} = \bar{\mathbf{e}}(\mathbf{x})$ and the body flux $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$ within the body Ω associated with the prescribed distributed body source $\mathbf{b} = \mathbf{b}(\mathbf{x})$, the given material properties $\{E_{\alpha I J \beta}\}$ and the prescribed boundary conditions $\mathbf{u}^0(\mathbf{x})$ and $\mathbf{t}^0(\mathbf{x})$ on $\partial\Omega^u$ and $\partial\Omega^t$, respectively

2.2.2 Basic governing field equations

A set of field equations governing all field quantities described in the problem description is formulated within a general framework to allow various classes of linear, second-order boundary value problems encountered in many areas (e.g., steady-state heat condition problems, steady-state flow in porous media, membrane problems, linear elasticity, problems associated with

multi-field materials such as piezoelectric and piezo-electromagnetic solids, etc.) to be treated in a unified manner. The integer Λ is used as a key parameter to indicate type of the problems. For instance, $\Lambda=1$ represents the heat condition problems, flow in porous media, and membrane problems and $\{\mathbf{u}(\mathbf{x}), \bar{\boldsymbol{\varepsilon}}(\mathbf{x}), \boldsymbol{\sigma}(\mathbf{x}), \mathbf{b}(\mathbf{x}), E_{\alpha I J \beta}, \mathbf{t}\}$ denotes {temperature, temperature gradient, heat flux, heat source, thermal conductivity, surface heat flux}, {fluid pressure, pressure gradient, fluid flux, source or sink, permeability, surface flux}, and {deflection, slopes, resultant shear force, distributed transverse load, membrane stiffness, end shear force}, respectively; $\Lambda=2$ represents linear elasticity problems and $\{\mathbf{u}(\mathbf{x}), \bar{\boldsymbol{\varepsilon}}(\mathbf{x}), \boldsymbol{\sigma}(\mathbf{x}), \mathbf{b}(\mathbf{x}), E_{\alpha I J \beta}, \mathbf{t}\}$ therefore denotes {displacement, displacement gradient, stress, body force, elastic constants, traction}; $\Lambda=3$ represents linear piezoelectric and piezo-magnetic problems and $\{\mathbf{u}(\mathbf{x}), \bar{\boldsymbol{\varepsilon}}(\mathbf{x}), \boldsymbol{\sigma}(\mathbf{x}), \mathbf{b}(\mathbf{x}), E_{\alpha I J \beta}, \mathbf{t}\}$ denotes {displacement and electric potential, displacement gradient and potential gradient, stress and electric induction, body force and charge, elastic and piezoelectric constants and dielectric permittivities, traction and surface charge} and {displacement and magnetic potential, displacement gradient and potential gradient, stress and magnetic induction, body force and magnetic body source, elastic and piezomagnetic constants and magnetic permeabilities, traction and surface magnetic induction}, respectively; and $\Lambda=4$ represents piezoelectromagnetic problems and $\{\mathbf{u}(\mathbf{x}), \bar{\boldsymbol{\varepsilon}}(\mathbf{x}), \boldsymbol{\sigma}(\mathbf{x}), \mathbf{b}(\mathbf{x}), E_{\alpha I J \beta}, \mathbf{t}\}$ denotes {displacement and electric and magnetic potentials, displacement gradient and potential gradients, stress and electric and magnetic inductions, body force and body charge and magnetic body source, elastic and piezoelectric and piezomagnetic and electromagnetic constants and dielectric permittivities and magnetic permeabilities, traction and surface charge and surface magnetic induction}.

The fundamental laws of conservation (e.g., conservation of linear and angular momentum, conservation of mass, conservation of heat flow, etc.), the linear constitutive laws (e.g., Darcy's law, Fourier's law, Hooke's law, generalized Hooke's law, etc), and the laws of kinematics (e.g., strain-displacement relations, electric potential-field relations, etc.) are employed to form the basic governing field equations and they are expressed in a single and concise form as indicated below. The body flux field $\boldsymbol{\sigma}(\mathbf{x})$ is related to the distributed body source field $\mathbf{b}(\mathbf{x})$ via a set of Λ -linear partial differential equations:

$$\sigma_{\alpha J, \alpha} + \mathbf{b}_J = 0 \quad \text{or} \quad \mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad (2.4)$$

where the superscript “ T ” denotes the transpose operator and $\mathbf{L}$ represents the linear differential operator defined, in terms of a $2\Lambda \times \Lambda$ -matrix, by

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_{\Lambda \times \Lambda} \\ \mathbf{0}_{\Lambda \times \Lambda} \end{bmatrix} \frac{\partial}{\partial x_1} + \begin{bmatrix} \mathbf{0}_{\Lambda \times \Lambda} \\ \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \frac{\partial}{\partial x_2} = \mathbf{L}_1 \frac{\partial}{\partial x_1} + \mathbf{L}_2 \frac{\partial}{\partial x_2}; \quad \mathbf{L}_1 = \begin{bmatrix} \mathbf{I}_{\Lambda \times \Lambda} \\ \mathbf{0}_{\Lambda \times \Lambda} \end{bmatrix}, \mathbf{L}_2 = \begin{bmatrix} \mathbf{0}_{\Lambda \times \Lambda} \\ \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \quad (2.5)$$

with $\mathbf{I}_{\Lambda \times \Lambda}$ and $\mathbf{0}_{\Lambda \times \Lambda}$ denoting a $\Lambda \times \Lambda$ -identity matrix and a $\Lambda \times \Lambda$ -zero matrix, respectively. The body flux $\boldsymbol{\sigma}(\mathbf{x})$ at any point $\mathbf{x}$ is directly related to the state-variable gradient $\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ at the same point via a local, linear constitutive law:

$$\sigma_{\alpha J} = E_{\alpha J K \beta} \bar{\varepsilon}_{\beta K} \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{D} \bar{\boldsymbol{\varepsilon}} \quad (2.6)$$

where $\mathbf{D}$ is a $2\Lambda \times 2\Lambda$ - termed matrix of the modulus matrix. It should be remarked that entries of the modulus matrix $\mathbf{D}$ can simply be obtained from the set $\{E_{\alpha IJ\beta}\}$ by properly considering the definition of the vectors $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\varepsilon}}$ (i.e., $E_{1JK1} = D_{JK}$, $E_{1JK2} = D_{J,K+\Lambda}$, $E_{2JK1} = D_{J+\Lambda,K}$ and $E_{2JK2} = D_{J+\Lambda,K+\Lambda}$) and, due to the symmetry of $E_{\alpha IJ\beta}$, the modulus matrix $\mathbf{D}$ is essentially symmetric. Finally, the state-variable gradient $\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ is related to the state variable $\mathbf{u}(\mathbf{x})$ by

$$\bar{\varepsilon}_{\beta K} = u_{K,\beta} \quad \text{or} \quad \bar{\boldsymbol{\varepsilon}} = \mathbf{L}\mathbf{u} \quad (2.7)$$

By applying the law of conservation at any smooth point $\mathbf{x}$ on the boundary $\partial\Omega$, the surface flux $\mathbf{t}(\mathbf{x})$ can be related to the body flux $\boldsymbol{\sigma}(\mathbf{x})$ and the outward unit normal vector $\mathbf{n}(\mathbf{x}) = \{n_1(\mathbf{x}) \ n_2(\mathbf{x})\}^T$ by

$$t_\alpha = \sigma_{\alpha J} n_\alpha \quad \text{or} \quad \mathbf{t} = [n_1 \mathbf{I} \ n_2 \mathbf{I}] \boldsymbol{\sigma} \quad (2.8)$$

where again $\mathbf{I}$ denotes the $\Lambda \times \Lambda$ -identity matrix. The relations (2.4), (2.6) and (2.7) constitute a set of 3Λ linear partial differential equations and 2Λ linear algebraic equations, involve 5Λ unknown functions (i.e., Λ components of the state variable, 2Λ components of the state-variable gradient, and 2Λ components of the body flux), and, as a result, form a complete set of governing equations for determining all field quantities. Other useful relations such as that directly relating the body flux and the state variable and one connecting the surface flux and the state variable on the boundary can readily be established. By combining (2.6) and (2.7), it simply leads to

$$\sigma_{\alpha J} = E_{\alpha JK\beta} u_{K,\beta} \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{D}(\mathbf{L}\mathbf{u}) \quad (2.9)$$

Then, by substituting (2.9) into (2.8), it yields

$$t_\alpha = n_\alpha E_{\alpha JK\beta} u_{K,\beta} \quad \text{or} \quad \mathbf{t} = [n_1 \mathbf{I} \ n_2 \mathbf{I}] \mathbf{D}(\mathbf{L}\mathbf{u}) \quad (2.10)$$

Now, the statement of the problem can be formulated mathematically by: find the state variable $\mathbf{u} = \mathbf{u}(\mathbf{x})$, the state-variable gradient $\bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ and the body flux $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$ that satisfies (2.4), (2.6)-(2.7) $\forall \mathbf{x} \in \text{int}\Omega$ and the boundary conditions $\mathbf{u}(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Omega^u$ and $\mathbf{t}(\mathbf{x}) = [n_1 \mathbf{I} \ n_2 \mathbf{I}] \boldsymbol{\sigma} = \mathbf{t}^0(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Omega^t$.

2.2.3 Weak formulation

The constitutive law established in a strong form (2.4) (i.e., in terms of partial differential equations) is then reformulated in a weak form well suited for the development of the scale boundary finite element equation presented further below. A standard weighted residual technique is adopted along with the integration by parts via the Gauss-divergence theorem to obtain the weak-form equation as follows.

By taking the inner product between (2.4) and any sufficiently smooth weight function $\mathbf{w}(\mathbf{x}) = \{w_1(\mathbf{x}) \ w_2(\mathbf{x}) \ ... \ w_\lambda(\mathbf{x})\}^T$ and then integrating the result over the body Ω , it leads to

$$\int_{\Omega} \mathbf{w}^T \mathbf{L}^T \boldsymbol{\sigma} dA + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA = 0 \quad (2.11)$$

By applying the identity $(w_J \sigma_{\alpha J})_{,\alpha} = \mathbf{w}^T \mathbf{L}^T \boldsymbol{\sigma} + (\mathbf{L}\mathbf{w})^T \boldsymbol{\sigma}$ to the first integral of (2.11), it yields

$$\int_{\Omega} (\mathbf{L}\mathbf{w})^T \boldsymbol{\sigma} dA = \int_{\Omega} (w_J \sigma_{\alpha J})_{,\alpha} dA + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA \quad (2.12)$$

From two-dimensional Gauss-divergence theorem, the first integral on the right hand side of (2.12) can be replaced by the boundary integral and the final result is given by

$$\int_{\Omega} (\mathbf{L}\mathbf{w})^T \boldsymbol{\sigma} dA = \int_{\partial\Omega} w_J (\sigma_{\alpha J} n_{\alpha}) dl + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA \quad (2.13)$$

where dl is an infinitesimal arc length. By applying the relation (2.8) to the boundary term, it finally yields

$$\int_{\Omega} (\mathbf{L}\mathbf{w})^T \boldsymbol{\sigma} dA = \int_{\partial\Omega} \mathbf{w}^T \mathbf{t} dl + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA \quad (2.14)$$

It is worth noting that the resulting weak-form (2.14) for the special case of linear elasticity is in fact the principle of virtual work with $\mathbf{w}(\mathbf{x})$ denoting the virtual displacement field. By further replacing the body flux $\boldsymbol{\sigma}$ in (2.14) by that associated with the state variable via the relation (2.9), it leads to

$$\int_{\Omega} (\mathbf{L}\mathbf{w})^T \mathbf{D}(\mathbf{L}\mathbf{u}) dA = \int_{\partial\Omega} \mathbf{w}^T \mathbf{t} dl + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA \quad (2.15)$$

It should be apparent from the above formulation that the weak-form equation (2.15) is valid for an arbitrary choice of the weight function $\mathbf{w}$. Only restriction placed to the weight function is the smoothness requirement to ensure the integrability of all integrals appearing in (2.15). This can be achieved by requiring the weight function and their first partial derivatives square integrable, i.e.

$$\int_{\Omega} [(\mathbf{L}\mathbf{w})^T (\mathbf{L}\mathbf{w}) + \mathbf{w}^T \mathbf{w}] dA < \infty \quad (2.16)$$

Now, let $\mathbf{W}$ be a space of all weight functions satisfying the condition (2.16). It can be readily verified that satisfaction of (2.4) and (2.9) $\forall \mathbf{x} \in \text{int } \Omega$ implies the weak statement (2.15) $\forall \mathbf{w} \in \mathbf{W}$. The converse is also valid if the data of the problem such as $\mathbf{D}$ and the distributed body source $\mathbf{b}$ is sufficiently smooth. Now, the problem statement can be formulated in terms of a weak-form equation by: find the state variable $\mathbf{u} = \mathbf{u}(\mathbf{x})$ such that the weak-form equation (2.15) is

satisfied $\forall \mathbf{w} \in \mathbf{W}$ and $\mathbf{u}(\mathbf{x}) = \mathbf{u}^0(\mathbf{x})$ is satisfied $\forall \mathbf{x} \in \partial\Omega^u$ and $\mathbf{t}(\mathbf{x}) = [\eta_1 I \quad \eta_2 I] \mathbf{D}(\mathbf{Lu}) = \mathbf{t}^0(\mathbf{x})$ is satisfied $\forall \mathbf{x} \in \partial\Omega^t$.

2.2.4 Scaled boundary coordinate transformation

Let $\mathbf{x}_0 = (x_{10}, x_{20})$ be a point in $\mathbf{R}^2$ and C be a simple, piecewise smooth curve in $\mathbf{R}^2$ parameterized by a function $\mathbf{r}: s \in [a, b] \rightarrow (x_{10} + \hat{x}_1(s), x_{20} + \hat{x}_2(s)) \in \mathbf{R}^2$ as shown in Figure 2.2. Let $\theta(s)$ be the circumferential angle of a point $\mathbf{r}(s)$ on the curve C measured from a straight line passing to $\mathbf{x}_0$ and $\mathbf{r}(a)$ to a straight line passing to $\mathbf{x}_0$ and $\mathbf{r}(s)$ (see Figure 2.2). The simple curve C considered here can be either closed (i.e., $\mathbf{r}(a) = \mathbf{r}(b)$) or not closed (i.e., $\mathbf{r}(a) \neq \mathbf{r}(b)$) and, in addition, it must not contain the point $\mathbf{x}_0$ and satisfies the condition $\theta \in [0, 2\pi]$ and $d\theta/ds > 0 \quad \forall s \in (a, b)$. Now, let us introduce the following coordinate transformation

$$\mathbf{x} = \mathbf{x}_0 + \xi \hat{\mathbf{x}}(s) \text{ or } x_\alpha = x_{\alpha 0} + \xi \hat{x}_\alpha(s) \quad (2.17)$$

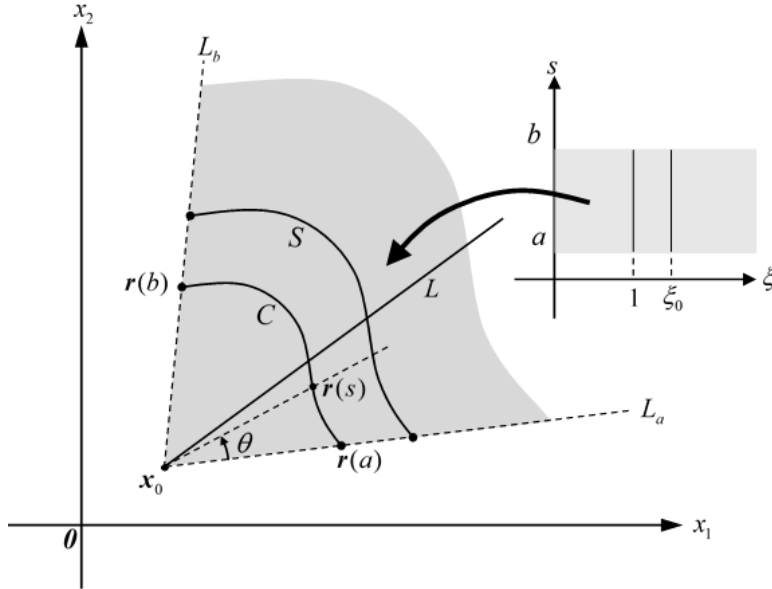


Figure 2.2: Schematic of a scaling center $\mathbf{x}_0$ and a defining curve C

where $\xi \geq 0$. It is evident from the coordinate transformation (2.17) that (i) any straight line $\xi = \xi_0, a \leq s \leq b$ in the $\xi - s$ plane is mapped to a curve S in the $x_1 - x_2$ plane which is simply a scaled version of the curve C about $\mathbf{x}_0$ and (ii) any straight line $\xi \geq 0, s = s_0 \in [a, b]$ in the $\xi - s$ plane is mapped to a semi-infinite straight line L in the $x_1 - x_2$ plane starting from $\mathbf{x}_0$ and passing through the point $\mathbf{r}(s_0)$ on the curve C (also see Figure 2.2). In addition, a straight line $\xi = 0, a \leq s \leq b$ in the $\xi - s$ plane is mapped to a single point $\mathbf{x}_0$ which is commonly termed the *scaling center* and a straight line $\xi = 1, a \leq s \leq b$ in the $\xi - s$ plane is mapped directly to the

curve C which is termed the *defining curve*. The coordinates ξ and s are termed the *scale boundary coordinates*. Clearly, the transformation (2.17) simply maps the region $\xi \geq 0, a \leq s \leq b$ in the $\xi - s$ plane into a region in the $x_1 - x_2$ plane bounded by the two straight lines L_a and L_b (i.e., a shaded region shown in Figure 2.2).

It is apparent from the coordinate mapping (2.17) that the differential line $d\xi = \{d\xi \ ds\}^T$ at any point (ξ, s) in the $\xi - s$ plane is related to the differential line $d\mathbf{x} = \{dx_1 \ dx_2\}^T$ at any point (x_1, x_2) in the $x_1 - x_2$ plane by

$$d\mathbf{x} = \mathbf{T} d\xi \quad (2.18)$$

where $\mathbf{T}$ is the Jacobian matrix of transformation given explicitly by

$$\mathbf{T} = \begin{bmatrix} \hat{x}_1 & \xi d\hat{x}_1/ds \\ \hat{x}_2 & \xi d\hat{x}_2/ds \end{bmatrix} \quad (2.19)$$

The inverse relation of (2.18) is given by

$$d\xi = \mathbf{T}^{-1} d\mathbf{x} = \mathbf{R} d\mathbf{x} \quad (2.20)$$

where the matrix $\mathbf{R}$ is the inverse of the Jacobian matrix $\mathbf{T}$ given by

$$\mathbf{R} = \frac{1}{J} \begin{bmatrix} d\hat{x}_2/ds & -d\hat{x}_1/ds \\ -\hat{x}_2/\xi & \hat{x}_1/\xi \end{bmatrix} \quad (2.21)$$

with J denoting

$$J = \hat{x}_1 \frac{d\hat{x}_2}{ds} - \hat{x}_2 \frac{d\hat{x}_1}{ds} \quad (2.22)$$

By setting $d\xi = (d\lambda)\mathbf{m}$ where $\mathbf{m}$ is a unit vector and $d\lambda$ is the length of $d\xi$, it can be readily verified that the length of $d\mathbf{x}$, denoted by dl , is given by

$$dl = \sqrt{dx_\alpha dx_\alpha} = \sqrt{T_{\alpha\beta} T_{\alpha\gamma} m_\beta m_\gamma} d\lambda \quad (2.23)$$

For following special cases (i) $\mathbf{m} = \{1 \ 0\}^T$, $d\lambda = d\xi$ and (ii) $\mathbf{m} = \{0 \ 1\}^T$, $d\lambda = ds$, the relation (2.23) reduces, respectively, to

$$dl = \sqrt{dx_\alpha dx_\alpha} = \sqrt{T_{\alpha\beta} T_{\alpha\gamma} m_\beta m_\gamma} d\lambda \quad (2.24)$$

$$dl = \sqrt{dx_\alpha dx_\alpha} = \sqrt{T_{\alpha\beta} T_{\alpha\gamma} m_\beta m_\gamma} d\lambda \quad (2.25)$$

Similarly, the differential area $d\xi ds$ at any point (ξ, s) in the $\xi - s$ plane can be related to the differential area dA at any point (x_1, x_2) in the $x_1 - x_2$ plane by

$$dA = J\xi d\xi ds \quad (2.26)$$

From the chain rule for differentiations, the partial derivative of any function with respect to the coordinate x_α can be related to those with respect to ξ and s via the following relation

$$\begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{Bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{d\hat{x}_2}{ds} & -\hat{x}_2 \\ -\frac{d\hat{x}_1}{ds} & \hat{x}_1 \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{1}{\xi} \frac{\partial}{\partial s} \end{Bmatrix} \quad (2.27)$$

The linear differential operator $\mathbf{L}$ given by (2.5) can now be expressed in terms of partial derivatives with respect to ξ and s by

$$\mathbf{L} = \mathbf{b}_1 \frac{\partial}{\partial \xi} + \mathbf{b}_2 \frac{1}{\xi} \frac{\partial}{\partial s} \quad (2.28)$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are $2\Lambda \times \Lambda$ -matrices defined by

$$\mathbf{b}_1 = \frac{1}{J} \left(\frac{d\hat{x}_2}{ds} \mathbf{L}_1 - \frac{d\hat{x}_1}{ds} \mathbf{L}_2 \right) = \frac{1}{J} \begin{bmatrix} \frac{d\hat{x}_2}{ds} \mathbf{I}_{\Lambda \times \Lambda} \\ -\frac{d\hat{x}_1}{ds} \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \quad (2.29)$$

$$\mathbf{b}_2 = \frac{1}{J} (\hat{x}_1 \mathbf{L}_2 - \hat{x}_2 \mathbf{L}_1) = \frac{1}{J} \begin{bmatrix} -\hat{x}_2 \mathbf{I}_{\Lambda \times \Lambda} \\ \hat{x}_1 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \quad (2.30)$$

Now, it is ready to pose the restriction on the geometry of the body Ω considered in the present investigation. For a given body Ω , there must exist the scaling center $\mathbf{x}_0$ and the defining curve C such that there exists a region $[\xi_1, \xi_2] \times [s_1, s_2]$ in the $\xi - s$ plane that is mapped to the region Ω in the $x_1 - x_2$ plane via the coordinate transformation (2.17). If the defining curve C is not closed, the body is said to be *opened* and portions of the boundary $\partial\Omega$ associated with $s = s_1$ and $s = s_2$ are termed the *side face* (see Figure 2.3). If the defining curve is closed, the body is said to be closed and the boundary $\partial\Omega$ contains no side face (see Figure 2.4). If ξ_1 is finite, the body Ω is said to be bounded (see Figures 2.3(a), 2.3(c), 2.4(a), 2.4(c)); otherwise (i.e., $\xi_2 = \infty$), it is said to be unbounded (see Figures 2.3(b), 2.3(d), 2.4(b), 2.4(d)). If $\xi_1 = 0$, the body Ω contains the scaling center $\mathbf{x}_0$ (see Figures 2.3(c), 2.3(d), 2.4(c), 2.4(d)); otherwise (i.e., $\xi_1 > 0$), the body Ω does not contain the scaling center $\mathbf{x}_0$ (see Figures 2.3(a), 2.3(b), 2.4(c), 2.4(d)). Portions of the boundary $\partial\Omega$ associated with $\xi = \xi_1 > 0$ and $\xi = \xi_2 < \infty$ are termed the inner and outer boundaries, respectively. It is remarked that for an unbounded body with the scaling center located outside, the inner boundary is commonly chosen as the defining curve (i.e., $\xi_1 = 1$)

whereas for a bounded body containing the scaling center, the outer boundary is chosen as the defining curve (i.e., $\xi_2 = 1$).

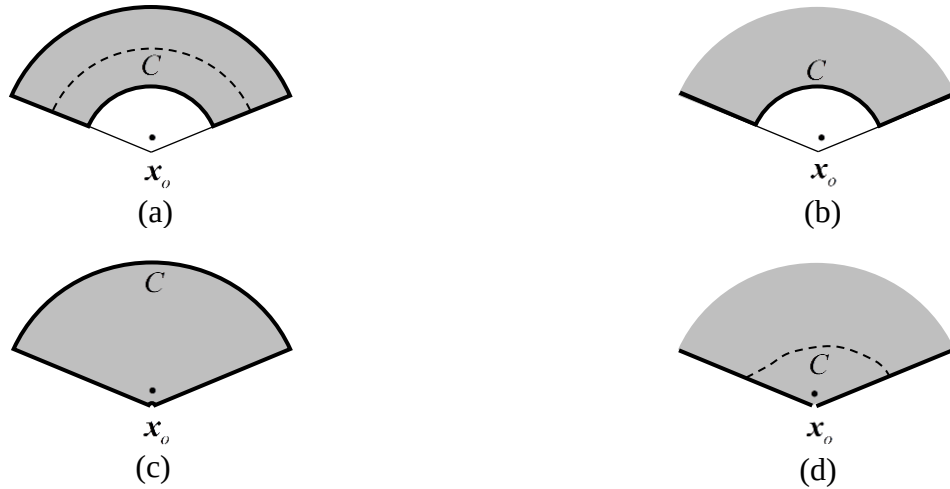


Figure 2.3: Examples of opened bodies: (a) bounded body containing no scaling center, (b) unbounded body containing no scaling center, (c) bounded body containing scaling center, and (d) unbounded body containing scaling center.

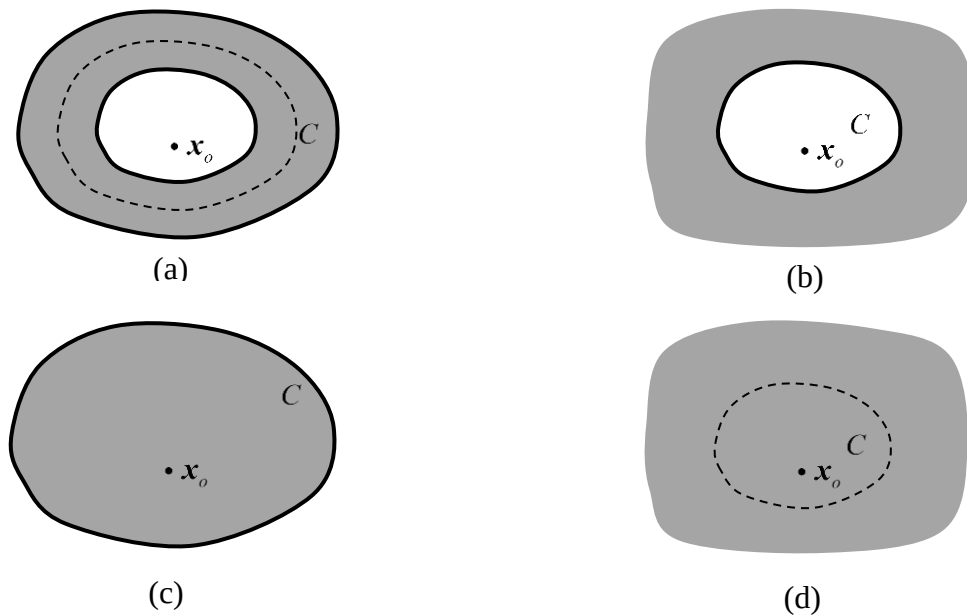


Figure 2.4: Examples of closed bodies: (a) bounded body containing a hole, (b) unbounded body containing a hole, (c) bounded body containing no hole, and (d) unbounded body containing no hole

2.2.5 Scaled boundary finite element approximation

In this section, three approximations, one associated with the geometry of the defining curve and the other two corresponding to the field quantities and the weight function, are introduced. The defining curve C is discretized into n elements with the total m nodes. The coordinates of points on C , denoted by $\mathbf{x} = \mathbf{x}_0 + \hat{\mathbf{x}}(s)$, is approximated by

$$\mathbf{x}_\alpha^h(s) = \mathbf{x}_{\alpha 0} + \hat{\mathbf{x}}_\alpha^h(s) = \mathbf{x}_{\alpha 0} + \sum_{i=1}^m \phi_{(i)}(s) \hat{\mathbf{x}}_{\alpha(i)} = \mathbf{x}_{\alpha 0} + \mathbf{N}^G \mathbf{X}_\alpha \quad (2.31)$$

where the superscript “ h ” is used to indicate the approximate quantities, $\phi_{(i)}(s)$ is the basis function associated with the i^{th} node, $\hat{\mathbf{x}}_{\alpha(i)} = \mathbf{x}_{\alpha(i)} - \mathbf{x}_{\alpha 0}$ denotes the coordinate of the i^{th} node relative to the scaling center $\mathbf{x}_0$, $\mathbf{N}^G = \{\phi_{(1)} \quad \phi_{(2)} \quad \cdots \quad \phi_{(m)}\}^T$ stands for a row-matrix containing all nodal basis functions, and $\mathbf{X}_\alpha = \{\hat{\mathbf{x}}_{\alpha(1)} \quad \hat{\mathbf{x}}_{\alpha(2)} \quad \cdots \quad \hat{\mathbf{x}}_{\alpha(m)}\}^T$ denotes a vector containing all nodal relative coordinates in which $\hat{\mathbf{x}}_{\alpha(i)} = \mathbf{x}_{\alpha(i)} - \mathbf{x}_{\alpha 0}$ represents the coordinate of the i^{th} node relative to the scaling center $\mathbf{x}_0$. It should be remarked that the global nodal basis function $\phi_{(i)}(s)$ is constructed in a standard fashion by simply patching the local element shape functions associated with the i^{th} node and, as a result, it satisfies the Kronecker-delta property, i.e., $\phi_{(i)}(s_{(j)}) = \delta_{ij}$ where $s_{(j)}$ is the value of the boundary coordinate s of the j^{th} node and δ_{ij} denotes the Kronecker-delta symbol. The resulting discretized (or approximate) defining curve is denoted by C^h and the region in the $x_1 - x_2$ plane described by the discretized defining curve C^h is then used as the approximation for the geometry of the body Ω and denoted by Ω^h . With the relation (2.31), the derivative $d\hat{\mathbf{x}}_\alpha/ds$ is then approximated by

$$\frac{d\hat{\mathbf{x}}_\alpha^h}{ds} = \mathbf{B}^G \mathbf{X}_\alpha \quad (2.32)$$

where $\mathbf{B}^G = d\mathbf{N}^G/ds = \{d\phi_{(1)}/ds \quad d\phi_{(2)}/ds \quad \cdots \quad d\phi_{(m)}/ds\}^T$. The approximations of J and the matrices $\mathbf{b}_1$ and $\mathbf{b}_2$ are given by

$$J^h = \mathbf{X}_1^T (\mathbf{N}^G)^T \mathbf{B}^G \mathbf{X}_2 - \mathbf{X}_2^T (\mathbf{N}^G)^T \mathbf{B}^G \mathbf{X}_1 \quad (2.33)$$

$$J^{\xi h} = \sqrt{\mathbf{X}_1^T (\mathbf{N}^G)^T \mathbf{N}^G \mathbf{X}_1 + \mathbf{X}_2^T (\mathbf{N}^G)^T \mathbf{N}^G \mathbf{X}_2} \quad (2.34)$$

$$J^{sh} = \sqrt{\mathbf{X}_1^T (\mathbf{B}^G)^T \mathbf{B}^G \mathbf{X}_1 + \mathbf{X}_2^T (\mathbf{B}^G)^T \mathbf{B}^G \mathbf{X}_2} \quad (2.35)$$

$$\mathbf{b}_1^h = \frac{1}{J} \begin{bmatrix} \mathbf{B}^G \mathbf{X}_2 \mathbf{I}_{\Lambda \times \Lambda} \\ -\mathbf{B}^G \mathbf{X}_1 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \quad (2.36)$$

$$\mathbf{b}_2^h = \frac{1}{J} \begin{bmatrix} -\mathbf{N}^G \mathbf{X}_2 \mathbf{I}_{\Lambda \times \Lambda} \\ \mathbf{N}^G \mathbf{X}_1 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \quad (2.37)$$

The approximate linear differential operator $\mathbf{L}$ is then given by

$$\mathbf{L}^h = \mathbf{b}_1^h \frac{\partial}{\partial \xi} + \mathbf{b}_2^h \frac{1}{\xi} \frac{\partial}{\partial s} \quad (2.38)$$

From the coordinate transformation (2.17) along with the approximation (2.32), the state variable $\mathbf{u}$ is now approximated by $\mathbf{u}^h$ in a form

$$\mathbf{u}^h = \mathbf{u}^h(\xi, s) = \sum_{i=1}^m \phi_{(i)}(s) \mathbf{u}_{(i)}^h(\xi) = \mathbf{N}^S \mathbf{U}^h \quad (2.39)$$

where $\mathbf{u}_{(i)}^h(\xi)$ denotes the value of the state variable along the line $s = s_{(i)}$ (i.e., a line passing through the scaling center and the i^{th} node), $\mathbf{N}^S = \{\phi_{(1)} \mathbf{I}_{\Lambda \times \Lambda} \quad \phi_{(2)} \mathbf{I}_{\Lambda \times \Lambda} \cdots \phi_{(m)} \mathbf{I}_{\Lambda \times \Lambda}\}$ is a $\Lambda \times m\Lambda$ -matrix containing all basis functions, and $\mathbf{U}^h = \{\mathbf{u}_{(1)}^h(\xi) \quad \mathbf{u}_{(2)}^h(\xi) \cdots \mathbf{u}_{(m)}^h(\xi)\}^T$, and $\mathbf{W}^h = \{\mathbf{w}_{(1)}^h(\xi) \cdots \mathbf{w}_{(m)}^h(\xi)\}^T$ denote vectors containing all functions $\mathbf{u}_{(i)}^h(\xi)$ and $\mathbf{w}_{(i)}^h(\xi)$, respectively. The approximation of the body flux σ at any point $\mathbf{x} \in \Omega^h$ in the $x_1 - x_2$ plane (or at any corresponding point (ξ, s) in the $\xi - s$ plane), denoted by σ^h , can be obtained directly by substituting (2.38) into the relation (2.9) along with the approximation (2.39) and the final result is given by

$$\sigma^h = \sigma^h(\xi, s) = \mathbf{D}(\mathbf{L}^h \mathbf{u}^h) = \mathbf{D} \left[\mathbf{b}_1^h \frac{\partial}{\partial \xi} + \mathbf{b}_2^h \frac{1}{\xi} \frac{\partial}{\partial s} \right] (\mathbf{N}^S \mathbf{U}^h) = \mathbf{D} \left[\mathbf{B}_1 \mathbf{U}_{,\xi}^h + \frac{1}{\xi} \mathbf{B}_2 \mathbf{U}^h \right] \quad (2.40)$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are given by

$$\mathbf{B}_1 = \mathbf{b}_1^h \mathbf{N}^S \quad (2.41)$$

$$\mathbf{B}_2 = \mathbf{b}_2^h \mathbf{B}^S \quad (2.42)$$

where $\mathbf{B}^S = d\mathbf{N}^S/ds = \{d\phi_{(1)}/ds \mathbf{I}_{\Lambda \times \Lambda} \quad d\phi_{(2)}/ds \mathbf{I}_{\Lambda \times \Lambda} \cdots d\phi_{(m)}/ds \mathbf{I}_{\Lambda \times \Lambda}\}$. It is evident that both the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are independent of the coordinate ξ . Similarly, the weight function $\mathbf{w}$ and its derivatives $\mathbf{L}\mathbf{w}$ can be approximated, in a similar fashion, by

$$\mathbf{w}^h = \mathbf{w}^h(\xi, s) = \sum_{i=1}^m \phi_{(i)}(s) \mathbf{w}_{(i)}^h(\xi) = \mathbf{N}^S \mathbf{W}^h \quad (2.43)$$

$$\mathbf{L}^h \mathbf{w}^h = \mathbf{L}^h \mathbf{w}^h(\xi, s) = \left[\mathbf{b}_1^h \frac{\partial}{\partial \xi} + \mathbf{b}_2^h \frac{1}{\xi} \frac{\partial}{\partial s} \right] (\mathbf{N}^S \mathbf{W}^h) = \left[\mathbf{B}_1 \mathbf{W}_{,\xi}^h + \frac{1}{\xi} \mathbf{B}_2 \mathbf{W}^h \right] \quad (2.44)$$

where $\mathbf{w}_{(i)}^h(\xi)$ denotes an arbitrary function of the coordinate ξ along the line $s = s_{(i)}$ and $\mathbf{W}^h = \{\mathbf{w}_{(1)}^h(\xi) \quad \mathbf{w}_{(2)}^h(\xi) \cdots \mathbf{w}_{(m)}^h(\xi)\}^T$ is a vector containing all functions $\mathbf{w}_{(i)}^h(\xi)$.

2.2.6 Scaled boundary finite element equations

In this section, a set of scaled boundary finite element equations is established for a generic, two-dimensional body Ω as shown in Figure 2.5. The boundary of the domain $\partial\Omega$ is assumed consisting of four parts resulting from the scale boundary coordinate transformation with the scaling center $\mathbf{x}_0$ and defining curve C : the inner boundary $\partial\Omega_1$, the outer boundary $\partial\Omega_2$, the side-face-1 $\partial\Omega_1^s$, and the side-face-2 $\partial\Omega_2^s$. The body is considered in this general setting to ensure that the resulting formulation is applicable to various cases. For certain special cases such as closed bodies without the side face, bodies containing the scaling center, and unbounded bodies, it simply takes $\partial\Omega_1^s = \partial\Omega_2^s = \emptyset$, $\partial\Omega_1^s = \emptyset$, and $\partial\Omega_2^s = \emptyset$, respectively, in the following formulation. The approximation of the given body Ω is achieved via the discretization of the defining curve C^h along with the mapping region $[\xi_1, \xi_2] \times [s_1, s_2]$ in the $\xi-s$ plane and the approximate body is denoted by Ω^h . In particular, the approximate inner boundary $\partial\Omega_1^h$, the approximate outer boundary $\partial\Omega_2^h$, the side-face-1 $\partial\Omega_1^s$, and the side-face-2 $\partial\Omega_2^s$ are fully described by a curve $\xi = \xi_1, s_1 \leq s \leq s_2$, a curve $\xi = \xi_2, s_1 \leq s \leq s_2$, a straight line $s = s_1, \xi_1 \leq \xi \leq \xi_2$, and a straight line $s = s_2, \xi_1 \leq \xi \leq \xi_2$, respectively.

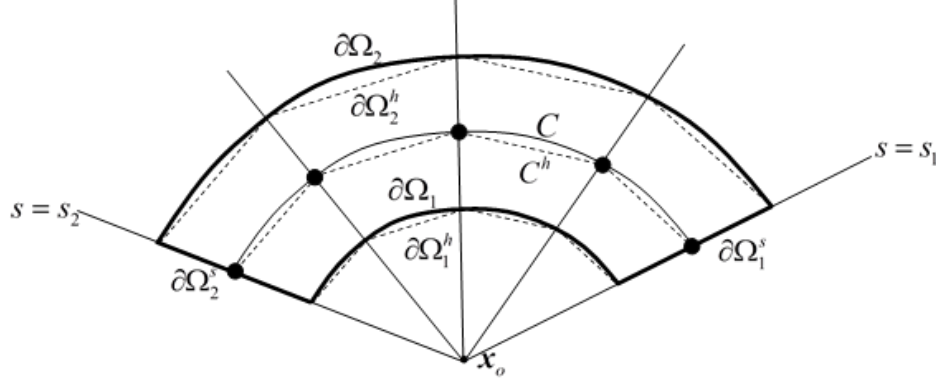


Figure 2.5: Schematic of a generic body Ω and its approximation Ω^h . The dashed lines are used to represent the approximation of the defining curve, the inner boundary and the outer boundary.

As a result of the boundary partition $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_1^s \cup \partial\Omega_2^s$, the weak-form equation (3.15) can be rewritten for the generic body Ω as

$$\int_{\Omega} (Lw)^T D(Lu) dA = \int_{\partial\Omega_1} w^T t dl + \int_{\partial\Omega_2} w^T t dl + \int_{\partial\Omega_1^s} w^T t dl + \int_{\partial\Omega_2^s} w^T t dl + \int_{\Omega} w^T b dA \quad (2.45)$$

By changing to the ξ, s -coordinates via the transformation (2.17) and using the relations (2.24)-(2.26), the weak-form (2.45) becomes

$$\begin{aligned}
\int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{L}\mathbf{w})^T \mathbf{D}(\mathbf{L}\mathbf{u}) J \xi d\xi ds &= \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} \mathbf{w}^T \mathbf{b} J \xi d\xi ds + \int_{s_1}^{s_2} \mathbf{w}_1^T \mathbf{t}_1(s) J^s(s) \xi_1 ds + \int_{s_1}^{s_2} \mathbf{w}_2^T \mathbf{t}_2(s) J^s(s) \xi_2 ds \\
&+ \int_{\xi_1}^{\xi_2} (\mathbf{w}_1^s)^T \mathbf{t}_1^s(\xi) J_1^\xi d\xi + \int_{\xi_1}^{\xi_2} (\mathbf{w}_2^s)^T \mathbf{t}_2^s(\xi) J_2^\xi d\xi
\end{aligned} \tag{2.46}$$

where $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_1^s$, $\mathbf{w}_2^s$ are restrictions of the weight function $\mathbf{w}$ on the boundaries $\partial\Omega_1, \partial\Omega_2$, $\partial\Omega_1^s, \partial\Omega_2^s$, respectively; $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_1^s$, $\mathbf{t}_2^s$ are surface flux on the boundaries $\partial\Omega_1, \partial\Omega_2, \partial\Omega_1^s, \partial\Omega_2^s$, respectively; and $J_1^\xi = J_\xi(s_1)$, $J_2^\xi = J_\xi(s_2)$. Next, by introducing the approximations of the body flux (2.40) and the derivatives of the weight function (2.44) along with the domain approximation, the integral on the left-hand side of (2.46), denoted for convenience by $\mathcal{J}_1$, becomes

$$\mathcal{J}_1 = \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} \left[\mathbf{B}_1 \mathbf{W}_{,\xi}^h \frac{\partial}{\partial \xi} + \frac{1}{\xi} \mathbf{B}_2 \mathbf{W}^h \right]^T \mathbf{D} \left[\mathbf{B}_1 \mathbf{U}_{,\xi}^h + \frac{1}{\xi} \mathbf{B}_2 \mathbf{U}^h \right] J^h \xi d\xi ds \tag{2.47}$$

By manipulating the involved matrix algebra, $\mathcal{J}_1$ can be expressed as

$$\begin{aligned}
\mathcal{J}_1 &= \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}_{,\xi}^h)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 \mathbf{U}_{,\xi}^h J^h \xi d\xi ds + \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}_{,\xi}^h)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 \mathbf{U}^h J^h d\xi ds \\
&+ \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 \mathbf{U}_{,\xi}^h J^h d\xi ds + \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 \mathbf{U}^h \frac{J^h}{\xi} d\xi ds
\end{aligned} \tag{2.48}$$

Further integrating the first two integrals by parts with respect to the coordinate ξ results in

$$\begin{aligned}
\mathcal{J}_1 &= - \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 (\xi \mathbf{U}_{,\xi}^h)_{,\xi} J^h d\xi ds - \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 \mathbf{U}_{,\xi}^h J^h d\xi ds \\
&+ \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 \mathbf{U}_{,\xi}^h J^h d\xi ds + \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 \mathbf{U}^h \frac{J^h}{\xi} d\xi ds \\
&+ \int_{s_1}^{s_2} (\mathbf{W}_2^h)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 \mathbf{U}_{,\xi}^h \Big|_{\xi=\xi_2} \xi_2 J^h ds - \int_{s_1}^{s_2} (\mathbf{W}_1^h)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 \mathbf{U}_{,\xi}^h \Big|_{\xi=\xi_1} \xi_1 J^h ds \\
&+ \int_{s_1}^{s_2} (\mathbf{W}_2^h)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 \mathbf{U}^h \Big|_{\xi=\xi_2} J^h ds - \int_{s_1}^{s_2} (\mathbf{W}_1^h)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 \mathbf{U}^h \Big|_{\xi=\xi_1} J^h d
\end{aligned} \tag{2.49}$$

where $\mathbf{w}_1^h = \mathbf{w}^h(\xi = \xi_1)$ and $\mathbf{w}_2^h = \mathbf{w}^h(\xi = \xi_2)$. By recalling that the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are independent of the coordinate ξ , $\mathbf{W}^h$ and $\mathbf{U}^h$ are independent of the coordinate s , and the matrix $\mathbf{D}$ is independent of both ξ and s , the integral $\mathcal{J}_1$ is simplified to

$$\begin{aligned} \mathcal{J}_1 = & \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \left[-\xi \mathbf{E}_0 \mathbf{U}_{,\xi\xi}^h + (\mathbf{E}_1 - \mathbf{E}_1^T - \mathbf{E}_0) \mathbf{U}_{,\xi}^h + \frac{1}{\xi} \mathbf{E}_2 \mathbf{U}^h \right] d\xi \\ & + (\mathbf{W}_2^h)^T \left\{ \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^h + \mathbf{E}_1^T \mathbf{U}^h \right\}_{\xi=\xi_2} - (\mathbf{W}_1^h)^T \left\{ \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^h + \mathbf{E}_1^T \mathbf{U}^h \right\}_{\xi=\xi_1} \end{aligned} \quad (2.50)$$

where the matrices $\mathbf{E}_0$, $\mathbf{E}_1$ and $\mathbf{E}_2$ are defined by

$$\mathbf{E}_0 = \int_{s_1}^{s_2} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 J^h ds \quad (2.51)$$

$$\mathbf{E}_1 = \int_{s_1}^{s_2} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 J^h ds \quad (2.52)$$

$$\mathbf{E}_2 = \int_{s_1}^{s_2} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 J^h ds \quad (2.53)$$

It is evident from (2.51) and (2.53) that both matrices $\mathbf{E}_0$ and $\mathbf{E}_2$ are symmetric, i.e., $(\mathbf{E}_0)^T = \mathbf{E}_0$ and $(\mathbf{E}_2)^T = \mathbf{E}_2$. Now, the boundary integrals appearing on the right-hand side of (2.46), denoted for convenience by $\mathcal{J}_2$, can also be approximated by

$$\begin{aligned} \mathcal{J}_2 = & (\mathbf{W}_1^h)^T \int_{s_1}^{s_2} (\mathbf{N}^S)^T \mathbf{t}_1(s) \xi_1 J^{sh}(s) ds + (\mathbf{W}_2^h)^T \int_{s_1}^{s_2} (\mathbf{N}^S)^T \mathbf{t}_2(s) \xi_2 J^{sh}(s) ds \\ & + \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T (\mathbf{N}_1^S)^T \mathbf{t}_1^s(\xi) J_1^{\xi h} d\xi + \int_{\xi_i}^{\xi_0} (\mathbf{W}^h)^T (\mathbf{N}_2^S)^T \mathbf{t}_2^s(\xi) J_2^{\xi h} d\xi \end{aligned} \quad (2.54)$$

where $\mathbf{N}_1^S = \mathbf{N}^S(s=s_1)$ and $\mathbf{N}_2^S = \mathbf{N}^S(s=s_2)$. By defining the following quantities

$$\mathbf{P}_1 = \int_{s_1}^{s_2} (\mathbf{N}^S)^T \mathbf{t}_1(s) \xi_1 J^{sh}(s) ds \quad (2.55)$$

$$\mathbf{P}_2 = \int_{s_1}^{s_2} (\mathbf{N}^S)^T \mathbf{t}_2(s) \xi_2 J^{sh}(s) ds \quad (2.56)$$

$$\mathbf{F}_1^t = \mathbf{F}_1^t(\xi) = (\mathbf{N}_1^S)^T \mathbf{t}_1^s(\xi) J_1^{\xi h} \quad (2.57)$$

$$\mathbf{F}_2^t = \mathbf{F}_2^t(\xi) = (\mathbf{N}_2^S)^T \mathbf{t}_2^s(\xi) J_2^{\xi h} \quad (2.58)$$

the integral $\mathcal{J}_2$ now becomes

$$\mathcal{J}_2 = (\mathbf{W}_1^h)^T \mathbf{P}_1 + (\mathbf{W}_2^h)^T \mathbf{P}_2 + \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \mathbf{F}_1^h d\xi + \int_{\xi_i}^{\xi_2} (\mathbf{W}^h)^T \mathbf{F}_2^h d\xi \quad (2.59)$$

Without loss of generality, the *first* node and the *last* node resulting from the discretization of opened bodies are taken as a node on the side-face-1 and a node on the side-face-2, respectively, and this applies in what follows. It should be remarked from the Kronecker property of the nodal basis functions that $\phi_{(1)}(s_1)=1$, $\phi_{(j)}(s_1)=0 \ \forall j \geq 2$ and $\phi_{(m)}(s_2)=1$, $\phi_{(j)}(s_2)=0 \ \forall j \leq m-1$. Now, both the matrices $\mathbf{N}_1^S$ and $\mathbf{N}_2^S$ clearly contain many zero entries and simply take the form

$$\mathbf{N}_1^S = \{\mathbf{I}_{\Lambda \times \Lambda} \quad \mathbf{0}_{\Lambda \times \Lambda} \quad \cdots \quad \mathbf{0}_{\Lambda \times \Lambda}\} \quad (2.60)$$

$$\mathbf{N}_2^S = \{\mathbf{0}_{\Lambda \times \Lambda} \quad \mathbf{0}_{\Lambda \times \Lambda} \quad \cdots \quad \mathbf{I}_{\Lambda \times \Lambda}\} \quad (2.61)$$

Substituting (2.60) into (2.57) and (2.61) into (2.58) leads to

$$\mathbf{F}_1^t = \mathbf{F}_1^t(\xi) = \{J_1^{\xi h} \mathbf{t}_1^s(\xi) \quad \mathbf{0} \quad \cdots \quad \mathbf{0}\}^T \quad (2.62)$$

$$\mathbf{F}_2^t = \mathbf{F}_2^t(\xi) = \{\mathbf{0} \quad \mathbf{0} \quad \cdots \quad J_2^{\xi h} \mathbf{t}_2^s(\xi)\}^T \quad (2.63)$$

where $\mathbf{0}$ is a zero Λ -component vector. Finally, the domain integral associated with the distributed body source on the right-hand side of (2.46), denoted by $\mathcal{J}_3$ can be approximated by

$$\mathcal{J}_3 = \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{N}^S \mathbf{W}^h)^T \mathbf{b} J^h \xi d\xi ds = \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \xi \mathbf{F}^b d\xi \quad (2.64)$$

where the matrix $\mathbf{F}^b$ is defined by

$$\mathbf{F}^b = \mathbf{F}^b(\xi) = \int_{s_1}^{s_2} (\mathbf{N}^S)^T \mathbf{b} J ds = \left\{ \int_{s_1}^{s_2} \phi_{(1)} \mathbf{b} J^h ds \quad \int_{s_1}^{s_2} \phi_{(2)} \mathbf{b} J^h ds \quad \cdots \quad \int_{s_1}^{s_2} \phi_{(m)} \mathbf{b} J^h ds \right\}^T \quad (2.65)$$

By combining the results (2.50), (2.59) and (2.64), the weak-form (2.46) is approximated by

$$\begin{aligned} & \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \left[-\xi \mathbf{E}_0 \mathbf{U}_{,\xi\xi}^h + (\mathbf{E}_1 - \mathbf{E}_1^T - \mathbf{E}_0) \mathbf{U}_{,\xi}^h + \frac{1}{\xi} \mathbf{E}_2 \mathbf{U}^h - \mathbf{F}_t - \xi \mathbf{F}_b \right] d\xi \\ & + (\mathbf{W}_2^h)^T \left[\left\{ \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^h + \mathbf{E}_1^T \mathbf{U}^h \right\}_{\xi=\xi_2} - \mathbf{P}_2 \right] - (\mathbf{W}_1^h)^T \left[\left\{ \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^h + \mathbf{E}_1^T \mathbf{U}^h \right\}_{\xi=\xi_1} + \mathbf{P}_1 \right] = 0 \end{aligned} \quad (2.66)$$

where $\mathbf{F}^t = \mathbf{F}_1^t + \mathbf{F}_2^t$. From the arbitrariness of the weight function $\mathbf{W}^h$, it can be deduced that

$$\xi^2 \mathbf{E}_0 \mathbf{U}_{,\xi\xi}^h + \xi (\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1) \mathbf{U}_{,\xi}^h - \mathbf{E}_2 \mathbf{U}^h + \xi \mathbf{F}^t + \xi^2 \mathbf{F}^b = \mathbf{0} \quad \forall \xi \in (\xi_1, \xi_2) \quad (2.67)$$

$$\mathbf{Q}^h(\xi_2) = \mathbf{P}_2 \quad (2.68)$$

$$\mathbf{Q}^h(\xi_1) = -\mathbf{P}_1 \quad (2.69)$$

where the vector $\mathbf{Q}^h = \mathbf{Q}^h(\xi)$ commonly known as the nodal internal flux is defined by

$$\mathbf{Q}^h(\xi) = \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^h + \mathbf{E}_1^T \mathbf{U}^h \quad (2.70)$$

Equations (2.67)-(2.69) form a set of the so-called scaled boundary finite element equations governing the function $\mathbf{U}^h = \mathbf{U}^h(\xi)$. It can be remarked that (2.67) forms a system of linear, second-order, nonhomogeneous, ordinary differential equations with respect to the coordinate ξ whereas (2.68) and (2.69) pose the boundary conditions on the inner and outer boundaries of the body. As previously pointed out, the governing equations (2.67)-(2.69) are formulated in a general context; as a result, for certain special cases, some terms or equations must be eliminated properly. For instances, the boundary condition (2.68) is ignored for unbounded bodies; the boundary condition (2.69) is ignored for bounded bodies containing the scaling center; the term $\mathbf{F}^t$ vanishes for closed bodies; and the term $\mathbf{F}^b$ vanishes for bodies free of distributed body source.

Now, the statement of an approximate problem can be formulated as follows: find the function $\mathbf{U}^h = \mathbf{U}^h(\xi)$ such that a system of linear ordinary differential equations (2.67) is satisfied $\forall \xi \in (\xi_1, \xi_2)$ and boundary conditions (2.68) and (2.69) are satisfied.

2.2.7 Treatment of prescribed conditions on side faces

It should be evident from (2.67)-(2.69) that the information associated with the prescribed distributed body source and the prescribed boundary conditions on both inner and outer boundaries can be integrated into the formulation via the term $\mathbf{F}^b$ and the conditions (2.68) and (2.69), respectively. However, the consideration of the prescribed conditions (either the surface flux or state variable) on the side face is still not apparent from the current formulation. To circumvent this situation, the system of linear differential equations (2.67) is further re-expressed in a form well-suited for the treatment of the prescribed state variable and surface flux on the side face.

First, bodies considered in the present investigation can be divided into five groups as follows: *Group-1* corresponding to closed bodies without the side-face, *Group-2* corresponding to opened bodies with prescribed surface flux on both side faces, *Group-3* corresponding to opened bodies with prescribed surface flux on the side-face-1 and prescribed state variable on the side-face-2, *Group-4* corresponding to opened bodies with prescribed surface flux on the side-face-2 and prescribed state variable on the side-face-1, and *Group-5* corresponding to opened bodies with prescribed state variable on both side faces. It should be evident that for bodies in *Group-1*, $\mathbf{U}^h$ contains only unknown functions and $\mathbf{t}_1^s, \mathbf{t}_2^s$ do not exist or, equivalently, vanish; for bodies in *Group-2*, $\mathbf{U}^h$ contains only unknown functions and $\mathbf{t}_1^s, \mathbf{t}_2^s$ are fully prescribed; for bodies in *Group-3*, $\mathbf{t}_1^s$ and $\mathbf{u}_{(m)}^h(\xi)$ are fully prescribed and $\mathbf{t}_2^s$ and the remaining functions $\mathbf{u}_{(i)}^h(\xi) \forall i \leq m-1$ in $\mathbf{U}^h$ are unknown; for bodies in *Group-4*, $\mathbf{t}_2^s$ and $\mathbf{u}_{(1)}^h(\xi)$ are fully prescribed and $\mathbf{t}_1^s$, and the remaining functions $\mathbf{u}_{(i)}^h(\xi) \forall i \geq 2$ in $\mathbf{U}^h$ are unknown; and for bodies in *Group-5*, both $\mathbf{u}_{(1)}^h(\xi)$ and $\mathbf{u}_{(m)}^h(\xi)$ are fully prescribed whereas $\mathbf{t}_1^s, \mathbf{t}_2^s$, and the remaining functions $\mathbf{u}_{(i)}^h(\xi) \forall 2 \leq i \leq m-1$ in $\mathbf{U}^h$ are unknown. To treat the prescribed conditions on the side-face for

all groups of bodies as a whole, the vector $\mathbf{U}^h$ is first partitioned and rearranged into known and unknown parts as $\mathbf{U}^h = \{\mathbf{U}^{hu} \quad \mathbf{U}^{hc}\}^T$ where $\mathbf{U}^{hu} = \mathbf{U}^{hu}(\xi)$ contains only unknown functions from a collection $\mathbf{u}_{(i)}^h(\xi), i=1,2,...,m$ and $\mathbf{U}^{hc} = \mathbf{U}^{hc}(\xi)$ contains the remaining known functions associated with the prescribed state variable on the side face. By denoting p the number of known functions $\mathbf{u}_{(i)}^h(\xi)$ contained in $\mathbf{U}^{hc}$, the number of remaining unknown functions $\mathbf{u}_{(i)}^h(\xi)$ contained in $\mathbf{U}^{hu}$ is then equal to $m-p$. Clearly, the value of p associated with bodies in *Group-1*, *Group-2*, *Group-3*, *Group-4*, and *Group-5* are 0, 0, 1, 1, and 2 respectively. Consistent with the partition of the vector $\mathbf{U}^h$, the vector $\mathbf{F}^t$ can also be partitioned into $\mathbf{F}^t = \{\mathbf{F}^{tu} \quad \mathbf{F}^{tc}\}^T$ where $\mathbf{F}^{tu} = \mathbf{F}^{tu}(\xi)$ contains many $\mathbf{0}$ functions and known functions associated with prescribed surface flux on the side face and has the same dimension as that of $\mathbf{U}^{hu}$ and $\mathbf{F}^{tc} = \mathbf{F}^{tc}(\xi)$ contains unknown functions associated with the unknown surface flux on the side face and has the same dimension as that of $\mathbf{U}^{hc}$. According to this partition, the system of differential equations (2.67) can be expressed as

$$\begin{aligned} \xi^2 \begin{bmatrix} \mathbf{E}_0^{uu} & \mathbf{E}_0^{uc} \\ (\mathbf{E}_0^{uc})^T & \mathbf{E}_0^{cc} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{,\xi\xi}^{hu} \\ \mathbf{U}_{,\xi\xi}^{hc} \end{Bmatrix} + \xi \begin{bmatrix} \mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} & \mathbf{E}_0^{uc} + (\mathbf{E}_1^{cu})^T - \mathbf{E}_1^{uc} \\ (\mathbf{E}_0^{uc})^T + (\mathbf{E}_1^{uc})^T - \mathbf{E}_1^{cu} & \mathbf{E}_0^{cc} + (\mathbf{E}_1^{cc})^T - \mathbf{E}_1^{cc} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{,\xi}^{hu} \\ \mathbf{U}_{,\xi}^{hc} \end{Bmatrix} \\ - \begin{bmatrix} \mathbf{E}_2^{uu} & \mathbf{E}_2^{uc} \\ (\mathbf{E}_2^{uc})^T & \mathbf{E}_2^{cc} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{hu} \\ \mathbf{U}^{hc} \end{Bmatrix} + \xi \begin{Bmatrix} \mathbf{F}^{tu} \\ \mathbf{F}^{tc} \end{Bmatrix} + \xi^2 \begin{Bmatrix} \mathbf{F}^{bu} \\ \mathbf{F}^{bc} \end{Bmatrix} = \mathbf{0} \end{aligned} \quad (2.71)$$

Similarly, equation (2.70) for the nodal internal flux can be expressed, in a partitioned form, as

$$\begin{Bmatrix} \mathbf{Q}^{hu} \\ \mathbf{Q}^{hc} \end{Bmatrix} = \xi \begin{bmatrix} \mathbf{E}_0^{uu} & \mathbf{E}_0^{uc} \\ (\mathbf{E}_0^{uc})^T & \mathbf{E}_0^{cc} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{,\xi}^{hu} \\ \mathbf{U}_{,\xi}^{hc} \end{Bmatrix} + \begin{bmatrix} (\mathbf{E}_1^{uu})^T & (\mathbf{E}_1^{cu})^T \\ (\mathbf{E}_1^{uc})^T & (\mathbf{E}_1^{cc})^T \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{hu} \\ \mathbf{U}^{hc} \end{Bmatrix} \quad (2.72)$$

Equation (2.71) can be separated into two systems:

$$\xi^2 \mathbf{E}_0^{uu} \mathbf{U}_{,\xi\xi}^{hu} + \xi \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] \mathbf{U}_{,\xi}^{hu} - \mathbf{E}_2^{uu} \mathbf{U}^{hu} = -\xi \mathbf{F}^{tu} - \xi^2 \mathbf{F}^{bu} - \mathbf{F}^{suu} \quad (2.73)$$

$$\xi \mathbf{F}^{tc} = -\xi^2 \mathbf{F}^{bc} - \mathbf{F}^{suc} - \mathbf{F}^{scc} \quad (2.74)$$

where the vectors $\mathbf{F}^{suu}$, $\mathbf{F}^{suc}$, and $\mathbf{F}^{scc}$ are defined by

$$\mathbf{F}^{suu} = \xi^2 \mathbf{E}_0^{uc} \mathbf{U}_{,\xi\xi}^{hc} + \xi (\mathbf{E}_0^{uc} + (\mathbf{E}_1^{cu})^T - \mathbf{E}_1^{uc}) \mathbf{U}_{,\xi}^{hc} - \mathbf{E}_2^{uc} \mathbf{U}^{hc} \quad (2.75)$$

$$\mathbf{F}^{suc} = \xi^2 (\mathbf{E}_0^{uc})^T \mathbf{U}_{,\xi\xi}^{hu} + \xi \left[(\mathbf{E}_0^{uc})^T + (\mathbf{E}_1^{uc})^T - \mathbf{E}_1^{cu} \right] \mathbf{U}_{,\xi}^{hu} - (\mathbf{E}_2^{uc})^T \mathbf{U}^{hu} \quad (2.76)$$

$$\mathbf{F}^{scc} = \xi^2 \mathbf{E}_0^{cc} \mathbf{U}_{,\xi\xi}^{hc} + \xi \left[\mathbf{E}_0^{cc} + (\mathbf{E}_1^{cc})^T - \mathbf{E}_1^{cc} \right] \mathbf{U}_{,\xi}^{hc} - \mathbf{E}_2^{cc} \mathbf{U}^{hc} \quad (2.77)$$

It can be remarked that $\mathbf{F}^{suu}$ and $\mathbf{F}^{scc}$ are known vectors obtained from the prescribed state variable on the side face whereas $\mathbf{F}^{suc}$ is an unknown vector involving the unknown vector $\mathbf{U}^{hu}$. By following the same procedure, (2.72) can be divided into two systems:

$$\mathbf{Q}^{hu}(\xi) = \xi \mathbf{E}_0^{uu} \mathbf{U}_{,\xi}^{hu} + (\mathbf{E}_1^{uu})^T \mathbf{U}^{hu} + \mathbf{Q}^{huc}(\xi) \quad (2.78)$$

$$\mathbf{Q}^{hc}(\xi) = \xi (\mathbf{E}_0^{uc})^T \mathbf{U}_{,\xi}^{hu} + (\mathbf{E}_1^{uc})^T \mathbf{U}^{hu} + \mathbf{Q}^{hcc}(\xi) \quad (2.79)$$

where the known vectors $\mathbf{Q}^{huc}(\xi)$ and $\mathbf{Q}^{hcc}(\xi)$ are defined by

$$\mathbf{Q}^{huc}(\xi) = \xi \mathbf{E}_0^{uc} \mathbf{U}_{,\xi}^{hc} + (\mathbf{E}_1^{cu})^T \mathbf{U}^{hc} \quad (2.80)$$

$$\mathbf{Q}^{hcc}(\xi) = \xi \mathbf{E}_0^{cc} \mathbf{U}_{,\xi}^{hc} + (\mathbf{E}_1^{cc})^T \mathbf{U}^{hc} \quad (2.81)$$

Now, a system of differential equations (2.73) along with the following two boundary conditions on the inner and outer boundaries:

$$\mathbf{Q}^{hu}(\xi_2) = \mathbf{P}_2^u \quad (2.82)$$

$$\mathbf{Q}^{hu}(\xi_1) = -\mathbf{P}_1^u \quad (2.83)$$

is sufficient for determining the general solution of $\mathbf{U}^{hu}$ for the given boundary value problem. The vectors $\mathbf{P}_2^u$ and $\mathbf{P}_1^u$ result directly from the partition of the vectors $\mathbf{P}_2$ and $\mathbf{P}_1$, respectively, i.e., $\mathbf{P}_2 = \{\mathbf{P}_2^u \quad \mathbf{P}_2^c\}^T$ and $\mathbf{P}_1 = \{\mathbf{P}_1^u \quad \mathbf{P}_1^c\}^T$. Once $\mathbf{U}^{hu}$ is solved, the unknown vectors $\mathbf{F}^{tc}$, $\mathbf{P}_1^c$, and $\mathbf{P}_2^c$ can be obtained, respectively, by (2.74) and

$$\mathbf{Q}^{hc}(\xi_2) = \mathbf{P}_2^c \quad (2.84)$$

$$\mathbf{Q}^{hc}(\xi_1) = -\mathbf{P}_1^c \quad (2.85)$$

Details of the solution procedure are provided in following sections.

2.3 Solution Methodology

This section presents the procedure for obtaining the analytical solution of a system of linear, second-order, nonhomogeneous, ordinary differential equations (2.73) and the boundary conditions (2.80) and (2.81). A corresponding eigenvalue problem is solved first to determine the homogeneous solution and the technique of assumed solution form is utilized to construct the particular solution associated with the distributed body source, the prescribed surface flux on the side face, and the prescribed state variable on the side face. Once the general solution is obtained, the boundary conditions on the inner and outer boundaries are enforced to determine all involved constants. Finally, the post-process for the field quantities such as the state variable and body flux is described.

2.3.1 Determination of Homogeneous Solution

In this section, a homogeneous solution of the system of linear differential equations (2.71), denoted by $\mathbf{U}_0^{hu}$, is derived following standard procedure from the theory of differential equations. The homogeneous solution $\mathbf{U}_0^{hu}$ must satisfy

$$\xi^2 \mathbf{E}_0^{uu} \mathbf{U}_{0,\xi\xi}^{hu} + \xi \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] \mathbf{U}_{0,\xi}^{hu} - \mathbf{E}_2^{uu} \mathbf{U}_0^{hu} = \mathbf{0} \quad (2.86)$$

The corresponding nodal internal flux, denoted by $\mathbf{Q}_0^{hu}(\xi)$, is given by

$$\mathbf{Q}_0^{hu}(\xi) = \xi \mathbf{E}_0^{uu} \mathbf{U}_{0,\xi}^{hu} + (\mathbf{E}_1^{uu})^T \mathbf{U}_0^{hu} \quad (2.87)$$

Since (2.86) is a set of $(m-p)\Lambda$ linear, second-order, Euler-Cauchy differential equations, the solution $\mathbf{U}_0^{hu}$ takes the following form

$$\mathbf{U}_0^{hu}(\xi) = \sum_{i=1}^{2(m-p)\Lambda} c_i \xi^{\lambda_i} \boldsymbol{\psi}_i^u \quad (2.88)$$

where a constant λ_i is termed the modal scaling factor, $\boldsymbol{\psi}_i$ is the $(m-p)\Lambda$ -component vector representing the i^{th} mode of the state variable, and c_i are arbitrary constants denoting the contribution of each mode to the solution. By substituting (2.88) into (2.87), it leads to

$$\mathbf{Q}_0^{hu}(\xi) = \sum_{i=1}^{2(m-p)\Lambda} c_i \xi^{\lambda_i} \left[\lambda_i \mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T \right] \boldsymbol{\psi}_i^u = \sum_{i=1}^{2(m-p)\Lambda} c_i \xi^{\lambda_i} \mathbf{q}_i^u \quad (2.89)$$

where $\mathbf{q}_i^u$ is termed the i^{th} modal internal flux given in terms of $\boldsymbol{\psi}_i^u$ by

$$\mathbf{q}_i^u = \left[\lambda_i \mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T \right] \boldsymbol{\psi}_i^u \quad \forall i \in \{1, 2, \dots, 2(m-p)\Lambda\} \quad (2.90)$$

By substituting (2.88) into (2.86), it results in

$$\sum_{i=1}^{2(m-p)\Lambda} c_i \xi^{\lambda_i} \left\{ \lambda_i (\lambda_i - 1) \mathbf{E}_0^{uu} + \lambda_i \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \boldsymbol{\psi}_i^u = \mathbf{0} \quad (2.91)$$

Since c_i are arbitrary, (2.91) implies that

$$\left\{ \lambda_i^2 \mathbf{E}_0^{uu} + \lambda_i \left[(\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \boldsymbol{\psi}_i^u = \mathbf{0} \quad \forall i \in \{1, 2, \dots, 2(m-p)\Lambda\} \quad (2.92)$$

By further rearranging terms in (2.90) such that

$$\lambda_i \boldsymbol{\psi}_i^u = -(\mathbf{E}_0^{uu})^{-1} (\mathbf{E}_1^{uu})^T \boldsymbol{\psi}_i^u + (\mathbf{E}_0^{uu})^{-1} \mathbf{q}_i^u \quad (2.93)$$

where $(\mathbf{E}_0^{uu})^{-1}$ denotes the inverse of $\mathbf{E}_0^{uu}$, and then substituting the result into (2.92), it gives rise to

$$\lambda_i \mathbf{q}_i^u = \left[\mathbf{E}_2^{uu} - \mathbf{E}_1^{uu} (\mathbf{E}_0^{uu})^{-1} (\mathbf{E}_1^{uu})^T \right] \boldsymbol{\psi}_i^u + \mathbf{E}_1^{uu} (\mathbf{E}_0^{uu})^{-1} \mathbf{q}_i^u \quad (2.94)$$

Now, by introducing a $2(m-p)\Lambda$ -component vector $\mathbf{X}_i$ such that $\mathbf{X}_i = \{\boldsymbol{\psi}_i^u \quad \mathbf{q}_i^u\}^T$, equations (2.93) and (2.94) can be combined into a system of linear algebraic equations

$$\mathbf{A}\mathbf{X}_i = \lambda_i \mathbf{X}_i \quad (2.95)$$

where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} -(\mathbf{E}_0^{uu})^{-1}(\mathbf{E}_1^{uu})^T & (\mathbf{E}_0^{uu})^{-1} \\ \mathbf{E}_2^{uu} - \mathbf{E}_1^{uu}(\mathbf{E}_0^{uu})^{-1}(\mathbf{E}_1^{uu})^T & \mathbf{E}_1^{uu}(\mathbf{E}_0^{uu})^{-1} \end{bmatrix} \quad (2.96)$$

Determination of all $2(m-p)\Lambda$ pairs $\{\lambda_i, \mathbf{X}_i\}$ is achieved by solving the eigenvalue problem (2.95) where λ_i denote the eigenvalues and $\mathbf{X}_i$ are associated eigenvectors. It should be remarked that since $\mathbf{A}$ is not symmetric, $\{\lambda_i, \mathbf{X}_i\}$ can involve complex numbers. In fact, only a half of the eigenvalues has the positive real part whereas the other half has negative real part. Let $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$ be $(m-p)\Lambda \times (m-p)\Lambda$ diagonal matrices containing eigenvalues with the positive real part and the negative real part, respectively. Also, let $\boldsymbol{\Phi}^{\psi^+}$ and $\boldsymbol{\Phi}^{q^+}$ be matrices whose columns containing, respectively, all vectors $\boldsymbol{\psi}_i^u$ and $\mathbf{q}_i^u$ obtained from the eigenvectors $\mathbf{X}_i = \{\boldsymbol{\psi}_i^u \quad \mathbf{q}_i^u\}^T$ associated with all eigenvalues contained in $\boldsymbol{\lambda}^+$ and let $\boldsymbol{\Phi}^{\psi^-}$ and $\boldsymbol{\Phi}^{q^-}$ be matrices whose columns containing, respectively, all vectors $\boldsymbol{\psi}_i^u$ and $\mathbf{q}_i^u$ obtained from the eigenvectors $\mathbf{X}_i = \{\boldsymbol{\psi}_i^u \quad \mathbf{q}_i^u\}^T$ associated with all eigenvalues contained in $\boldsymbol{\lambda}^-$, i.e.,

$$\mathbf{A} \begin{bmatrix} \boldsymbol{\Phi}^{\psi^+} \\ \boldsymbol{\Phi}^{q^+} \end{bmatrix} = \boldsymbol{\lambda}^+ \begin{bmatrix} \boldsymbol{\Phi}^{\psi^+} \\ \boldsymbol{\Phi}^{q^+} \end{bmatrix} \quad (2.97)$$

$$\mathbf{A} \begin{bmatrix} \boldsymbol{\Phi}^{\psi^-} \\ \boldsymbol{\Phi}^{q^-} \end{bmatrix} = \boldsymbol{\lambda}^- \begin{bmatrix} \boldsymbol{\Phi}^{\psi^-} \\ \boldsymbol{\Phi}^{q^-} \end{bmatrix} \quad (2.98)$$

Now, the homogeneous solutions $\mathbf{U}_0^{hu}$ and $\mathbf{Q}_0^{hu}(\xi)$ are given by

$$\mathbf{U}_0^{hu}(\xi) = \boldsymbol{\Phi}^{\psi^+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{\psi^-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- \quad (2.99)$$

$$\mathbf{Q}_0^{hu}(\xi) = \boldsymbol{\Phi}^{q^+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{q^-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- \quad (2.100)$$

where $\boldsymbol{\Pi}^+$ and $\boldsymbol{\Pi}^-$ are diagonal matrices obtained by simply replacing the diagonal entries λ_i of the matrices $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$ by the a function ξ^{λ_i} , respectively; and $\mathbf{C}^+$ and $\mathbf{C}^-$ are vectors containing arbitrary constants representing the contribution of each mode. It is apparent that the diagonal entries of $\boldsymbol{\Pi}^+$ become infinite when $\xi \rightarrow \infty$ whereas those of $\boldsymbol{\Pi}^-$ is unbounded when $\xi \rightarrow 0$. As a result, $\mathbf{C}^+$ is taken to $\mathbf{0}$ to ensure the boundedness of the solution for unbounded bodies and, similarly, the condition $\mathbf{C}^- = \mathbf{0}$ is enforced for bodies containing the scaling center.

2.3.2 Determination of Particular Solution

In this section, a particular solution of the system of linear differential equations (2.71), denoted by $\mathbf{U}_1^{hu}$, is established. In the present study, the distributed body source $\mathbf{b}$, the surface flux on the side-face-1 $\mathbf{t}_1^s$ (if it is prescribed), the surface flux on the side-face-2 $\mathbf{t}_2^s$ (if it is prescribed), and the prescribed state variable on the side face $\mathbf{U}^{hc}$ are assumed to admit the form

$$\mathbf{b}(\xi, s) = \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j} \bar{\mathbf{b}}_j(s) \quad (2.101)$$

$$\mathbf{t}_1^s(\xi) = \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j} \bar{\mathbf{t}}_j^{s1} \quad (2.102)$$

$$\mathbf{t}_2^s(\xi) = \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j} \bar{\mathbf{t}}_j^{s2} \quad (2.103)$$

$$\mathbf{U}^{hc}(\xi) = \sum_{\omega_j \in \mathbb{R}^*} \xi^{\omega_j} \bar{\mathbf{U}}_j^{hc} \quad (2.104)$$

where $\mathbb{R}^*$ denotes a set of non-negative real numbers, $\bar{\mathbf{b}}_j(s)$ are given vectors which are a function of s , and $\bar{\mathbf{t}}_j^{s1}$, $\bar{\mathbf{t}}_j^{s2}$, $\bar{\mathbf{U}}_j^{hc}$ are given constant vectors. Substituting (2.101) into (2.63), (2.102) into (2.60), (2.103) into (2.61), and (2.104) into (2.73) yields

$$\mathbf{F}^b = \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j} \bar{\mathbf{F}}_j^b \quad (2.105)$$

$$\mathbf{F}_1^t = \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j} \bar{\mathbf{F}}_j^{t1} \quad (2.106)$$

$$\mathbf{F}_2^t = \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j} \bar{\mathbf{F}}_j^{t2} \quad (2.107)$$

$$\mathbf{F}^{suu} = \sum_{\omega_j \in \mathbb{R}^*} \xi^{\omega_j} \bar{\mathbf{F}}_j^{suu} \quad (2.108)$$

where $\bar{\mathbf{F}}_j^b$, $\bar{\mathbf{F}}_j^{t1}$, $\bar{\mathbf{F}}_j^{t2}$, and $\bar{\mathbf{F}}_j^{suu}$ are constant vectors defined, in terms of prescribed data, by

$$\bar{\mathbf{F}}_j^b = \left\{ \int_{s_i}^{s_o} \phi_{(1)} \bar{\mathbf{b}}_j(s) J ds \quad \int_{s_i}^{s_o} \phi_{(2)} \bar{\mathbf{b}}_j(s) J ds \quad \cdots \quad \int_{s_i}^{s_o} \phi_{(m)} \bar{\mathbf{b}}_j(s) J ds \right\}^T \quad (2.109)$$

$$\bar{\mathbf{F}}_j^{t1} = \{J_1^{\xi} \bar{\mathbf{t}}_j^{s1} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}\} \quad (2.110)$$

$$\bar{\mathbf{F}}_j^{t2} = \{\mathbf{0} \quad \mathbf{0} \quad \cdots \quad J_2^{\xi} \bar{\mathbf{t}}_j^{s2}\}^T \quad (2.111)$$

$$\bar{\mathbf{F}}_j^{suu} = \{\omega_j(\omega_j - 1)\mathbf{E}_0^{uc} + \omega_j(\mathbf{E}_0^{uc} + (\mathbf{E}_1^{cu})^T - \mathbf{E}_1^{uc}) - \mathbf{E}_2^{uc}\} \bar{\mathbf{U}}_j^{hc} \quad (2.112)$$

The vectors $\mathbf{F}^b$ and $\mathbf{F}^t = \mathbf{F}_1^t + \mathbf{F}_2^t$ can be further partitioned to obtain $\mathbf{F}^b = \{\mathbf{F}^{bu} \quad \mathbf{F}^{bc}\}^T$ and $\mathbf{F}^t = \{\mathbf{F}^{tu} \quad \mathbf{F}^{tc}\}^T = \{\mathbf{F}_1^{tu} \quad \mathbf{F}_1^{tc}\}^T + \{\mathbf{F}_2^{tu} \quad \mathbf{F}_2^{tc}\}^T$ where

$$\mathbf{F}^{bu} = \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j} \bar{\mathbf{F}}_j^{bu}; \quad \mathbf{F}^{bc} = \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j} \bar{\mathbf{F}}_j^{bc}; \quad \bar{\mathbf{F}}_j^b = \left\{ \bar{\mathbf{F}}_j^{bu} \quad \bar{\mathbf{F}}_j^{bc} \right\}^T \quad (2.113)$$

$$\mathbf{F}_1^{tu} = \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j} \bar{\mathbf{F}}_j^{tu1}; \quad \mathbf{F}_1^{tc} = \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j} \bar{\mathbf{F}}_j^{tc1}; \quad \bar{\mathbf{F}}_j^{t1} = \left\{ \bar{\mathbf{F}}_j^{tu1} \quad \bar{\mathbf{F}}_j^{tc1} \right\}^T \quad (2.114)$$

$$\mathbf{F}_2^{tu} = \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j} \bar{\mathbf{F}}_j^{tu2}; \quad \mathbf{F}_2^{tc} = \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j} \bar{\mathbf{F}}_j^{tc2}; \quad \bar{\mathbf{F}}_j^{t2} = \left\{ \bar{\mathbf{F}}_j^{tu2} \quad \bar{\mathbf{F}}_j^{tc2} \right\}^T \quad (2.115)$$

Based on this form of prescribed data, the particular solution $\mathbf{U}_1^{hu}$ must take the form

$$\mathbf{U}_1^{hu}(\xi) = \mathbf{U}_1^{hub}(\xi) + \mathbf{U}_1^{hut1}(\xi) + \mathbf{U}_1^{hut2}(\xi) + \mathbf{U}_1^{huu}(\xi) \quad (2.116)$$

where

$$\mathbf{U}_1^{hub}(\xi) = \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j+2} \mathbf{c}_j^b \quad (2.117)$$

$$\mathbf{U}_1^{hut1}(\xi) = \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j+1} \mathbf{c}_j^{t1} \quad (2.118)$$

$$\mathbf{U}_1^{hut2}(\xi) = \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j+1} \mathbf{c}_j^{t2} \quad (2.119)$$

$$\mathbf{U}_1^{huu}(\xi) = \sum_{\omega_j \in \mathbb{R}^*} \xi^{\omega_j} \mathbf{c}_j^{uc} \quad (2.120)$$

with $\mathbf{c}_j^b$, $\mathbf{c}_j^{t1}$, $\mathbf{c}_j^{t2}$, and $\mathbf{c}_j^{uc}$ being vectors of unknown constants. By substituting (2.105)-(2.108) and (2.116) along with (2.117)-(2.120) into (2.71), it leads to

$$\begin{aligned} & \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j+2} \left[\left\{ (\kappa_j+2)(\kappa_j+1) \mathbf{E}_0^{uu} + (\kappa_j+2) \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \mathbf{c}_j^b + \bar{\mathbf{F}}_j^{bu} \right] + \\ & \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j+1} \left[\left\{ (\tau_j+1)\tau_j \mathbf{E}_0^{uu} + (\tau_j+1) \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \mathbf{c}_j^{t1} + \bar{\mathbf{F}}_j^{tu1} \right] + \\ & \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j+1} \left[\left\{ (\gamma_j+1)\gamma_j \mathbf{E}_0^{uu} + (\gamma_j+1) \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \mathbf{c}_j^{t2} + \bar{\mathbf{F}}_j^{tu2} \right] + \\ & \sum_{\omega_j \in \mathbb{R}^*} \xi^{\omega_j} \left[\left\{ \omega_j(\omega_j-1) \mathbf{E}_0^{uu} + (\omega_j-1) \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \mathbf{c}_j^{uc} + \bar{\mathbf{F}}_j^{suu} \right] = 0 \end{aligned} \quad (2.121)$$

Equation (2.121) can be satisfied $\forall \xi \in [\xi_1, \xi_2]$ if and only if

$$\left\{ (\kappa_j+2)(\kappa_j+1) \mathbf{E}_0^{uu} + (\kappa_j+2) \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \mathbf{c}_j^b + \bar{\mathbf{F}}_j^{bu} = \mathbf{0} \quad (2.122)$$

$$\left\{ (\tau_j+1)\tau_j \mathbf{E}_0^{uu} + (\tau_j+1) \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \mathbf{c}_j^{t1} + \bar{\mathbf{F}}_j^{tu1} = \mathbf{0} \quad (2.123)$$

$$\left\{ (\gamma_j + 1)\gamma_j \mathbf{E}_0^{uu} + (\gamma_j + 1) \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \mathbf{c}_j^{t2} + \bar{\mathbf{F}}_j^{tu2} = \mathbf{0} \quad (2.124)$$

$$\left\{ \omega_j(\omega_j - 1)\mathbf{E}_0^{uu} + (\omega_j - 1) \left[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu} \right] - \mathbf{E}_2^{uu} \right\} \mathbf{c}_j^{uc} + \bar{\mathbf{F}}_j^{suu} = \mathbf{0} \quad (2.125)$$

A system of linear algebraic equations (2.122)-(2.125) is sufficient for determining all unknown constant vectors $\mathbf{c}_j^b$, $\mathbf{c}_j^{t1}$, $\mathbf{c}_j^{t2}$, and $\mathbf{c}_j^{uc}$. One the particular solution $\mathbf{U}_1^{hu}(\xi)$ is obtained, the corresponding particular nodal internal flux, denoted by $\mathbf{Q}_1^{hu}(\xi)$, is given by

$$\mathbf{Q}_1^{hu}(\xi) = \xi \mathbf{E}_0^{uu} \mathbf{U}_{1,\xi}^{hu} + (\mathbf{E}_1^{uu})^T \mathbf{U}_1^{hu} \quad (2.126)$$

2.3.3 Final General Solution

The general solution of (2.71) $\mathbf{U}^{hu}(\xi)$ and the corresponding nodal internal flux $\mathbf{Q}^{hu}(\xi)$ are then given by

$$\mathbf{U}^{hu}(\xi) = \mathbf{U}_0^{hu}(\xi) + \mathbf{U}_1^{hu}(\xi) = \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- + \mathbf{U}_1^{hu}(\xi) \quad (2.127)$$

$$\mathbf{Q}^{hu}(\xi) = \mathbf{Q}_0^{hu}(\xi) + \mathbf{Q}_1^{hu}(\xi) = \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- + \mathbf{Q}_1^{hu}(\xi) \quad (2.128)$$

To determine the constants contained in $\mathbf{C}^+$ and $\mathbf{C}^-$, the boundary conditions on both inner and outer boundaries are enforced. By applying the condition (2.78) and (2.79), it results in

$$\boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) \mathbf{C}^+ + \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \mathbf{C}^- + \mathbf{Q}_1^{hu}(\xi_1) = -\mathbf{P}_1^u \quad (2.129)$$

$$\boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) \mathbf{C}^+ + \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \mathbf{C}^- + \mathbf{Q}_1^{hu}(\xi_2) = \mathbf{P}_2^u \quad (2.130)$$

It is worth noting that for bounded bodies containing the scaling center, the first system (4.41) is ignored with $\mathbf{C}^- = \mathbf{0}$ whereas, for unbounded bodies, the second system is ignored with $\mathbf{C}^+ = \mathbf{0}$. The two systems (2.129) and (2.130) can be combined to obtain

$$\begin{bmatrix} \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix} \begin{Bmatrix} \mathbf{C}^+ \\ \mathbf{C}^- \end{Bmatrix} = \begin{Bmatrix} -\mathbf{P}_1^u \\ \mathbf{P}_2^u \end{Bmatrix} - \begin{Bmatrix} \mathbf{Q}_1^{hu}(\xi_1) \\ \mathbf{Q}_1^{hu}(\xi_2) \end{Bmatrix} \quad (2.131)$$

The system (2.131) can be inverted to obtain

$$\begin{Bmatrix} \mathbf{C}^+ \\ \mathbf{C}^- \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix}^{-1} \left(\begin{Bmatrix} -\mathbf{P}_1^u \\ \mathbf{P}_2^u \end{Bmatrix} - \begin{Bmatrix} \mathbf{Q}_1^{hu}(\xi_1) \\ \mathbf{Q}_1^{hu}(\xi_2) \end{Bmatrix} \right) \quad (2.132)$$

From (2.127), it can be readily obtained

$$\begin{Bmatrix} \mathbf{U}^{hu}(\xi_1) \\ \mathbf{U}^{hu}(\xi_2) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix} \begin{Bmatrix} \mathbf{C}^+ \\ \mathbf{C}^- \end{Bmatrix} + \begin{Bmatrix} \mathbf{U}_1^{hu}(\xi_1) \\ \mathbf{U}_1^{hu}(\xi_2) \end{Bmatrix} \quad (2.133)$$

Substituting (2.132) into (2.133) yields

$$\mathbf{K} \begin{Bmatrix} \mathbf{U}^{hu}(\xi_1) \\ \mathbf{U}^{hu}(\xi_2) \end{Bmatrix} = \begin{Bmatrix} -\mathbf{P}_1^u \\ \mathbf{P}_2^u \end{Bmatrix} + \mathbf{K} \begin{Bmatrix} \mathbf{U}_1^{hu}(\xi_1) \\ \mathbf{U}_1^{hu}(\xi_2) \end{Bmatrix} - \begin{Bmatrix} \mathbf{Q}_1^{hu}(\xi_1) \\ \mathbf{Q}_1^{hu}(\xi_2) \end{Bmatrix} \quad (2.134)$$

where the coefficient matrix $\mathbf{K}$, commonly termed the stiffness matrix, is given by

$$\mathbf{K} = \begin{bmatrix} \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix}^{-1} \quad (2.135)$$

By applying the prescribed surface flux and the state variable on both inner and outer boundaries, a system of linear algebraic equations (2.134) is sufficient for determining all unknowns contained in the vectors $\{\mathbf{U}^{hu}(\xi_1) \ \mathbf{U}^{hu}(\xi_2)\}^T$ and $\{-\mathbf{P}_1^u \ \mathbf{P}_2^u\}^T$. Once the unknowns on both inner and outer boundaries are solved, the constant vectors $\mathbf{C}^+$ and $\mathbf{C}^-$ can be obtained from (2.132) and, then, the general solution for both $\mathbf{U}^{hu}(\xi)$ and $\mathbf{Q}^{hu}(\xi)$ are obtained from (2.127) and (2.128), respectively.

2.3.4 Post-process for field quantities

Once the approximate general solution $\mathbf{U}^{hu}(\xi)$ is obtained, the approximate field quantities such as the state variable and the surface flux within the body can be determined. For instance, the approximate state variable associated with a point (ξ, s) can be obtained from (2.38) as

$$\mathbf{u}^h(\xi, s) = \mathbf{N}^S(s) \mathbf{U}^h(\xi) = \begin{bmatrix} \mathbf{N}^{Su}(s) & \mathbf{N}^{Sc}(s) \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{hu}(\xi) \\ \mathbf{U}^{hc}(\xi) \end{Bmatrix} = \mathbf{N}^{Su}(s) \mathbf{U}^{hu}(\xi) + \mathbf{N}^{Sc}(s) \mathbf{U}^{hc}(\xi) \quad (2.136)$$

where $\mathbf{N}^{Su}$ and $\mathbf{N}^{Sc}$ are matrices resulting from the partition of $\mathbf{N}^S$. Similarly, the approximated body flux can be computed from (2.39) as

$$\begin{aligned} \sigma^h(\xi, s) &= \mathbf{D} \left[\mathbf{B}_1(s) \mathbf{U}_{,\xi}^h + \frac{1}{\xi} \mathbf{B}_2(s) \mathbf{U}^h \right] \\ &= \mathbf{D} \begin{bmatrix} \mathbf{B}_1^u(s) & \mathbf{B}_1^c(s) \end{bmatrix} \begin{Bmatrix} \mathbf{U}_{,\xi}^{hu}(\xi) \\ \mathbf{U}_{,\xi}^{hc}(\xi) \end{Bmatrix} + \frac{1}{\xi} \mathbf{D} \begin{bmatrix} \mathbf{B}_2^u(s) & \mathbf{B}_2^c(s) \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{hu}(\xi) \\ \mathbf{U}^{hc}(\xi) \end{Bmatrix} \\ &= \mathbf{D} \left[\mathbf{B}_1^u(s) \mathbf{U}_{,\xi}^{hu}(\xi) + \frac{1}{\xi} \mathbf{B}_2^u(s) \mathbf{U}^{hu}(\xi) \right] + \mathbf{D} \left[\mathbf{B}_1^c(s) \mathbf{U}_{,\xi}^{hc}(\xi) + \frac{1}{\xi} \mathbf{B}_2^c(s) \mathbf{U}^{hc}(\xi) \right] \end{aligned} \quad (2.137)$$

where $\mathbf{B}_1^u$, $\mathbf{B}_1^c$ and $\mathbf{B}_2^u$, $\mathbf{B}_2^c$ are matrices resulting from the partition of the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$, respectively. It is emphasized here again that the solutions (2.136) and (2.137) also apply to the special cases of bounded and unbounded bodies. For bounded bodies containing the scaling center, $\mathbf{C}^-$ simply vanishes and, for unbounded bodies, $\mathbf{C}^+ = \mathbf{0}$.

2.3.5 Errors of approximations

To investigate the error of the scaled boundary finite element approximation, an error vector-valued function $\mathbf{e} = \mathbf{e}(x)$ is first defined by

$$\mathbf{e}(x) = \mathbf{u}^{exact}(x) - \mathbf{u}^h(x) \quad (2.138)$$

where $\mathbf{u}^{exact}$ and $\mathbf{u}^h$ denotes the exact solution and scaled boundary finite element solution of the state variables, respectively. The following standard L^2 -norm is employed to measure the magnitude of the error function, i.e.,

$$\|\mathbf{e}\|_{L^2} = \left(\int_{\Omega} \mathbf{e}^T(x) \mathbf{e}(x) \right)^{1/2} \quad (2.139)$$

Note that the evaluation of the above integral can be achieved efficiently in the $\xi-s$ space. Finally, the relative prevent error, denoted by $error$, is obtained by normalizing the L^2 -norm of the error function by the L^2 -norm of the exact state variable, i.e.,

$$error = \frac{\|\mathbf{e}\|_{L^2}}{\|\mathbf{u}^{exact}\|_{L^2}} \times 100 \quad (2.140)$$

where

$$\|\mathbf{u}^{exact}\|_{L^2} = \left(\int_{\Omega} [\mathbf{u}^{exact}(x)]^T \mathbf{u}^{exact}(x) \right)^{1/2} \quad (2.141)$$

It is remarked that for problems without the exact solution, the converged solution obtained from a particular, sufficiently fine mesh can be used to estimate the error for any level of meshes used in the approximation.

2.4 Numerical Results

Numerical results of four representative problems are presented, in this section, to verify both the formulation and implementation of the proposed technique. To demonstrate its capability to treat a variety of boundary value problems, general boundary conditions, and prescribed data on the side faces, three different types of problems associated with steady-state heat conduction ($\Lambda = 1$), linear elasticity ($\Lambda = 2$), and linear piezoelectricity ($\Lambda = 3$), under various scenarios are considered. In the approximation, standard linear finite elements are employed to discretize both the defining curve and the trial and test functions. The accuracy and convergence of numerical solutions are confirmed by benchmarking with available analytical solutions and carrying out the analysis via a series of meshes.

2.4.1 Heat conduction in rectangular domain

The proposed technique is first tested with a representative problem associated with the steady-state heat conduction ($\Lambda=1$). Let us consider a two-dimensional rectangular domain of dimensions l_1 and l_2 as shown schematically in Figure 2.6(a). The domain is made of a medium with an isotropic heat conductivity k_0 (or, equivalently, $D_{11} = D_{22} = k_0, D_{12} = D_{21} = 0$) and subjected to a uniform heat source $b_1 = Q_0$. A zero temperature is prescribed along the boundary AD (i.e., $u_1^{AD} = 0$) and a uniform surface heat flux q_0 is prescribed along the boundary BC (i.e., $t_1^{BC} = q_0$) whereas, along the boundaries AB and CD , the surface heat flux vanishes (i.e., $t_1^{AB} = t_1^{CD} = 0$). For this particular case, the exact solution for the temperature field (i.e., $u_1(x)$) and body heat flux (i.e., $\sigma_{11}(x)$ and $\sigma_{21}(x)$) is given by

$$u_1 = \frac{1}{k_0} \left[(q_0 + Q_0 l_1) x_1 - \frac{1}{2} Q_0 x_1^2 \right] \quad (2.142)$$

$$\sigma_{11} = q_0 + Q_0(l_1 - x_1), \quad \sigma_{21} = 0 \quad (2.143)$$

The domain geometry is described by the scaling center at point D and the defining curve ABC ; as a result, the boundaries AD and CD become the side faces (see Figure 2.6(b)). In the numerical study, q_0 and Q_0 are chosen such that $Q_0 l_1 = 3q_0$ and the defining curve is discretized by a series of N identical linear elements.

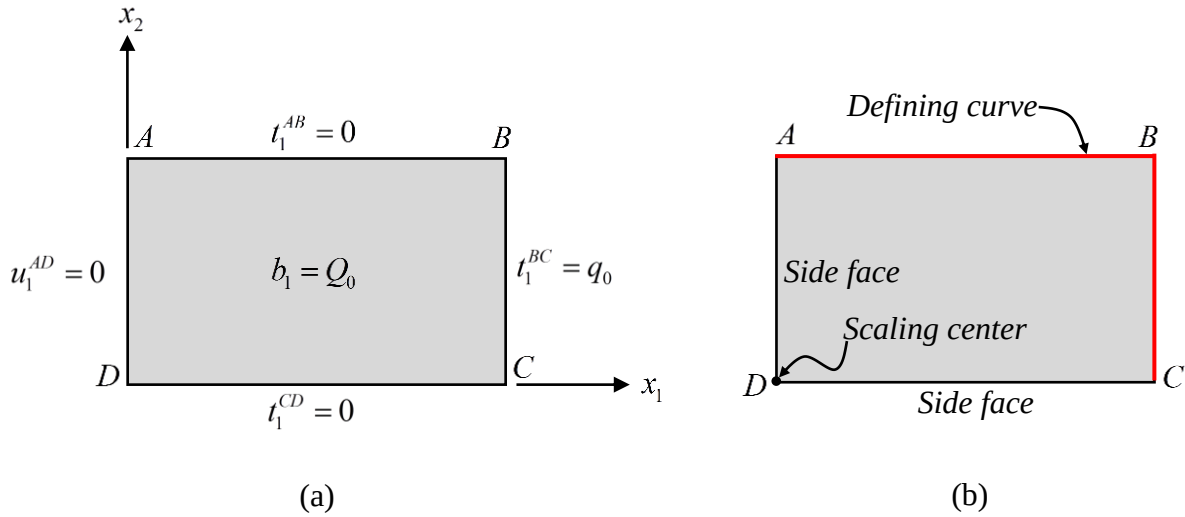


Figure 2.6: Schematic of (a) rectangular domain under body heat source and mixed boundary conditions and (b) scaling center and defining curve used in scale boundary finite element analysis.

The normalized temperature $u_1 k_0 / q_0 l_1$ and the non-zero normalized body heat flux σ_{11} / q_0 along the boundary AB are reported in Table 2.1 for various values of normalized coordinate $\bar{x}_1 = x_1 / l_1$ and meshes. Clearly, computed numerical solutions exhibit an excellent convergence behavior as the number of elements increases and, in addition, accurate results in comparison with the

benchmark solution can be obtained even when few elements are employed to discretize the defining curve and solution along the boundary direction.

Table 2.1: Normalized temperatures and body heat flux along the boundary AB of a rectangular domain subjected to body heat source and mixed boundary conditions.

	$u_1 k_0 / q_0 l_1$					
$\bar{x}_1$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 4$	0.0000	0.7196	1.3287	1.8290	2.2175	2.4684
SBFEM, $N = 8$	0.0000	0.7352	1.3520	1.8512	2.2332	2.4950
SBFEM, $N = 16$	0.0000	0.7388	1.3580	1.8577	2.2382	2.4986
SBFEM, $N = 32$	0.0000	0.7397	1.3595	1.8594	2.2395	2.4996
Exact solution	0.0000	0.7400	1.3600	1.8600	2.2400	2.5000
	σ_{11} / q_0					
$\bar{x}_1$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 4$	3.8735	3.3211	2.7724	2.2297	1.6359	0.8099
SBFEM, $N = 8$	3.9709	3.3802	2.7890	2.2037	1.6121	1.0222
SBFEM, $N = 16$	3.9929	3.3950	2.7970	2.2005	1.6034	1.0006
SBFEM, $N = 32$	3.9982	3.3988	2.7992	2.2001	1.6008	1.0000
Exact solution	4.0000	3.4000	2.8000	2.2000	1.6000	1.0000

2.4.2 Linear elastic hollowed cylinder under uniform pressure

Consider, next, a hollowed cylinder of the inner radius R_1 and outer radius R_2 (see Figure 2.7(a)). The cylinder is made of a homogeneous, linearly elastic, isotropic material with Young's modulus E and Poisson's ratio ν and is subjected to a plane-strain condition and uniform pressure p_1 and p_2 on the inner and outer boundary, respectively, (i.e., $\Lambda = 2$ and the modulus matrix $\mathbf{D}$ with non-zero entries $D_{11} = D_{44} = (1-\nu)E/(1+\nu)(1-2\nu)$, $D_{14} = D_{41} = \nu E/(1+\nu)(1-2\nu)$, $D_{22} = D_{23} = D_{32} = D_{33} = E/2(1+\nu)$). Due to the symmetry, it is sufficient to model this problem using only a quarter of the cylinder (see Figure 2.7(b)) with appropriate conditions on both side faces (i.e., the normal displacement and tangential traction on the side faces vanish). To describe the geometry, the scaling center is chosen at the center of the cylinder whereas the inner boundary is treated as the defining curve. In a numerical study, $R_2 / R_1 = 1.5$, $p_2 / p_1 = 2$, $\nu = 0.3$, and meshes with N identical linear elements are employed.

Results for the normalized radial displacement ($u_r / (p_1 R_1 / E)$), normalized radial stress (σ_{rr} / p_1) and normalized hoop stress ($\sigma_{\theta\theta} / p_1$) are reported along with existing analytical solutions [36] in Tables 2.2, 2.3 and 2.4, respectively, for four meshes (i.e., $N = 4, 8, 16, 32$). It is seen that numerical solutions generated by the proposed technique converge and exhibit excellent agreement with the benchmark solution. It is worth noting that the discretization with only few linear elements can capture numerical solution with sufficient accuracy.

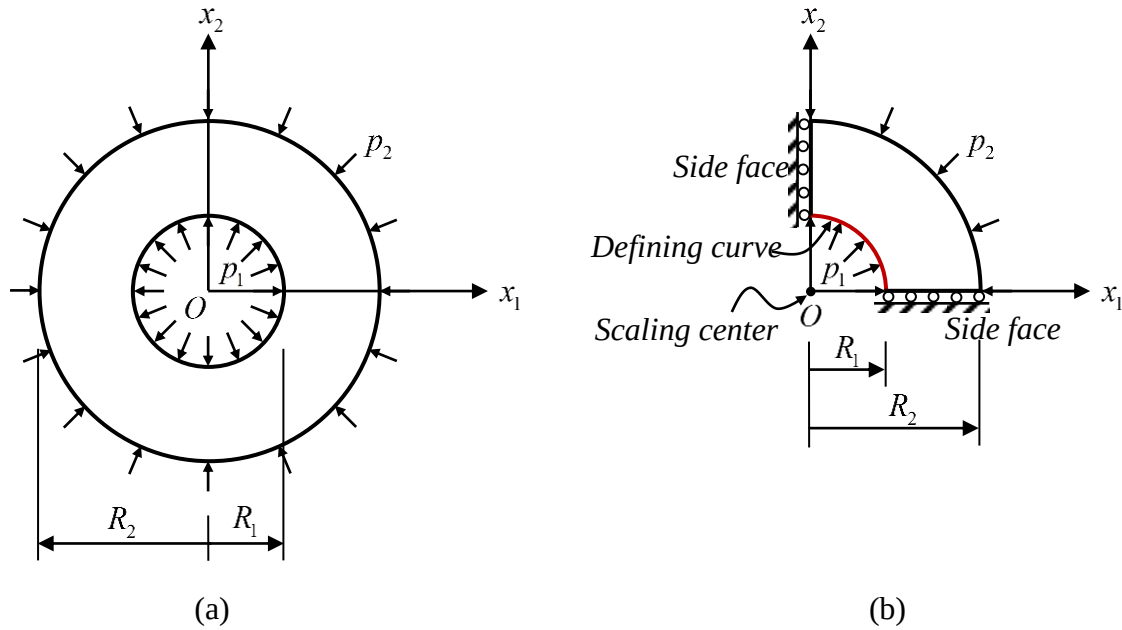


Figure 2.7: Schematic of (a) hollowed cylinder under uniform internal and external pressure and (b) quarter of cylinder used in the analysis.

Table 2.2: Normalized radial displacement of hollowed cylinder under internal and external uniform pressure. Results are reported at different values of radial coordinate $r = \sqrt{x_1^2 + x_2^2}$ for four meshes.

	$u_r / (p_1 R_1 / E)$					
r / R_1	1.0	1.1	1.2	1.3	1.4	1.5
SBFEM, $N = 4$	1.0958	0.9864	0.8943	0.8155	0.7473	0.6875
SBFEM, $N = 8$	1.1124	1.0014	0.9080	0.8282	0.7590	0.6984
SBFEM, $N = 16$	1.1166	1.0052	0.9115	0.8313	0.7619	0.7011
SBFEM, $N = 32$	1.1177	1.0061	0.9123	0.8321	0.7627	0.7018
Exact solution	1.1180	1.0064	0.9126	0.8324	0.7629	0.7020

Table 2.3: Normalized radial stress of hollowed cylinder under internal and external uniform pressure. Results are reported at different values of radial coordinate $r = \sqrt{x_1^2 + x_2^2}$ for four meshes.

	σ_r / p_1					
r / R_1	1.0	1.1	1.2	1.3	1.4	1.5
SBFEM, $N = 4$	-0.9821	-0.8289	-0.7124	-0.6217	-0.5497	-0.4917
SBFEM, $N = 8$	-0.9955	-0.8401	-0.7218	-0.6298	-0.5568	-0.4979
SBFEM, $N = 16$	-0.9989	-0.8429	-0.7242	-0.6319	-0.5586	-0.4995
SBFEM, $N = 32$	-0.9997	-0.8436	-0.7248	-0.6324	-0.5590	-0.4999
Exact solution	-1.0000	-0.8438	-0.7250	-0.6325	-0.5592	-0.5000

Table 2.4: Normalized hoop stress of hollowed cylinder under internal and external uniform pressure. Results are reported at different values of radial coordinate $r = \sqrt{x_1^2 + x_2^2}$ for four meshes.

	$\sigma_{\theta\theta} / p_1$					
r / R_1	1.0	1.1	1.2	1.3	1.4	1.5
SBFEM, $N = 4$	0.7833	0.6301	0.5136	0.4229	0.3510	0.2929
SBFEM, $N = 8$	0.7958	0.6404	0.5222	0.4301	0.3571	0.2982
SBFEM, $N = 16$	0.7990	0.6429	0.5243	0.4319	0.3587	0.2996
SBFEM, $N = 32$	0.7997	0.6436	0.5248	0.4324	0.3591	0.2999
Exact solution	0.8000	0.6438	0.5250	0.4325	0.3592	0.3000

2.4.3 Linear elastic square plate under mixed boundary conditions

Another representative problem for linear elasticity ($\Lambda = 2$) is chosen to demonstrate the capability of the implemented procedure to treat problems with distributed body source and prescribed state variable and surface flux on the boundary. Consider a plane-strain, square plate $ABCD$ made of a homogeneous, isotropic, linearly elastic material of Young's modulus E and Poisson's ratio ν as shown schematically in Figure 2.8(a). Note that the modulus matrix $\mathbf{D}$ for this particular problem is the same as the previous case. The plate is subjected to a linear body force field $b_1 = 2b_0(1 + \bar{x}_2)$, $b_2 = 2b_0(1 - \bar{x}_1)$ with b_0 denoting a constant and $\bar{x}_1 = x_1/l$, $\bar{x}_2 = x_2/l$ denoting the normalized coordinates whereas the nonuniform traction and homogeneous displacement boundary conditions are prescribed on its four sides as follows:

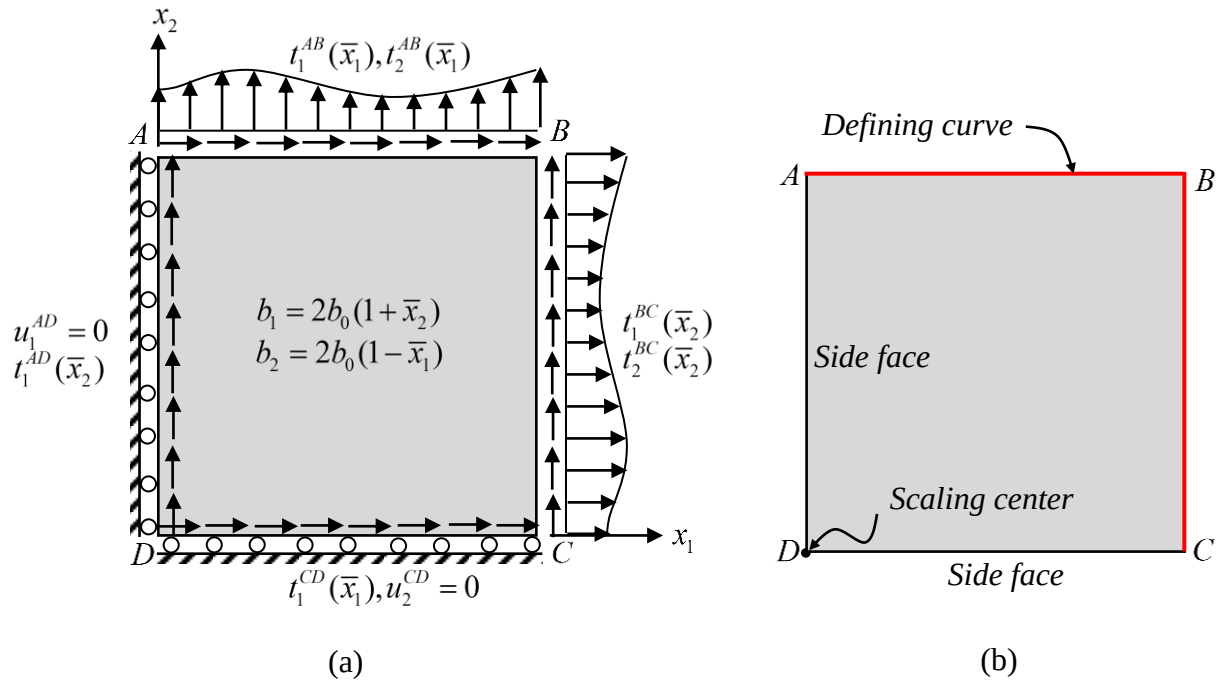


Figure 2.8: Schematic of (a) elastic square plate under mixed boundary conditions and (b) scaling center and defining curve used in scale boundary finite element analysis.

Side AB: $t_1^{AB} = b_0 l (3 + 14\bar{x}_1 - \bar{x}_1^2)$ and $t_2^{AB} = 2b_0 l (-4 + 4\bar{x}_1 + 3\bar{x}_1^2)$

Side BC: $t_1^{BC} = 6b_0 l (\bar{x}_2^2 - 2)$ and $t_2^{BC} = b_0 l (1 + 14\bar{x}_2 + \bar{x}_2^2)$

Side AD: $u_1^{AD} = 0$ and $t_2^{AD} = -b_0 l (2\bar{x}_2 + \bar{x}_2^2)$

Side CD: $t_1^{CD} = b_0 l (\bar{x}_1^2 - 2\bar{x}_1)$ and $u_2^{CD} = 0$

The exact solution for this particular problem under the plane strain condition can be readily obtained from a classical theory of linear elasticity and final results are given by

$$Eu_1^{exact} = -2b_0 l^2 (1 + \nu) (1 + \bar{x}_1 - 2\bar{x}_2 + \bar{x}_1^2 - 3\bar{x}_2^2 + \bar{x}_1 \bar{x}_2) \bar{x}_1 \quad (2.144)$$

$$Eu_2^{exact} = -2b_0 l^2 (1 + \nu) (-1 - 2\bar{x}_1 + \bar{x}_2 - 3\bar{x}_1^2 + \bar{x}_2^2 - \bar{x}_1 \bar{x}_2) \bar{x}_2 \quad (2.145)$$

$$\sigma_{11}^{exact} = -\sigma_{22}^{exact} = -2b_0 l (1 + 2\bar{x}_1 - 2\bar{x}_2 + 3\bar{x}_1^2 - 3\bar{x}_2^2 + 2\bar{x}_1 \bar{x}_2) \quad (2.146)$$

$$\sigma_{12}^{exact} = -b_0 l (-2\bar{x}_1 - 2\bar{x}_2 + \bar{x}_1^2 - \bar{x}_2^2 - 12\bar{x}_1 \bar{x}_2) \quad (2.147)$$

The geometry of the plate is fully described by the scaling center at point D and the defining curve ABC and, as a result, the boundaries AD and CD become the side faces (see Figure 2.8(b)). In the analysis, Poisson's ratio is taken as $\nu = 0.3$ and the defining curve is discretized by N identical linear elements. The normalized displacements $Eu_1/b_0 l^2$ and $Eu_2/b_0 l^2$ along the boundaries AB and BC are reported in Tables 2.5 and 2.6 for various locations and meshes. It can be seen that the numerical solutions converges to the exact solution as the number of elements N used to discretize the defining curve increases and, in addition, only few number of degrees of freedom is sufficient to obtain accurate displacements. The normalized normal stress components $\sigma_{11}/b_0 l$ and $\sigma_{22}/b_0 l$ along the boundaries CD and AD are also reported in Figures 2.9 and 2.10 along with the exact solution. Similar to the displacements, the present method also yields highly accurate stress components and the good convergent behavior; in particular, results obtained from all meshes are nearly indistinguishable from the benchmark solution.

Table 2.5: Normalized displacements along the boundary AB of elastic square plate subjected to mixed boundary conditions.

	$Eu_1/b_0 l^2$					
$\bar{x}_1$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 8$	0.0000	1.8132	3.0677	3.6989	3.5973	2.6088
SBFEM, $N = 16$	0.0000	1.8340	3.1482	3.7916	3.6349	2.6036
SBFEM, $N = 32$	0.0000	1.8489	3.1558	3.7996	3.6578	2.6013
SBFEM, $N = 64$	0.0000	1.8501	3.1608	3.8055	3.6592	2.6004
Exact solution	0.0000	1.8512	3.1616	3.8064	3.6608	2.6000
	$Eu_2/b_0 l^2$					
$\bar{x}_1$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 8$	-2.6000	-0.7080	1.8267	4.9752	8.6765	12.8377
SBFEM, $N = 16$	-2.6000	-0.7152	1.7717	4.8997	8.6582	12.9368
SBFEM, $N = 32$	-2.6000	-0.7274	1.7713	4.8933	8.6364	12.9780
SBFEM, $N = 64$	-2.6000	-0.7272	1.7682	4.8887	8.6337	12.9929
Exact solution	-2.6000	-0.7280	1.7680	4.8880	8.6320	13.0000

Table 2.6: Normalized displacements along the boundary BC of elastic square plate subjected to mixed boundary conditions.

	Eu_1/b_0l^2					
$\bar{x}_2$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 8$	-7.8000	-6.9689	-5.4890	-3.3779	-0.6650	2.6088
SBFEM, $N = 16$	-7.8000	-6.9603	-5.5174	-3.4306	-0.7012	2.6036
SBFEM, $N = 32$	-7.8000	-6.9686	-5.5110	-3.4293	-0.7242	2.6013
SBFEM, $N = 64$	-7.8000	-6.9676	-5.5124	-3.4320	-0.7265	2.6004
Exact solution	-7.8000	-6.9680	-5.5120	-3.4320	-0.7280	2.6000

	Eu_2/b_0l^2					
$\bar{x}_2$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 8$	0.0000	3.1006	6.0281	8.7072	11.0261	12.8377
SBFEM, $N = 16$	0.0000	3.0981	6.0659	8.7820	11.1097	12.9368
SBFEM, $N = 32$	0.0000	3.0993	6.0709	8.7928	11.1408	12.9780
SBFEM, $N = 64$	0.0000	3.0991	6.0732	8.7974	11.1463	12.9929
Exact solution	0.0000	3.0992	6.0736	8.7984	11.1488	13.0000

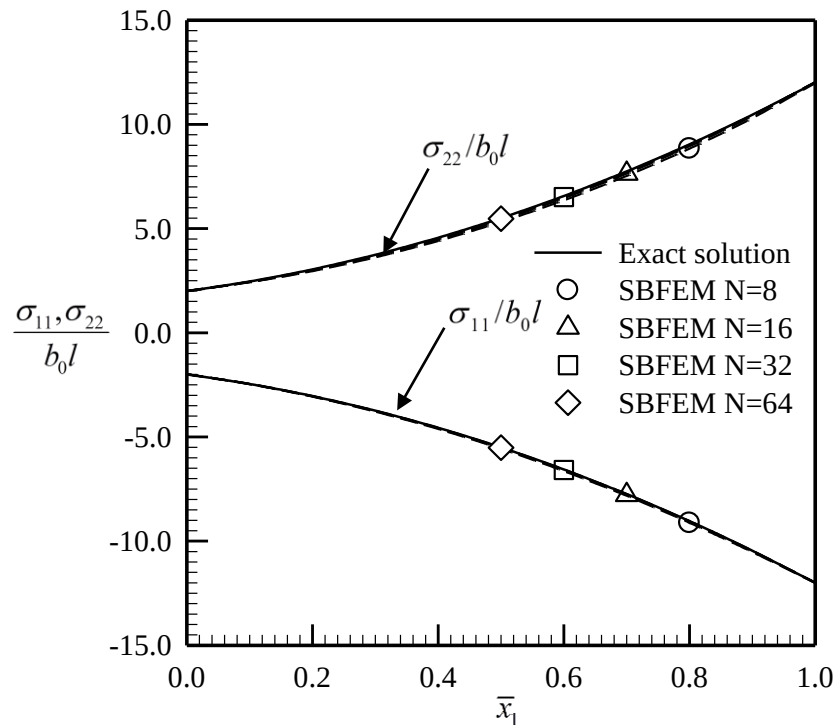


Figure 2.9: Normalized normal stress component along the boundary CD of elastic square plate subjected to mixed boundary conditions

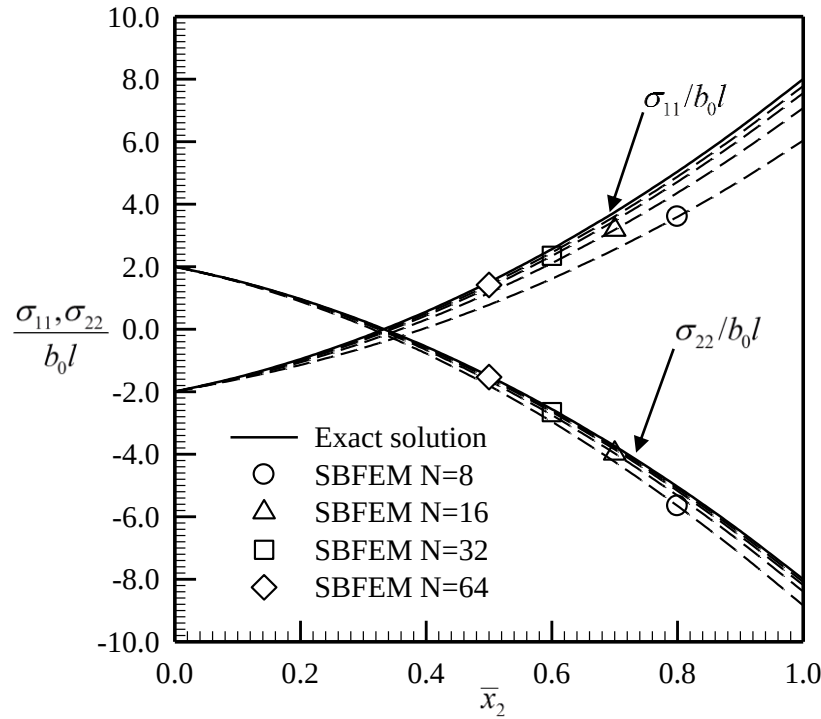


Figure 2.10: Normalized normal stress component along the boundary AD of elastic square plate subjected to mixed boundary conditions

2.4.4 Linear piezoelectric square plate

As a final example, a representative boundary value problem associated with the linear piezoelectricity ($\Lambda = 3$) is investigated to further highlight the capability of the proposed technique to treat linear multi-field problems. Let us consider a two-dimensional square plate of length l and made of a linear piezoelectric solid with all material constants taken from PZT-4 [5] as shown schematically in Figure 2.11(a). In particular, all non-zero entries of the modulus matrix $\mathbf{D}$ are given explicitly by $D_{11} = 139\text{GPa}$, $D_{15} = D_{51} = 74.3\text{GPa}$, $D_{16} = D_{61} = -6.98\text{Cm}^{-2}$, $D_{22} = D_{24} = D_{42} = D_{44} = 25.6\text{GPa}$, $D_{33} = 6.0 \times 10^{-9}\text{C(Vm)}^{-1}$, $D_{34} = D_{43} = 13.44\text{Cm}^{-2}$, $D_{55} = 113\text{GPa}$, $D_{56} = D_{65} = 13.84\text{Cm}^{-2}$, and $D_{66} = 5.47 \times 10^{-9}\text{C(Vm)}^{-1}$. For a purpose of verification, a set of prescribed data such as the distributed body source and boundary conditions on four sides is chosen such that the exact solution for the displacements (u_1 and u_2) and the electric potential (u_3) takes the following form

$$u_1 = u_{10}\bar{x}_1^2, u_2 = u_{20}\bar{x}_2^2, u_3 = \phi_0\bar{x}_2 \quad (2.148)$$

where u_{10} , u_{20} , ϕ_0 are given constants and $\bar{x}_1 = x_1/l$, $\bar{x}_2 = x_2/l$ are normalized coordinates. The corresponding exact stress field $(\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22})$ and electrical induction vector $(\sigma_{13}, \sigma_{23})$ is given by

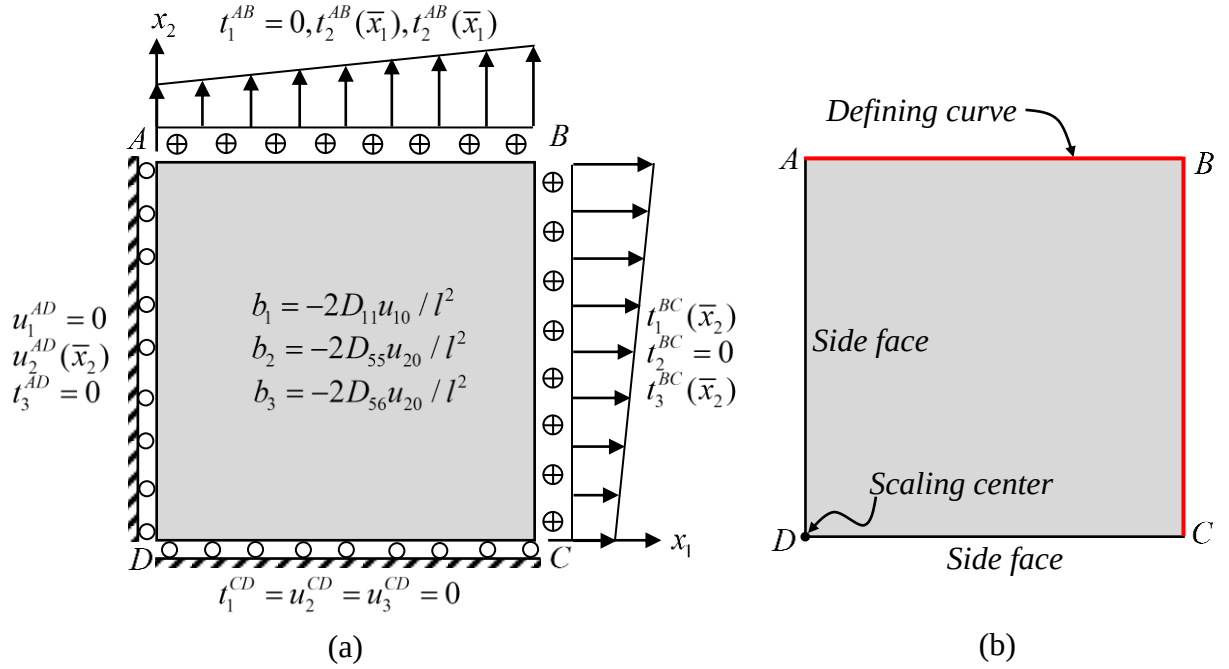


Figure 2.11: Schematic of (a) linear piezoelectric square plate under mixed boundary conditions and (b) scaling center and defining curve used in scale boundary finite element analysis.

$$\sigma_{11} = (2D_{11}u_{10}\bar{x}_1 + 2D_{15}u_{20}\bar{x}_2 + D_{16}\phi_0)/l, \quad \sigma_{12} = \sigma_{21} = 0 \quad (2.149)$$

$$\sigma_{22} = (2D_{15}u_{10}\bar{x}_1 + 2D_{55}u_{20}\bar{x}_2 + D_{56}\phi_0)/l \quad (2.150)$$

$$\sigma_{13} = 0, \quad \sigma_{23} = (2D_{16}u_{10}\bar{x}_1 + 2D_{56}u_{20}\bar{x}_2 - D_{66}\phi_0)/l \quad (2.151)$$

The distributed body source that is in equilibrium with the above stress and electric induction can readily be obtained as

$$b_1 = -2D_{11}u_{10}/l^2, \quad b_2 = -2D_{55}u_{20}/l^2, \quad b_3 = -2D_{56}u_{20}/l^2 \quad (2.152)$$

In the modeling, the scaling center is chosen at a point D and the prescribed conditions on the side faces AD and CD and the boundaries AB and BC are given below

Side AB: $t_1^{AB} = 0, t_2^{AB} = (2D_{15}u_{10}\bar{x}_1 + 2D_{55}u_{20} + D_{56}\phi_0)/l, t_3^{AB} = (2D_{16}u_{10}\bar{x}_1 + 2D_{56}u_{20} - D_{66}\phi_0)/l$

Side BC: $t_1^{BC} = (2D_{11}u_{10} + 2D_{15}u_{20}\bar{x}_2 + D_{16}\phi_0)/l, t_2^{BC} = 0, t_3^{BC} = 0$

Side AD: $u_1 = 0, u_2 = u_{20}\bar{x}_2^2, t_3^{AD} = 0$

Side CD: $t_1^{CD} = 0, u_2 = 0, u_3 = 0$

It is worth noting that the boundary conditions shown above are chosen to represent the general prescribed data on the side faces and boundaries of the domain. In the numerical study, a series of meshes with N identical linear elements is constructed to discretize the defining curve and

solution along the scale boundary direction and $u_{20}/u_{10} = 2$, $D_{16}\phi_0/D_{11}u_{10} = 1$ are employed. Computed displacements and electric potential along the diagonal line BD are reported in Table 2.7 for various meshes. It is seen for this particular problem that the proposed technique yield highly accurate results even when relatively coarse meshes containing only few degrees of freedom are employed. In addition, the improvement of solutions as the mesh is refined is clearly observed. The good quality of numerical solutions is also confirmed for the body flux as indicate in Figures 2.12 and 2.13. Computed stresses and electrical induction along the diagonal line BD show an excellent agreement with the exact solution; in particular, only slight difference between solutions can be seen for very coarse meshes while almost indistinguishable results are obtained for fine meshes.

Table 2.7: Normalized displacements and electrical potential along the diagonal line BD of linear piezoelectric square plate subjected to mixed boundary conditions. Results are reports as a function of normalized length $\bar{s} = s/\sqrt{2}l$ where s is the length along the line BD measured from point D .

	u_1/u_{10}					
$\bar{s}$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 4$	0.0000	0.0369	0.1351	0.3278	0.6291	1.0459
SBFEM, $N = 8$	0.0000	0.0393	0.1531	0.3503	0.6360	1.0252
SBFEM, $N = 16$	0.0000	0.0398	0.1582	0.3574	0.6385	1.0098
SBFEM, $N = 32$	0.0000	0.0400	0.1596	0.3593	0.6396	1.0033
Exact solution	0.0000	0.0400	0.1600	0.3600	0.6400	1.0000
	u_2/u_{20}					
$\bar{s}$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 4$	0.0000	0.0821	0.3272	0.7222	1.2554	1.9134
SBFEM, $N = 8$	0.0000	0.0803	0.3220	0.7213	1.2739	1.9679
SBFEM, $N = 16$	0.0000	0.0801	0.3205	0.7204	1.2786	1.9891
SBFEM, $N = 32$	0.0000	0.0800	0.3201	0.7201	1.2797	1.9965
Exact solution	0.0000	0.0800	0.3200	0.7200	1.2800	2.0000
	u_3/ϕ_0					
$\bar{s}$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N = 4$	0.0000	0.2013	0.4019	0.6013	0.7991	0.9938
SBFEM, $N = 8$	0.0000	0.2003	0.4004	0.6003	0.7998	0.9976
SBFEM, $N = 16$	0.0000	0.2001	0.4001	0.6001	0.7999	0.9992
SBFEM, $N = 32$	0.0000	0.2000	0.4000	0.6000	0.8000	0.9997
Exact solution	0.0000	0.2000	0.4000	0.6000	0.8000	1.0000

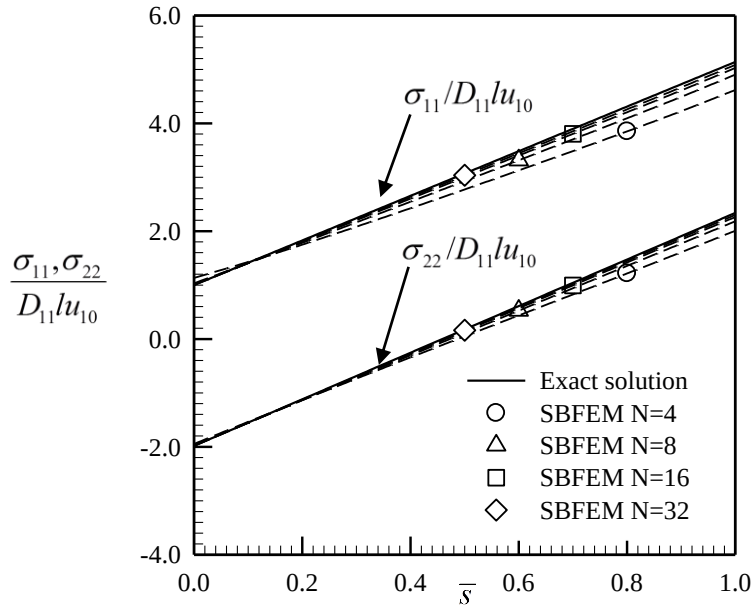


Figure 2.12: Normalized non-zero stress components along the diagonal line BD of a piezoelectric square plate subjected to mixed boundary conditions. Results are reports as a function of normalized length $\bar{s} = s/\sqrt{2}l$ where s is the length along the line BD measured from point D .

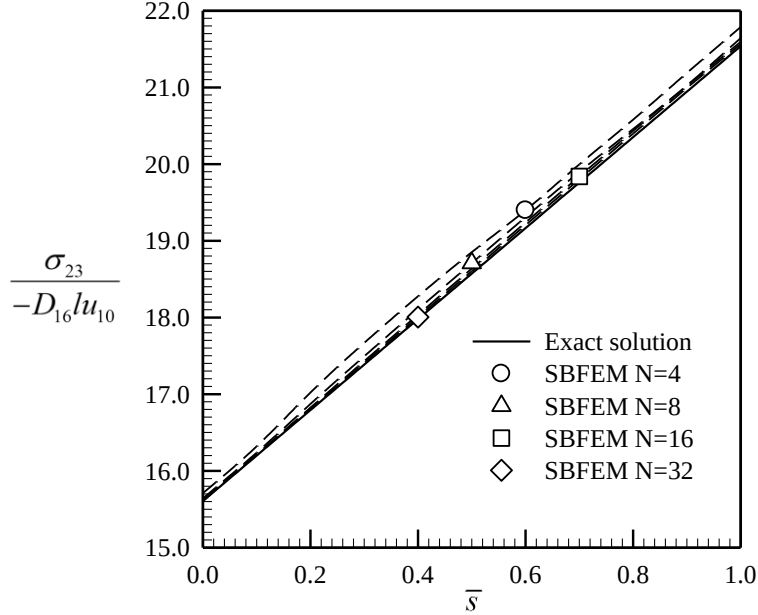


Figure 2.13: Normalized a non-zero electrical induction component along the diagonal line BD of a piezoelectric square plate subjected to mixed boundary conditions. Results are reports as a function of normalized length $\bar{s} = s/\sqrt{2}l$ where s is the length along the line BD measured from point D .

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CHAPTER 3

SOLUTION OF GENERALIZED T-STRESS OF PENNY-SHAPED CRACKS IN LINEAR PIEZOELECTRIC WHOLE SPACE

This chapter presents the closed-form solution of the generalized T-stress components of a penny-shaped crack in a transversely isotropic, linear piezoelectric, infinite medium under different crack-face conditions including electrically permeable, impermeable, semi-permeable and energetically consistent conditions. The solution for the case of uniform remote and crack-face mechanical/electrical loadings are derived for all four crack-face conditions whereas that for a crack under a pair of self-equilibrated, crack-face concentrated loads is obtained for permeable and impermeable cases. In the derivation, existing analytical solutions of the generalized stress field are employed together with an explicit expression resulting from the asymptotic expansion of the near-front field. All involved differentiations and limits are carried out in a careful manner following standard procedures and the final solution of the generalized T-stress components involves only elementary functions. In addition, the established Green's function for the generalized T-stress is utilized to form an integral formula for computing generalized T-stress for impermeable and permeable cracks under general crack-face loading. The analytical/semi-analytical feature of the derived solutions play a key role in the investigation of the influence of various parameters such as the crack-face conditions, electrical/mechanical far field, the crack-face loading, and material properties on the value of the generalized T-stress components along the crack front. A selected set of results is reported and significant findings are concluded.

3.1 Background and Review

Piezoelectric materials have been found possessing unique properties which enable to transform mechanical energy into electrical energy or vice versa. Such strong electro-mechanical coupling effect is widely known as the piezoelectric effect (including both the direct and converse piezoelectric effects). Owing to these abilities, piezoelectric materials have been utilized in many applications in various disciplines such as aerospace and automotive industries, acoustic field, engineering and medical applications. Additionally, piezoelectric materials have also been found as constituting materials of the key components of many recent advanced devices and structures including piezoelectric power supplies, the electro-mechanical sensors and actuators, the sonar transducers, and smart and self-adaptive structures. However, one obvious drawback of this particular class of materials is their brittle characteristics associated with possessing relatively low fracture toughness and high sensitivity to flaw inducing. In addition, these piezoelectric-based devices often operate under various circumstances that may be prone to fracture-induced damages and failures. For instance, vibration detector sensors are designed to capture the earthquake motion; loud speakers receive electric signals and convert them into mechanical vibrations which create sound waves to produce the desirable acoustic; and the piezoelectric floors, which have been trailed in some stations in Japan, generate the electricity from walking

pedestrian for automatic ticket gates and electronic display systems. As a direct consequence of their key properties and applications, cracks/defects are unavoidably found inside piezoelectric bodies under usages. Therefore, when subjected to mechanical, electrical, or electrical-mechanical excitations, these piezoelectric components and devices can encounter the degradations, damages, and ultimate failure as a result of preexisting or loaded induced cracks/defects. Understanding of fundamental fracture characteristics and fracture-induced failure mechanism is obviously essential and required in the design of piezoelectric components and devices to maintain their safety and integrity.

Mathematical modeling and simulations, based mainly on the classical theory of linear piezoelectricity, have been vastly applied in literature and found effective and efficient for the generalized stress analysis of piezoelectric cracked bodies. The generalized stress field in the neighborhood of the crack boundary (including the mechanical stress and the electric induction), predicted from the linear theory, dominates characteristics of the cracked medium and has been found essential as the basic information for simulating cracks initiations and advances. Most of existing studies in the context of linear piezoelectric fracture mechanics focused mainly on the dominant singular terms in the near-front expansion of the generalized stress field and, the corresponding parameters such as the electric and stress intensity factors and generalized strain energy release rate have been often used in the growth criteria. Numerous researches have been carried out to demonstrate the significance of these crack-front parameters, for instance, the crack formation, the subsequent crack propagation, and the damage and failure (Zhou et al., 2013). Nevertheless, existing experimental and theoretical evidences have also indicated the significant contribution of the non-singular term of the near-tip generalized stress, known as the generalized T-stress, and the necessity to integrate such information in the fracture modeling. Within the context of elastic media, the T-stress is viewed as the boundary effect by following the study of Shahani and Tabatabaei (2009). In addition, the T-stress also found to strongly influence the shape and size of plastic zones, the crack-front stress constraint and tri-axiality, and the directional stability of advancing crack (Larsson & Carlsson, 1973; Cotterell & Rice, 1980; Du & Hancock, 1991; Matvienko & Pochinkov, 2013; Matvienko, 2014). The experimental study by Ayatollahi and Safari (2003) also confirmed that the rotation of the isochromatic fringes pattern is also affected by the sign of the T-stress. In addition, the shielding or anti-shielding characteristics of the plastic-zone size and shape at the crack front are dependent on the negative and positive values of the T-stress (Zhou et al., 2011). For piezoelectric media, several investigators also concluded that the generalized T-stress is essential quantities affecting the crack-kinking behavior and the plastic zone shapes (Zhu & Yang, 1999; Hao & Biao, 2004; Li & Lee, 2004a,b; Viola et al., 2008). These past evidences confirmed the significance of the generalized T-stress and ignorance of such crucial fracture information can, in fact, lead to the inaccurate prediction of fracture responses. As an essential step to integrate the information of the generalized T-stress in the fracture modeling, analytical and computational procedures must be properly adopted to accurately and efficiently determine those quantities along the crack front. From limited advances of researches in this particular area as supported by a careful survey of literature in the following section, the development of a solution methodology to extract the fracture data of the generalized T-stress still requires further investigations and its merit to the community of fracture mechanics should be remarkable.

Regarding to linear elastic cracks media, non-singular (or finite) part of stress in the near-front expansion of the stress field have been well investigated and a vast amount of researches have been published regarding to the calculations of the T-stresses and the study of their effect

on the crack-tip stress field and fracture process. For instance, Sladek and Sladek (1997) developed a boundary integral equation method to compute the T-stress for interface cracks of two-dimensional, semi-infinite, dissimilar materials subjected to a point force at the crack tip. Fett (1997) also derived T-stress green's functions due to a pair of normal concentrated forces and then used the obtained result to develop a boundary collocation technique to determine the T-stress of an edged-crack embedded in a rectangular plate made of a linear isotropic elastic material and subjected to general normal traction. Later, Fett (1998) extended his previous research to investigate linearly elastic rectangular plate and circular disks containing various crack configurations such as edged cracks and centered cracks and subjected to both tensile and bending loads. Yang and Ravi (1999) proposed the stress-difference method together with an iterative dual boundary integral equation method along with the tip-node displacement jump to compute the T-stress at a crack tip for thermo-elastic crack problems. Later, Chen (2000) applied the complex-potential-function theory to develop the solution of the T-stress for four types of cracks including a line crack, a symmetric airfoil crack, circular-arc crack, and a symmetric lip crack in a two-dimensional, linear elastic, infinite plate. Zhao et al. (2001) applied the domain integral and interaction integral technique to obtain numerical solution for the T-stresses of a quarter-circular crack and a tunnel-corner crack in an elastic square plate under remote tension. Their results indicated the good agreement with experimental observations, for instance, the discrepancies of the fatigue crack growth rate between CN (corner notch) and CT (compact tension) test specimens, and the crack tunneling in CN specimens under pre-dominantly sustained load. Fett (2001) utilized the same technique as that by Fett (1997) to investigate a circular disk containing an internal crack under different types of boundary conditions such as pure displacement boundary conditions, pure traction boundary conditions and mix boundary conditions. Fett (2002) extended his previous work to treat different crack configurations in both single and double edge cracked circular disks. Wang (2002) used the finite element technique together with the weight-function approach to compute the T-stress for several test specimens, for instance, CCP (a centered crack plate), SECP (a single edged crack plate) and DECP (a double edged crack plate). The applied crack-face normal traction with either uniform, linear, parabolic, or cubic variation was investigated. Fett and Rizzi (2006) made use of the finite element procedure and the weight-function approach to determine the T-stresses for CT (a compact tension crack), DCC (a double cantilever crack), and ECB (an edge cracked bar) under self-equilibrated general normal traction near the crack tip. Also, Fett et al. (2006) employed Green's function technique to obtain the T-stress of a forked crack and a kinked crack in a plated made of an isotropic, linearly elastic material and subjected to in-plane shear and normal tractions. Recently, the T-stress of an edged crack embedded in a plate which is made of two different isotropic linearly elastic materials was studied by Zhou et al. (2013). A symplectic expansion method was employed in this particular work, and it was found efficient for treating complex boundary conditions.

While various researches concerning the two-dimensional problems have been carried out, simplified assumptions used in the modeling can lead to an inaccurate prediction and the loss of certain essential information and, as a consequence, the full three-dimensional analysis have been continuously proposed. Various existing methods for calculating the T-stress of cracks in three-dimensional media are briefly summarized and discussed below. Wang (2003) generalized the work of Wang (2002) to treat a three-dimensional crack problem by utilizing the finite element method together with the interaction integral formula to obtain the T-stresses of a surface-breaking, semi-elliptical crack in an isotropic, linearly elastic finite thick plate under

bending and tensile loadings. Later, Wang and Bell (2004) further extended his previous research to treat other types of loading with different spatial variations (e.g., uniform, linear, parabolic, and cubic distribution). Wang (2004) applied the potential-theory-based method and the Hankel integral transformation to determine an exact solution for the T-stress of a penny-shaped crack contained in an isotropic, linearly elastic, infinite body under applied remote tension and bending loads. Qu and Wang (2006) determined the T-stress of a corner-quarter-elliptical crack in an isotropic, linearly elastic, thick plate via the finite element method and the interaction integral formula. Both tensile and bending loads applied at both ends of the plate were considered in their study. Kirilyuk and Levchuk (2007) presented the T-stress solution of a flat-elliptical crack contained in an isotropic, linearly elastic, infinite body under the remote axial tension and bending loads. This research is the extension of the work of Wang (2004) by utilizing a special set of harmonic functions instead of the Hankel integrals. Zhou and Li (2007) and Zhou et al. (2011) also presented the analytical solution of the T-stress under both mode-I and mode-II conditions for the crack-inclusion interaction by using Eshelby equivalent inclusion method. Most recently, Rungamornrat and Pinitpanich (2016) utilized the existing stress field from Fabrikant (1989) together with conventional differentiations and limit procedure to derive Green's functions for the T-stress of a circular crack under a pair of opposite, normal point forces. A superposition method along with a numerical quadrature was then adopted to construct an integral formula capable of computing the T-stress components for a circular crack in a three-dimensional, transversely isotropic, linear elastic medium subjected to general applied normal traction on the crack surface.

While significant advances and progress of the modeling and analysis for the T-stress have been well recognized in the literature for various crack geometries and loading conditions, work concerning the generalized T-stress in piezoelectric cracked bodies is still limited. This may result from not only the complexity of electro-mechanical coupling effect but also the lack of recognition of the generalized T-stress and its contribution. The brief overview of relevant literatures concerning the determination of the generalized T-stress solution is presented and discussed in a chronological order as follows. Zhu and Yang (1999) confirmed the contribution of the generalized T-stress on the crack-kinking behavior for a straight crack embedded in a piezoelectric, two-dimensional body under prescribed mechanical loading conditions. In their study, the continuous dislocation theory and a boundary integral equation method were employed. Hao and Biao (2004) derived the exact solution of the generalized T-stresses of an impermeable straight crack embedded in a transversely isotropic, linear piezoelectric whole space under applied mechanical and electrical loading conditions. In their study, the principle of superposition along the Plemelj formulation was utilized. Results from this investigation also indicated that the value of the generalized T-stress depends mainly on the elastic, piezoelectric constants, and dielectric permittivity. In the same year, Li and Lee (2004a) applied Fourier transform to formulate a pair of dual integral equations and then derived the analytical solution of a semi-permeable Griffith crack oriented normal to the poling direction and subjected to uniform electro-mechanical loading conditions. Later, Zhong and Li (2008) reported the closed-form solution of the generalized T-stress for a semi-permeable Griffith crack in a two-dimensional, transversely isotropic, linear piezoelectric solid under the remote uniform mechanical tension, electrical induction and magnetic effect. In their analytical study, the Fourier integral transform along with the standard procedure was applied to solve the dual integral equation to obtain the complete solution. Liu et al. (2012) studied an elliptical hole embedded in a two-dimensional, transversely isotropic, linear piezoelectric, infinite body subjected to the

uniform pressure at the surface of the hole and the remote electro-mechanical loads. The complete analytical solution of both mechanical and electrical fields for the electrical permeable boundary condition was obtained by applying the complex variable function approach. They also pointed out that the behavior of the electric induction and stress in the vicinity of the crack front depends strongly on the value of the non-singular term. Most recently, Subsathaphol (2013) developed a numerical procedure based upon the weakly singular, boundary integral equation technique and standard Galerkin procedure to extract the generalized T-stress components of an impermeable isolated cracks contained in three-dimensional, anisotropic, piezoelectric, infinite media under general mechanical/electrical loading conditions.

Another important issue effecting the calculation of the generalized T-stress components of cracks in piezoelectric media is the model used to simulate the crack-face conditions. Various models have been proposed to simulate such crack-face conditions and the key difference among those models stems directly from the assumption of the dielectric permittivity of a medium filled within crack cavity. Suitability of all existing models is still questionable, highly problem dependent, and, therefore, requires further investigations. The electrical impermeable and electrical permeable crack-face conditions are two idealized models that have been widely employed in the linear piezoelectric fracture modeling due to their mathematical simplicity. The permeable model, proposed by Parton (1976), assumes the continuity of a normal component of the electrical induction and the electrical potential across the geometrically identical crack surfaces. This is opposite to the impermeable model where the electric induction normal to surface of the crack is fully prescribed whereas the crack-face electrical potential is unknown a priori and discontinuous. It should be remarked that these two crack-face models could provide an inaccurate prediction of solution due to the ignorance of the influence of permittivity. In order to improve the modeling, Hao and Shen (1994) developed a semi-permeable electrical boundary condition which includes the effect of the dielectric permittivity of the medium within the crack cavity on the continuity of the electric field across the crack gap. In this point of view, the electrical impermeable and electrical permeable conditions are simply two extreme cases of the semi-permeable crack-face condition. Later, McMeeking (2004) found that the total energy and the crack-tip energy release rates of a Griffith crack resulting from the use of the semi-permeable condition are not consistent. To overcome the energy inconsistency, Landis (2004) further modified the original semi-permeable model to obtain an energetically consistent model by integrating the traction normal to the crack face.

From the comprehensive review of involved literatures, most the existing solutions for the T-stress of cracks in elastic bodies have been well-established in the context of both two-dimensional and three-dimensional boundary value problems. However, the analytical and numerical solutions of the generalized T-stress have been very limited to very simple settings. In addition, the integration of more suitable electrical crack-face conditions such as those described by energetically consistent and semi-permeable models in the determination of the generalized T-stress components and the study of their effects on that essential fracture data have not been well recognized. These existing gaps of knowledge directly motivate the proposed research.

3.2 Problem Formulation

This section begins with describing basic field equations essential for formulating boundary value problems of linear piezoelectricity and, also, providing the definition of all involved parameters. Following sections present the clear description of the crack-face conditions

considered in the present study, the near-front generalized stress field and essential formula for extracting the generalized T-stress, the superposition method employed to formulate the integral formula of the generalized T-stress components, and, finally, existing solutions of the generalized stress field for certain fundamental cases.

3.2.1 Basic field equations

Basic equations for a medium with zero body forces and body electric charges are adopted from the classical theory of linear piezoelectricity. The laws of conservation are expressed by

$$\begin{aligned}\sigma_{ij,i} &= 0 \\ D_{i,i} &= 0\end{aligned}\tag{3.1}$$

where σ_{ij} denotes components of a stress tensor; D_i denotes components of an electrical induction vector; the comma notation $f_{,i}$ represents a partial derivative of any function f with respect to a Cartesian coordinate x_i ; and standard indicial notations apply. Here and in what follows, a standard indicial notation applies; specifically, lower-case subscripts range from 1 to 3 and the repeated lower-case index indicates the summation over its range. Components of the stress tensor and electric induction vector are linearly related to the electric field with its components denoted by E_i and the strain tensor with its components denoted by $\bar{\varepsilon}_{ij}$ via a set of fully coupled linear algebraic equations

$$\begin{aligned}\sigma_{ij} &= E_{ijkm}\bar{\varepsilon}_{km} - e_{mij}E_m \\ D_i &= e_{ikm}\bar{\varepsilon}_{km} + \varepsilon_{im}E_m\end{aligned}\tag{3.2}$$

where E_{ijkm} denotes the elastic constants; e_{mij} denotes the piezoelectric constants; and ε_{im} denotes the dielectric permittivity. The electric field E_i and the strain tensor $\bar{\varepsilon}_{ij}$ are given in terms of components of the displacement vector, denoted by u_i , and the electric potential, denoted by ϕ , via the following linearized kinematics

$$\begin{aligned}\bar{\varepsilon}_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \\ E_i &= -\phi_{,i}\end{aligned}\tag{3.3}$$

Components of the mechanical traction, denoted by t_i , and the surface electric charge, denoted by D , at any point on the smooth boundary can be computed in terms of the stress field and the electric induction vector by

$$\begin{aligned}t_i &= \sigma_{ij}n_j \\ D &= D_jn_j\end{aligned}\tag{3.4}$$

where n_j denote components of an outward unit normal vector to the smooth surface. All field equations (3.1)-(3.3) can also be expressed in a more concise form suggested by Rungamornrat and Mear (2008) as follows:

$$\sigma_{iJ,i} = 0 \quad (3.5)$$

$$\sigma_{iJ} = E_{iJKm} u_{K,m} \quad (3.6)$$

where upper-case index ranges from 1 to 4 and the repeated upper-case index indicates the summation over its range; σ_{iJ} is termed the generalized stress which contains components of the stress tensor σ_{ij} and components of the electrical induction vector $\sigma_{i4} \equiv D_i$; u_J is termed the generalized displacement which contains components of the displacement u_j and the electrical potential $u_4 \equiv \phi$; and E_{iJKl} is termed the generalized moduli which contains elastic moduli E_{ijkl} , piezoelectric constants $E_{ij4l} \equiv E_{l4ij} \equiv e_{lij}$, and dielectric permittivity $E_{i44l} \equiv E_{l44i} \equiv -\epsilon_{il}$. Consistent with this new notation, the generalized traction, denoted by t_j and computed from $t_j \equiv \sigma_{iJ} n_i$, contains the mechanical traction $t_j = \sigma_{ij} n_i$ and the surface electric charge $t_4 \equiv \sigma_{i4} n_i = D$. It can be remarked that the generalized moduli E_{iJKl} defined above possesses the following symmetric property $E_{iJKl} = E_{lJKi}$ provided that E_{l4ji} is selected to be identical to E_{ij4l} .

3.2.2 Crack-face conditions

On the surface of the crack, the condition on the crack-face data such as the generalized displacement on the upper crack surface (S_c^+) and the lower crack surface (S_c^-), denoted by u_j^+ and u_j^- , respectively, and the generalized traction on both the upper and lower crack surfaces, denoted by t_j^+ and t_j^- , respectively, must be properly prescribed. In the current investigation, four types of crack-face models including electrically permeable, impermeable, semi-permeable, and energetically consistent crack-face conditions are treated.

For an impermeable crack model, the crack-face generalized traction t_j^+ and t_j^- are fully prescribed whereas the crack-face generalized displacements u_j^+ and u_j^- are unknown a priori. For a permeable crack model, the mechanical crack-face tractions t_j^+ and t_j^- are fully prescribed and the jump in the crack-face electrical potential and the sum of the crack-face surface electrical charge vanishes (i.e., $\Delta u_4 \equiv u_4^+ - u_4^- = 0$ and $\Sigma t_4 \equiv t_4^+ + t_4^- = 0$) whereas the sum of the crack-face generalized displacement $\Sigma u_j \equiv u_j^+ + u_j^-$, the jump in the crack-face displacement $\Delta u_j \equiv u_j^+ - u_j^-$, and the jump in the crack-face surface electrical charge $\Delta t_4 \equiv t_4^+ - t_4^-$ are unknown a priori. For a semi-permeable crack model, the crack-face generalized displacement u_j^+ and u_j^- , and the jump in the crack-face surface electric charge Δt_4 are unknown a priori whereas the mechanical crack-

face tractions t_j^+ and t_j^- are fully known, the sum of the crack-face surface electrical charge vanishes (i.e., $\Sigma t_4 = 0$), and the following nonlinear relationship is satisfied

$$t_4^+ \Delta u_i n_i^+ = -\kappa_c \Delta u_4 \quad (3.7)$$

where κ_c denotes the dielectric permittivity of a medium filled within the crack cavity. Finally, for an energetically consistent crack model, the crack-face generalized displacement u_j^+ and u_j^- , the jump in the crack-face surface electrical charge Δt_4 , and the jump in the mechanical crack-face normal traction, denoted by $\Delta \sigma_i \equiv \sigma_i^+ - \sigma_i^-$ where $\sigma_i^+ = t_j^+ n_j n_i$ and $\sigma_i^- = t_j^- n_j n_i$, are unknown a priori whereas the mechanical crack-face shear tractions, denoted by $\tau_i^+ = t_i^+ - \sigma_i^+$ and $\tau_i^- = t_i^- - \sigma_i^-$ are fully known, the sum of the crack-face surface electrical charge and the sum of the mechanical crack-face shear traction vanish (i.e., $\Sigma t_4 = 0$ and $\Sigma \sigma_i \equiv \sigma_i^+ + \sigma_i^- = 0$), and the following two nonlinear relations are satisfied

$$\begin{aligned} t_4^+ \Delta u_i n_i^+ &= -\kappa_c \Delta u_4 \\ \sigma_j^+ &= (1/2) \kappa_c n_j^+ (\Delta u_4)^2 / (\Delta u_i n_i^+)^2 \end{aligned} \quad (3.8)$$

It is emphasized that the superscripts “+” and “-” are used only to designate quantities on the upper crack surface and the lower crack surface, respectively.

3.2.3 Problem description

Consider a three-dimensional body occupying the whole space and containing a circular (or penny-shaped) crack with the radius a as shown in Figure 3.1. For convenience, the reference Cartesian coordinate system $\{\mathbf{0}; x_1, x_2, x_3\}$ and the associated cylindrical reference coordinate system $\{\mathbf{0}; r, \theta, z\}$ are introduced such that an origin $\mathbf{0}$ is located at the crack center, the x_3 -axis directs upward, and the remaining axes follow the right-hand rule. A material constituting the body is assumed to be homogeneous, transversely isotropic, and linear piezoelectric. All involved material constants are fully prescribed whereas the axis of material symmetry and poling direction are assumed to direct normal to the crack surface.

Besides the basic field equations (3.1)-(3.3), the prescribed crack-face boundary conditions have been found strongly affecting responses of the piezoelectric cracked medium. Thus, the proper understanding of the role of the crack-face condition on the fracture data along the crack boundary such as the generalized intensity factors and generalized T-stress is essential and requires the full investigation. In the present study, all four models of crack-face boundary conditions indicated above are considered and the different types of mechanical/electrical loading conditions are considered as follows.

For a circular crack under the impermeable condition, six following loading cases are considered: (i) a pair of opposite, normal concentrated forces acting to the crack surface (see Figure 3.2), (ii) a pair of opposite, concentrated electrical charge acting to the crack surface (see

Figure 3.3), (iii) a self-equilibrated, crack-face, uniform normal traction t_3^o and a self-equilibrated, crack-face, uniform surface charge d_3^o (see Figure 3.4(a-b)), (iv) uniform remote triaxial stress $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and a uniform remote electrical induction $\{d_1^\infty, d_2^\infty, d_3^\infty\}$ (see Figure 3.5), (v) general, self-equilibrated, crack-face normal traction t_3 (see Figure 3.6(a)), and (vi) general, self-equilibrated, crack-face surface electrical charge d_3 (see Figure 3.6(b)).

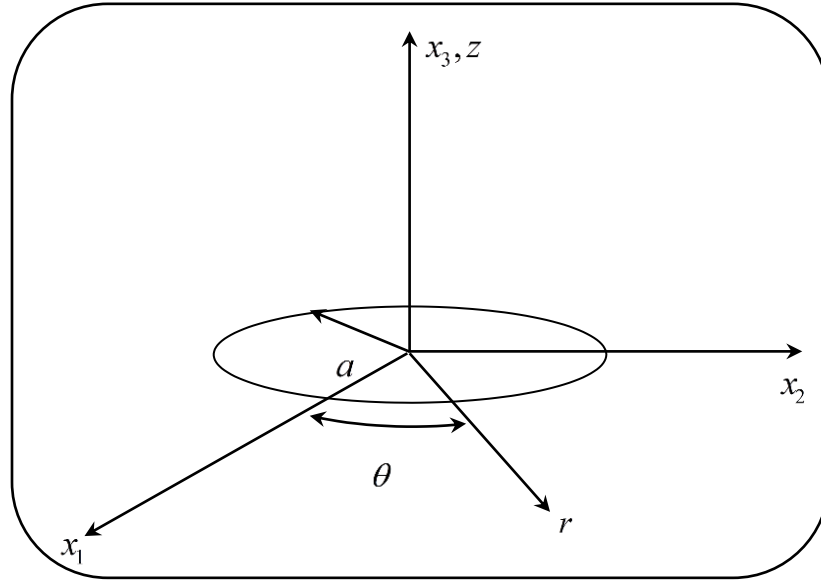


Figure 3.1: Schematic of circular crack with radius a contained in transversely isotropic, linear piezoelectric, infinite medium

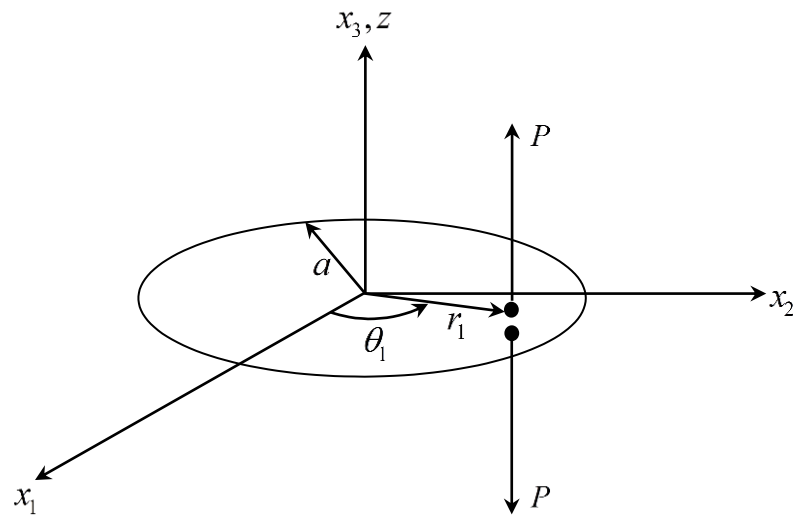


Figure 3.2: Schematic of circular crack under pair of opposite normal concentrated forces acting to crack surface

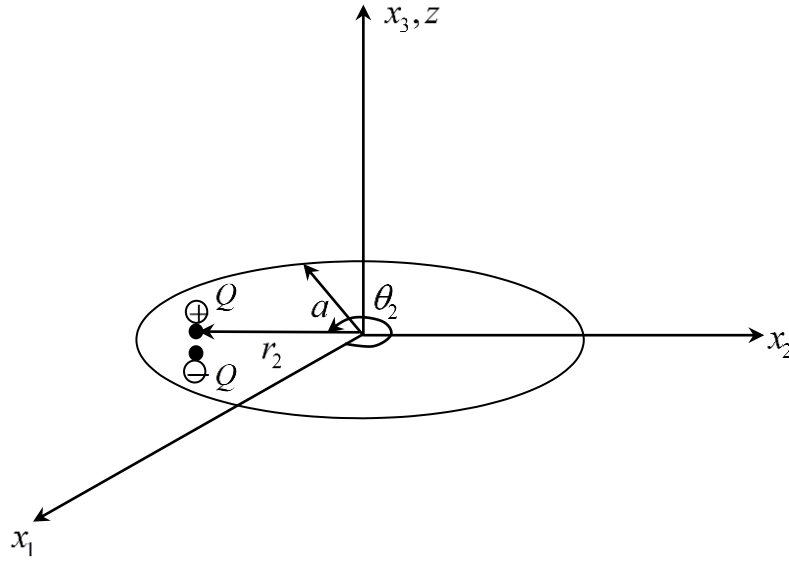


Figure 3.3: Schematic of circular crack under pair of opposite, concentrated electrical charge acting to crack surface

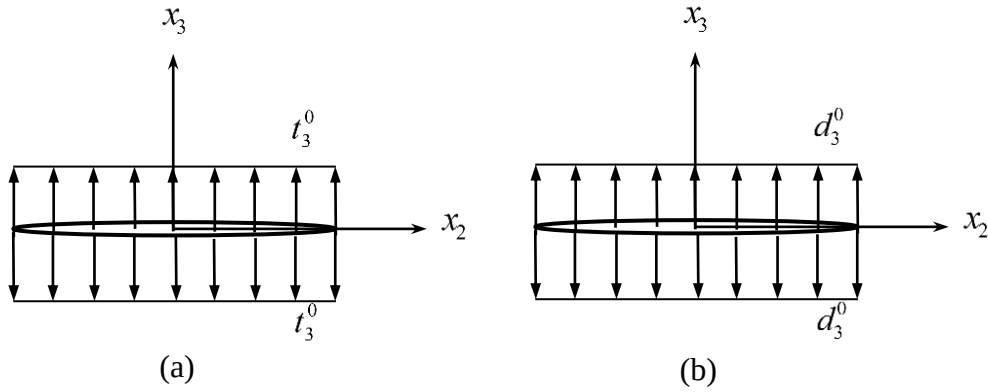


Figure 3.4: Schematic of circular crack under (a) self-equilibrated, crack-face, uniform normal traction t_3^0 and (b) self-equilibrated, crack-face, uniform surface charge d_3^0

For a permeable circular crack, four loading cases are considered: (i) a pair of opposite, normal concentrated forces acting to the crack surface (see Figure 3.2), (ii) self-equilibrated, crack-face, uniform normal traction t_3^0 (see Figure 3.4(a)), (iii) uniform remote triaxial stress $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and a uniform remote electrical induction $\{d_1^\infty, d_2^\infty, d_3^\infty\}$ (see Figure 3.5), and (iv) general, self-equilibrated, crack-face normal traction t_3 (see Figure 3.6(a)). For a semi-permeable penny shaped crack, two following loading cases are considered: (i) self-equilibrated, uniform normal traction t_3^0 (see Figure 3.4(a)) and (ii) a uniform remote triaxial stress $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and a uniform remote electrical induction $\{d_1^\infty, d_2^\infty, d_3^\infty\}$ (see Figure 3.5). Finally, for an energetically consistent circular crack, only the constant remote triaxial stress

$\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and a uniform remote electrical induction $\{d_1^\infty, d_2^\infty, d_3^\infty\}$ (see Figure 3.5) are considered.

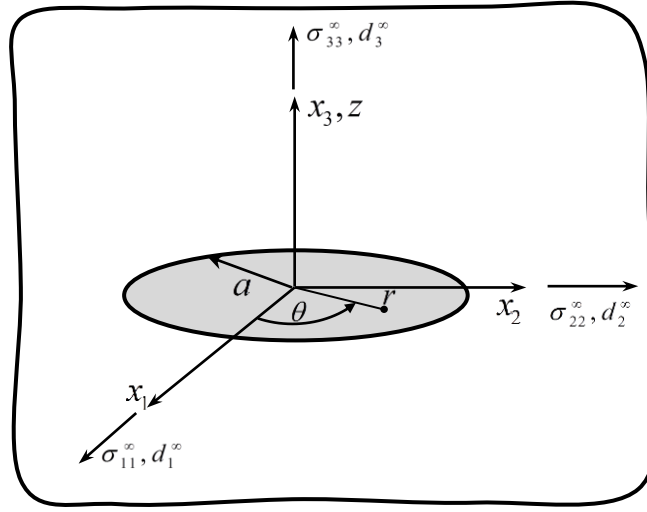


Figure 3.5: Schematic of circular crack under uniform remote triaxial stress $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and uniform remote electrical induction $\{d_1^\infty, d_2^\infty, d_3^\infty\}$

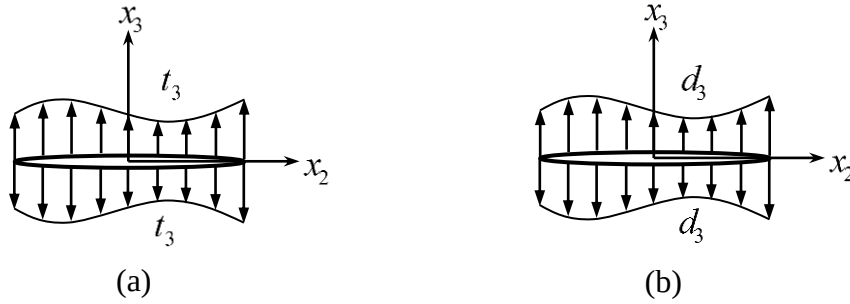


Figure 3.6: Schematic of circular crack under (a) general, self-equilibrated, crack-face, normal traction and (b) general, self-equilibrated, crack-face, surface charge

3.2.4 Near-front generalized stress field

By following an analogy as that employed by Williams (1957), the local generalized stress field in a region containing the point $\mathbf{x}_c$ on the crack front can be expressed as

$$\bar{\sigma}_{ij}(\mathbf{x}_c; \bar{\mathbf{r}}, \bar{\theta}) = \frac{1}{\sqrt{r}} \bar{\sigma}_{ij}^K(\mathbf{x}_c; \theta) + \bar{\sigma}_{ij}^T(\mathbf{x}_c; \theta) + \sum_{m=1}^{\infty} \bar{r}^{m/2} \bar{\sigma}_{ij}^{(m)}(\mathbf{x}_c; \theta) \quad (3.9)$$

where the “bar” is used to designate quantities referring to the local reference coordinate system $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ with the origin at $\mathbf{x}_c$ and the corresponding orthonormal basis $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ such that $\bar{x}_1 - \bar{x}_3$ is a tangent plane to the crack surface at $\mathbf{x}_c$ whereas the $\bar{x}_1 - \bar{x}_2$ plane is normal to the crack front at $\mathbf{x}_c$ as shown in Figure 3.7; $(\bar{r}, \bar{\theta})$ denotes the polar coordinates of a point in the $\bar{x}_1 - \bar{x}_2$ plane; $\bar{\sigma}_{ij}(\mathbf{x}_c; \bar{r}, \bar{\theta})$ denotes the generalized stress at any point in the $\bar{x}_1 - \bar{x}_2$ plane in the neighborhood of the point $\mathbf{x}_c$; $\bar{\sigma}_{ij}^K$, $\bar{\sigma}_{ij}^T$, and $\bar{\sigma}_{ij}^{(m)}$ are functions independent of the radial coordinate $\bar{r}$. This asymptotic expansion of the near-front field is expressed in the same form as that appearing in Rungamornrat and Pinitpanich (2016). Regarding to the near-front expansion (3.9), it is evident that the first term represents the dominant part of the field which is singular of order $1/\sqrt{\bar{r}}$ at the point $\mathbf{x}_c$; the second term represent the constant field that is independent of the radial coordinate $\bar{r}$, and the third term represents the non-singular part that vanishes identically at the point $\mathbf{x}_c$ along the crack front.

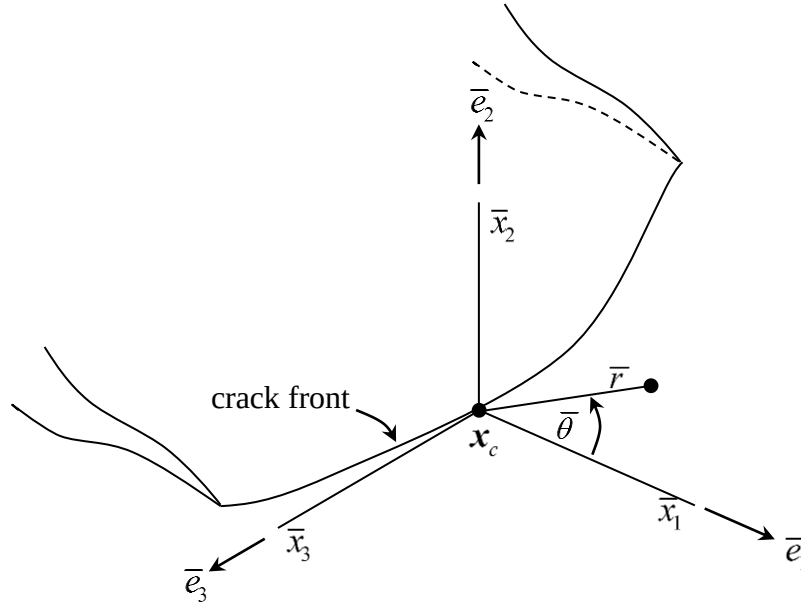


Figure 3.7: Schematic of crack front and local reference coordinate system for defining generalized T-stress components

Now, let us define the components $T_{ij} = T_{ij}(\mathbf{x}_c)$ referring to the local reference coordinate system $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ such that

$$T_{ij}(\mathbf{x}_c) = \bar{\sigma}_{ij}^T(\mathbf{x}_c; \bar{\theta} = 0) \quad (3.10)$$

From the definition (3.10) and the definition of the generalized stress, it is apparent that $T_{ij} = T_{ji}$ and, as a result, only nine components are independent. In addition, the components $T_{ij}(\mathbf{x}_c)$ represent the non-singular part of the generalized stress field at the point $\mathbf{x}_c$ and not all these

components are unknown a priori. From the continuity of the finite part of the generalized stress at the point $\mathbf{x}_c$, it can be readily verified that the four components T_{12} , T_{22} , T_{23} and T_{24} can be obtained in terms of the generalized crack-face traction at a limiting point $\mathbf{x}_c$ on the surface of the crack. As a result, $\{T_{12}, T_{22}, T_{23}, T_{24}\}$, $\{T_{12}, T_{22}, T_{23}\}$, $\{T_{12}, T_{22}, T_{23}\}$, and $\{T_{12}, T_{23}\}$ are known a priori for electrically impermeable, electrically permeable, electrically semi-permeable, and energetically consistent cracks, respectively. The components T_{22} and T_{24} can be readily obtained once the unknown crack-face generalized tractions are solved. The five independent components $\{T_{11}, T_{13}, T_{33}, T_{14}, T_{34}\}$ are always unknown and they are commonly termed the generalized T-stress components. The first three components $\{T_{11}, T_{13}, T_{33}\}$ are the mechanical T-stress components associated with the stress field similar to the elastic case (see also the work of Rungamornrat and Pinitpanich (2016)) whereas the last two components $\{T_{14}, T_{34}\}$ are the electrical T-stress components associated with the electric induction field. The generalized T-stress can be obtained if the generalized stress field at least in the region embedding the crack front is known.

From the near-front asymptotic expansion of the generalized stress field (3.9) and the definition (3.10), components of the generalized T-stress are given explicitly in terms of the generalized stress field by

$$T_Q(\mathbf{x}_c) = 2 \lim_{\bar{r} \rightarrow 0} \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \sqrt{\bar{r}} \bar{\sigma}_Q(\mathbf{x}_c; \bar{r}, \bar{\theta} = 0) \quad (3.11)$$

where $Q \in \{11, 13, 33, 14, 34\}$. According to the formula (3.11), it becomes evident that the generalized T-stress components can be calculated from existing generalized stress fields of a cracked body via standard differentiations and limiting processes without directly carrying the series expansion.

3.2.5 Existing generalized stress fields

It is apparent from the earlier section that the generalized stress field is the key ingredient for extracting the generalized T-stress. To construct the complete analytical solution for the generalized stress field, various analytical techniques based on the integral transforms, the representation theories, and potential-function-based theories have been applied successfully for cracked bodies with simple crack and domain geometries and loading conditions as recognized in the literature. In the present investigation, existing complete solutions of the generalized stress field for a circular crack embedded in a three-dimensional, transversely isotropic, linear piezoelectric, whole space subjected to four different crack-face conditions are utilized as the basis for the derivation of the generalized T-stress components for various scenarios:

- An impermeable crack: Chen and Shioya (1999) applied the potential-function-based theory to derive the closed-form solution of the generalized stress field of a penny-shaped crack in a three-dimensional piezoelectric whole space under a pair of self-equilibrated, normal concentrated forces and a pair of self-equilibrated concentrated electric charges acting on both crack surfaces. For the special case of uniform remote mechanical and electrical loading, Chen et al. (2000) utilized the potential-function-based theory together

with superposition method to derive the closed-form solution of the generalized stress field.

- A permeable crack: Chen and Lim (2005) applied the potential-function-based theory to obtain the analytical solution of the generalized stress field of a circular crack in a three-dimensional, piezoelectric whole space under a pair of opposite, unit normal, concentrated forces and uniform remote mechanical and electrical loading conditions.
- A semi-permeable crack: Li and Lee (2004b) employed the Hankel integral transform technique to obtain the exact solution of the generalized stress field of a circular crack in a three-dimensional piezoelectric whole space under uniform remote mechanical and electrical loading conditions.
- An energetically consistent crack: Li et al. (2011) applied the classical complex potential theory to derive the generalized stress field of a circular crack contained in a three-dimensional, piezoelectric whole space under uniform remote electrical and mechanical loading conditions.

3.2.6 Generalized T-stress for general generalized traction

The generalized T-stress Green's functions obtained from equation (3.11) for the electrically impermeable and permeable circular cracks are utilized along with the superposition method to construct an integral formula for computing the generalized T-stress components of circular cracks under general, crack-face, normal traction and general, crack-face, surface electrical charge for the impermeable case and under arbitrarily distributed, surface electric charge for the permeable case.

For the impermeable case, let us consider a circular crack under general, self-equilibrated, crack-face, normal traction $t_3^+ = -t_3^-$ and surface electrical charge $t_4^+ = -t_4^-$. The generalized T-stress components for this case, denoted by T_Q for $Q \in \{11, 13, 33, 14, 34\}$, can be obtained in an integral form as

$$T_Q(\theta) = \int_{-\pi}^{\pi} \int_0^a T_Q^{GP}(\theta; r_1, \theta_1) t_3^+(t_1, \theta_1) r_1 dr_1 d\theta_1 + \int_{-\pi}^{\pi} \int_0^a T_Q^{GE}(\theta; r_2, \theta_2) t_4^+(r_2, \theta_2) r_2 dr_2 d\theta_2 \quad (3.12)$$

where T_Q^{GP} is the generalized T-stress Green's function due to a pair of opposite, unit normal concentrated forces under the impermeable crack-face boundary condition and T_Q^{GE} is the generalized T-stress Green's function due to a pair of opposite, unit concentrated electrical charges under the impermeable crack-face boundary condition.

For the permeable case, consider a circular crack under a general, self-equilibrated, crack-face, normal traction $t_3^+ = -t_3^-$. By following the same methodology as that for the impermeable case, the integral formula of the generalized T-stress components is then obtained as

$$T_Q(\theta) = \int_{-\pi}^{\pi} \int_0^a \bar{T}_Q^{GP}(\theta; r_1, \theta_1) t_3^+(t_1, \theta_1) r_1 dr_1 d\theta \quad (3.13)$$

where $\bar{T}_Q^{GP}$ is the generalized T-stress Green's function due to a pair of opposite, unit normal concentrated forces under the electrically permeable crack-face boundary condition.

3.3 Determination of Generalized T-stress Components

By applying the formula (3.11) along with existing solutions of the generalized stress field indicated in Section 3.2, the direct differentiations, and proper limiting procedure, the generalized T-stress components can be derived in an exact form involving only elementary functions. It is noted that the generalized stress field of cracked media under the remote loading can be readily related to that of the cracked medium under an equivalent crack-face generalized traction via the method of superposition. This chapter mainly presents the exact solutions of the generalized T-stress components under four different crack-face conditions and the Green's functions for the generalized T-stress components for both impermeable and permeable crack-face conditions. Finally, the integral formula of the generalized T-stress components for both electrically impermeable and permeable cracks under general loading conditions is presented.

3.3.1 Generalized T-stress of impermeable crack under uniform load

The generalized stress fields proposed by Chen and Shioya (1999) are applied together with the methodology described in the previous chapter to provide the generalized T-stress components for an impermeable circular crack under the uniform remote electrical inductions $\{d_1^\infty, d_2^\infty, d_3^\infty\}$, the uniform remote triaxial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$, and the self-equilibrated, uniform normal traction t_3^0 and self-equilibrated uniform electrical charge d_3^0 on the crack surface. The final set of results is given explicitly as

$$T_{11} = A \left\{ \sum_{i=1}^3 [(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2}] p_i \right\} + \bar{\sigma}_{11}^\infty \quad (3.14)$$

$$T_{33} = A \left\{ \sum_{i=1}^3 [(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2}] p_i \right\} + \bar{\sigma}_{33}^\infty \quad (3.15)$$

$$T_{13} = \bar{\sigma}_{13}^\infty, \quad T_{14} = \bar{d}_1^\infty, \quad T_{34} = \bar{d}_3^\infty \quad (3.16)$$

where $\{\bar{\sigma}_{11}^\infty, \bar{\sigma}_{13}^\infty, \bar{\sigma}_{33}^\infty\}$ and $\{\bar{d}_1^\infty, \bar{d}_3^\infty\}$ are components of the remote triaxial stresses and remote electrical inductions referring to the local coordinate system defined in section 3.4 and all involved quantities are given by

$$A = 1/(g_2g_3 - g_1g_4) \quad (3.17)$$

$$p_i = (d_i g_3 - c_i g_4)(\sigma_{33}^\infty + t_3^0) + (c_i g_2 - d_i g_1)(d_3^\infty + d_3^0) \quad (3.18)$$

$$g_1 = -\sum_{i=1}^3 c_i \gamma_{i1} \quad (3.19)$$

$$g_2 = -\sum_{i=1}^3 d_i \gamma_{i1} \quad (3.20)$$

$$g_3 = -\sum_{i=1}^3 c_i \gamma_{i2} \quad (3.21)$$

$$g_4 = -\sum_{i=1}^3 d_i \gamma_{i2} \quad (3.22)$$

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{1}{2\pi} \begin{bmatrix} s_1 & s_2 & s_3 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \quad (3.23)$$

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \frac{1}{2\pi} \begin{bmatrix} s_1 & s_2 & s_3 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{bmatrix}^{-1} \begin{Bmatrix} e_{15}/c_{44} \\ 0 \\ -1 \end{Bmatrix} \quad (3.24)$$

$$\gamma_{i1} = -c_{13} + c_{33}s_i\alpha_{i1} + e_{33}s_i\alpha_{i2} \quad (3.25)$$

$$\gamma_{i2} = -e_{31} + e_{33}s_i\alpha_{i1} - \varepsilon_{33}s_i\alpha_{i2} \quad (3.26)$$

$$\alpha_{i1} = \frac{c_{11}\varepsilon_{11} - m_3s_i^2 + c_{44}\varepsilon_{33}s_i^4}{(m_1 - m_2s_i^2)s_i} \quad (3.27)$$

$$\alpha_{i2} = \frac{c_{11}e_{15} - m_4s_i^2 + c_{44}e_{33}s_i^4}{(m_1 - m_2s_i^2)s_i} \quad (3.28)$$

$$m_1 = \varepsilon_{11}(c_{13} + c_{44}) + e_{15}(e_{15} + e_{31}) \quad (3.29)$$

$$m_2 = \varepsilon_{33}(c_{13} + c_{44}) + e_{33}(e_{15} + e_{31}) \quad (3.30)$$

$$m_3 = c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2 \quad (3.31)$$

$$m_4 = c_{11}e_{33} + c_{44}e_{15} - (c_{13} + c_{44})(e_{15} + e_{31}) \quad (3.32)$$

$$s_4^2 = c_{66}/c_{44} \quad (3.33)$$

All involved material constants are defined by $c_{11} = E_{1111}$, $c_{13} = E_{1133}$, $c_{33} = E_{3333}$, $c_{44} = E_{1313}$, $c_{66} = E_{1212}$, $e_{15} = E_{1341}$, $e_{31} = E_{1143}$, $e_{33} = E_{3343}$, $\varepsilon_{11} = -E_{1441}$, $\varepsilon_{33} = -E_{3443}$ and the three parameters $\{s_1^2, s_2^2, s_3^2\}$ are solutions of the nonlinear equation

$$as^6 - bs^4 + cs^2 - d = 0 \quad (3.34)$$

where constants a, b, c, d are defined by

$$a = c_{44}(e_{33}^2 + c_{33}\varepsilon_{33}) \quad (3.35)$$

$$b = c_{33}m_3 + \varepsilon_{33}[c_{44}^2 - (c_{13} + c_{44})^2] + e_{33}(2m_4 - c_{11}e_{33}) \quad (3.36)$$

$$c = c_{44}m_3 + \varepsilon_{11}[c_{11}c_{33} - (c_{13} + c_{44})^2] + e_{15}(2m_4 - c_{44}e_{15}) \quad (3.37)$$

$$d = c_{11}(e_{15}^2 + c_{44}\varepsilon_{11}) \quad (3.38)$$

All involved parameters given above can be also found in the work of Chen and Shioya (1999).

3.3.2 Generalized T-stress of permeable crack under uniform load

By using the generalized stress field proposed by Chen and Lim (2005) along with the same procedure utilized in Section 3.3.1, the closed-form solution of the generalized T-stress components for an electrically permeable circular crack under the uniform remote triaxial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$, the uniform remote electrical inductions $\{d_1^\infty, d_2^\infty, d_3^\infty\}$, and the self-equilibrated, crack-face, uniform normal traction t_3^0 can be obtained as

$$T_{11} = \frac{\sigma_{33}^\infty + t_3^0}{g_1} \sum_{i=1}^3 [(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2}]c_i + \bar{\sigma}_{11}^\infty \quad (3.39)$$

$$T_{33} = \frac{\sigma_{33}^\infty + t_3^0}{g_1} \sum_{i=1}^3 [(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2}]c_i + \bar{\sigma}_{33}^\infty \quad (3.40)$$

$$T_{13} = \bar{\sigma}_{13}^\infty, \quad T_{14} = \bar{d}_1^\infty, \quad T_{34} = \bar{d}_3^\infty \quad (3.41)$$

where parameter c_i are given by

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = -\frac{1}{2\pi} \begin{bmatrix} \gamma_{11}s_1 & \gamma_{21}s_2 & \gamma_{31}s_3 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad (3.42)$$

and all remaining parameters are defined in the same fashion as those provided in Section 3.3.1 (the definition of all parameters can be also found in the work of Chen and Lim (2005)).

3.3.3 Generalized T-stress of semi-permeable crack under uniform load

By starting with results reported in Li and Lee (2004b), the generalized T-stress components for an electrically semi-permeable circular crack under the uniform remote tri-axial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$, the uniform remote electrical inductions $\{d_1^\infty, d_2^\infty, d_3^\infty\}$, and the self-equilibrated, crack-face, uniform normal traction t_3^0 can be obtained explicitly as

$$T_{11} = \frac{P}{2\kappa} \sum_{i=1}^3 [2\beta_{0i}a_i + (c_{11} - c_{12})a_i] + \bar{\sigma}_{11}^\infty \quad (3.43)$$

$$T_{33} = \frac{P}{2\kappa} \sum_{i=1}^3 [2\beta_{5i}a_i - (c_{11} - c_{12})a_i] + \bar{\sigma}_{33}^\infty \quad (3.44)$$

$$T_{13} = \bar{\sigma}_{13}^\infty, \quad T_{14} = \bar{d}_1^\infty, \quad T_{34} = \bar{d}_3^\infty \quad (3.45)$$

where all involved parameters are defined by

$$P = \begin{cases} d_3^\infty - d_3^c, & \sigma_{33}^\infty = t_3^0 = 0 \\ \sigma_{33}^\infty + t_3^0, & \sigma_{33}^\infty \neq 0 \text{ or } t_3^0 \neq 0 \end{cases} \quad (3.46)$$

$$\kappa = \begin{cases} \sum_{j=1}^3 \beta_{4j} a_j, & \sigma_{33}^\infty = t_3^0 = 0 \\ \sum_{j=1}^3 \beta_{1j} a_j, & \sigma_{33}^\infty \neq 0 \text{ or } t_3^0 \neq 0 \end{cases} \quad (3.47)$$

$$\begin{Bmatrix} a_2/a_1 \\ a_3/a_1 \end{Bmatrix} = -(\mathbf{A}^T)^{-1} \mathbf{B} \quad (3.48)$$

$$\mathbf{A} = \begin{bmatrix} \beta_{22} & (\sigma_{33}^\infty + t_3^0)\beta_{42} + (d_3^c - d_3^\infty)\beta_{12} \\ \beta_{23} & (\sigma_{33}^\infty + t_3^0)\beta_{43} + (d_3^c - d_3^\infty)\beta_{13} \end{bmatrix} \quad (3.49)$$

$$\mathbf{B} = \begin{Bmatrix} \beta_{21} \\ (\sigma_{33}^\infty + t_3^0)\beta_{41} + (d_3^c - d_3^\infty)\beta_{11} \end{Bmatrix} \quad (3.50)$$

$$d_3^c = \frac{-m_1 \pm \sqrt{m_1^2 - 4m_0 m_2}}{2m_2} \quad (3.51)$$

$$m_0 = \varepsilon_c (\sigma_{33}^\infty + t_3^0) \det[\boldsymbol{\beta}_4 \quad \boldsymbol{\beta}_2 \quad \boldsymbol{\eta}_2] - \varepsilon_c d_3^\infty \det[\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad \boldsymbol{\eta}_2] \quad (3.52)$$

$$m_1 = \varepsilon_c \det[\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad \boldsymbol{\eta}_2] + (\sigma_{33}^\infty + t_3^0) \det[\boldsymbol{\beta}_4 \quad \boldsymbol{\beta}_2 \quad \boldsymbol{\eta}_1] - d_3^\infty \det[\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad \boldsymbol{\eta}_1] \quad (3.53)$$

$$m_2 = \det[\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad \boldsymbol{\eta}_1] \quad (3.54)$$

$$\boldsymbol{\beta}_k = \begin{Bmatrix} \beta_{k1} \\ \beta_{k2} \\ \beta_{k3} \end{Bmatrix}, k = 1, 2, 4 \quad (3.55)$$

$$\boldsymbol{\eta}_{k-2} = \begin{Bmatrix} \eta_{k1}\gamma_1 \\ \eta_{k2}\gamma_2 \\ \eta_{k3}\gamma_3 \end{Bmatrix}, k = 3, 4 \quad (3.56)$$

$$\beta_{0j} = (c_{13}n_{3j} + e_{31}n_{4j})\gamma_j^2 - c_{11} \quad (3.57)$$

$$\beta_{1j} = (c_{33}n_{3j} + e_{33}n_{4j})\gamma_j^2 - c_{13} \quad (3.48)$$

$$\beta_{2j} = [c_{44}(1 + \eta_{3j}) + e_{15}n_{4j}]\gamma_j \quad (3.59)$$

$$\beta_{3j} = [e_{15}(1 + \eta_{3j}) - \varepsilon_{11}n_{4j}]\gamma_j \quad (3.60)$$

$$\beta_{4j} = (e_{33}n_{3j} - \varepsilon_{33}n_{4j})\gamma_j^2 - e_{31} \quad (3.61)$$

$$\beta_{5j} = (c_{13}n_{3j} + e_{31}n_{4j})\gamma_j^2 - c_{12} \quad (3.62)$$

with ε_c denoting the dielectric permittivity of a medium filled within the crack cavity and γ_j^2 , η_{3j} and η_{4j} for $j = 1, 2, 3$ being solved from the two nonlinear algebraic equations

$$\gamma_j^2 = \frac{c_{11}}{c_{44} + (c_{13} + c_{44})\eta_{3j} + (e_{31} + e_{15})\eta_{4j}} = \frac{c_{13} + c_{44} + c_{44}\eta_{3j} + e_{15}\eta_{4j}}{c_{33}\eta_{3j} + e_{33}\eta_{4j}} \quad (3.63)$$

$$\gamma_j^2 = \frac{c_{11}}{c_{44} + (c_{13} + c_{44})\eta_{3j} + (e_{31} + e_{15})\eta_{4j}} = \frac{e_{31} + e_{15} + e_{15}\eta_{3j} - \varepsilon_{11}\eta_{4j}}{e_{33}\eta_{3j} - \varepsilon_{33}\eta_{4j}} \quad (3.64)$$

The equations (3.63) and (3.64) can be further reduced to a cubic equation in terms of γ_j^2 :

$$a_0(\gamma_j^2)^3 + b_0(\gamma_j^2)^2 + c_0(\gamma_j^2) + d_0 = 0 \quad (3.65)$$

where a_0, b_0, c_0, d_0 are constants given by

$$a_0 = c_{44}(c_{33}\varepsilon_{33} + e_{33}^2) \quad (3.66)$$

$$b_0 = -c_{33}c_{44}\varepsilon_{11} + c_{13}^2\varepsilon_{33} - c_{11}c_{33}\varepsilon_{33} + 2c_{13}c_{44}\varepsilon_{33} - c_{33}e_{15}^2 - c_{11}e_{33}^2 - 2c_{33}e_{31}e_{15} - c_{33}e_{31}^2 + 2c_{13}e_{33}e_{15} + 2c_{13}e_{33}e_{31} + 2c_{44}e_{33}e_{31} \quad (3.67)$$

$$c_0 = -c_{13}^2\varepsilon_{11} + c_{11}c_{33}\varepsilon_{11} - 2c_{13}c_{44}\varepsilon_{11} + c_{11}c_{44}\varepsilon_{33} - 2c_{13}e_{15}^2 - 2c_{13}e_{15}e_{31} + c_{44}e_{31}^2 + 2c_{11}e_{15}e_{33} \quad (3.68)$$

$$d_0 = -c_{11}(c_{44}\varepsilon_{11} + e_{15}^2) \quad (3.69)$$

It should be remarked that only the value of d_3^c computed from (3.53) that falls between those of the two limiting cases (i.e., electrically permeable and electrically impermeable cases) is acceptable. Remark that the closed-form solution of the semi-permeable model can be further reduced to the special cases of impermeable and permeable models by taking the dielectric permittivity of the crack medium within the crack cavity ε_c to zero and infinity, respectively.

3.3.4 Generalized T-stress of energetically consistent crack under uniform load

The generalized T-stress components for an energetically consistent circular crack under the uniform remote triaxial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and the uniform remote electrical inductions $\{d_1^\infty, d_2^\infty, d_3^\infty\}$ can be derived using the same procedure and the analytical solution presented by Li et al. (2011). Final results are given explicitly by

$$T_{11} = \frac{\pi i}{4} \sum_{j=1}^3 A_j \left[(c_{11} + c_{12}) - \frac{2(c_{13}k_{2j} - e_{31}k_{1j})}{v_j^2} \right] + \bar{\sigma}_{11}^\infty \quad (3.70)$$

$$T_{33} = \frac{\pi i}{4} \sum_{j=1}^3 A_j \left[(c_{11} + c_{12}) - \frac{2(c_{13}k_{2j} - e_{31}k_{1j})}{v_j^2} \right] + \bar{\sigma}_{33}^\infty \quad (3.71)$$

$$T_{13} = \bar{\sigma}_{13}^\infty, \quad T_{14} = \bar{d}_1^\infty, \quad T_{34} = \bar{d}_3^\infty \quad (3.72)$$

where all involved parameters are defined by (also see the work of Li et al. (2011))

$$A_j = \Lambda_{j2} \left[\sigma_{33}^\infty - \frac{(d_3^c)^2}{2\varepsilon_c} \right] + \Lambda_{j3} (d_3^\infty - d_3^c) \quad (3.73)$$

$$H_1 = \pi \sum_{j=1}^3 \frac{k_{2j}}{v_j} \Lambda_{j2} \quad (3.74)$$

$$H_2 = \pi \sum_{j=1}^3 \frac{k_{2j}}{v_j} \Lambda_{j3} \quad (3.75)$$

$$H_3 = -\pi \sum_{j=1}^3 \frac{k_{1j}}{v_j} \Lambda_{j2} \quad (3.76)$$

$$H_4 = -\pi \sum_{j=1}^3 \frac{k_{1j}}{v_j} \Lambda_{j3} \quad (3.77)$$

$$d_3^c = -\varepsilon_c \frac{H_3 \left[\sigma_{33}^\infty - \frac{(d_3^c)^2}{2\varepsilon_c} \right] + H_4 (d_3^\infty - d_3^c)}{H_1 \left[\sigma_{33}^\infty - \frac{(d_3^c)^2}{2\varepsilon_c} \right] + H_2 (d_3^\infty - d_3^c)} \quad (3.78)$$

$$\mathbf{\Lambda} = \mathbf{B}^{-1} \quad (3.79)$$

$$B_{1j} = \frac{1}{v_j} [c_{44}(1 + k_{2j}) - e_{15}k_{1j}] \quad (3.80)$$

$$B_{2j} = \frac{\pi i}{2} [c_{44}(1 + k_{2j}) - e_{15}k_{1j}] \quad (3.81)$$

$$B_{3j} = \frac{\pi i}{2} [e_{44}(1 + k_{2j}) + \varepsilon_{11}k_{1j}] \quad (3.82)$$

with v_j , k_{1j} and k_{2j} obtained from

$$v_j^2 = \frac{c_{44} + (c_{13} + c_{44})k_{2j} - (e_{31} + e_{15})k_{1j}}{c_{11}} = \frac{c_{33}k_{2j} - e_{33}k_{1j}}{(c_{13} + c_{44}) + c_{44}k_{2j} - e_{15}k_{1j}} \quad (3.83)$$

$$v_j^2 = \frac{c_{44} + (c_{13} + c_{44})k_{2j} - (e_{31} + e_{15})k_{1j}}{c_{11}} = \frac{e_{33}k_{2j} + \varepsilon_{33}k_{1j}}{(e_{15} + e_{31}) + \varepsilon_{11}k_{1j} + e_{15}k_{2j}} \quad (3.84)$$

It is also important to remark that d_3^c computed from (3.78) is acceptable if it induces a positive value of the crack-opening displacement. Remark that the analytical solution associated with the

energetically consistent model can be further specialized to the solutions of impermeable and permeable crack-face conditions by taking the dielectric permittivity of a medium within the crack cavity ε_c to zero and infinity, respectively.

3.3.5 Generalized T-stress Green's function for impermeable crack

Consider a problem concerning a circular crack under a pair of opposite, unit normal point forces and electric point charges as reported by Chen and Shioya (1999). The closed-form solution of the generalized-stress Green's function is given by

$$\sigma_1^G = 8A \sum_{j=1}^3 \left[(c_{66} - c_{11}) + c_{13}s_j\alpha_{j1} + e_{31}s_j\alpha_{j2} \right] \left[\beta_{j1}f_{31}(z_j) + \beta_{j2}f_{32}(z_j) \right] \quad (3.85)$$

$$\sigma_2^G = 8Ac_{66} \sum_{j=1}^3 \left[\beta_{j1}f_{41}(z_j) + \beta_{j2}f_{42}(z_j) \right] \quad (3.86)$$

$$\tau_z^G = 4A \sum_{j=1}^3 \left[c_{44}(s_j + \alpha_{j1}) + e_{15}\alpha_{j2} \right] \left[\beta_{j1}f_{51}(z_j) + \beta_{j2}f_{52}(z_j) \right] \quad (3.87)$$

$$D^G = 4A \sum_{j=1}^3 \left[e_{15}(s_j + \alpha_{j1}) - \varepsilon_{11}\alpha_{j2} \right] \left[\beta_{j1}f_{51}(z_j) + \beta_{j2}f_{52}(z_j) \right] \quad (3.88)$$

in which a superscript “G” is used to designate Green's function, $i = \sqrt{-1}$, $z_j = s_j z$,

$$\sigma_1^G = \sigma_{rr}^G + \sigma_{\theta\theta}^G \quad (3.89)$$

$$\sigma_2^G = (\sigma_{rr}^G - \sigma_{\theta\theta}^G + 2i\sigma_{r\theta}^G) e^{2i\theta} \quad (3.90)$$

$$\tau_z^G = (\sigma_{rz}^G + \sigma_{\theta z}^G) e^{i\theta} \quad (3.91)$$

$$D^G = (D_r^G + D_\theta^G) e^{i\theta} \quad (3.92)$$

$$\beta_{j1} = c_j g_4 - d_j g_3 \quad (3.93)$$

$$\beta_{j2} = d_j g_1 - c_j g_2 \quad (3.94)$$

All other involved parameters defined in Section 3.3.1 apply here and the complex functions $f_{3j}(z_j)$, $f_{4j}(z_j)$, $f_{5j}(z_j)$ are defined by

$$f_{3j}(z) = \frac{z}{R_j^3} \tan^{-1} \left(\frac{h_j}{R_j} \right) - \frac{h_j}{z(R_j^2 + h_j^2)} \left[\frac{r^2 - l_1^2}{l_2^2 - l_1^2} - \frac{z^2}{R_j^2} \right] \quad (3.95)$$

$$\begin{aligned} f_{4j}(z) = & \frac{\sqrt{a^2 - r_j^2}}{\bar{t}_j \bar{s}_j} \left(\frac{2}{\bar{t}_j} - \frac{r_j e^{i\theta_j}}{\bar{s}_j^2} \right) \tan^{-1} \left[\frac{\bar{s}_j}{\sqrt{l_2^2 - a^2}} \right] - \frac{z(3R_j^2 - z^2)}{\bar{t}_j^2 R_j^3} \tan^{-1} \left(\frac{h_j}{R_j} \right) \\ & + \frac{\sqrt{a^2 - r_j^2} \sqrt{l_2^2 - a^2} r_j e^{i\theta_j}}{\bar{t}_j \bar{s}_j^2 [l_2^2 - r r_j e^{-i(\theta - \theta_j)}]} - \frac{z h_j}{R_j^2 + h_j^2} \left[\frac{t_j}{\bar{t}_j R_j^2} - \frac{r^2 e^{2i\theta}}{(l_2^2 - l_1^2)(l_2^2 - r^2)} \right] \end{aligned} \quad (3.96)$$

$$f_{5j}(z) = \frac{t_j}{R_j^3} \tan^{-1} \left(\frac{h_j}{R_j} \right) + \frac{h_j}{R_j^2 + h_j^2} \left[\frac{re^{i\theta}}{l_2^2 - l_1^2} + \frac{t_j}{R_j^2} \right] \quad (3.97)$$

with all involved parameters defined explicitly by

$$l_1 = \frac{1}{2} \left[\sqrt{(r+a)^2 + z^2} - \sqrt{(r-a)^2 + z^2} \right] \quad (3.98)$$

$$l_2 = \frac{1}{2} \left[\sqrt{(r+a)^2 + z^2} + \sqrt{(r-a)^2 + z^2} \right] \quad (3.99)$$

$$h_j = \frac{\sqrt{a^2 - l_1^2} \sqrt{a^2 - r_j^2}}{a} \quad (3.100)$$

$$t_j = re^{i\theta} - r_j e^{i\theta_j} \quad (3.101)$$

$$\bar{t}_j = re^{-i\theta} - r_j e^{-i\theta_j} \quad (3.102)$$

$$\bar{s}_j = \sqrt{a^2 - rr_j e^{-i(\theta - \theta_j)}} \quad (3.103)$$

$$R_j = \sqrt{r^2 + r_j^2 - 2rr_j \cos(\theta - \theta_j) + z^2} \quad (3.104)$$

By first noting following properties (see also the work of Rungamornrat and Pinitpanich (2016))

$$\lim_{z \rightarrow 0} \frac{\sqrt{a - l_1}}{z} = \frac{\sqrt{a}}{\sqrt{2(r^2 - a^2)}} \quad (3.105)$$

$$\lim_{z \rightarrow 0} \frac{zh_j}{l_2 - r} = \frac{2\sqrt{r^2 - a^2} \sqrt{a^2 - r_j^2}}{r} \quad (3.106)$$

and then substituting these results into (3.95)-(3.97) and (3.85)-(3.88), the Green's function for the generalized stress at any point on the plane $z = 0$ is given explicitly by

$$\sigma_1^G(r, \theta, 0; r_\alpha, \theta_\alpha) = 8A \sum_{j=1}^3 \left[(c_{66} - c_{11}) + c_{13}s_j \alpha_{j1} + e_{31}s_j \alpha_{j2} \right] \left[\beta_{j1} f_{31}(0) + \beta_{j2} f_{32}(0) \right] \quad (3.107)$$

$$\sigma_2^G(r, \theta, 0; r_\alpha, \theta_\alpha) = 8Ac_{66} \sum_{j=1}^3 \left[\beta_{j1} f_{41}(0) + \beta_{j2} f_{42}(0) \right] \quad (3.108)$$

$$\tau_z^G(r, \theta, 0; r_\alpha, \theta_\alpha) = 0 \quad (3.109)$$

$$D^G(r, \theta, 0; r_\alpha, \theta_\alpha) = 0 \quad (3.110)$$

where $f_{3\alpha}(0)$, $f_{4\alpha}(0)$, $f_{5\alpha}(0)$ for $\alpha = 1, 2$ are given by

$$f_{3\alpha}(0) = -\frac{\sqrt{a^2 - r_\alpha^2}}{\sqrt{r^2 - a^2} R_{0\alpha}^2} \quad (3.111)$$

$$f_{4\alpha}(0) = \frac{\sqrt{a^2 - r_\alpha^2}}{\bar{t}_\alpha \bar{s}_\alpha} \left(\frac{2}{\bar{t}_\alpha} - \frac{r_\alpha e^{i\theta_\alpha}}{\bar{s}_\alpha^2} \right) \tan^{-1} \left(\frac{\bar{s}_\alpha}{\sqrt{r^2 - a^2}} \right) + \frac{\sqrt{r^2 - a^2} \sqrt{a^2 - r_\alpha^2} r_\alpha e^{i\theta_\alpha}}{\bar{t}_\alpha \bar{s}_\alpha^2 (r^2 - r r_\alpha e^{-i(\theta - \theta_\alpha)})} + \frac{e^{2i\theta} \sqrt{a^2 - r_\alpha^2}}{R_{0\alpha}^2 \sqrt{r^2 - a^2}} \quad (3.112)$$

$$f_{5\alpha}(0) = 0 \quad (3.113)$$

with the parameter $R_{0\alpha}$ given by

$$R_{0\alpha} = R_\alpha(z=0) = \sqrt{r^2 + r_\alpha^2 - 2r r_\alpha \cos(\theta - \theta_\alpha)} \quad (3.114)$$

By introducing the variable transformation $\bar{r} = r - a$ together with the proper limiting process, for $(r_\alpha, \theta_\alpha) \neq (a, \theta)$, it is readily to show that

$$\lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \hat{r}} \left[\sqrt{\bar{r}} \left(\frac{\sqrt{a^2 - r_\alpha^2}}{\sqrt{r^2 - a^2}} \right) \right] \right\} = 0 \quad (3.115)$$

$$\lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \hat{r}} \left[\sqrt{\bar{r}} \left(\frac{\sqrt{r^2 - a^2} \sqrt{a^2 - r_\alpha^2}}{\bar{t}_\alpha \bar{s}_\alpha^2 (r^2 - r r_\alpha e^{-i(\theta - \theta_\alpha)})} \right) \right] \right\} = 0 \quad (3.116)$$

$$\lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \hat{r}} \left[\sqrt{\bar{r}} \left(\frac{\sqrt{a^2 - r_\alpha^2}}{\bar{t}_\alpha \bar{s}_\alpha} \left(\frac{2}{\bar{t}_\alpha} - \frac{r_\alpha e^{i\theta_\alpha}}{\bar{s}_\alpha^2} \right) \tan^{-1} \left(\frac{\bar{s}_\alpha}{\sqrt{r^2 - a^2}} \right) \right) \right] \right\} = \pi \sqrt{a^2 - r_\alpha^2} \left(\frac{1}{2\bar{t}_{0\alpha}^2 \bar{s}_{0\alpha}} - \frac{r_\alpha e^{i\theta_\alpha}}{4\bar{t}_{0\alpha} \bar{s}_{0\alpha}^3} \right) \quad (3.117)$$

where $\bar{t}_{0\alpha} = a e^{-i\theta} - r_\alpha e^{-i\theta_\alpha}$ and $\bar{s}_{0\alpha} = \sqrt{a^2 - a r_\alpha e^{-i(\theta - \theta_\alpha)}}$. By substituting (3.115)-(3.117) into (3.110)-(3.113) and (3.11) along with the relations (3.89)-(3.92), the generalized T-stress components are obtained as

$$T_{11}^G(\theta; r_\alpha, \theta_\alpha) = 4\pi A c_{66} \sum_{j=1}^3 \sum_{\alpha=1}^2 \Re \left[\beta_{j\alpha} \left(\frac{\sqrt{a^2 - r_\alpha^2}}{\sqrt{a} (a e^{-i\theta} - r_\alpha e^{-i\theta_\alpha}) \sqrt{a - r_\alpha e^{-i(\theta - \theta_\alpha)}}} \right) \times \left(\frac{1}{a - r_\alpha e^{-i(\theta_\alpha - \theta)}} - \frac{r_\alpha e^{i(\theta_\alpha - \theta)}}{2a(a - r_\alpha e^{-i(\theta - \theta_\alpha)})} \right) \right] \quad (3.118)$$

$$T_{33}^G(\theta; r_\alpha, \theta_\alpha) = 4\pi A c_{66} \sum_{j=1}^3 \sum_{\alpha=1}^2 \Re \left[\beta_{j\alpha} \left(\frac{\sqrt{a^2 - r_\alpha^2}}{\sqrt{a}(ae^{-i\theta} - r_\alpha e^{-i\theta_\alpha}) \sqrt{a - r_\alpha e^{-i(\theta - \theta_\alpha)}}} \right) \right. \\ \left. \times \left(\frac{r_\alpha e^{i(\theta_\alpha - \theta)}}{2a(a - r_\alpha e^{-i(\theta - \theta_\alpha)})} - \frac{1}{a - r_\alpha e^{-i(\theta_\alpha - \theta)}} \right) \right] \quad (3.119)$$

$$T_{13}^G(\theta; r_\alpha, \theta_\alpha) = 4\pi A c_{66} \sum_{j=1}^3 \sum_{\alpha=1}^2 \Im \left[\beta_{j\alpha} \left(\frac{\sqrt{a^2 - r_\alpha^2}}{\sqrt{a}(ae^{-i\theta} - r_\alpha e^{-i\theta_\alpha}) \sqrt{a - r_\alpha e^{-i(\theta - \theta_\alpha)}}} \right) \right. \\ \left. \times \left(\frac{1}{a - r_\alpha e^{-i(\theta_\alpha - \theta)}} - \frac{r_\alpha e^{i(\theta_\alpha - \theta)}}{2a(a - r_\alpha e^{-i(\theta - \theta_\alpha)})} \right) \right] \quad (3.120)$$

$$T_{14}^G = 0 \quad (3.121)$$

$$T_{34}^G = 0 \quad (3.122)$$

where $\Re(f)$ and $\Im(f)$ represent real and imaginary parts of a complex function f , respectively. The expressions (3.118)-(3.120) can be reduced further by introducing new variables l_1, l_2, ϕ_1, ϕ_2 satisfying

$$r_1 \sin(\theta_1 - \theta) = l_1 \sin \phi; \quad r_1 \cos(\theta_1 - \theta) + l_1 \cos \phi = a \quad (3.123)$$

$$r_2 \sin(\theta_2 - \theta) = l_2 \sin \phi; \quad r_2 \cos(\theta_2 - \theta) + l_2 \cos \phi = a \quad (3.124)$$

and $\beta_{j\alpha} = A_{j\alpha} + iB_{j\alpha}$ where $A_{j\alpha}$ and $B_{j\alpha}$ are real and imaginary parts of $\beta_{j\alpha}$. The graphical interpretation of the new variables l_1, l_2, ϕ_1, ϕ_2 is indicated in Figure 3.8. The final form of the generalized T-stress Green's function is now given by

$$T_Q^G = T_Q^{GP} + T_Q^{GE} \quad (3.125)$$

where T_Q^{GP} and T_Q^{GE} are components of the generalized T-stress Green's function due to a pair of opposite, unit normal concentrated forces and a pair of opposite, unit concentrated, electrical charges, respectively. The explicit expressions of both T_Q^{GP} and T_Q^{GE} are given by

$$T_{11}^{GP}(\theta, l_1, \phi_1) = -4\pi A c_{66} \frac{1}{l_1^2} \sqrt{2 \cos \phi_1 - \frac{l_1}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l_1}{a} \right) \cos \frac{\phi_1}{2} - \cos \frac{3\phi_1}{2} \right] \sum_{j=1}^3 A_{j1} \right. \\ \left. - \left[\frac{1}{2} \left(1 + \frac{l_1}{a} \right) \sin \frac{\phi_1}{2} + \sin \frac{3\phi_1}{2} \right] \sum_{j=1}^3 B_{j1} \right\} \quad (3.126)$$

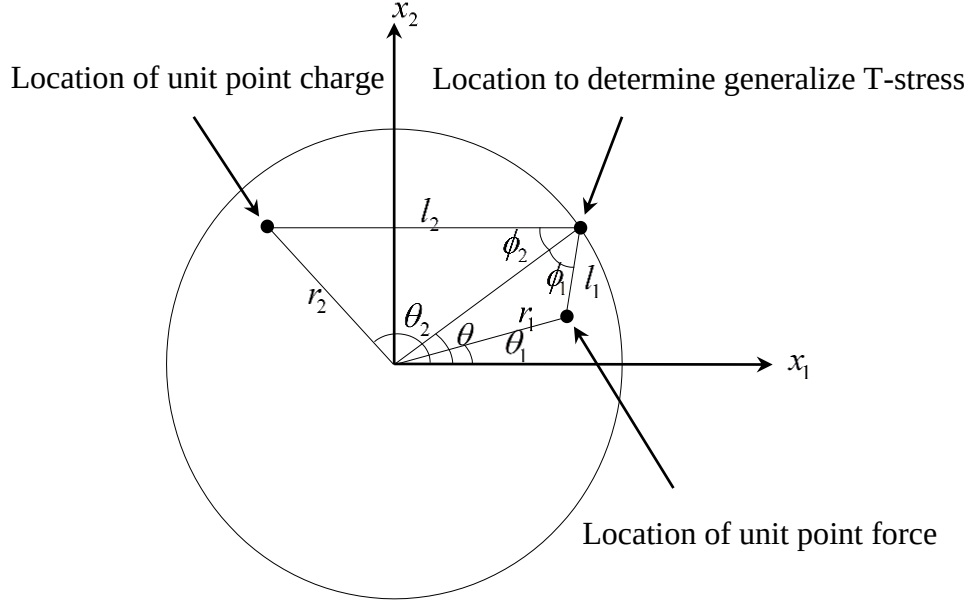


Figure 3.8: Graphical interpretation of variables r_1 , r_2 , l_1 , l_2 , θ_1 , θ_2 , θ , ϕ_1 , and ϕ_2

$$T_{33}^{GP}(\theta, l_1, \phi_1) = 4\pi A c_{66} \frac{1}{l_1^2} \sqrt{2 \cos \phi_1 - \frac{l_1}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l_1}{a} \right) \cos \frac{\phi_1}{2} - \cos \frac{3\phi_1}{2} \right] \sum_{j=1}^3 A_{j1} - \left[\frac{1}{2} \left(1 + \frac{l_1}{a} \right) \sin \frac{\phi_1}{2} + \sin \frac{3\phi_1}{2} \right] \sum_{j=1}^3 B_{j1} \right\} \quad (3.127)$$

$$T_{13}^{GP}(\theta, l_1, \phi_1) = -4\pi A c_{66} \frac{1}{l_1^2} \sqrt{2 \cos \phi_1 - \frac{l_1}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l_1}{a} \right) \cos \frac{\phi_1}{2} - \cos \frac{3\phi_1}{2} \right] \sum_{j=1}^3 B_{j1} + \left[\frac{1}{2} \left(1 + \frac{l_1}{a} \right) \sin \frac{\phi_1}{2} + \sin \frac{3\phi_1}{2} \right] \sum_{j=1}^3 A_{j1} \right\} \quad (3.128)$$

$$T_{11}^{GE}(\theta, l_2, \phi_2) = -4\pi A c_{66} \frac{1}{l_2^2} \sqrt{2 \cos \phi_2 - \frac{l_2}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l_2}{a} \right) \cos \frac{\phi_2}{2} - \cos \frac{3\phi_2}{2} \right] \sum_{j=1}^3 A_{j2} - \left[\frac{1}{2} \left(1 + \frac{l_2}{a} \right) \sin \frac{\phi_2}{2} + \sin \frac{3\phi_2}{2} \right] \sum_{j=1}^3 B_{j2} \right\} \quad (3.129)$$

$$T_{33}^{GE}(\theta, l_2, \phi_2) = 4\pi A c_{66} \frac{1}{l_2^2} \sqrt{2 \cos \phi_2 - \frac{l_2}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l_2}{a} \right) \cos \frac{\phi_2}{2} - \cos \frac{3\phi_2}{2} \right] \sum_{j=1}^3 A_{j2} - \left[\frac{1}{2} \left(1 + \frac{l_2}{a} \right) \sin \frac{\phi_2}{2} + \sin \frac{3\phi_2}{2} \right] \sum_{j=1}^3 B_{j2} \right\} \quad (3.130)$$

$$T_{13}^{GE}(\theta, l_2, \phi_2) = -4\pi A c_{66} \frac{1}{l_2^2} \sqrt{2 \cos \phi_2 - \frac{l_2}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l_2}{a} \right) \cos \frac{\phi_2}{2} - \cos \frac{3\phi_2}{2} \right] \sum_{j=1}^3 B_{j2} + \left[\frac{1}{2} \left(1 + \frac{l_2}{a} \right) \sin \frac{\phi_2}{2} + \sin \frac{3\phi_2}{2} \right] \sum_{j=1}^3 A_{j2} \right\} \quad (3.131)$$

From the formula (3.126)-(3.131), it is apparent that the generalized T-stress Green's functions of an impermeable crack depend on material properties and are singular only at $l = 0$ of $\mathcal{O}(1/l^2)$.

3.3.6 Generalized T-stress Green's function for permeable crack

Consider, next, a circular crack under a pair of opposite, unit normal point forces presented by Chen and Lim (2005), the exact solution of the generalized stress Green's function is given by

$$\sigma_{z3}^G = \frac{1}{g_1 \pi^2} \sum_{j=1}^3 \gamma_{j3} c_j f_3(z_j) \quad (3.132)$$

$$\sigma_2^G = \frac{2c_{66}}{g_1 \pi^2} \sum_{j=1}^3 c_j f_4(z_j) \quad (3.133)$$

$$\tau_{z1}^G = \frac{1}{g_1 \pi^2} \sum_{j=1}^3 \gamma_{j1} s_j c_j f_5(z_j) \quad (3.134)$$

$$\tau_{z2}^G = \frac{1}{g_1 \pi^2} \sum_{j=1}^3 \gamma_{j2} s_j c_j f_5(z_j) \quad (3.135)$$

where $\sigma_1^G, \sigma_2^G, \tau_{z1}^G, \tau_{z2}^G$ and $\gamma_{j1}, \gamma_{j2}, \gamma_{j3}$ are defined by

$$\sigma_1^G = \sigma_{rr}^G + \sigma_{\theta\theta}^G \quad (3.136)$$

$$\sigma_2^G = (\sigma_{rr}^G - \sigma_{\theta\theta}^G + 2i\sigma_{r\theta}^G) e^{2i\theta} \quad (3.137)$$

$$\tau_{z1}^G = (\sigma_{rz}^G + \sigma_{\theta z}^G) e^{i\theta} \quad (3.138)$$

$$\tau_{z2}^G = (D_r^G + D_\theta^G) e^{i\theta} \quad (3.139)$$

$$\gamma_{j1} = -c_{13} + c_{33} s_j \alpha_{j1} + e_{33} s_j \alpha_{j2} \quad (3.140)$$

$$\gamma_{j2} = -e_{31} + e_{33} s_j \alpha_{j1} - \varepsilon_{33} s_j \alpha_{j2} \quad (3.141)$$

$$\gamma_{j3} = 2 \left[(c_{66} - c_{11}) + c_{13} s_j \alpha_{j1} + e_{31} s_j \alpha_{j2} \right] \quad (3.142)$$

All other involved parameters defined in Section 3.3.1 apply here except c_j are defined in Section 3.3.2 and the complex functions $f_3(z_j), f_4(z_j), f_5(z_j)$ are given explicitly by

$$f_3(z) = \frac{z}{R_0^3} \tan^{-1} \left(\frac{h}{R_0} \right) - \frac{h}{z(R_0^2 + h^2)} \left[\frac{r^2 - l_1^2}{l_2^2 - l_1^2} - \frac{z^2}{R_0^2} \right] \quad (3.143)$$

$$f_4(z) = \frac{\sqrt{a^2 - r_0^2}}{\bar{t} \bar{s}} \left(\frac{2}{\bar{t}} - \frac{r_0 e^{i\theta_0}}{\bar{s}^2} \right) \tan^{-1} \left[\frac{\bar{s}}{\sqrt{l_2^2 - a^2}} \right] - \frac{z(3R_0^2 - z^2)}{\bar{t}^2 R_0^3} \tan^{-1} \left(\frac{h}{R_0} \right) \\ + \frac{\sqrt{a^2 - r_0^2} \sqrt{l_2^2 - a^2} r_0 e^{i\theta_0}}{\bar{t} \bar{s}^2 [l_2^2 - r r_0 e^{-i(\theta - \theta_0)}]} - \frac{zh}{R_0^2 + h^2} \left[\frac{t}{\bar{t} R_0^2} - \frac{r^2 e^{2i\theta}}{(l_2^2 - l_1^2)(l_2^2 - r^2)} \right] \quad (3.144)$$

$$f_4(z) = \frac{t}{R_0^3} \tan^{-1} \left(\frac{h}{R_0} \right) + \frac{h}{R_0^2 + h^2} \left[\frac{r e^{i\theta}}{l_2^2 - l_1^2} + \frac{t}{R_0^2} \right] \quad (3.145)$$

where

$$h = \frac{\sqrt{a^2 - l_1^2} \sqrt{a^2 - r_0^2}}{a} \quad (3.146)$$

$$t = r e^{i\theta} - r_0 e^{i\theta_0} \quad (3.147)$$

$$\bar{s} = \sqrt{a^2 - r r_0 e^{-i(\theta - \theta_0)}} \quad (3.148)$$

$$R_0 = \sqrt{r^2 + r_0^2 - 2r r_0 \cos(\theta - \theta_0) + z^2} \quad (3.149)$$

By applying the properties (3.105)-(3.106) to the relations (3.143)-(3.145) and (3.132)-(3.135), the Green's function of the generalized stress at any point on the plane $z = 0$ is given by

$$\sigma_{z3}^G(r, \theta, 0; r_0, \theta_0) = \frac{1}{g_1 \pi^2} \sum_{j=1}^3 \gamma_{j3} c_j f_3(0) \quad (3.150)$$

$$\sigma_2^G(r, \theta, 0; r_0, \theta_0) = \frac{2c_{66}}{g_1 \pi^2} \sum_{j=1}^3 c_j f_4(0) \quad (3.151)$$

$$\tau_{z1}^G(r, \theta, 0; r_0, \theta_0) = 0 \quad (3.152)$$

$$\tau_{z2}^G(r, \theta, 0; r_0, \theta_0) = 0 \quad (3.153)$$

where g_1 is defined in Section 3.3.2, and $f_3(0)$, $f_4(0)$, $f_5(0)$ are obtained as

$$f_3(0) = -\frac{\sqrt{a^2 - r_0^2}}{\sqrt{r^2 - a^2} R_{00}^2} \quad (3.154)$$

$$f_4(0) = \frac{\sqrt{a^2 - r_0^2}}{\bar{t}_0 \bar{s}_0} \left(\frac{2}{\bar{t}_0} - \frac{r_0 e^{i\theta_0}}{\bar{s}_0^2} \right) \tan^{-1} \left(\frac{\bar{s}_0}{\sqrt{r^2 - a^2}} \right) + \frac{\sqrt{r^2 - a^2} \sqrt{a^2 - r_0^2} r_0 e^{i\theta_0}}{\bar{t}_0 \bar{s}_0^2 (r^2 - r r_0 e^{-i(\theta - \theta_0)})} + \frac{e^{2i\theta} \sqrt{a^2 - r_0^2}}{R_{00}^2 \sqrt{r^2 - a^2}} \quad (3.155)$$

$$f_5(0) = 0 \quad (3.156)$$

with R_{00} defined by

$$R_{00} = R_0(z=0) = \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)} \quad (3.157)$$

By introducing, again, the variable transformation $\bar{r} = r - a$ along with the proper limiting process, it can be proved, for $(r_0, \theta_0) \neq (a, \theta)$, that

$$\lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \hat{r}} \left[\sqrt{\bar{r}} \left(\frac{\sqrt{a^2 - r_0^2}}{\sqrt{r^2 - a^2}} \right) \right] \right\} = 0 \quad (3.158)$$

$$\lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \hat{r}} \left[\sqrt{\bar{r}} \left(\frac{\sqrt{r^2 - a^2} \sqrt{a^2 - r_0^2}}{\bar{t} \bar{s}^2 (r^2 - rr_0 e^{-i(\theta - \theta_0)})} \right) \right] \right\} = 0 \quad (3.159)$$

$$\lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \left(\frac{\sqrt{a^2 - r_0^2}}{\bar{t} \bar{s}} \left(\frac{2}{\bar{t}} - \frac{r_0 e^{i\theta_0}}{\bar{s}^2} \right) \tan^{-1} \left(\frac{\bar{s}}{\sqrt{r^2 - a^2}} \right) \right) \right] \right\} = \pi \sqrt{a^2 - r_0^2} \left(\frac{1}{2\bar{t}_0^2 \bar{s}_0} - \frac{r_0 e^{i\theta_0}}{4\bar{t}_0 \bar{s}_0^3} \right) \quad (3.160)$$

where $\bar{t}_0 = ae^{-i\theta} - r_0 e^{-i\theta_0}$ and $\bar{s}_0 = \sqrt{a^2 - ar_0 e^{-i(\theta - \theta_0)}}$. By substituting (3.158)-(3.160) into (3.150)-(3.153), (3.11) and (3.136)-(3.139), the generalized T-stress Green's functions are obtained as

$$T_{11}^G(\theta, r_0, \theta_0) = \frac{c_{66}}{g_1 \pi} \sum_{j=1}^3 \Re \left[c_j \left(\frac{\sqrt{a^2 - r_0^2}}{\sqrt{a} (ae^{-i\theta} - r_0 e^{-i\theta_0}) \sqrt{a - r_0 e^{-i(\theta - \theta_0)}}} \right) \times \left(\frac{1}{a - r_0 e^{-i(\theta_0 - \theta)}} - \frac{r_0 e^{i(\theta_0 - \theta)}}{2a(a - r_0 e^{-i(\theta - \theta_0)})} \right) \right] \quad (3.161)$$

$$T_{33}^G(\theta, r_0, \theta_0) = \frac{c_{66}}{g_1 \pi} \sum_{j=1}^3 \Re \left[c_j \left(\frac{\sqrt{a^2 - r_0^2}}{\sqrt{a} (ae^{-i\theta} - r_0 e^{-i\theta_0}) \sqrt{a - r_0 e^{-i(\theta - \theta_0)}}} \right) \times \left(\frac{r_0 e^{i(\theta_0 - \theta)}}{2a(a - r_0 e^{-i(\theta - \theta_0)})} - \frac{1}{a - r_0 e^{-i(\theta_0 - \theta)}} \right) \right] \quad (3.162)$$

$$T_{13}^G(\theta, r_0, \theta_0) = \frac{c_{66}}{g_1 \pi} \sum_{j=1}^3 \Im \left[c_j \left(\frac{\sqrt{a^2 - r_0^2}}{\sqrt{a} (ae^{-i\theta} - r_0 e^{-i\theta_0}) \sqrt{a - r_0 e^{-i(\theta - \theta_0)}}} \right) \times \left(\frac{1}{a - r_0 e^{-i(\theta_0 - \theta)}} - \frac{r_1 e^{i(\theta_0 - \theta)}}{2a(a - r_0 e^{-i(\theta - \theta_0)})} \right) \right] \quad (3.163)$$

$$T_{14}^G = 0 \quad (3.164)$$

$$T_{34}^G = 0 \quad (3.165)$$

The expressions (3.161)-(3.163) can be further reduced by introducing the following variable transformations

$$r_0 \sin(\theta_0 - \theta) = l \sin \phi; \quad r_0 \cos(\theta_0 - \theta) + l \cos \phi = a \quad (3.166)$$

and the representation $c_j = A_j + iB_j$ where A_j and B_j are real and imaginary parts of c_j . The graphical interpretation of the variables l, ϕ is indicated in Figure 3.9. The generalized T-stress Green's function can be given explicitly by

$$T_{11}^G(\theta, l, \phi) = -\frac{c_{66}}{g_1 \pi l^2} \sqrt{2 \cos \phi - \frac{l}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l}{a} \right) \cos \frac{\phi}{2} - \cos \frac{3\phi}{2} \right] \sum_{j=1}^3 A_j - \left[\frac{1}{2} \left(1 + \frac{l}{a} \right) \sin \frac{\phi}{2} + \sin \frac{3\phi}{2} \right] \sum_{j=1}^3 B_j \right\} \quad (3.167)$$

$$T_{33}^G(\theta, l, \phi) = \frac{c_{66}}{g_1 \pi l^2} \sqrt{2 \cos \phi - \frac{l}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l}{a} \right) \cos \frac{\phi}{2} - \cos \frac{3\phi}{2} \right] \sum_{j=1}^3 A_j - \left[\frac{1}{2} \left(1 + \frac{l}{a} \right) \sin \frac{\phi}{2} + \sin \frac{3\phi}{2} \right] \sum_{j=1}^3 B_j \right\} \quad (3.168)$$

$$T_{13}^G(\theta, l, \phi) = -\frac{c_{66}}{g_1 \pi l^2} \sqrt{2 \cos \phi - \frac{l}{a}} \left\{ \left[\frac{1}{2} \left(1 - \frac{l}{a} \right) \cos \frac{\phi}{2} - \cos \frac{3\phi}{2} \right] \sum_{j=1}^3 B_j + \left[\frac{1}{2} \left(1 + \frac{l}{a} \right) \sin \frac{\phi}{2} + \sin \frac{3\phi}{2} \right] \sum_{j=1}^3 A_j \right\} \quad (3.169)$$

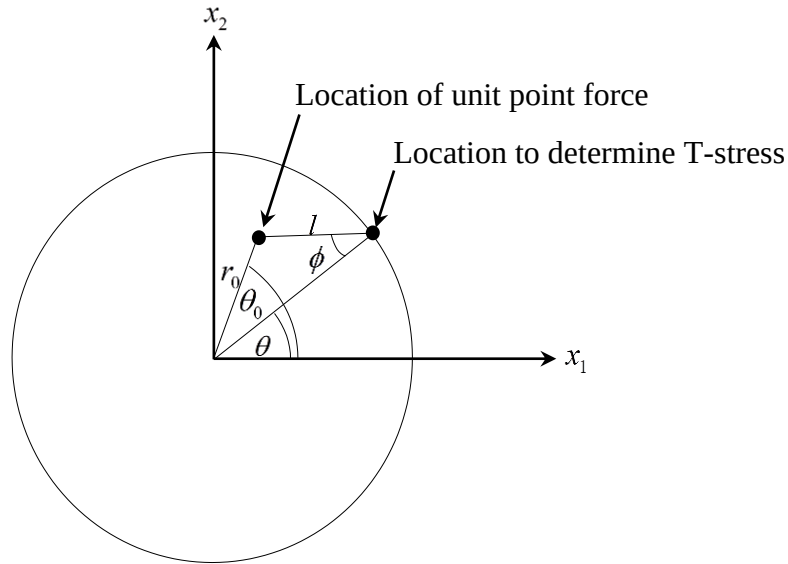


Figure 3.9: Graphical interpretation of variables r_0, l, θ_0, θ and ϕ

It is clear that the generalized T-stress Green's functions of an electrically permeable crack also depend on material properties and are singular at $l = 0$ of $\mathcal{O}(1/l^2)$.

3.3.7 Generalized T-stress components for general crack-face loading

Results of the generalized T-stress Green's function established in Section 3.3.5 and Section 3.3.6 are employed as a basis to formulate the general integral formula for computing the generalized T-stress components for a circular crack subjected to general crack-face loads.

3.3.7.1 Generalized T-stress components for impermeable crack

For an impermeable circular crack under the uniform remote triaxial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$, the uniform remote electrical inductions $\{d_1^\infty, d_2^\infty, d_3^\infty\}$, and the self-equilibrated, crack-face, general normal traction t_3 and the self-equilibrated, crack-face, general electrical charge d_3 , the integral formula for the generalized T-stress components is given implicitly by

$$\begin{aligned} \bar{\sigma}_{ij}(\theta, \bar{r}, \bar{\theta}) = & \bar{\sigma}_{ij}^\infty + \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{ij}^{GP}(\theta, \bar{r}, \bar{\theta}; r_1, \theta_1) \{t_3(r_1, \theta_1) + \sigma_{33}^\infty\} r_1 dr_1 d\theta_1 \\ & + \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{ij}^{GE}(\theta, \bar{r}, \bar{\theta}; r_2, \theta_2) \{d_3(r_2, \theta_2) + d_3^\infty\} r_2 dr_2 d\theta_2 \end{aligned} \quad (3.170)$$

where $\bar{\sigma}_{ij}$, $\bar{\sigma}_{ij}^\infty$, $\bar{\sigma}_{ij}^{GP}$, and $\bar{\sigma}_{ij}^{GE}$ are local components of the generalized stress field, local components of the uniform remote triaxial generalized stresses σ_{ij}^∞ (including the remote triaxial stress $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and remote electrical inductions $\{d_1^\infty, d_2^\infty, d_3^\infty\}$), components of the generalized stress Green's function σ^{GP} and σ^{GE} under a pair of opposite, unit normal point forces and opposite, unit point charges, respectively. By following the work of Rungamornrat and Pinitpanich (2016), i.e., employing equation (3.11) together with the expression (3.170), the generalized T-stress components at any point $(a, \theta, 0)$ along the crack front is given by

$$T_Q(\theta) = T_Q^\infty + T_Q^0(\theta) + T_Q^1(\theta) \quad (3.171)$$

with T_Q^∞ , $T_Q^0(\theta)$, and $T_Q^1(\theta)$ defined by

$$\begin{aligned} T_Q^\infty = & \bar{\sigma}_Q^\infty + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GP}(\theta, \bar{r}, \bar{\theta}; r_1, \theta_1) \sigma_{33}^\infty r_1 dr_1 d\theta_1 \right] \right\} \\ & + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GE}(\theta, \bar{r}, \bar{\theta}; r_2, \theta_2) d_3^\infty r_2 dr_2 d\theta_2 \right] \right\} \end{aligned} \quad (3.172)$$

$$T_Q^0(\theta) = 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GP}(\theta, \bar{r}, \bar{\theta}; r_1, \theta_1) t_3(r_1, \theta_1) r_1 dr_1 d\theta_1 \right] \right\} \\ + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GE}(\theta, \bar{r}, \bar{\theta}; r_2, \theta_2) d_3(r_2, \theta_2) r_2 dr_2 d\theta_2 \right] \right\} \quad (3.173)$$

$$T_Q^1(\theta) = 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GP}(\theta, \bar{r}, \bar{\theta}; r_1, \theta_1) [t_3(r_1, \theta_1) - t_3(a, \theta)] r_1 dr_1 d\theta_1 \right] \right\} \\ + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GE}(\theta, \bar{r}, \bar{\theta}; r_2, \theta_2) [d_3(r_2, \theta_2) - d_3(a, \theta)] r_2 dr_2 d\theta_2 \right] \right\} \quad (3.174)$$

where T_Q^∞ represents the generalized T-stress components associated with the uniform remote triaxial stress σ_{ij}^∞ ; $T_Q^0(\theta)$ are the generalized T-stress components corresponding to the uniform portion of the general applied traction $t_3(r_1, \theta_1)$ and general electrical charge $d_3(r_2, \theta_2)$ with the magnitude equal to $t_3(r_1 = a, \theta_1 = \theta)$ and $d_3(r_2 = a, \theta_2 = \theta)$, respectively; and $T_Q^1(\theta)$ represent the generalized T-stress components associated with the remaining portion of the general applied traction $t_3(r_1, \theta_1)$ and the general electrical charge $d_3(r_2, \theta_2)$ with its magnitude equal to zero at the point (a, θ) , respectively.

By employing the condition $t_3(r_1, \theta_1) - t_3(a, \theta) = 0$, $d_3(r_2, \theta_2) - d_3(a, \theta) = 0$ at the point (a, θ) and the fact that the singularity of $\bar{\sigma}_Q^G(r, \theta, 0; r_\alpha, \theta_\alpha)$ is present only at a location where the concentrated load is applied together with interchanging the order of limits, integrations and differentiations, the generalized T-stress $T_Q^1(\theta)$ can be obtained as

$$T_Q^1(\theta) = \int_{-\pi}^{\pi} \int_0^a \left\{ 2 \lim_{\bar{r} \rightarrow 0} \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \bar{\sigma}_Q^{GP}(\theta, \bar{r}, \bar{\theta}; r_1, \theta_1) \right] \right\} [t_3(r_1, \theta_1) - t_3(a, \theta)] r_1 dr_1 d\theta_1 \\ + \int_{-\pi}^{\pi} \int_0^a \left\{ 2 \lim_{\bar{r} \rightarrow 0} \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \bar{\sigma}_Q^{GE}(\theta, \bar{r}, \bar{\theta}; r_2, \theta_2) \right] \right\} [d_3(r_2, \theta_2) - d_3(a, \theta)] r_2 dr_2 d\theta_2 \quad (3.175)$$

By applying the definition of the generalized T-stress, the relation (3.174) reduces to

$$T_Q^1(\theta) = \int_{-\pi}^{\pi} \int_0^a T_Q^{GP}(\theta; r_1, \theta_1) p_1(r_1, \theta_1) r_1 dr_1 d\theta_1 \\ + \int_{-\pi}^{\pi} \int_0^a T_Q^{GE}(\theta; r_2, \theta_2) p_2(r_2, \theta_2) r_2 dr_2 d\theta_2 \quad (3.176)$$

where $p_1(r_1, \theta_1) = t_3(r_1, \theta_1) - t_3(a, \theta)$ and $p_2(r_2, \theta_2) = d_3(r_2, \theta_2) - d_3(a, \theta)$. By applying the variable transformations (3.122)-(3.123), it leads to the alternative form of (3.175)

$$\begin{aligned}
T_Q^1(\theta) = & \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \phi} \left[\left(l \hat{T}_Q^{GP}(\theta; l_1, \phi_1) \right) \hat{p}_1(l_1, \phi_1) dl_1 d\phi_1 \right] \\
& + \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \phi} \left[\left(l \hat{T}_Q^{GE}(\theta; l_2, \phi_2) \right) \hat{p}_2(l_2, \phi_2) dl_2 d\phi_2 \right]
\end{aligned} \tag{3.177}$$

where $\hat{p}_1(l_1, \phi_1) = p_1(r_1(l_1, \phi_1), \theta_1(l_1, \phi_1))$ and $\hat{p}_2(l_2, \phi_2) = p_2(r_2(l_2, \phi_2), \theta_2(l_2, \phi_2))$.

It should be evident from the formula (3.126)-(3.131) that the function $l \hat{T}_Q^{GP}(\theta; l_1, \phi_1)$ and $l \hat{T}_Q^{GE}(\theta; l_2, \phi_2)$ appearing in (3.177) is singular only at $l=0$ of $\mathcal{O}(1/l)$, $\hat{p}_1(0, \phi_1)=0$ and $\hat{p}_2(0, \phi_2)=0$. Moreover, the normal traction $t_3(r_1, \theta_1)$ and the normal electric induction $d_3(r_2, \theta_2)$ are the most important factors for the existence of the integral (3.177). If the functions $t_3(r_1, \theta_1)$ and $d_3(r_2, \theta_2)$ are sufficiently smooth at a point (a, θ) with $\hat{p}_1(l_1, \phi_1) = \mathcal{O}(l^{\lambda_1})$, $\hat{p}_2(l_2, \phi_2) = \mathcal{O}(l^{\lambda_2})$, and $\lambda_1, \lambda_2 > 0$, the existence of all integrals appearing in (3.177) is confirmed since all involved integrands are of $\mathcal{O}(l^{\lambda_1-1})$ and $\mathcal{O}(l^{\lambda_2-1})$. For $\lambda_1, \lambda_2 \geq 1$, all involved integrands are obviously nonsingular and all integrals can be efficiently evaluated by Gaussian quadrature. For $0 < \lambda_1, \lambda_2 < 1$, the involved integrands are weakly singular and a numerical integration technique indicated below is employed. By introducing the change of variables $l_1 = l_1(\bar{l}_1) = \bar{l}_1^{\gamma_1}$ and $l_2 = l_2(\bar{l}_2) = \bar{l}_2^{\gamma_2}$ where γ_1 and γ_2 are selected constants, the integral (3.176) can be modified as

$$\begin{aligned}
T_Q^1(\theta) = & \gamma_1 \int_{-\pi/2}^{\pi/2} \int_0^{(2a \cos \phi)^{1/\gamma_1}} \left[l_1(\bar{l}_1) \hat{T}_Q^{GP}(\theta; l_1(\bar{l}_1), \phi_1) \right] \hat{p}_1(l_1(\bar{l}_1), \phi_1) [l_1(\bar{l}_1)]^{(\gamma_1-1)/\gamma_1} d\bar{l}_1 d\phi_1 \\
& + \gamma_2 \int_{-\pi/2}^{\pi/2} \int_0^{(2a \cos \phi)^{1/\gamma_2}} \left[l_2(\bar{l}_2) \hat{T}_Q^{GE}(\theta; l_2(\bar{l}_2), \phi_2) \right] \hat{p}_2(l_2(\bar{l}_2), \phi_2) [l_2(\bar{l}_2)]^{(\gamma_2-1)/\gamma_2} d\bar{l}_2 d\phi_2
\end{aligned} \tag{3.178}$$

By setting $\gamma_1 = 1/\lambda_1$ and $\gamma_2 = 1/\lambda_2$, the integral (3.178) now becomes

$$\begin{aligned}
T_Q^1(\theta) = & \frac{1}{\lambda_1} \int_{-\pi/2}^{\pi/2} \int_0^{(2a \cos \phi)^{\lambda_1}} \left[l_1(\bar{l}_1) \hat{T}_Q^{GP}(\theta; l_1(\bar{l}_1), \phi_1) \right] \hat{p}_1(l_1(\bar{l}_1), \phi_1) [l_1(\bar{l}_1)]^{(1-\lambda_1)} d\bar{l}_1 d\phi_1 \\
& + \frac{1}{\lambda_2} \int_{-\pi/2}^{\pi/2} \int_0^{(2a \cos \phi)^{\lambda_2}} \left[l_2(\bar{l}_2) \hat{T}_Q^{GE}(\theta; l_2(\bar{l}_2), \phi_2) \right] \hat{p}_2(l_2(\bar{l}_2), \phi_2) [l_2(\bar{l}_2)]^{(1-\lambda_2)} d\bar{l}_2 d\phi_2
\end{aligned} \tag{3.179}$$

It is apparent that all integrals appearing in (3.172) can now be efficiently integrated by standard Gaussian Quadrature since the weak singularity at $l=0$ is completely removed by the specific choice of γ_1, γ_2 .

Since the closed-form solution of the generalized T-stress of an impermeable circular crack under the uniform remote and crack-face loadings is already established in Section 3.3.1, the generalized T-stress components T_Q^∞ and $T_Q^0(\theta)$ are then given by

$$T_{11}^\infty = A \left\{ \sum_{i=1}^3 \left[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2} \right] p_i^\infty \right\} + \bar{\sigma}_{11}^\infty \quad (3.180)$$

$$T_{33}^\infty = A \left\{ \sum_{i=1}^3 \left[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2} \right] p_i^\infty \right\} + \bar{\sigma}_{33}^\infty \quad (3.181)$$

$$T_{13}^\infty = \bar{\sigma}_{13}^\infty, \quad T_{14}^\infty = \bar{d}_1^\infty, \quad T_{34}^\infty = \bar{d}_3^\infty \quad (3.182)$$

$$T_{11}^0 = T_{33}^0 = A \left\{ \sum_{i=1}^3 \left[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2} \right] p_i^0 \right\} \quad (3.183)$$

$$T_{13}^0 = 0, \quad T_{14}^0 = 0, \quad T_{34}^0 = 0 \quad (3.184)$$

where the loading parameters p_i^0 and p_i^∞ are defined in a fashion similar to those indicated in (3.18), i.e., $p_i^\infty = (d_i g_3 - c_i g_4)\sigma_{33}^\infty + (c_i g_2 - d_i g_1)d_3^\infty$ and $p_i^0 = (d_i g_3 - c_i g_4)\sigma_{33}^0 + (c_i g_2 - d_i g_1)d_3^0$.

3.3.7.2 Generalized T-stress components for permeable crack

For a permeable circular crack under the uniform remote triaxial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$, the uniform remote electrical inductions $\{d_1^\infty, d_2^\infty, d_3^\infty\}$, and the self-equilibrated, crack-face, general normal traction t_3 , the integral formula of the generalized T-stress components are given implicitly by

$$\bar{\sigma}_{ij}(\theta, \bar{r}, \bar{\theta}) = \bar{\sigma}_{ij}^\infty + \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{ij}^{GP}(\theta, \bar{r}, \bar{\theta}; r_0, \theta_0) \{t_3(r_0, \theta_0) + \sigma_{33}^\infty\} r_0 dr_0 d\theta_0 \quad (3.185)$$

By following the procedure similar to that employed by Rungamornrat and Pinitpanich (2016), components of the generalized T-stress at any point $(a, \theta, 0)$ on the crack boundary can be also written as

$$T_Q(\theta) = T_Q^\infty + T_Q^0(\theta) + T_Q^1(\theta) \quad (3.186)$$

where $T_Q^\infty, T_Q^0(\theta), T_Q^1(\theta)$ are defined by

$$\begin{aligned} T_Q^\infty = & \bar{\sigma}_Q^\infty + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GP}(\theta, \bar{r}, \bar{\theta}; r_1, \theta_1) \sigma_{33}^\infty r_1 dr_1 d\theta_1 \right] \right\} \\ & + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GE}(\theta, \bar{r}, \bar{\theta}; r_2, \theta_2) d_3^\infty r_2 dr_2 d\theta_2 \right] \right\} \end{aligned} \quad (3.187)$$

$$T_Q^0(\theta) = 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GP}(\theta, \bar{r}, \bar{\theta}; r_1, \theta_1) t_3(r_1, \theta_1) r_1 dr_1 d\theta_1 \right] \right\} \\ + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GE}(\theta, \bar{r}, \bar{\theta}; r_2, \theta_2) d_3(r_2, \theta_2) r_2 dr_2 d\theta_2 \right] \right\} \quad (3.188)$$

$$T_Q^1(\theta) = 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GP}(\theta, \bar{r}, \bar{\theta}; r_1, \theta_1) [t_3(r_1, \theta_1) - t_3(a, \theta)] r_1 dr_1 d\theta_1 \right] \right\} \\ + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_Q^{GE}(\theta, \bar{r}, \bar{\theta}; r_2, \theta_2) [d_3(r_2, \theta_2) - d_3(a, \theta)] r_2 dr_2 d\theta_2 \right] \right\} \quad (3.189)$$

Note that the meaning of T_Q^∞ , $T_Q^0(\theta)$, and $T_Q^1(\theta)$ is similar to that described in Section 3.3.7.1.

Again, by employing the condition $t_3(r_1, \theta_1) - t_3(a, \theta) = 0$ at the point (a, θ) and the fact that the singularity of $\bar{\sigma}_Q^G(r, \theta, 0; r_\alpha, \theta_\alpha)$ is present only at a location where the concentrated force is applied along with the interchange of the order of limits, differentiations and integrations, the generalized T-stress $T_Q^1(\theta)$ can be obtained as

$$T_Q^1(\theta) = \int_{-\pi}^{\pi} \int_0^a \left\{ 2 \lim_{\bar{r} \rightarrow 0} \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \bar{\sigma}_Q^{GP}(\theta, \bar{r}, \bar{\theta}; r_0, \theta_0) \right] \right\} [t_3(r_0, \theta_0) - t_3(a, \theta)] r_0 dr_0 d\theta_0 \quad (3.190)$$

By utilizing the definition of the generalized T-stress, the expression (3.190) can be simplified to

$$T_Q^1(\theta) = \int_{-\pi}^{\pi} \int_0^a T_Q^{GP}(\theta; r_0, \theta_0) p(r_0, \theta_0) r_0 dr_0 d\theta_0 \quad (3.191)$$

where $p(r_0, \theta_0) = t_3(r_0, \theta_0) - t_3(a, \theta)$. By applying the variable transformations (3.123)-(3.124) to the integral relation (3.176), it leads to an alternative integral formula

$$T_Q^1(\theta) = \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \phi} \left[(l \hat{T}_Q^{GP}(\theta; l, \phi)) \hat{p}(l, \phi) dl d\phi \right] \quad (3.192)$$

where $\hat{p}(l, \phi) = p(r_0(l, \phi), \theta_0(l, \phi))$. It should be evident from the formula (3.167)-(3.169) that the function $l \hat{T}_Q^{GP}(\theta; l, \phi)$ appearing in (3.192) is singular only at $l=0$ of $\mathcal{O}(1/l)$ and $\hat{p}(0, \phi) = 0$. If the traction data $t_3(r_0, \theta_0)$ is sufficiently smooth at a point (a, θ) such that $\hat{p}(l, \phi) = \mathcal{O}(l^\lambda)$ with $\lambda > 0$, the integrand of (3.192) is obviously of $\mathcal{O}(l^{\lambda-1})$ and, as a result, the existence of the integral in (3.192) is confirmed. For $\lambda \geq 1$, the integral appearing in (3.192) is clearly not singular and can be integrated using standard Gaussian quadrature while for $0 < \lambda < 1$, the integral contains a weakly singular integrand and the same integration scheme described in

Section 3.3.7.1 is applied. Specifically, by utilizing the variable transformation $l = l(\bar{l}) = \bar{l}^\gamma$ where γ denotes a pre-selected constant, the formula (3.192) can be modified as

$$T_Q^1(\theta) = \gamma \int_{-\pi/2}^{\pi/2} \int_0^{(2a \cos \phi)^{1/\gamma}} \left[l(\bar{l}) \hat{T}_Q^{GP}(\theta; l(\bar{l}), \phi) \right] \hat{p}(l(\bar{l}), \phi) \left[l(\bar{l}) \right]^{(\gamma-1)/\gamma} d\bar{l}_0 d\phi_0 \quad (3.193)$$

By setting $\gamma = 1/\lambda$, (3.193) now becomes

$$T_Q^1(\theta) = \frac{1}{\lambda} \int_{-\pi/2}^{\pi/2} \int_0^{(2a \cos \phi)^\lambda} \left[l(\bar{l}) \hat{T}_Q^{GP}(\theta; l(\bar{l}), \phi) \right] \hat{p}(l(\bar{l}), \phi) \left[l(\bar{l}) \right]^{(1-\lambda)} d\bar{l} d\phi \quad (3.194)$$

It is evident that for the chosen γ , the singularity of the integrand at $l = 0$ is fully eliminated. Now, the resulting integral can be integrated accurately and efficiently by standard Gaussian quadrature. By using the exact solution of the components of the generalized T-stress for a permeable circular crack under uniform remote and crack-face loading presented in Section 3.3.2, the generalized T-stress components T_Q^∞ and $T_Q^0(\theta)$ can be given explicitly by

$$T_{33}^\infty = (\sigma_{33}^\infty / g_1) \sum_{i=1}^3 \left[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2} \right] c_i + \bar{\sigma}_{11}^\infty \quad (3.195)$$

$$T_{33}^\infty = (\sigma_{33}^\infty / g_1) \sum_{i=1}^3 \left[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2} \right] c_i + \bar{\sigma}_{33}^\infty \quad (3.196)$$

$$T_{13}^\infty = \bar{\sigma}_{13}^\infty, \quad T_{14}^\infty = \bar{d}_1^\infty, \quad T_{34}^\infty = \bar{d}_3^\infty \quad (3.197)$$

$$T_{11}^0 = T_{33}^0 = (t_3^0 / g_1) \sum_{i=1}^3 \left[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2} \right] c_i \quad (3.198)$$

$$T_{13}^0 = 0, \quad T_{14}^0 = 0, \quad T_{34}^0 = 0 \quad (3.199)$$

3.4 Numerical Results

Verification of the integral formula of the generalized T-stress components derived in the previous section for impermeable and permeable circular cracks subjected to general crack-face loading conditions is presented in this section. In addition, the influence of the crack-face applied loads on the generalized T-stress along the crack boundary is reported.

3.4.1 Verification of integral formula for generalized T-stress

Consider impermeable and permeable circular cracks under only the linearly distributed, self-equilibrated, crack-face normal traction $t_3 = \sigma_0(1 + x_1/a)/2$ along the global x_1 -axis. The body is made from a representative transversely isotropic, piezoelectric material with the properties similar to those of PZT-5H, shown in Table 3.1. To verify the integral formula developed in Section 3.3.7, two benchmark cases are considered. First, the formula (3.171) for the impermeable crack is specialized to an elastic case by simply eliminating the electric-mechanical coupling effect (i.e., $e_{15} = e_{31} = e_{33} = 0$) and results are then compared with the analytical solution

reported by Rungamornrat and Pinitpanich (2016) as shown in Figure 3.10. Next, results obtained from the integral formula (3.171) and (3.186) are benchmarked with solutions obtained from a numerical technique presented by Limwibul et al. (2016) in Figure 3.11 and Figure 3.12 for impermeable and permeable cracks, respectively.

Table 3.1: Elastic moduli, piezoelectric constants and dielectric permittivities of representative, transversely isotropic, piezoelectric solid identical to PZT-5H (eg. Li and Lee (2004b))

Elastic constants ($\times 10^9$ Pa)	c_{11}	126.00
	c_{13}	53.00
	c_{33}	117.00
	c_{44}	35.30
	c_{66}	35.50
Piezoelectric constants (C/m ²)	e_{15}	17.00
	e_{31}	-6.50
	e_{33}	23.30
Dielectric permittivities ($\times 10^{-9}$ C/(Vm))	κ_{11}	15.10
	κ_{33}	13.00

It is evident from these results that numerical solutions generated by the proposed integral formula exhibit an excellent agreement with the reference solutions for both cases. This good agreement of results should ensure the correctness of both the proposed integral formula and the implemented numerical quadrature.

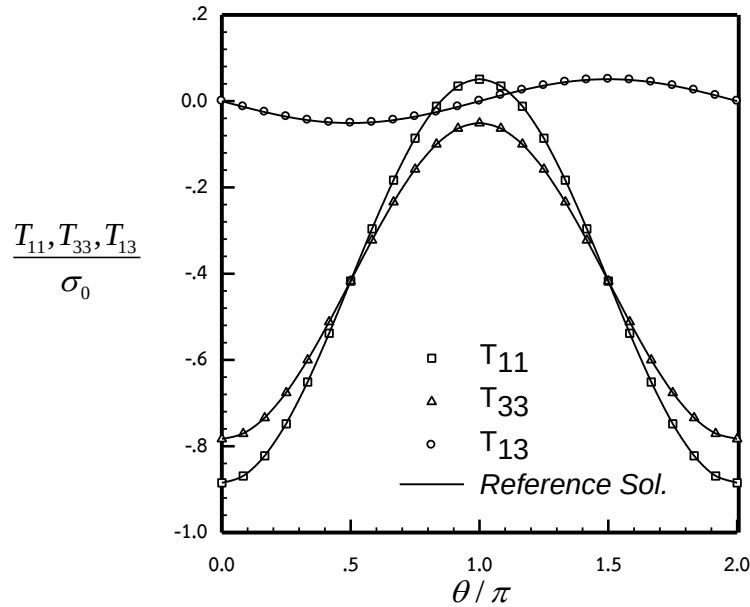


Figure 3.10: Normalized T-stress components of circular crack due to linearly distributed, self-equilibrated, crack-face, normal traction $\sigma_0(1+x_1/a)/2$ by ignoring the electro-mechanical effect. The reference solution is taken from Rungamornrat and Pinitpanich (2016).

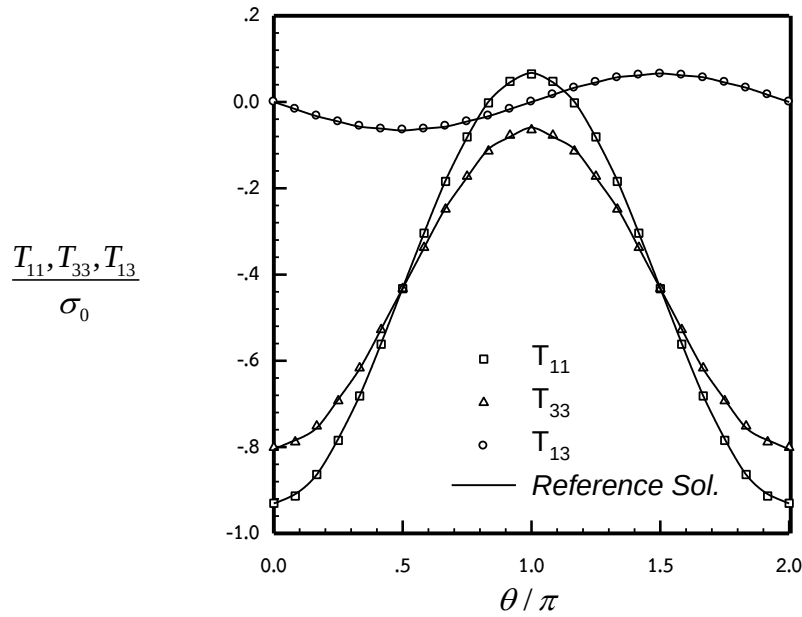


Figure 3.11: Normalized generalized T-stress components of impermeable circular crack subjected to linearly distributed, self-equilibrated, crack-face, normal traction $\sigma_0(1+x_1/a)/2$. The reference solution is generated by a technique proposed by Limwibul et al. (2016).

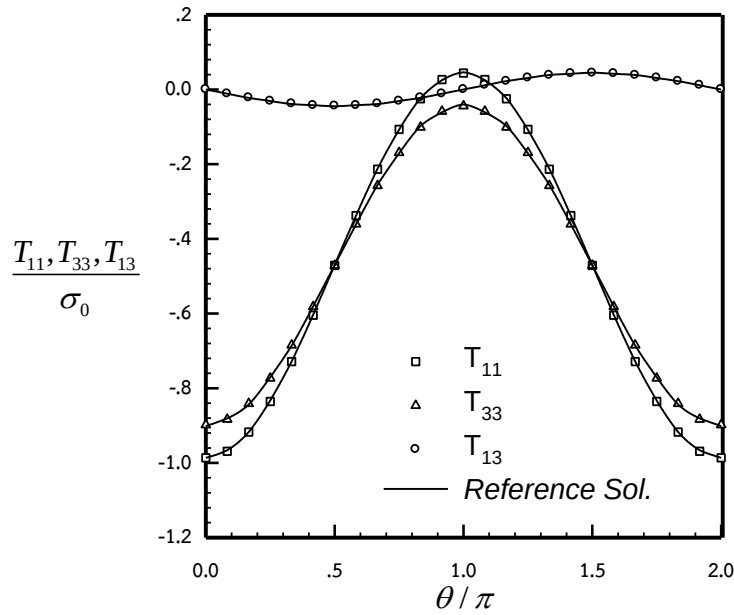


Figure 3.12: Normalized generalized T-stress components of permeable circular crack subjected to linearly distributed, self-equilibrated, crack-face, normal traction $\sigma_0(1+x_1/a)/2$. The benchmark solutions are obtained from numerical technique developed by Limwibul et al. (2016).

3.4.2 Generalized T-stress components under uniform loading conditions

From the closed-form solutions presented in Sections 3.3.1-3.3.4, it is apparent that the value of the generalized T-stress components T_{13}, T_{14}, T_{34} is independent of the uniform crack-face, normal traction and uniform crack-face electrical charge, dependent only on the in-plane components of the remote triaxial stresses and remote electrical inductions, and independent of the electrical crack-face conditions. In addition, for the permeable case, the out-of-plane electrical induction d_3^∞ shows no contribution to both the mechanical T-stress components T_{11}, T_{33}, T_{13} and the electrical T-stress components T_{14}, T_{34} . By integrating the effect of the dielectric permittivity of a medium within the crack gap, both electrically semi-permeable and energetically consistent cases introduce a new parameter d_3^c which significantly influences the values of the generalized T-stress components T_{11} and T_{33} .

To demonstrate the effect of the crack-face conditions and the remote mechanical/electrical loading, the generalized T-stress components of a circular crack under the uniform remote triaxial stresses $2\sigma_{11}^\infty = \sigma_0$, $\sigma_{22}^\infty = \sigma_0$, $2\sigma_{33}^\infty = \sigma_0$ and the uniform remote electrical inductions $2d_1^\infty = d_0$, $d_2^\infty = d_0$, $2d_3^\infty = d_0$ are computed and reported in Figure 3.13 to Figure 3.20 for electrically impermeable, electrically permeable, electrically semi-permeable, and energetically consistent cracks. In the numerical study, the dielectric permittivity of a medium within the crack cavity for both electrically semi-permeable and energetically consistent cracks is taken to be $\varepsilon_c / \varepsilon_0 = 0.1$ where $\varepsilon_0 = 8.85 \times 10^{-12} \text{ C/(Vm)}$ is the value of the dielectric permittivity of the air. In addition, the normalized generalized T-stress components at $\theta = 0$ are also reported as a function of the normalized dielectric permittivity $\varepsilon_c / \varepsilon_0$ for the remote mechanical load and the combined remote mechanical/electrical loads in Figure 3.19 and Figure 3.20, respectively.

It is evident from these results that for a medium subjected only to the remote triaxial stresses (i.e., $d_0 = 0$), the electrically impermeable and electrically permeable crack models yield, respectively, the upper and lower bound solutions of the generalized T-stress components T_{11} and T_{33} on the entire crack boundary (see Figures 3.13 and 3.14). In addition, the generalized T-stress components T_{11} , T_{33} predicted by the semi-permeable model are only slightly different from the lower bound solution generated by the permeable case while solutions obtained from the energetically consistent crack model fall in between the electrically impermeable and permeable solutions. For this particular loading case, the generalized T-stress component T_{13} is independent of the crack-face conditions and its variation along the crack boundary due to the influence of the in-plane remote stresses is clearly indicated in Figure 3.17 whereas the components T_{14} and T_{34} identically vanish.

Unlike the previous case, for a combined remote mechanical/electrical loading (i.e., $\sigma_0 = 1 \times 10^6 \text{ Pa}$ and $d_0 = 1 \times 10^{-3} \text{ C/m}^2$), the upper and lower bounds of the generalized T-stress components T_{11} and T_{33} change to those generated by the energetically consistent and impermeable models, respectively (see Figures 3.15 and 3.16). It can be also seen from these results that T_{11} and T_{33} generated by the impermeable model are significantly different from solutions predicted by the other three crack-face models. In addition, T_{13} is still identical to the previous case whereas T_{14} and T_{34} are now non-zero but still independent of the crack-face

conditions. The variation of T_{14} and T_{34} along the crack front is due mainly to the influence of the non-zero in-plane electric inductions d_1^∞, d_2^∞ as indicated in Figure 3.18.

To further demonstrate the influence of the dielectric permittivity of the medium within the crack cavity on the value of the generalized T-stress components, results are obtained for a range of $\varepsilon_c/\varepsilon_0$ from 0 to 2. Since the behavior of both components T_{11} and T_{33} is similar for the entire crack boundary as indicated above and T_{13}, T_{14}, T_{34} are independent of the crack-face conditions, it is sufficient to report only the generalized T-stress component T_{11} at a representative location at $\theta = 0$ along the crack-front as indicated in Figures 3.19 and 3.20. In the absence of the remote electrical loading (see Figures 3.19), it is apparent that the electrically permeable and impermeable solutions constitute the lower and upper bounds, respectively. In addition, as the dielectric permittivity increases from zero, semi-permeable and energetically consistent solutions start deviating from the impermeable solution and gradually converging to that of the permeable case for a sufficiently large value of $\varepsilon_c/\varepsilon_0$. For the case of combined remote triaxial stresses and electric inductions, the permeable and impermeable models yield the upper and lower bounds of the semi-permeable solution for the entire range of $\varepsilon_c/\varepsilon_0$ considered (see Figures 3.20). However, results of the energetically consistent case show that as the dielectric permittivity increases from zero, T_{11} increases rapidly to its peak value above the permeable solution and then decreases monotonically to that predicted by the permeable model.

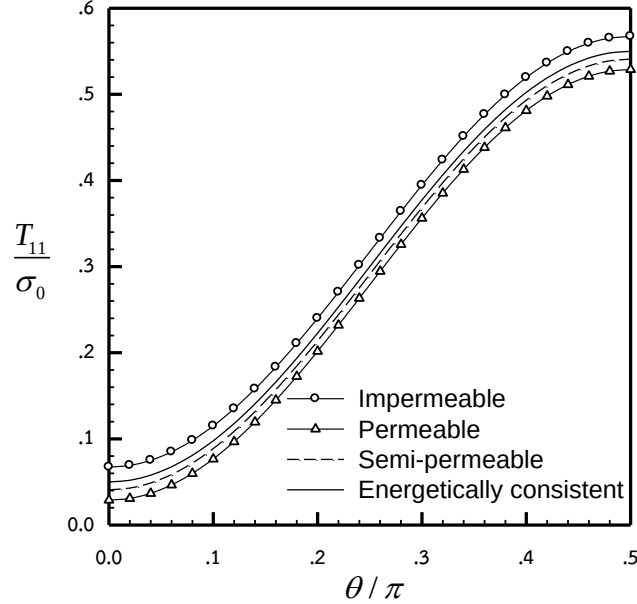


Figure 3.13: Normalized generalized T-stress component T_{11} along the crack front. Results are obtained for $\varepsilon_c / \varepsilon_0 = 0.1$, $\sigma_0 = 1 \times 10^6$ Pa and $d_0 = 0$.

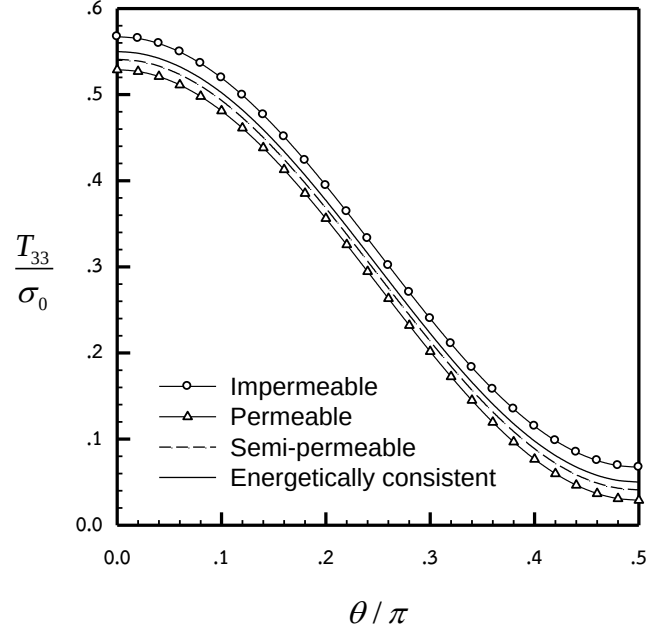


Figure 3.14: Normalized generalized T-stress component T_{33} on crack boundary. Results are obtained for $\varepsilon_c / \varepsilon_0 = 0.1$, $\sigma_0 = 1 \times 10^6$ Pa and $d_0 = 0$.

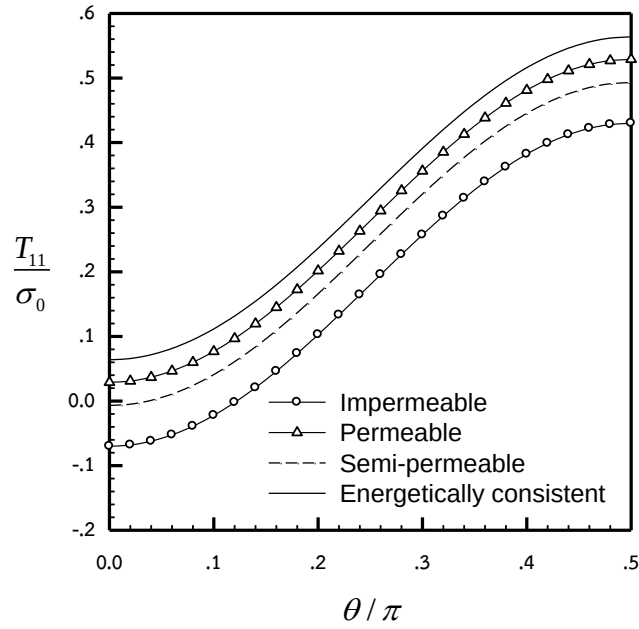


Figure 3.15: Normalized generalized T-stress component T_{11} on crack boundary. Results are obtained for $\varepsilon_c / \varepsilon_0 = 0.1$, $\sigma_0 = 1 \times 10^6$ Pa and $d_0 = 1 \times 10^{-3}$ C/m².

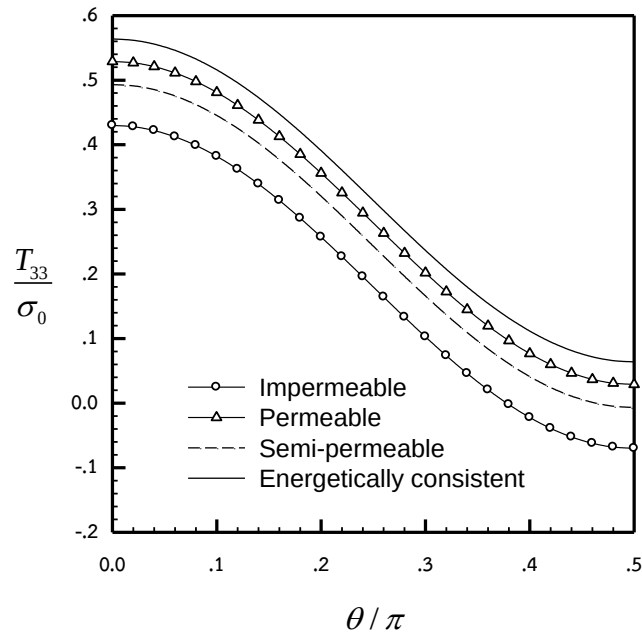


Figure 3.16: Normalized generalized T-stress component T_{33} along the crack front. Results are obtained for $\varepsilon_c / \varepsilon_0 = 0.1$, $\sigma_0 = 1 \times 10^6$ Pa and $d_0 = 1 \times 10^{-3}$ C/m².

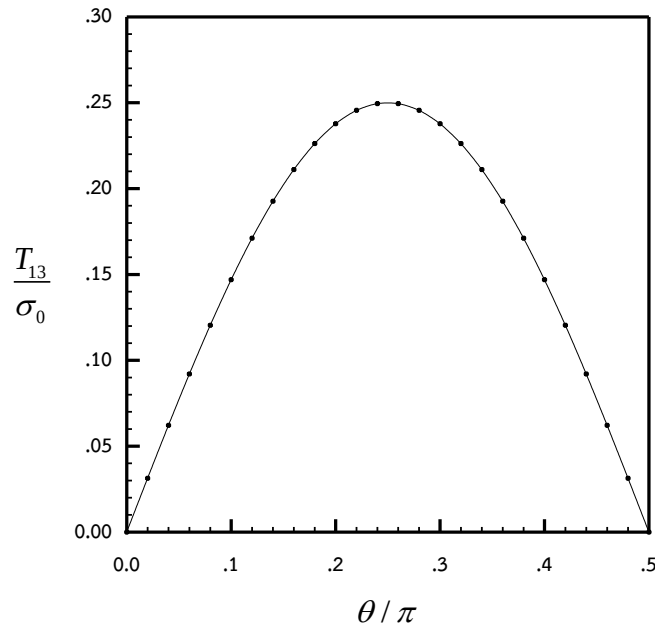


Figure 3.17: Normalized generalized T-stress component T_{13} along the crack front for all crack-face conditions and $\sigma_0 = 1 \times 10^6$ Pa

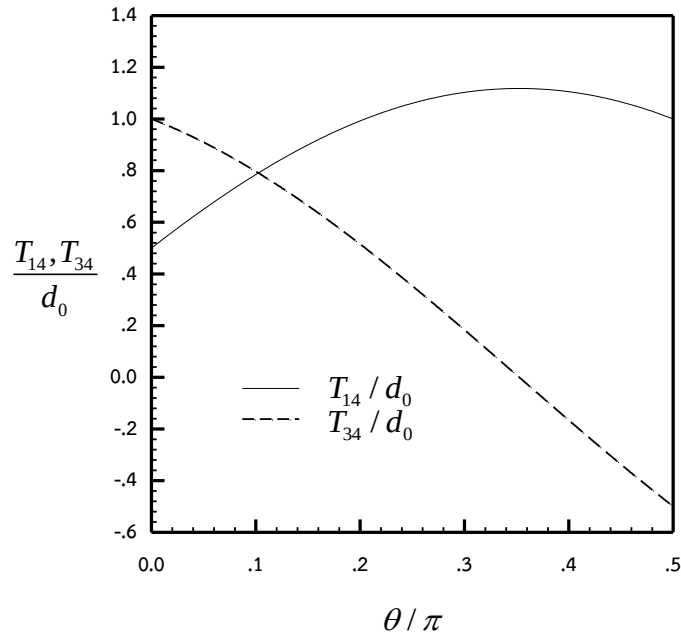


Figure 3.18: Normalized generalized T-stress components T_{14} and T_{34} along the crack front for all crack-face conditions and $\sigma_0 = 1 \times 10^6 \text{ Pa}$ and $d_0 = 1 \times 10^{-3} \text{ C/m}^2$

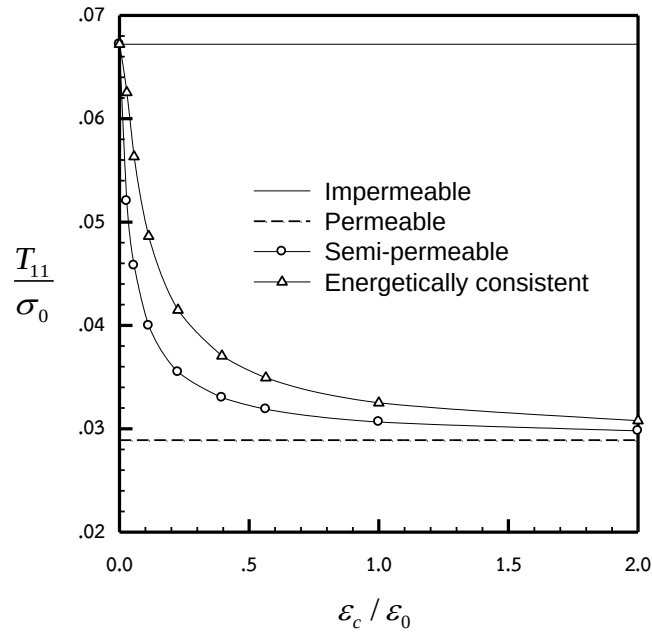


Figure 3.19: Normalized generalized T-stress component T_{11} at $\theta = 0$ versus the normalized dielectric permittivity ϵ_c / ϵ_0 for $\sigma_0 = 1 \times 10^6 \text{ Pa}$ and $d_0 = 0$

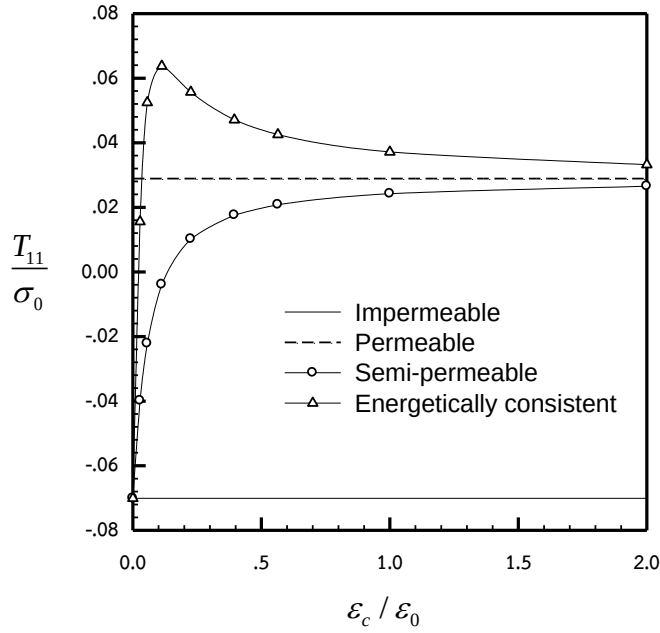


Figure 3.20: Normalized generalized T-stress component T_{11} at $\theta=0$ versus the normalized dielectric permittivity $\varepsilon_c/\varepsilon_0$ for $\sigma_0=1\times 10^6$ Pa and $d_0=1\times 10^{-3}$ C/m²

3.4.3 Influence of crack-face mechanical loading on generalized T-stress

In this final section, the influence the distribution of the self-equilibrated, normal traction acting to the crack surface on the generalized T-stress components for both electrically permeable and impermeable cracks is studied.

Consider a circular crack under an axisymmetric, self-equilibrated, crack-face normal traction $t_3 = \sigma_0 (r/a)^n$ where σ_0 is a constant representing the maximum value of the traction along the crack front and n is an exponent indicating the distribution of traction across the crack surface (see Figure 3.21). The resultant force of this traction can be readily computed and denoted by $T_0 = 2\pi\sigma_0 a^2/(n+2)$. It can be remarked that the exact solution of the generalized T-stress is available for the special case of $n=0$ in Section 3.3.1 and Section 3.3.2. For this particular problem, only the generalized T-stress components T_{11} and T_{33} are non-zero and are apparently independent of the position along the crack front for impermeable and permeable cases. The computed generalized T-stress components T_{11} and T_{33} , normalized either by the maximum value of the traction σ_0 or the resultant force T_0 , are reported as a function of the exponent n in Figure 4-14 to Figure 4-17. As indicated by results in Figures 3.22 and 3.23, the normalized generalized T-stress component T_{11} decreases monotonically in magnitude as the exponent n whereas the reverse effect is observed for the component T_{33} for both crack-face conditions. Note also that both T_{11} and T_{33} are negative and, for a fixed value of the exponent n , the permeable model yields higher T_{11} and T_{33} in magnitude than those generated by the

impermeable model. By changing the means of normalization (from σ_0 to $T_0 / \pi a^2$), it is clear from Figures 3.24 and 3.25 that the normalized generalized T-stress T_{11} and T_{33} depends linearly on the exponent n . In particular, as n increases, the normalized T-stress components increase in magnitude.

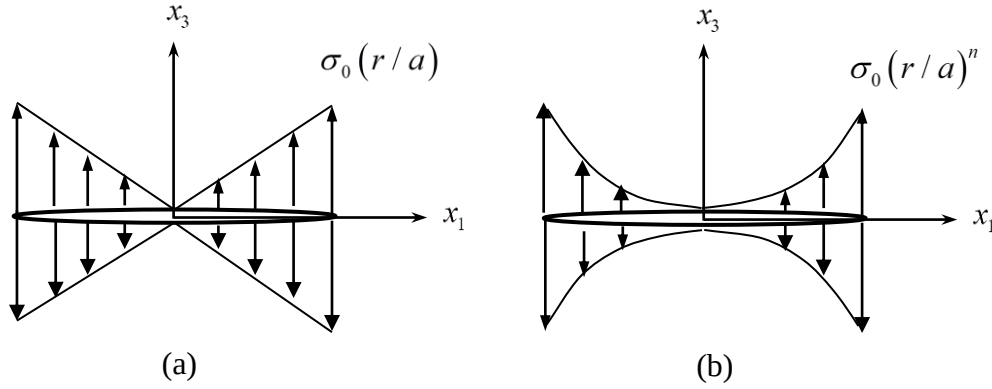


Figure 3.21: Circular crack under axisymmetric, self-equilibrated, crack-face, normal traction $\sigma_0 (r/a)^n$: (a) $n=1$ and (b) any generic value of n

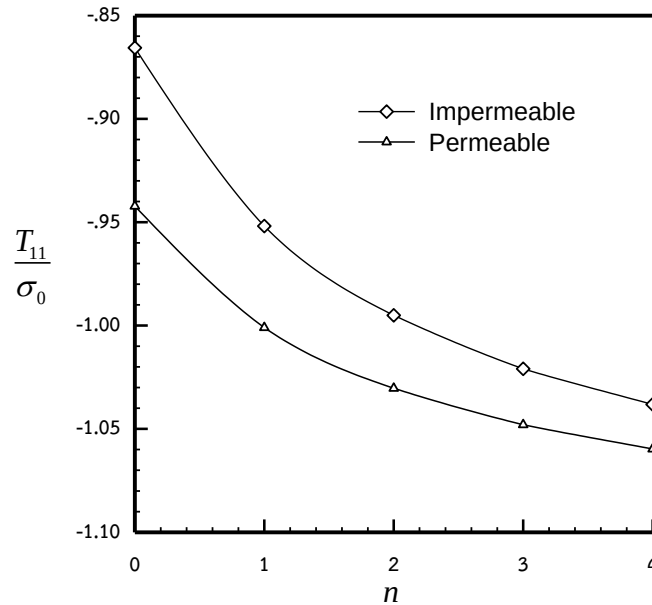


Figure 3.22: Normalized generalized T-stress components T_{11} of circular crack under axisymmetric, self-equilibrated, crack-face, normal traction $t_3 = \sigma_0 (r/a)^n$. Results are normalized by maximum value of traction σ_0 .

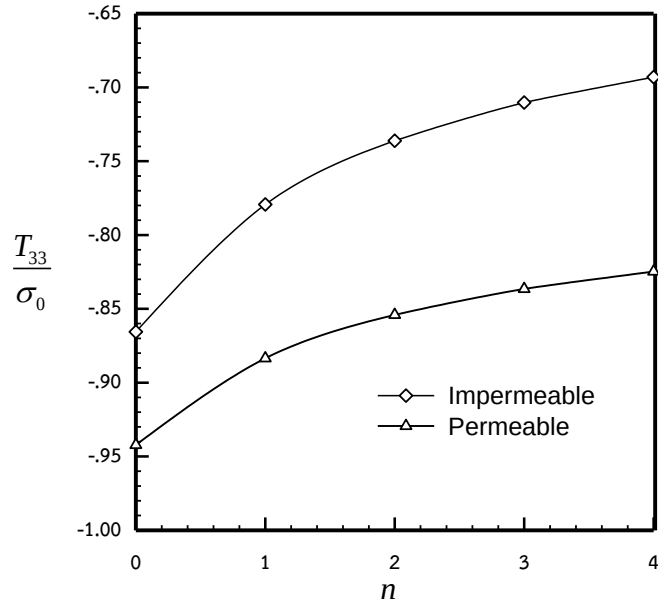


Figure 3.23: Normalized generalized T-stress components T_{33} of circular crack under axisymmetric, self-equilibrated, crack-face, normal traction $t_3 = \sigma_0 (r/a)^n$. Results are normalized by maximum value of traction σ_0 .

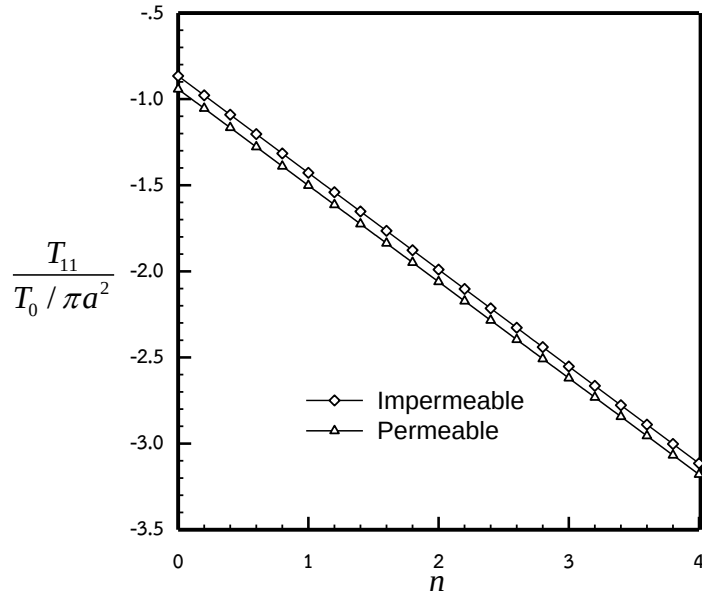


Figure 3.24: Normalized generalized T-stress components T_{11} of circular crack under axisymmetric, self-equilibrated, crack-face, normal traction $t_3 = \sigma_0 (r/a)^n$. Results are normalized by resultant force of traction T_0 .

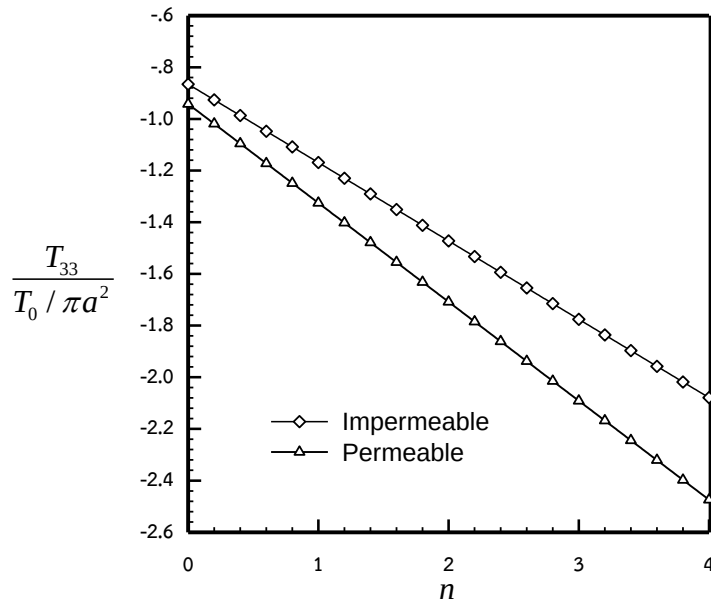


Figure 3.25: Normalized generalized T-stress components T_{33} of circular crack under axisymmetric, self-equilibrated, crack-face, normal traction $t_3 = \sigma_0 (r/a)^n$. Results are normalized by resultant force of traction T_0

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CHAPTER 4

WEAKLY SINGULAR BIEM FOR ANALYSIS OF CRACKS IN 3D MULTI-FIELD MEDIA

This chapter presents the development of a weakly singular boundary integral equation method for the analysis of cracks embedded in a coupled-field whole space. A pair of weak-form integral equations, one for crack-face generalized traction and the other for the gradient of the crack-face generalized displacement, is established in a general framework allowing various types of materials including elastic, piezoelectric, piezomagnetic and piezoelectromagnetic solids, general crack geometry and loading conditions to be handled in a unified fashion. In addition, the final governing integral equations contain only weakly singular kernels and this, as a result, renders the integral interpretation in terms of Riemann sum and the use of continuous basis functions in the approximation possible. Both symmetric Galerkin boundary element method and the Galerkin technique are employed to solve the pair of weak-form integral equations. Special near-front approximation is also employed to enhance the quality of solution near the crack front with the use of relatively coarse meshes. Explicit formula in terms of the gradient of the sum of the crack-face generalized displacement along the crack front is proposed to extract the generalized T-stress. Extensive results are reported not only to validate the present technique but also to demonstrate its capability and computational robustness.

4.1 Background and Review

The classical theory of linear fracture mechanics has been successfully employed, for the past several decades, in the modeling, analysis, and assessment of fracture-induced failure mechanisms of components and parts made of various types of materials such as elastic solids (e.g., William, 1957; Paris and Erdogan, 1960; Irwin, 1961; Foreman *et al.*, 1967; Nakamura and Park, 1990) and fully coupled-field media (e.g., Pak, 1992; Park and Sun, 1995; Zhang *et al.*, 2002; Gao *et al.*, 2003; Ma *et al.*, 2007; Ma *et al.*, 2015). The majority of those existing works is based primarily on the well-known assumptions that the inelastic deformation and nonlinear processes induced due to the presence of discontinuity are only pertained in a sufficiently small region surrounding the fracture front and are negligible (e.g., small scale yielding concept) and the fracture-related responses (e.g., crack growth criteria and the direction of crack advances) are governed or dictated only by a dominant, singular part of the near-front stress and generalized stress fields. It is evident from the primitive work of William (1957) for linear elastic media and its analogous extension for coupled-field materials that such the singular term contained in the near-front field expansion can be completely described by a single set of parameters known as the intensity factors. The knowledge of this fracture data is, therefore, essential and, at the same time, sufficient for establishing single-parameter-based fracture models relying only upon the singular field. However, various past evidences have indicated that integrating the nonsingular part of the near-front field into those models can further enhance or improve their capability to simulate fracture phenomena (e.g., Ayatollahi *et al.*, 2002; Sedinghiani *et al.*, 2011; Cheng *et al.*, 2012; Cheng *et al.*, 2012). Within the context of linear elasticity, a nonsingular part of stress

existing along the crack front, commonly known as the T-stress, has been found to significantly affect the hydrostatic tri-axiality ahead of the crack front, crack growth resistance curve, direction and stability of crack advances, and size and shape of the inelastic region surrounding the crack front (e.g., Cotterell, 1966; Williams and Ewing, 1972; Rice, 1974; Du and Hancock, 1991; Ayatollahi *et al.*, 1998; Smith *et al.*, 2001; Tvergaard, 2008). In addition, Larsson and Carlsson (1973) found that the conventional boundary layer technique integrating only the stress intensity factors was not a suitable approach for modeling real crack problems. However, once enhanced by adding the information of the T-stress, the modified boundary layer approach can provide more accurate prediction of the stress intensity factor at the first plastic yielding stage for various types of specimens. Similar to the elastic case, the important role of nonsingular terms in simulations of cracks in coupled-field media such as piezoelectric materials has also been investigated and confirmed (e.g., Zhu and Yang, 1999; Hao and Biao, 2004; Li and Lee, 2004; Viola *et al.*, 2008; Liu *et al.*, 2012). Two-parameter-based fracture models, integrating both the singular and nonsingular data along the crack front, have been found promising and increasingly employed by many investigators, in the past three decades, to study cracks in various aspects.

To facilitate the use of two-parameter-based fracture models in fracture simulations, the analysis technique capable of extracting the information of both intensity factors and the nonsingular terms along the crack front must be supplied. For instance, in the simulation of crack advances in an elastic medium, both the stress intensity factors and T-stress must be accurately calculated at any analysis step and then used as the basic information in the growth law to propagate the crack. To achieve such a crucial task, a powerful and robust solution procedure must be employed to handle not only the inherent complexity associated with the problem at the initial stage but also the general crack geometry resulting from the crack evolution. On the basis of an extensive literature survey, the computational technology for determining the intensity factors of cracks in both single and coupled-field media has been well established and various existing techniques have been found to have the vast capabilities to treat problems in a general context (e.g., Xu and Ortiz, 1993; Li *et al.*, 1998; Xu, 2000; Haas and Kuhn, 2002; Frangi *et al.*, 2002; Sanz *et al.*, 2005; Qin *et al.*, 2007; Rungamornrat and Mear, 2008b,c; Phongtinnaboot *et al.*, 2011; Rungamornrat *et al.*, 2015). Unlike the intensity factors, existing solution procedures for calculating the nonsingular terms such as the T-stress and generalized T-stress are relatively few and most of them are of limited capabilities. While several analytical schemes were proposed to establish explicit solutions of the T-stress and generalized T-stress along the crack front (e.g., Wang, 2004; Hao and Biao, 2004; Li and Lee, 2004; Kirilyuk and Levchuk, 2007; Zhong and Li, 2008; Viola *et al.*, 2008; Liu *et al.*, 2012; Rungamornrat and Pinitpanich, 2016; Rungamornrat *et al.*, 2018), such derived solutions are of limited uses due mainly to relatively simple settings and assumptions considered when the problems were formulated (e.g., geometries of cracks and bodies, material properties, loading conditions, crack-face conditions, etc.). To circumvent the limitation of analytical procedures, computational packages have been continuously proposed for determining the nonsingular fracture data (e.g., Nakamura and Parks, 1992; Henry and Luxmoore, 1995; Sladek *et al.*, 1997; Zhu and Yang, 1999; Fett *et al.*, 2006; Tran, 2010; Subsathapol *et al.*, 2014; Limwibul *et al.*, 2016). Those offered analysis packages commonly consist of two main routines, one associated with the analysis of field equations to obtain primary unknowns and the other corresponding to the post-process of solved data to attain the nonsingular terms along the crack front. The accuracy and computational efficiency of each technique depends primarily on those two routines.

A group of techniques based upon boundary integral equations has been proved efficient for fracture modeling and analysis since the first routine involves field equations of reduced spatial dimensions and primary unknowns are associated with quantities on the boundary of a body and crack surfaces (e.g., Cruse, 1988; Xu and Ortiz, 1993; Li *et al.*, 1998; Xu, 2000; Rungamornrat and Mear, 2008b). Such positive feature renders the boundary integral equation methods (BIEMs) computationally attractive when employed to handle evolving cracks. It is important to remark, however, that due to the suppression of primary unknowns to those on the boundary and cracks, field quantities within the domain must be obtained later via the post-process algorithm. In addition, the extraction of the fracture data such as the intensity factors and nonsingular terms directly from the interior near-front field is non-trivial in terms of computational efficiency and accuracy. For instance, the post-process for the T-stress using the path independent integrals (e.g., Sladek *et al.*, 1997; Shah *et al.*, 2005) requires the nontrivial evaluation of interior fields along the computational path. To maintain the benefit of BIEMs, alternative but efficient post-process schemes requiring only information on the crack surface have been sought. Recently, a technique based upon applying the additional displacement integral equation for cracks to determine the sum of the crack-face displacement and using such information to directly post-process for the T-stress was proposed by Tran (2010) and Rungamornrat *et al.* (preparation for publication) for cracks in two- and three-dimensional media, respectively. Such technique was also extended to treat cracks in an elastic half-space (Pham *et al.*, 2015) and cracks in couple-field media (Subsathapol *et al.*, 2014; Limwibul *et al.*, 2016). While this proposed technique already resolved the post-process issue regarding to the use of interior near-front field, the explicit formula used to extract the information of the T-stress and generalized T-stress still involves derivatives of the sum of the crack-face displacement. It is commonly known from the finite element approximation theory that the post-process for derivatives of approximate solutions can further degrade their accuracy. No evidence, to the best knowledge of the investigator, has been shown in the literature to support further enhancement of such technique. This gap of knowledge is considered significant and it, as a result, provides the underlying motivation of the present study

Various analytical techniques were proposed in the literature and successfully utilized to derive explicit solutions of the T-stress for cracks in elastic media (e.g., Wang, 2004; Kirilyuk and Levchuk, 2007; Schutte and Molla-Abbasi, 2007; Wang and Chen, 2014; Hua *et al.*, 2015; Rungamornrat and Pinitpanich, 2016) and the generalized T-stress for cracks in piezoelectric media (e.g., Hao and Biao, 2004; Li and Lee, 2004; Zhong and Li, 2008; Viola *et al.*, 2008; Liu *et al.*, 2012; Rungamornrat *et al.*, 2018). In those studies, Stroh formalism, solution representations (e.g., real and complex potential theories) and methods based upon integral transforms (e.g., Hankel integral transform and Fourier integral transform) were mainly employed to obtain basic field quantities and such information in the vicinity of the crack front is then used to obtain the nonsingular fracture data via the post-process algorithm such as the stress difference method (e.g., Wang, 2004) and the formula resulting from the asymptotic near-front expansion (e.g., Rungamornrat and Pinitpanich, 2016; Rungamornrat *et al.*, 2018). Derived explicit solutions for simple cracks under certain fundamental loading conditions (e.g., concentrated forces and charges) were further utilized as Green's function to establish post-process formula for computing the T-stress and generalized T-stress of the same crack under general loading conditions (e.g., Fett, 1997; Schutte and Molla-Abbasi, 2007; Rungamornrat and Pinitpanich, 2016; Rungamornrat *et al.*, 2018). Other explicit expressions were also proposed based upon the known solutions for certain fundamental cases together with the weight-function

technique (e.g., Hua *et al.*, 2015). While a set of explicit solutions/expressions/formula for nonsingular terms was derived, they are clearly of limited uses in the fracture modeling due primarily to the relatively simple and very specific settings considered in the problem formulation. The capability to handling general crack geometry, loading conditions and material properties is an essential issue especially when simulations of crack advances are of interest.

Advance in the development of numerical procedures for the analysis of nonsingular terms of cracks in various types of materials has been continuously grown in the past three decades. The key objective of most of existing studies was to seek a technique that yields sufficiently accurate results and, at the same time, is computationally robust and user friendly. Methods based upon the finite element approximation have been widely utilized as the basic tool for performing the analysis of basic fields (e.g., displacements and stresses) within the cracked media. Such finite element solutions were then post-processed for the T-stress data along the crack front by various investigators via Eshelby's method, interaction integrals and path independent contour integrals based on Betti-Rayleigh reciprocal theorem (e.g., Cardew *et al.*, 1985; Kfoury, 1986; Nakamura and Parks, 1992; Sladek *et al.*, 1997; Zhao *et al.*, 2001; Chen *et al.*, 2001; Kim and Paulino, 2003; Wang, 2003; Wang and Bell, 2004). Other approaches such as the method of Green's function, reciprocal integrals and the weight function scheme (e.g., Sham, 1991; Wang, 2002a; Wang, 2002b; Fett and Rizzi, 2006; Lewis and Wang, 2008) and the method of superposition (e.g. Wang and Bell, 2004; Molla-Abbasi and Schutte, 2008; Meshii *et al.*, 2010) were also utilized along with the finite element results to extract the T-stress solutions for various cases. While the FE-based techniques have circumvented the limited capability of analytical and semi-analytical schemes, the analysis of field problems still requires the discretization of solution over the whole domain and this poses serious issue on computational efficiency and simplicity when used to handle crack evolution. In addition, standard finite element procedure generally requires relatively very fine mesh in the region surrounding the cracks in order to achieve the high accuracy of approximate solutions especially when the fracture data along the crack front is of primary interest (e.g., Ayhan *et al.*, 2006). It is remarked also that the post-process algorithm mentioned above does not directly involve the information of near-front fields and still requires nontrivial evaluation of involved path integrals. The techniques, which directly exercise the near-front solutions such as a boundary-layer method based on the subtraction of tangential stress components (e.g., Lasson and Carlsson, 1973; Leever and Radon, 1982) and a method using the near-front crack-face displacement (e.g. Al-Ani and Hancock, 1991; Ayatollahi *et al.*, 1998) were also recognized. The performance of such two methods when applied to three-dimensional crack problems was investigated by Henry and Luxmoore (1995). Recently, a group of techniques based upon scaled boundary finite element methods which combine both analytical feature and the finite element approximation was proposed for the analysis of T-stress of cracks in two-dimensional elastic media (e.g., Saputra *et al.*, 2015; Chowdhury *et al.*, 2015). While such numerical procedures reduce one spatial dimension in the discretization and provide a direct mean to calculate the fracture data at the crack tip, their computational efficiency is still questionable when dealing with non-planar and multiple cracks and large scale problems.

Alternative numerical techniques based upon boundary integral equations have been extensively established for solving linear boundary value problems in mechanics (e.g., Liggett and Liu, 1983; Cruse, 1988; Brebbia and Dominguez, 1989) and they have been proved computationally efficient for linear fracture analysis (e.g., Gu and Yew, 1988; Xu and Ortiz, 1993; Li *et al.*, 1998; Pan and Yuan, 2000; Xu, 2000; Frangi *et al.*, 2002; Ariza and Dominguez,

2004; Rungamornrat and Mear, 2008b,c; Phongtinnaboot *et al.*, 2011; Rungamornrat *et al.*, 2015). One of the positive features of those methods is that the key governing equations can be formulated in terms of unknowns on the boundary and the crack surface, thus rendering the reduction of one spatial dimension in the solution discretization procedure. The less number of degrees of freedom is generally required, in comparison with the domain-based schemes such as the finite element method, especially when an unbounded medium is to be treated. In addition, such low dimension of discretization also renders the techniques in this group suitable for simulating crack evolutions. Within the context of linear fracture analysis, studies related to the development and use of boundary integral equation methods in the analysis of nonsingular fracture data along the crack front are significantly less in comparison with those for determining the intensity factors (e.g., Li *et al.*, 1998; Xu, 2000; Frangi *et al.*, 2002; Rungamornrat and Mear, 2008b,c; Phongtinnaboot *et al.*, 2011; Rungamornrat *et al.*, 2015). Results from a careful review of relevant research articles are summarized as follows. Sladek *et al.* (1997) proposed a conventional boundary element method together with the nonsingular integral formula for computing the T-stress of cracks in a two-dimensional, isotropic elastic finite media. In their work, the post-process algorithm still requires the knowledge of the field along a selected path and such data at any point must be obtained before applying the contour integral. Sutradhar and Paulino (2004) developed a hypersingular symmetric Galerkin boundary element method (SGBEM) to solve cracks in two-dimensional, isotropic, linear elastic infinite and finite bodies. Similar to the work of Sladek *et al.* (1997), the same type of path independent integral was employed to post-process for the T-stress. Later, Yu *et al.* (2006) investigated crack emanating from a circular hole in a two-dimensional, rectangular elastic plate subjected to uniform tension and bending loads. In the analysis, the boundary element method was employed to solve for the key unknowns on the boundary and crack and the M-contour integral scheme was adopted to obtain the T-stress at the crack-tip. Again, in their technique, the additional post-process for stresses within the body must be carried out before applying the contour integral. To enhance the efficiency of the post-process algorithm, Tran (2010) employed the weak-form displacement integral equation for cracks in addition to the weak-form traction integral equation used in the weakly singular symmetric Galerkin boundary element method (SGBEM) to compute the sum of the crack-face displacement. Such additional crack-face data was then employed to compute the T-stress at the crack tip. The explicit formula of the T-stress proposed by Tran (2010) does not require the information of interior fields and the evaluation of any integral but still involves the calculation of derivatives of the sum of the crack-face displacement. Another approach was introduced by Phan (2011) in the analysis of cracks in two-dimensional, elastic infinite and finite bodies. In his work, a hypersingular SGBEM was used to solve the boundary value problem for the boundary data and the relative crack-face displacement, and a novel non-singular boundary integral formula containing only the crack-face displacement data was established and used in the post-process for the T-stress. While such proposed formula does not require the information of elastic fields within the body, the evaluation of all involved integrals over the entire crack can be computationally demanding.

Due to attractive features of the technique proposed by Tran (2010), it was further extended by Rungamornrat *et al.* (preparation for publication) for cracks in three-dimensional, anisotropic linear elastic infinite media, by Subsathaphol (2013) and Subsathaphol *et al.* (2015) for impermeable cracks in three-dimensional, linear piezoelectric infinite media, and later by Limwibul (2015) and Limwibul *et al.* (2016) to take the influence of different crack-face electrical/mechanical conditions into account. The generalization to handle cracks in linear

elastic half-space under several types of boundary conditions was also recognized in the work of Pham *et al.* (2015). While this particular technique was successfully implemented for various scenarios, the post-process for extracting the T-stress or generalized T-stress data still requires the calculation of derivatives of the sum of the crack-face displacement or generalized displacement along the crack front and this can lead to the degradation of the solution accuracy (e.g., Pham *et al.*, 2015). To overcome such difficulty, Tran and Mear (2018) recently proposed an alternative by replacing the weak-form displacement integral equation for cracks by that involving the surface gradient of the sum of the crack-face displacement. Such the novel integral equation can provide the direct crack-face data to be employed in the post-process formula without the requirement for differentiations. While the technique developed by Tran and Mear (2018) is computationally promising for the analysis of nonsingular fracture data, the formulation and implementation are still restricted to cracks in two-dimensional media. The generalization to handle three-dimensional cracks problems for both single and couple-field media has not been recognized in the literature.

4.2 Problem Formulation

This section mainly presents the formulation of a general boundary value problem and the development of boundary integral equations governing cracks in a general, linear coupled-field infinite medium. A set of basic field equations is established first in a sufficiently general form allowing a variety of problems to be treated and boundary integral relations of Somigliana-type are then developed for the surface of discontinuity embedded in a whole space. A systematic regularization technique is then applied to derive a pair of weakly singular weak-form boundary integral equations governing the unknown crack-face data.

4.2.1 Basic field equations

Consider a three-dimensional body Ω occupying a whole Euclidean space $\mathbb{R}^3$ (i.e., $\Omega = \mathbb{R}^3$) as illustrated in Figure 4.1. Let $\{O; x_1, x_2, x_3\}$ be a Cartesian coordinate system with O denoting the origin and x_i denoting the coordinate along the direction of a unit base vector $\mathbf{e}_i$. A state of the body is described by a collection of α vector fields $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \dots, \mathbf{u}^{(\alpha)}$ and β scalar fields $\phi^{[1]}, \phi^{[2]}, \dots, \phi^{[\beta]}$ and this collection is termed the state variables. It is remarked that α and β are nonnegative integers and $\alpha + \beta > 0$. Components of the state variables referring to the coordinate system $\{O; x_1, x_2, x_3\}$ can be stored, for convenience in further development, in a single $3\alpha + \beta$ -component vector $\hat{\mathbf{u}}$ such that $u_i^{(k)} = \hat{u}_{3(k)+i-3}$ and $\phi^{[k]} = \hat{u}_{3\alpha+[k]}$. Here and in what follows, lower case indices with the parenthesis range from 1 to α ; lower case indices with the bracket range from 1 to β ; and lower case indices without the parenthesis and bracket range from 1 to 3. The measure of change of all state variables across the entire body is denoted by a collection of α symmetric second-order tensor fields $\boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)}, \dots, \boldsymbol{\varepsilon}^{(\alpha)}$ and β vector fields $\mathbf{g}^{[1]}, \mathbf{g}^{[2]}, \dots, \mathbf{g}^{[\beta]}$ and these quantities are termed the state-variable variations. In the present study, the state-variable variations are assumed to be linearly related to the state variables by

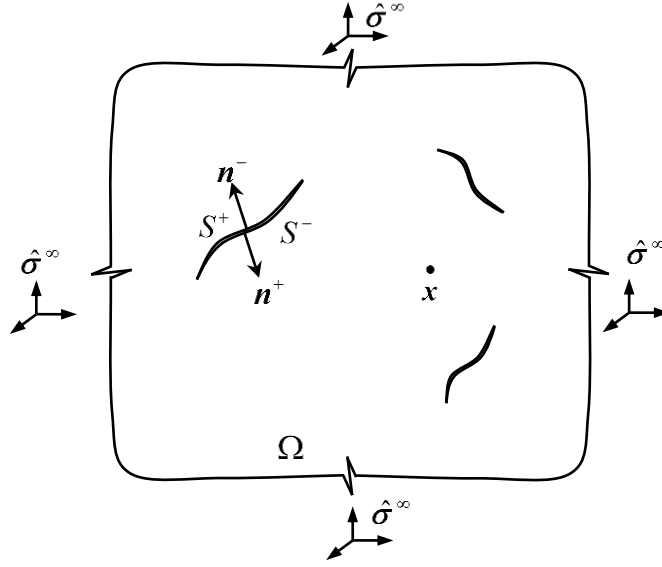


Figure 4.1: Schematic of isolated cracks in linear whole space under crack-face and remote loading

$$\boldsymbol{\varepsilon}^{(k)} = \frac{1}{2} \left[\nabla \mathbf{u}^{(k)} + \nabla^T \mathbf{u}^{(k)} \right] \Leftrightarrow \varepsilon_{ij}^{(k)} = \frac{1}{2} \left[\frac{\partial u_i^{(k)}}{\partial x_j} + \frac{\partial u_j^{(k)}}{\partial x_i} \right] \quad (4.1)$$

$$\mathbf{g}^{[k]} = \nabla \phi^{[k]} \Leftrightarrow g_i^{[k]} = \frac{\partial \phi^{[k]}}{\partial x_i} \quad (4.2)$$

where ∇ denotes a standard gradient operator and ∇^T denotes its transpose. Consistent with the case of state variables, components of all state-variable variations referring to the coordinate system $\{O; x_1, x_2, x_3\}$ are stored in a single $3 \times (3\alpha + \beta)$ -matrix $\hat{\boldsymbol{\varepsilon}}$ such that $\varepsilon_{ij}^{(k)} = \hat{\varepsilon}_{i:3(k)+j-3}$ and $g_i^{[k]} = \hat{\varepsilon}_{i:3\alpha+k}$ where the semicolon “:” is used here with the only purpose to separate the indices of rows and columns of the matrix. With the definition of $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\varepsilon}}$, the relations (4.1) and (4.2) can be now rewritten in a more concise indicial form as

$$\hat{\varepsilon}_{iJ} = \begin{cases} \frac{1}{2} \left[\frac{\partial \hat{u}_J}{\partial x_i} + \frac{\partial \hat{u}_{3(k)+i-3}}{\partial x_J} \right], & J = 3(k) + j - 3 \leq 3\alpha, j \in \{1, 2, 3\} \\ \frac{\partial \hat{u}_J}{\partial x_i}, & J > 3\alpha \end{cases} \quad (4.3)$$

where the upper case indices, used here and in what follows, range from 1 to $3\alpha + \beta$. The internal flux induced within the body due to the non-zero state-variable variations (or, equivalently, the nonhomogeneous state variables), termed the body flux, is measured by a collection of α symmetric second-order tensor fields $\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \dots, \boldsymbol{\sigma}^{(\alpha)}$ and β vector fields $\mathbf{s}^{[1]}$

, $\mathbf{s}^{[2]}$, ..., $\mathbf{s}^{[\beta]}$. The intensity of the body flux depends primarily on the behavior of a material constituting the body and, in the present study, the following linear constitutive laws are considered:

$$\boldsymbol{\sigma}^{(k)} = \mathbf{E}^{(k)(r)} \boldsymbol{\varepsilon}^{(r)} + \mathbf{D}^{(k)[r]} \mathbf{g}^{(r)} \Leftrightarrow \sigma_{ij}^{(k)} = E_{ijpq}^{(k)(r)} \varepsilon_{pq}^{(r)} + D_{ijp}^{(k)[r]} g_p^{[r]} \quad (4.4)$$

$$\mathbf{s}^{[k]} = \mathbf{D}^{[k](r)} \boldsymbol{\varepsilon}^{(r)} + \mathbf{C}^{[k][r]} \mathbf{g}^{(r)} \Leftrightarrow s_i^{[k]} = D_{pqi}^{[k](r)} \varepsilon_{pq}^{(r)} + C_{ip}^{[k][r]} g_p^{[r]} \quad (4.5)$$

where $\mathbf{E}^{(k)(r)}$, $\mathbf{D}^{(k)[r]}$, and $\mathbf{C}^{(k)[r]}$ are fourth-order, third-order, and second-order tensors of material constants, respectively, and repeated indices imply the summation over their range. In addition, the material is postulated to possess following symmetries: $E_{ijpq}^{(k)(r)} = E_{jipq}^{(k)(r)} = E_{ijqp}^{(k)(r)} = E_{pqij}^{(k)(r)}$, $E_{ijpq}^{(k)(r)} = E_{ijpq}^{(r)(k)}$, $D_{ijp}^{(k)[r]} = D_{jip}^{(k)[r]}$, $D_{ijp}^{[k](r)} = D_{jip}^{[k](r)}$, $D_{ijp}^{(k)[r]} = D_{ijp}^{[r][k]}$, $C_{ip}^{[k][r]} = C_{pi}^{[k][r]}$, and $C_{ip}^{[k][r]} = C_{ip}^{[r][k]}$. Components of the body flux can then be stored, in a similar fashion, in a single $3\alpha(3\alpha+\beta)$ -matrix $\hat{\boldsymbol{\sigma}}$ such that $\sigma_{ij}^{(k)} = \hat{\sigma}_{i:3(k)+j-3}$ and $s_i^{[k]} = \hat{\sigma}_{i:3\alpha+k}$. By combining the relations (4.1)-(4.2) and the constitutive relations (4.4) and (4.5) and then using the introduced concise notations, it leads to the relation between the body flux and the state variables of the form

$$\hat{\sigma}_{iJ} = \hat{E}_{iJKm} \hat{u}_{K,m} \quad (4.6)$$

where $\hat{E}_{iJKm}$ is termed a generalized moduli and defined in terms of $\mathbf{E}^{(k)(r)}$, $\mathbf{D}^{(k)[r]}$, and $\mathbf{C}^{(k)[r]}$ in a manner consistent with the body flux $\hat{\sigma}_{iJ}$ and the state variable $\hat{u}_K$. It is important to note that due to the symmetric properties of $\mathbf{E}^{(k)(r)}$, $\mathbf{D}^{(k)[r]}$, and $\mathbf{C}^{(k)[r]}$, it can readily be verified that $\hat{E}_{iJKm} = \hat{E}_{mKJi}$. Finally, the body flux must satisfy the following laws of conservations:

$$\nabla \cdot (\boldsymbol{\sigma}^{(k)})^T + \mathbf{b}^{(k)} = \mathbf{0} \Leftrightarrow \sigma_{ij,i}^{(k)} + b_j^{(k)} = 0 \quad (4.7)$$

$$\nabla \cdot \mathbf{s}^{[k]} + \mathbf{a}^{[k]} = 0 \Leftrightarrow s_{i,i}^{[k]} + a^{[k]} = 0 \quad (4.8)$$

where $\nabla \cdot$ denotes the divergence operator and a collection of α prescribed vector fields $\mathbf{b}^{(1)}$, $\mathbf{b}^{(2)}$, ..., $\mathbf{b}^{(\alpha)}$ and β prescribed scalar fields $a^{[1]}$, $a^{[2]}$, ..., $a^{[\beta]}$ is termed the distributed body source. By storing components of the distributed body source in a single $3\alpha+\beta$ -component vector $\hat{\mathbf{b}}$ such that $b_i^{(k)} = \hat{b}_{3(k)+i-3}$ and $a^{[k]} = \hat{b}_{3\alpha+[k]}$, the laws of conservation (4.7) and (4.8) simply reduces in form to

$$\hat{\sigma}_{iJ,i} + \hat{b}_J = 0 \quad (4.9)$$

The field equations (2.3), (2.6) and (2.9) constitute a complete set of equations governing the state variables $\hat{\mathbf{u}}$, the state-variable variations $\hat{\boldsymbol{\varepsilon}}$, and the body flux $\hat{\boldsymbol{\sigma}}$.

At any point on a smooth surface, the surface flux described by a collection of α vector fields $\mathbf{t}^{(1)}$, $\mathbf{t}^{(2)}$, ..., $\mathbf{t}^{(\alpha)}$ and β scalar fields $d^{[1]}$, $d^{[2]}$, ..., $d^{[\beta]}$ can be related to the body flux and the outward unit normal vector to the surface $\mathbf{n}$ via the laws of conservation by

$$\mathbf{t}^{(k)} = (\boldsymbol{\sigma}^{(k)})^T \mathbf{n} \Leftrightarrow t_i^{(k)} = \sigma_{ji}^{(k)} n_j \quad (4.10)$$

$$\mathbf{d}^{[k]} = \mathbf{s}^{[k]} \mathbf{n} \Leftrightarrow d^{[k]} = s_j^{[k]} n_j \quad (4.11)$$

Again, the relations (4.10) and (4.11) can also be expressed in a concise form as

$$\hat{t}_j = \hat{\sigma}_{ij} n_i \quad (4.12)$$

It is worth noting that the field equations (4.3), (4.6) and (4.9) along with the relation (4.12) and the involved field quantities, via the notation introduced above, are of the same form as that for problems in linear elasticity. In addition, this form of general field equations can be applied to various classes of linear problems via the proper change of the two integers α and β . For instance, field problems governed by Laplace's or generalized Laplace's equation can be handled by setting $\alpha = 0, \beta = 1$; linear elasticity problems are obtained by taking $\alpha = 1, \beta = 0$; linear piezoelectric and linear piezomagnetic problems are handled by setting $\alpha = 1, \beta = 1$; and linear piezoelectromagnetic are obtained by taking $\alpha = 1, \beta = 2$.

Now, it is ready to further provide the clear description of a problem to be investigated in the present study. The body Ω shown in Figure 4.1 is made of a material obeying the constitutive law (4.6) with the generalized moduli $\hat{E}_{iJKm}$ constant throughout and is free of the distributed body source (i.e., $\hat{b}_j = 0$). The medium contains isolated surfaces across which the state variables experience the jump. The surface of discontinuity is represented by two geometrically identical surfaces S^+ and S^- whose outward unit normal vectors at the coincident points, denoted respectively by $\mathbf{n}^+$ and $\mathbf{n}^-$, satisfy $\mathbf{n}^+ = -\mathbf{n}^-$. In the present study, the attention is restricted only to the case that the surface of discontinuity is piecewise smooth, free of the jump in the state variables along its boundary, and subjected to fully prescribed surface flux. From now on, a term "crack" is used, for convenience and establishing the connection to problems in linear elasticity, to designate this type of discontinuity. In addition to the surface flux applied to the crack surface, the body is subjected to the constant remote body flux $\hat{\sigma}^\infty$.

4.2.2 Standard integral relations

As an essential component in the development of standard integral relations for cracks in an infinite medium, fundamental solutions of an uncracked whole space subjected to a particular distributed body source $\hat{b}_j^p(\boldsymbol{\xi} - \mathbf{x}) = \delta_{jp} \delta(\boldsymbol{\xi} - \mathbf{x})$ where δ_{jp} denotes the generalized Kronecker symbol and $\delta(\boldsymbol{\xi} - \mathbf{x})$ is the Dirac-delta distribution center at $\mathbf{x}$ is established first. A system of partial differential equation governing the state-variable fundamental solution $U_j^p(\boldsymbol{\xi} - \mathbf{x})$ and the associated body flux $\Lambda_{ij}^p(\boldsymbol{\xi} - \mathbf{x})$ is given by

$$\Lambda_{ij,i}^p(\boldsymbol{\xi} - \mathbf{x}) + \delta_{jp} \delta(\boldsymbol{\xi} - \mathbf{x}) = 0 \quad (4.13)$$

$$\Lambda_{ij}^p(\boldsymbol{\xi} - \mathbf{x}) = \hat{E}_{iJKm} U_{K,m}^p(\boldsymbol{\xi} - \mathbf{x}) \quad (4.14)$$

By applying the same solution technique via the Radon integral transform suggested by Rungamornrat and Mear (2008a), the state-variable fundamental solution $U_J^P(\xi - \mathbf{x})$ can be obtained explicitly as

$$U_J^P(\xi - \mathbf{x}) = \frac{1}{8\pi^2 r} \oint_{\mathbf{z} \cdot \mathbf{r} = 0} (\mathbf{z}, \mathbf{z})_{JP}^{-1} d\mathbf{z} \quad (4.15)$$

where $\mathbf{r} = \xi - \mathbf{x}$, $r = \|\mathbf{r}\|$, $\mathbf{z}$ is a unit vector along the plane $\mathbf{z} \cdot \mathbf{r} = 0$, $(\mathbf{z}, \mathbf{z})$ is a symmetric matrix defined by $(\mathbf{z}, \mathbf{z})_{JP} = z_i E_{iJPj} z_j$, and $(\mathbf{z}, \mathbf{z})^{-1}$ stands for the inverse of $(\mathbf{z}, \mathbf{z})$. The body-flux fundamental solution $\Lambda_{ij}^P(\xi - \mathbf{x})$ can readily be obtained by substituting (4.15) into the constitutive relation (4.14). It is obviously seen that $U_J^P(\xi - \mathbf{x})$ and $\Lambda_{ij}^P(\xi - \mathbf{x})$ are singular only at $\xi = \mathbf{x}$ of order $1/r$ and $1/r^2$, respectively, and, in addition, $U_J^P(\xi - \mathbf{x}) = U_P^J(\xi - \mathbf{x})$.

By combining (4.6) and (4.9) and setting $\hat{b}_j = 0$, it yields a system of partial differential equations governing the state variables of a cracked medium shown in Figure 4.1:

$$\hat{E}_{iJKm} \hat{u}_{K,mi} = 0 \quad (4.16)$$

Taking the inner product between (4.16) and $U_J^P(\xi - \mathbf{x})$ and then integrating the result over the entire body Ω lead to

$$\int_{\Omega} U_J^P(\xi - \mathbf{x}) \hat{E}_{iJKm} \hat{u}_{K,mi}(\xi) dV(\xi) = 0 \quad (4.17)$$

By performing the integration by parts via Gauss-divergence theorem and then using the constitutive law (4.6), it gives rise to

$$\begin{aligned} \int_{\Omega} \hat{E}_{iJKm} U_{J,i}^P(\xi - \mathbf{x}) \hat{u}_{K,m}(\xi) dV(\xi) &= \int_{S^+} U_J^P(\xi - \mathbf{x}) \hat{t}_J^+(\xi) dS(\xi) + \int_{S^-} U_J^P(\xi - \mathbf{x}) \hat{t}_J^-(\xi) dS(\xi) \\ &\quad + \int_{\partial\Omega} U_J^P(\xi - \mathbf{x}) \hat{\sigma}_{ij}(\xi) n_i(\xi) dS(\xi) \end{aligned} \quad (4.18)$$

where $\hat{t}_J^+(\xi) = \hat{\sigma}_{ij}(\xi) n_i^+(\xi)$ and $\hat{t}_J^-(\xi) = \hat{\sigma}_{ij}(\xi) n_i^-(\xi)$ denote the prescribed surface fluxes on the crack surfaces S^+ and S^- , respectively. By recalling the symmetry $\hat{E}_{iJKm} = \hat{E}_{mKji}$ and (4.14), the relation (4.18) reduces to

$$\begin{aligned} \int_{\Omega} \Lambda_{mK}^P(\xi - \mathbf{x}) \hat{u}_{K,m}(\xi) dV(\xi) &= \int_{S^+} U_J^P(\xi - \mathbf{x}) \hat{t}_J^+(\xi) dS(\xi) + \int_{S^-} U_J^P(\xi - \mathbf{x}) \hat{t}_J^-(\xi) dS(\xi) \\ &\quad + \int_{\partial\Omega} U_J^P(\xi - \mathbf{x}) \hat{\sigma}_{ij}(\xi) n_i(\xi) dS(\xi) \end{aligned} \quad (4.19)$$

By performing the integration by parts of the integral on the left hand side via Gauss-divergence theorem, it yields

$$\begin{aligned}
\int_{\Omega} \Lambda_{mK,m}^P(\xi - \mathbf{x}) \hat{u}_K(\xi) dV(\xi) = & - \int_{S^+} U_J^P(\xi - \mathbf{x}) \hat{t}_J^+(\xi) dS(\xi) - \int_{S^-} U_J^P(\xi - \mathbf{x}) \hat{t}_J^-(\xi) dS(\xi) \\
& + \int_{S^+} \Lambda_{mK}^P(\xi - \mathbf{x}) n_m^+(\xi) \hat{u}_K^+(\xi) dS(\xi) + \int_{S^-} \Lambda_{mK}^P(\xi - \mathbf{x}) n_m^-(\xi) \hat{u}_K^-(\xi) dS(\xi) \\
& - \int_{\partial\Omega} U_J^P(\xi - \mathbf{x}) \hat{\sigma}_{ij}(\xi) n_i(\xi) dS(\xi) + \int_{\partial\Omega} \Lambda_{mK}^P(\xi - \mathbf{x}) n_m(\xi) \hat{u}_K(\xi) dS(\xi)
\end{aligned} \tag{4.20}$$

where $\hat{u}_K^+$ and $\hat{u}_K^-$ denote the values of state variables on the crack surfaces S^+ and S^- , respectively. From the definition of the body-flux fundamental solution and properties of δ_{jp} and $\delta(\xi - \mathbf{x})$, the integral relation (4.20) becomes

$$\begin{aligned}
\hat{u}_p(\mathbf{x}) = & \int_{S^+} U_J^P(\xi - \mathbf{x}) \hat{t}_J^+(\xi) dS(\xi) + \int_{S^-} U_J^P(\xi - \mathbf{x}) \hat{t}_J^-(\xi) dS(\xi) \\
& - \int_{S^+} \Lambda_{mK}^P(\xi - \mathbf{x}) n_m^+(\xi) \hat{u}_K^+(\xi) dS(\xi) - \int_{S^-} \Lambda_{mK}^P(\xi - \mathbf{x}) n_m^-(\xi) \hat{u}_K^-(\xi) dS(\xi) \\
& + \int_{\partial\Omega} U_J^P(\xi - \mathbf{x}) \hat{\sigma}_{ij}(\xi) n_i(\xi) dS(\xi) - \int_{\partial\Omega} \Lambda_{mK}^P(\xi - \mathbf{x}) n_m(\xi) \hat{u}_K(\xi) dS(\xi)
\end{aligned} \tag{4.21}$$

By using the remote condition, the identical geometry of S^+ and S^- , $\mathbf{n}^+ = -\mathbf{n}^-$, and the continuity of $U_J^P(\xi - \mathbf{x})$ and $\Lambda_{mK}^P(\xi - \mathbf{x})$ across the cracks, it leads to the integral relation for the state variables

$$\hat{u}_p(\mathbf{x}) = \hat{u}_p^\infty(\mathbf{x}) + \int_{S^+} U_J^P(\xi - \mathbf{x}) \Sigma \hat{t}_J(\xi) dS(\xi) - \int_{S^+} \Lambda_{mK}^P(\xi - \mathbf{x}) n_m^+(\xi) \Delta \hat{u}_K(\xi) dS(\xi) \tag{4.22}$$

where $\Sigma \hat{t}_J = \hat{t}_J^+ + \hat{t}_J^-$ and $\Delta \hat{u}_K = \hat{u}_K^+ - \hat{u}_K^-$ are the sum of the crack-face surface flux and the jump in the crack-face state variables, respectively. The integral relation (4.22) is of the same form as that of Somigliana's identity. To establish the integral relation for the body flux, the integral relation (4.22) is first substituted into (4.6) and then using the constitutive law, the reciprocity $U_J^P(\xi - \mathbf{x}) = U_P^J(\xi - \mathbf{x})$, translational properties $\partial U_J^P / \partial \xi_i = -\partial U_J^P / \partial x_i$ and $\partial \Lambda_{mK}^P / \partial \xi_i = -\partial \Lambda_{mK}^P / \partial x_i$, it finally yields

$$\hat{\sigma}_{ij}(\mathbf{x}) = \hat{\sigma}_{ij}^\infty(\mathbf{x}) - \int_{S^+} \Lambda_{ij}^P(\xi - \mathbf{x}) \Sigma \hat{t}_p(\xi) dS(\xi) + \int_{S^+} \Sigma_{ij}^{IK}(\xi - \mathbf{x}) n_i^+(\xi) \Delta \hat{u}_K(\xi) dS(\xi) \tag{4.23}$$

where the function $\Sigma_{ij}^{IK}(\xi - \mathbf{x})$ is defined by

$$\Sigma_{ij}^{IK}(\xi - \mathbf{x}) = \hat{E}_{ijPm} \frac{\partial \Lambda_{IK}^P(\xi - \mathbf{x})}{\partial \xi_m} \quad (4.24)$$

From the singularity behavior of $\Lambda_{ij}^P(\xi - \mathbf{x})$, the function $\Sigma_{ij}^{IK}(\xi - \mathbf{x})$ is singular at $\xi = \mathbf{x}$ of order $1/r^3$. It is noted that the integral relation for the state variable (4.22) involves the weakly singular kernel $U_j^P(\xi - \mathbf{x})$ and the strongly kernel $\Lambda_{ij}^P(\xi - \mathbf{x})$ whereas the integral relation for the body flux (4.23) contains the strongly singular kernel $\Lambda_{ij}^P(\xi - \mathbf{x})$ and the hypersingular kernel $\Sigma_{ij}^{IK}(\xi - \mathbf{x})$.

4.2.3 Singularity-reduced integral relations

By following the same strategy proposed by Rungamornrat and Mear (2008a), the strongly singular kernel $\Lambda_{ij}^P(\xi - \mathbf{x})$ and the hypersingular kernel $\Sigma_{ij}^{IK}(\xi - \mathbf{x})$ can be decomposed into

$$\Lambda_{ij}^P(\xi - \mathbf{x}) = H_{ij}^P(\xi - \mathbf{x}) + \varepsilon_{ism} \frac{\partial G_{mj}^P(\xi - \mathbf{x})}{\partial \xi_s} \quad (4.25)$$

$$\Sigma_{IK}^{ij}(\xi - \mathbf{x}) = -\hat{E}_{sKji} \delta(\xi - \mathbf{x}) + \varepsilon_{lqm} \frac{\partial}{\partial \xi_q} \varepsilon_{irt} \frac{\partial}{\partial \xi_r} C_{mK}^{tj}(\xi - \mathbf{x}) \quad (4.26)$$

where $G_{mj}^P(\xi - \mathbf{x})$ and $C_{mK}^{tj}(\xi - \mathbf{x})$ are unknown weakly singular functions and $H_{ij}^P(\xi - \mathbf{x})$ is a prescribed function defined by

$$H_{ij}^P(\xi - \mathbf{x}) = -\frac{\delta_{JP}}{4\pi r^3}(\xi_i - x_i) \quad (4.27)$$

The existence of the functions $G_{mj}^P(\xi - \mathbf{x})$ and $C_{mK}^{tj}(\xi - \mathbf{x})$ in the decompositions (4.25) and (4.26) and the corresponding solution procedure follow directly the work of Rungamornrat and Mear (2008a). The explicit expression of $G_{mj}^P(\xi - \mathbf{x})$ and $C_{mK}^{tj}(\xi - \mathbf{x})$ is given by

$$G_{mj}^P(\xi - \mathbf{x}) = \frac{\varepsilon_{mqa} E_{qJKl}}{8\pi^2 r} \oint_{\mathbf{z} \cdot \mathbf{r}=0} z_a z_l (\mathbf{z}, \mathbf{z})_{KP}^{-1} ds(\mathbf{z}) \quad (4.28)$$

$$C_{mK}^{tj}(\xi - \mathbf{x}) = \frac{A_{mKQs}^{tJPl}}{8\pi^2 r} \oint_{\mathbf{z} \cdot \mathbf{r}=0} z_s z_l (\mathbf{z}, \mathbf{z})_{PQ}^{-1} ds(\mathbf{z}) \quad (4.29)$$

where ε_{mqa} is a standard permutation symbol and the constants A_{mKQs}^{tJPl} are defined in terms of the generalized moduli by

$$A_{mKQs}^{IJPl} = \varepsilon_{ptd} \varepsilon_{pmq} \left\{ \hat{E}_{dJPl} \hat{E}_{qKQs} - \frac{1}{3\alpha + \beta} \hat{E}_{lQP_s} \hat{E}_{dJKq} \right\} \quad (4.30)$$

It is evident from (4.27), (4.28) and (4.29) that $H_{ij}^P(\xi - \mathbf{x})$, $G_{mJ}^P(\xi - \mathbf{x})$ and $C_{mK}^{IJ}(\xi - \mathbf{x})$ are singular at $\xi = \mathbf{x}$ of order $1/r^2$, $1/r$ and $1/r$, respectively.

By employing the decomposition (4.25) in the integral relation for the state variables (4.22) and then performing the integration by parts of an integral involving the kernel $G_{mJ}^P(\xi - \mathbf{x})$ via Stokes' theorem, it leads to an alternative, singularity-reduced integral representation for the state variables:

$$\begin{aligned} \hat{u}_p(\mathbf{x}) = & \hat{u}_p^\infty(\mathbf{x}) + \int_{S^+} U_J^P(\xi - \mathbf{x}) \Sigma \hat{t}_J(\xi) dS(\xi) - \int_{S^+} H_{mJ}^P(\xi - \mathbf{x}) n_m^+(\xi) \Delta \hat{u}_J(\xi) dS(\xi) \\ & + \int_{S^+} G_{mJ}^P(\xi - \mathbf{x}) D_m \Delta \hat{u}_J(\xi) dS(\xi) \end{aligned} \quad (4.31)$$

where $D_m = n_i \varepsilon_{ism} \partial / \partial \xi_s$ is the surface differential operator and the contribution of the line integral along the boundary of the crack vanishes due to the closure condition (i.e., $\Delta \hat{u}_J = 0$ on ∂S^+). Since $H_{mJ}^P n_m^+$ is singular at $\xi = \mathbf{x}$ of order $1/r$ (see the work of Xiao, 1998), the integral relation (4.31) involves only weakly singular kernels.

Similarly, the singularity-reduced body-flux integral relation can be derived as follow: first, the decompositions of $\Lambda_{ij}^P(\xi - \mathbf{x})$ and $\Sigma_{ij}^{IK}(\xi - \mathbf{x})$ given by (4.25) and (4.26) are substituted into the body-flux integral relation (4.23); next, an integral containing the kernel $C_{mK}^{IJ}(\xi - \mathbf{x})$ is integrated by parts via Stokes' theorem; and, finally, the translational properties of both $G_{mJ}^P(\xi - \mathbf{x})$ and $C_{mK}^{IJ}(\xi - \mathbf{x})$ are utilized. The resulting integral relation is given by

$$\begin{aligned} \hat{\sigma}_{IK}(\mathbf{x}) = & \hat{\sigma}_{IK}^\infty + \varepsilon_{Irt} \frac{\partial}{\partial x_r} \left\{ \int_{S^+} G_{IJ}^K(\xi - \mathbf{x}) \Sigma t_J(\xi) dS(\xi) + \int_{S^+} C_{mJ}^{IK}(\xi - \mathbf{x}) D_m \Delta \hat{u}_J(\xi) dS(\xi) \right\} \\ & - \int_{S^+} H_{IK}^J(\xi - \mathbf{x}) \Sigma t_J(\xi) dS(\xi) \end{aligned} \quad (4.32)$$

It is seen that the alternative, body-flux integral relation (4.32) contains only strongly singular kernels of order $1/r^2$.

A pair of singularity-reduced integral relations (4.31) and (4.32) can be used in the post-process for the state variables and body flux at any interior point and, at the same time, constitutes an essential component in the development of integral equations governing the unknown crack-face data as described in the following section.

4.2.4 Weakly singular weak-form integral equations

By taking limit, as point $\mathbf{x}$ approaches a smooth point $\mathbf{y} \in S^+$, of the state-variable integral relation (4.31) and noting the contribution of a source embedded in the strongly singular kernel $H_{ij}^P(\boldsymbol{\xi} - \mathbf{x})$, it leads to a state-variable integral equation

$$\begin{aligned} \frac{1}{2} \Sigma \hat{u}_p(\mathbf{y}) - \hat{u}_p^\infty(\mathbf{y}) = & \int_{S^+} U_J^P(\boldsymbol{\xi} - \mathbf{y}) \Sigma \hat{t}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) - \int_{S^+} H_{mJ}^P(\boldsymbol{\xi} - \mathbf{y}) n_m^+(\boldsymbol{\xi}) \Delta \hat{u}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) \\ & + \int_{S^+} G_{mJ}^P(\boldsymbol{\xi} - \mathbf{y}) D_m \Delta \hat{u}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) \end{aligned} \quad (4.33)$$

where $\Sigma \hat{u}_p = \hat{u}_p^+ + \hat{u}_p^-$ denotes the sum of crack-face state variables. The integral equation for the surface gradient of the sum of the crack-face state variable can now be obtained by directly applying the surface differential operator D_s to (4.33) and the final result is given by

$$\frac{1}{2} D_s \Sigma \hat{u}_p(\mathbf{y}) - D_s \hat{u}_p^\infty(\mathbf{y}) = D_s \Psi_p(\mathbf{y}) \quad (4.34)$$

where the function $\Psi_p(\mathbf{y})$ is defined by

$$\begin{aligned} \Psi_p(\mathbf{y}) = & \int_{S^+} U_J^P(\boldsymbol{\xi} - \mathbf{y}) \Sigma \hat{t}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) - \int_{S^+} H_{mJ}^P(\boldsymbol{\xi} - \mathbf{y}) n_m^+(\boldsymbol{\xi}) \Delta \hat{u}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) \\ & + \int_{S^+} G_{mJ}^P(\boldsymbol{\xi} - \mathbf{y}) D_m \Delta \hat{u}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) \end{aligned} \quad (4.35)$$

The weak-form statement of (4.33), established by applying a standard weight residual technique, takes the form

$$\begin{aligned} \int_{S^+} \tilde{t}_p(\mathbf{y}) \left\{ \frac{1}{2} \Sigma \hat{u}_p(\mathbf{y}) - \hat{u}_p^\infty(\mathbf{y}) \right\} dS(\mathbf{y}) = & \int_{S^+} \tilde{t}_p(\mathbf{y}) \int_{S^+} U_J^P(\boldsymbol{\xi} - \mathbf{y}) \Sigma \hat{t}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \\ & - \int_{S^+} \tilde{t}_p(\mathbf{y}) \int_{S^+} H_{mJ}^P(\boldsymbol{\xi} - \mathbf{y}) n_m^+(\boldsymbol{\xi}) \Delta \hat{u}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \\ & + \int_{S^+} \tilde{t}_p(\mathbf{y}) \int_{S^+} G_{mJ}^P(\boldsymbol{\xi} - \mathbf{y}) D_m \Delta \hat{u}_J(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \end{aligned} \quad (4.36)$$

where $\tilde{t}_p(\mathbf{y})$ is a vector containing sufficiently smooth test functions. The weak-form integral equation (4.35) contains only weakly singular kernels of order $1/r$ and involves two sets of unknown crack-face data, one associated with the sum of the crack-face state variables $\Sigma \hat{u}_p$ and the other corresponding to the jump in the crack-face surface flux $\Delta \hat{u}_J$. An alternative weak-form integral equation for the crack-face data $\Sigma \hat{u}_p$, expressed in terms of its surface gradients

$D_s \Sigma \hat{u}_p$, can be established from the integral equation (4.34). Applying the weight residual technique to (4.34) yields

$$\int_{S^+} \tilde{\tau}_{sp}(\mathbf{y}) \left\{ \frac{1}{2} D_s \Sigma \hat{u}_p(\mathbf{y}) - D_s \hat{u}_p^\infty(\mathbf{y}) \right\} dS(\mathbf{y}) = \int_{S^+} \tilde{\tau}_{sp}(\mathbf{y}) D_s \Psi_p(\mathbf{y}) dS(\mathbf{y}) \quad (4.37)$$

where $\tilde{\tau}_{sp}(\mathbf{y})$ denotes a collection of sufficiently smooth test functions. Note that the resulting integral equation (4.34) still involves strongly singular kernels due to the surface differential operator D_s . To further regularize such strong singularity, the integral on the right hand side of (4.37) is integrated by parts via Stokes' theorem and the final weak-form equation becomes

$$\int_{S^+} \tilde{\tau}_{sp}(\mathbf{y}) \left\{ \frac{1}{2} D_s \Sigma \hat{u}_p(\mathbf{y}) - D_s \hat{u}_p^\infty(\mathbf{y}) \right\} dS(\mathbf{y}) = - \int_{S^+} D_s \tilde{\tau}_{sp}(\mathbf{y}) \Psi_p(\mathbf{y}) dS(\mathbf{y}) + \int_{\partial S^+} \tilde{\tau}_{sp}(\mathbf{y}) \Psi_p(\mathbf{y}) dy_s \quad (4.38)$$

From the structure of $\Psi_p(\mathbf{y})$ defined in (4.35), the weak-form integral equation (4.38) is apparently weakly singular in the sense that all involved kernels are singular of order $1/r$. Similar to the integral equation (4.36), the weak-form equation (4.38) contains two sets of unknown crack-face data, the surface gradient of the sum of the crack-face state variables $D_s \Sigma \hat{u}_p$ and the jump in the crack-face surface flux $\Delta \hat{u}_j$.

In the derivation of the weak-form integral equation for the surface flux on the crack, the body-flux integral relation is employed by first forming the product $\hat{\sigma}_{IK}(\mathbf{x}) n_l^+(\mathbf{y})$ for any smooth point $\mathbf{y} \in S^+$ and then taking the limit an interior point $\mathbf{x} \rightarrow \mathbf{y} \in S^+$ to obtain

$$\begin{aligned} \frac{1}{2} \Delta \hat{t}_K(\mathbf{y}) - \hat{\sigma}_{IK}^\infty n_l^+(\mathbf{y}) = D_t \left\{ \int_{S^+} G_{IJ}^K(\xi - \mathbf{x}) \Sigma t_J(\xi) dS(\xi) + \int_{S^+} C_{mJ}^{IK}(\xi - \mathbf{x}) D_m \Delta \hat{u}_J(\xi) dS(\xi) \right\} \\ - n_l^+(\mathbf{y}) \int_{S^+} H_{IK}^J(\xi - \mathbf{x}) \Sigma t_J(\xi) dS(\xi) \end{aligned} \quad (4.39)$$

where $\Delta \hat{t}_K = \hat{t}_p^+ - \hat{t}_p^-$ denotes the jump in the crack-face surface flux and the contribution of the source contained in the kernel $H_{IK}^J(\xi - \mathbf{x})$ is already taken into account. Taking the inner product between (4.39) and a vector of sufficiently well-behaved test functions $\tilde{v}_K$, then integrating the result over the crack, and finally performing the integration by parts of terms involving the surface differential operator D_t yield

$$\begin{aligned} \int_{S^+} \tilde{v}_K(\mathbf{y}) \left\{ \frac{1}{2} \Delta \hat{t}_K(\mathbf{y}) - \hat{\sigma}_{IK}^\infty n_l^+(\mathbf{y}) \right\} dS(\mathbf{y}) = - \int_{S^+} D_t \tilde{v}_K(\mathbf{y}) \int_{S^+} G_{IJ}^K(\xi - \mathbf{y}) \Sigma t_J(\xi) dS(\xi) dS(\mathbf{y}) \\ - \int_{S^+} D_t \tilde{v}_K(\mathbf{y}) \int_{S^+} C_{mJ}^{IK}(\xi - \mathbf{y}) D_m \Delta \hat{u}_J(\xi) dS(\xi) dS(\mathbf{y}) \\ - \int_{S^+} \tilde{v}_K(\mathbf{y}) \int_{S^+} H_{IK}^J(\xi - \mathbf{y}) n_l^+(\mathbf{y}) \Sigma t_J(\xi) dS(\xi) dS(\mathbf{y}) \end{aligned} \quad (4.40)$$

In the development of (4.40), the test functions $\tilde{v}_K$ are chosen specifically to satisfy the same closure condition along the crack boundary as that of $\Delta\hat{u}_J$ and this, as a result, eliminates the contribution of all line integrals resulting from Stokes' theorem. Clearly, the weak-form equation (4.40) contains only weakly singular kernels of order $1/r$ and clearly involves a single set of unknown crack-face data, i.e., the jump in the crack-face state variables.

A pair of weakly singular, weak-form integral equations (4.36) and (4.40) or (4.38) and (4.40) is adequate for formulating a boundary value problem of isolated cracks in a whole space made of general coupled-field materials. Note in particular that those weak-form integral equations involve the unknown sum of and jump in the state variable across the crack surface while containing the prescribed crack-face surface flux, in terms of the sum of and jump in the crack-face surface flux, and the prescribed remote body flux.

4.3 Solution Implementations

In this section, a numerical technique for solving a system of integral equations governing cracks in a general coupled-field whole space is proposed. Components essential for the development of such solution procedure including the solution discretization, evaluation of involved kernels and double surface integrals, and the post-process algorithm for generalized T-stress, are briefly addressed.

4.3.1 Key governing integral equations

A pair of weakly singular, weak-form integral equations (4.38) and (4.40) derived in Section 4.2 is utilized as the key governing equations of a boundary value problem of cracks in linear coupled-field media. Those integral equations can be re-expressed, in a more concise form suiting further reference and discussion, as

$$\mathcal{D}(\tilde{\tau}, \Sigma\hat{u}) + \mathcal{H}(\tilde{\tau}, \Delta\hat{u}) + \mathcal{G}(\tilde{\tau}, \Delta\hat{u}) = 2\mathcal{D}(\tilde{\tau}, \hat{u}^\infty) + \mathcal{U}(\tilde{\tau}, \Sigma\hat{t}) \quad (4.41)$$

$$\mathcal{C}(\tilde{v}, \Delta\hat{u}) = -\bar{\mathcal{D}}(\tilde{v}, \Delta\hat{t}) + 2\bar{\mathcal{D}}(\tilde{v}, \hat{\sigma}^\infty n^+) - \bar{\mathcal{H}}(\tilde{v}, \Sigma\hat{t}) - \bar{\mathcal{G}}(\tilde{v}, \Sigma\hat{t}) \quad (4.42)$$

where all involved linear and bilinear integral operators are defined, for any $3\alpha+\beta$ -component vectors $\mathbf{U}$, $\mathbf{V}$ and any $3 \times (3\alpha+\beta)$ -matrix $\mathbf{T}$, by

$$\mathcal{D}(\mathbf{T}, \mathbf{V}) = \frac{1}{2} \int_{S^+} T_{sp}(\mathbf{y}) D_s V_p(\mathbf{y}) dS(\mathbf{y}) \quad (4.43)$$

$$\bar{\mathcal{D}}(\mathbf{U}, \mathbf{V}) = \frac{1}{2} \int_{S^+} U_K(\mathbf{y}) V_K(\mathbf{y}) dS(\mathbf{y}) \quad (4.44)$$

$$\begin{aligned} \mathcal{U}(\mathbf{T}, \mathbf{V}) = & \int_{\partial S^+} T_{sp}(\mathbf{y}) \int_{S^+} U_J^p(\xi - \mathbf{y}) V_J(\xi) dS(\xi) dy_s \\ & - \int_{S^+} D_s T_{sp}(\mathbf{y}) \int_{S^+} U_J^p(\xi - \mathbf{y}) V_J(\xi) dS(\xi) dS(\mathbf{y}) \end{aligned} \quad (4.45)$$

$$\mathcal{C}(\mathbf{U}, \mathbf{V}) = \int_{S^+} D_t U_K(\mathbf{y}) \int_{S^+} C_{mJ}^{tK}(\xi - \mathbf{x}) D_m V_J(\xi) dS(\xi) dS(\mathbf{y}) \quad (4.46)$$

$$\begin{aligned}\mathcal{H}(T, V) = & \int_{\partial S^+} T_{sP}(\mathbf{y}) \int_{S^+} H_{mJ}^P(\xi - \mathbf{y}) n_m^+(\xi) V_J(\xi) dS(\xi) d\mathbf{y}_s \\ & - \int_{S^+} D_s T_{sP}(\mathbf{y}) \int_{S^+} H_{mJ}^P(\xi - \mathbf{y}) n_m^+(\xi) V_J(\xi) dS(\xi) dS(\mathbf{y})\end{aligned}\quad (4.47)$$

$$\bar{\mathcal{H}}(U, V) = \int_{S^+} U_K(\mathbf{y}) \int_{S^+} H_{lK}^J(\xi - \mathbf{y}) n_l^+(\mathbf{y}) V_J(\xi) dS(\xi) dS(\mathbf{y}) \quad (4.48)$$

$$\begin{aligned}\mathcal{G}(T, V) = & \int_{S^+} D_s T_{sP}(\mathbf{y}) \int_{S^+} G_{mJ}^P(\xi - \mathbf{y}) D_m V_J(\xi) dS(\xi) dS(\mathbf{y}) \\ & - \int_{\partial S^+} T_{sP}(\mathbf{y}) \int_{S^+} G_{mJ}^P(\xi - \mathbf{y}) D_m V_J(\xi) dS(\xi) d\mathbf{y}_s\end{aligned}\quad (4.49)$$

$$\bar{\mathcal{G}}(U, V) = \int_{S^+} D_l U_K(\mathbf{y}) \int_{S^+} G_{lJ}^K(\xi - \mathbf{y}) V_J(\xi) dS(\xi) dS(\mathbf{y}) \quad (4.50)$$

It is evident that the weak-form integral equation (4.41) involve two sets of unknown crack-face data (i.e., the sum of the crack-face state variables $\Sigma \hat{\mathbf{u}}$ which appear in terms of its surface gradients and the jump in the crack-face state variables $\Delta \hat{\mathbf{u}}$) and two sets of prescribed data (i.e., the state variables $\hat{\mathbf{u}}^\infty$ associated with the prescribed remote body flux $\hat{\sigma}^\infty$ and the sum of the crack-face surface flux $\Sigma \hat{\mathbf{t}}$). Unlike (4.41), the weak-form integral equation (4.42) contains only the unknown jump in the crack-face state variables $\Delta \hat{\mathbf{u}}$ and the prescribed data associated with the crack-face surface flux $\Sigma \hat{\mathbf{t}}$, $\Delta \hat{\mathbf{t}}$ and the remote body flux $\hat{\sigma}^\infty$. Due to the one-way coupling feature of (4.41) and (4.42), it is clearly inefficient to solve this system of equations simultaneously. In the present study, the weak-form equation (4.42) will be solved first to obtain the unknown crack-face data $\Delta \hat{\mathbf{u}}$ and such solution will serves as the known information in the weak-form equation (4.41). The other set of unknown crack-face data, the surface gradients of $\Sigma \hat{\mathbf{u}}$, can subsequently be obtained by solving (4.41). It should be clear from (3.31) and (3.32) that once the jump in the crack-face state variables $\Delta \hat{\mathbf{u}}$ is solved, the state variables and the body flux at any interior point of the medium can be calculated from those integral formulae. In addition, by following the same strategy proposed by Rungamornrat and Mear (2008b, c) and Rungamornrat and Senjuntichai (2009), the solved data $\Delta \hat{\mathbf{u}}$ can be further employed to extract the information of the intensity factors along the edge of the crack. In the post-process of those quantities, the information of $\Sigma \hat{\mathbf{u}}$ and its surface gradients is clearly not required and, as a result, the weak-form integral equation (4.41) is not necessary to be solved. In the present study, the crack-face data $\Sigma \hat{\mathbf{u}}$ is of primary interest since such information **can** be further exploited to directly extract the information of the generalized T-stress along the boundary of the crack.

4.3.2 Solution procedure for $\Delta \hat{\mathbf{u}}_J$

The weakly singular, symmetric Galerkin boundary element method (SGBEM) similar to that reported by Li et al. (1998), Rungamornrat and Mear (2008b, c), and Rungamornrat and Senjuntichai (2009) is extended to solve the weak-form integral equation (4.42) for the unknown jump in the crack-face state variables $\Delta \hat{\mathbf{u}}_J$. The solution discretization follows directly the

Galerkin strategy and standard finite element approximation; more specifically, the test function $\tilde{v}_K$ and the trial function $\Delta\hat{u}_J$ are discretized as

$$\Delta\hat{u}_J(\xi) = \sum_{h=1}^N \Delta u_J^{*(h)} \phi^{(h)}(\xi); \quad \tilde{v}_K(y) = \sum_{h=1}^N v_K^{*(h)} \phi^{(h)}(y) \quad (4.51)$$

where $\Delta u_J^{*(h)}$ are nodal degrees of freedom associated with the h^{th} node; $v_K^{*(h)}$ are arbitrary nodal constants associated with the h^{th} node; $\phi^{(h)}$ denotes the nodal basis functions associated with the h^{th} node; and N is the number of nodes resulting from the discretization. By substituting (4.51) into (4.42) and then employing the arbitrariness of $v_K^{*(h)}$, it yields a system of linear algebraic equations

$$\mathbf{C}\Delta\mathbf{U} = \mathbf{D}^\infty - \mathbf{D}^0 - \mathbf{H}^0 - \mathbf{G}^0 \quad (4.52)$$

where $\Delta\mathbf{U}$ is a vector consisting of unknown nodal degrees of freedom defined by $[\Delta\mathbf{U}]_{\rho(h-1)+J} = \Delta u_J^{*(h)}$ with $\rho = 3\alpha + \beta$ and the coefficient matrix $\mathbf{C}$ and the known vectors $\mathbf{D}^\infty, \mathbf{D}^0, \mathbf{H}^0, \mathbf{G}^0$ are given by

$$[\mathbf{C}]_{\rho(h-1)+K, \rho(r-1)+J} = \int_{S^+} D_t \phi^{(h)}(y) \int_{S^+} C_{mJ}^{tK}(\xi - x) D_m \phi^{(r)}(\xi) dS(\xi) dS(y) \quad (4.53)$$

$$[\mathbf{D}^\infty]_{\rho(h-1)+K} = \int_{S^+} \phi^{(h)}(y) \hat{\sigma}_{lK}^\infty n_l^+(y) dS(y) \quad (4.54)$$

$$[\mathbf{D}^0]_{\rho(h-1)+K} = \frac{1}{2} \int_{S^+} \phi^{(h)}(y) \Delta \hat{t}_K(y) dS(y) \quad (4.55)$$

$$[\mathbf{H}^0]_{\rho(h-1)+K} = \int_{S^+} \phi^{(h)}(y) \int_{S^+} H_{lK}^J(\xi - y) n_l^+(y) \Sigma \hat{t}_J(\xi) dS(\xi) dS(y) \quad (4.56)$$

$$[\mathbf{G}^0]_{\rho(h-1)+K} = \int_{S^+} D_t \phi^{(h)}(y) \int_{S^+} G_{tJ}^K(\xi - y) \Sigma \hat{t}_J(\xi) dS(\xi) dS(y) \quad (4.57)$$

In particular, nodal basis functions generated systematically by a finite element mesh are employed and a near-front approximation of $\Delta\hat{u}_J$ is enhanced by using special crack-tip elements proposed by Li *et al.* (1998) and Rungamornrat and Mear (2008b) for cracks in elastic media and by Rungamornrat and Mear (2008c) for cracks in piezoelectric media. To construct a system of linear algebraic equations resulting from the solution discretization, the evaluation of all involved kernels and all types of double surface integrals are properly treated using the strategy described in Rungamornrat and Mear (2008b) and Xiao (1998), respectively. The linear solution of such the system is then obtained via a preconditioning conjugate gradient algorithm.

4.3.3 Solution procedure for $\Sigma\hat{u}_J$

Once the jump in the crack-face state variables $\Delta\hat{u}_J$ is solved, it is then substituted into (3.1) and the resulting weak-form equation contains the only unknown in terms of the surface gradients of

$\Sigma\hat{\mathbf{u}}$, i.e., $D_s\Sigma\hat{\mathbf{u}}_p$. Note, in addition, that the bilinear operator $\mathcal{D}$ is not only kernel free but also defined in terms of a single surface integral. To solve this weak-form integral equation, a standard procedure based on Galerkin technique and finite element approximation is adopted. In particular, the test function $\tilde{\tau}_{sp}$ and trial function $D_s\Sigma\hat{\mathbf{u}}_p$ are approximated by

$$D_s\Sigma\hat{\mathbf{u}}_p(\mathbf{y}) - 2D_s\hat{\mathbf{u}}_p^\infty(\mathbf{y}) \equiv D_s\Sigma\tilde{\mathbf{u}}_p(\mathbf{y}) = \sum_{h=1}^N (D\Sigma\tilde{\mathbf{u}})_{sp}^{*(h)} \phi^{(h)}(\mathbf{y}); \quad \tilde{\tau}_{sp}(\mathbf{y}) = \sum_{h=1}^N \tau_{sp}^{*(h)} \phi^{(h)}(\mathbf{y}) \quad (4.58)$$

where $D_s\Sigma\tilde{\mathbf{u}}_p$ denotes the surface gradient of the sum of the crack-face state variable less that inducing by the remote loading in the un-cracked case; $\phi^{(h)}$ denotes the nodal basis shape function associated with the h^{th} node; $(D\Sigma\tilde{\mathbf{u}})_{sp}^{*(h)}$ are unknown nodal quantities associated with the h^{th} node; and $\tau_{sp}^{*(h)}$ are arbitrary constants corresponding to the h^{th} node. Due to the regular behavior of $D_s\Sigma\tilde{\mathbf{u}}_p$ on the entire crack surface, continuous basis functions constructed on a mesh of standard isoparametric elements are utilized in the approximation of both solution and test functions. By substituting (4.58) into (4.41), it results in

$$\mathbf{K}[\mathbf{D\Sigma U}] = \mathbf{T}^0 - \mathbf{B}\mathbf{A}\mathbf{U} \quad (4.59)$$

where $\mathbf{D\Sigma U}$ is a vector of nodal unknowns defined by $[\mathbf{D\Sigma U}]_{3\rho(r-1)+\rho(J-1)+k} = (D\Sigma\hat{\mathbf{u}})_{kJ}^{*(r)}$ and the coefficient matrices $\mathbf{K}$, $\mathbf{B}$ and the known vector $\mathbf{T}^0$ are defined by

$$[\mathbf{K}]_{3\rho(h-1)+\rho(P-1)+s, 3\rho(r-1)+\rho(J-1)+k} = \frac{1}{2} \int_{S^+} \psi^{(h)}(\mathbf{y}) \psi^{(r)}(\mathbf{y}) \delta_{sk} \delta_{PJ} dS(\mathbf{y}) \quad (4.60)$$

$$[\mathbf{T}^0]_{3\rho(h-1)+\rho(P-1)+s} = \frac{1}{2} \int_{\partial S^+} \psi^{(h)}(\mathbf{y}) \int_{S^+} U_J^P(\xi - \mathbf{y}) \Sigma \hat{\mathbf{t}}_J(\xi) dS(\xi) dy_s \\ - \int_{S^+} D_s \psi^{(h)}(\mathbf{y}) \int_{S^+} U_J^P(\xi - \mathbf{y}) \Sigma \hat{\mathbf{t}}_J(\xi) dS(\xi) dS(\mathbf{y}) \quad (4.61)$$

$$[\mathbf{B}]_{3\rho(h-1)+\rho(P-1)+s, 3\rho(r-1)+J} = \int_{\partial S^+} \psi^{(h)}(\mathbf{y}) \int_{S^+} H_{mJ}^P(\xi - \mathbf{y}) n_m^+(\xi) \psi^{(r)}(\xi) dS(\xi) dy_s \\ - \int_{S^+} D_s \psi^{(h)}(\mathbf{y}) \int_{S^+} H_{mJ}^P(\xi - \mathbf{y}) n_m^+(\xi) \psi^{(r)}(\xi) dS(\xi) dS(\mathbf{y}) \\ + \int_{S^+} D_s \psi^{(h)}(\mathbf{y}) \int_{S^+} G_{mJ}^P(\xi - \mathbf{y}) D_m \psi^{(r)} dS(\xi) dS(\mathbf{y}) \\ - \int_{\partial S^+} \psi^{(h)}(\mathbf{y}) \int_{S^+} G_{mJ}^P(\xi - \mathbf{y}) D_m \psi^{(r)} dS(\xi) dy_s \quad (4.62)$$

According to (4.60), it is clear that the coefficient matrix $\mathbf{K}$ is symmetric. The numerical integration of both path and surface integrals follows the same strategy as that reported by Rungamornrat and Mear (2008b) and Limwibul *et al.* (2016). Certain components used in the development of SGBEM are also adopted, here, to treat the double surface integrals containing

the prescribed or solved data. The resulting system of linear algebraic equations (4.59) is then solved by the preconditioning conjugate gradient.

4.3.4 Post-process for fracture data

Once the surface gradient of the sum of the crack-face state variables ($D_s \Sigma \hat{u}_p$) is solved, a generalized T-stress, defined as the finite part of the body flux along the boundary of the crack, can be determined from this set of data as described below.

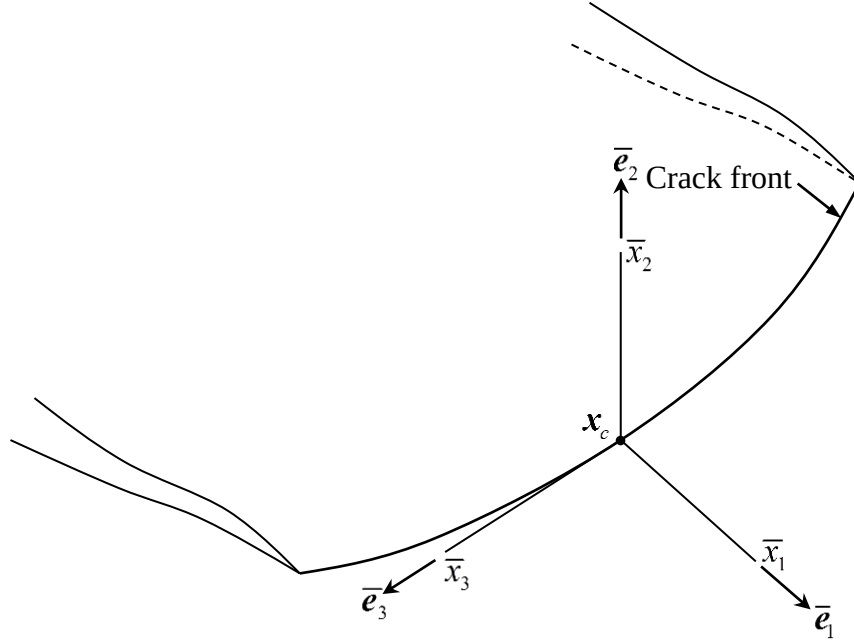


Figure 4.2: Schematic of crack boundary and local coordinate system used for calculating generalized T-stress

Let x_c denote a nodal point located at the crack boundary and $\{x_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ be a local Cartesian coordinate system with the origin at x_c and the unit base vectors $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ as shown in Figure 4.1. In particular, $\bar{e}_2 = -n^+$ whereas $\bar{e}_3$ and $\bar{e}_1$ are chosen tangent and normal to the crack-front at x_c , respectively. By extending the work of Subsathaphol *et al.* (2015) and Limwibul *et al.* (2016), the nonsingular part of the body flux along the boundary of the crack, represented by a collection of α symmetric second-order tensors $T_{ij}^{(1)}, T_{ij}^{(2)}, \dots, T_{ij}^{(\alpha)}$ and β vectors $Q_i^{[1]}, Q_i^{[2]}, \dots, Q_i^{[\beta]}$ can be related to the finite part of the state-variable variation at the same point, denoted by a collection of α symmetric second-order tensors $\bar{\varepsilon}_{ij}^{0(1)}, \bar{\varepsilon}_{ij}^{0(2)}, \dots, \bar{\varepsilon}_{ij}^{0(\alpha)}$ and β vectors $\bar{g}_i^{0[1]}, \bar{g}_i^{0[2]}, \dots, \bar{g}_i^{0[\beta]}$, via the constitutive relations (2.4) and (2.5), i.e.,

$$T_{ij}^{(k)} = \bar{E}_{ijpq}^{(k)(r)} \bar{\varepsilon}_{pq}^{0(r)} + \bar{D}_{ijp}^{(k)[r]} \bar{g}_p^{0[r]} \quad (4.63)$$

$$Q_i^{[k]} = \bar{D}_{pqi}^{[k](r)} \bar{\varepsilon}_{pq}^{0(r)} + \bar{C}_{ip}^{[k][r]} \bar{g}_p^{0[r]} \quad (4.64)$$

where the “bar” superscript is used to designate components in the local coordinate system $\{\bar{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$. From the continuity of the finite part of the body flux at the point x_c , it can be verified that the following out-of-plane components $\{T_{12}^{(1)}, T_{22}^{(1)}, T_{23}^{(1)}\}$, $\{T_{12}^{(2)}, T_{22}^{(2)}, T_{23}^{(2)}\}$, ..., $\{T_{12}^{(\alpha)}, T_{22}^{(\alpha)}, T_{23}^{(\alpha)}\}$ and $Q_2^{[1]}$, $Q_2^{[2]}$, ..., $Q_2^{[\beta]}$ are known and can be obtained in terms of the prescribed surface flux on the crack at x_c . The remaining components, i.e., $\{T_{11}^{(1)}, T_{33}^{(1)}, T_{13}^{(1)}\}$, $\{T_{11}^{(2)}, T_{33}^{(2)}, T_{13}^{(2)}\}$, ..., $\{T_{11}^{(\alpha)}, T_{33}^{(\alpha)}, T_{13}^{(\alpha)}\}$ and $\{Q_1^{[1]}, Q_3^{[1]}\}$, $\{Q_1^{[2]}, Q_3^{[2]}\}$, ..., $\{Q_1^{[\beta]}, Q_3^{[\beta]}\}$ are unknown a priori and termed, here, the generalized T-stress components. From the relations (4.1) and (4.2), the finite part of the state-variable variations are related to the sum of the crack-face state variables at the point x_c by

$$\bar{\varepsilon}_{ij}^{0(k)} = \frac{1}{4} \left[\frac{\partial \Sigma \bar{u}_i^{(k)}}{\partial \bar{x}_j} + \frac{\partial \Sigma \bar{u}_j^{(k)}}{\partial \bar{x}_i} \right] \quad (4.65)$$

$$\bar{g}_i^{0[k]} = \frac{1}{2} \frac{\partial \Sigma \bar{\phi}^{[k]}}{\partial \bar{x}_i} \quad (4.66)$$

By first applying the coordinate transformation to the solved data $D_s \Sigma \hat{u}_p$ in the global coordinate system, it leads to the surface gradients $\bar{D}_s \Sigma \hat{u}_p$ defined in the local coordinate system $\{\bar{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$. The in-plane components of $\bar{\varepsilon}_{ij}^{0(k)}$ and $\bar{g}_i^{0[k]}$ (i.e., $\{\bar{\varepsilon}_{11}^{0(k)}, \bar{\varepsilon}_{33}^{0(k)}, \bar{\varepsilon}_{13}^{0(k)}\}$ and $\{\bar{g}_1^{0[k]}, \bar{g}_3^{0[k]}\}$) can then be obtained directly from the information $\bar{D}_s \Sigma \hat{u}_p$; by following relations below:

$$\bar{\varepsilon}_{11}^{0(k)} = \frac{1}{2} \bar{a}_{3s} \bar{a}_{1p} D_s \Sigma \hat{u}_{3(k)+P-3} \quad (4.67)$$

$$\bar{\varepsilon}_{33}^{0(k)} = -\frac{1}{2} \bar{a}_{1s} \bar{a}_{3p} D_s \Sigma \hat{u}_{3(k)+P-3} \quad (4.68)$$

$$\bar{\varepsilon}_{13}^{0(k)} = \frac{1}{4} (\bar{a}_{3s} \bar{a}_{3p} D_s \Sigma \hat{u}_{3(k)+P-3} - \bar{a}_{1s} \bar{a}_{1p} D_s \Sigma \hat{u}_{3(k)+P-3}) \quad (4.69)$$

$$\bar{g}_1^{0[k]} = \frac{1}{2} \bar{a}_{3s} D_s \Sigma \hat{u}_{3\alpha+[k]} \quad (4.70)$$

$$\bar{g}_3^{0[k]} = -\frac{1}{2} \bar{a}_{1s} D_s \Sigma \hat{u}_{3\alpha+[k]} \quad (4.71)$$

where $\bar{a}_{ij}$ denotes the cosine of the angle between i^{th} and j^{th} coordinate axis. By combining the known information of the in-plane components of the finite part of the state-variable variation and the out-of-plane components of the finite part of the body flux, (4.63) and (4.64) yield a system of linear algebraic equations sufficient for determining the generalized T-stress

components. According to (4.67)-(4.61), It clearly seen that the surface gradient of sum of crack-face state variable can be directly used for obtaining the generalized T-stresses by using generalized constitutive relations that without additional approximation in the post-process procedure.

4.4 Numerical Results and Discussion

This section presents numerical results of the generalized T-stresses obtained from the proposed technique. Computational performance including the accuracy and convergence is carefully investigated through various examples involving both linear single and couple-field media. Various type of loading conditions (e.g., uniformly distributed crack-face loading, linearly distributed crack-face loading, and uniform remote loading) and planar, non-planar and multiple cracks are considered in the numerical study to demonstrate the capability and versatility of the technique. As the essential step in the verification of the integral formulation and the implementation of solution procedure and post-process algorithm for determining the generalized T-stress, results for a penny-shaped crack in both single and couple-field media are obtained and then compared with reliable benchmark solutions. Then, results for more complicated fracture problems involving non-planar and multiple cracks are reported.

In the numerical study, standard 6-node and 8-node isoparametric C^0 -elements are employed to discretize both the sum of and jump in the crack-face generalized displacement on the majority of the crack surface whereas standard 9-node isoparametric C^0 -elements and special 9-node crack-tip elements are utilized to discretize, respectively, the sum of and jump in the crack-face generalized displacement in a region adjacent to the crack front. A series of meshes, with significant difference of element size, is adopted to investigate the convergence of numerical solutions. Three representative material models including a transversely isotropic linear elastic material (Mat-1), a transversely isotropic linear piezoelectric material (Mat-2), and a transversely isotropic linear piezoelectromagnetic material (Mat-3) with all material parameters given in Tables 4.1-4.3 are utilized in the numerical simulations.

Table 4.1: Material parameters of a representative transversely isotropic linear elastic material (Mat-1) used in numerical study

Elastic constants (x 10 ⁹ Pa)	E_{1111}	126.00
	E_{1122}	55.00
	E_{1133}	53.00
	E_{3333}	117.00
	E_{1313}	35.30

Table 4.2: Material parameters of a transversely isotropic linear piezoelectric material (Mat-2)

Elastic constants ($\times 10^9$ Pa)	E_{1111}	126.00
	E_{1122}	55.00
	E_{1133}	53.00
	E_{3333}	117.00
	E_{1313}	35.30
Piezoelectric constants (C/m ²)	E_{1134}	-6.50
	E_{3334}	23.30
	E_{1314}	17.00
Dielectric permittivities ($\times 10^{-9}$ C/(Vm))	$-E_{1414}$	15.10
	$-E_{3434}$	13.00

Table 4.3: Material parameters of a representative transversely isotropic linear piezoelectromagnetic material BaTiO₃-CoFe₂O₄ (Mat-3)

Elastic constants ($\times 10^9$ Pa)	E_{1111}	226.00
	E_{1122}	125.00
	E_{1133}	124.00
	E_{3333}	216.00
	E_{2323}	44.00
Piezoelectric constants (C/m ²)	E_{1134}	-2.20
	E_{3334}	9.30
	E_{1314}	5.80
Piezomagnetic constants (N/Am)	E_{1135}	290.00
	E_{3335}	350.00
	E_{1315}	275.00
Magneto-electric coefficients ($\times 10^{-12}$ Ns/(Am))	$-E_{1415}$	5.37
	$-E_{3435}$	2737.50
Dielectric permittivities ($\times 10^{-10}$ C/(Vm))	$-E_{1414}$	56.40
	$-E_{3434}$	63.50
Magnetic permittivities ($\times 10^{-6}$ Ns ² /C ²)	$-E_{1515}$	297.00
	$-E_{3535}$	83.50

4.4.1 Verification

In order to verify the proposed technique, simple crack problems with available analytical solution are considered. From an extensive literature survey, it has been found that the closed form solutions of the generalized T-stresses of an isolated penny-shaped crack were reported by Rungamornrat and Pinitpanich (2016) for a transversely isotropic linear elastic infinite medium and by Rungamornrat *et al.* (2018) for a transversely isotropic linear piezoelectric medium. For the case of a penny-shaped crack embedded in a transversely isotropic linear piezoelectromagnetic medium, the complete field was obtained by Zhao *et al.* (2006) and such results can then be used to directly post-process for the generalized T-stress via the technique proposed by Rungamornrat and Pinitpanich (2016) and Rungamornrat *et al.* (2018). Solutions for those three cases constitute the sufficient basis for the verification of the present technique.

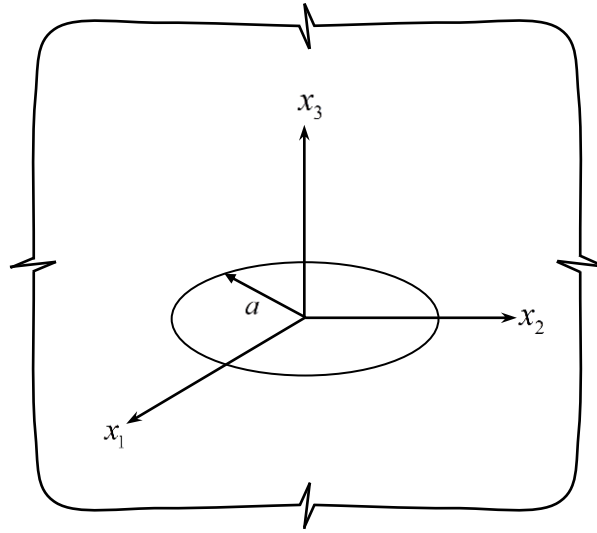


Figure 4.3: Schematic of a penny-shaped crack of radius a embedded in a couple-field whole space

Consider a penny shaped crack of radius a embedded in a whole space with the axis of material symmetry directing perpendicular to the plane of the crack as shown in Figure 4.3. The crack front is parameterized by

$$x_1 = a \cos \theta, \quad x_2 = a \sin \theta, \quad x_3 = 0 \quad (4.72)$$

where $\theta \in [0, 2\pi]$ denotes the angular position. The whole space is made of Mat-1, Mat-2 or Mat-3 with material parameters given in Tables 4.1-4.3. Following six loading conditions are considered: (i) uniformly distributed normal mechanical crack-face traction $t_3^+ = -t_3^- = \sigma_0$ (see Figure 4.4(a)), (ii) linearly distributed normal mechanical crack-face traction $t_3^+ = -t_3^- = \sigma_0(1 + x_1/a)/2$ (see Figure 4.4(b)), (iii) nonlinearly distributed normal mechanics crack-face traction $t_3^+ = -t_3^- = \sigma_0(1 + x_1/a)^2/4$ (see Figure 4.4(c)), (iv) uniform remote mechanical

loading $\sigma_{33}^{\infty} = 2\sigma_{11}^{\infty} = 2\sigma_0$ (see Figure 4.4(d)), (v) uniform remote electrical loading $\sigma_{34}^{\infty} = 2\sigma_{14}^{\infty} = 2d_0$ (see Figure 4.4(e)), and (vi) combined uniform remote mechanical-electromagnetic loading in x_3 -direction $\sigma_{33}^{\infty} = \sigma_0$, $\sigma_{34}^{\infty} = d_0$, $\sigma_{35}^{\infty} = b_0$ (see Figure 4.4(f)) where $\sigma_0 = 1 \times 10^6 \text{ N/m}^2$, $d_0 = 1 \times 10^{-3} \text{ C/m}^2$ and $b_0 = 1 \times 10^{-3} \text{ N/Am}$ are taken in the numerical study. The loading conditions (i)-(iv) are applied for the whole space made of Mat-1, the loading conditions (i)-(v) are applied for the whole space made of Mat-2, and the loading conditions (i), (iv) and (vi) are applied for the whole space made of Mat-3.

In the analysis, three meshes of a penny-shaped crack shown in Figure 4.5 are adopted; in particular, Mesh-1 contains 8 elements with 4 crack-tip elements, Mesh-2 contains 36 elements with 12 crack-tip elements and Mesh-3 contains 144 elements with 24 elements.

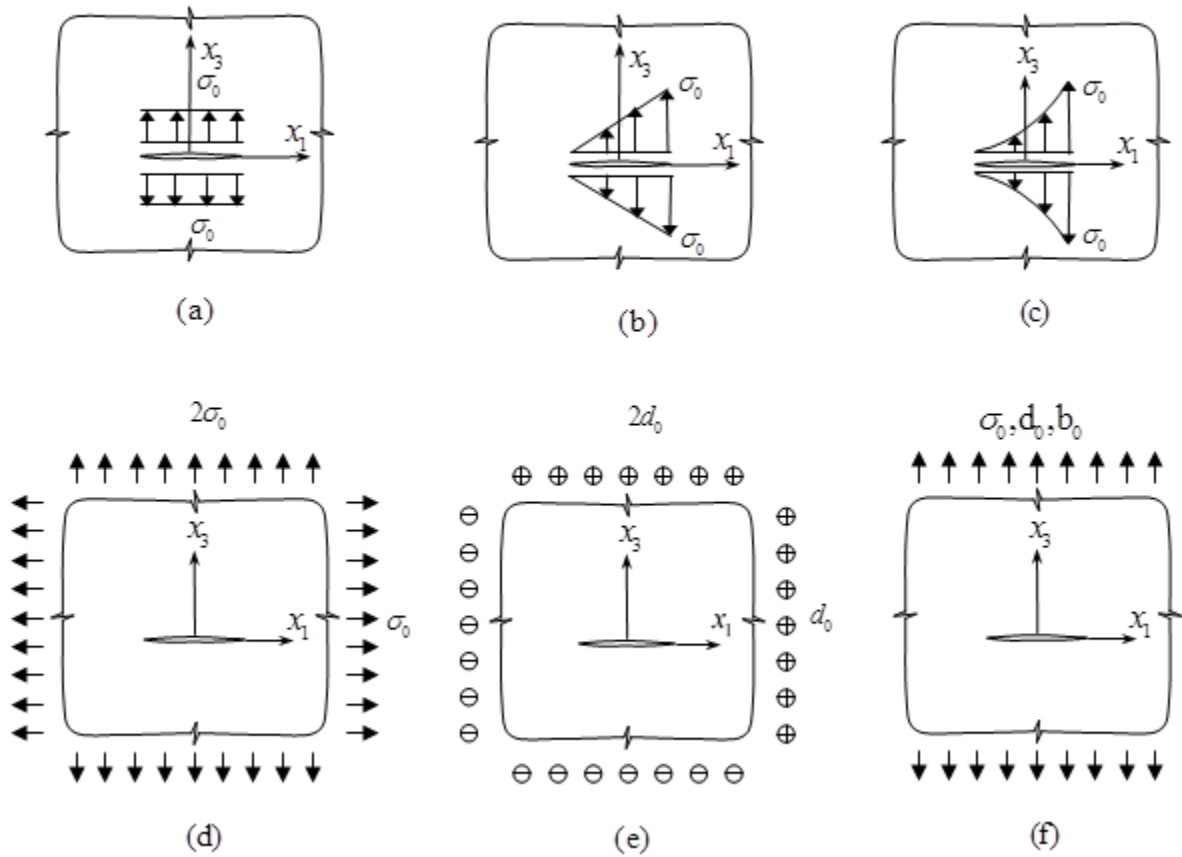


Figure 4.4: Schematic of six loading conditions considered in the analysis of a penny-shaped crack in a couple-field whole space

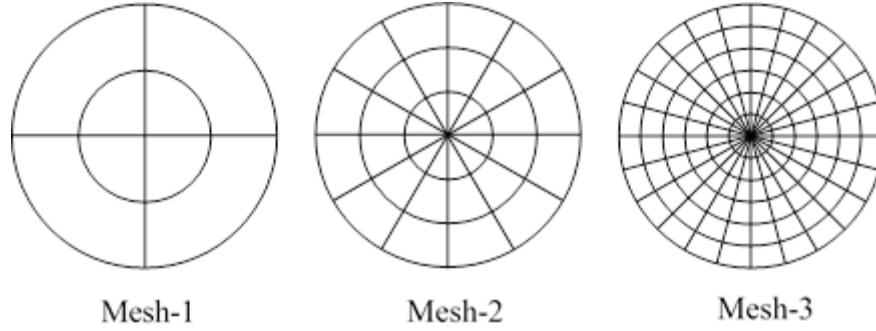


Figure 4.5: Schematic of three meshes of a penny-shaped crack adopted in the analysis

4.4.1.1 Uniformly distributed normal mechanical crack-face traction

For this particular loading condition, the whole space made of elastic, piezoelectric and piezoelectromagnetic materials (i.e., Mat-1, Mat-2 and Mat-3) are considered. Numerical results for the generalized T-stresses are normalized by the existing exact solution and then reported in Table 4.4 for all three meshes. Obtained results indicate that for all three types of materials considered, only T_{11} and T_{33} are non-zero and they constant along the crack front due to the axisymmetry. In addition, numerical results generated by the proposed technique exhibit excellent agreement with the exact solutions and are slightly dependent on the level of mesh refinement. Clearly, the difference, in comparison with the exact solution, does not exceed 0.25% for results obtained from the Mesh-1 and 0.05% for those obtained from the Mesh-2 and Mesh-3.

Table 4.4 Normalized generalized T-stress components T_{11} and T_{33} for a penny shaped crack subjected to uniformly distributed normal mechanical crack-face traction

Mesh	Mat-1		Mat-2		Mat-3	
	T_{11}/T_{11}^{exact}	T_{33}/T_{33}^{exact}	T_{11}/T_{11}^{exact}	T_{33}/T_{33}^{exact}	T_{11}/T_{11}^{exact}	T_{33}/T_{33}^{exact}
1	0.9980	1.0008	0.9975	1.0009	0.9983	1.0004
2	0.9996	0.9997	0.9995	0.9997	0.9996	0.9997
3	0.9996	0.9999	0.9995	0.9999	0.9995	0.9998

4.4.1.2 Linearly distributed normal mechanical crack-face traction

For this loading case, results are obtained for the whole space made of Mat-1 and Mat-2. Only the mechanical T-stress components (T_{11}, T_{33}, T_{13}) are non-zero for both types of materials and they vary as a function of angular position along the crack front. Computed T-stress components from the three meshes are first normalized by σ_0 and then reported versus the angular position

θ in Figure 4.6 for the elastic case and Figure 4.7 for the piezoelectric case. As can be seen from this set of results, the present technique yields solutions of excellent agreement with the benchmark solutions for all three meshes. Results are nearly indistinguishable from the reference solution although the relatively coarse mesh containing few elements is employed. In addition, the generalized T-stress components are quite weak dependent on the material constituting the body; in particular, both the values and distribution of the generalized T-stress along the crack front are not significantly different.

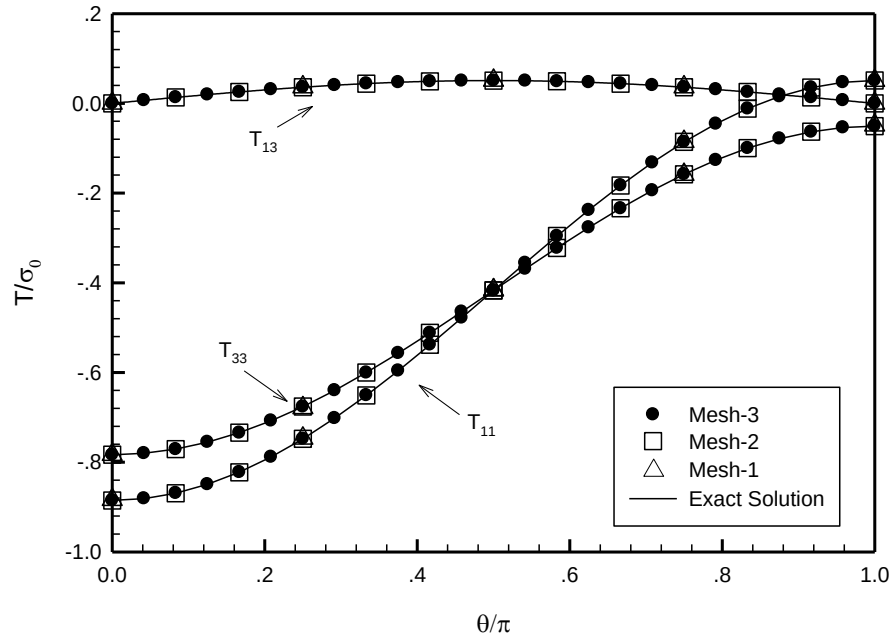


Figure 4.6: Normalized generalized T-stress components T_{11} , T_{33} and T_{13} for a penny-shaped crack embedded in a linear elastic whole space subjected to linearly distributed normal mechanical crack-face traction

4.4.1.3 Nonlinearly distributed normal mechanical crack-face traction

For this particular loading case, the applied normal mechanical crack-face traction assumes the parabolic distribution along the x_1 -direction and both types of materials constituting the medium (i.e., Mat-1 and Mat-2) are considered. Similar to the previous case, only the mechanical T-stress components (T_{11} , T_{33} , T_{13}) are non-zero for both elastic and piezoelectric cases and they vary along the crack front. Computed results from the three meshes are normalized and reported, along with the exact solutions, in Figure 4.8 for Mat-1 and Figure 4.9 for Mat-2. As can be seen from the numerical solutions, the proposed technique can capture the variation of the generalized T-stress components along the crack front with the high accuracy even when the coarse mesh is utilized. This should result directly from the use of special crack-tip elements to model the near-front relative crack-face generalized displacement together with the direct means to post-process the generalized T-stress components from the gradient of the sum of the generalized displacement along the crack front.

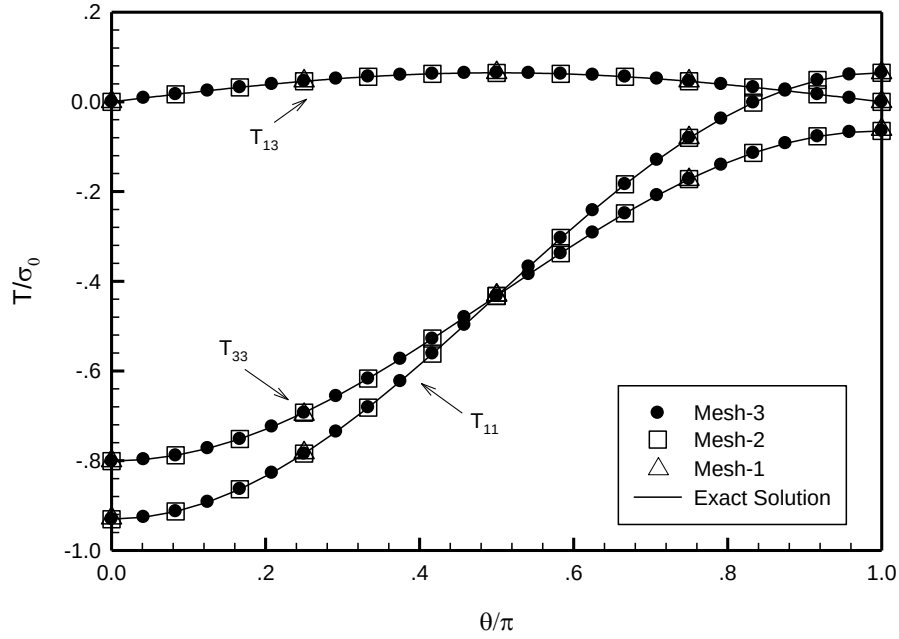


Figure 4.7: Normalized generalized T-stress components T_{11} , T_{33} and T_{13} for a penny-shaped crack embedded in a linear piezoelectric whole space subjected to linearly distributed normal mechanical crack-face traction

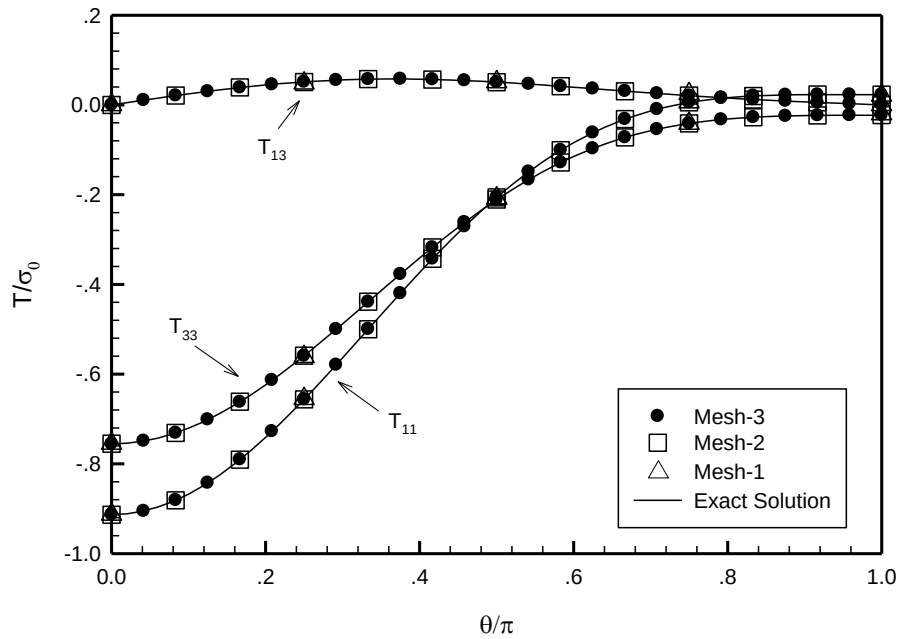


Figure 4.8: Normalized generalized T-stress components T_{11} , T_{33} and T_{13} for a penny-shaped crack embedded in a linear elastic whole space subjected to nonlinear distributed normal mechanical crack-face traction

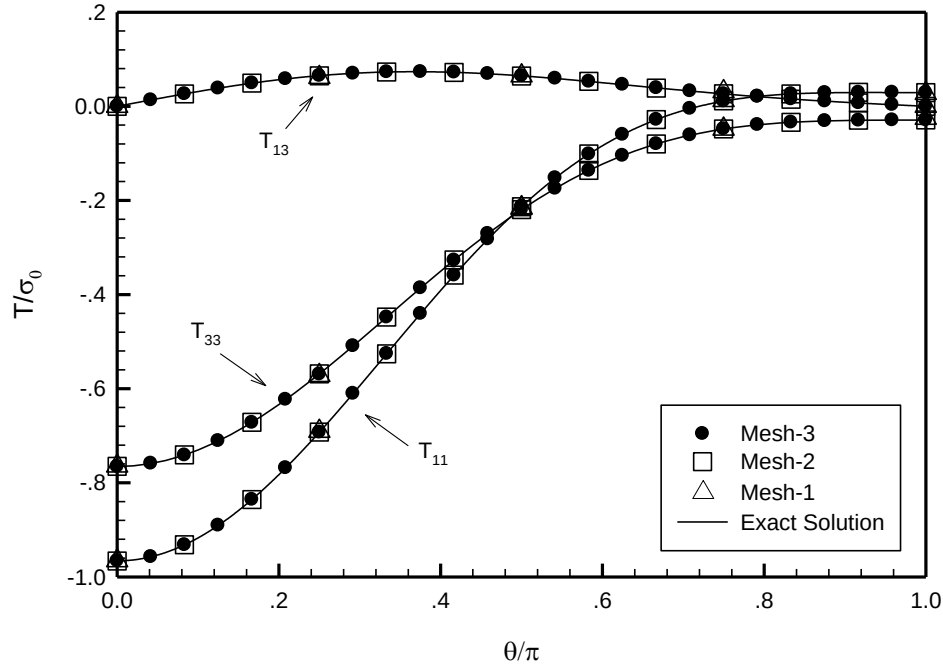


Figure 4.9: Normalized generalized T-stress components T_{11} , T_{33} and T_{13} for a penny-shaped crack embedded in a linear piezoelectric whole space subjected to nonlinear distributed normal mechanical crack-face traction

4.4.1.4 Uniform remote mechanical loading

Next, let us consider the action of the uniform remote mechanical loading on the value and distribution of the generalized T-stress for a penny-shaped crack embedded in elastic (Mat-1), piezoelectric (Mat-2), and piezoelectromagnetic (Mat-3) medium. Numerical results obtained from the three meshes are reported in Figures 4.10, 4.11, and 4.12 for the generalized T-stress components T_{11} , T_{33} and T_{13} , respectively. It is seen that results generated by the proposed technique converge to the analytical solutions and, in addition, results from the three meshes are almost indistinguishable and those obtained from the coarsest mesh are slightly different from the reference solutions. In particular, it is found that errors of the solution from Mesh-1 are less than 0.5% and those associated with the Mesh-2 and Mesh-3 are less than 0.3%. As also clear from obtained results, the uniform remote mechanical loading does not induce the electrical and magnetic T-stress although the medium is made of the coupled-field materials. By comparing results for different material models, it is found that the mechanical T-stress component T_{13} are identical whereas the remaining two components of the mechanical T-stress (i.e., T_{11} and T_{33}) possess the same variation characteristic along the crack front but only slight difference in magnitude.

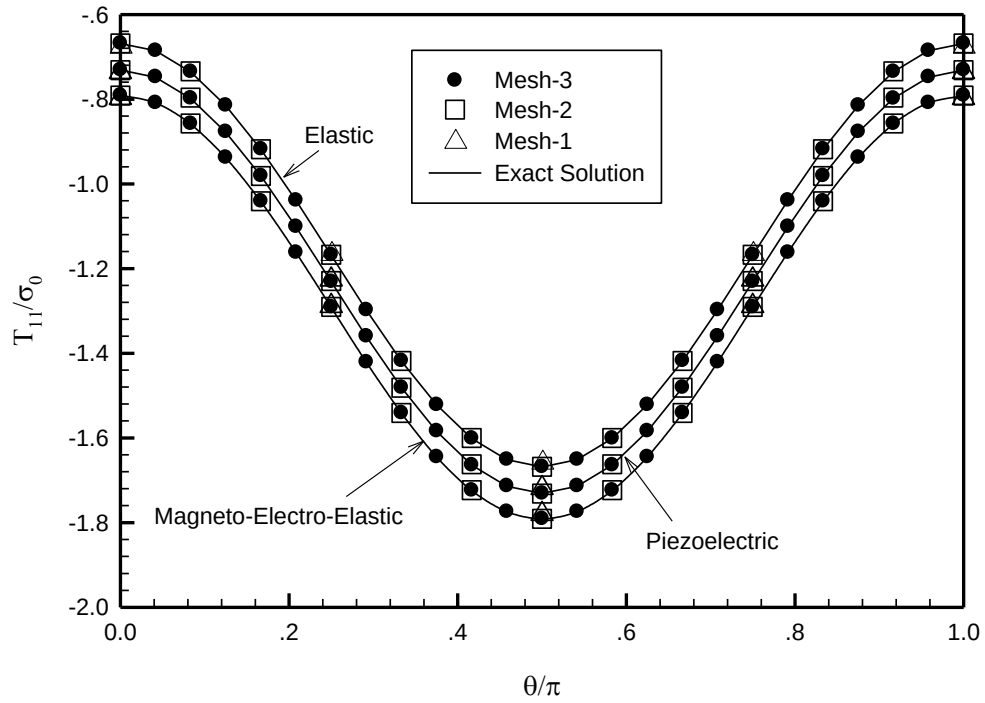


Figure 4.10: Normalized generalized T-stress component T_{11} for a penny-shaped crack embedded in a linear whole space subjected to uniform remote mechanical loading

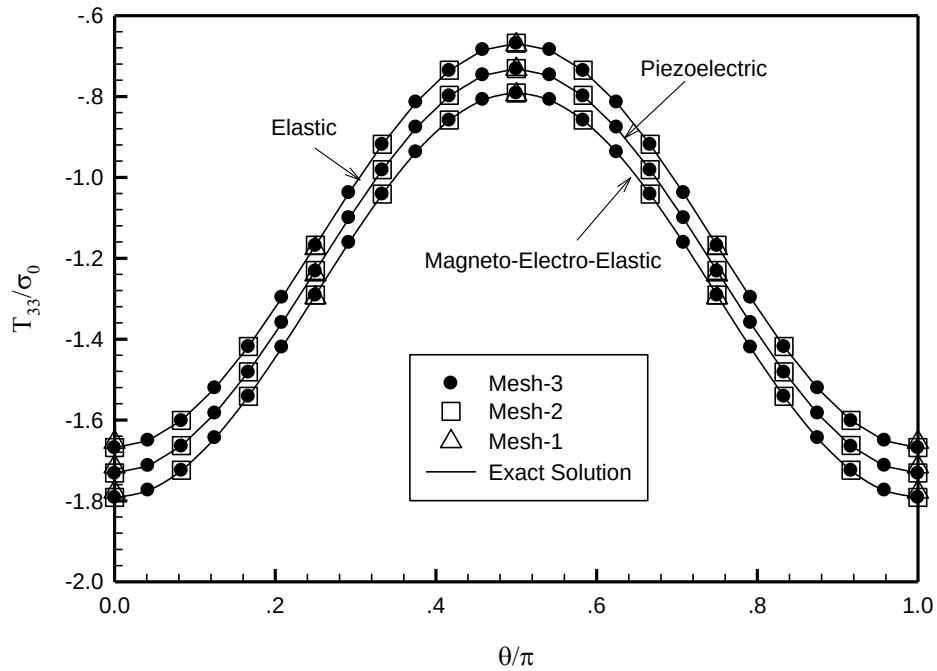


Figure 4.11: Normalized generalized T-stress component T_{33} for a penny-shaped crack embedded in a linear whole space subjected to uniform remote mechanical loading

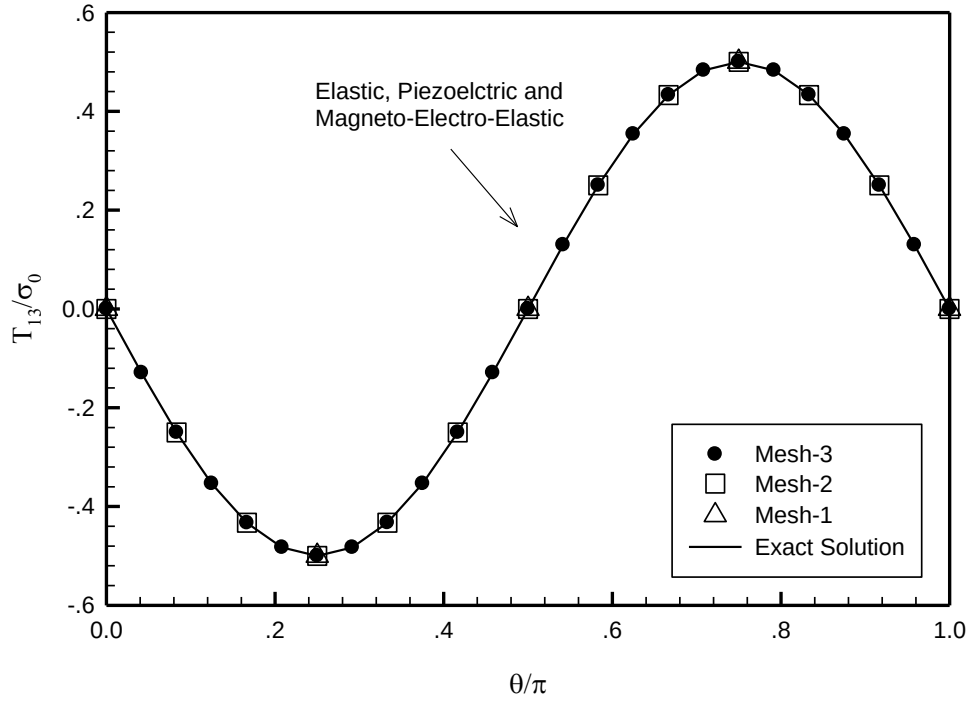


Figure 4.12: Normalized generalized T-stress component T_{13} for a penny-shaped crack embedded in a linear whole space subjected to uniform remote mechanical loading

4.4.1.5 Uniform remote electrical loading

Here, a penny-shaped crack embedded within the piezoelectric whole space (Mat-2) under the uniform remote electrical loading $\sigma_{34}^\infty = 2\sigma_{14}^\infty = 2d_0$ is examined. Non-zero generalized T-stress components obtained from the three meshes are normalized and reported in Figure 4.13 together with the analytical solutions. Again, the proposed technique yields the converged numerical solution of very high accuracy when compared with the reference solutions; in particular, results from the Mesh-1 show only slight difference from the exact solutions whereas those from the Mesh-2 and Mesh-3 are nearly identical to the benchmark solution. For this particular loading case, the mechanical T-stress component T_{13} vanishes identically; the non-zero mechanical T-stress components T_{11} and T_{33} are independent of the position along the crack front; and the non-zero electrical T-stress components T_{14} and T_{34} are a function of position along the crack front.

4.4.1.6 Combined uniform remote mechanical-electro-magnetic loading

As the final case, consider a penny-shaped crack in a linear piezoelectromagnetic whole space (Mat-3) subjected to the combined remote mechanical-electro-magnetic loading $\sigma_{33}^\infty = \sigma_0$, $\sigma_{34}^\infty = d_0$, $\sigma_{35}^\infty = b_0$. For this particular case, the mechanical T-stress component T_{13} , the electrical T-stress component T_{14} and T_{34} , and the magnetic T-stress component T_{15} and T_{35} identically

vanish along the crack front whereas the non-zero mechanical T-stress components T_{11} and T_{33} are constant along the crack front due to the axisymmetry. Computed T-stress components T_{11} and T_{33} obtained from Mesh-1, Mesh-2 and Mesh-3 are normalized by the exact solution and then reported in Table 4.5. As can be seen from this set of results, the proposed technique can capture the solution with the high accuracy even when the coarsest mesh is employed in the discretization. In particular, the discrepancy between the computed and reference solutions is less than 0.2% for the Mesh-1 and 0.1% for the Mesh-2 and Mesh-3.

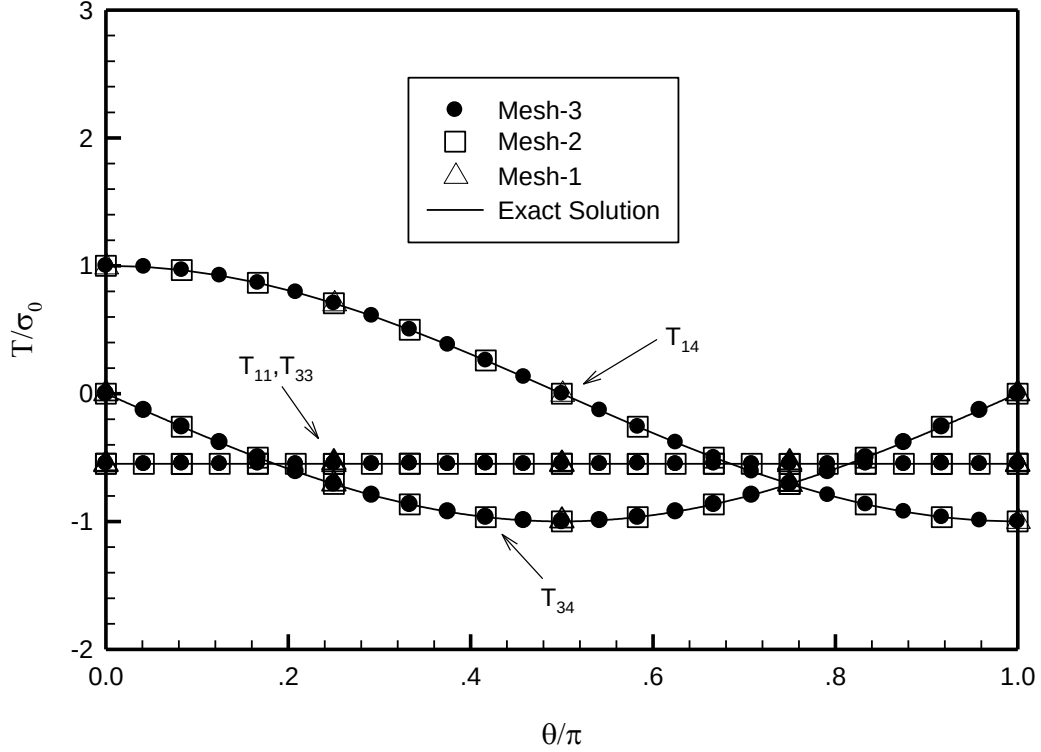


Figure 4.13: Normalized generalized T-stress components T_{11}, T_{33}, T_{14} and T_{34} for a penny-shaped crack embedded in a linear piezoelectric whole space subjected to uniform remote electrical loading

Table 4.5 Normalized generalized T-stress components T_{11} and T_{33} for a penny-shaped crack embedded in a linear piezoelectromagnetic whole space subjected to combined uniform mechanical-electro-magnetic loading

Normalized generalized T-stress	Mesh-1	Mesh-2	Mesh-3
T_{11}/T_{11}^{exact}	1.0003	0.9993	0.9991
T_{33}/T_{33}^{exact}	0.9986	0.9994	0.9992

4.4.2 Capability of proposed technique

While the convergence and accuracy of the proposed technique have been extensively verified with reliable benchmark solutions, all boundary value problems considered in such numerical experiments are limited to relatively simple scenarios. In order to further demonstrate the computational capability and versatility of the developed technique, more complex problems involving non-planar and multiple cracks are also investigated in this section. It should be remarked that according to the complexity of the boundary value problems, the exact or analytical solutions for all representative examples have not been found. A series of meshes with the significant difference in mesh size is utilized in the numerical study to ascertain the convergence of numerical solutions.

4.4.2.1 Inclined elliptical crack

First, consider an inclined elliptical crack that embedded in a linear whole space made of elastic (Mat-2), piezoelectric (Mat-2), and piezoelectromagnetic (Mat-3) materials as shown in Figure 4.14. The crack is oriented with respect to the reference Cartesian coordinate system such that the crack front can be parameterized by

$$x_1 = a \cos \theta \cos \beta, \quad x_2 = b \sin \theta, \quad x_3 = a \cos \theta \sin \beta \quad (4.73)$$

where a and b denote the major and minor semi-axes; $\theta \in [0, 2\pi]$ denotes the angular position along the crack front; and β denotes the angle between the x_3 -axis and the unit normal vector to the surface of the crack. The axis of material symmetry is assumed directing along the x_3 -axis.

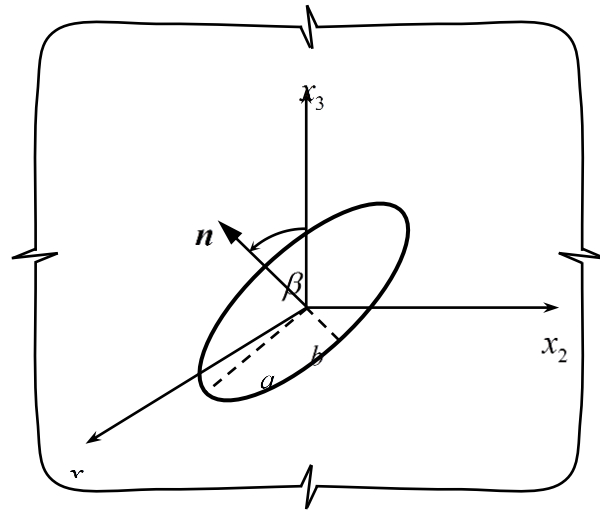


Figure 4.14: Schematic of an inclined elliptical crack embedded in a linear whole space

In the analysis, the aspect ratio $b/a = 2$, the orientation angle $\beta = 45^\circ$, and three meshes adopted in Figure 4.15 are employed. In particular, the Mesh-1, Mesh-2 and Mesh-3 contain 8, 36, and 144 elements with 4, 12, and 24 special crack-tip elements, respectively.

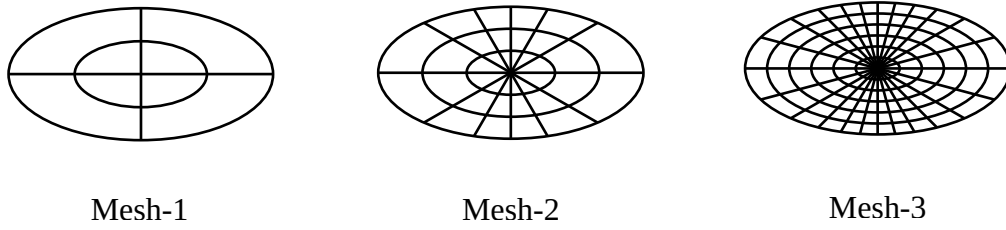


Figure 4.15: Three meshes of an inclined elliptical crack used in numerical study

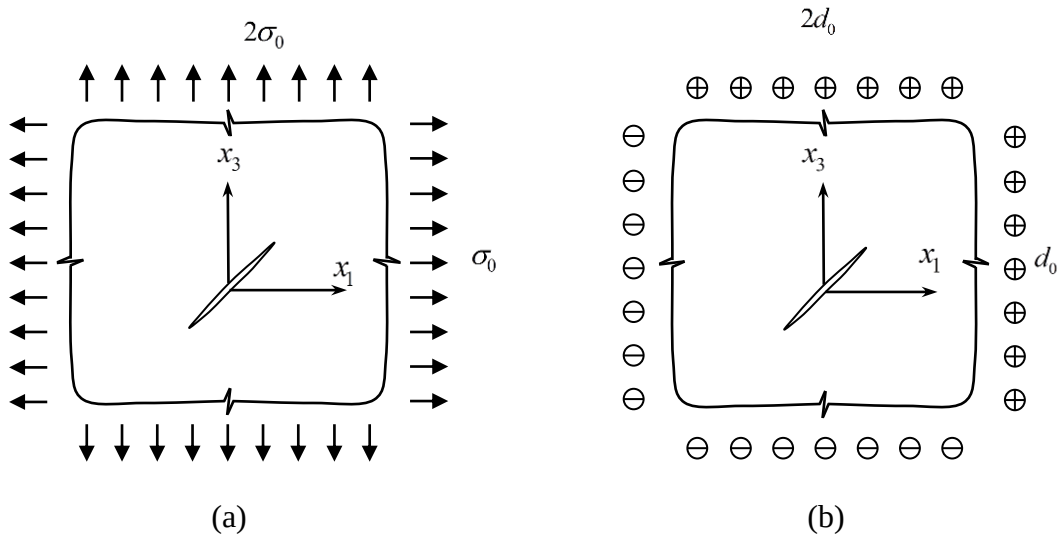


Figure 4.16: Schematics of cracked whole space subjected to (a) uniform remote mechanical loading and (b) uniform remote electrical loading

4.4.2.1.1 Uniform remote mechanical loading

First, consider the cracked whole space subjected only to the uniform remote mechanical loading $\sigma_{33}^\infty = 2\sigma_{11}^\infty = 2\sigma_0$ with $\sigma_0 = 1 \times 10^6 \text{ N/m}^2$ (see Figure 4.16(a)). Due to the nonalignment between the normal to the crack surface and the axis of material symmetry, all components of the generalized T-stress are non-zero and vary as a function of position along the crack front. The computed mechanical T-stress components T_{11}, T_{33}, T_{13} obtained from the three meshes for three types of materials are reported in Figures 4.17, 4.18, and 4.19, respectively. Besides the good convergence behavior of the numerical solutions with only weak dependence on the level of mesh refinement, the variation of the mechanical T-stress components for all three-types of materials are similar and their values are only slightly different. Results for the electrical T-stress components T_{14}, T_{34} are reported in Figures 4.20 and 4.21, respectively for both piezoelectric and piezoelectromagnetic materials. Similar to the mechanical T-stress components, the good

convergence behavior is observed. However, for this particular case, the electrical T-stress components obtained from the two material models are significantly different. The computed magnetic T-stress components T_{15}, T_{35} for cracked whole space made of the piezoelectromagnetic material are also reported in Figure 4.22 for all three meshes. Similar to the previous case, the convergence of numerical solutions is confirmed and, in addition, only coarse meshes with few degrees of freedom can be used to obtain sufficiently accurate generalized T-stress components.

4.4.2.1.2 Uniform remote electrical loading

Next, consider the cracked whole space made of Mat-2 or Mat-3 and subjected only to the uniform remote electrical loading $\sigma_{34}^{\infty} = 2\sigma_{14}^{\infty} = 2d_0$ with $\sigma_0 = 1 \times 10^{-3} C/m^2$ (see Figure 4.16(b)). Similar to the previous loading case, all components of the generalized T-stress are non-zero and vary as a function of position along the crack front. The computed mechanical and electrical T-stress components $T_{11}, T_{33}, T_{13}, T_{14}, T_{34}$ for both types of materials are reported in Figures 4.23-4.27, respectively, whereas those for the magnetic T-stress components T_{15}, T_{35} for Mat-3 are reported in Figure 4.28. As clearly indicated from results generated from three different meshes, the proposed technique yields converged solutions and also exhibits the weak dependence on the mesh size. It is worth noting that when the cracked whole space is subjected to the uniform remote electrical loading, the mechanical T-stress components for both types of materials are significantly different (see Figures 4.23-4.25) but the electrical T-stress components are almost independent of the material properties (see Figures 4.26-4.27). In addition, the presence of the magnetic coupling effect also induces the non-zero magnetic T-stress components although there is no magnetic loading applied.

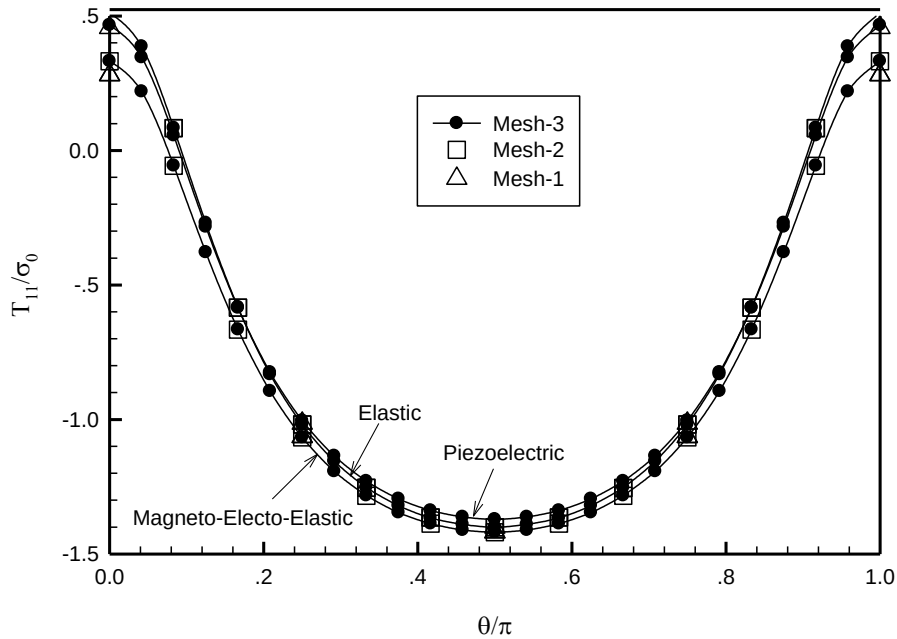


Figure 4.17: Normalized mechanical T-stress component T_{11} of inclined elliptical crack subjected to uniform remote mechanical loading for Mat-1, Mat-2 and Mat-3

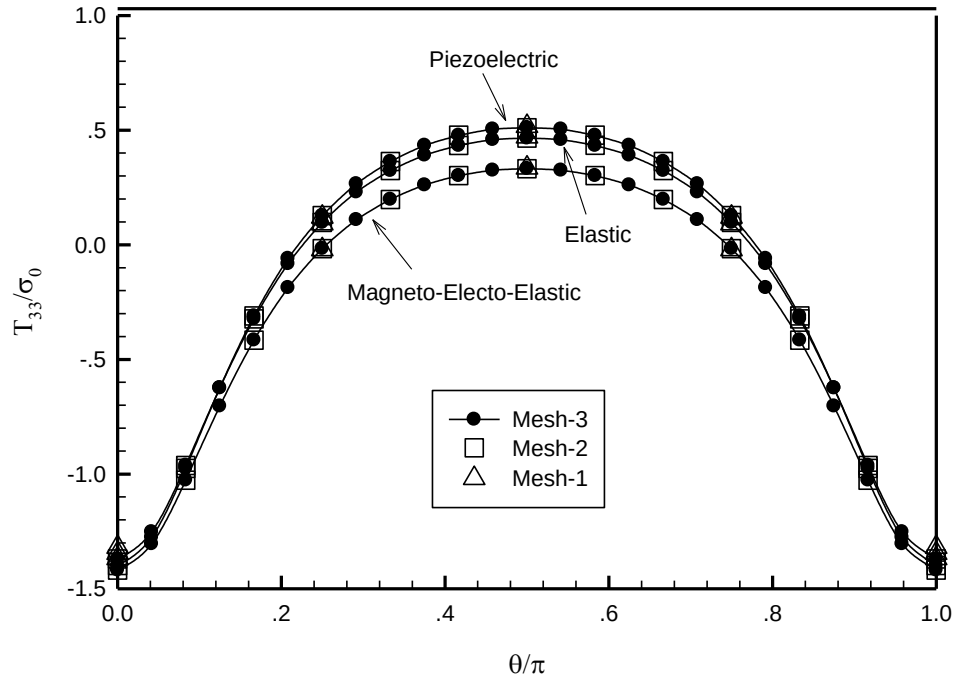


Figure 4.18: Normalized mechanical T-stress component T_{33} of inclined elliptical crack subjected to uniform remote mechanical loading for Mat-1, Mat-2 and Mat-3

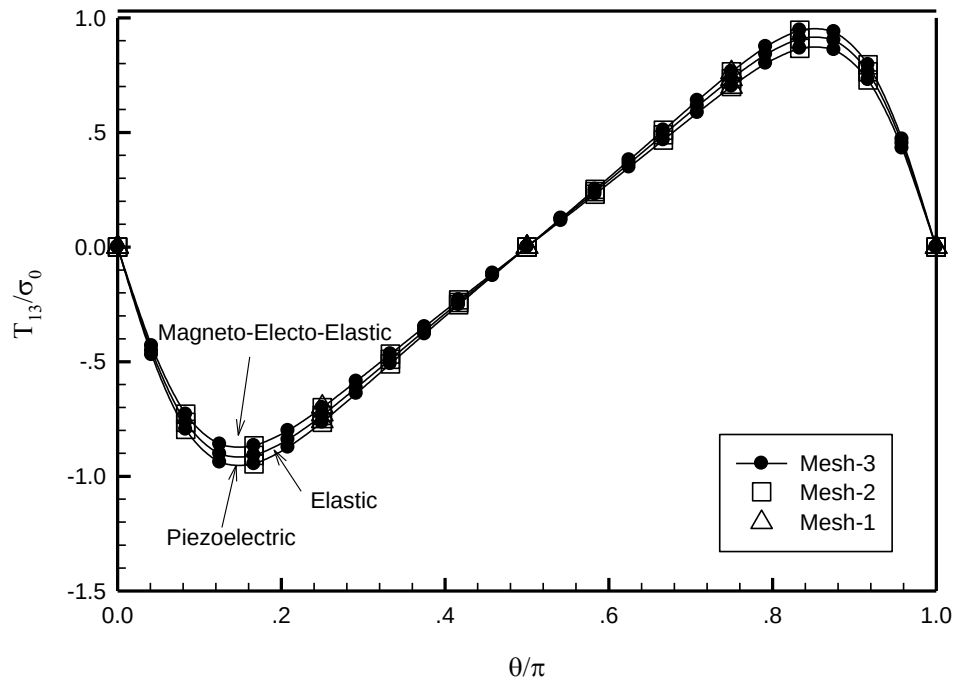


Figure 4.19: Normalized mechanical T-stress component T_{13} of inclined elliptical crack subjected to uniform remote mechanical loading for Mat-1, Mat-2 and Mat-3

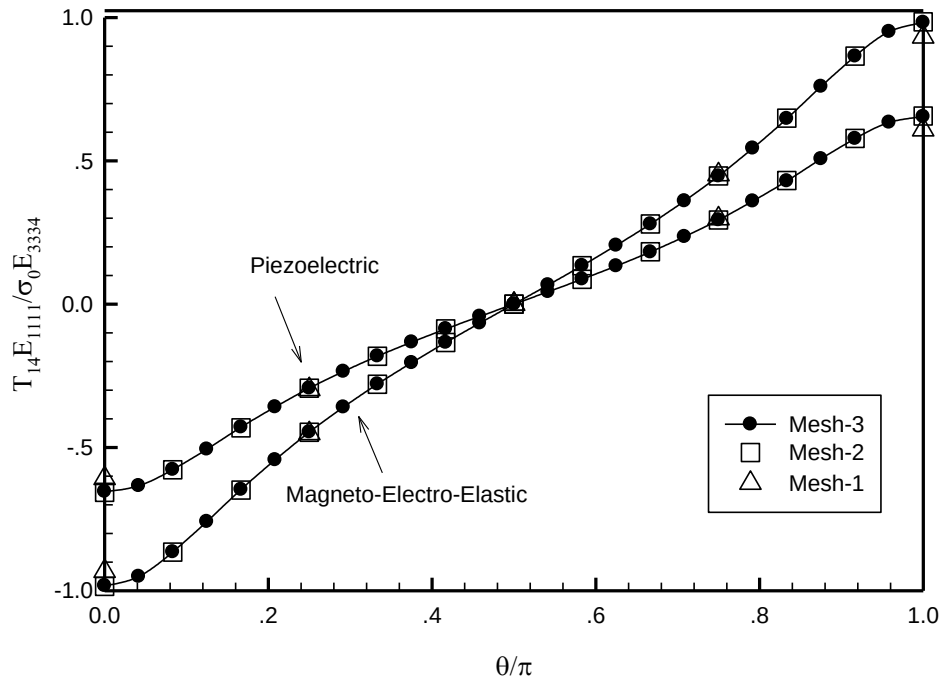


Figure 4.20: Normalized electrical T-stress component T_{14} of inclined elliptical crack subjected to uniform remote mechanical loading for Mat-2 and Mat-3

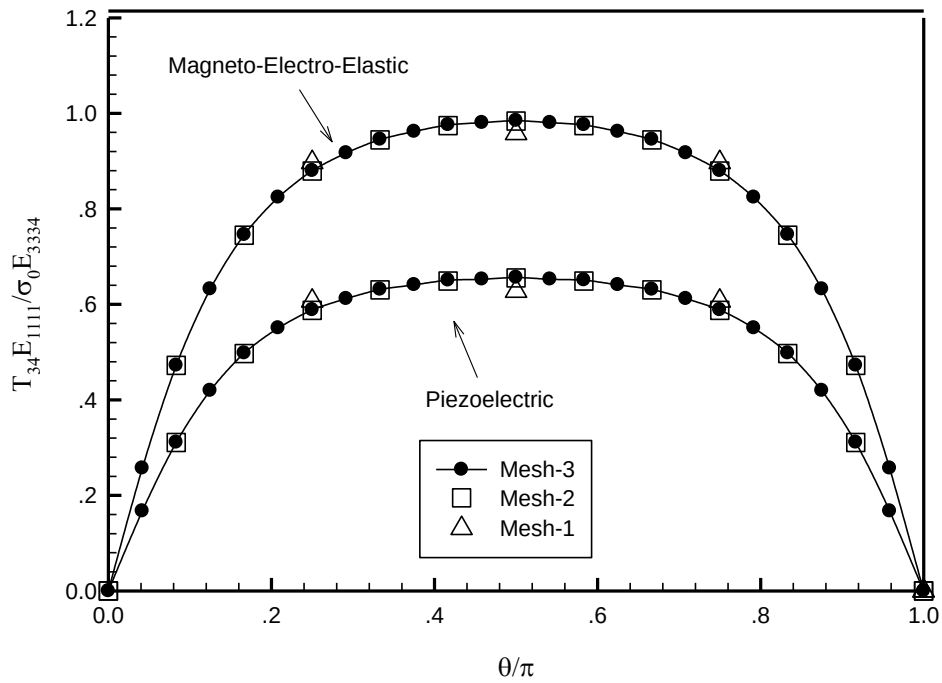


Figure 4.21: Normalized electrical T-stress component T_{34} of inclined elliptical crack subjected to uniform remote mechanical loading for Mat-2 and Mat-3

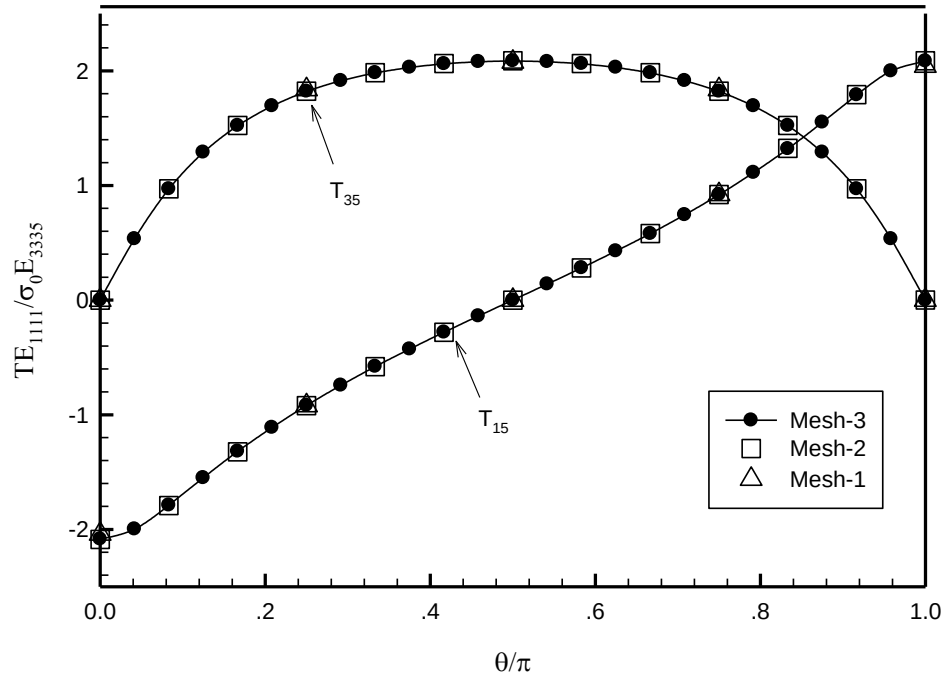


Figure 4.22: Normalized magnetic T-stress components T_{15} and T_{35} of inclined elliptical crack subjected to uniform remote mechanical loading for Mat-3

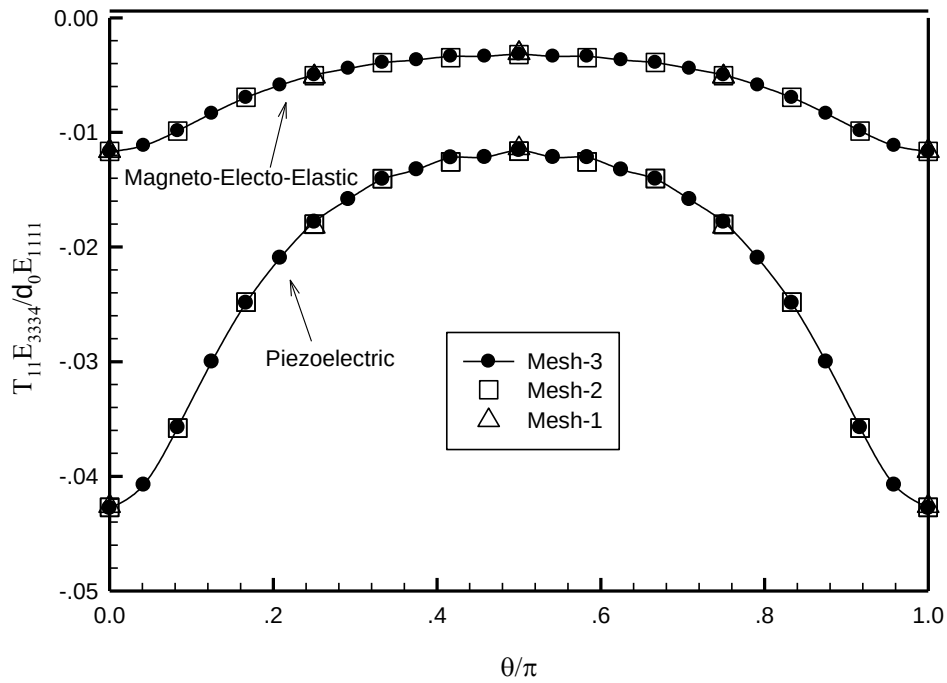


Figure 4.23: Normalized mechanical T-stress component T_{11} of inclined elliptical crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

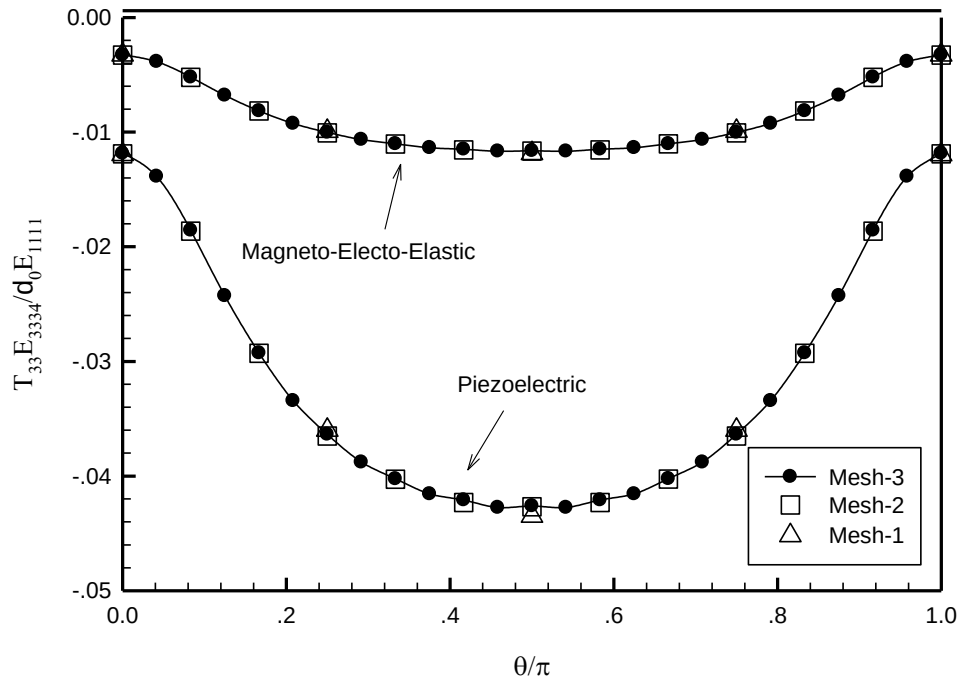


Figure 4.24: Normalized mechanical T-stress component T_{33} of inclined elliptical crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

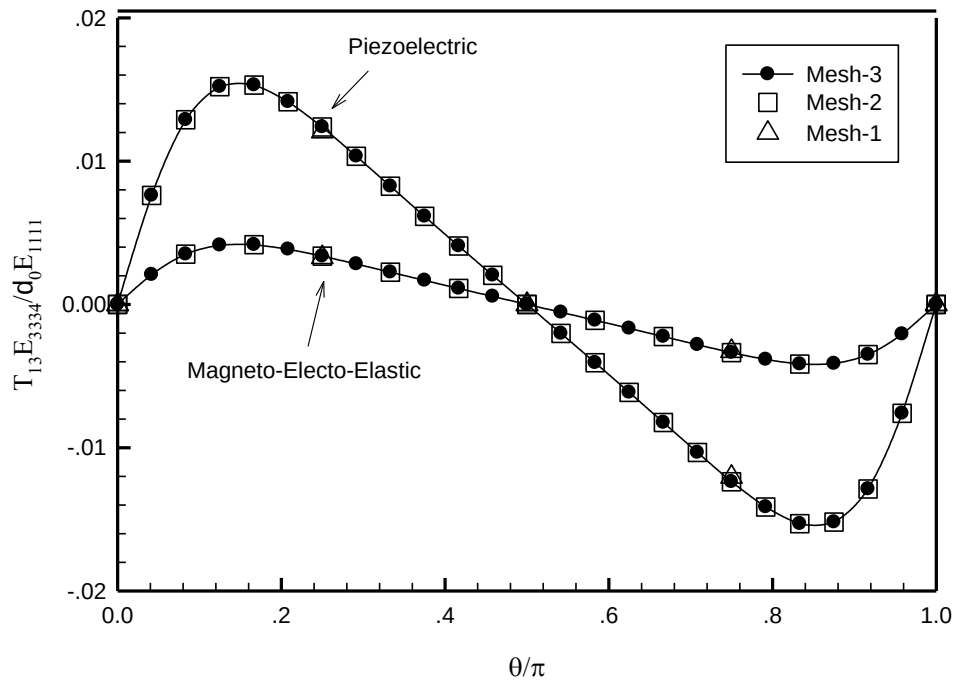


Figure 4.25: Normalized mechanical T-stress component T_{13} of inclined elliptical crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

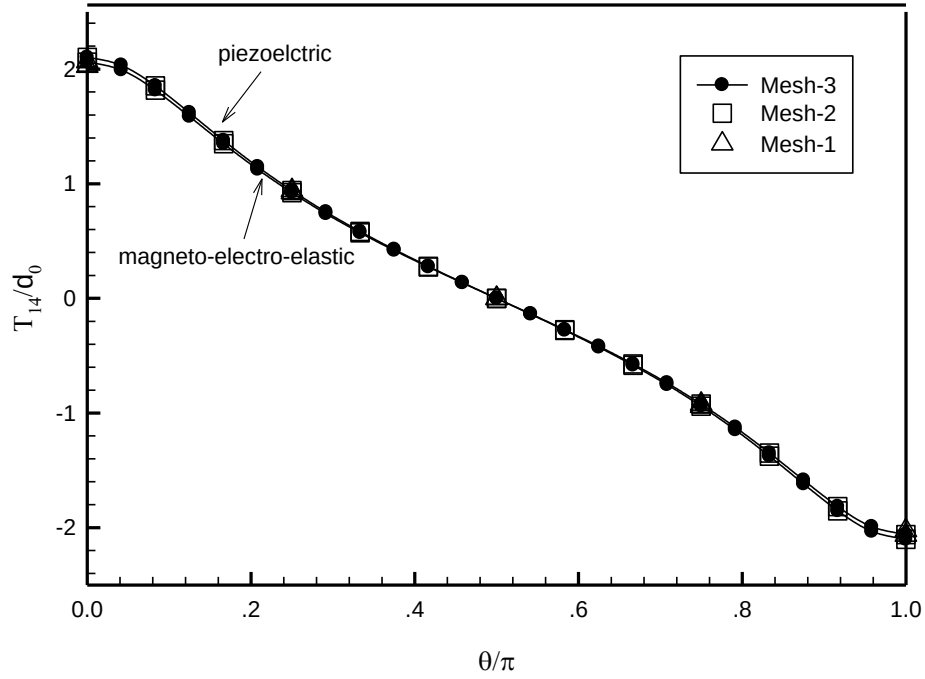


Figure 4.26: Normalized electrical T-stress component T_{14} of inclined elliptical crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

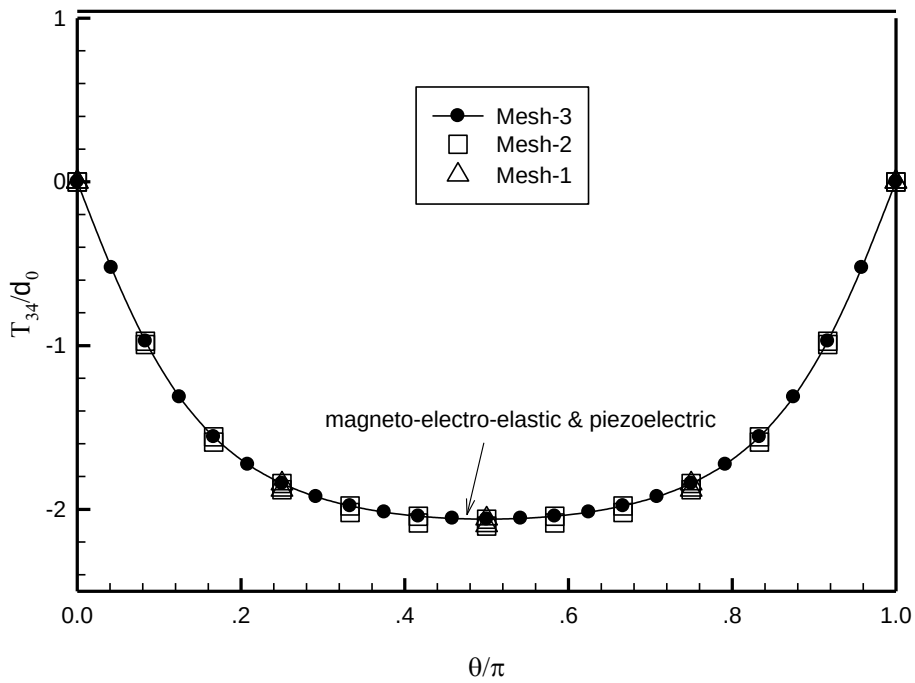


Figure 4.27: Normalized electrical T-stress component T_{34} of inclined elliptical crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

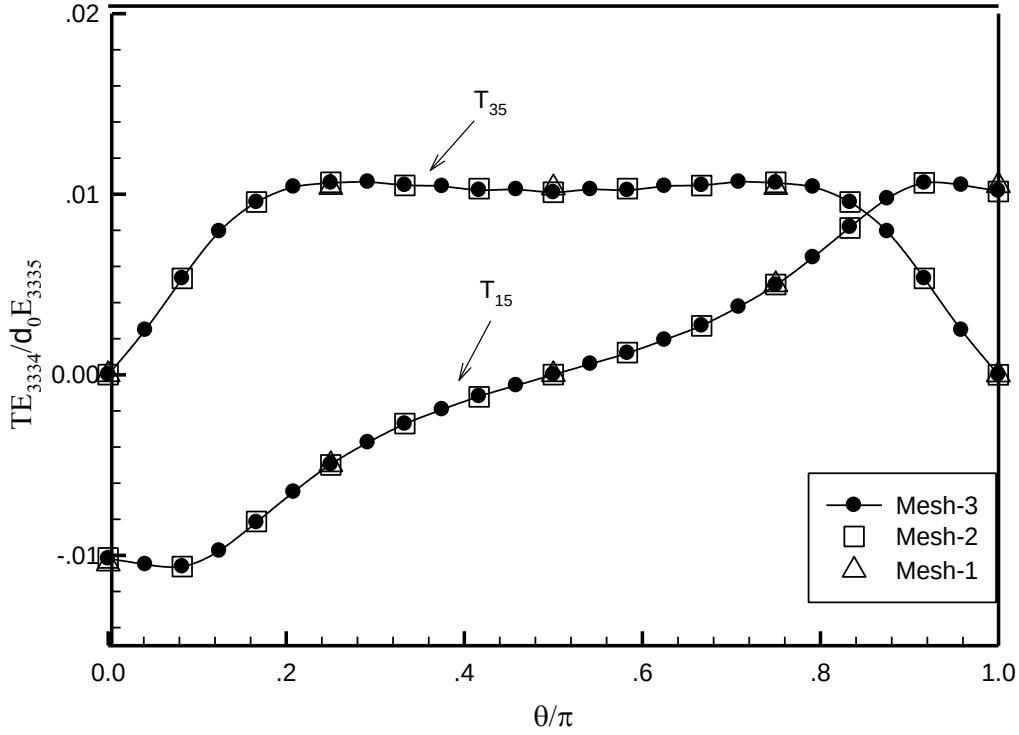


Figure 4.28: Normalized magnetic T-stress components T_{15}, T_{35} of inclined elliptical crack subjected to uniform remote electrical loading for Mat-3

4.4.2.2 Spherical cap crack

As a representative example to demonstrate the capability of the proposed technique to treat non-planar cracks, consider a spherical cap crack of radius a embedded in a linear whole space as shown in Figure 4.29. The crack is oriented such that its boundary (i.e., the crack front) can be parameterized by

$$x_1 = a \sin \gamma \cos \beta, \quad x_2 = a \sin \gamma \sin \beta, \quad x_3 = a \cos \gamma, \quad \beta \in [0, 2\pi], \quad \gamma \in [0, \phi] \quad (4.74)$$

where ϕ denotes the half-subtended angle of a spherical crack surface and β represents an angular position of point along the crack front. The medium is made of either elastic (Mat-1), piezoelectric (Mat-2), or piezoelectromagnetic (Mat-3) material with the axis of material symmetry directing along the x_3 coordinate direction. In the numerical study, the half subtended angle $\phi = 30^\circ$ is considered and three meshes as shown in Figure 4.30 are adopted. In particular, the Mesh-1, Mesh-2 and Mesh-3 consist of 16, 64 and 208 elements with 8, 16 and 32 special crack-tip elements, respectively.

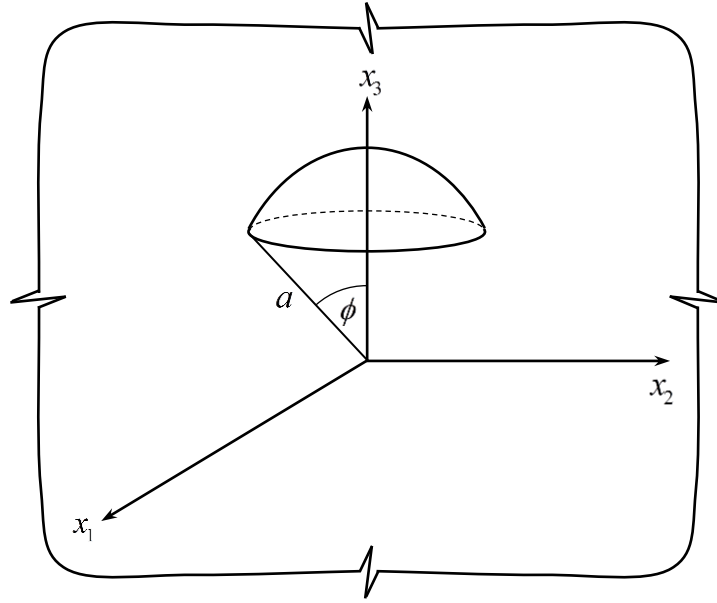


Figure 4.29: Schematic of a spherical cap crack with crack radius a and half subtended angle ϕ embedded in a linear whole space

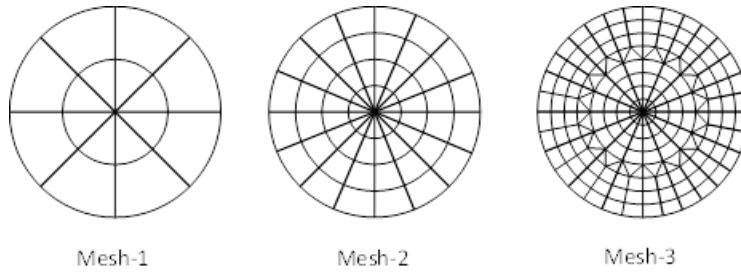


Figure 4.30: Three meshes of a spherical cap crack adopted in numerical simulations

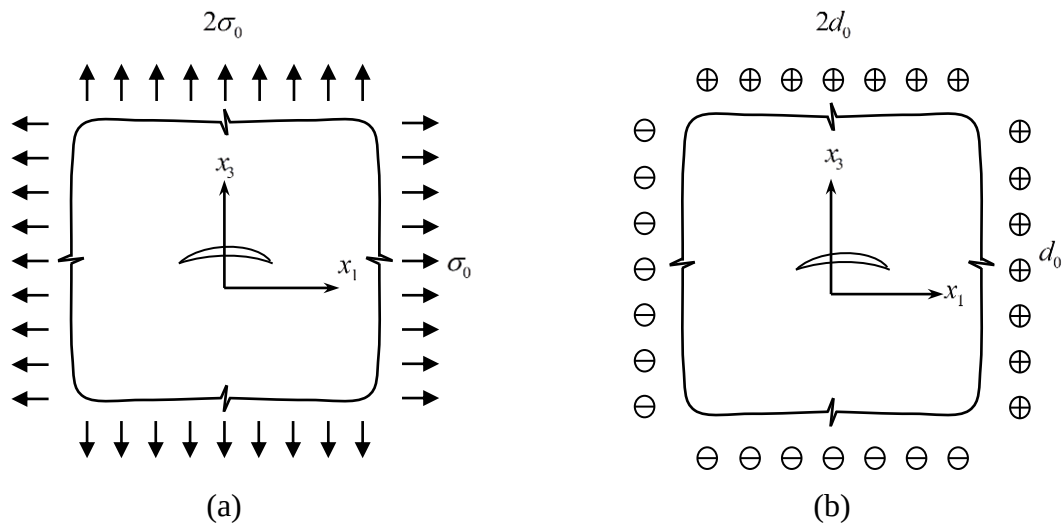


Figure 4.31: Schematics of a cracked whole space subjected to (a) uniform remote mechanical loading and (b) uniform remote electrical loading

4.4.2.2.1 Uniform remote mechanical loading

First, consider the cracked whole space made of Mat-1, Mat-2 or Mat-3 and subjected to the uniform remote mechanical loading $\sigma_{33}^{\infty} = 2\sigma_{11}^{\infty} = 2\sigma_0$ with $\sigma_0 = 1 \times 10^6 \text{ N/m}^2$ as shown schematically in Figure 4.31(a). Computed generalized T-stress components obtained for all types of materials and from three different meshes are reported in Figures 4.32-4.37. As expected, the proposed technique yields the converged solutions for all generalized T-stress components and all types of materials; in particular, the weak dependence on the mesh size of the computed solutions is clearly observed except for the magnetic T-stress components T_{15} where results obtained from the coarsest mesh is significantly different from those generated by the finest mesh. In addition, the mechanical T-stress components obtained for three material models are nearly identical whereas the electrical T-stress components are quite different for both Mat-2 and Mat-3. It should also point out that the coupling effect through the constitutive relations induces both electrical and magnetic T-stress components although the remote loading is purely mechanical.

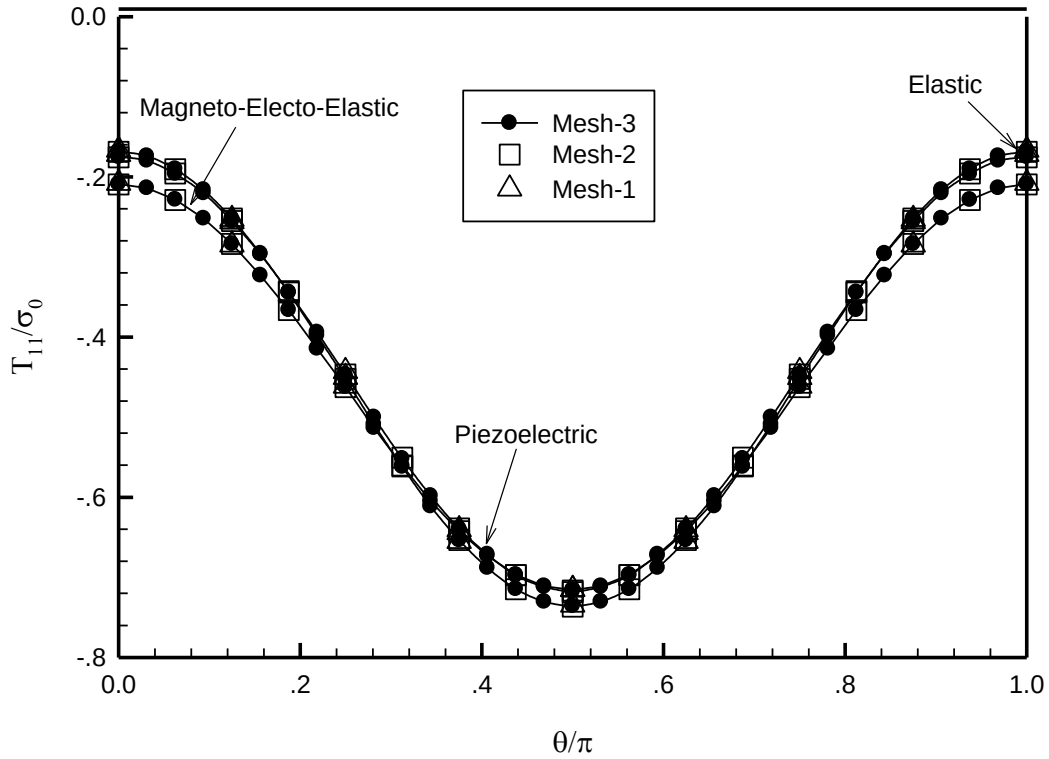


Figure 4.32: Normalized mechanical T-stress component T_{11} of a spherical cap crack subjected to uniform remote mechanical loading for Mat-1, Mat-2 and Mat-3

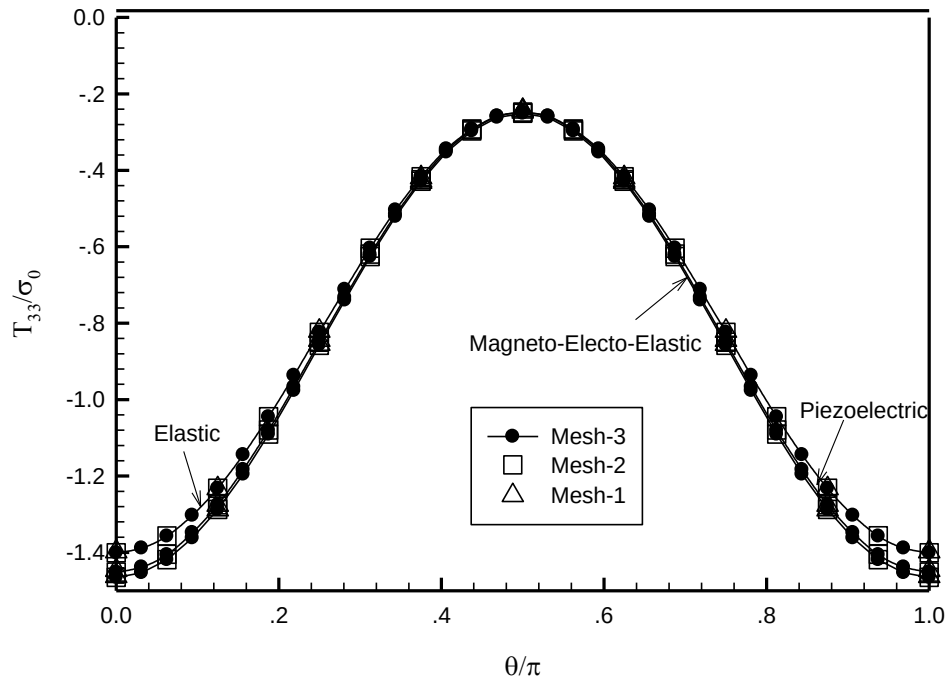


Figure 4.33: Normalized mechanical T-stress component T_{33} of a spherical cap crack subjected to uniform remote mechanical loading for Mat-1, Mat-2 and Mat-3

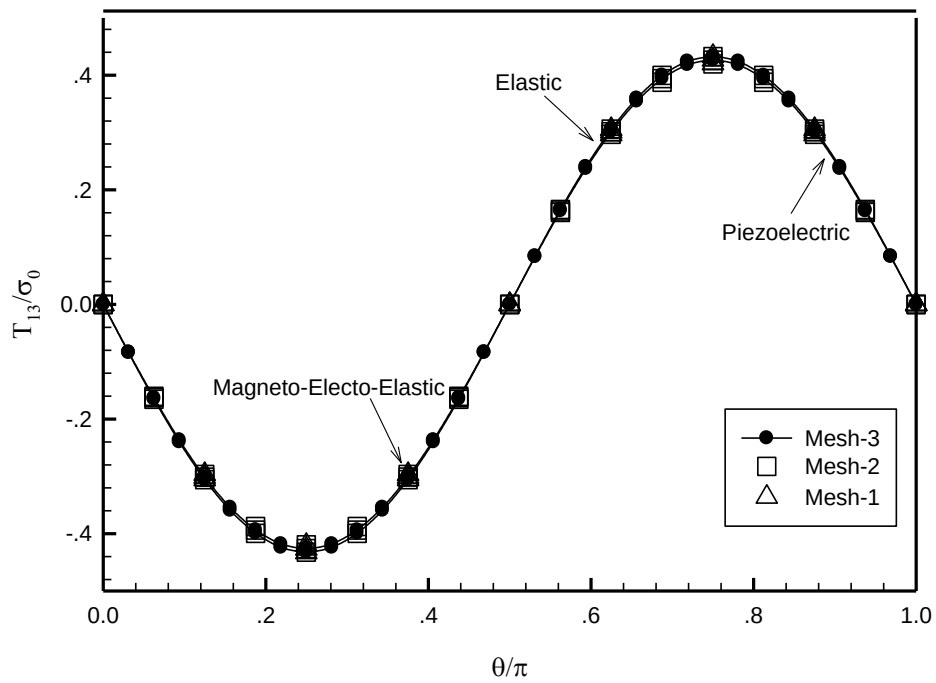


Figure 4.34: Normalized mechanical T-stress component T_{13} of a spherical cap crack subjected to uniform remote mechanical loading for Mat-1, Mat-2 and Mat-3

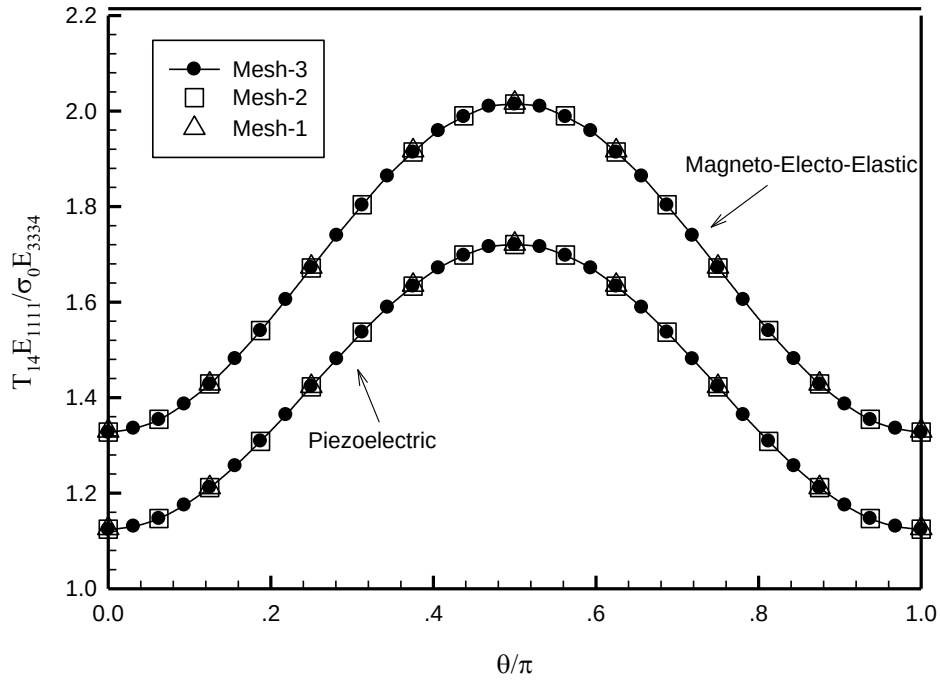


Figure 4.35: Normalized electrical T-stress component T_{14} of a spherical cap crack subjected to uniform remote mechanical loading for Mat-2 and Mat-3

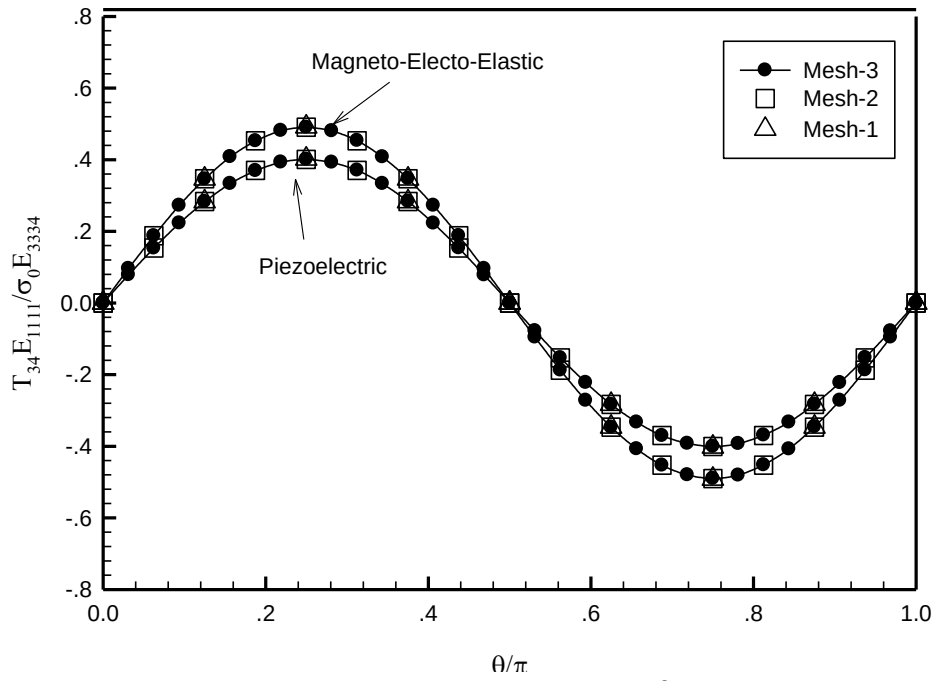


Figure 4.36: Normalized electrical T-stress component T_{34} of a spherical cap crack subjected to uniform remote mechanical loading for Mat-2 and Mat-3

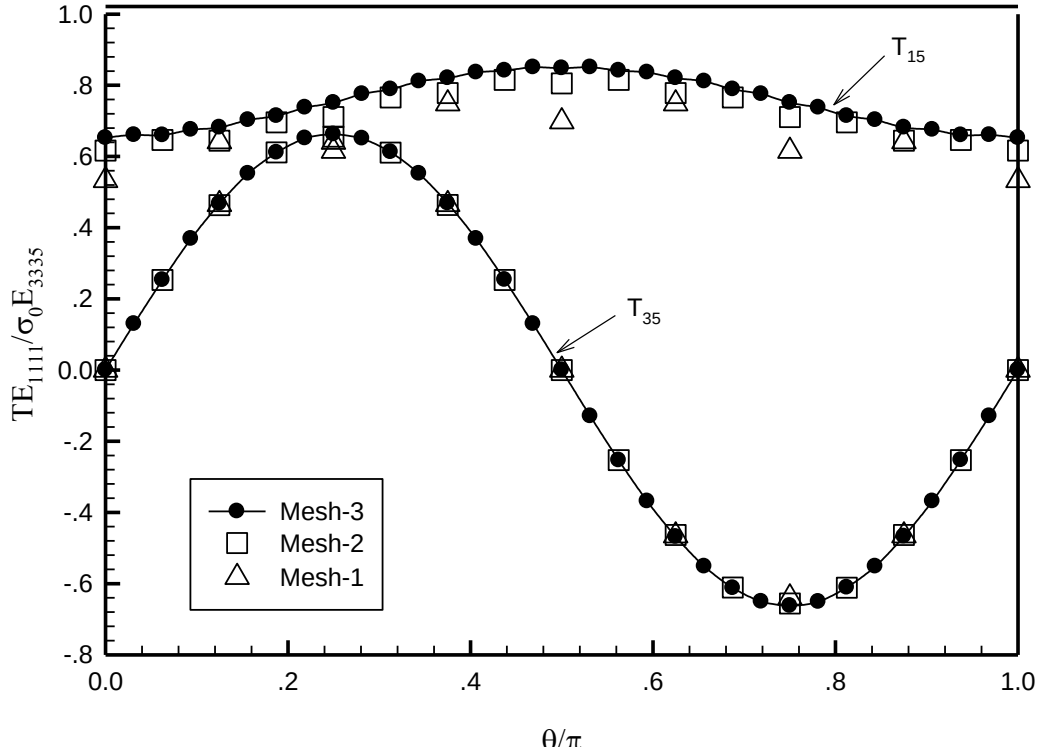


Figure 4.37: Normalized magnetic T-stress components T_{15}, T_{35} of a spherical cap crack subjected to uniform remote mechanical loading for Mat-3

4.4.2.2.2 Uniform remote electrical loading

Consider, next, the cracked whole space that is made of either Mat-2 or Mat-3 and subjected to the uniform remote electrical loading $\sigma_{34}^{\infty} = 2\sigma_{14}^{\infty} = 2d_0$ with $d_0 = 1 \times 10^{-3} \text{C/m}^2$ as illustrated in Figure 4.31 (b). All generalized T-stress components for this loading case are non-zero and their values obtained from the proposed technique for all three meshes are shown in Figures 4.38-4.43. Similar to all cases presented previously, the convergence of numerical solutions is confirmed and relatively coarse meshes can be used to obtain results comparable to the converged solution. Obtained results also indicate that when the cracked whole space is subjected to the uniform remote electrical loading, the type of coupled-field materials strongly influences the values of mechanical T-stress components but insignificantly affects the electrical T-stress components. It is also evident that electro-mechanical and magneto-electro-mechanical couplings via the constitutive relations induce both the mechanical and magnetic responses although the excitation is purely electrical.

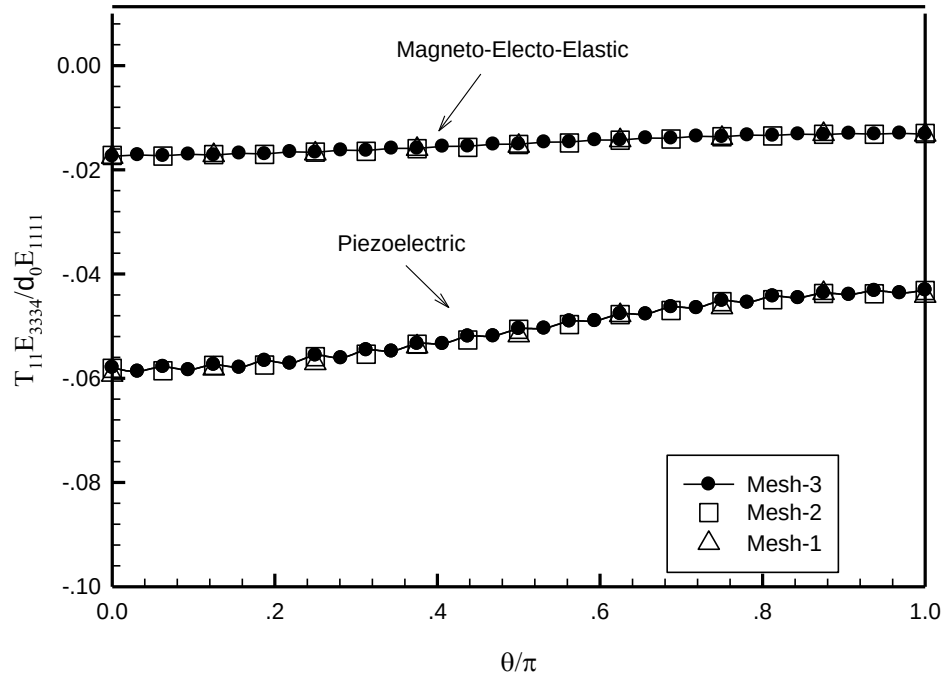


Figure 4.38: Normalized mechanical T-stress component T_{11} of a spherical cap crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

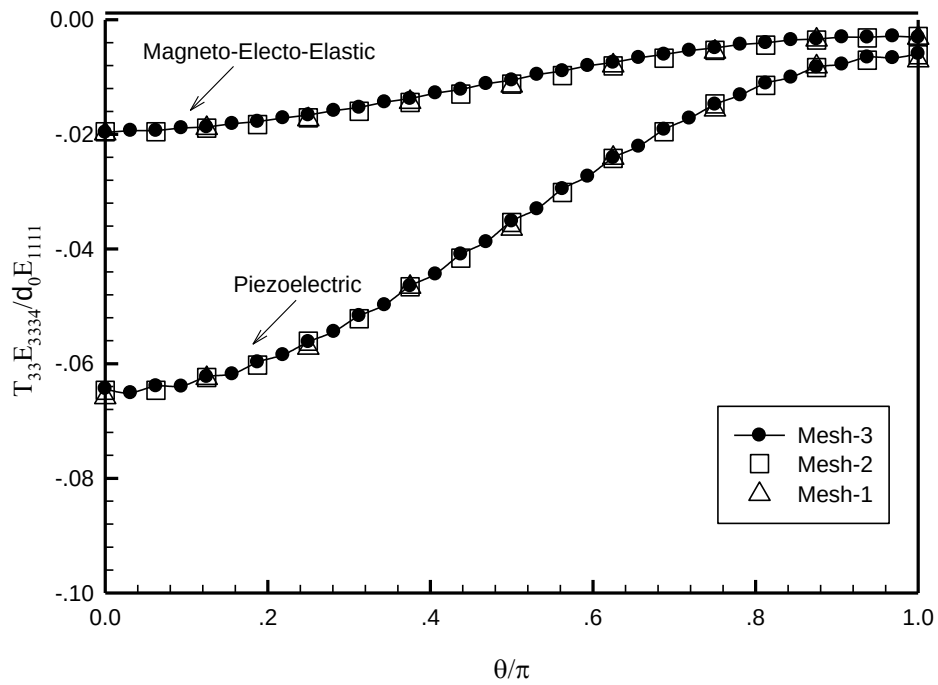


Figure 4.39: Normalized mechanical T-stress component T_{33} of a spherical cap crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

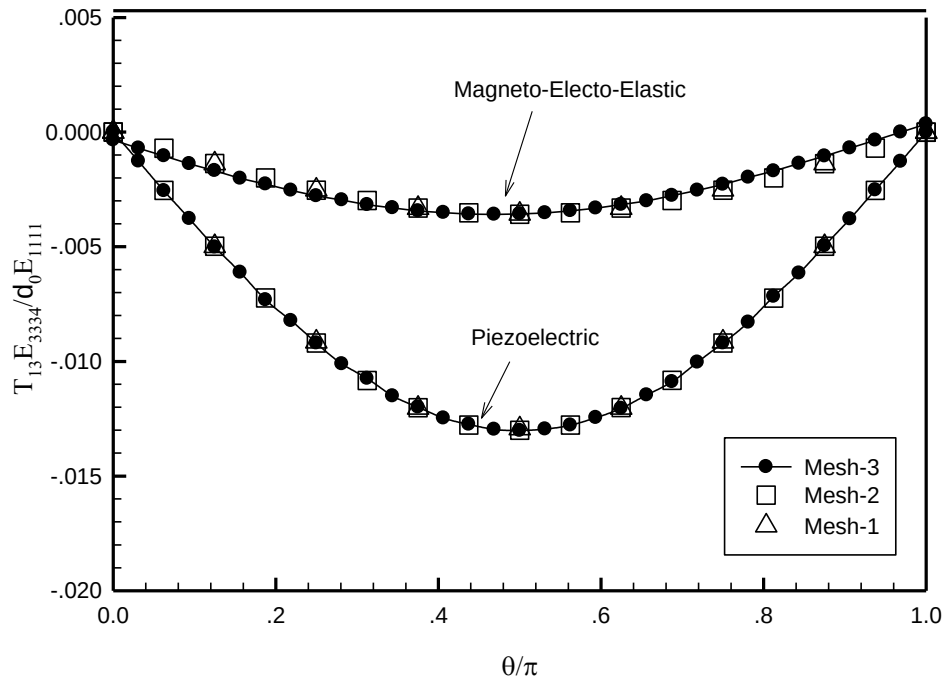


Figure 4.40: Normalized mechanical T-stress component T_{13} of a spherical cap crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

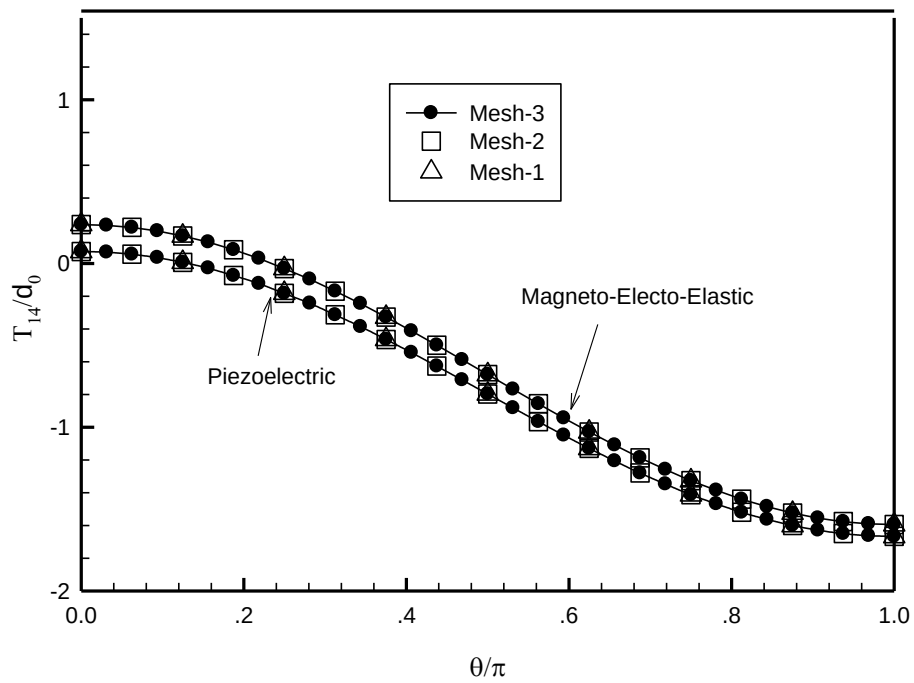


Figure 4.41: Normalized electrical T-stress component T_{14} of a spherical cap crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

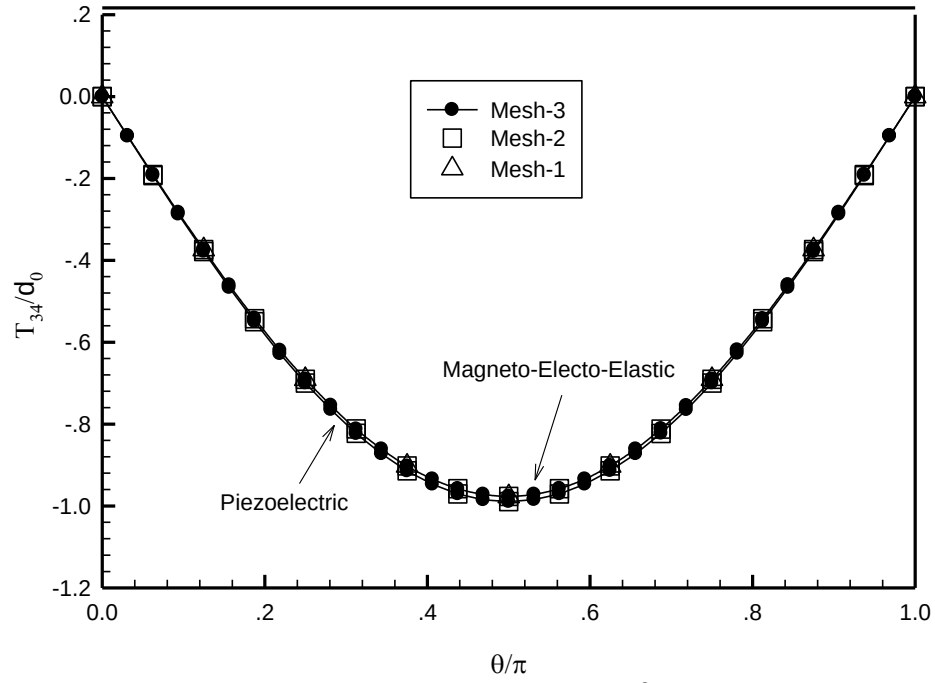


Figure 4.42: Normalized electrical T-stress component T_{34} of a spherical cap crack subjected to uniform remote electrical loading for Mat-2 and Mat-3

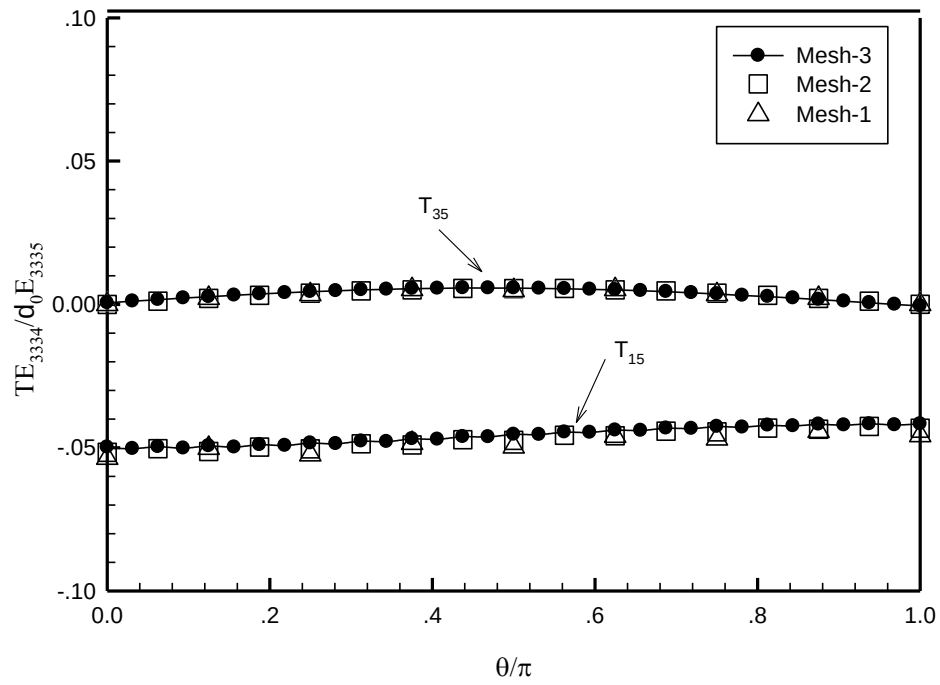


Figure 4.43: Normalized magnetic T-stress components T_{15}, T_{35} of a spherical cap crack subjected to uniform remote electrical loading for Mat-3

4.4.2.3 A pair of penny- shaped cracks

To finally demonstrate the capability of the proposed technique to treat multiple cracks, let us consider a pair of identical penny-shaped cracks of radius a embedded in a linear whole space made of Mat-1, Mat-2, or Mat-3 as shown in Figure 4.44. The two cracks are oriented such that the unit normal vectors to both crack surfaces direct along the x_3 -axis and the center of each crack is located at $(0,0,-d)$ and $(0,0,d)$. The crack front of the two cracks can be described by (4.72) except that the coordinate x_3 changes to d and $-d$ for the upper and lower cracks, respectively. The axis of material symmetry is assumed directing along the x_3 coordinate direction and the medium is subjected to a uniform remote mechanical loading in the x_3 direction, i.e., $\sigma_{33}^\infty = \sigma_0$. Here, the interaction between the two cracks is investigated by varying the ratio d/a and results are then compared with those of a single penny shaped crack in which the exact solutions for the generalized T-stress are available or can be readily constructed. It is worth noting that due to the axisymmetry, the generalized T-stress components are constant along the crack front.

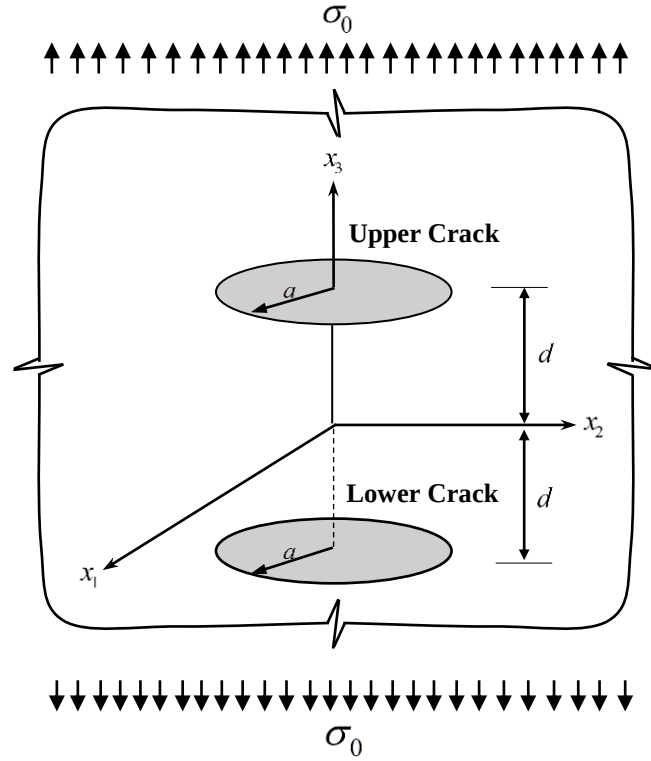


Figure 4.44: Schematic of a pair of identical penny-shaped cracks in a linear whole space under a uniform remote mechanical loading in x_3 direction

In the numerical study, three different meshes shown in Figure 4.5 are adopted for both upper and lower cracks. Results for the case $d/a = 1$ are obtained for the three meshes and three material models and the non-zero mechanical T-stress components are then reported in Table 4.6. It is clear that as the mesh is refined from the Mesh-1 to the Mesh-3, the converged solutions

for the non-zero generalized T-stress components are obtained. This implies that the Mesh-3 is sufficiently refined and can be used to generate results in the following parametric study.

Table 4.6 Normalized mechanical T-stress components T_{11} and T_{33} for a pair of penny-shaped cracks in linear whole space subjected to uniform remote mechanical loading in x_3 direction

Mesh	Elastic		Piezoelectric		Magneto-Electro-Elastic	
	$\frac{T_{11}}{T_{11}^{Ref}}$	$\frac{T_{33}}{T_{33}^{Ref}}$	$\frac{T_{11}}{T_{11}^{Ref}}$	$\frac{T_{33}}{T_{33}^{Ref}}$	$\frac{T_{11}}{T_{11}^{Ref}}$	$\frac{T_{33}}{T_{33}^{Ref}}$
1	1.0045	0.8562	0.9819	0.8571	0.9961	0.8678
2	0.9987	0.8588	0.9773	0.8586	0.9905	0.8699
3	0.9968	0.8584	0.9755	0.8581	0.9885	0.8693

To investigate the influence of the distance between the two cracks on the value of the generalized T-stress component, results are generated using the Mesh-3 for various values of the aspect ratio d/a and then reported together with the results for the single crack in Figures 4.45-4.48 for the generalized T-stress components T_{11} , T_{33} , T_{14} , and T_{15} , respectively. As the ratio d/a increases, the mechanical T-stress components T_{11} and T_{33} decrease rapidly in magnitude to attain the minimum value and then gradually increase in magnitude to approach the solution of the single crack when $d/a \geq 5$. For the electric and magnetic T-stress components T_{14} and T_{15} , as the ratio d/a increases, they monotonically decrease in magnitude and then approach zero (the solution of the single crack) when $d/a \geq 3$.

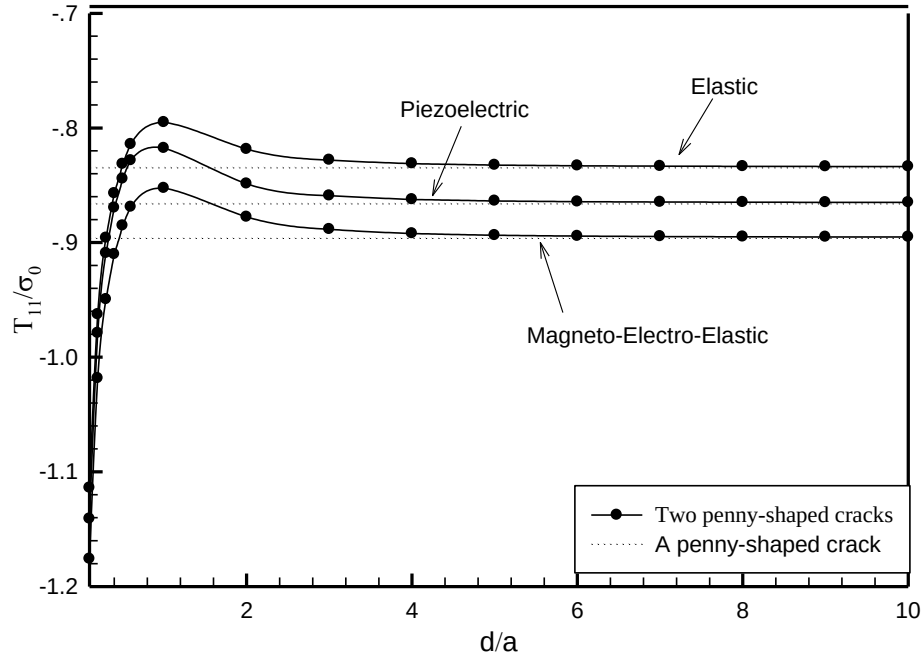


Figure 4.45: Normalized mechanical T-stress component T_{11} of a pair of penny-shaped cracks under uniform remote mechanical loading in x_3 direction for Mat-1, Mat-2, and Mat-3

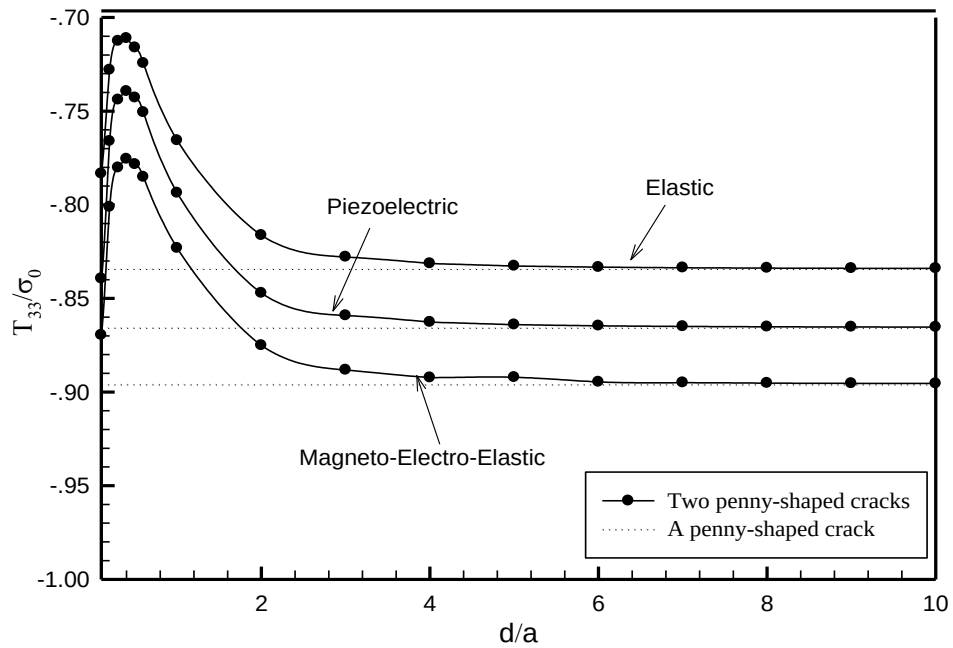


Figure 4.46: Normalized mechanical T-stress component T_{33} of a pair of penny-shaped cracks under uniform remote mechanical loading in x_3 direction for Mat-1, Mat-2, and Mat-3

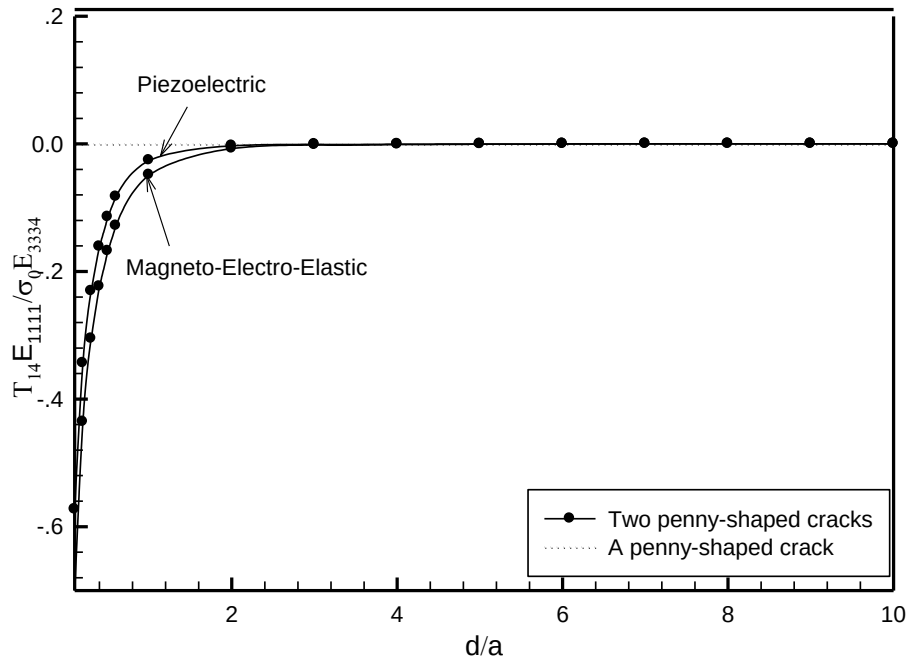


Figure 4.47: Normalized electric T-stress component T_{14} of a pair of penny-shaped cracks under uniform remote mechanical loading in x_3 direction for Mat-2 and Mat-3

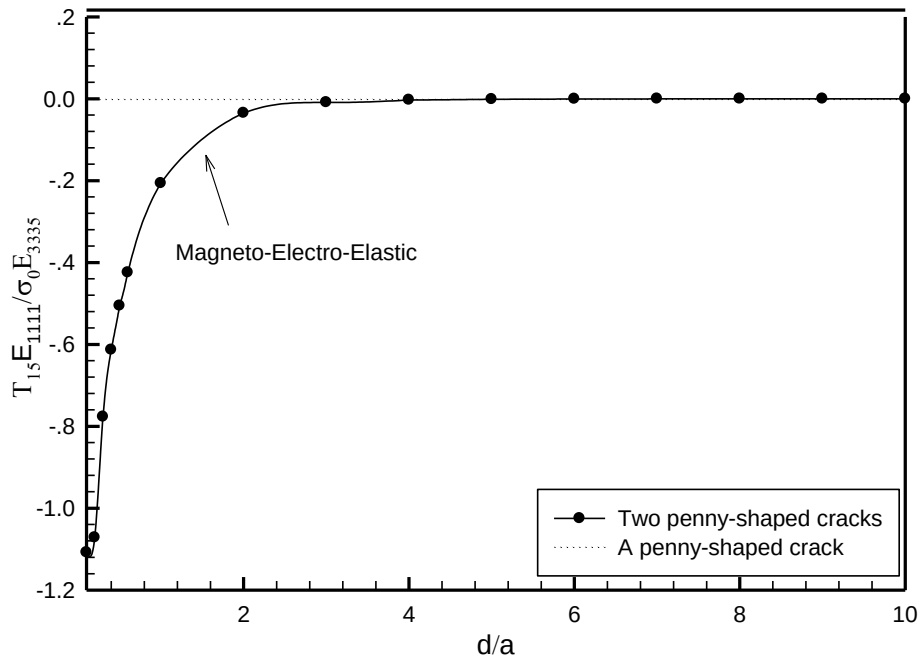


Figure 4.48: Normalized magnetic T-stress component T_{15} of a pair of penny-shaped cracks under uniform remote mechanical loading in x_3 direction for Mat-3

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CHAPTER 5

WEAKLY SINGULAR BIEM FOR ANALYSIS OF CRACKS IN ELASTIC HALFSPACE

This chapter presents an efficient numerical technique for the analysis of stress intensity factors and T-stress components for arbitrary-shaped cracks in a homogeneous, linear elastic half-space under various conditions on the free surface. The key governing equations are established in a form of weakly singular boundary integral equations involving both unknown relative and sum of the crack-face displacements. A systematic regularization technique based on the integration by parts and special decompositions of singular kernels is adopted to regularize all involved strongly singular and hyper-singular integrals to those containing only weakly singular kernels and requiring only continuous crack-face data for their validity. Besides the direct consequence of the weakly singular nature, the governing integral equations also possess several positive features such as no requirement of free-surface discretization and the capability to treat material anisotropy, non-planar crack geometry and general crack-face loading conditions. In numerical implementations, a weakly singular, symmetric Galerkin boundary element method along with the special near-front approximation is employed to solve the traction integral equation for the relative crack-face displacement. The sum of the crack-face displacement is then obtained by solving the displacement integral equation via standard Galerkin method. The stress intensity factors and the T-stress components along the crack are extracted directly from the near-front relative and sum of the crack-face displacement data. Obtained numerical results for various scenarios clearly demonstrate the accuracy, convergence and capability of the proposed technique.

5.1 Background and Review

A half-space containing a surface of discontinuities is one of mathematical domains commonly used to represent a physical body in the simulation of localized, near-surface flaws, defects, and impurities if their sizes are sufficiently small in comparison with the characteristic dimensions of the body. Such idealization significantly simplifies the real geometry, reduces the computational effort, and also yields the reasonably accurate prediction relative to the full analysis of the whole body. To perform such theoretical simulations, solution techniques have played an important role in the solution accuracy and computational performance and they must be properly selected to suit each involved scenario. Many investigations based on the conventional theories of linear elasticity and linear elastic fracture mechanics have been well recognized in the literature and various techniques including analytical and semi-analytical approaches, conventional domain-based numerical techniques, and boundary integral equation methods have been proposed to construct solutions of mathematical models of cracked half-spaces. In particular, those existing solution techniques were based on integral-transform-based and series-representation method (e.g., Srivastava and Singh, 1969; Feng *et al.*, 2007; Monastyrskyy and Kaczyński, 2010; Bogdanov, 2011; Eskandari-Ghadi *et al.*, 2013), the body force method (e.g., Murakami, 1985; Noguchi and Smith, 1995; Noguchi *et al.*, 1997), the iterative-based alternating method (e.g.,

Smith and Alavi, 1971; Shah and Kobayashi, 1973; Dhondt, 1995), the inverse method (e.g., Keat *et al.*, 1998), the standard finite element method (e.g., Ghajar and Alizadeh, 2013) and the boundary integral equation methods (e.g., Lo, 1979; Hayashi and Abé, 1980; Mayrhofer and Fischer, 1989; Martin *et al.*, 1993; Khaj and Sushko, 1994; Khaj and Sushko, 1996; Sushko and Khaj, 1996; Movchan and Willis, 2000; Kit *et al.*, 2000; Hrylyts'kyi *et al.*, 2003; Gordeliy and Detournay, 2011; Skalsky *et al.*, 2013).

Analytical and semi-analytical techniques based on integral transforms, dual integral equation theories, and series representations were successfully established in the investigation of the near-front elastic field and stress intensity factors of some simple-shaped cracks (e.g., a penny-shaped crack) in an isotropic elastic half-space (e.g., Srivastava and Singh, 1969; Feng *et al.*, 2007; Bogdanov, 2011). While those methodologies can yield very accurate results comparable to the exact solution, their capability has been found very limited and they cannot be applied to treat various practical scenarios due to the high complexity posed by the general prescribed information such as crack geometries, material behavior, boundary and loading conditions.

To further enhance and broaden the capability for modeling elastic cracked half-spaces, a variety of numerical procedures have continuously been developed in the past several decades. Based on the linear feature of the elastic field and a standard superposition technique, the body force method was proposed by several investigators to perform the stress analysis and estimate the stress intensity factors along the crack front of planar cracks contained in an elastic half-space (e.g., Murakami, 1985; Noguchi and Smith, 1995; Noguchi *et al.*, 1997). Although fully mixed-mode problems have been included in the work of Murakami (1985), it was still limited to planar cracks, the traction-free surface and isotropic materials. Other approaches such as the alternating techniques, which are based on an iterative solution procedure to remove the fictitious residual stress, were also established to solve cracked half-space problems (e.g., Smith and Alavi, 1971; Shah and Kobayashi, 1973; Dhondt, 1995). Due to the nature of the solution strategy, the alternating technique was found yielding results of relatively low accuracy (e.g., Mayrhofer and Fischer, 1989). An experimental approach was also proposed by Keat *et al.* (1998) in which the algorithm of inversion for identifying embedded crack geometries and locations in a half-space from field measurements of the surface displacement was presented. Nonetheless, this technique can only be applied for simple cases of crack configurations and loading conditions. Recently, a standard finite element technique was also developed to investigate cracked half-space problems (e.g., Ghajar and Alizadeh, 2013); however, it has been found that substantially fine mesh is required in the discretization especially in the near-front zone and the truncation of the domain must be properly considered to treat the unbounded dimensions. It should also be remarked that all studies indicated above were still limited to certain crack configurations, isotropic materials, and determination of stress intensity factors. The analysis for nonsingular terms such as the T-stress components and the treatment of material anisotropy were still not addressed. The consideration of material anisotropy was found in the work of Monastyrskyy and Kaczyński (2010). In their study, the material is assumed to be transversely isotropic with the axis of material symmetry directing perpendicular to the free surface and only a crack of penny-shape oriented horizontally is modeled. A technique of Hankel integral transforms along with a selected numerical procedure was adopted to determine the stress intensity factors along the crack front. Recently, Eskandari-Ghadi *et al.* (2013) applied the method of integral transform to analytically solve a horizontal, penny-shaped crack in a transversely isotropic, linearly elastic half-space under time-harmonic asymmetric tractions.

Among various existing solution techniques, ones based on boundary integral equations have been found attractive and extensively employed in the analysis procedure due to their simplicity to treat the unbounded geometry. With the proper incorporation of fundamental solutions to satisfy the conditions on the free surface of the half-space, the key governing integral equations simply involve unknowns only on the crack surface and this, as a result, significantly reduces the number of degrees of freedom resulting from the discretization. In 1979, Lo applied Mindlin's point force solution to derive the strongly singular boundary integral equation for an embedded planar crack oriented parallel to the free surface of an isotropic, linear elastic half-space. While the integral equation was successfully obtained, the implementation to solve cracked half-space problem was not found in this study. Hayashi and Abé (1980) developed the collocation technique based on a strongly singular traction boundary integral equation to determine the mode-I stress intensity factors of a uniformly pressurized, vertical, semi-elliptical, surface-breaking crack in an isotropic elastic half-space with the traction-free surface. Martin *et al.* (1993) employed Mindlin's fundamental solution to derive a system of singular boundary integral equations for a pressurized crack oriented perpendicular to the free surface of an isotropic elastic half-space with the stress-free surface. Although the implementation of those equations was not included in their study, the solvability and the regularity of solutions of the developed integral equations were proven. Later, Khaj and Sushko (1994) investigated the interaction of two identical, near-surface, coplanar, circular cracks in an isotropic elastic half space with the stress-free surface and loaded by forces normal to the crack surface. In the analysis, the opening of the cracks was obtained by solving a system of hypersingular boundary integral equations and those results were employed to extract the stress intensity factors. Sushko and Khaj (1996) developed a system of hypersingular integro-differential equations governing the opening displacement of a planar surface-breaking crack in an isotropic elastic half-space. Such equations were solved numerically for the special case of cracks with their geometry is a part of a circle. In the same year, Khaj and Sushko (1996) employed a conformal mapping along with a semi-analytical technique to solve a system of hypersingular boundary integral equations in the investigation of the influence of the depth and orientation of a planar crack with the shape of a limaçon of Pascal. Later, Kit *et al.* (2000) generalized the work of Khaj and Sushko (1994) to investigate the interaction of multiple, planar, surface-breaking cracks of arbitrary shapes and orientations. The solution procedure was based primary on solving a system of boundary integral equations for the relative crack-face displacement and used that data to post-process for the stress intensity factors. Movchan and Willis (2000) extended the work of Martin *et al.* (1993) to further explore the corner singularity at locations where the crack intersects the free boundary. In their analysis, a hypersingular boundary integral equation formulated in terms of the relative crack-face displacement is employed. Later, Hrylyts'kyi *et al.* (2003) implemented a hypersingular boundary integral equation method to study the interaction of planar cracks in an isotropic elastic half-space with rigidly restrained surface. The potential method was employed to derive boundary integral equations in terms of unknown relative crack-face displacement. The stress intensity factors were also calculated for a circular crack perpendicular to rigid surface. Gordeliy and Detournay (2011) developed the displacement discontinuity method to model axisymmetric cracks in an isotropic, elastic half-space. Their key formulation was based primarily on the hypersingular integral equations in terms of the displacement jump across the crack surface and the combined numerical integration scheme and recursive procedure was adopted to directly integrate involved hyper and Cauchy singular integrals. In addition, the tip-elements were also employed to enhance the accuracy of the near-front approximation. Recently, Skalsky *et al.*

(2013) investigated an axisymmetric crack problem involving the mode-III loading condition for an isotropic, linearly elastic half-space. The displacement-field functions were numerically determined by the frequency-domain boundary integral equations. Still, this particular work focuses only a boundary value problem concerning a penny-shaped crack under the action of torsion loading. It should be remarked that, among various existing studies described above, the key formulations were still based primarily on strongly and/or hypersingular integral equations with limitation to isotropic materials and such equations were implemented mainly to treat planar cracks under simple crack-faced loading and determine the stress intensity factors.

It is well known that the displacement integral equation derived directly from Somigliana's identity is insufficient for solving cracked bodies within the context of a single-domain-based formulation (e.g., Cruse, 1988). An alternative means to circumvent such difficulty besides inefficient multi-domain techniques is to employ, in addition, the traction boundary integral equation to treat cracks. Unfortunately, the conventional traction integral equation established directly from the stress integral relation contains both strongly singular and hyper-singular kernels and their treatment requires special care (e.g., Guiggiani *et al.*, 1991). In addition, the smoothness requirement of the relative crack-face displacement data for the validity of the hyper-singular integral equation implies the need of C^1 -elements in the solution discretization (e.g., Gray *et al.*, 1990, Martin and Rizzo, 1996). To further alleviate such requirement and facilitate the numerical treatment of involved integrals, a regularization technique was proposed by Li (1996) to derive the singularity-reduced boundary integral equations for discontinuities embedded in a three-dimensional, isotropic, linear elastic half-space under various boundary conditions. Such technique was later enhanced and utilized by Li and Mear (1998), Li *et al.* (1998), Rungamornrat and Mear (2008a) in the derivation of the weakly singular boundary integral equations for cracks in both isotropic and anisotropic, linear elastic infinite and finite media. However, the similar nontrivial extension to treat material anisotropy for cracks in an elastic half-space has not been recognized in the literature. In addition, the implementation of those weakly singular boundary integral equations to solve cracked half-spaces for essential fracture data such as stress intensity factors and T-stress components has also not been found.

Various regularization techniques have been continuously proposed to construct singularity-reduced boundary integral equations with the primary objective to reduce the smoothness requirement in solution discretization. Within the context of the stress analysis of cracked elastic media, a set of singularity-reduced boundary integral equations has been recognized for past several decades and some of those relevant investigations are briefly presented here to indicate the recent advances in the area. Bui (1977) and Weaver (1977) simultaneously developed the singularity-reduced traction boundary integral equation for mode-I planar cracks embedded in an infinite elastic medium. In their work, all hypersingular integrals were regularized with the final integrands containing only Cauchy-singular kernels. Later, Sládek and Sládek (1982) generalized the work of Bui (1977) and Weaver (1977) to treat non-planar cracks under general mixed-mode loading conditions. The key component used to establish the singularity-reduced traction integral equation indicated above was the derivative transferring between the hypersingular kernels and the crack-face data via the integration by parts. It is worth noting that the obtained strongly singular traction boundary integral equation still require the continuity of the relative crack-face displacement for its validity and, in the numerical implementation perspective, the computational cost associated with the use of C^1 -interpolation functions is still significant (see also the work of Bonnet and Bui, 1993).

In past several decades, efforts have been invested to develop the completely regularized boundary integral equations containing only weakly singular kernels instead of ones involving Cauchy singular integrals. This is due mainly to the simplicity associated with the interpretation and numerical evaluation of singular integrals and the low smoothness requirement of data. Within the context of linear elasticity applications, the work of Gu and Yew (1988) was recognized as the first development of weakly singular boundary integral equations for crack problems. However, their formulation was still restricted to the case of isolated, planar cracks in an isotropic elastic whole space under the pure mode-I loading condition. Later Xu and Ortiz (1993) extended the work of Gu and Yew (1988) to treat non-planar cracks and mixed-mode loading conditions. In their work, the relative crack-face displacements were represented as a continuous distribution of dislocation loops.

To allow the treatment of cracks of arbitrary shapes in an isotropic elastic half-space under various types of boundary conditions on the free surface, Li (1996) proposed a systematic regularization technique based on the special representations of singular kernels and the integration by parts via Stokes' theorem. It is remarked that the formulation was limited to isotropic materials and the implementation of those integral equations to solve crack problems was not considered. Li and Mear (1998), and Li *et al.* (1998) generalized the work of Li (1996) to the case of arbitrary cracks in isotropic, linearly elastic, unbounded and finite media. The implementation of those equations based on the weakly singular SGBEM was also successfully established to solve crack problems under various scenarios. Later, Xu (2000) employed the dislocation theory to establish the weak singular traction boundary integral equation for cracks in a medium made from generally anisotropic materials. While his formulation is applicable to treat arbitrary-shaped cracks and general loading conditions, it was still restricted to cracks in the whole space. The treatment of general material anisotropy in the formulation of boundary integral equations and the implementation of numerical procedure via the weakly singular SGBEM was also investigated by Rungamornrat and Mear (2008a) and Rungamornrat and Mear (2008b) for cracks in both unbounded and finite media. While various regularization procedures have been proposed in the literature, on the basis of an extensive literature survey, work within the context of cracks in an anisotropic, linear elastic half-space has not been recognized. In addition, the development of weakly singular SGBEM for the special case of cracks in an isotropic, linearly elastic half-space under various conditions on the free surface is also not available in the literature.

In the present study, the systematic regularization procedure proposed by Li and Mear (1998); Li *et al.* (1998); Rungamornrat and Mear (2008a); Rungamornrat and Mear (2008b); Rungamornrat and Senjuntichai (2009) is further extended to derive a set of regularized integral relations for a half-space containing cracks and made of anisotropic materials possessing a plane of material symmetry parallel to the free surface. In addition, the implementation of the derived equations in the context of weakly singular SGBEM and standard Galerkin technique is also established. The proposed work is carried out within a general framework allowing the treatment of arbitrary shaped cracks and half-spaces under various conditions on the free surface and the determination of both stress intensity factors and the T-stresses along the crack front.

5.2 Problem Formulation

This section summarizes the clear problem description, basic field equations from a classical theory of linear elasticity, fundamental solutions of the displacement and stress within an un-

cracked elastic half-space under symmetrical and anti-symmetrical boundary conditions, standard integral relations for both stress and displacement within an elastic half-space containing cracks, special decompositions of strongly singular and hyper-singular functions, and the development of completely regularized boundary integral equations essential for analysis of cracks within the half-space.

5.2.1 Problem description

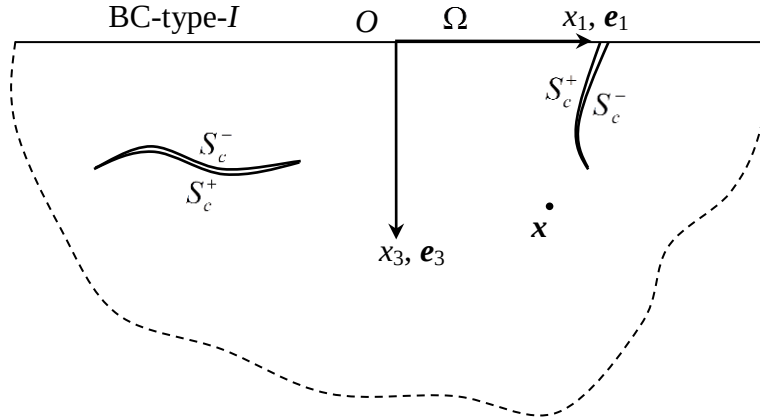


Figure 5.1: Schematic of elastic half-space containing isolated cracks and subjected to BC-type-*I* boundary conditions on the free surface.

Let us consider an elastic half-space Ω which contains an embedded crack and a surface breaking crack as shown schematically in Figure 3.1. For convenience in further development, a reference Cartesian coordinate system $\{O; x_1, x_2, x_3\}$ with the orthonormal base vectors $\{e_1, e_2, e_3\}$ is chosen such that the origin O is located on the free surface; the x_3 -axis directs downward; and the x_1 - and x_2 -axes follow the right hand rule. The body is made of a homogeneous, anisotropic, linearly elastic material with the elastic constants E_{ijkl} referring to the selected coordinate system and possessing $x_3 = 0$ as the plane of material symmetry. The crack surfaces in the reference or undeformed state can be represented by two geometrically identical surfaces S_c^+ and S_c^- (see Figure 5.1). The outward unit normal vectors at a smooth point $\xi^+ \in S_c^+$ and its coincident point $\xi^- \in S_c^-$ are denoted, respectively, by n^+ and n^- and they clearly satisfy $n^+ = -n^-$. On both crack surfaces S_c^+ and S_c^- , the traction data is fully prescribed (i.e., $t(x) = t^{0+}(x) \forall x \in S_c^+$ and $t(x) = t^{0-}(x) \forall x \in S_c^-$ where t^{0+} and t^{0-} are known functions) whereas, on the surface $x_3 = 0$, one of the following four types of boundary conditions, termed BC-type-*I* for $I = 1, 2, 3$ and 4 , is assumed. For the BC-type-1, the normal component of the displacement and the shear tractions vanish identically (i.e., $u_3 = 0, \sigma_{13} = \sigma_{23} = 0$); for the BC-type-2, the tangential components of the displacement and the normal traction vanish (i.e., $u_1 = u_2 = 0, \sigma_{33} = 0$); for the BC-type-3, all components of the traction vanish (i.e.,

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$); and for the BC-type-4, all components of the displacement vanish (i.e., $u_1 = u_2 = u_3 = 0$). In the present study, the crack surfaces S_c^+ and S_c^- are assumed to be sufficiently smooth (i.e., the unit normal vector is well defined everywhere) and the body is free of the body force and remote loading.

The statement of the current research problem is to determine the complete elastic field (including the displacement and stress fields) of the cracked elastic half-space Ω and essential fracture data such as the relative crack-face displacement, the stress intensity factors and T-stress components along the crack front.

5.2.2 Basic field equations

Mechanical responses of the given half-space Ω (including the displacement, stress, and strain fields) under external excitation are mathematically modeled by a classical theory of linear elasticity and the concept of linear elastic fracture mechanics. All involved field quantities and associated field equations are introduced and listed below.

From the conservation of linear and angular momentum and in the absence of the body force field, the stress tensor σ (with its components referring to the reference Cartesian coordinate system $\{\mathbf{O}; x_1, x_2, x_3\}$ denoted by σ_{ij}) must be symmetric (i.e., $\sigma_{ij} = \sigma_{ji}$) and divergence free everywhere, i.e.,

$$\sigma_{ij,j} = 0 \quad (5.1)$$

where $f_{,j}$ denotes the partial derivative of a function f with respect to the coordinate x_j . Here and in what follows, all standard Latin indices take the values 1, 2 and 3 and repeated indices imply the summation over their range. For a body made of a linearly elastic material, the stress tensor σ can be related to the strain tensor ε (with its components denoted by ε_{ij}) via the generalized Hooke's law, i.e.,

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl} \quad (5.2)$$

where E_{ijkl} are prescribed constant elastic moduli satisfying following symmetries: $E_{ijkl} = E_{ijlk} = E_{jikl} = E_{klij}$. For the special case of isotropic materials, the elastic moduli involve only two independent material parameters and take a simple form

$$E_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl} \quad (5.3)$$

where μ and λ are Lamé's constants and δ_{ij} is a standard Kronecker-delta symbol. The strain tensor ε and the displacement vector $\mathbf{u}$ are related by the linearized kinematics

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (5.4)$$

where u_i denotes the components of the displacement vector $\mathbf{u}$. In addition, the traction vector $\mathbf{t}$ at any point on a sufficiently smooth surface can be related to the stress tensor at the same point by

$$t_i = \sigma_{ij}n_j \quad (5.5)$$

where t_i and n_i are components of the traction vector $\mathbf{t}$ and a unit normal vector $\mathbf{n}$, respectively. The relations (5.1), (5.2) and (5.4) form the basic field equations governing all unknown elastic fields $\mathbf{u} = \mathbf{u}(\mathbf{x})$, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{x})$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$.

5.2.3 Fundamental solutions of half-space under BC-type-1 and BC-type-2

To construct fundamental solutions for the stress and displacement of an un-cracked elastic half-space subjected to boundary conditions of BC-type-1 and BC-type-2, the existing fundamental solutions of an un-cracked elastic whole space (e.g., Ting and Lee, 1997; Wang, 1997) can be used along with the symmetrical and anti-symmetrical conditions as indicated below. It should be remarked that results presented here are restricted only to the case that the material constituting the half-space is homogeneous and linearly elastic, and possesses $x_3 = 0$ as a plane of material symmetry.

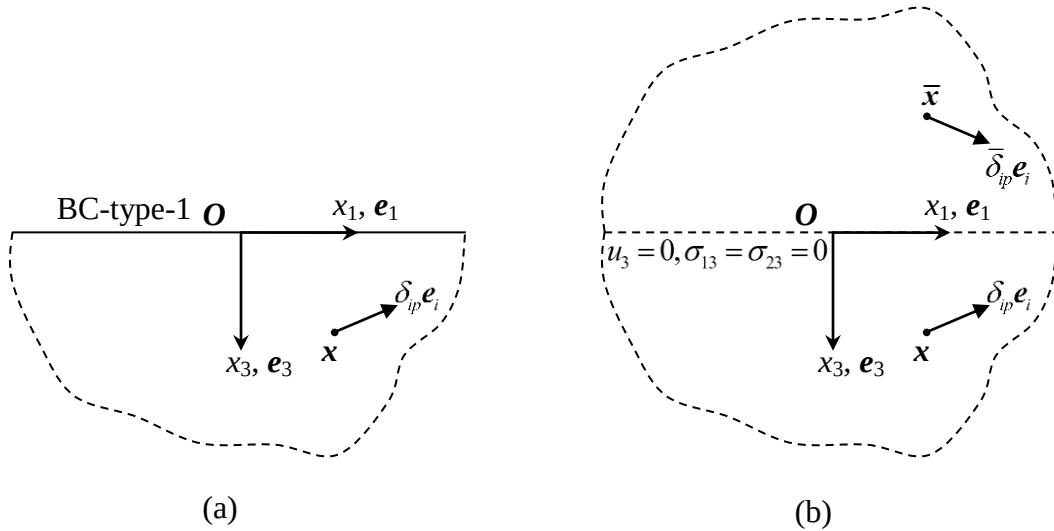


Figure 5.2: Schematic of (a) un-cracked elastic half-space with boundary condition of BC-type-1 and subjected to unit concentrated force at point $\mathbf{x}$ and (b) un-cracked elastic whole space subjected to a pair of symmetrical unit concentrated forces at a point $\mathbf{x}$ and its image point $\bar{\mathbf{x}}$ with respect to the surface $x_3 = 0$.

First, consider the un-cracked elastic half-space with the boundary condition of BC-type-1 and subjected to a unit concentrated force $\delta_{ip}\mathbf{e}_i$ at a source point $\mathbf{x} = x_k\mathbf{e}_k$ as illustrated in Figure 5.2(a). It can readily be verified that an elastic field of this elastic half-space is identical to that of the bottom half of an elastic whole space subjected to a unit concentrated force $\delta_{ip}\mathbf{e}_i$

acting at a point $\mathbf{x} = x_k \mathbf{e}_k$ and a unit concentrated force $\bar{\delta}_{ip} \mathbf{e}_i$ (where $\bar{\delta}_{ip}$ is a constant tensor defined by $\bar{\delta}_{11} = \bar{\delta}_{22} = -\bar{\delta}_{33} = 1$, $\bar{\delta}_{ij} = 0$ for $i \neq j$) acting at a point $\bar{\mathbf{x}} = \bar{x}_k \mathbf{e}_k$ with $\bar{x}_k = \bar{\delta}_{kp} x_p$ as shown in Figure 5.2(b). Note, in particular, that the loading condition of the elastic whole space is clearly symmetrical with respect to the plane $x_3 = 0$ and, as a result, the conditions $u_3 = 0, \sigma_{13} = \sigma_{23} = 0$ are automatically satisfied. Upon using the symmetry, the displacement and stress at any field point $\boldsymbol{\xi}$ within the half-space, denoted respectively by $U_j^{1p}(\boldsymbol{\xi}, \mathbf{x})$ and $S_{ij}^{1p}(\boldsymbol{\xi}, \mathbf{x})$, can therefore be obtained as

$$U_j^{1p}(\boldsymbol{\xi}, \mathbf{x}) = U_j^p(\boldsymbol{\xi} - \mathbf{x}) + \bar{\delta}_{pk} U_j^k(\boldsymbol{\xi} - \bar{\mathbf{x}}) \quad (5.6)$$

$$S_{ij}^{1p}(\boldsymbol{\xi}, \mathbf{x}) = S_{ij}^p(\boldsymbol{\xi} - \mathbf{x}) + \bar{\delta}_{pk} S_{ij}^k(\boldsymbol{\xi} - \bar{\mathbf{x}}) \quad (5.7)$$

where the superscript “1” is used to emphasize that the fundamental solutions are associated with the boundary condition of BC-type-1 and $U_j^p(\boldsymbol{\xi} - \mathbf{x})$ and $S_{ij}^p(\boldsymbol{\xi} - \mathbf{x})$ denote the displacement and stress at any point $\boldsymbol{\xi}$ of the elastic whole space subjected to a unit concentrated force $\delta_{ip} \mathbf{e}_i$ at a point $\mathbf{x} = x_k \mathbf{e}_k$. The explicit form of the displacement and stress fundamental solutions $U_j^p(\boldsymbol{\xi} - \mathbf{x})$ and $S_{ij}^p(\boldsymbol{\xi} - \mathbf{x})$ for generally anisotropic materials is given by (see details in the work of Rungamornrat and Mear, 2008a)

$$U_j^p(\boldsymbol{\xi} - \mathbf{x}) = \frac{1}{8\pi^2 r} \oint_{\mathbf{z} \cdot \mathbf{r} = 0} (\mathbf{z}, \mathbf{z})_{jp}^{-1} ds(\mathbf{z}) \quad (5.8)$$

$$S_{ij}^p(\boldsymbol{\xi} - \mathbf{x}) = E_{ijkl} \frac{\partial U_k^p(\boldsymbol{\xi} - \mathbf{x})}{\partial \xi_l} \quad (5.9)$$

where $\mathbf{r} = \boldsymbol{\xi} - \mathbf{x}$, $r = \|\mathbf{r}\|$, $(\mathbf{z}, \mathbf{z})_{kl} = z_i E_{iklj} z_j$ and the closed contour integral is to be evaluated over a unit circle $\|\mathbf{z}\| = 1$ on a plane defined by $\mathbf{z} \cdot \mathbf{r} = 0$. It is evident from (5.6)-(5.7) and the properties of the fundamental solutions $U_j^p(\boldsymbol{\xi} - \mathbf{x})$ and $S_{ij}^p(\boldsymbol{\xi} - \mathbf{x})$ that $U_j^{1p}(\boldsymbol{\xi}, \mathbf{x})$ and $S_{ij}^{1p}(\boldsymbol{\xi}, \mathbf{x})$ are singular only at $\boldsymbol{\xi} = \mathbf{x}$ of order $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r^2)$, respectively. Explicit form of the fundamental solutions for the isotropic case can be found in available literatures (e.g., Xiao, 1998).

Let us consider, next, the elastic half-space with the boundary condition of BC-type-2 and subjected to a unit concentrated force $\delta_{ip} \mathbf{e}_i$ at a source point $\mathbf{x} = x_k \mathbf{e}_k$ as shown in Figure 5.3(a). With the same analogy as the previous case, the elastic solution of the half-space is identical to that of the bottom half of an elastic whole space subjected to a unit concentrated force $\delta_{ip} \mathbf{e}_i$ acting at a point $\mathbf{x} = x_k \mathbf{e}_k$ and a unit concentrated force $-\bar{\delta}_{ip} \mathbf{e}_i$ acting at the image point $\bar{\mathbf{x}} = \bar{x}_k \mathbf{e}_k$ as indicated in Figure 5.3(b). For this particular case, the elastic whole space apparently subjected to anti-symmetrical loading conditions with respect to the plane $x_3 = 0$. As a

result, the displacement and stress at any field point ξ of the half-space, denoted respectively by $U_j^{2p}(\xi, \mathbf{x})$ and $S_{ij}^{2p}(\xi, \mathbf{x})$, are given, in terms of $U_j^p(\xi - \mathbf{x})$ and $S_{ij}^p(\xi - \mathbf{x})$, by

$$U_j^{2p}(\xi, \mathbf{x}) = U_j^p(\xi - \mathbf{x}) - \bar{\delta}_{pk} U_j^k(\xi - \bar{\mathbf{x}}) \quad (5.10)$$

$$S_{ij}^{2p}(\xi, \mathbf{x}) = S_{ij}^p(\xi - \mathbf{x}) - \bar{\delta}_{pk} S_{ij}^k(\xi - \bar{\mathbf{x}}) \quad (5.11)$$

where the superscript “2” is introduced to clearly indicate that the fundamental solutions belong to the half-space with the boundary condition of BC-type-2. Similar to the previous case, the singularity behavior of the fundamental solutions $U_j^{2p}(\xi, \mathbf{x})$ and $S_{ij}^{2p}(\xi, \mathbf{x})$ follows directly that of $U_j^p(\xi - \mathbf{x})$ and $S_{ij}^p(\xi - \mathbf{x})$.

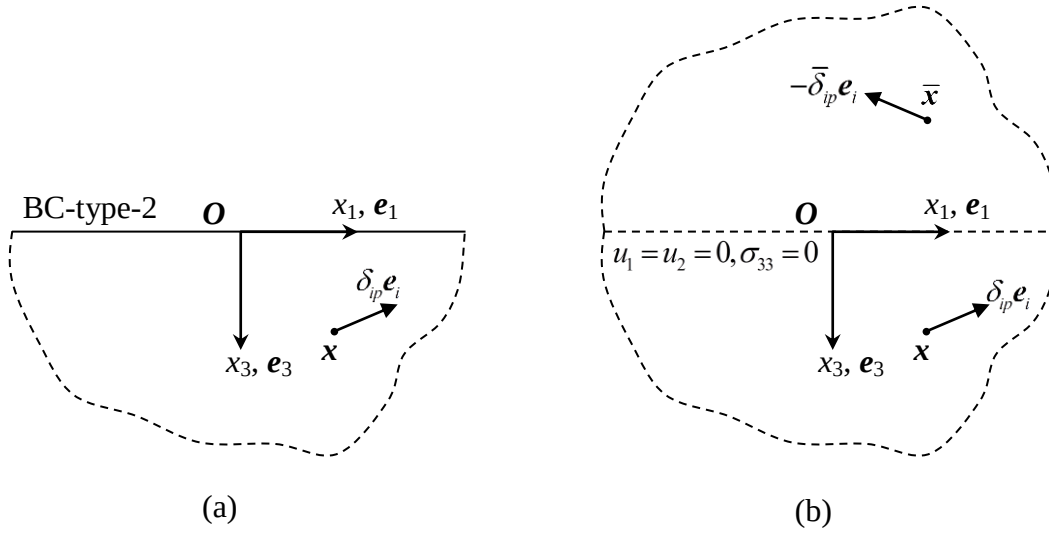


Figure 5.3: Schematic of (a) un-cracked elastic half-space with boundary condition of BC-type-2 and subjected to unit concentrated force at point $\mathbf{x}$ and (b) un-cracked elastic whole space subjected to a pair of anti-symmetrical unit concentrated forces at a point $\mathbf{x}$ and its image point $\bar{\mathbf{x}}$ with respect to the surface $x_3 = 0$.

5.2.4 Standard integral relations for cracked half-space under BC-type-1 and BC-type-2

By applying the reciprocal theorem to an elastic half-space Ω containing cracks and subjected to the boundary condition of BC-type- α ($\alpha = 1$ or 2) as shown in Figure 5.1 along with another elastic state associated with a fundamental problem of the half-space with the same boundary condition of BC-type- α (constructed in section 5.2.3), it leads to a boundary integral relation for the displacement at any interior point $\mathbf{x}$ of the cracked half-space with the boundary condition of BC-type- α :

$$u_p^\alpha(\mathbf{x}) = \int_{S_c^+} U_j^{\alpha p}(\xi, \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) - \int_{S_c^+} S_{ij}^{\alpha p}(\xi, \mathbf{x}) n_i^+(\xi) \Delta u_j^\alpha(\xi) dA(\xi) \quad (5.12)$$

where $\Sigma t_j^0(\xi) = t_j^{0+}(\xi) + t_j^{0-}(\xi)$ denotes the sum of tractions on both crack surfaces; $\Delta u_j^\alpha(\xi) = u_j^{\alpha+}(\xi) - u_j^{\alpha-}(\xi)$ denotes the jump in the crack-face displacements; and the fundamental solutions $U_j^{\alpha p}(\xi, \mathbf{x})$ and $S_{ij}^{\alpha p}(\xi, \mathbf{x})$ are expressed in terms of $U_j^p(\xi - \mathbf{x})$ and $S_{ij}^p(\xi - \mathbf{x})$ in a concise form as

$$U_j^{\alpha p}(\xi, \mathbf{x}) = \Gamma_\alpha U_j^p(\xi - \mathbf{x}) + \Lambda_\alpha \bar{\delta}_{pk} U_j^k(\xi - \bar{\mathbf{x}}) \quad (5.13)$$

$$S_{ij}^{\alpha p}(\xi, \mathbf{x}) = \Gamma_\alpha S_{ij}^p(\xi - \mathbf{x}) + \Lambda_\alpha \bar{\delta}_{pk} S_{ij}^k(\xi - \bar{\mathbf{x}}) \quad (5.14)$$

where Γ_α and Λ_α are constants defined by $\Gamma_1 = \Gamma_2 = 1$, $\Lambda_1 = 1$ and $\Lambda_2 = -1$. Here and in what follows, the Greek index α takes the values 1 and 2 and the summation does not apply to the repeated α . It is important to remark that the reduction to integrals over a single crack surface S_c^+ results directly from the continuity of the fundamental solutions $U_j^{\alpha p}(\xi, \mathbf{x})$ and $S_{ij}^{\alpha p}(\xi, \mathbf{x})$.

By taking spatial derivative of (5.12) along with the relation (5.4) to obtain the strain tensor and then employing the constitutive law (5.2), it yields a boundary integral relation for the stress at any interior point $\mathbf{x}$ of the cracked elastic half-space under the boundary condition of BC-type- α :

$$\sigma_{lk}^\alpha(\mathbf{x}) = \int_{S_c^+} E_{lkpq} \frac{\partial U_j^{\alpha p}(\xi, \mathbf{x})}{\partial x_q} \Sigma t_j^0(\xi) dA(\xi) - \int_{S_c^+} E_{lkpq} \frac{\partial S_{ij}^{\alpha p}(\xi, \mathbf{x})}{\partial x_q} n_i^+(\xi) \Delta u_j^\alpha(\xi) dA(\xi) \quad (5.15)$$

From the relations (5.13) along with (5.9), the kernel $E_{lkpq} \partial U_j^{\alpha p}(\xi, \mathbf{x}) / \partial x_q$ can be obtained explicitly in terms of the stress fundamental solution $S_{ij}^p(\xi - \mathbf{x})$ for the un-cracked elastic whole space as follows:

$$\begin{aligned} E_{lkpq} \frac{\partial U_j^{\alpha p}(\xi, \mathbf{x})}{\partial x_q} &= \Gamma_\alpha E_{lkpq} \frac{\partial U_j^p(\xi - \mathbf{x})}{\partial x_q} + \Lambda_\alpha \bar{\delta}_{pm} E_{lkpq} \frac{\partial U_j^m(\xi - \bar{\mathbf{x}})}{\partial x_q} \\ &= -\Gamma_\alpha E_{lkpq} \frac{\partial U_p^j(\xi - \mathbf{x})}{\partial \xi_q} + \Lambda_\alpha \bar{\delta}_{pm} \bar{\delta}_{ma} \bar{\delta}_{jb} E_{lkpq} \frac{\partial U_b^a(\bar{\xi} - \mathbf{x})}{\partial x_q} \\ &= -\Gamma_\alpha E_{lkpq} \frac{\partial U_p^j(\xi - \mathbf{x})}{\partial \xi_q} + \Lambda_\alpha \bar{\delta}_{jb} E_{lkpq} \frac{\partial U_p^b(\mathbf{x} - \bar{\xi})}{\partial x_q} \\ &= -\Gamma_\alpha S_{lk}^j(\xi - \mathbf{x}) + \Lambda_\alpha \bar{\delta}_{jb} S_{lk}^b(\mathbf{x} - \bar{\xi}) \end{aligned} \quad (5.16)$$

where the following crucial properties have been used:

$$U_j^p(\xi - \mathbf{x}) = U_p^j(\xi - \mathbf{x}) \quad (5.17)$$

$$U_j^m(\xi - \bar{\mathbf{x}}) = \bar{\delta}_{ma} \bar{\delta}_{jb} U_b^a(\bar{\xi} - \mathbf{x}) \quad (5.18)$$

$$\partial U_p^j(\xi - \mathbf{x}) / \partial x_q = -\partial U_p^j(\xi - \mathbf{x}) / \partial \xi_q \quad (5.19)$$

$$\bar{\delta}_{ia}\bar{\delta}_{aj} = \delta_{ij} \quad (5.20)$$

It should be evident from (5.16) that the kernel $E_{lkpq}\partial U_j^{\alpha p}(\xi, \mathbf{x})/\partial x_q$ is singular only at $\xi = \mathbf{x}$ of order $\mathcal{O}(1/r^2)$. From the relation (5.14), the kernel $E_{lkpq}\partial S_{ij}^{\alpha p}(\xi, \mathbf{x})/\partial x_q$ in (3.15) can be further simplified, in a similar fashion, to

$$\begin{aligned} E_{lkpq} \frac{\partial S_{ij}^{\alpha p}(\xi, \mathbf{x})}{\partial x_q} &= \Gamma_\alpha E_{lkpq} \frac{\partial S_{ij}^p(\xi - \mathbf{x})}{\partial x_q} + \Lambda_\alpha \bar{\delta}_{pm} E_{lkpq} \frac{\partial S_{ij}^m(\xi - \bar{\mathbf{x}})}{\partial x_q} \\ &= -\Gamma_\alpha E_{lkpq} \frac{\partial S_{ij}^p(\xi - \mathbf{x})}{\partial \xi_q} + \Lambda_\alpha \bar{\delta}_{pm} \bar{\delta}_{ia} \bar{\delta}_{jb} \bar{\delta}_{mc} E_{lkpq} \frac{\partial S_{ab}^c(\bar{\xi} - \mathbf{x})}{\partial x_q} \\ &= -\Gamma_\alpha E_{lkpq} \frac{\partial S_{ij}^p(\xi - \mathbf{x})}{\partial \xi_q} - \Lambda_\alpha \bar{\delta}_{ia} \bar{\delta}_{jb} E_{lkpq} \frac{\partial S_{ab}^p(\mathbf{x} - \bar{\xi})}{\partial x_q} \\ &= -\Gamma_\alpha \Sigma_{ij}^{lk}(\xi - \mathbf{x}) - \Lambda_\alpha \bar{\delta}_{ia} \bar{\delta}_{jb} \Sigma_{ab}^{lk}(\mathbf{x} - \bar{\xi}) \end{aligned} \quad (5.21)$$

where the kernel $\Sigma_{ij}^{lk}(\xi - \mathbf{x})$ is defined in terms of the stress fundamental solution $S_{ij}^p(\xi - \mathbf{x})$ of the un-cracked elastic whole space by

$$\Sigma_{ij}^{lk}(\xi - \mathbf{x}) = E_{lkpq} \frac{\partial S_{ij}^p(\xi - \mathbf{x})}{\partial \xi_q} \quad (5.22)$$

In addition, to obtain the final expression (5.21), the relation (5.20) and the following properties have also been employed:

$$\partial S_{ij}^p(\xi - \mathbf{x}) / \partial x_q = -\partial S_{ij}^p(\xi - \mathbf{x}) / \partial \xi_q \quad (5.23)$$

$$S_{ij}^m(\xi - \bar{\mathbf{x}}) = \bar{\delta}_{ia} \bar{\delta}_{jb} \bar{\delta}_{mc} S_{ab}^c(\bar{\xi} - \mathbf{x}) \quad (5.24)$$

$$S_{ab}^c(\bar{\xi} - \mathbf{x}) = -S_{ab}^c(\mathbf{x} - \bar{\xi}) \quad (5.25)$$

From (5.21), (5.22) and the singularity behavior of the fundamental solution $S_{ij}^p(\xi - \mathbf{x})$, the kernel $E_{lkpq}\partial S_{ij}^{\alpha p}(\xi, \mathbf{x})/\partial x_q$ is singular only at $\xi = \mathbf{x}$ of order $\mathcal{O}(1/r^3)$. It is worth noting that the boundary integral relations (5.12) and (5.15) can be used to determine the displacement and stress at any interior point $\mathbf{x}$ provided that all unknown data of the displacement and traction on the crack surface is completely known. By taking an appropriate limit process of the boundary integral relation (5.12) to any point on the crack surface, it can yield the standard boundary integral equation for the sum of the crack-face displacement. However, such the displacement integral equation is mathematically degenerate and not sufficient for solving the primary unknowns on the crack surface. This is due to the geometrical coincidence of the two crack surfaces S_c^+ and S_c^- that renders the final displacement boundary integral equation containing incomplete information of the traction on the crack surface (i.e., it contains only the sum of the crack-face tractions). To overcome this mathematical difficulty, the boundary integral equation

for the jump in the traction across the crack surfaces is additionally required in the formulation of the governing equation. Such the integral equation can readily be obtained from the boundary integral relation of the stress (5.15) by taking the appropriate limit to any point on the crack surface S_c^+ . While the conventional traction boundary integral equation is sufficient for performing the stress analysis of cracked bodies, it still contains both the strongly singular kernels $S_{ij}^{\alpha p}(\xi, \mathbf{x})$ and $E_{lkpq} \partial U_j^{\alpha p}(\xi, \mathbf{x}) / \partial x_q$ of order $\mathcal{O}(1/r^2)$ and the hyper-singular kernel $E_{lkpq} \partial S_{ij}^{\alpha p}(\xi, \mathbf{x}) / \partial x_q$ of order $\mathcal{O}(1/r^3)$ that render the involved integrals difficult to be interpreted theoretically and treated numerically.

5.2.5 Decompositions of strongly singular and hypersingular kernels

To aid the regularization procedure of the boundary integral relations (5.12) and (5.15), the strongly singular kernels $S_{ij}^{\alpha p}(\xi, \mathbf{x})$ and $E_{lkpq} \partial U_j^{\alpha p}(\xi, \mathbf{x}) / \partial x_q$ and the hypersingular kernel $E_{lkpq} \partial S_{ij}^{\alpha p}(\xi, \mathbf{x}) / \partial x_q$ are, first, decomposed into a form suiting the integration by parts via Stokes' theorem described in the next section. The key component to achieve such crucial task is based on the special decompositions of the strongly singular kernel $S_{ij}^p(\xi - \mathbf{x})$ and the hypersingular kernel $\Sigma_{ij}^{lk}(\xi - \mathbf{x})$ associated with the un-cracked elastic whole space proposed by Rungamornrat and Mear (2008a):

$$S_{ij}^p(\xi - \mathbf{x}) = H_{ij}^p(\xi - \mathbf{x}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} G_{mj}^p(\xi - \mathbf{x}) \quad (5.26)$$

$$\Sigma_{ij}^{lk}(\xi - \mathbf{x}) = -E_{ijkl} \delta(\xi - \mathbf{x}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} \varepsilon_{lrt} \frac{\partial}{\partial \xi_r} C_{mj}^{tk}(\xi - \mathbf{x}) \quad (5.27)$$

where ε_{ism} is a standard alternating symbol, $\delta(\xi - \mathbf{x})$ is a Dirac-delta distribution centered at a point $\mathbf{x}$, and the functions $H_{ij}^p(\xi - \mathbf{x})$, $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{tk}(\xi - \mathbf{x})$ are defined by

$$H_{ij}^p(\xi - \mathbf{x}) = -\delta_{jp} \frac{\xi_i - x_i}{4\pi r^3} \quad (5.28)$$

$$G_{mj}^p(\xi - \mathbf{x}) = \frac{\varepsilon_{mqa} E_{qjkl}}{8\pi^2 r} \oint_{\mathbf{z} \cdot \mathbf{r}=0} (\mathbf{z}, \mathbf{z})_{kp}^{-1} z_a z_l ds(\mathbf{z}) \quad (5.29)$$

$$C_{mj}^{tk}(\xi - \mathbf{x}) = \frac{A_{mtsl}^{kjap}}{8\pi^2 r} \oint_{\mathbf{z} \cdot \mathbf{r}=0} (\mathbf{z}, \mathbf{z})_{ap}^{-1} z_s z_l ds(\mathbf{z}), \quad A_{mtsl}^{kjap} = \varepsilon_{ptd} \varepsilon_{pmq} \left\{ E_{djpl} E_{qkms} - \frac{1}{3} E_{lmps} E_{djks} \right\} \quad (5.30)$$

For the special case of isotropic materials, the kernels $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{tk}(\xi - \mathbf{x})$ are provided in a closed form (see the work of Li, 1996)

$$G_{mj}^p(\xi - \mathbf{x}) = \frac{1}{8\pi(1-\nu)r} \left[(1-2\nu) \varepsilon_{mpj} + \varepsilon_{ijm} \frac{(\xi_i - x_i)(\xi_p - x_p)}{r^2} \right] \quad (5.31)$$

$$C_{mj}^{tk}(\xi - \mathbf{x}) = \frac{\mu}{4\pi(1-\nu)r} \left[(1-\nu)\delta_{tk}\delta_{mj} + 2\nu\delta_{ij}\delta_{km} - \delta_{tm}\delta_{kj} - \delta_{tm} \frac{(\xi_k - x_k)(\xi_j - x_j)}{r^2} \right] \quad (5.32)$$

Note, in addition, that the function $H_{ij}^p(\xi - \mathbf{x})$ is independent of material constants and singular only at $\xi = \mathbf{x}$ of order $\mathcal{O}(1/r^2)$ whereas $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{tk}(\xi - \mathbf{x})$ depends primarily on material properties and are singular only at $\xi = \mathbf{x}$ of $\mathcal{O}(1/r)$.

By using the decomposition (5.26) along with the relation (5.14), it leads to the decomposition of the strongly singular kernel $S_{ij}^{\alpha p}(\xi, \mathbf{x})$:

$$S_{ij}^{\alpha p}(\xi, \mathbf{x}) = H_{ij}^{\alpha p}(\xi, \mathbf{x}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} G_{mj}^{\alpha p}(\xi, \mathbf{x}) \quad (5.33)$$

where the functions $H_{ij}^{\alpha p}(\xi, \mathbf{x})$ and $G_{mj}^{\alpha p}(\xi, \mathbf{x})$ are defined by

$$H_{ij}^{\alpha p}(\xi, \mathbf{x}) = \Gamma_\alpha H_{ij}^p(\xi - \mathbf{x}) + \Lambda_\alpha \bar{\delta}_{pk} H_{ij}^k(\xi - \bar{\mathbf{x}}) \quad (5.34)$$

$$G_{mj}^{\alpha p}(\xi, \mathbf{x}) = \Gamma_\alpha G_{mj}^p(\xi - \mathbf{x}) + \Lambda_\alpha \bar{\delta}_{pk} G_{mj}^k(\xi - \bar{\mathbf{x}}) \quad (5.35)$$

Similarly, by using again the decomposition (5.26) along with the expression (5.16), it results in

$$E_{lkpq} \frac{\partial U_j^{\alpha p}(\xi, \mathbf{x})}{\partial x_q} = H_{lk}^{\alpha j}(\mathbf{x}, \xi) + \varepsilon_{lrt} \frac{\partial}{\partial x_r} G_{tk}^{\alpha j}(\mathbf{x}, \xi) \quad (5.36)$$

where the fact that $H_{lk}^j(\xi - \mathbf{x})$ and $G_{mj}^p(\xi - \mathbf{x})$ are odd and even functions, respectively, has been utilized. Finally, by applying the decomposition (5.27) along with the relation (5.21), it yields the decomposition of the hypersingular kernel $E_{lkpq} \partial S_{ij}^{\alpha p}(\xi, \mathbf{x}) / \partial x_q$:

$$E_{lkpq} \frac{\partial S_{ij}^{\alpha p}(\xi, \mathbf{x})}{\partial x_q} = \Gamma_\alpha E_{ijkl} \delta(\xi - \mathbf{x}) - \Lambda_\alpha \bar{\delta}_{ia} \bar{\delta}_{jb} E_{abkl} \delta(\mathbf{x} - \bar{\xi}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} \varepsilon_{lrt} \frac{\partial}{\partial x_r} C_{mj}^{\alpha tk}(\xi, \mathbf{x}) \quad (5.37)$$

where the kernel $C_{mj}^{\alpha tk}(\xi, \mathbf{x})$ is given by

$$C_{mj}^{\alpha tk}(\xi, \mathbf{x}) = \Gamma_\alpha C_{mj}^{tk}(\xi - \mathbf{x}) - \Lambda_\alpha \bar{\delta}_{am} \bar{\delta}_{bj} C_{ab}^{tk}(\mathbf{x} - \bar{\xi}) \quad (5.38)$$

and following important properties have been employed:

$$\partial C_{mj}^{tk}(\xi - \mathbf{x}) / \partial \xi_q = -\partial C_{mj}^{tk}(\xi - \mathbf{x}) / \partial x_q \quad (5.39)$$

$$\partial C_{mj}^{tk}(\mathbf{x} - \bar{\xi}) / \partial x_q = -\partial C_{mj}^{tk}(\mathbf{x} - \bar{\xi}) / \partial \bar{\xi}_q \quad (5.40)$$

$$\partial(\cdot) / \partial \bar{\xi}_q = \bar{\delta}_{mq} \partial(\cdot) / \partial \xi_q \quad (5.41)$$

$$\varepsilon_{ijk} = -\bar{\delta}_{ia} \bar{\delta}_{jb} \bar{\delta}_{kc} \varepsilon_{abc} \quad (5.42)$$

5.2.6 Weakly singular integral equation for BC-type-1 and BC-type-2

In this section, a pair of singularity-reduced boundary integral equations, one for the displacement and the other for the traction on the crack surface, is established. The special decompositions (5.33), (5.36) and (5.37) are employed along with Stokes' theorem to carry out the integration by parts of strongly singular and hypersingular integrals in order to shift the derivatives from involved kernels to the crack-face data. This regularization procedure is analogous to that used by Rungamornrat and Mear (2008a) and Rungamornrat and Senjuntichai (2009).

5.2.6.1 Weakly singular integral equation for displacement

To establish the singularity reduced boundary integral relation for the displacement at any interior point $\mathbf{x} \in \Omega$, the special decomposition (5.33) is first substituted into the integral relation (5.12) to obtain

$$u_p^\alpha(\mathbf{x}) = \int_{S_c^+} U_j^{\alpha p}(\xi, \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) - \int_{S_c^+} H_{ij}^{\alpha p}(\xi, \mathbf{x}) n_i^+(\xi) \Delta u_j^\alpha(\xi) dA(\xi) - \int_{S_c^+} \Delta u_j^\alpha(\xi) D_m G_{mj}^{\alpha p}(\xi, \mathbf{x}) dA(\xi) \quad (5.43)$$

where $D_m(\cdot)$ denotes a surface differential operator defined by

$$D_m(\cdot) = n_i \varepsilon_{ism} \frac{\partial(\cdot)}{\partial \xi_s} \quad (5.44)$$

Upon performing the integration by parts of a term containing curl of the function $G_{mj}^p(\xi - \mathbf{x})$ via Stokes' theorem and then using the fact that the relative crack-face displacement Δu_j^α identically vanishes along the crack front (i.e., $\Delta u_j^\alpha = 0$ on ∂S_c^+), it finally yields

$$u_p^\alpha(\mathbf{x}) = \int_{S_c^+} U_j^{\alpha p}(\xi, \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) - \int_{S_c^+} H_{ij}^{\alpha p}(\xi, \mathbf{x}) n_i^+(\xi) \Delta u_j^\alpha(\xi) dA(\xi) + \int_{S_c^+} G_{mj}^{\alpha p}(\xi, \mathbf{x}) D_m \Delta u_j^\alpha(\xi) dA(\xi) \quad (5.45)$$

By taking limit $\mathbf{x} \rightarrow \mathbf{y} \in S_c^+$ of (5.45), we then obtain the boundary integral equation for the sum of the crack-face displacement as

$$c(\mathbf{y}) \Sigma u_p^\alpha(\mathbf{y}) = \int_{S_c^+} U_j^{\alpha p}(\xi, \mathbf{y}) \Sigma t_j^0(\xi) dA(\xi) - \int_{S_c^+} H_{ij}^{\alpha p}(\xi, \mathbf{y}) n_i^+(\xi) \Delta u_j^\alpha(\xi) dA(\xi) + \int_{S_c^+} G_{mj}^{\alpha p}(\xi, \mathbf{y}) D_m \Delta u_j^\alpha(\xi) dA(\xi) \quad (5.46)$$

where c is a function defined such that $c(\mathbf{y})=1/2$ if the surface is sufficiently smooth at a point $\mathbf{y}$ (i.e., the unit normal vector $\mathbf{n}$ at $\mathbf{y}$ is well-defined), otherwise $c(\mathbf{y}) \in (0, 1)$ and $\Sigma u_p^\alpha(\mathbf{y}) = u_p^{\alpha+}(\mathbf{y}) + u_p^{\alpha-}(\mathbf{y})$ denotes the sum of the crack-face displacement. By multiplying (5.46) with a sufficiently smooth test function $\tilde{t}_p(\mathbf{y})$ and then integrating the result over the entire crack surface S_c^+ , it finally results in a weakly-singular, weak-form boundary integral equation for the displacement as

$$\begin{aligned} \frac{1}{2} \int_{S_c^+} \tilde{t}_p(\mathbf{y}) \Sigma u_p^\alpha(\mathbf{y}) dA(\mathbf{y}) &= \int_{S_c^+} \tilde{t}_p(\mathbf{y}) \int_{S_c^+} U_j^{\alpha p}(\xi, \mathbf{y}) \Sigma t_j^0(\xi) dA(\xi) dA(\mathbf{y}) \\ &+ \int_{S_c^+} \tilde{t}_p(\mathbf{y}) \int_{S_c^+} G_{mj}^{\alpha p}(\xi, \mathbf{y}) D_m \Delta u_j^\alpha(\xi) dA(\xi) dS(\mathbf{y}) \\ &- \int_{S_c^+} \tilde{t}_p(\mathbf{y}) \int_{S_c^+} H_{ij}^{\alpha p}(\xi, \mathbf{y}) n_i^+(\xi) \Delta u_j^\alpha(\xi) dA(\xi) dS(\mathbf{y}) \end{aligned} \quad (5.47)$$

Note that the function $c=c(\mathbf{y})$ simply reduces to $1/2$ since the crack surface is assumed piecewise smooth (i.e., a set of points $\mathbf{y}$ where the unit normal vector $\mathbf{n}$ is not well-defined is of measure zero). It is worth noting that the weak-form boundary integral equation (5.47) contains only weakly singular kernels $U_j^{\alpha p}(\xi, \mathbf{y})$, $G_{mj}^{\alpha p}(\xi, \mathbf{y})$ and $H_{ij}^{\alpha p}(\xi, \mathbf{y}) n_i^+(\xi)$ of $\mathcal{O}(1/r)$.

5.2.6.2 Weakly singular integral equation for traction

To obtain a weakly singular, weak-form boundary integral equation for the traction, a similar procedure as described in the previous subsection is employed. First, the special decompositions (5.36) and (5.37) are substituted into the boundary integral relation for the stress (5.15) to obtain

$$\begin{aligned} \sigma_{lk}^\alpha(\mathbf{x}) &= \int_{S_c^+} H_{lk}^{\alpha j}(\mathbf{x}, \xi) \Sigma t_j^0(\xi) dA(\xi) + \varepsilon_{lrt} \frac{\partial}{\partial x_r} \left\{ \int_{S_c^+} G_{tk}^{\alpha j}(\mathbf{x}, \xi) \Sigma t_j^0(\xi) dA(\xi) \right\} \\ &- \int_{S_c^+} D_m \left\{ \varepsilon_{lrt} \frac{\partial}{\partial x_r} C_{mj}^{\alpha tk}(\xi, \mathbf{x}) \right\} \Delta u_j^\alpha(\xi) dA(\xi) \end{aligned} \quad (5.48)$$

By performing the integration by parts of the third integral on the right hand side via Stokes' theorem, it gives rise to the singularity-reduced boundary integral relation for the stress at any interior point $\mathbf{x} \in \Omega$:

$$\begin{aligned} \sigma_{lk}^\alpha(\mathbf{x}) &= \varepsilon_{lrt} \frac{\partial}{\partial x_r} \left\{ \int_{S_c^+} G_{tk}^{\alpha j}(\mathbf{x}, \xi) \Sigma t_j^0(\xi) dA(\xi) + \int_{S_c^+} C_{mj}^{\alpha tk}(\xi, \mathbf{x}) D_m \Delta u_j^\alpha(\xi) dA(\xi) \right\} \\ &+ \int_{S_c^+} H_{lk}^{\alpha j}(\mathbf{x}, \xi) \Sigma t_j^0(\xi) dA(\xi) \end{aligned} \quad (5.49)$$

It should be remarked that the boundary integral relation for the stress (5.49) still contains strongly singular kernels of $\mathcal{O}(1/r^2)$. By further forming the product $n_l^+(\mathbf{y})\sigma_{lk}^\alpha(\mathbf{x})$ where $\mathbf{y} \in S_c^+$ and then taking appropriate limit $\mathbf{x} \rightarrow \mathbf{y}$, it leads to a boundary integral equation for the jump in the crack-face traction as

$$\begin{aligned} \rho(\mathbf{y})\Delta t_k^0(\mathbf{y}) = D_t \left\{ \int_{S_c^+} G_{tk}^{\alpha j}(\mathbf{y}, \boldsymbol{\xi}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) + \int_{S_c^+} C_{mj}^{\alpha tk}(\boldsymbol{\xi}, \mathbf{y}) D_m \Delta u_j^\alpha(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) \right\} \\ + n_l^+(\mathbf{y}) \int_{S_c^+} H_{lk}^{\alpha j}(\mathbf{y}, \boldsymbol{\xi}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) \end{aligned} \quad (5.50)$$

where ρ is a function defined such that $\rho(\mathbf{y}) = 1/2$ if the surface is sufficiently smooth at $\mathbf{y}$ otherwise $\rho(\mathbf{y}) \in (0, 1)$ and $\Delta t_k^0(\boldsymbol{\xi}) = t_k^{0+}(\boldsymbol{\xi}) - t_k^{0-}(\boldsymbol{\xi})$ denotes the jump in the traction across the crack surface. Upon multiplying the boundary integral equation (5.50) by a sufficiently smooth test function $\Delta \tilde{u}_k(\mathbf{y})$, forming the integration of the result over the crack surface S_c^+ , performing integration by parts via Stokes' theorem, and using the fact that the test function $\Delta \tilde{u}_k(\mathbf{y})$ satisfies the homogeneous condition along the crack front, it finally yields a weakly singular, weak-form integral equation for the jump in the crack-face traction:

$$\begin{aligned} \frac{1}{2} \int_{S_c^+} \Delta \tilde{u}_k(\mathbf{y}) \Delta t_k^0(\mathbf{y}) dA(\mathbf{y}) = \int_{S_c^+} \Delta \tilde{u}_k(\mathbf{y}) \int_{S_c^+} H_{lk}^{\alpha j}(\mathbf{y}, \boldsymbol{\xi}) n_l^+(\mathbf{y}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \\ - \int_{S_c^+} D_t \Delta \tilde{u}_k(\mathbf{y}) \int_{S_c^+} G_{tk}^{\alpha j}(\mathbf{y}, \boldsymbol{\xi}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \\ - \int_{S_c^+} D_t \Delta \tilde{u}_k(\mathbf{y}) \int_{S_c^+} C_{mj}^{\alpha tk}(\boldsymbol{\xi}, \mathbf{y}) D_m \Delta u_j^\alpha(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \end{aligned} \quad (5.51)$$

Again, from the assumption that the crack surface is piecewise smooth, the function $\rho(\mathbf{y})$ simply reduces to 1/2. It should be remarked that the boundary integral equation (5.51) contains only weakly singular kernels $H_{lk}^{\alpha j}(\mathbf{y}, \boldsymbol{\xi}) n_l^+(\mathbf{y})$, $G_{tk}^{\alpha j}(\mathbf{y}, \boldsymbol{\xi})$ and $C_{mj}^{\alpha tk}(\boldsymbol{\xi}, \mathbf{y})$ of $\mathcal{O}(1/r)$.

While the two equations (5.47) and (5.51) form a sufficient system of governing equations for determining the crack-face unknown data, the two relations (5.45) and (5.49) can be used to compute the displacement and stress at any interior point of the body once the primary unknowns on the crack surface are solved.

5.2.7 Treatment of cracked halfspace under BC-type-I (for $I = 3, 4$)

By applying the reciprocal theorem to the cracked elastic half-space Ω subjected to the boundary condition of BC-type-I ($I = 1, 2, 3, 4$) along with the elastic state associated with a fundamental problem of the un-cracked half-space under the same type of boundary conditions on the free surface, the standard integral relations for the displacement and stress at any interior point of the cracked half-space can be generalized to

$$u_p^I(\mathbf{x}) = \int_{S_c^+} U_j^{Ip}(\boldsymbol{\xi}, \mathbf{x}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) - \int_{S_c^+} S_{ij}^{Ip}(\boldsymbol{\xi}, \mathbf{x}) n_i^+(\boldsymbol{\xi}) \Delta u_j^I(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) \quad (5.52)$$

$$\sigma_{lk}^I(\mathbf{x}) = \int_{S_c^+} E_{lkpq} \frac{\partial U_j^{Ip}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) - \int_{S_c^+} E_{lkpq} \frac{\partial S_{ij}^{Ip}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} n_i^+(\boldsymbol{\xi}) \Delta u_j^I(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) \quad (5.53)$$

where $U_j^{Ip}(\boldsymbol{\xi}, \mathbf{x})$ and $S_{ij}^{Ip}(\boldsymbol{\xi}, \mathbf{x})$ are the displacement and stress fundamental solutions of an un-cracked half space under BC-type- I and the summation does not apply to the repeated index I .

To completely regularize the boundary integral relations (3.52) and (3.53) for the two remaining boundary conditions BC-type- I ($I = 3, 4$), the same regularization procedure can be used and the special decompositions of the strongly singular kernels $S_{ij}^{Ip}(\boldsymbol{\xi}, \mathbf{x})$ and $E_{lkpq} \partial U_j^{Ip}(\boldsymbol{\xi}, \mathbf{x}) / \partial x_q$ and the hypersingular kernel $E_{lkpq} \partial S_{ij}^{Ip}(\boldsymbol{\xi}, \mathbf{x}) / \partial x_q$ play an important role in such development. To derive those decompositions, the superposition technique similar to that employed by Li (1996) can be adopted. More specifically, the problem of a cracked half-space under BC-type-3 can be treated as a linear combination of two sub-problems, a sub-problem-1 associated with the original cracked half-space with the boundary condition BC-type-3 on the free surface being replaced by the boundary condition BC-type-1 and a sub-problem-2 corresponding to an un-cracked half-space subjected only to the normal traction opposite to that generated on the free surface of the sub-problem-1. Similarly, the problem of a cracked half-space under the boundary condition BC-type-4 can also be treated as a linear combination of two sub-problems: a sub-problem-1 associated with the original cracked half-space with the boundary condition BC-type-4 on the free surface being replaced by the boundary condition BC-type-2 and a sub-problem-2 corresponding to an un-cracked half-space subjected only to the shear traction opposite to that generated on the free surface of the sub-problem-2. Based on above linear combinations, the kernels $S_{ij}^{Ip}(\boldsymbol{\xi}, \mathbf{x})$, $E_{lkpq} \partial U_j^{Ip}(\boldsymbol{\xi}, \mathbf{x}) / \partial x_q$, and $E_{lkpq} \partial S_{ij}^{Ip}(\boldsymbol{\xi}, \mathbf{x}) / \partial x_q$ for $I = 3$ and 4 can be expressed in terms of the known results associated with BC-type- α and the correction terms as

$$S_{ij}^{3p}(\boldsymbol{\xi}, \mathbf{x}) = S_{ij}^{1p}(\boldsymbol{\xi}, \mathbf{x}) + \hat{S}_{ij}^{3p}(\boldsymbol{\xi}, \mathbf{x}) \quad (5.54)$$

$$S_{ij}^{4p}(\boldsymbol{\xi}, \mathbf{x}) = S_{ij}^{1p}(\boldsymbol{\xi}, \mathbf{x}) + \hat{S}_{ij}^{4p}(\boldsymbol{\xi}, \mathbf{x}) \quad (5.55)$$

$$E_{lkpq} \frac{\partial U_j^{3p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} = E_{lkpq} \frac{\partial U_j^{1p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} + E_{lkpq} \frac{\partial \hat{U}_j^{3p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} \quad (5.56)$$

$$E_{lkpq} \frac{\partial U_j^{4p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} = E_{lkpq} \frac{\partial U_j^{2p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} + E_{lkpq} \frac{\partial \hat{U}_j^{4p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} \quad (5.57)$$

$$E_{lkpq} \frac{\partial S_{ij}^{3p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} = E_{lkpq} \frac{\partial S_{ij}^{1p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} + E_{lkpq} \frac{\partial \hat{S}_{ij}^{3p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} \quad (5.58)$$

$$E_{lkpq} \frac{\partial S_{ij}^{4p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} = E_{lkpq} \frac{\partial S_{ij}^{2p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} + E_{lkpq} \frac{\partial \hat{S}_{ij}^{4p}(\boldsymbol{\xi}, \mathbf{x})}{\partial x_q} \quad (5.59)$$

where the correction terms $\hat{S}_{ij}^{lp}(\xi, \mathbf{x})$, $E_{lkpq} \partial \hat{U}_j^{lp}(\xi, \mathbf{x}) / \partial x_q$, and $E_{lkpq} \partial \hat{S}_{ij}^{lp}(\xi, \mathbf{x}) / \partial x_q$ are obtained by solving the sub-problem-2 for each type of boundary conditions. From the regularity of the sub-problem-2, it can readily be verified from the divergence-free condition that all the functions $\hat{S}_{ij}^{lp}(\xi, \mathbf{x})$, $E_{lkpq} \partial \hat{U}_j^{lp}(\xi, \mathbf{x}) / \partial x_q$ and $E_{lkpq} \partial \hat{S}_{ij}^{lp}(\xi, \mathbf{x}) / \partial x_q$ admit the following representations

$$\hat{S}_{ij}^{lp}(\xi, \mathbf{x}) = \varepsilon_{ism} \frac{\partial}{\partial \xi_s} \hat{G}_{mj}^{lp}(\xi, \mathbf{x}) \quad (5.60)$$

$$E_{lkpq} \frac{\partial \hat{U}_j^{lp}(\xi, \mathbf{x})}{\partial x_q} = \varepsilon_{lrt} \frac{\partial}{\partial x_r} \hat{G}_{tk}^{lj}(\mathbf{x}, \xi) \quad (5.61)$$

$$E_{lkpq} \frac{\partial \hat{S}_{ij}^{lp}(\xi, \mathbf{x})}{\partial x_q} = \varepsilon_{ism} \frac{\partial}{\partial \xi_s} \varepsilon_{lrt} \frac{\partial}{\partial x_r} \hat{C}_{mj}^{ltk}(\xi, \mathbf{x}) \quad (5.62)$$

where $\hat{G}_{mj}^{lp}(\xi, \mathbf{x})$ and $\hat{C}_{mj}^{ltk}(\xi, \mathbf{x})$ are functions that are singular only at a point $\xi = \bar{\mathbf{x}}$ of $\mathcal{O}(1/r)$. While the existence of the representations (5.60)-(5.62) can be ensured, determination of both functions $\hat{G}_{mj}^{lp}(\xi, \mathbf{x})$ and $\hat{C}_{mj}^{ltk}(\xi, \mathbf{x})$ is nontrivial and generally requires solving a system of partial differential equations.

For the special case of an isotropic elastic half-space, the functions $\hat{G}_{mj}^{lp}(\xi, \mathbf{x})$ and $\hat{C}_{mj}^{ltk}(\xi, \mathbf{x})$ were successfully developed by Li (1996). In his study, the sub-problem-2 for both types of boundary conditions indicated above was solved first using the method of displacement representation in terms of Papovich-Neuber potentials and the functions $\hat{G}_{mj}^{lp}(\xi, \mathbf{x})$ and $\hat{C}_{mj}^{ltk}(\xi, \mathbf{x})$ in the representations (5.60)-(5.62) were derived indirectly and equivalently using the method of stress functions. The explicit expressions for BC-type-3 are given by

$$\hat{G}_{mj}^{3p}(\xi, \mathbf{x}) = \frac{1}{2\mu} \left[x_3 \frac{\partial M_{mj}(\xi, \mathbf{x})}{\partial x_p} - (3-4\nu) \delta_{3p} M_{mj}(\xi, \mathbf{x}) + \frac{\partial N_{mj}(\xi, \mathbf{x})}{\partial x_p} \right] \quad (5.63)$$

$$\hat{C}_{mj}^{3tk}(\xi, \mathbf{x}) = \varepsilon_{tak} x_3 \frac{\partial M_{mj}(\xi, \mathbf{x})}{\partial x_a} - \frac{1}{(1-2\nu)} \varepsilon_{ta\beta} \lambda_{ak} \frac{\partial N_{mj}(\xi, \mathbf{x})}{\partial x_\beta} \quad (5.64)$$

where $M_{mj}(\xi, \mathbf{x})$ and $N_{mj}(\xi, \mathbf{x})$ are Papkovitch-Neuber potentials (see closed-form expressions in Li (1996)), and λ_{ak} denotes a constant tensor defined by

$$\lambda_{ak} = \begin{cases} (1-2\nu) & a = k = 1, 2 \\ -1 & a = k = 3 \\ 0 & a \neq k \end{cases} \quad (5.65)$$

Similarly, the functions $\hat{G}_{mj}^{4p}(\xi, \mathbf{x})$ and $\hat{C}_{mj}^{4tk}(\xi, \mathbf{x})$ for BC-type-4 can also be obtained in a closed form as

$$\hat{G}_{mj}^{4p}(\xi, \mathbf{x}) = \frac{1}{2\mu} [x_3 \frac{\partial P_{mj}(\xi, \mathbf{x})}{\partial x_p} - (3-4\nu)\delta_{3p}P_{mj}(\xi, \mathbf{x})] \quad (5.66)$$

$$\hat{C}_{mj}^{4tk}(\xi, \mathbf{x}) = (1-2\nu)\varepsilon_{tk3}P_{mj}(\xi, \mathbf{x}) + \varepsilon_{tak}x_3 \frac{\partial P_{mj}(\xi, \mathbf{x})}{\partial x_a} + 2(1-\nu)\delta_{3k}\varepsilon_{3\gamma t} \frac{\partial Q_{mj}(\xi, \mathbf{x})}{\partial x_\gamma} \quad (5.67)$$

where $P_{mj}(\xi, \mathbf{x})$ is the Papkovitch-Neuber potential and $Q_{mj}(\xi, \mathbf{x})$ is obtained directly by integrating the potential $P_{mj}(\xi, \mathbf{x})$. The closed-form expressions of $P_{mj}(\xi, \mathbf{x})$ and $Q_{mj}(\xi, \mathbf{x})$ can also be found in Li (1996).

Now, based on the relations (5.54)-(5.59) and the representations (5.60)-(5.62), the decompositions (5.33), (5.36) and (5.37) can be now generalized to all four types of boundary conditions on the free surface:

$$S_{ij}^{Ip}(\xi, \mathbf{x}) = H_{ij}^{Ip}(\xi, \mathbf{x}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} G_{mj}^{Ip}(\xi, \mathbf{x}) \quad (5.68)$$

$$E_{lkpq} \frac{\partial U_j^{Ip}(\xi, \mathbf{x})}{\partial x_q} = H_{lk}^{Ij}(\mathbf{x}, \xi) + \varepsilon_{lrt} \frac{\partial}{\partial x_r} G_{tk}^{Ij}(\mathbf{x}, \xi) \quad (5.69)$$

$$E_{lkpq} \frac{\partial S_{ij}^{Ip}(\xi, \mathbf{x})}{\partial x_q} = \Gamma_I E_{ijkl} \delta(\xi - \mathbf{x}) - \Lambda_I \bar{\delta}_{ia} \bar{\delta}_{jb} E_{abkl} \delta(\mathbf{x} - \bar{\xi}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} \varepsilon_{lrt} \frac{\partial}{\partial x_r} C_{mj}^{Itk}(\xi, \mathbf{x}) \quad (5.70)$$

where $\Gamma_3 = \Gamma_1$, $\Gamma_4 = \Gamma_2$, $\Lambda_3 = \Lambda_1$, $\Lambda_4 = \Lambda_2$ and the functions $H_{ij}^{Ip}(\xi, \mathbf{x})$, $G_{mj}^{Ip}(\xi, \mathbf{x})$ and $C_{mj}^{Itk}(\xi, \mathbf{x})$ for $I = 3$ and 4 are defined by

$$H_{ij}^{3p}(\xi, \mathbf{y}) = H_{ij}^{1p}(\xi, \mathbf{y}) \quad (5.71)$$

$$H_{ij}^{4p}(\xi, \mathbf{x}) = H_{ij}^{2p}(\xi, \mathbf{x}) \quad (5.72)$$

$$G_{mj}^{3p}(\xi, \mathbf{y}) = G_{mj}^{1p}(\xi, \mathbf{y}) + \hat{G}_{mj}^{3p}(\xi, \mathbf{y}) \quad (5.73)$$

$$G_{mj}^{4p}(\xi, \mathbf{x}) = G_{mj}^{2p}(\xi, \mathbf{x}) + \hat{G}_{mj}^{4p}(\xi, \mathbf{x}) \quad (5.74)$$

$$C_{mj}^{3tk}(\xi, \mathbf{y}) = C_{mj}^{1tk}(\xi, \mathbf{y}) + \hat{C}_{mj}^{3tk}(\xi, \mathbf{y}) \quad (5.75)$$

$$C_{mj}^{4tk}(\xi, \mathbf{x}) = C_{mj}^{2tk}(\xi, \mathbf{x}) + \hat{C}_{mj}^{4tk}(\xi, \mathbf{x}) \quad (5.76)$$

Upon using the decompositions (5.68)-(5.70) along with the regularization technique presented for BC-type-1 and BC-type-2, the singularity-reduced boundary integral relations for displacement and stress and a pair of weakly singular weak-form boundary integral equations for the crack-face displacement and crack-face traction of the cracked half-space under one of the four types of boundary conditions can be obtained in the same form as

$$\begin{aligned} u_p^I(\mathbf{x}) = & \int_{S_c^+} U_j^{Ip}(\xi, \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) - \int_{S_c^+} H_{ij}^{Ip}(\xi, \mathbf{x}) n_i^+(\xi) \Delta u_j^I(\xi) dA(\xi) \\ & + \int_{S_c^+} G_{mj}^{Ip}(\xi, \mathbf{x}) D_m \Delta u_j^I(\xi) dA(\xi) \end{aligned} \quad (5.77)$$

$$\sigma_{lk}^I(\mathbf{x}) = \varepsilon_{lrt} \frac{\partial}{\partial x_r} \left\{ \int_{S_c^+} G_{tk}^{lj}(\mathbf{x}, \xi) \Sigma t_j^0(\xi) dA(\xi) + \int_{S_c^+} C_{mj}^{ltk}(\xi, \mathbf{x}) D_m \Delta u_j^I(\xi) dA(\xi) \right\} \\ + \int_{S_c^+} H_{lk}^{lj}(\mathbf{x}, \xi) \Sigma t_j^0(\xi) dA(\xi) \quad (5.78)$$

$$\frac{1}{2} \int_{S_c^+} \tilde{t}_p(\mathbf{y}) \Sigma u_p^I(\mathbf{y}) dA(\mathbf{y}) = \int_{S_c^+} \tilde{t}_p(\mathbf{y}) \int_{S_c^+} U_j^{lp}(\xi, \mathbf{y}) \Sigma t_j^0(\xi) dA(\xi) dA(\mathbf{y}) \\ + \int_{S_c^+} \tilde{t}_p(\mathbf{y}) \int_{S_c^+} G_{mj}^{lp}(\xi, \mathbf{y}) D_m \Delta u_j^I(\xi) dA(\xi) dS(\mathbf{y}) \\ - \int_{S_c^+} \tilde{t}_p(\mathbf{y}) \int_{S_c^+} H_{ij}^{lp}(\xi, \mathbf{y}) n_i^+(\xi) \Delta u_j^I(\xi) dA(\xi) dS(\mathbf{y}) \quad (5.79)$$

$$\frac{1}{2} \int_{S_c^+} \Delta \tilde{u}_k(\mathbf{y}) \Delta t_k^0(\mathbf{y}) dA(\mathbf{y}) = \int_{S_c^+} \Delta \tilde{u}_k(\mathbf{y}) \int_{S_c^+} H_{lk}^{lj}(\mathbf{y}, \xi) n_l^+(\mathbf{y}) \Sigma t_j^0(\xi) dA(\xi) dA(\mathbf{y}) \\ - \int_{S_c^+} D_t \Delta \tilde{u}_k(\mathbf{y}) \int_{S_c^+} G_{tk}^{lj}(\mathbf{y}, \xi) \Sigma t_j^0(\xi) dA(\xi) dA(\mathbf{y}) \\ - \int_{S_c^+} D_t \Delta \tilde{u}_k(\mathbf{y}) \int_{S_c^+} C_{mj}^{ltk}(\xi, \mathbf{y}) D_m \Delta u_j^I(\xi) dA(\xi) dA(\mathbf{y}) \quad (5.80)$$

The last two boundary integral equations form a sufficient system of governing equations for determining the unknown crack-face data whereas the first two relations can be used to compute the displacement and stress at any interior point of the body once the primary unknowns on the crack surface are solved.

5.3 Solution Methodology

This section mainly presents an efficient and accurate numerical procedure for determining the unknown crack-face data and related fracture information such as the stress intensity factors and the T-stress components. A formulation of the boundary value problem based on a pair of singularity-reduced boundary integral equations established in the previous chapter is obtained first and, then, the solution procedure based on the weakly singular SGBEM and standard Galerkin method is briefly outlined. Essential ingredients to enhance the computational efficiency and accuracy such as the approximation of the near-front relative crack-face displacement, the numerical integration, and the evaluations of kernels for generally anisotropic materials are also discussed. Finally, an explicit formula for extracting the stress intensity factors and the T-stress components along the crack front is presented.

5.3.1 Solution procedure

To obtain the complete elastic field of the given problem described in section 5.2.1 (i.e., the displacement and stress fields), it is sufficient to determine the unknown data on the crack surface including the sum of and jump in the crack-face displacement. The displacement and

stress at any interior point can be post-processed from a pair of singularity-reduced boundary integral relations (5.77) and (5.78).

In the present study, a pair of weakly singular weak-form boundary integral equations for the displacement and traction (5.79) and (5.80) is utilized to form a complete set of integral equations governing the sum of and jump in the crack-face displacements. Those two equations can be expressed in a more concise form as

$$\mathcal{B}(\tilde{\mathbf{t}}, \Sigma \mathbf{u}^I) = \mathcal{U}(\tilde{\mathbf{t}}, \Sigma \mathbf{t}^0) + \mathcal{G}(\tilde{\mathbf{t}}, \Delta \mathbf{u}^I) + \mathcal{H}(\tilde{\mathbf{t}}, \Delta \mathbf{u}^I) \quad (5.81)$$

$$\mathcal{C}(\Delta \tilde{\mathbf{u}}, \Delta \mathbf{u}^I) = \mathcal{G}(\Sigma \mathbf{t}^0, \Delta \tilde{\mathbf{u}}) + \mathcal{H}(\Sigma \mathbf{t}^0, \Delta \tilde{\mathbf{u}}) + \mathcal{B}(\Delta \tilde{\mathbf{u}}, \Delta \mathbf{t}^0) \quad (5.82)$$

where the linear integral operator $\mathcal{B}, \mathcal{U}, \mathcal{G}, \mathcal{H}$ and $\mathcal{C}$ are defined by

$$\mathcal{B}(\mathbf{X}, \mathbf{Y}) = \frac{1}{2} \int_{S_c^+} X_p(\mathbf{y}) Y_p(\mathbf{y}) dS(\mathbf{y}) \quad (5.83)$$

$$\mathcal{U}(\mathbf{X}, \mathbf{Y}) = \int_{S_c^+} X_p(\mathbf{y}) \int_{S_c^+} U_j^{lp}(\xi, \mathbf{y}) Y_j(\xi) dS(\xi) dS(\mathbf{y}) \quad (5.84)$$

$$\mathcal{G}(\mathbf{X}, \mathbf{Y}) = \int_{S_c^+} X_p(\mathbf{y}) \int_{S_c^+} G_{mj}^{lp}(\xi, \mathbf{y}) D_m Y_j(\xi) dS(\xi) dS(\mathbf{y}) \quad (5.85)$$

$$\mathcal{H}(\mathbf{X}, \mathbf{Y}) = - \int_{S_c^+} X_p(\mathbf{y}) \int_{S_c^+} H_{ij}^{lp}(\xi, \mathbf{y}) n_i^+(\xi) Y_j(\xi) dS(\xi) dS(\mathbf{y}) \quad (5.86)$$

$$\mathcal{C}(\mathbf{X}, \mathbf{Y}) = - \int_{S_c^+} D_t X_k(\mathbf{y}) \int_{S_c^+} C_{mj}^{ltk}(\xi, \mathbf{y}) D_m Y_j(\xi) dS(\xi) dS(\mathbf{y}) \quad (5.87)$$

where $\mathbf{X}$ and $\mathbf{Y}$ are argument vectors and U_j^{lp} , G_{mj}^{lp} , $H_{ij}^{lp} n_i^+$ and C_{mj}^{ltk} are weakly singular kernels associated with the elastic half-space under the boundary condition BC-type-I on the free surface. It should be evident from the symmetry of the kernel C_{mj}^{ltk} and the integral form of (5.83) and (5.87) that the linear integral operators $\mathcal{B}$ and $\mathcal{C}$ satisfy the relations $\mathcal{B}(\mathbf{X}, \mathbf{Y}) = \mathcal{B}(\mathbf{Y}, \mathbf{X})$ and $\mathcal{C}(\mathbf{X}, \mathbf{Y}) = \mathcal{C}(\mathbf{Y}, \mathbf{X})$. As a result, the boundary integral equations (5.81) and (5.82) are in a symmetric form with respect to the unknown sum of and jump in the crack-face displacements, respectively. Since the integral equation for the jump in the crack-face traction (5.82) is independent of the sum of the crack-face displacement, it can be solved first to obtain the jump in the crack-face displacement. Once $\Delta \mathbf{u}^I$ is fully determined for the entire crack surface, the sum of the crack-face displacement $\Sigma \mathbf{u}^I$ can be obtained by solving the boundary integral equation (5.81).

To construct the numerical solution of the crack-face traction boundary integral equation (5.82), a weakly singular, symmetric Galerkin boundary element method (SGBEM) is employed (also see details in the work of Li *et al.*, 1998 and Rungamornrat and Mear, 2008b). Both the jump in the crack-face displacement $\Delta \mathbf{u}^I$ and the test function $\Delta \tilde{\mathbf{u}}$ on the entire crack surface are approximated by

$$\Delta \mathbf{u}^I(\xi) = \sum_{i=1}^N \Delta \mathbf{u}^{I(i)} \phi_i(\xi) \quad (5.88)$$

$$\Delta \tilde{\mathbf{u}}(\xi) = \sum_{i=1}^N \Delta \tilde{\mathbf{u}}^{(i)} \phi_i(\xi) \quad (5.89)$$

where N is the number of nodes; $\phi_i(\xi)$ is a selected nodal basis function; $\Delta \mathbf{u}^{I(i)}$ denotes the unknown nodal quantities; and $\Delta \tilde{\mathbf{u}}^{(i)}$ are arbitrary nodal constants. In the present study, the basis function $\phi_i(\xi)$ is constructed locally in an element-wise fashion by using standard, two-dimensional, C^0 -isoparametric elements on the majority of the crack surface and special crack-tip elements on the remaining region adjacent to the crack front. The special crack-tip element was originally developed by Li *et al.* (1998) to properly integrate the square-root-type behavior into the shape functions in the modeling of cracks in isotropic elastic media and later modified by Rungamornrat and Mear (2008b) to treat cracks in anisotropic bodies. It has been found in their study that use of such special crack-tip elements to model the near-front relative crack-face displacement can significantly enhance both the accuracy of numerical solutions and the computational efficiency regarding to the significant reduction of a level of mesh refinement. To allow the treatment of general prescribed traction data on the crack surfaces, the sum of and jump in the crack-face tractions are interpolated from

$$\Sigma \mathbf{t}^0(\xi) = \sum_{i=1}^N \Sigma \mathbf{t}^{0(i)} \psi_i(\xi) \quad (5.90)$$

$$\Delta \mathbf{t}^0(\xi) = \sum_{i=1}^N \Delta \mathbf{t}^{0(i)} \psi_i(\xi) \quad (5.91)$$

where $\psi_i(\xi)$ is a nodal basis function constructed by standard, two-dimensional, C^0 -isoparametric elements for the entire crack front; $\Sigma \mathbf{t}^{0(i)}$ denotes the prescribed sum of the crack-face traction at the i^{th} node; and $\Delta \mathbf{t}^{0(i)}$ denotes the prescribed jump in the crack-face traction at the i^{th} node. By substituting (5.88)-(5.91) into (5.82) and then invoking the arbitrariness of the constant $\Delta \tilde{\mathbf{u}}^{(i)}$, it results in a system of linear algebraic equations:

$$\mathbf{C} \Delta \mathbf{U}^I = (\mathbf{G} + \mathbf{H}) \Sigma \mathbf{T}^0 + \mathbf{B} \Delta \mathbf{T}^0 \quad (5.92)$$

where $\Delta \mathbf{U}^I$ is a vector containing the unknown nodal quantities $\Delta \mathbf{u}^{I(i)}$ with its entries given by $[\Delta \mathbf{U}^I]_{3(i-1)+k} = [\Delta \mathbf{u}^{I(i)}]_k$; $\Sigma \mathbf{T}^0$ is a vector containing the prescribed sum of nodal tractions $\Sigma \mathbf{t}^{0(i)}$ with its entries given by $[\Sigma \mathbf{T}^0]_{3(i-1)+k} = [\Sigma \mathbf{t}^{0(i)}]_k$; $\Delta \mathbf{T}^0$ is a vector containing the prescribed jump in nodal tractions $\Delta \mathbf{t}^{0(i)}$ with its entries given by $[\Delta \mathbf{T}^0]_{3(i-1)+k} = [\Delta \mathbf{t}^{0(i)}]_k$; and $\mathbf{C}$, $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{B}$ are known coefficient matrices associated with the linear integral operators $\mathcal{C}$, $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{B}$, respectively, and their entries are given explicitly by

$$[\mathbf{C}]_{3(i-1)+k, 3(j-1)+l} = - \int_{S_c^+} D_t \phi_i(\mathbf{y}) \int_{S_c^+} C_{ml}^{Itk}(\xi, \mathbf{y}) D_m \phi_j(\xi) dS(\xi) dS(\mathbf{y}) \quad (5.93)$$

$$[\mathbf{G}]_{3(i-1)+k, 3(j-1)+l} = - \int_{S_c^+} D_t \phi_i(\mathbf{y}) \int_{S_c^+} G_{tk}^l(\mathbf{y}, \xi) \psi_j(\xi) dA(\xi) dA(\mathbf{y}) \quad (5.94)$$

$$[\mathbf{H}]_{3(i-1)+k, 3(j-1)+l} = \int_{S_c^+} \phi_i(\mathbf{y}) \int_{S_c^+} H_{mk}^l(\xi, \mathbf{y}) n_m^+(\xi) \psi_j(\xi) dS(\xi) dS(\mathbf{y}) \quad (5.95)$$

$$[\mathbf{B}]_{3(i-1)+k, 3(j-1)+l} = \frac{1}{2} \int_{S_c^+} \phi_i(\mathbf{y}) \psi_j(\mathbf{y}) \delta_{kl} dS(\mathbf{y}) \quad (5.96)$$

It is evident that terms on the right hand side of (5.92) are completely known and the coefficient matrix $\mathbf{C}$ is symmetric.

To discretize the weakly singular, weak-form displacement boundary integral equation (5.81), a similar strategy following Galerkin approximation is employed. Specifically, the sum of the crack-face displacement $\Sigma \mathbf{u}^I$ and the test function $\tilde{\mathbf{t}}$ are approximated by

$$\Sigma \mathbf{u}^I(\xi) = \sum_{i=1}^N \Sigma \mathbf{u}^{I(i)} \psi_i(\xi) \quad (5.97)$$

$$\tilde{\mathbf{t}}(\xi) = \sum_{i=1}^N \tilde{\mathbf{t}}^{(i)} \psi_i(\xi) \quad (5.98)$$

where $\Sigma \mathbf{u}^{I(i)}$ and $\tilde{\mathbf{t}}^{(i)}$ denote the unknown sum of the crack-face displacement and an arbitrary constant vector at the i^{th} node, respectively. It is worth noting that the same nodal basis functions $\psi_i(\xi)$ as those employed in (5.90) and (5.91) are utilized in the discretization (5.97) and (5.98) since the unknown $\Sigma \mathbf{u}^I$ is well-behaved for the entire crack surface. By substituting (5.88), (5.90) and (5.97)-(5.98) into (5.81), it yields a system of linear algebraic equations

$$\bar{\mathbf{B}} \Sigma \mathbf{U}^I = \mathbf{D} \Sigma \mathbf{T}^0 + (\bar{\mathbf{G}} + \bar{\mathbf{H}}) \Delta \mathbf{U}^I \quad (5.99)$$

where $\Sigma \mathbf{U}^I$ is a vector containing the unknown nodal quantities $\Sigma \mathbf{u}^{I(i)}$ with its entries given by $[\Sigma \mathbf{U}^I]_{3(i-1)+k} = [\Sigma \mathbf{u}^{I(i)}]_k$ and $\bar{\mathbf{B}}$, $\mathbf{D}$, $\bar{\mathbf{G}}$ and $\bar{\mathbf{H}}$ are known matrices associated with the linear integral operators $\mathcal{B}$, $\mathcal{U}$, $\mathcal{G}$ and $\mathcal{H}$, respectively, and their entries are given explicitly by

$$[\bar{\mathbf{B}}]_{3(i-1)+k, 3(j-1)+l} = \frac{1}{2} \int_{S_c^+} \psi_i(\mathbf{y}) \psi_j(\mathbf{y}) \delta_{kl} dS(\mathbf{y}) \quad (5.100)$$

$$[\mathbf{D}]_{3(i-1)+k, 3(j-1)+l} = \int_{S_c^+} \psi_i(\mathbf{y}) \int_{S_c^+} U_l^{lk}(\xi, \mathbf{y}) \psi_j(\xi) dS(\xi) dS(\mathbf{y}) \quad (5.101)$$

$$[\bar{\mathbf{G}}]_{3(i-1)+k, 3(j-1)+l} = \int_{S_c^+} \psi_i(\mathbf{y}) \int_{S_c^+} G_{tk}^{ll}(\mathbf{y}, \xi) D_t \phi_j(\xi) dA(\xi) dA(\mathbf{y}) \quad (5.102)$$

$$[\bar{\mathbf{H}}]_{3(i-1)+k, 3(j-1)+l} = - \int_{S_c^+} \psi_i(\mathbf{y}) \int_{S_c^+} H_{ml}^{lk}(\xi, \mathbf{y}) n_m^+(\xi) \phi_j(\xi) dS(\xi) dS(\mathbf{y}) \quad (5.103)$$

It should be remarked that the matrix $\mathbf{B}$ is symmetric and, once the unknown $\Delta \mathbf{U}^I$ is obtained from (5.92), the system (5.99) can be subsequently solved for the unknown $\Sigma \mathbf{U}^I$.

5.3.2 Computation of coefficient matrices

To form the two systems of linear algebraic equations (5.92) and (5.99), the coefficient matrices $\mathbf{B}$, $\bar{\mathbf{B}}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{G}$, $\bar{\mathbf{G}}$, $\mathbf{H}$ and $\bar{\mathbf{H}}$ are computed numerically as described below.

All entries of the matrices $\mathbf{B}$ and $\bar{\mathbf{B}}$ clearly involve the single surface integral of regular elementary functions and the numerical evaluation of their value can be achieved efficiently and accurately using standard Gaussian quadrature. On the contrary, calculations of all entries of the matrices $\mathbf{C}$, $\mathbf{D}$, $\mathbf{G}$, $\bar{\mathbf{G}}$, $\mathbf{H}$ and $\bar{\mathbf{H}}$ are more computationally challenging since it requires to

evaluate the double surface integrals containing weakly singular kernels. Resulting from the discretization, double surface integrals over a pair of finite elements can be separated into three groups depending on the behavior of the integrand. A regular double surface integral over a pair of remote elements can be efficiently integrated by using standard Gaussian quadrature since the involved integrand is sufficiently well-behaved. For a weakly singular double surface integral over a pair of identical elements, a systematic numerical integration scheme based on a series of special variable transformations is employed to regularize the weakly singular integrand allowing the resulting integrals to be efficiently integrated by standard Gaussian quadrature. For a double surface integral over a pair of adjacent elements, the integrand possesses a rapid variation behavior introduced by pairs of relatively close source and field points and, as a result, it cannot efficiently be integrated by standard quadrature (see also the work of Xiao (1998)). In the present study, a family of logarithmic variable transformation is introduced to eliminate the nearly singular feature of the integrand and the final integral can be computed using standard Gaussian quadrature. Details of special numerical quadrature for both weakly singular and nearly singular double surface integrals can be found in the work of Xiao (1998).

In addition to the numerical integration scheme described above, it still requires the efficient computation of all involved kernels $H_{ij}^{lp}(\xi, y)$, $U_j^{lp}(\xi, y)$, $G_{mj}^{lp}(\xi, y)$ and $C_{mj}^{Itk}(\xi, y)$ for every pair of points (ξ, y) . Since the kernel $H_{ij}^{lp}(\xi, y)$ involves only elementary functions and is independent of the material properties, its value can readily and efficiently be obtained by a direct substitution. Unlike the kernel $H_{ij}^{lp}(\xi, y)$, the kernels $U_j^{lp}(\xi, y)$, $G_{mj}^{lp}(\xi, y)$ and $C_{mj}^{Itk}(\xi, y)$ are given in terms of a closed contour integral over a unit circle and the direct computation of such integral for every pair of source and field points (ξ, y) can be computationally massive. In the present study, an interpolation technique similar to that employed by Rungamornrat and Mear (2008b) is implemented. In this scheme, the direct evaluation of the closed contour integral is only required at the interpolation grid points and this can be achieved efficiently via standard Gaussian quadrature. Values of each kernel at any pair of points (ξ, y) within the interpolation grid can be approximated from their nodal-based interpolants. The accuracy of such approximation can readily be controlled by increasing the number of grid points and the order of the interpolation functions employed.

5.3.3 Determination of stress intensity factors and T-stress components

Let $\mathbf{x}_c$ be a point on the crack front and $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ be a local Cartesian coordinate system with the origin at $\mathbf{x}_c$ and the corresponding orthonormal base vectors $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$. In particular, the $\bar{x}_3$ -axis is selected to be tangent to the crack front; the $\bar{x}_1$ -axis directs outward and normal to the crack front such that $\bar{x}_1$ - $\bar{x}_3$ plane forms a local tangent plane of the crack surface at $\mathbf{x}_c$; and $\bar{x}_2$ -axis follows the right hand rule as illustrated in Figure 5.4. From the asymptotic stress analysis, the stress field in the vicinity of the point $\mathbf{x}_c$ takes the form

$$\sigma_{ij}(\mathbf{x}_c; r, \theta) = \sum_{p=1}^3 \frac{K_p(\mathbf{x}_c)}{\sqrt{r}} f_{ij}^p(\theta) + T_{ij}(\mathbf{x}_c) g^{ij}(\theta) + \sum_{m=1}^{\infty} r^{m/2} \hat{\sigma}_{ij}^m(\mathbf{x}_c; \theta) \quad (5.104)$$

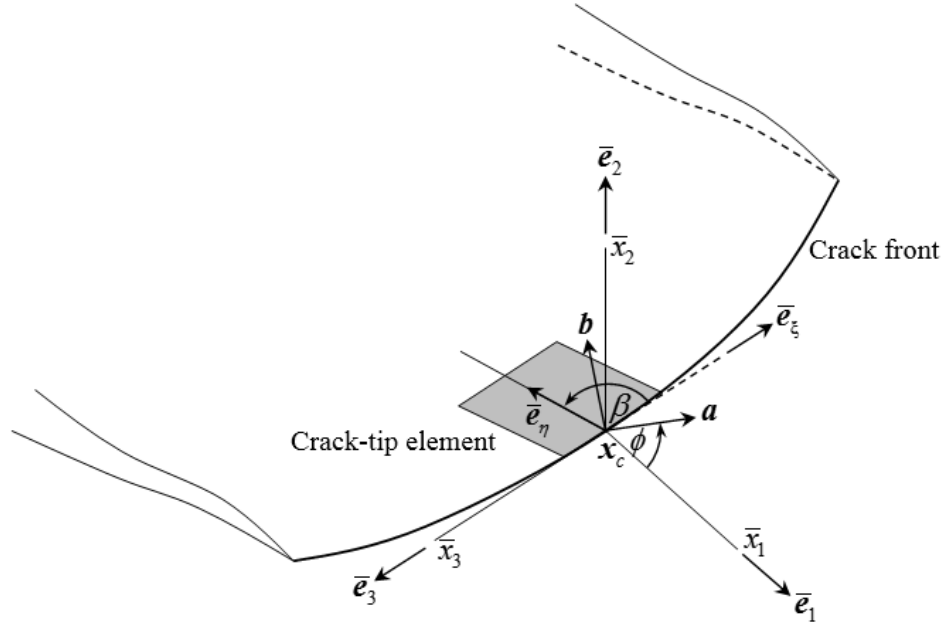


Figure 5.4: Local Cartesian coordinate system and all involved parameters for determining stress intensity factors and T-stress components.

where (r, θ) are polar coordinates of a point in the $\bar{x}_1 - \bar{x}_2$ plane; K_I , K_{II} and K_{III} denote the mode-I, mode-II and mode-III stress intensity factors, respectively; T_{ij} denotes the T-stress tensor; and f_{ij}^p , g^{ij} and $\hat{\sigma}_{ij}^m$ are functions independent of the radial coordinate r . It is important to remark that the angular dependent function f_{ij}^p and g^{ij} can be fully determined from the eigen analysis of the near-front field and, as a result, the singular term and the first non-singular term of the near-tip stress field are completely known once the stress intensity factors and the T-stress tensor are computed. Unlike the functions f_{ij}^p and g^{ij} , both K_p and T_{ij} cannot be obtained directly from the asymptotic analysis but requires solving the full boundary value problem.

In the present study, an explicit formula proposed by Rungamornrat and Mear (2008b) for cracks in anisotropic infinite and finite media is utilized to determine the stress intensity factors. This formula only requires the information of extra nodal degrees of freedom along the crack front as follows:

$$k_i(\mathbf{x}_c) = \sqrt{\frac{\pi}{2J_\eta \sin \beta}} B_{il} \{ \hat{\mathbf{u}}(\mathbf{x}_c) \cdot \bar{\mathbf{e}}_l(\mathbf{x}_c) \} \quad (5.105)$$

where $k_1 \equiv K_{II}$, $k_2 \equiv K_I$ and $k_3 \equiv K_{III}$; $J_\eta = \|\partial \mathbf{r} / \partial \eta(\xi_c, -1)\|$; $J_\xi = \|\partial \mathbf{r} / \partial \xi(\xi_c, -1)\|$; $\mathbf{r} = \mathbf{r}(\xi, \eta) = \mathbf{x}(\xi, \eta) - \mathbf{x}_c(\xi_c, -1)$; (ξ, η) are natural coordinates of any point $\mathbf{x}$ on the crack-tip element; $(\xi_c, -1)$ are natural coordinates of a point $\mathbf{x}_c$; β denotes an angle between the unit vectors $\bar{\mathbf{e}}_\eta = \{\partial \mathbf{r} / \partial \eta(\xi_c, -1)\} / J_\eta$ and $\bar{\mathbf{e}}_\xi = \{\partial \mathbf{r} / \partial \xi(\xi_c, -1)\} / J_\xi$; $\hat{\mathbf{u}}(\mathbf{x}_c) = \sum_i \Delta \mathbf{u}^{(i)} \varphi_i(\xi_c, -1)$ with $\Delta \mathbf{u}^{(i)}$

denoting the extra degree of freedom of the special crack-tip element obtained from equation (5.92), φ_i denoting the standard shape functions, and the summation taken over all nodes along the crack front of the crack-tip element containing the point $\mathbf{x}_c$; and

$$B_{il} = \frac{-1}{2\pi} \int_0^{2\pi} \left[(\mathbf{a}, \mathbf{b})_{im} (\mathbf{b}, \mathbf{b})_{mn}^{-1} (\mathbf{b}, \mathbf{a})_{nl} - (\mathbf{a}, \mathbf{a})_{il} \right] d\phi \quad (5.106)$$

with $\mathbf{a}$ and $\mathbf{b}$ denoting orthonormal vectors in the plane $\bar{x}_3 = 0$, ϕ denoting the angle between $\mathbf{a}$ and the unit vector $\bar{\mathbf{e}}_3$ as indicated in Figure 4.1, $(\mathbf{a}, \mathbf{b})_{ij} = a_m E_{imnj} b_n$, and $(\mathbf{b}, \mathbf{b})^{-1}$ denoting the inverse of $(\mathbf{b}, \mathbf{b})$.

The unknown T-stress components along the crack front can also be extracted from the sum of the crack-face displacement as described below. First, components of the T-stress tensor at the point $\mathbf{x}_c$ are related to the finite part of the strain tensor at the same point via the following linear constitutive relation

$$T_{ij}(\mathbf{x}_c) = \bar{E}_{ijkl}(\mathbf{x}_c) \bar{\varepsilon}_{kl}(\mathbf{x}_c) \quad (5.107)$$

where $\bar{E}_{ijkl}(\mathbf{x}_c)$ and $\bar{\varepsilon}_{kl}(\mathbf{x}_c)$ are elastic constants and components of the finite strain tensor relative to the local coordinate system at $\mathbf{x}_c$. It can readily be verified from the continuity of the tensor $T_{ij}(\mathbf{x}_c)$ that its components T_{12} , T_{22} , and T_{23} are known a priori and equal, respectively, to the $\bar{x}_1$ -, $\bar{x}_2$ -, and $\bar{x}_3$ -components of the applied traction on the crack surface at the limiting point of $\mathbf{x}_c$. The finite strain components $\bar{\varepsilon}_{11}$, $\bar{\varepsilon}_{33}$ and $\bar{\varepsilon}_{13}$ at $\mathbf{x}_c$ can also be calculated from the sum of the crack-face displacement in the vicinity of the point $\mathbf{x}_c$ via the following relations

$$\bar{\varepsilon}_{11}(\mathbf{x}_c) = \frac{1}{2} \frac{\partial \Sigma u_1^y(\mathbf{x}_c)}{\partial \bar{x}_1} \quad (5.108)$$

$$\bar{\varepsilon}_{33}(\mathbf{x}_c) = \frac{1}{2} \frac{\partial u_3^y(\mathbf{x}_c)}{\partial \bar{x}_3} \quad (5.109)$$

$$\bar{\varepsilon}_{13}(\mathbf{x}_c) = \frac{1}{4} \left(\frac{\partial \Sigma u_1^y(\mathbf{x}_c)}{\partial \bar{x}_3} + \frac{\partial \Sigma u_3^y(\mathbf{x}_c)}{\partial \bar{x}_1} \right) \quad (5.110)$$

The derivatives involved in the expressions (5.108)-(5.110) can be carried out directly within elements along the crack front. From the prescribed information of T_{12} , T_{22} , and T_{23} and the computed strain components $\bar{\varepsilon}_{11}$, $\bar{\varepsilon}_{33}$ and $\bar{\varepsilon}_{13}$, the unknown components T_{11} , T_{13} and T_{33} at any point $\mathbf{x}_c$ along the crack front, commonly termed the *T-stress components*, can be obtained by solving a system of six independent linear algebraic equations (5.107).

5.4 Numerical Results and Discussion

In this section, a selected set of boundary value problems concerning cracks in an elastic half-space under various scenarios is solved using the implemented solution procedure. First, the proposed technique and the underlying integral formulation is fully verified by comparing

computed numerical results with existing reliable reference solutions for both an isotropic cracked half-space under all four types of boundary conditions and an anisotropic cracked half-space under symmetrical and anti-symmetrical boundary conditions on the free surface. Then, more complex boundary value problems involving non-planar and multiple cracks are also investigated to illustrate the capability, versatility, and robustness of the developed numerical procedure.

In the numerical study, both isotropic and transversely isotropic materials with the elastic constants given in Table 5.1 are considered. It is remarked that for the transversely isotropic case, the axis of material symmetry is taken to direct perpendicular to the free surface of the half-space. To investigate the convergence behavior of numerical solutions, a series of meshes with different levels of refinement is adopted in the analysis. Special 9-node crack-tip elements are employed along the crack front whereas standard 8-node quadrilateral and 6-node triangular elements are utilized to discretize the majority of the crack surface.

Table 5.1: Elastic constants for isotropic material (associated with Poisson's ratio $\nu = 0.3$ and $E = 2.6\text{GPa}$) and transversely isotropic material with the axis of material symmetry perpendicular to free surface of half-space (Kassir and Sih, 1975).

Materials	Elastic constants (GPa)				
	E_{1111}	E_{1122}	E_{1133}	E_{2222}	E_{1313}
Isotropic material	3.500	1.500	1.500	3.500	1.000
Transversely isotropic material	16.090	3.350	5.010	6.100	3.830

5.4.1 Verification

This section aims to verify the integral formulation and the implemented numerical scheme. Results for both isotropic and anisotropic cracked half-space under various conditions on the free surface are reported and compared with the benchmark solutions. Both embedded cracks and surface breaking cracks under mode-I and mixed-mode conditions are investigated.

5.4.1.1 Isotropic Cracked Half-space under BC-type-I

In this particular sub-section, the proposed technique is validated for the case of an isotropic cracked half-space under all four types of boundary conditions on the free surface (i.e., BC-type-1: symmetrical boundary condition; BC-type-2: anti-symmetrical boundary condition; BC-type-3: traction-free boundary condition; and BC-type-4: rigidly-fixed boundary condition).

5.4.1.1.1 Horizontal penny-shaped crack

Consider a penny-shaped crack embedded in a half-space with the depth h and the crack surface parallel to the free surface as shown in Figure 5.5(a). The crack radius is denoted by a and the crack front is parameterized $x_1 = a \cos \beta$, $x_2 = -a \sin \beta$, $x_3 = h$ for $\beta \in [0, 2\pi]$. The crack is

subjected only to a self-equilibrated, uniform traction $t_1^+ = -t_1^- = \sigma_0$, $t_2^+ = -t_2^- = 0$, $t_3^+ = -t_3^- = \sigma_0$ (see Figure 5.5(b)) whereas the free surface of the half-space is under the boundary conditions BC-type-1 and BC-type-2. In the analysis, four meshes are adopted as indicated in Figure 5.5(d) and the normalized depth $h/a = 0.5$ is considered.

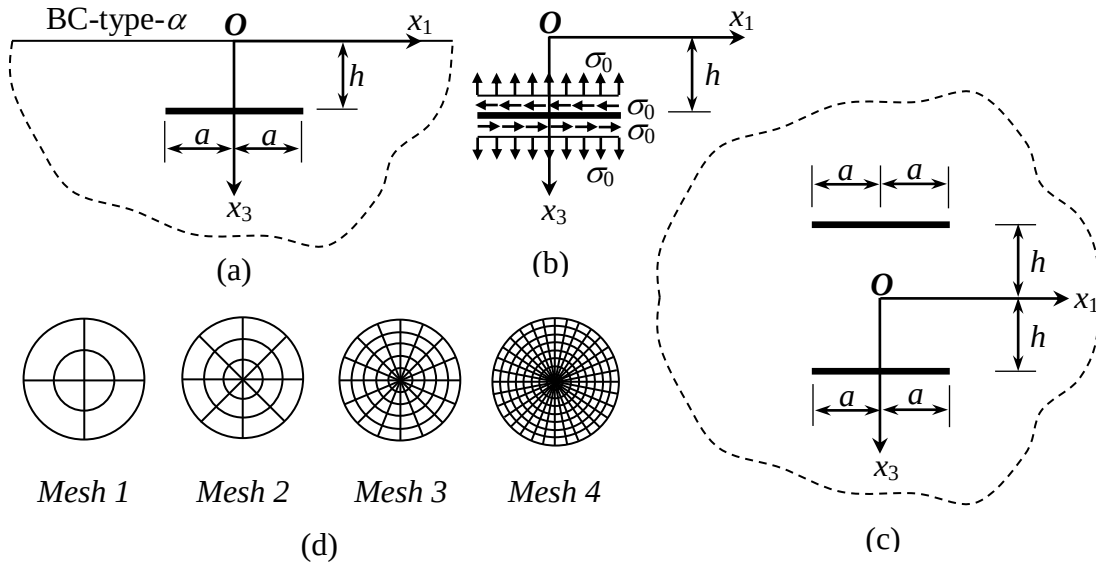


Figure 5.5: (a) Schematic of horizontal penny-shaped crack embedded in half-space under BC-type-1 or BC-type-2, (b) tractions acting on crack surfaces, (c) equivalent whole space problem, and (d) four meshes adopted in the analysis.

Computed results for the stress intensity factors and T-stress components at $\beta = \{0^\circ, 90^\circ, 180^\circ\}$ of both boundary conditions are normalized by the reference solution and reported in Table 5.2 and Table 5.3, respectively. It is remarked that the reference solution is taken as the solution of an equivalent whole space containing a pair of symmetric penny-shaped cracks about the surface $x_3 = 0$ shown in Figure 5.5(c), which is generated by the SGBEM proposed by Rungamornrat and Mear (2008b) with use of the Mesh 4. As can be seen from results in Table 5.2, numerical solutions exhibit excellent agreement with the reference solution for the first three meshes and they are weakly dependent on the level of mesh refinement. In particular, the discrepancy between the stress intensity factors generated by the coarsest and intermediate meshes and the reference solution is less than 2.4% and 0.2%, respectively, whereas results obtained from the Mesh 3 are nearly identical to the reference solution. The high quality of the numerical solutions, while employing relatively coarse meshes, is the direct consequence of the use of special crack-tip elements to enhance the near-front approximation of the relative crack-face displacement.

Similar convergence behavior can also be observed for results of the T-stress components shown in Table 5.3; however, it is evident that the difference between the computed solutions T_{13} from the Mesh 1 and Mesh 2 and the reference solution is larger than the case of the stress intensity factors. The reduction in the accuracy results directly from the fact that the derivatives of the sum of the crack-face displacement are required in the calculation of the T-stress components.

Table 5.2: Normalized stress intensity factors at $\beta=0^0$, 90^0 and 180^0 for horizontal penny-shaped crack embedded in half-space under BC-type-1 or BC-type-2 and $h/a = 0.5$.

β	Mesh	BC-type-1			BC-type-2		
		K_I / K_I^{ref}	K_{II} / K_{II}^{ref}	K_{III} / K_{III}^{ref}	K_I / K_I^{ref}	K_{II} / K_{II}^{ref}	K_{III} / K_{III}^{ref}
0^0	1	0.9949	0.9896	-	0.9844	0.9806	-
	2	1.0004	1.0001	-	0.9987	0.9987	-
	3	1.0003	1.0002	-	0.9999	1.0001	-
90^0	1	0.9935	1.0011	0.9817	0.9861	0.9901	0.9947
	2	1.0002	1.0016	0.9990	0.9988	0.9998	1.0007
	3	1.0003	1.0004	1.0000	0.9999	1.0001	1.0002
180^0	1	0.9916	0.9911	-	0.9874	0.9761	-
	2	1.0000	1.0003	-	0.9990	0.9982	-
	3	1.0002	1.0002	-	1.0000	1.0000	-

Table 5.3: Normalized T-stress components at $\beta=0^0$, 90^0 and 180^0 for horizontal penny-shaped crack embedded in half-space under BC-type-1 or BC-type-2 and $h/a = 0.5$.

β	Mesh	BC-type-1			BC-type-2		
		T_{11} / T_{11}^{ref}	T_{33} / T_{33}^{ref}	T_{13} / T_{13}^{ref}	T_{11} / T_{11}^{ref}	T_{33} / T_{33}^{ref}	T_{13} / T_{13}^{ref}
0^0	1	1.0359	1.0252	-	0.9322	0.9593	-
	2	1.0186	1.0166	-	0.9625	0.9789	-
	3	1.0055	1.0049	-	0.9909	0.9944	-
90^0	1	1.0037	0.9988	0.9626	0.9895	0.9906	1.0874
	2	1.0042	1.0013	1.1549	0.9878	0.9968	1.0928
	3	1.0021	1.0008	1.0662	0.9969	0.9993	1.0337
180^0	1	0.9658	0.9603	-	1.0334	1.0136	-
	2	0.9873	0.9790	-	1.0072	1.0099	-
	3	0.9980	0.9947	-	1.0016	1.0029	-

5.4.1.1.2 Vertical penny-shaped crack

Consider, next, a vertical penny-shaped crack of radius a embedded in a half-space with a depth h (measured from the center of the crack to the free surface) as shown in Figure 5.6(a). The crack front is parameterized by $x_1 = 0$, $x_2 = -a \cos \beta$, $x_3 = h - a \sin \beta$ for $\beta \in [0, 2\pi]$. The crack is subjected only to the self-equilibrated uniform normal traction $t_1^+ = -t_1^- = \sigma_0$, $t_2^+ = -t_2^- = 0$, $t_3^+ = -t_3^- = 0$ (see Figure 5.6(b)) whereas the free surface of the half-space is under all four types of boundary conditions, BC-type-I. It is apparent from the symmetry of the crack-face loading about the plane $x_1 = 0$ that the mode-II and mode-III stress intensity factors identically vanish

along the crack front. In the numerical study, four meshes of the penny-shaped crack are adopted as illustrated in Figure 5.6(d) and the aspect ratio $h/a = 1.5$ is considered.

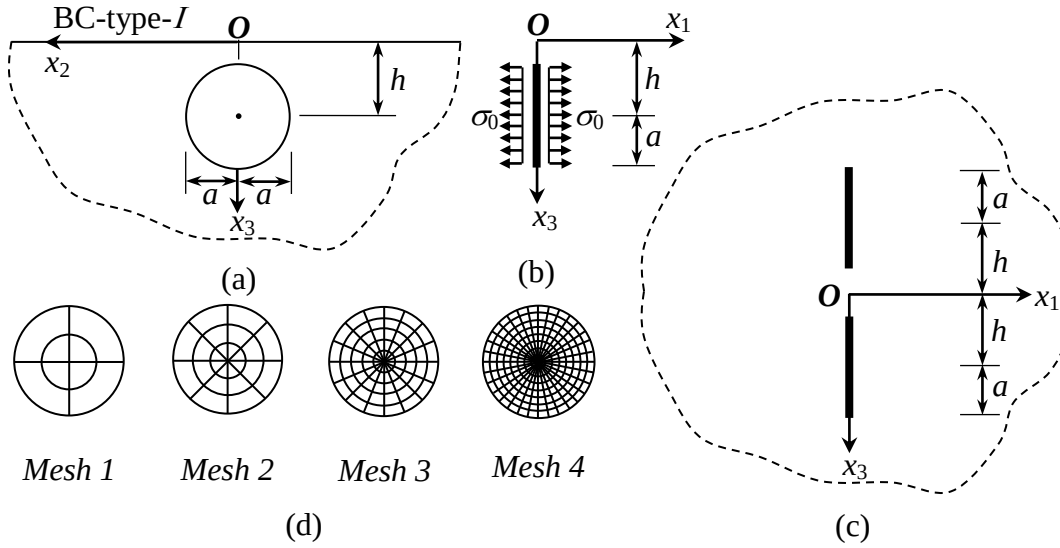


Figure 5.6: Schematic of vertical penny-shaped crack embedded in half-space under BC-type-I, (b) uniform normal traction acting to crack surfaces, (c) equivalent whole space problem for BC-type- α , and (d) four meshes adopted in the analysis.

For the first two types of boundary conditions, BC-type- α , the computed stress intensity factors and T-stress components are normalized by the reference solution (generated from an equivalent whole space containing a pair of geometrically symmetric cracks with respect to the surface $x_3 = 0$ shown in Figure 5.6(c) and subjected to the symmetric loading condition for BC-type-1 and the anti-symmetric loading condition for BC-type-2 by using the SGBEM with the Mesh 4) and reported in Table 5.4 and Table 5.5, respectively. Results of the stress intensity factors for the last two boundary conditions (i.e., BC-type-3 and BC-type-4) normalized by $K_I^{ref} = 2\sigma_0\sqrt{a/\pi}$ are compared with those obtained by Hrylyts'kyi *et al.* (2003) in Table 5.6. As indicated by results shown in Table 5.4, the stress intensity factors obtained from the first three meshes for BC-type-1 and BC-type-2 are only slightly different from those obtained from the equivalent whole space problem. The discrepancy between the two solutions is small even though the coarsest mesh is utilized in the analysis. However, as shown in Table 5.5, results of the T-stress components indicate that the discrepancy between T_{13} generated by the Mesh 1 and the reference solution is larger than the case of the stress intensity factors. The reduction of accuracy results directly from that the derivatives of the sum of the crack-face displacement are required in the calculation of the T-stress components. For BC-type-3 and BC-type-4, the numerical solutions for the mode-I stress intensity factors are in good agreement with those obtained by Hrylyts'kyi *et al.* (2003) and the weak dependence on the level of mesh refinement is observed for this particular case. Only coarse meshes with few elements can be used to obtain reasonably accurate results. This high quality of the computed solutions is due primarily to the use of special crack-tip elements to capture the field in the neighborhood of the crack front.

Table 5.4: Normalized stress intensity factors at $\beta = \{0^0, 90^0\}$ for vertical penny-shaped crack embedded in half-space under BC-type-1 or BC-type-2 and $h/a = 1.5$.

β	Mesh	BC-type-1	BC-type-2
		K_I / K_I^{ref}	K_I / K_I^{ref}
0^0	1	0.9909	0.9918
	2	0.9997	0.9998
	3	1.0001	1.0001
90^0	1	0.9916	0.9912
	2	1.0000	0.9996
	3	1.0002	1.0001

Table 5.5: Normalized T-stress components at $\beta = \{0^0, 90^0\}$ for vertical penny-shaped crack embedded in half-space under BC-type-1 or BC-type-2 and $h/a = 1.5$.

β	Mesh	BC-type-1			BC-type-2		
		T_{11} / T_{11}^{ref}	T_{33} / T_{33}^{ref}	T_{13} / T_{13}^{ref}	T_{11} / T_{11}^{ref}	T_{33} / T_{33}^{ref}	T_{13} / T_{13}^{ref}
0^0	1	1.0033	0.9970	1.1834	1.0084	0.9992	1.1848
	2	1.0024	1.0009	1.0287	1.0025	1.0000	1.0264
	3	1.0014	1.0006	1.0029	1.0013	1.0003	1.0029
90^0	1	1.0038	0.9904	-	1.0078	1.0060	-
	2	1.0016	1.0011	-	1.0033	0.9998	-
	3	1.0015	1.0012	-	1.0012	0.9996	-

Table 5.6: Normalized stress intensity factors at $\beta = \{0^0, 90^0\}$ for vertical penny-shaped crack embedded in half-space under BC-type-3 or BC-type-4 and $h/a = 1.5$.

β	Mesh	Present results		Hrylyts'kyi	
		BC-type-3	BC-type-4	BC-type-3	BC-type-4
0^0	1	1.0114	0.9763	1.0200 [†]	0.9790 [†]
	2	1.0209	0.9838		
	3	1.0213	0.9841		
90^0	1	1.0440	0.9477	1.0500 [†]	0.9530 [†]
	2	1.0524	0.9560		
	3	1.0524	0.9567		

[†]Results obtained by means of extraction from certain figures reported by Hrylyts'kyi *et al.* (2003)

5.4.1.1.3 Elliptical crack perpendicular to half-space surface

Consider, next, an elliptical crack oriented vertically within the half-space as shown in Figure 5.7(a). The semi major and minor axes of the crack are denoted by a and b , respectively; the orientation of the major axis relative to the free surface is denoted by the angle α ; the depth of the crack measured from the free surface to its center is denoted by h ; and $\beta \in [0, 2\pi]$ is an angle used for parameterizing the position along the crack front. The crack is loaded by uniformly distributed pressure σ_0 whereas the free surface of the half-space is under the traction-free condition (i.e., BC-type-3). In the analysis, two scenarios of the crack are considered: type A associated with $\alpha = 0$, $a/b = 2$ and type B corresponding to $\alpha \neq 0$, $a/b = 2$, $h/b = 1.8$, and three meshes as shown in Figure 5.7(b) are employed.

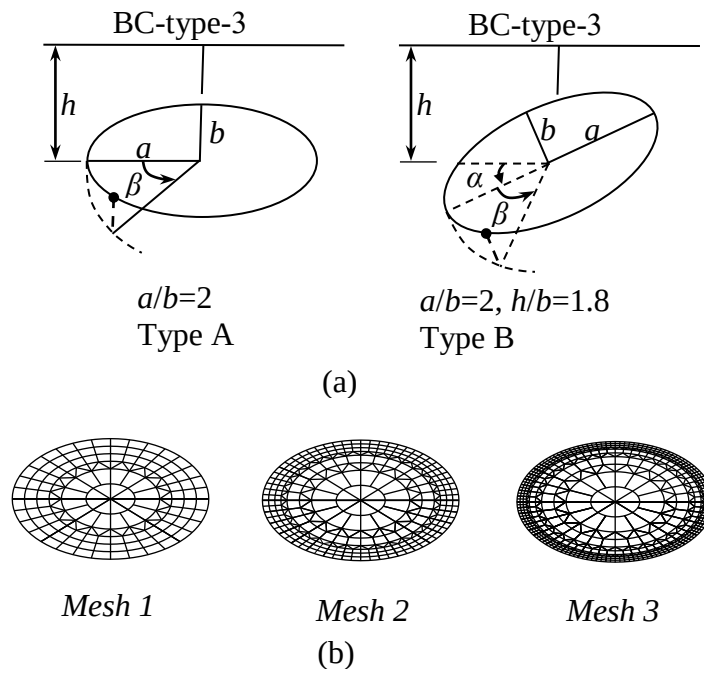


Figure 5.7: (a) Schematic of elliptical crack oriented vertically within the half-space and (b) three meshes used in analysis (mesh 1 containing 168 elements and 32 crack-tip elements; mesh 2 containing 392 elements and 64 crack-tips elements; and mesh 3 containing 840 elements and 128 crack-tip elements).

Results of the computed stress intensity factors for both type A and type B are first normalized by $K_I^{ref} = \sigma_0 \sqrt{\pi b}$ and then compared with those obtained by Noguchi *et al.* (1997) as shown in Figure 5.8(a) and Figure 5.8(b), respectively. It can be seen from this set of results that numerical solutions obtained from the proposed technique are highly accurate and in good agreement with the reference solution for both type A and type B. In particular, the stress intensity factors in the present study are weakly dependent on the level of mesh refinement. Again, this high quality of numerical solutions results directly from the use of special interpolation functions to capture the near-front relative crack-face displacement.

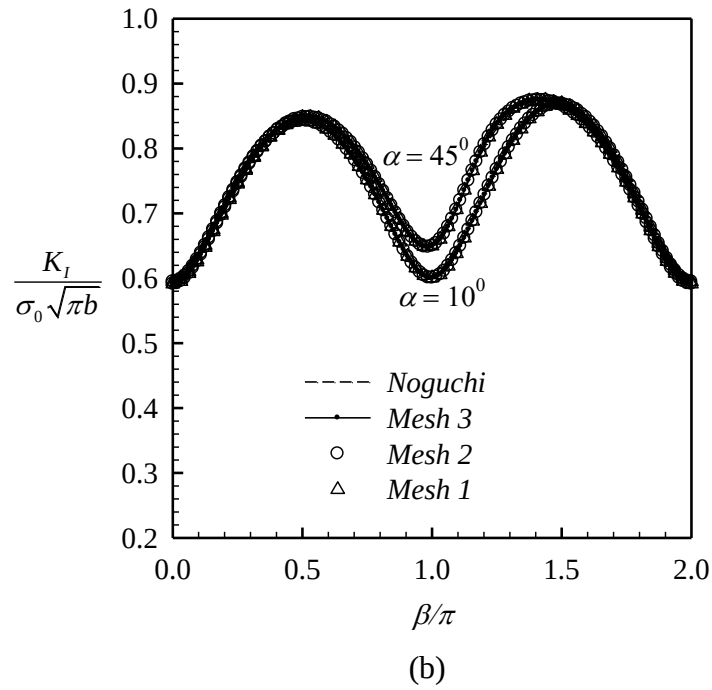
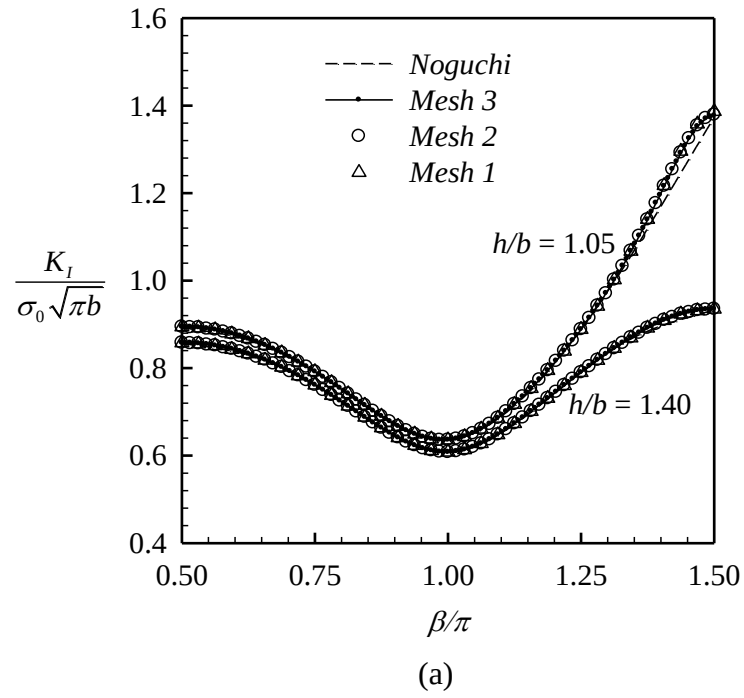


Figure 5.8: Normalized mode-I stress intensity factors for vertical elliptical crack of (a) type A and (b) type B, in half-space under BC-type-3.

5.4.1.1.4 Surface breaking crack

Consider another example associated with a semi-circular surface breaking crack of radius a and oriented vertically in a half-space as shown in Figure 5.9(a). The crack front is parameterized by $x_1 = -a \cos \beta$, $x_2 = 0$, $x_3 = a \sin \beta$ for $\beta \in [0, \pi]$. The half-space is subjected to three cases of loading and boundary conditions: (i) half-space under BC-type-1 and crack subjected to uniform normal traction $t_1^+ = 0$, $t_2^+ = -\sigma_0$, $t_3^+ = 0$; (ii) half-space under BC-type-2 and crack subjected to linear normal traction $t_1^+ = 0$, $t_2^+ = -\sigma_0(x_3/a)$, $t_3^+ = 0$; and (iii) half-space under BC-type-3 and crack subjected to three types of loading conditions: uniform distributed pressure (denoted by **0**), linear distributed pressure (denoted by **1**), and quadratic distributed pressure (denoted by **2**) as shown in Figure 5.9(b). In the numerical study, three meshes with different levels of refinement (Mesh 1 containing 8 elements and 4 crack-tip elements; Mesh 2 containing 32 elements and 8 crack-tips elements; and Mesh 3 containing 72 elements and 12 crack-tip elements) are adopted as shown in Figure 5.9(c).

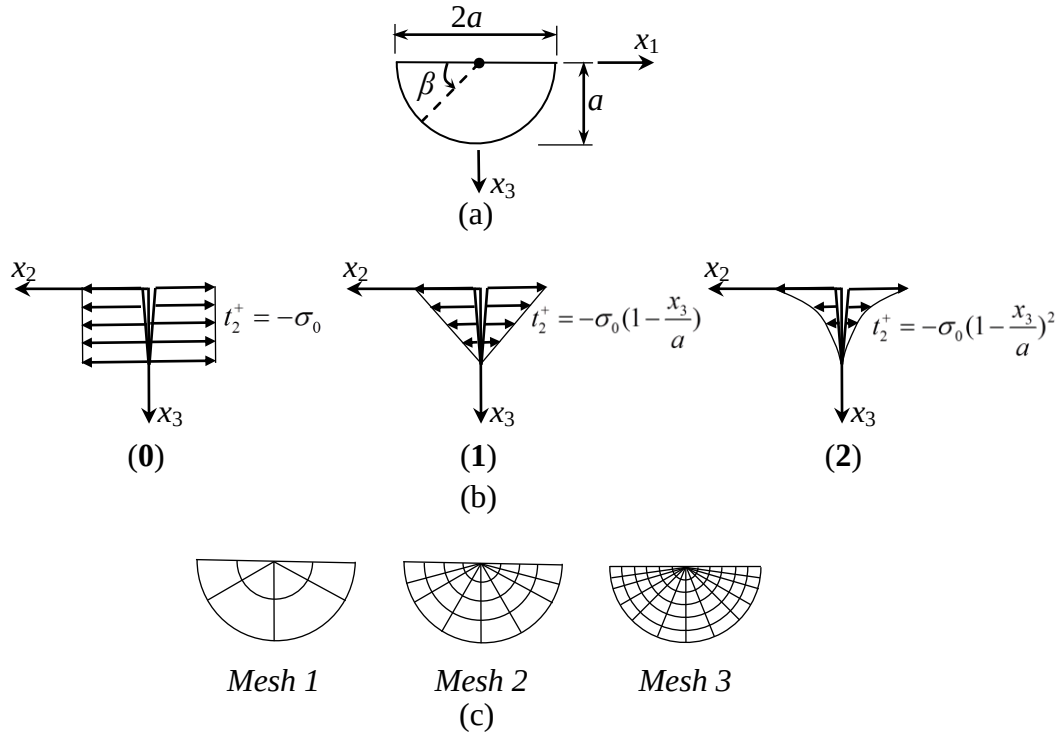


Figure 5.9: (a) Schematic of semi-circular surface-breaking crack in half-space under BC-type-3, (b) crack under three loading conditions, and (c) three meshes used in analysis.

For the first two cases, the computed stress intensity factors and non-zero T-stress components are compared with the existing analytical solutions. As indicated in Table 5.7, numerical results obtained for the uniform normal traction under BC-type-1 are normalized by $K_I^{exact} = 2\sigma_0\sqrt{a/\pi}$ for the mode-I stress intensity factor and those proposed by Wang (2004) for the non-zero T-stress components. Clearly, the present results agree very well with the exact solutions. The maximum difference is less than 0.2% for the stress intensity factor and 0.57% for the T-stress components. For the case of linear normal traction under BC-type-2, the normalized mode-I

stress intensity factor and T-stress components are compared with the benchmark solution obtained from Shah and Kobayashi (1971) and Rungamornrat and Pinitpanich (2016) (also found in Wang, 2004), respectively, and reported in Figure 5.10. It is evident that the present numerical results are nearly identical to the analytical solution.

For the last case, as shown in Table 5.8, the computed stress intensity factors normalized by $K_I^{ref} = \sigma_0 \sqrt{\pi a}$ at the intersection point between the crack front and free surface (a surface breaking point) and the deepest point are compared with those obtained by Noguchi and Smith (1995) and Murakami (1985). It can be seen that results obtained from the present technique exhibit good agreement with the benchmark solutions. It can be seen that the discrepancy between the computed stress intensity factors and the solution reported by Noguchi and Smith (1995) and Murakami (1985) at the surface breaking point is larger than that at the deepest point. In addition, results obtained from the coarse and intermediate meshes are nearly indistinguishable from those generated by Mesh 3 (with maximum discrepancy less than 2%).

Table 5.7: Normalized mode-I stress intensity factors and T-stresses at $\beta = \{0^\circ, 90^\circ\}$ for semi-circular surface-breaking crack in half-space under BC-type-1 and uniform normal traction.

β	Mesh	Stress intensity factor	T-stress	
		K_I / K_I^{exact}	T_{11} / T_{11}^{exact}	T_{33} / T_{33}^{exact}
0°	1	0.9995	0.9989	0.9944
	2	1.0004	0.9971	0.9989
	3	1.0005	0.9964	0.9989
30°	1	0.9992	1.0018	1.0013
	2	1.0011	0.9970	0.9993
	3	1.0009	0.9954	0.9987
90°	1	0.9981	1.0023	1.0006
	2	1.0010	0.9965	0.9990
	3	1.0011	0.9951	0.9987

Table 5.8: Normalized mode-I stress intensity factors for semi-circular surface-breaking crack within half-space under BC-type-3 and subjected to three types of loading conditions.

Loading condition	Mesh	$\beta = 0^\circ$			$\beta = 90^\circ$		
		Present results	Noguchi	Murakami	Present results	Noguchi	Murakami
0	1	0.783	0.760	0.748	0.660	0.656	0.666
	2	0.777			0.659		
	3	0.772			0.659		
1	1	0.662	0.644		0.193	0.191	
	2	0.658			0.192		
	3	0.655			0.192		
2	1	0.588	0.573		0.111	0.111	
	2	0.585			0.113		
	3	0.583			0.113		

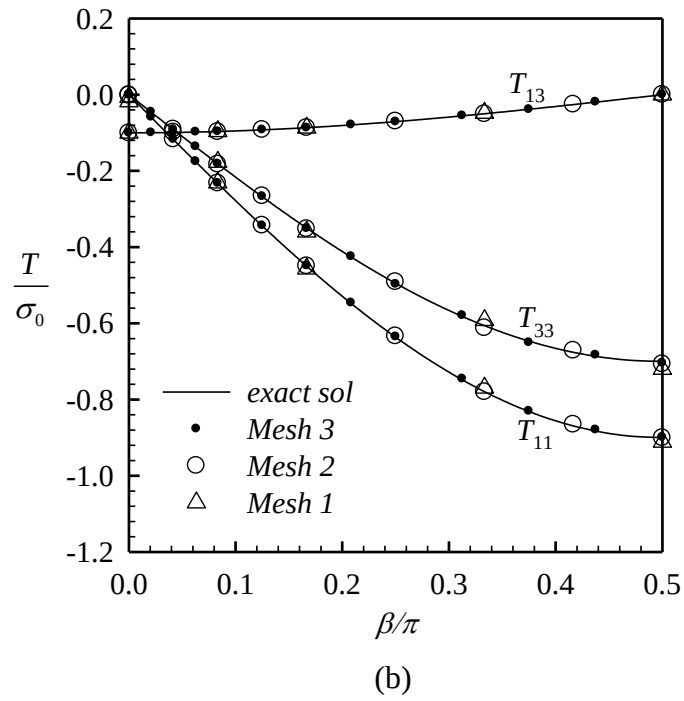
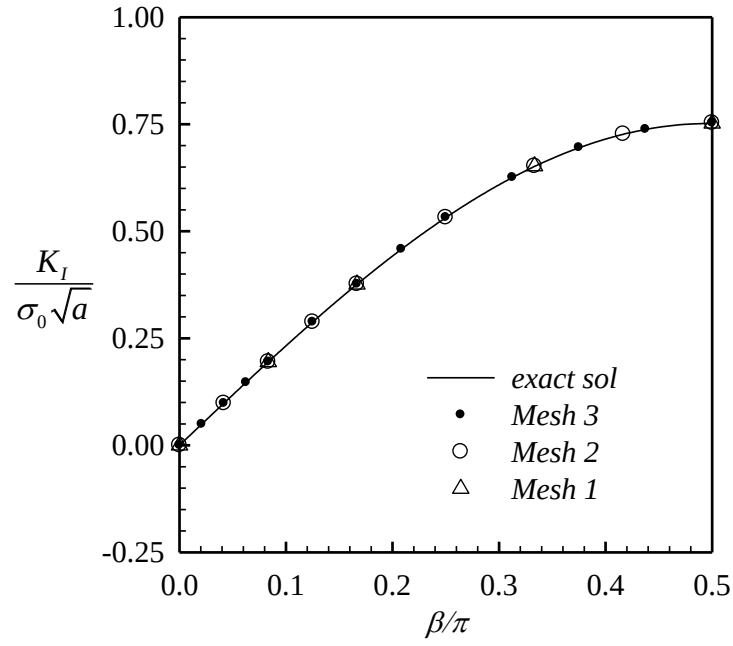


Figure 5.10: (a) Normalization of mode-I stress intensity factor and (b) normalized T-stress for semi-circular surface-braking crack in half-space under BC-type-2 and subjected to linear normal traction.

5.4.1.2 Anisotropic cracked half-space under BC-type- α

In this particular sub-section, the integral formulation and the implemented numerical procedure are verified for the case of an anisotropic cracked half space under symmetric boundary condition (BC-type-1) and anti-symmetric boundary condition (BC-type-2). Results are reported for a selected representative material which is transversely isotropic with the elastic constants shown in Table 5.1 and compared with the reference solution generated from an equivalent whole space problem (obtained from invoking the symmetrical and anti-symmetrical conditions) by the weakly singular SGBEM proposed by Rungamornrat and Mear (2008b).

5.4.1.2.1 Horizontal penny-shaped crack

Consider the same problem associated with a horizontal penny-shaped crack indicated in Figure 5.5(a) except that the half-space is made of the transversely isotropic material. The normalized stress intensity factors and the T-stress components along the crack front are reported and compared with the reference solutions in Figure 5.11 and Figure 5.12 for BC-type-1 and BC-type-2, respectively. As can be seen from these results, the computed stress intensity factors and T-stress components for both BC-type-1 and BC-type-2 are highly accurate when compared with the reference solutions and weakly dependent on the level of mesh refinement.

5.4.1.2.2 Vertical penny-shaped crack

Next, consider the same problem of a vertical penny-shaped crack under the uniform normal pressure as shown in Figure 5.2 with the material being replaced by one that is transversely isotropic. In the analysis, the ratio $h/a = 1.25$ is utilized and the same series of meshes is employed. The stress intensity factors and the T-stress components are normalized and compared with the reference solutions in the same manner as the case of the horizontal penny-shaped crack.

As can be seen from results in Figure 5.13, the computed mode-I stress intensity factor show very good agreement with the reference solution for all three meshes and both types of boundary conditions. Similarly, numerical solutions for the T-stress components shown in Figure 5.14 also indicate the good convergence behavior and the weak dependence on the level of mesh refinement. Clearly, results generated by the coarsest, intermediate and finest meshes are nearly indistinguishable from the benchmark solution. The high quality of the numerical solutions results directly, again, from the use of special crack-tip elements to capture the near-front relative crack-face displacement. It should also be remarked that, for this particular case, the symmetric and anti-symmetric boundary conditions have only weak influence on both the value and distribution of the stress intensity factor and T-stress components along the crack front.

5.4.1.2.3 Surface-breaking crack

As a final example, let us consider a semi-circular surface breaking crack of radius a and oriented vertically in a half-space as shown in Figure 5.15(a)-(b). The crack front is parameterized by $x_1 = -a \cos \beta$, $x_2 = 0$, $x_3 = a \sin \beta$ for $\beta \in [0, \pi]$. The surface $x_3 = 0$ of the half-space is subjected to the boundary conditions BC-type-1 and BC-type-2 whereas the crack is subjected to three types of loading conditions: the non-uniform normal traction $t_1^+ = t_3^+ = 0$, $t_2^+ = -\sigma_0(x_3/a)^2$, the non-uniform horizontal shear traction $t_2^+ = t_3^+ = 0$, $t_1^+ = \sigma_0(x_3/a)^2$, and the non-uniform vertical shear traction $t_1^+ = t_2^+ = 0$, $t_3^+ = \sigma_0(x_3/a)^2$. Three meshes as indicated in Figure

5.15(d) are adopted in the analysis and the reference solution obtained from the Mesh 4 in Figure 5.6(d) is constructed based on an equivalent whole space containing a vertical penny-shaped crack of radius a with the plane $x_3 = 0$ passing through its center and subjected to symmetric and anti-symmetric tractions on the crack surface (see Figure 5.15(c)).

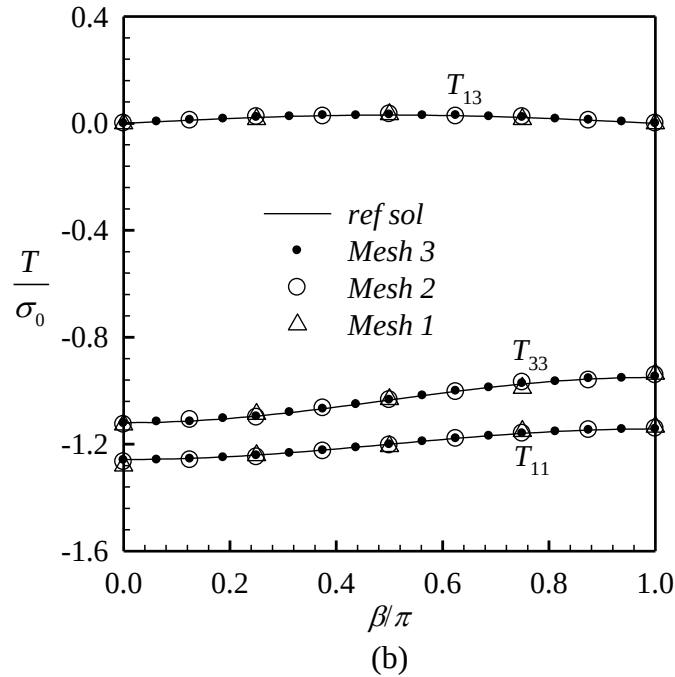
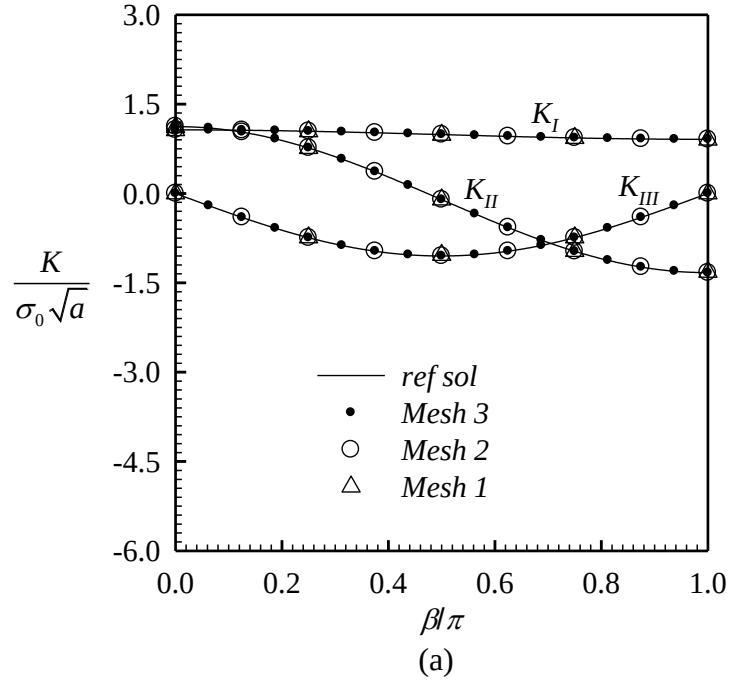


Figure 5.11: (a) Normalized stress intensity factors and (b) normalized T-stress components for horizontal penny-shaped crack in transversely isotropic half-space under BC-type-1.

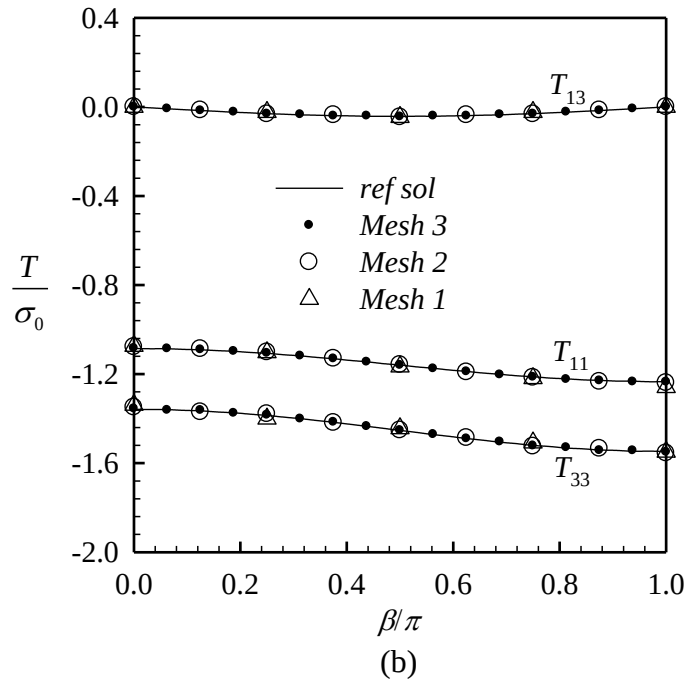
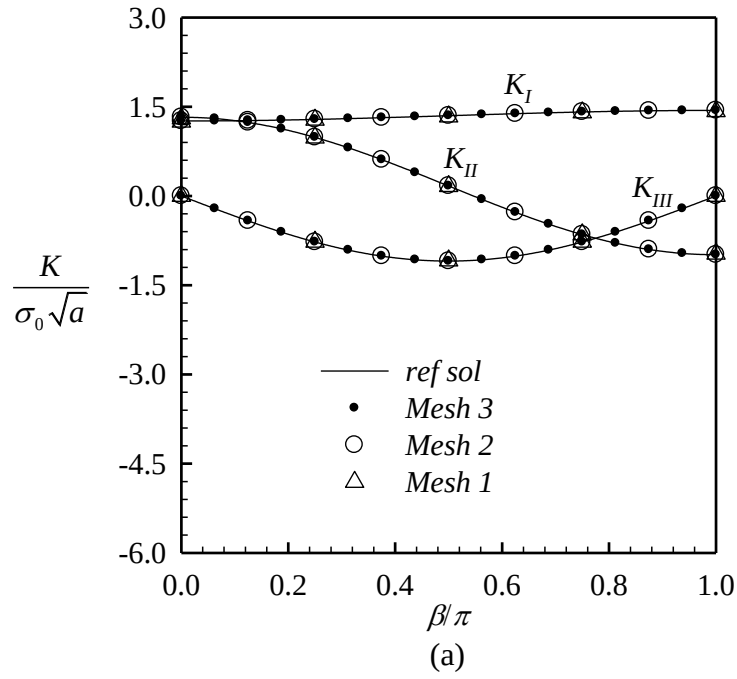


Figure 5.12: (a) Normalized stress intensity factors and (b) normalized T-stress components for horizontal penny-shaped crack in transversely isotropic half-space under BC-type-2.

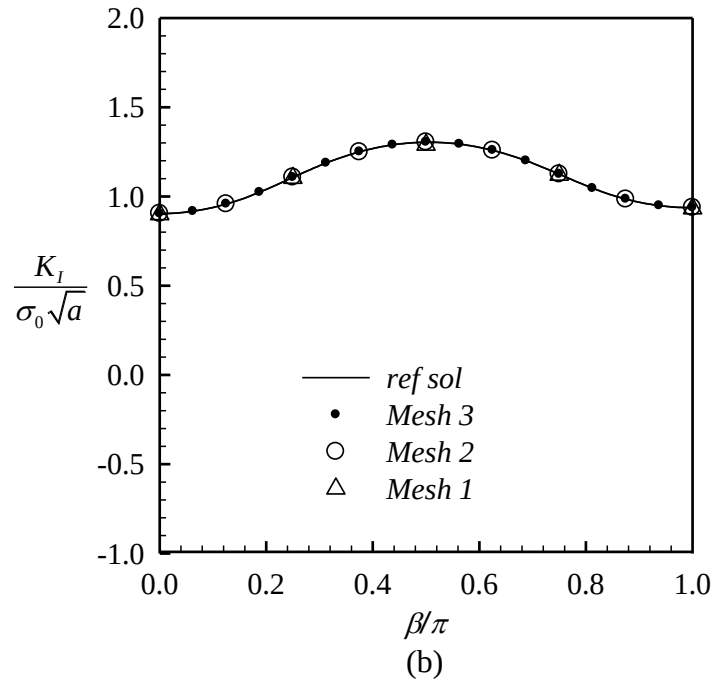
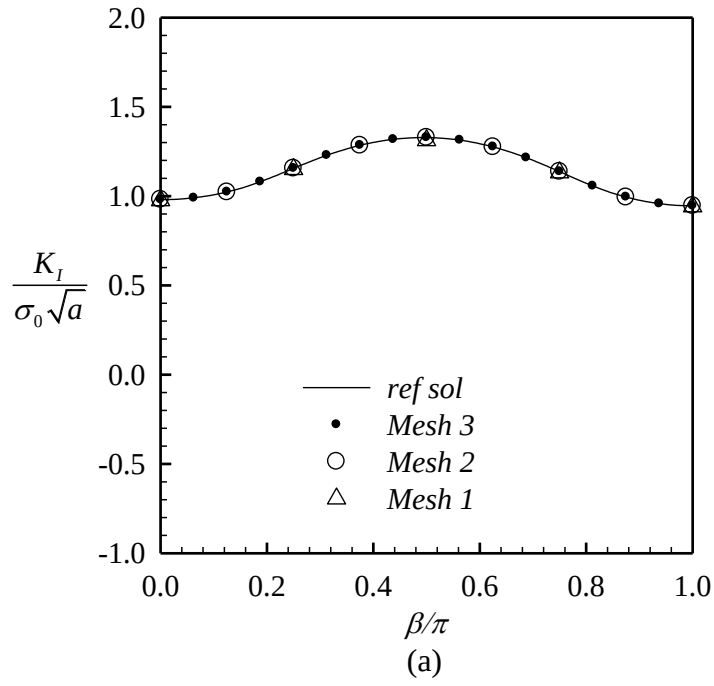


Figure 5.13: Normalized mode-I stress intensity factor for vertical penny-shaped crack in transversely isotropic half-space under (a) BC-type-1 and (b) BC-type-2.

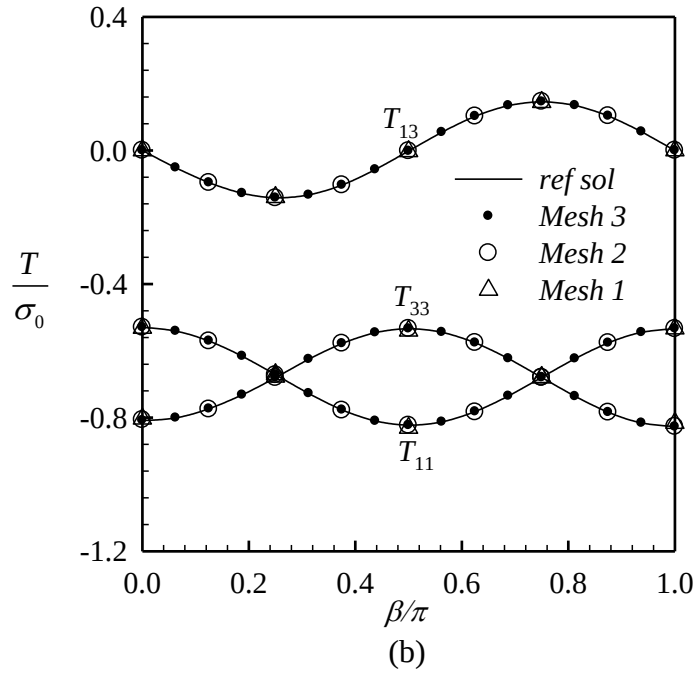
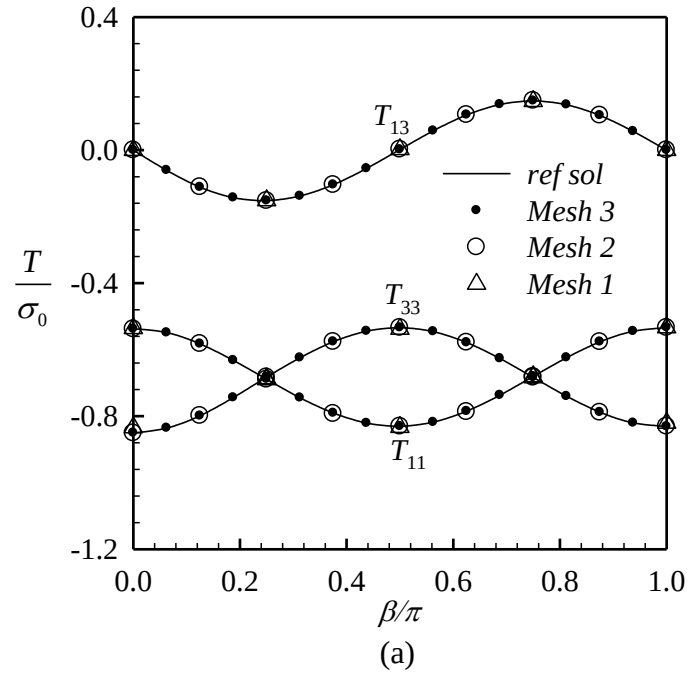


Figure 5.14: Normalized T-stress components for vertical penny-shaped crack in transversely isotropic half-space under (a) BC-type-1 and (b) BC-type-2.

For the first loading condition, the mode-II and mode-III stress intensity factors vanish whereas the mode-I stress intensity factor and the T-stress components vary along the crack front. The computed mode-I stress intensity factor and the non-zero T-stress components are reported in Figures 5.16 and 5.17, respectively. In addition to the good agreement between the computed results and the reference solution and the weak dependence on the mesh refinement, it can also be seen that the discrepancy between mode-I stress intensity factor and all T-stress components for BC-type-1 and BC-type-2 are not significant.

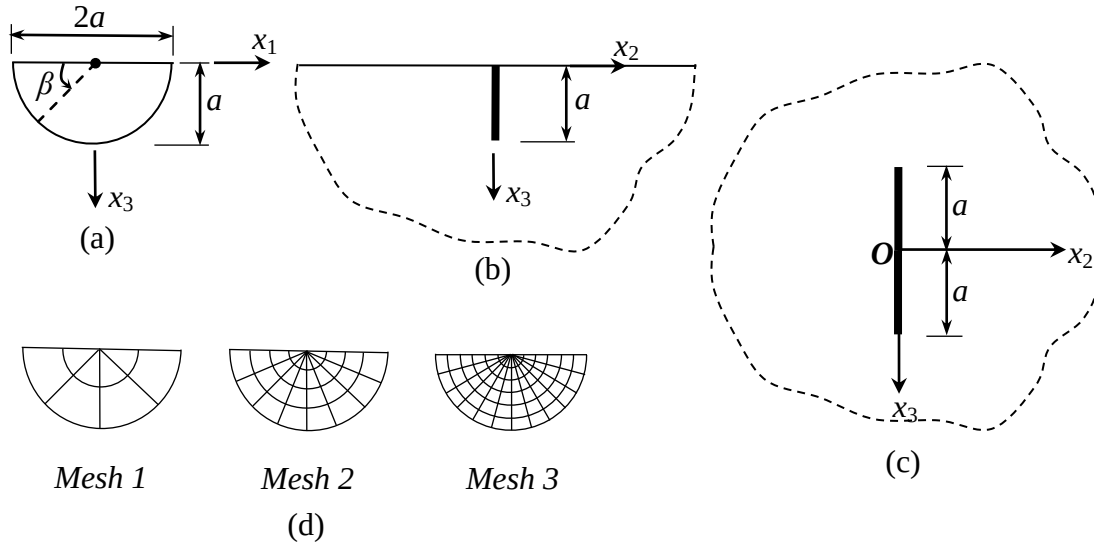


Figure 5.15: (a) Schematic semi-circular surface breaking crack, (b) semi-circular surface-breaking crack in half-space, (c) equivalent whole space problem, and (d) three meshes used in analysis.

Unlike the first loading case, the applied shear traction in either x_1 - or x_3 - direction yields zero mode-I stress intensity factor and T-stress components along the crack front. The non-zero mode-II and mode-III stress intensity factors for the second and third loading conditions are reported for all three meshes along with the reference solution in Figures 5.18 and 5.19, respectively. Similar to all previous cases, it is evident from this set of results that the proposed technique yields highly accurate numerical results which are comparable to the reference solution for three meshes employed. In addition, obtained results also suggest that both the value and distribution of the stress intensity factors are weakly dependent on the boundary condition posed on the free surface of the half-space

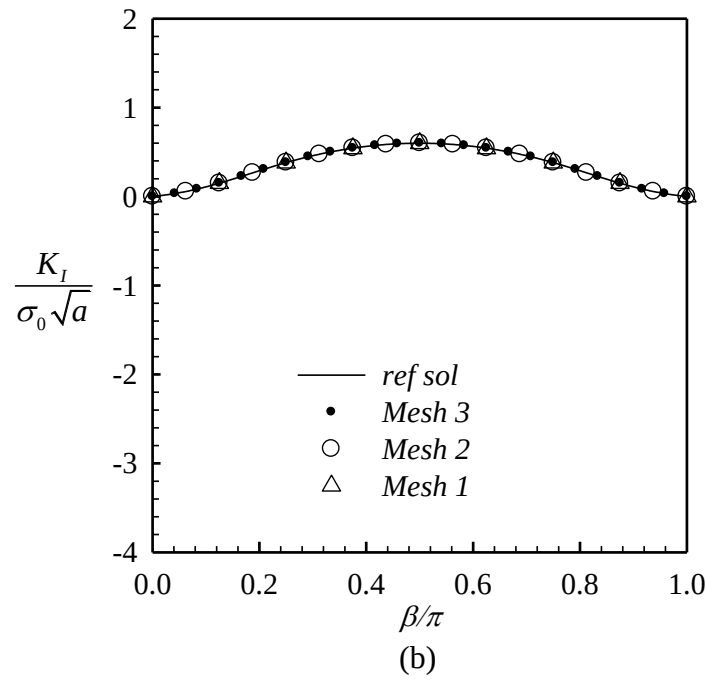
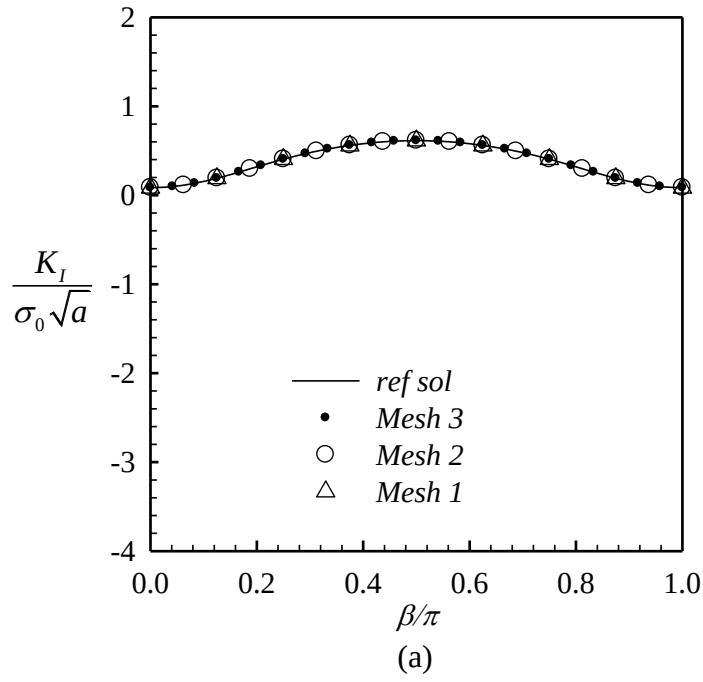


Figure 5.16: Normalized mode-I stress intensity factors for semi-circular surface-breaking crack subjected to non-uniform normal traction in transversely isotropic half-space under (a) BC-type-1 and (b) BC-type-2.

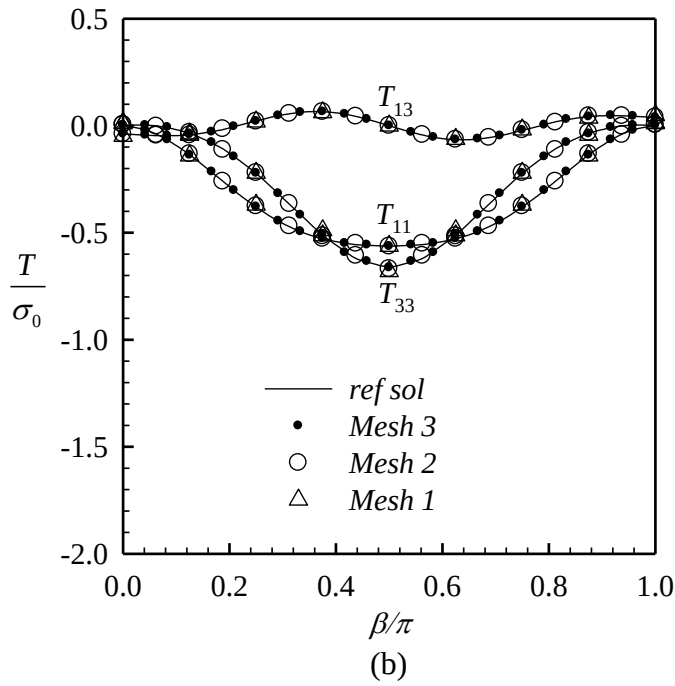
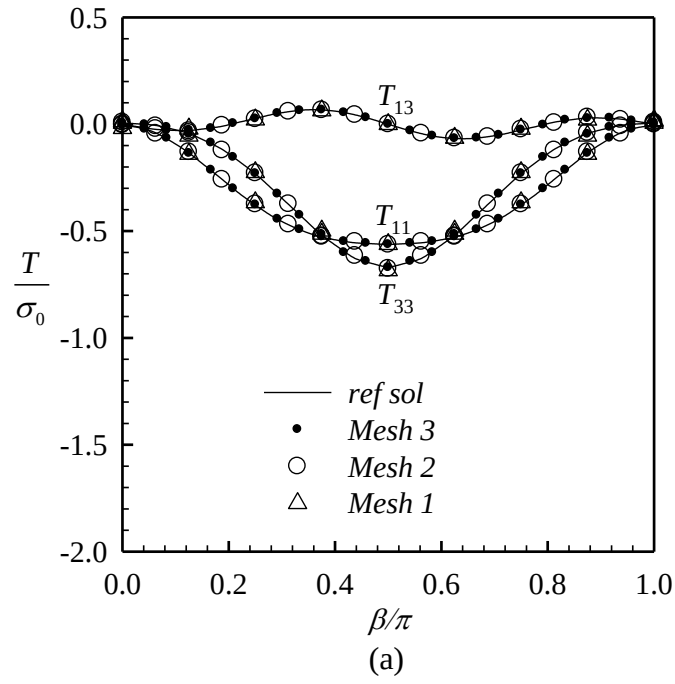


Figure 5.17: Normalized T-stress components for semi-circular surface-breaking crack subjected to non-uniform normal traction in transversely isotropic half-space under (a) BC-type-1 and (b) BC-type-2.

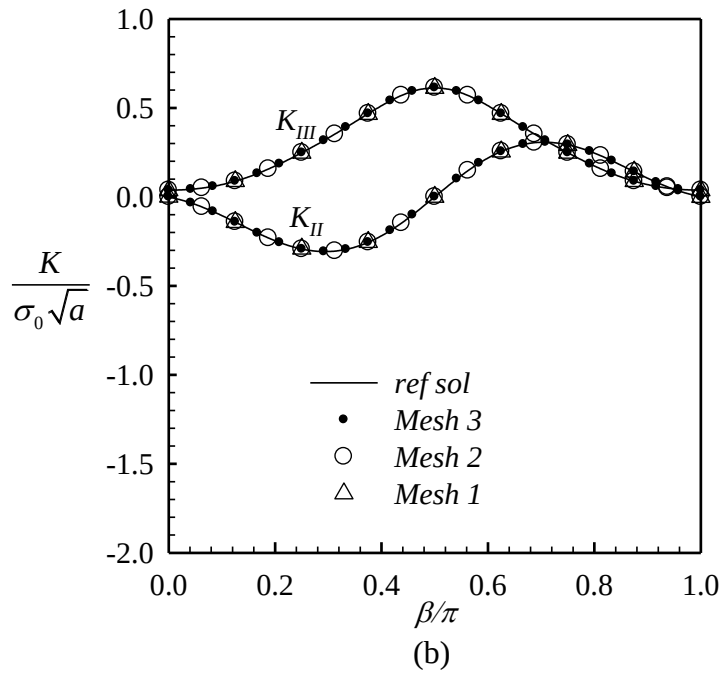
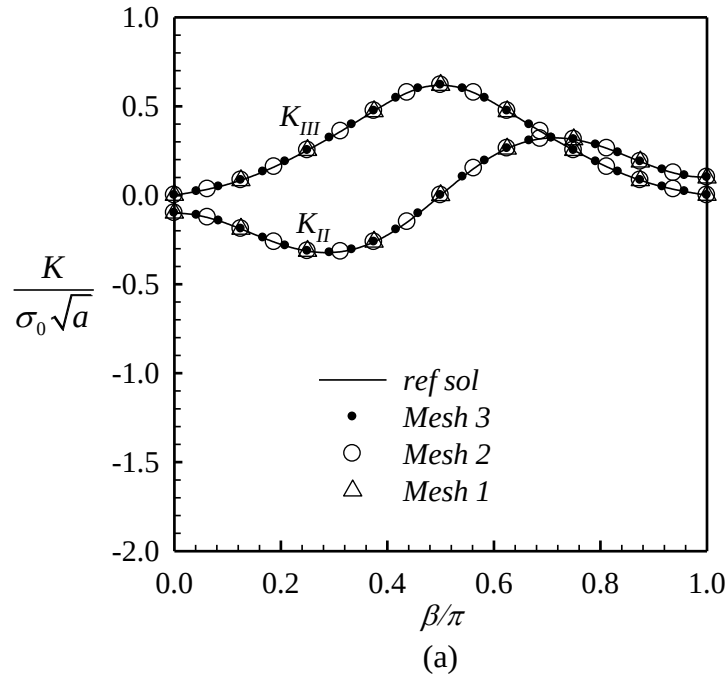


Figure 5.18: Normalized stress intensity factors for semi-circular surface-breaking crack subjected to non-uniform shear traction in x_1 -direction in transversely isotropic half-space under (a) BC-type-1 and (b) BC-type-2.

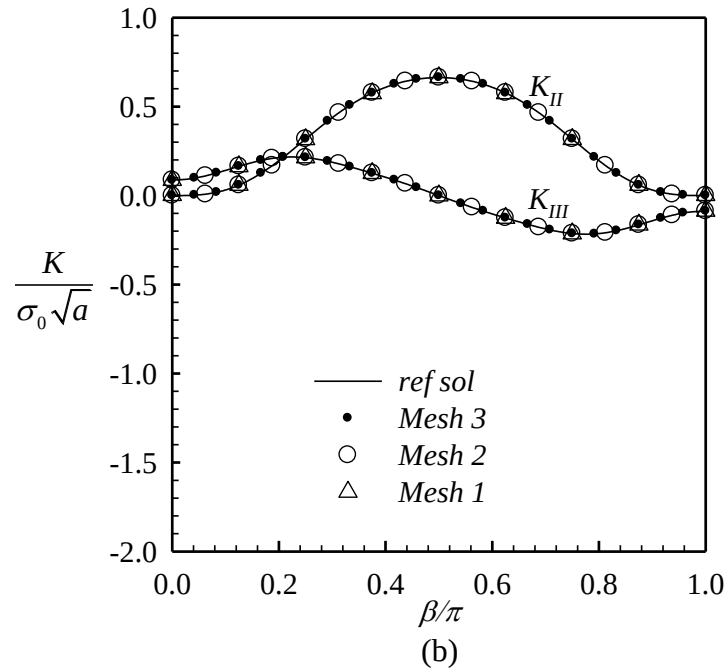
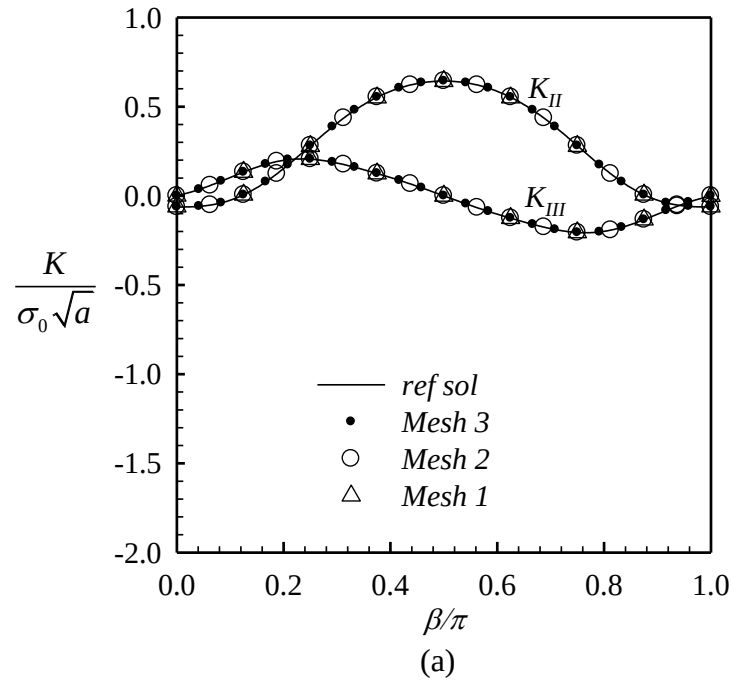


Figure 5.19: Normalized stress intensity factors for semi-circular surface-breaking crack subjected to non-uniform shear traction in x_3 -direction in transversely isotropic half-space under (a) BC-type-1 and (b) BC-type-2.

5.4.2 Capability of proposed technique

Once the integral formulation and implemented solution procedure are fully tested, more complex boundary value problems involving an embedded non-flat crack and multiple embedded cracks are presented here to demonstrate the capability and robustness of the proposed technique. The convergence behavior of numerical results is investigated by performing the simulations using a series of meshes.

5.4.2.1 Embedded spherical cap crack

Consider an elastic half-space containing a spherical cap crack of radius a and a half-subtended angle α as shown in Figure 5.20(a). The crack is oriented such that its surface is parameterized by $x_1 = a \sin \theta \cos \beta$, $x_2 = -a \sin \theta \sin \beta$, $x_3 = h + a - a \cos \theta$ where $\theta \in [0, \alpha]$, $\beta \in [0, 2\pi]$ and h is the distance from the free surface to the top of the spherical cap crack. The half-space is subjected to the remote uniform biaxial tensions $\sigma_{11} = \sigma_{33} = \sigma_0$ as indicated in Figure 5.20(b). In the analysis, the half-subtended angle is taken as $\alpha = 45^\circ$; the depth is taken as $h/a = 0.5$; and four meshes shown in Figure 5.20(c) are adopted. Similar to the previous section, two types of materials (e.g., isotropic and transversely isotropic materials with the elastic constants shown in Table 5.1) are chosen in the simulations.

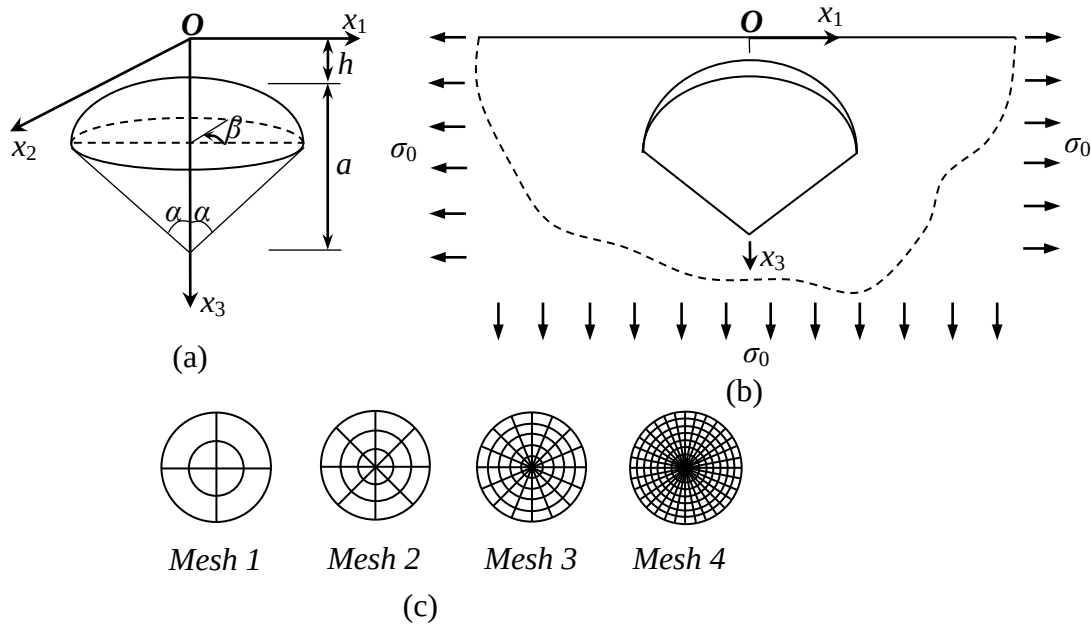


Figure 5.20: (a) Schematic of spherical cap crack embedded in an elastic half-space, (b) remote uniform biaxial tensions, and (c) three meshes adopted in analysis.

For isotropic case, the computed stress intensity factors and T-stress components under all four types of boundary conditions are first normalized by $K_{ref} = \sigma_0 \sqrt{a}$ and $T_{ref} = \sigma_0$, respectively, and then reported in Figure 5.21 and Figure 5.24 for all the first three meshes. It can

be observed from this set of results that the numerical solutions generated from the coarsest and intermediate meshes are nearly identical to that generated by the Mesh 3 for all cases. Similarly, the computed stress intensity factors and T-stresses for the transversely isotropic elastic half-space under the boundary conditions BC-type-1 and BC-type-2 are first normalized by $K_{ref} = \sigma_0 \sqrt{a}$ and $T_{ref} = \sigma_0$, respectively, and then reported in Figure 5.21 and Figure 5.22 for all the first three meshes. It can be seen, again, for this relatively complex boundary value problem posed by the material anisotropy and non-planar crack geometry that results obtained from the three different meshes are still in excellent agreement. This good convergence behavior and the requirement of using relatively coarse meshes to accurately capture the numerical solutions should result directly from the selection of suitable shape functions for crack-tip elements to approximate the near-front relative crack face displacement. Besides, results from the first two boundary conditions are also compared with the reference solutions generated from an equivalent whole space containing a pair of geometrically symmetric cracks with respect to the plane $x_3 = 0$ using the SGBEM with the Mesh 4. The good agreement between obtained results and the reference solution is observed for the stress intensity factors and T-stress components (see Figures 5.21-5.22 and Figures 5.25-5.26).

5.4.2.2 Two embedded elliptical cracks

Consider an elastic half-space containing two identical, vertical elliptical cracks as shown schematically in Figure 5.27(a). The half-space is subjected to a uniform remote uniaxial tension σ_0 as shown in Figure 5.27(b). The crack front of both is parameterized by

$$\text{Crack 1: } x_1 = 0; \quad x_2 = d - a \sin \beta; \quad x_3 = h + b \cos \beta \quad (5.111)$$

$$\text{Crack 2: } x_1 = 0; \quad x_2 = -d - a \sin \beta; \quad x_3 = h + b \cos \beta \quad (5.112)$$

where d is the distance between the center of each crack and x_3 -axis; a and b are the major and minor semi-axis of the elliptical cracks; and $\beta \in [0, 2\pi]$. In the analysis, the aspect ratio $a/b=1.5$ and $d/a=1.1, 2.0$ and two types of materials including isotropic and transversely isotropic solids are considered and three meshes shown in Figure 5.27(c) are employed. Due to the symmetry, the computed stress intensity factors and the T-stress components are reported only for the crack 1.

The stress intensity factors and T-stress components generated from the three meshes for all four types of boundary conditions along the front of the crack 1 are reported in Figures 5.28-5.35. It is evident that the numerical solutions for the stress intensity factors and the T-stress components exhibit the good convergence behavior; in particular, the discrepancy between results obtained from the Mesh 1 and Mesh 2 and that from the Mesh 3 is insignificant for the two crack depths $d/a=1.1$ and $d/a=2.0$ considered in the simulations. In addition, the dependence of results along the crack front on the ratio d/a is also investigated. It can be seen that the variation of both the stress intensity factors and the T-stress components at points on the crack front relatively close to those on the other crack is significant and strongly dependent on the ratio d/a . Clearly, the higher value of the ratio d/a , the less variation of the stress intensity factors and the T-stress components.

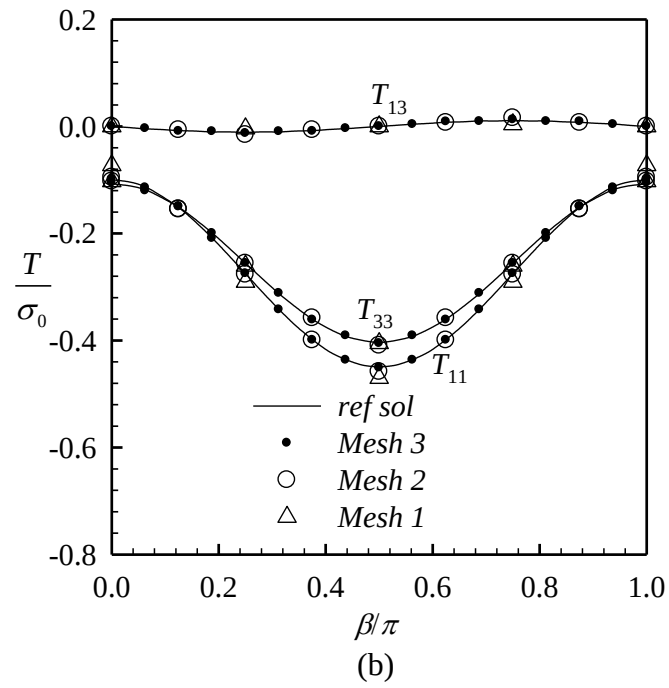
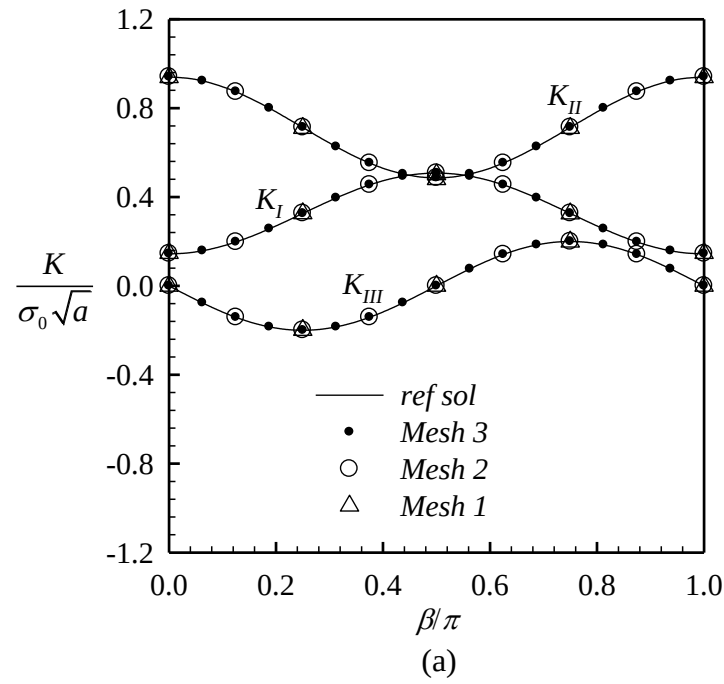


Figure 5.21: (a) Normalized stress intensity factors and (b) normalized T-stress components for spherical cap crack in isotropic half-space under BC-type-1.

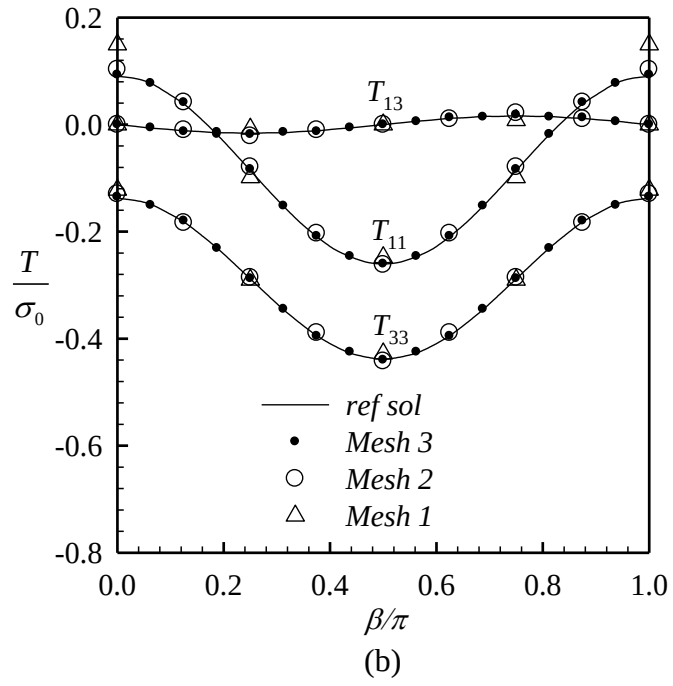
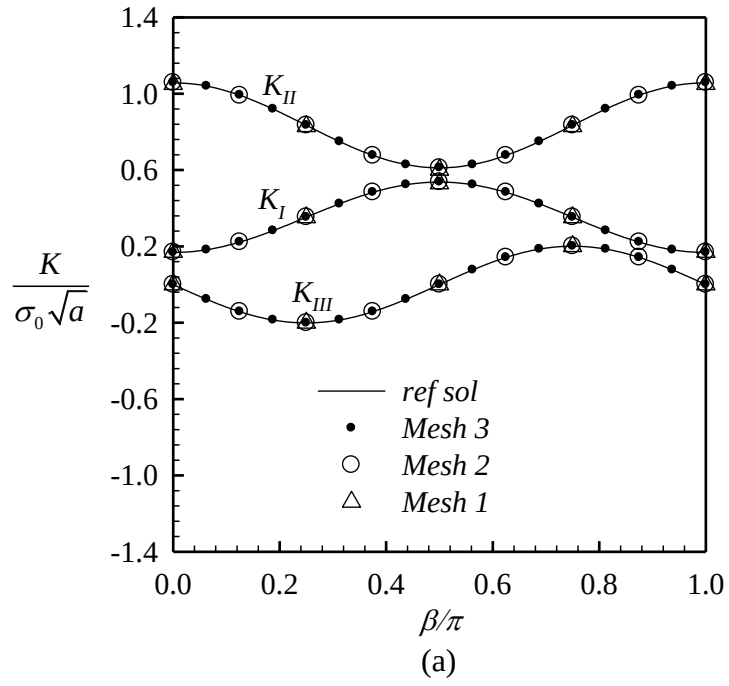


Figure 5.22: (a) Normalized stress intensity factors and (b) normalized T-stress components for spherical cap crack in isotropic half-space under BC-type-2.

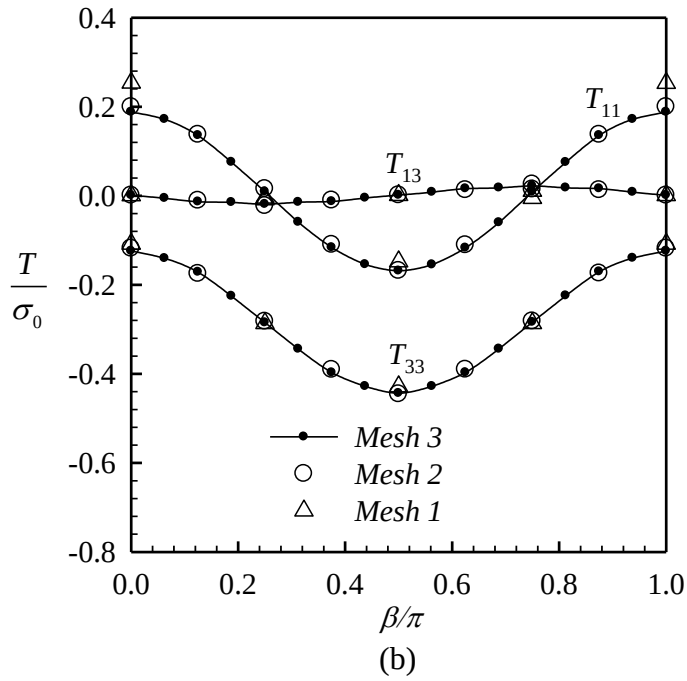
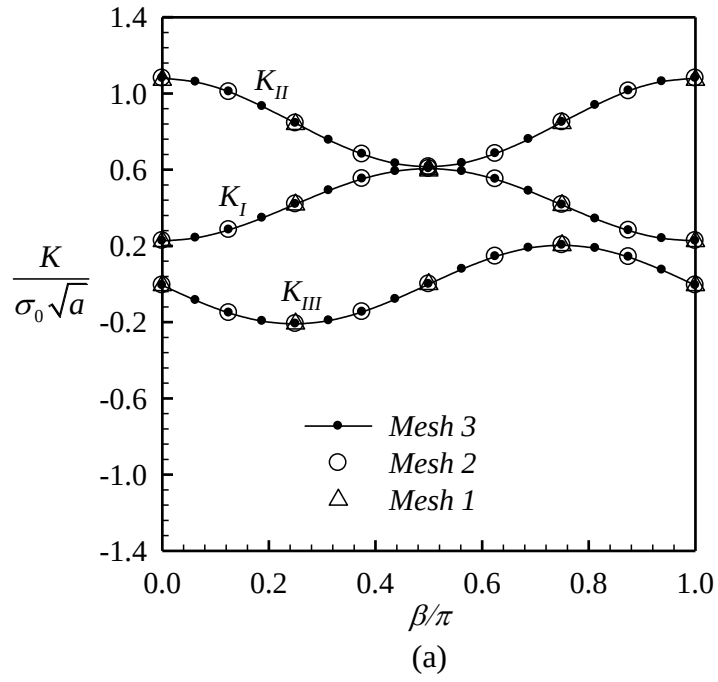


Figure 5.23: (a) Normalized stress intensity factors and (b) normalized T-stress components for spherical cap crack in isotropic half-space under BC-type-3.

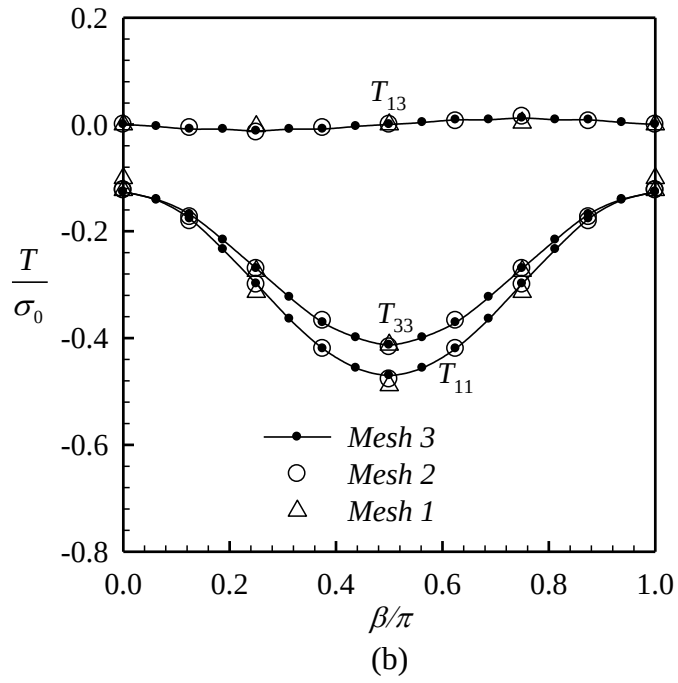
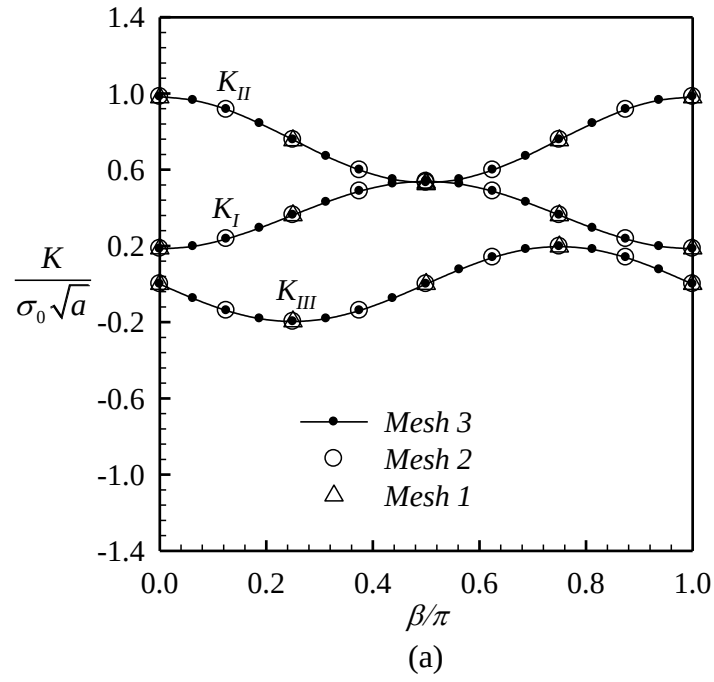
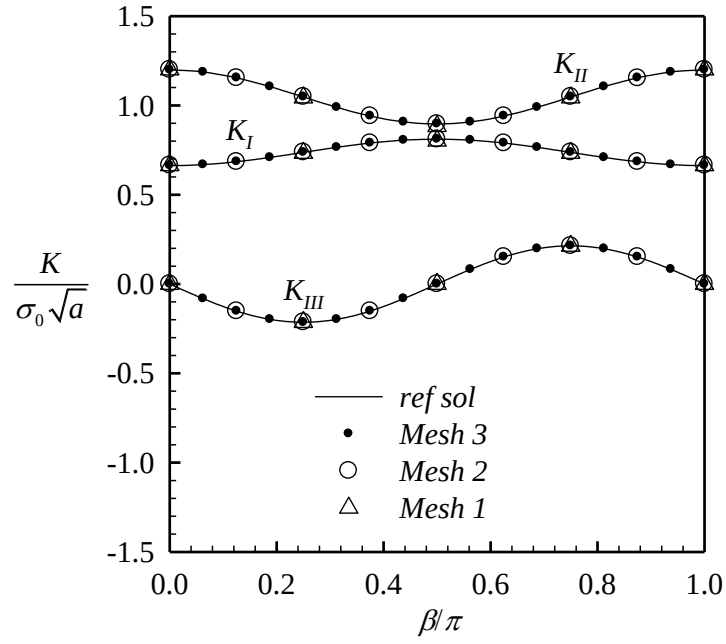
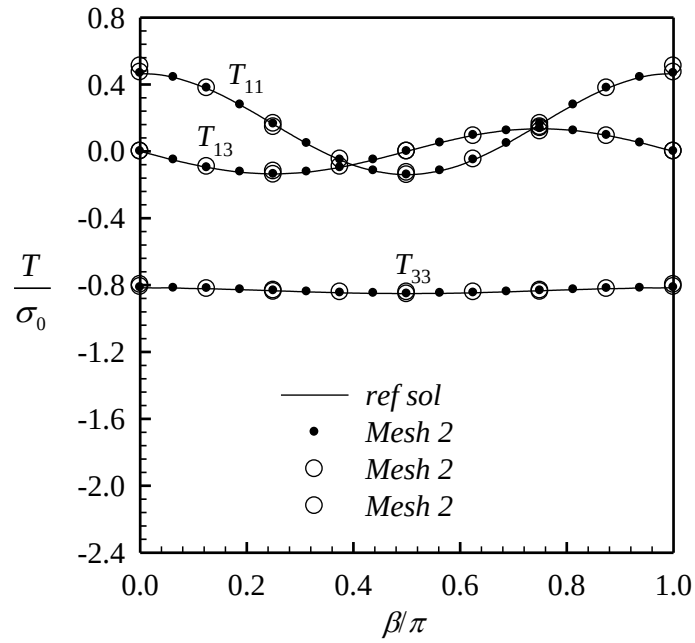


Figure 5.24: (a) Normalized stress intensity factors and (b) normalized T-stress components for spherical cap crack in isotropic half-space under BC-type-4.



(a)



(b)

Figure 5.25: (a) Normalized stress intensity factors and (b) normalized T-stress components for spherical cap crack in transversely isotropic half-space under BC-type-1.

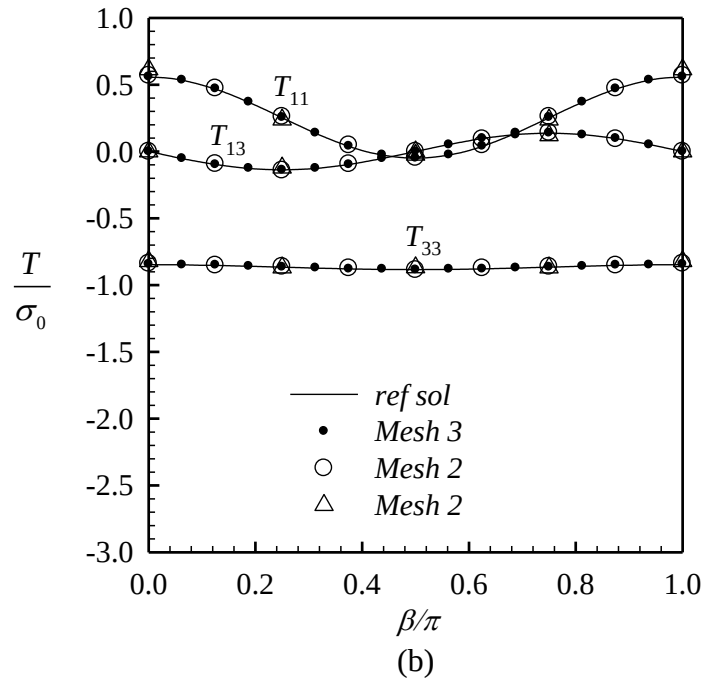
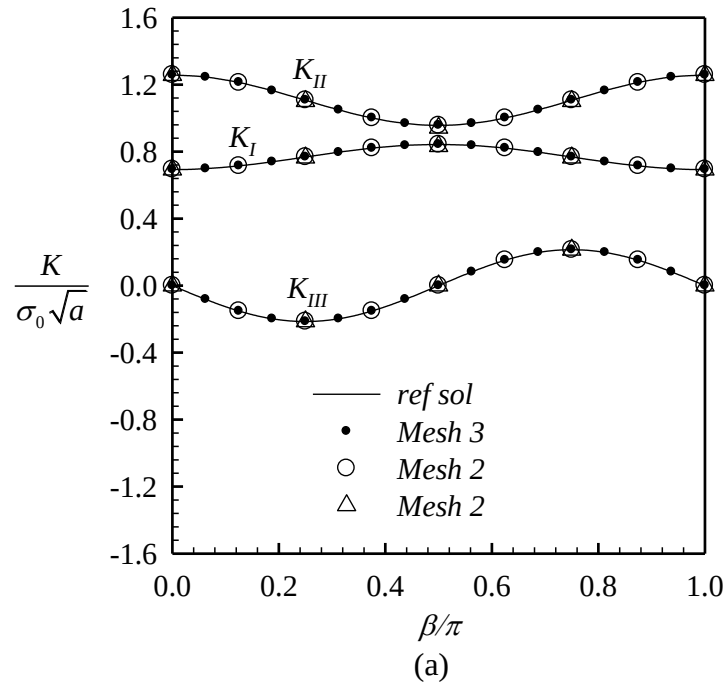


Figure 5.26: (a) Normalized stress intensity factors and (b) normalized T-stress components for spherical cap crack in transversely isotropic half-space under BC-type-2.

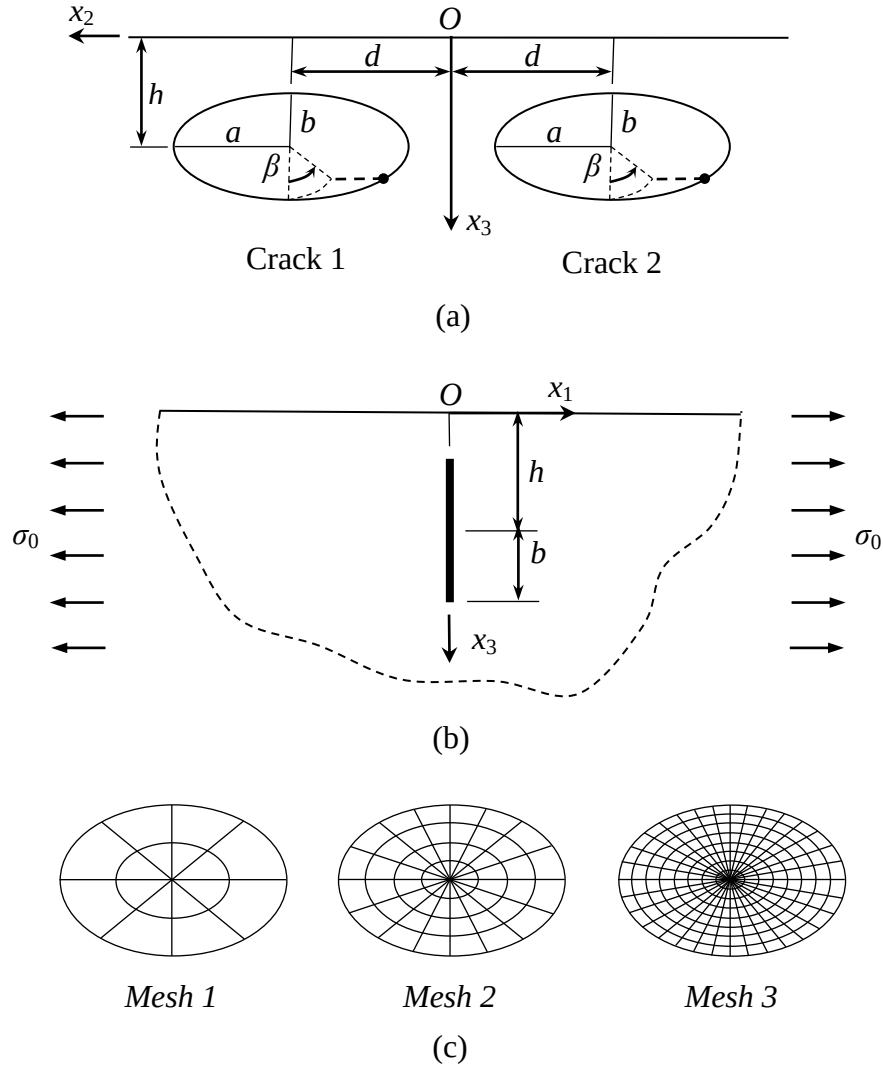


Figure 5.27: (a) Schematic of two identical elliptical cracks embedded in elastic half-space, (b) half-space under remote uniaxial tension in x_1 -direction, and (c) three meshes adopted in analysis.

The normalized stress intensity factors and T-stress components obtained from the transversely isotropic half-space under BC-type-1 and BC-type-2 along the front of the crack 1 are reported in Figures 5.36-5.39. Similar to results of the isotropic case, the good convergence of both stress intensity factors and T-stress components for $d/a = 1.1$ and 2.0 are observed. In particular, results obtained from the coarse and intermediate meshes are almost indistinguishable from those computed from the fine mesh. Results shown in Figures 5.36-5.39 also indicate that the interaction of the two elliptical cracks and its influence on the stress intensity factors and T-stress components is obviously dependent on the ratio d/a . The variation of the stress intensity factors and the T-stress components is found significant along the region of the crack front where the two cracks are relatively close. In particular, this variation is more rapid when the value of d is close to a .

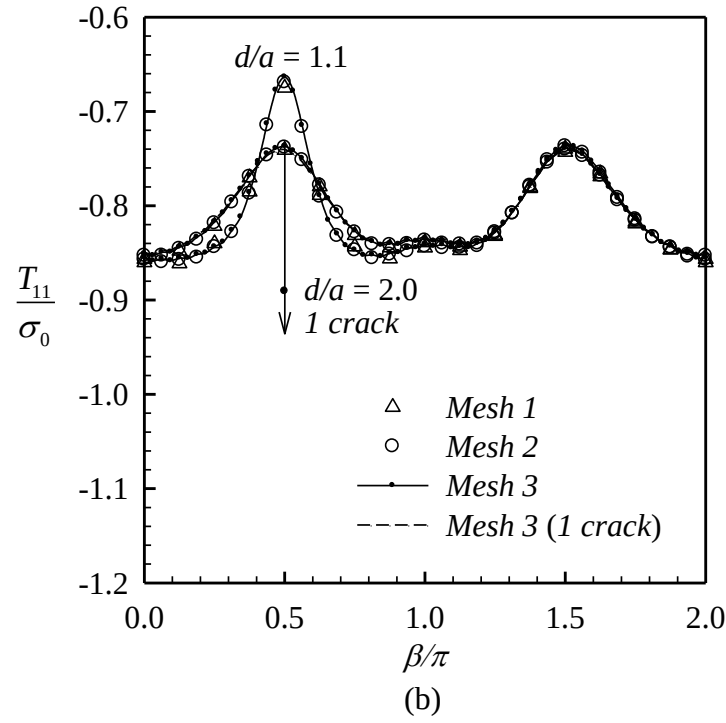
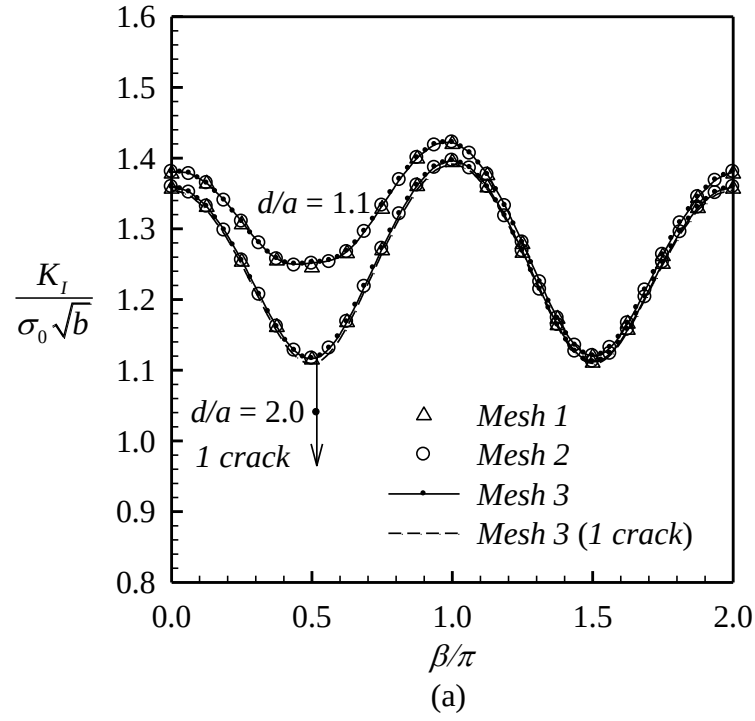


Figure 5.28: (a) Normalized mode-I stress intensity factor and (b) normalized T-stress component T_{11} for elliptical crack 1 in isotropic half-space under BC-type-1.

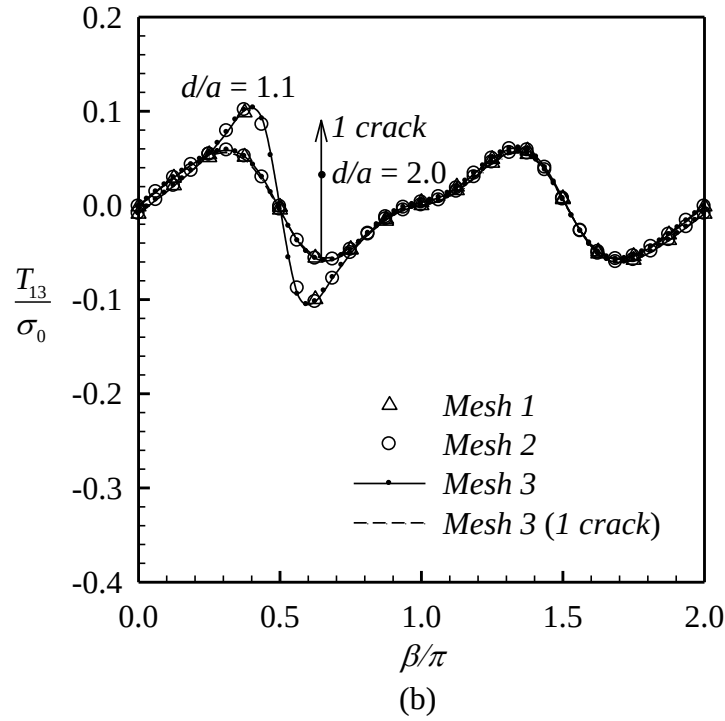
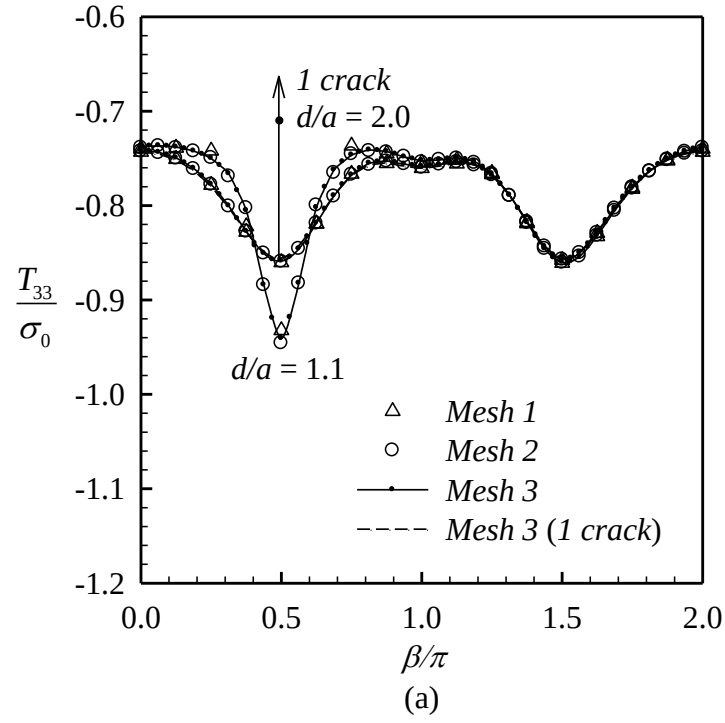


Figure 5.29: (a) Normalized T-stress component T_{33} and (b) normalized T-stress component T_{13} for elliptical crack 1 in isotropic half-space under BC-type-1.

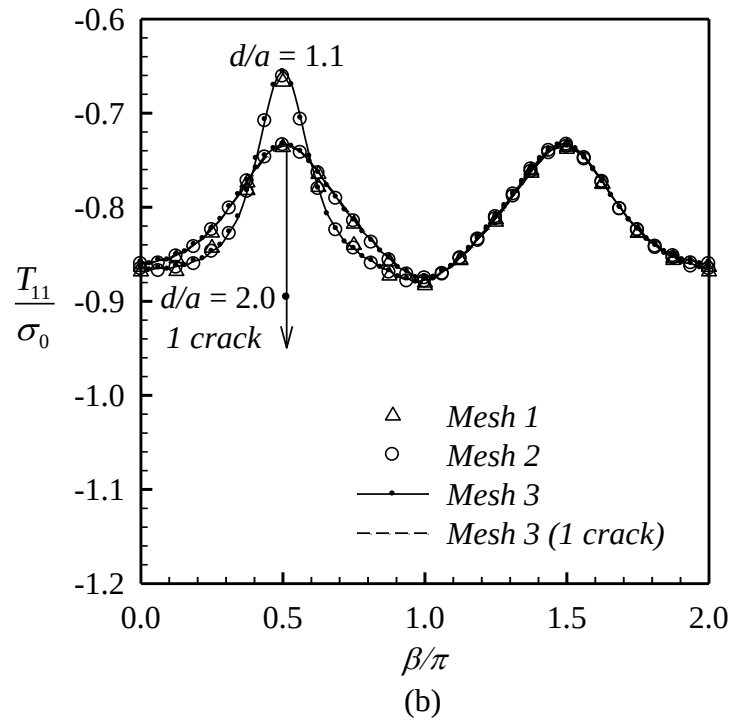
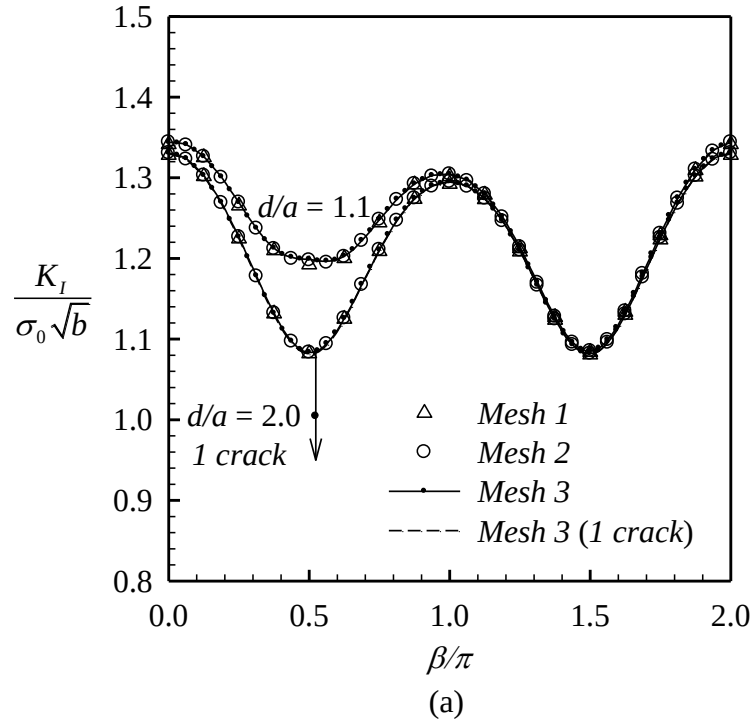


Figure 5.30: (a) Normalized mode-I stress intensity factor and (b) normalized T-stress component T_{11} for elliptical crack 1 in isotropic half-space under BC-type-2.

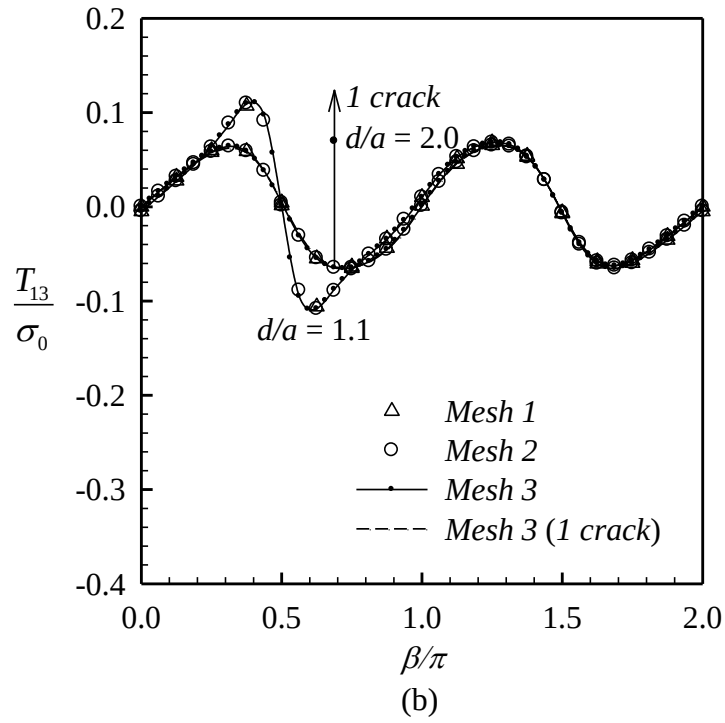
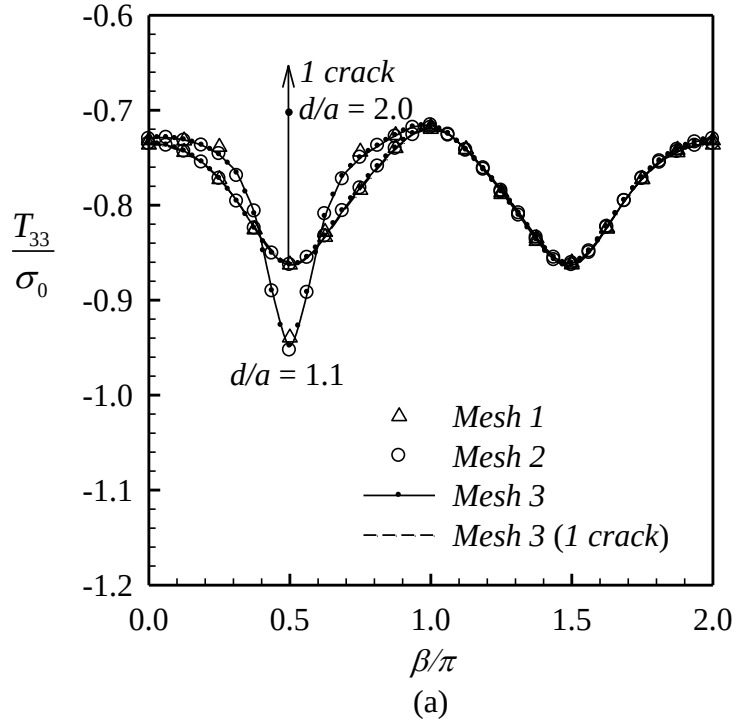


Figure 5.31: (a) Normalized T-stress component T_{33} and (b) normalized T-stress component T_{13} for elliptical crack 1 in isotropic half-space under BC-type-2.

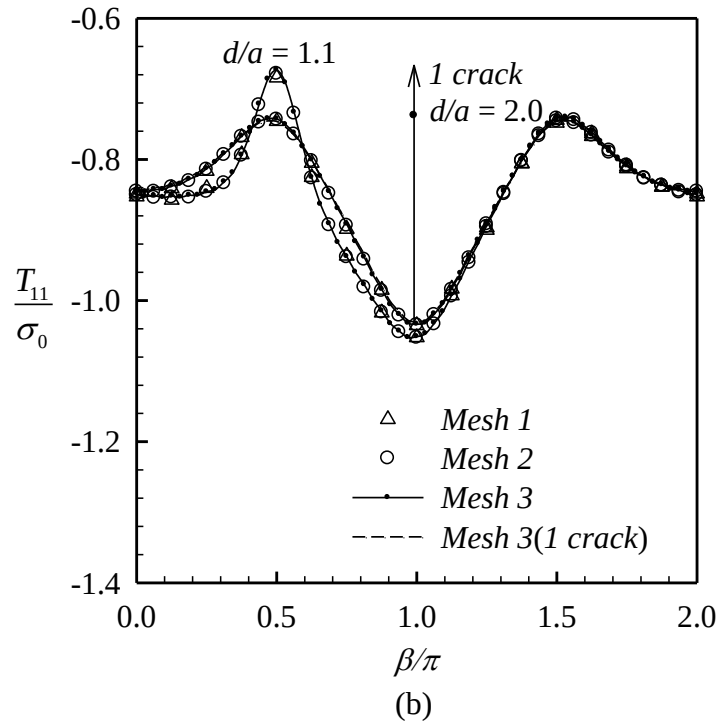
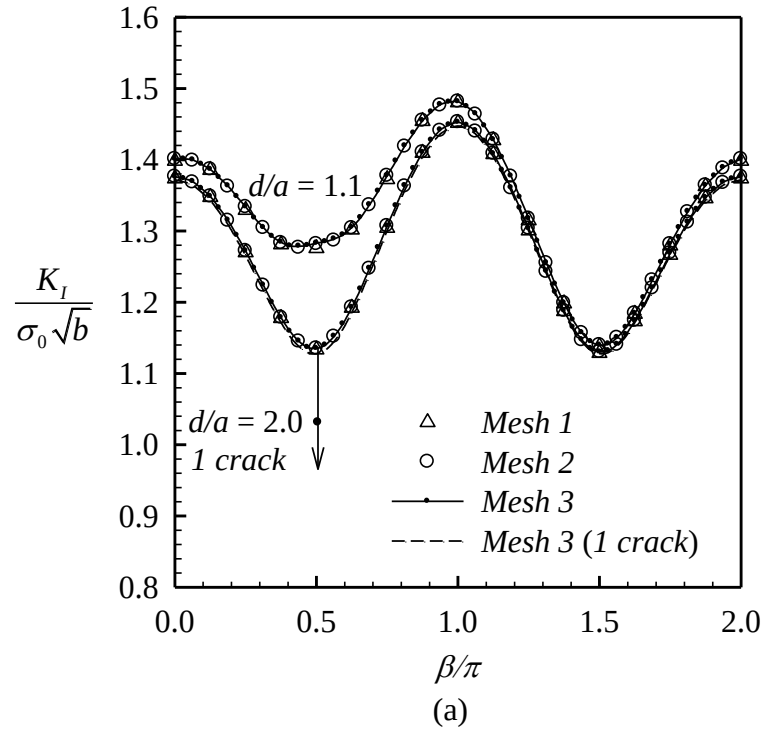


Figure 5.32: (a) Normalized mode-I stress intensity factor and (b) normalized T-stress component T_{11} for elliptical crack 1 in isotropic half-space under BC-type-3.

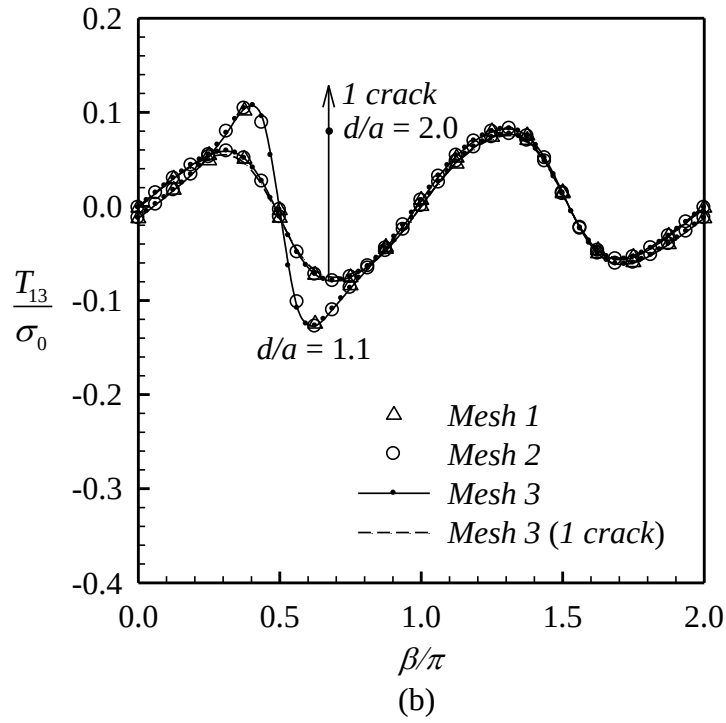
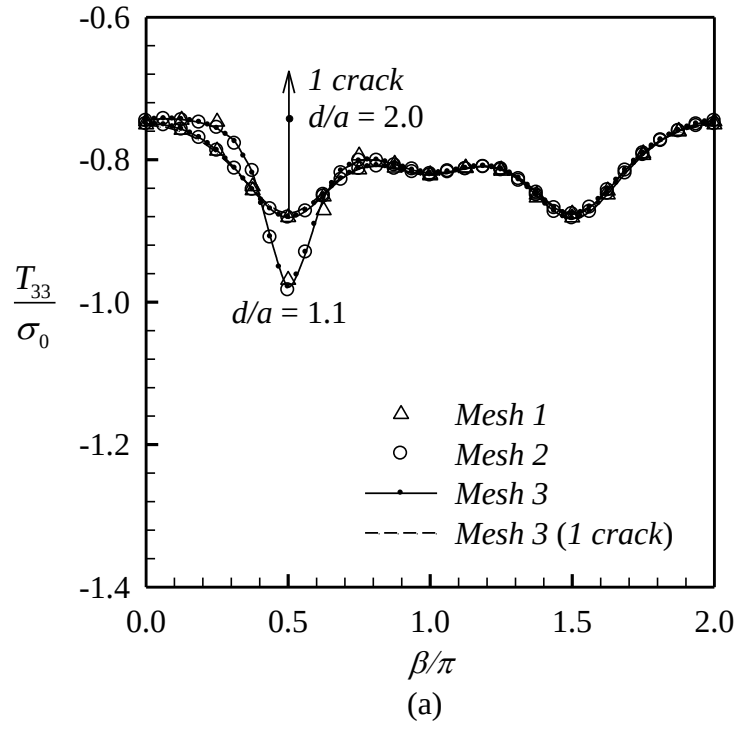


Figure 5.33: (a) Normalized T-stress component T_{33} and (b) normalized T-stress component T_{13} for elliptical crack 1 in isotropic half-space under BC-type-3.

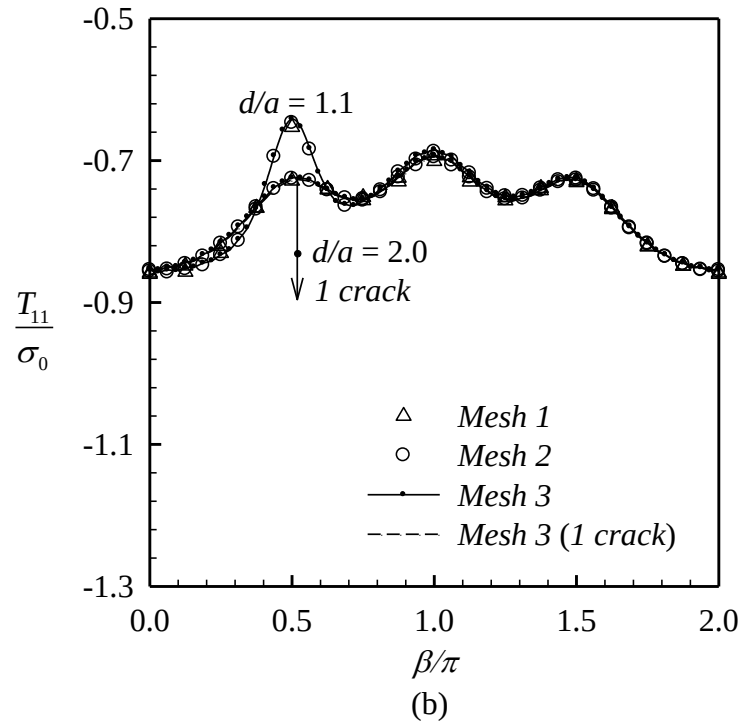
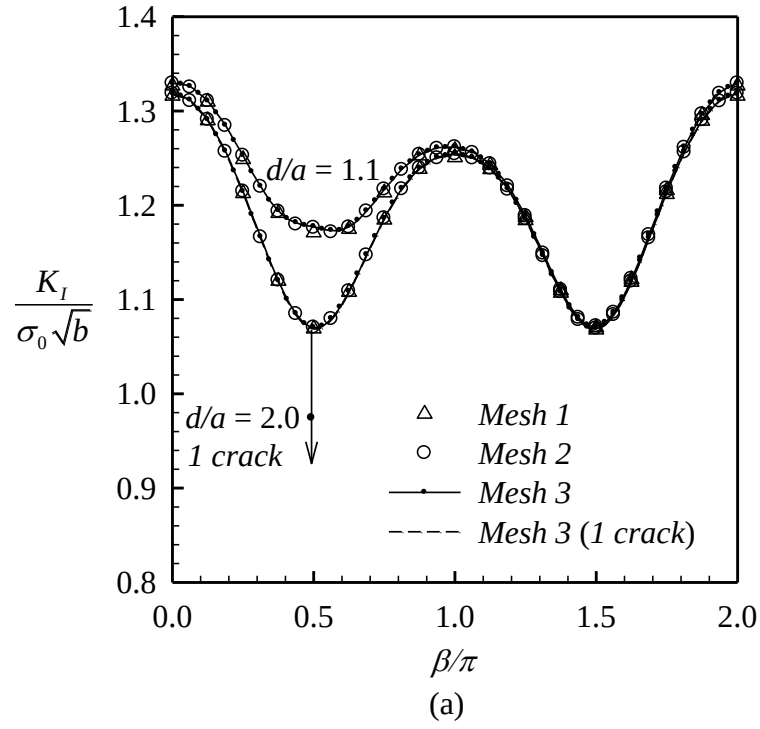


Figure 5.34: (a) Normalized mode-I stress intensity factor and (b) normalized T-stress component T_{11} for elliptical crack 1 in isotropic half-space under BC-type-4.

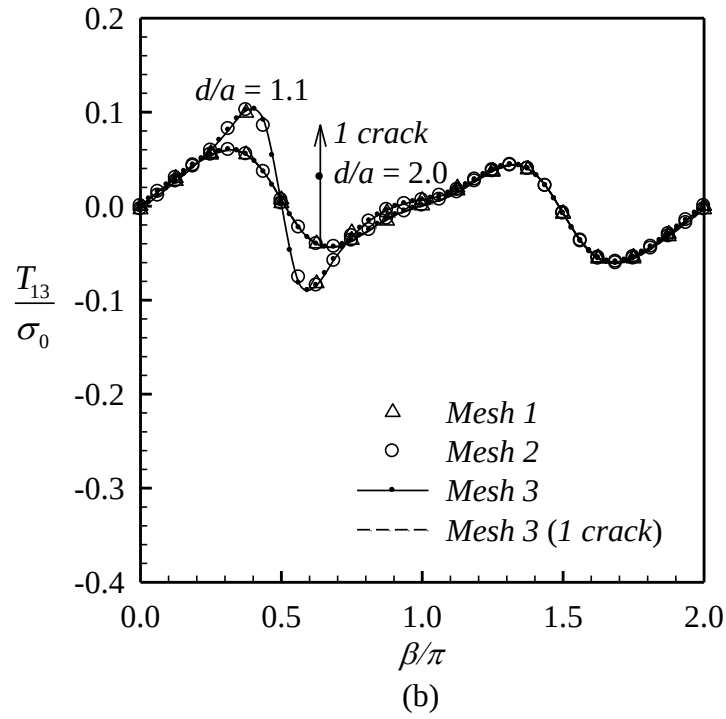
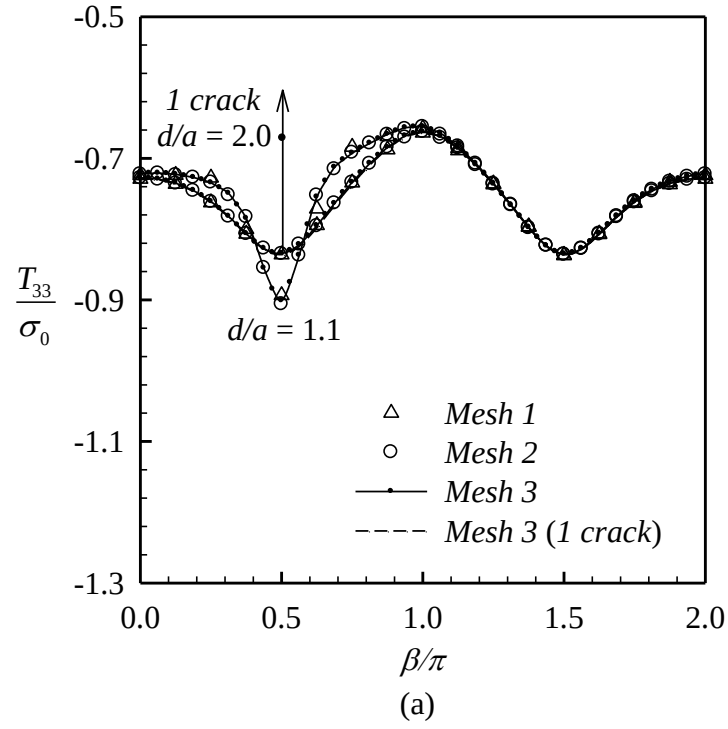


Figure 5.35: (a) Normalized T-stress component T_{33} and (b) normalized T-stress component T_{13} for elliptical crack 1 in isotropic half-space under BC-type-4.

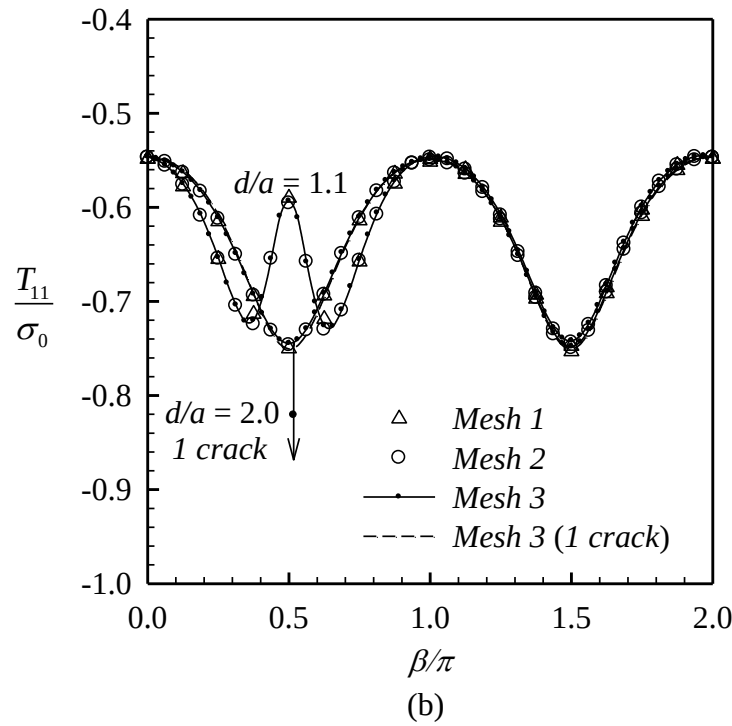
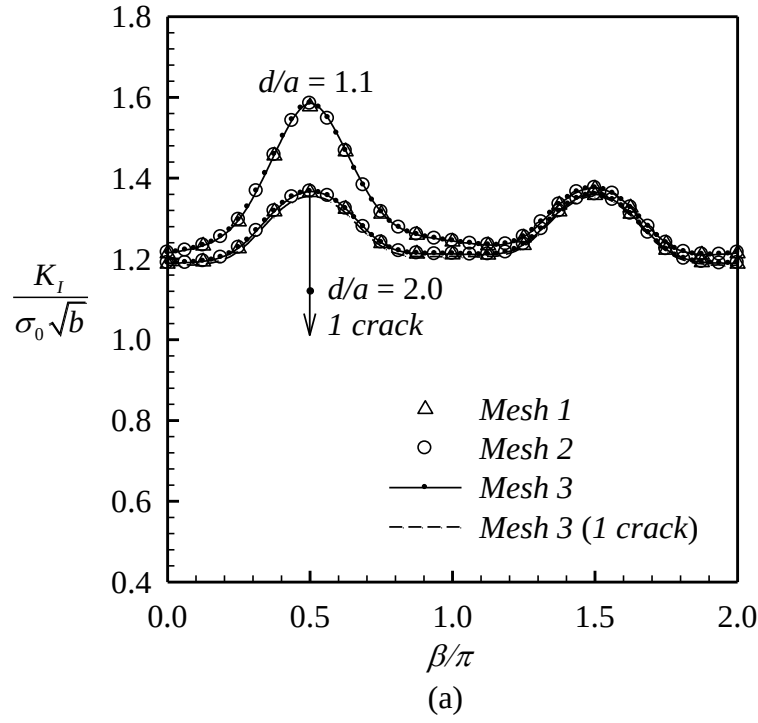


Figure 5.36: (a) Normalized mode-I stress intensity factor and (b) normalized T-stress component T_{11} for elliptical crack 1 in transversely isotropic half-space under BC-type-1.

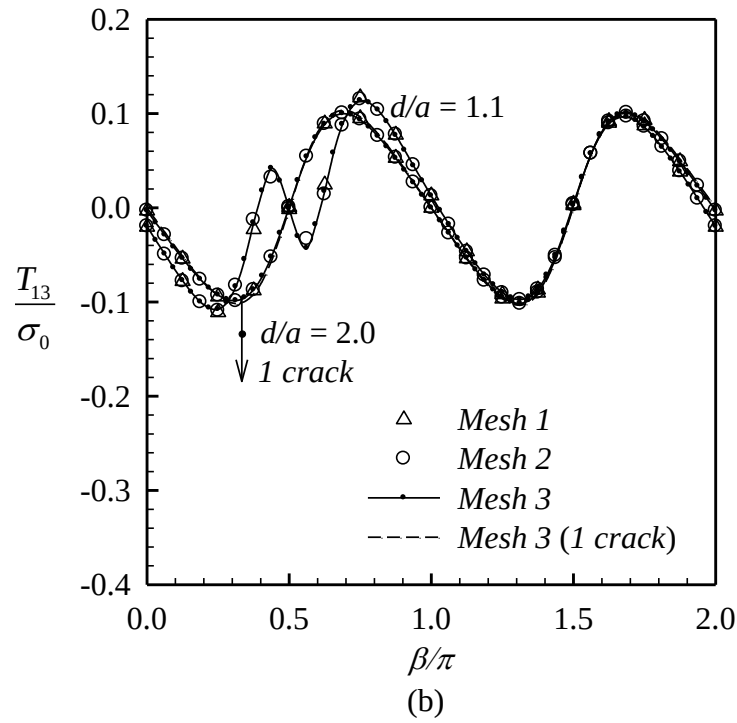
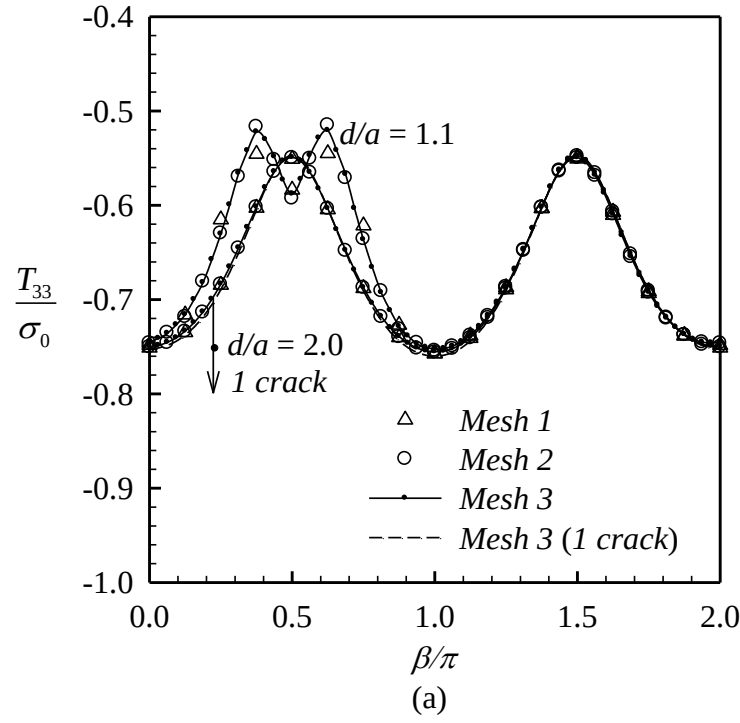


Figure 5.37: (a) Normalized T-stress component T_{33} and (b) normalized T-stress component T_{13} for elliptical crack 1 in transversely isotropic half-space under BC-type-1.

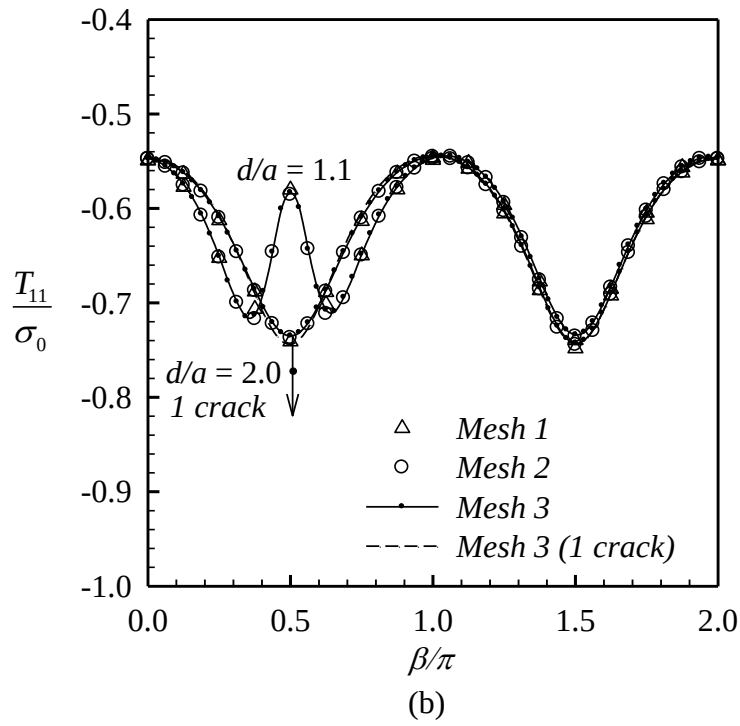
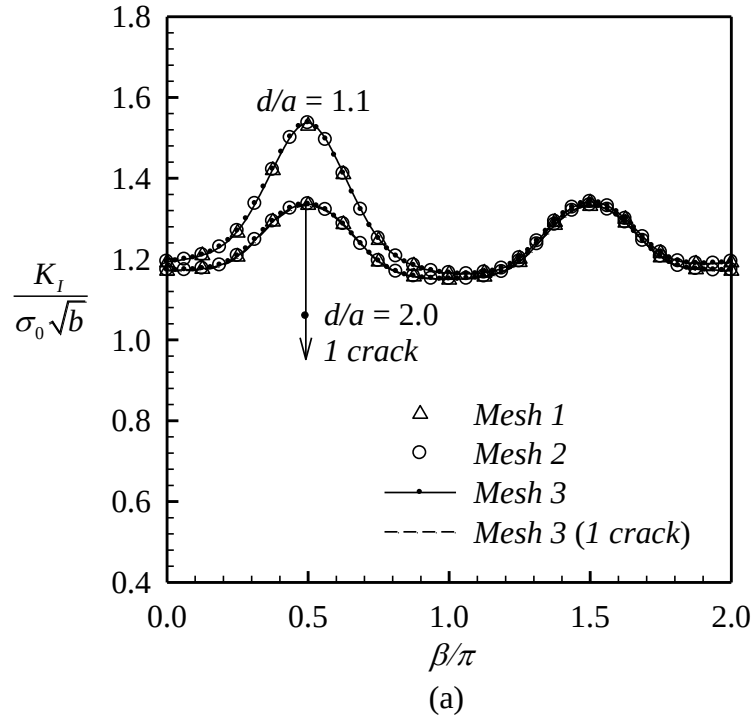


Figure 5.38: (a) Normalized mode-I stress intensity factor and (b) normalized T-stress component T_{11} for elliptical crack 1 in transversely isotropic half-space under BC-type-2.

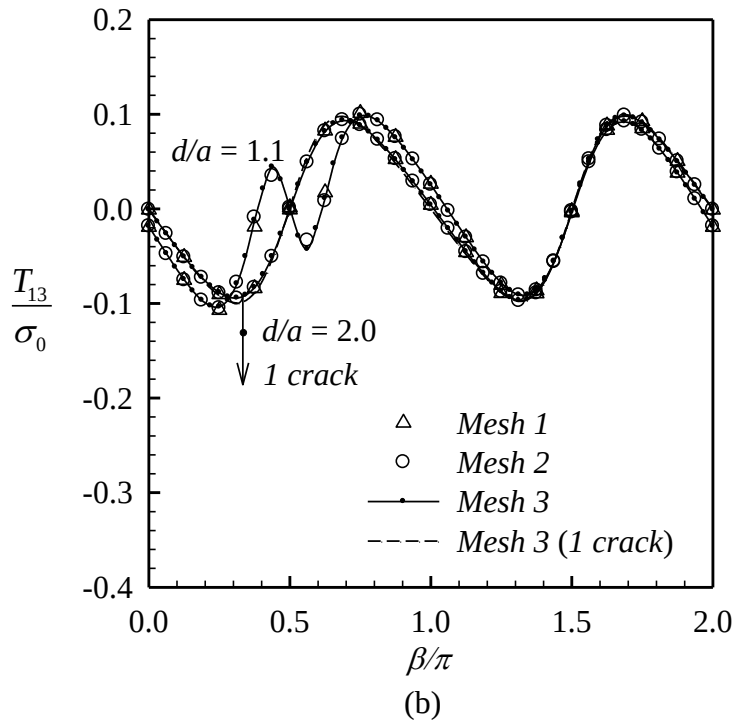
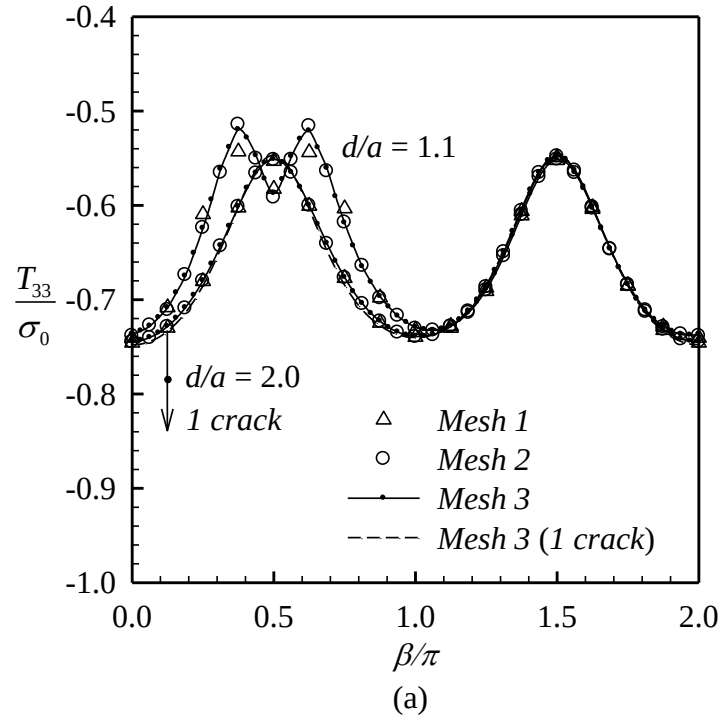


Figure 5.39: (a) Normalized T-stress component T_{33} and (b) normalized T-stress component T_{13} for elliptical crack 1 in transversely isotropic half-space under BC-type-2.

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CHAPTER 6

CONCLUSIONS AND REMARKS

This chapter summarizes both analytical and numerical techniques (e.g., scaled boundary finite element method and boundary integral equation method) established in the present investigation for the analysis of fundamental problems in solid mechanics. Major findings regarding to both the computational performance of the developed techniques (e.g., convergence and accuracy) and the extensive parametric study on the influence of various parameters on predicted solutions are also concluded. Limitations of the current development and the possible extensions for each particular problem are also indicated.

6.1 SBFEM for Linear Multi-field Problems

A numerical technique based upon the scaled boundary finite element method (SBFEM) has been successfully developed for solving two-dimensional, multi-field boundary value problems. Both the formulation and implementations have been established in a general framework allowing various classes of linear boundary value problems (e.g., steady-state heat conduction problems, Laplace's equation, linear elasticity, linear piezoelectricity, etc.) and a set of general data such as the domain geometry, the prescribed distributed body source, the prescribed boundary conditions, and the contribution of the side-face conditions to be treated in a single, unified fashion. Results from an extensive numerical study for various scenarios have revealed that the proposed SBFEM yields highly accurate numerical solutions with the good convergence behavior. Relatively coarse meshes containing only few degrees of freedom have been found to accurately capture both the state variable and the body flux. In particular, both prescribed state variables and prescribed surface flux along the side faces and the general mixed boundary conditions along the scale boundary direction have been implemented into the proposed procedure and this, therefore, provides the flexibility in the selection of the scaling center and defining curve for describing the domain geometry.

Several numerical examples have been solved and obtained results have been compared with analytical solutions to validate the proposed technique. Linear and quadratic shape functions have been employed in the approximation of the defining curve and the trial and test functions. From the convergence study of numerical solutions, it has been found that the rate of convergences resulting from the use of quadratic elements in the approximation is higher than that of the linear elements. In addition, the capability and robustness of the proposed technique to handle the general boundary conditions, general geometry, prescribed state variables, and the flexible choices of the scaling center have been demonstrated. Extensive analysis of problems in various fields such as steady-state heat conduction problems, linear elasticity problems, and linear piezoelectric problems, have also indicated that the proposed technique is promising in terms of accuracy and computational efficiency and can handle problems in various different scenarios in an efficient and unified fashion.

Although the proposed technique has been implemented within the context of two-dimensional boundary value problems with domains described by a single scaling center, its underlying formulation and computational procedure are sufficiently general and should provide an essential basis for an extension to treat more general bodies such as those requiring multiple

scaling centers to fully describe their geometry. Another potential extension is to generalize the present unified framework to three-dimensional multi-field problems.

6.2 Generalized T-stress of Penny-shaped Crack in Piezoelectric Whole Space

An exact solution of the generalized T-stress components for a circular crack contained in a three-dimensional, transversely isotropic, linear piezoelectric, infinite body under electrically permeable, impermeable, semi-permeable and energetically consistent crack-face conditions and subjected to both a uniform generalized far field and uniformly distributed crack-face mechanical/electrical loading has been derived. In addition, for both permeable and impermeable crack-face conditions, the generalized T-stress Green's function due to a pair of opposite, unit concentrated crack-face loads and the integral formula equipped with the numerical quadrature for calculating the generalized T-stress components under general loading conditions have been also established. In the derivation, existing generalized stress fields for various cases have been employed together with the near-front expansion and standard differentiations and proper limiting procedures. The obtained analytical solutions are explicit and involve only elementary functions, and they can be used as a basis for the parametric study to explore the effect of loading conditions, crack-face conditions and material properties on the generalized T-stress components along the crack boundary. In addition, the derived solutions can be also employed as the reliable benchmark solutions in the validation procedure of newly developed numerical schemes (e.g., boundary element and finite element methods) for analysis of general and complex crack problems.

The implemented integral formula using the proposed numerical quadrature has been also tested for several scenarios and it has been indicated that obtained results are in very good agreement with benchmark solutions. Additionally, results from a preliminary parametric study have shown that both the crack-face conditions and crack-face loading play an important role on the generalized T-stress along the crack boundary. To gain an in-depth understanding of the effect of many parameters on the generalized T-stress, a more extensive parametric study is still required and solutions and formula obtained from the present study should provide a useful basis for such investigation.

6.3 Weakly Singular BIEM for Analysis of Cracks in Multi-field Media

An accurate and efficient numerical technique based upon a weakly singular, boundary integral equation method together with the direct post-process procedure has been successfully implemented for the analysis of the generalized T-stress components for isolated cracks in a whole space made of linear coupled-field materials. The formulation and implementation have been established in a general framework allowing various scenarios (e.g., various classes of material models, general crack geometries, and general loading conditions) to be handled in a unified fashion. A pair of weakly singular, weak-form integral equations, one governing the jump in the crack-face generalized displacement and the other governing the gradient of the sum of the crack-face generalized displacement, has been derived and then used to form the key governing equations. In addition to its unified feature, this pair of integral equations contains only weakly singular kernels allowing all involved integrals to be interpreted in the sense of Riemann sum and continuous basis functions to be employed in the solution approximation.

A well-known symmetric Galerkin boundary element method has been implemented within the context of weakly singular formulation to solve the weak-form integral equation for the crack-face generalized traction. In addition to the careful treatment of the numerical integration and evaluation of all involved kernels, special near-front approximation has been utilized to enhance the solution quality of the jump in the crack-face generalized displacement in the neighborhood of the crack front. The gradient of the sum of the crack-face generalized displacement has been solved separately, once the jump in the crack-face generalized displacement was determined, from the remaining weak-form integral equation for the crack via a standard Galerkin technique. An explicit formula has been established, based on the asymptotic near-front field expansion together with the constitutive laws, for extracting the generalized T-stress directly in terms of the gradient of the sum of the crack-face generalized displacement along the crack front.

Extensive numerical experiments together with benchmarking with reliable reference solutions for various cases have confirmed the validity of the integral formulation, solution procedure and post-process algorithm. From the convergence study via a series of meshes, it has been found that the proposed technique yields the converged numerical solutions and accurate results can be obtained even when relatively coarse meshes are adopted in the discretization. The highly accurate feature of the present technique stems directly from the use of special crack-tip elements to improve the approximation of the near-front relative crack-face generalized displacement and the post-process formula for extracting the generalized T-stress directly in terms of primary unknowns solved from the governing integral equations. The computational capability and versatility of the technique to treat boundary value problems with general data (e.g., non-planar and multiple cracks, various types of linear coupled-field materials, general loading conditions, etc.) have also been elucidated via a selected set of representative scenarios.

It should be remarked that the computational performance of the proposed technique in comparison with other numerical procedures such as those proposed by Rungamornrat *et al.* (preparation for publication) for cracks in elastic media and Subsathapol *et al.* (2014) and Limwibul *et al.* (2016) for cracks in piezoelectric media is still required further investigation. In addition, the framework of the proposed technique is still limited to isolated cracks in an infinite media. The potential extension of the present technique to treat cracks in a finite domain should significantly enhance its capability to handle more practical problems.

6.4 Weakly Singular BIEM for Analysis of Cracks in Elastic Half-space

A set of singularity-reduced boundary integral relations and equations has been established for a generally anisotropic, linearly elastic half-space containing arbitrary shaped cracks with the free surface subjected to symmetrical and anti-symmetrical boundary conditions on the free surface. The systematic regularization procedure based on the integration by parts via Stokes' theorem and special decompositions for both strongly singular and hypersingular kernels has been developed and the final completely regularized integral equations containing only weakly singular kernels. In addition, results proposed by Li (1996) has been adopted to form a set of singularity-reduced boundary integral relations and equations for an isotropic, linearly elastic half-space subjected to the traction-free boundary condition and the fully-restrained boundary condition on the free surface. Another key feature of the developed integral equations is the automatic treatment of the free surface via the use of the fundamental solutions of the half-space with the same type of boundary conditions. This therefore avoids the discretization of the free

surface in the solution procedure. A weakly singular symmetric Galerkin boundary element method has been successfully implemented to numerically solve the weak-form traction boundary integral equation for the relative crack-face displacement and such information has then been utilized as the known data to determine the sum of the crack-face displacement from the weak-form displacement boundary integral equation by standard Galerkin method. Special crack-tip elements have also been employed to enhance the approximation of the near-front field. The fracture data along the crack front such as the stress intensity factors and T-stresses has been directly extracted from the relative and sum of crack-face displacement using the explicit formula.

Results from extensive numerical experiments and the comparison with several benchmarked cases have revealed that the proposed numerical procedure is highly accurate and computationally robust for the analysis of anisotropic cracked half-spaces under symmetric and anti-symmetric boundary conditions and isotropic cracked half-space under various types of boundary conditions. Use of the special crack-tip elements along the crack front has indicated that the stress intensity factors and the T-stress components can be accurately captured using relatively very coarse meshes and this therefore renders the technique more suitable for linear fracture analysis than the standard finite element method which generally requires sufficiently fine mesh to capture the near-front field and experiences difficulty in the treatment of an unbounded domain.

While the proposed technique has been successfully implemented, it is still restricted, for an anisotropic cracked half-space, to symmetric and anti-symmetric boundary conditions and the material must possess the plane of material symmetry parallel to the free surface. The potential extension of the current work to treat other types of boundary conditions such as the traction-free and rigid surfaces and other types of materials such as multi-field and smart solids is considered essential. It is important to emphasize that besides the reduction of the computational cost directly gained from using the half-space model instead of the full treatment of a cracked whole space under symmetric and anti-symmetric conditions, the key ingredients and results established in the present study also form the useful and essential basis for such nontrivial generalization.

Appendix A

List of Publications

Published/Accepted Articles in International Journals

1. Nguyen C.V., **Rungamornrat J.**, Pheinsusom P. Scaled boundary finite element method for two-dimensional linear multi-field media. *Engineering Journal*, 2017, **21(7)**, pp.333-360. (ISI/SCOPUS, IF = NA, SJR = 0.194)
2. **Rungamornrat J.**, Nguyen T.B., Pich V.C., Phongtinnaboot W., Wijeyewickrema A.C. Generalized T-stress solutions for penny-shaped cracks in transversely isotropic piezoelectric media. *Engineering Fracture Mechanics*, 2018, **192**, pp. 225-241. (ISI/SCOPUS, IF = 2.908, SJR = 1.331)
3. Pham T.N., **Rungamornrat J.**, Pansuk W., Sato Y. Weakly singular BIEM for analysis of cracks embedded in symmetric elastic whole space. *Lowland Technology International*, 2018, **20(2)**, pp. 169-186. (SCOPUS, SJR = 0.192)
4. **Rungamornrat J.**, Nguyen V.C. Scaled boundary finite element method with exact defining curves for two-dimensional linear multi-field media. *Frontiers of Structural and Civil Engineering*, 2019, **13(1)**, pp. 201-214. (ISI/SCOPUS, IF = 1.272, SJR = 0.401)
5. **Rungamornrat J.**, Sukulthanasorn N., Mear M.E. Analysis for T-stress of cracks in 3D anisotropic elastic media by weakly singular integral equation method. *Computer Methods in Applied Mechanics and Engineering*, 2019, **347**, pp. 1004-1029. (ISI/SCOPUS, IF = 4.821, SJR = 2.996)
6. **Rungamornrat J.**, Chansavang B, Phongtinnaboot W, Nguyen VC. Investigation of generalized SIFs of cracks in 3D piezoelectric media under various crack-face conditions. *Frontiers of Structural and Civil Engineering*, **Accepted for publication**. (ISI/SCOPUS, IF = 1.272, SJR = 0.401)

Articles Published in International Conference Proceedings

1. Limwibul V, Phongtinnaboot W, **Rungamornrat J**. Analysis for generalized intensity factors and T-stresses of permeable cracks in 3D linear piezoelectric media. Proceedings of the Fifteenth East Asia-Pacific Conference on Structural Engineering & Construction (EASEC-15), Xi'an, China, October 11-13, 2017
2. Sukulthanasorn N, **Rungamornrat J**, Furukawa A. Efficient BIEM for analysis of SIFs and T-stresses for cracks in 3D linear elastic whole space. Proceedings of the Thirtieth KKHTCNN Symposium on Civil Engineering, Taiwan, November 2-4, 2017
3. Le TM, **Rungamornrat J**, Bui TQ. Influence of surface and couple stresses on response of surface-loaded elastic half-plane. Proceedings of the 10th International Conference on Computational Methods (ICCM2019), Singapore, July 9-13, 2019
4. Pham BV, Nguyen TB, **Rungamornrat J**. Analysis of cracked body strengthened by adhesively bonded patches by BEM-FEM coupling. Proceedings of the 10th International Conference on Computational Methods (ICCM2019), Singapore, July 9-13, 2019

Articles Published in National Conference Proceedings

1. Watanavit P, **Rungamornrat J**, Samittakorn W. Analysis of near interface cracks by weakly singular boundary integral equation method. Proceedings of the 22nd National Convention on Civil Engineering, Khao Yai, Nakhon Ratchasima, Thailand, July 18-20, 2017
2. Sengcompan C, **Rungamornrat J**. Calculations of SIFs for sub-interface cracks in bi-material. Proceedings of the 24th National Convention on Civil Engineering, Udonthani, Thailand, July 10-12, 2019
3. Wongviboonsin W, **Rungamornrat J**. Size-dependence elastic responses of layered media under surface loading. Proceedings of the 24th National Convention on Civil Engineering, Udonthani, Thailand, July 10-12, 2019

Appendix B

Reprints/Manuscripts

Journal Article No. 1

Title: Scaled boundary finite element method for two-dimensional linear multi-field media

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Article

Scaled Boundary Finite Element Method for Two-Dimensional Linear Multi-Field Media

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Abstract. This paper presents an efficient and accurate numerical technique, based on a scaled boundary finite element method (SBFEM) that is capable of solving two-dimensional, second-order, linear, multi-field boundary value problems. Basic governing equations are established in a general, unified context allowing the treatment of various classes of linear problems such as steady-state heat conduction problems, steady-state flow in porous media, linear elasticity, linear piezoelectricity, and linear piezomagnetic and piezoelectromagnetic problems. A scaled boundary finite element approximation is also formulated within a general framework integrating the influence of the distributed body source, general boundary conditions, contributions of the general side-face data, and the flexibility of scaled boundary approximations. Standard procedures for numerical integration, search of eigenvalues and eigenvectors, determination of particular solutions, and solving a system of linear algebraic equations are adopted. After fully tested with available benchmark solutions, the proposed SBFEM is applied to solve various classes of linear problems under different scenarios to demonstrate its vast capability, computational efficiency and robustness.

Keywords: Multi-field problems, SBFEM, surface flux, state variables, scaled boundary coordinates.

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1. Introduction

A scaled boundary finite element method (SBFEM) has been found an attractive alternative analysis tool in the modelling of various problems in applied mechanics for the past two decades [1-12]. The method is recognized as a semi-analytical technique combining features of both analytical schemes and the finite element approximation and, due to the reduction of a spatial dimension in the discretization, it can be also categorized as a particular type of boundary element methods (BEMs). However, the key difference between the SBFEM and conventional BEMs is that the former is generally free of fundamental solutions; as a direct consequence, the non-trivial numerical treatment of singular integrals is not required. The concept of the SBFEM was originally introduced by Wolf and Song [13-15] using the mechanical-based approach to properly and efficiently model linear elastic unbounded media in the dynamic analysis of soil-structure interactions. To further reduce the complexity of the formulation presented in the original work, other techniques such as the standard weighted residual procedure and the principle of virtual work were also utilized to obtain the scaled boundary finite element equations [16-18]. As a result of the computational efficiency of the SBFEM in modelling unbounded media and the reduction of the discretization cost, various researches have been continuously and increasingly conducted, since its first emergence, to introduce other novel aspects and further enhance its capability.

In recent years, the SBFEM has been also applied to solve various boundary value problems. One salient feature of the SBFEM is that the whole domain can be generated by scaling its boundary with respect to a single point termed the scaling center. The SBFEM only requires meshing on a representative boundary of the domain and does not involve fundamental solutions. For a domain with the complex geometries, its interior can be discretized into sub-domains to satisfy the scaling requirement. For fracture problems, the scaling center is commonly located at the crack tip and, as a result, the stress field can be expressed analytically along the direction radiating from the crack tip. As a result, the strength of singularity and associated information can be directly and accurately calculated from the obtained solution [19, 20]. Based on this positive feature, the method has been extensively utilized in the investigation and simulations of fracture problems under various scenarios such as crack formation, static and dynamic crack propagation, and transient responses of bodies containing interfacial cracks, stress singularities and bi-material interfaces [21, 22]. In the modelling of wave propagations, the SBFEM has been also widely employed to study elastic guided waves in an unbounded domain and the radiation and diffraction of linear water waves in shallow water with sidewalls [23-25]. Results from those investigations demonstrated that the SBFEM generally yields more accurate numerical solutions and is more computationally efficient, in terms of the number of degrees of freedom involved, than other existing approaches. Li et al. [26] applied the SBFEM to investigate the structural behavior of offshore monopoles with the influence of the ocean wave loads. The basic concept of the SBFEM was adopted to formulate the governing monopole's equation and the analytical wave equation. Later, Li et al. [6, 7] extended the SBFEM to study three-dimensional wave-pile problems with the emphasis on the wave behavior and pile-group responses. In their formulation, the scaled boundary finite element approximation is applied to Helmholtz equation by separating the vertical component from the velocity potential. The two-dimensional SBFEM was adopted to examine the wave field near the free surface level. Results from these studies indicated that the SBFEM is an accurate and efficient computational tool for the analysis of wave problems and wave-structure interactions. However, their formulation and implementations were still limited to particular settings.

To further enhance the capability of the technique, Deeks and Wolf [27] also integrated an *h*-hierarchical adaptive procedure to the standard SBFEM. This technique exploited the ability of the SBFEM to model the stress singularities at the scaling center and to avoid the discretization of certain adjacent segments of the boundary. Later, Vu and Deeks [28] investigated the performance of high-order elements in the SBFEM. In their study, both the spectral element and hierarchical approaches were employed and they found that the spectral element approach was better than the hierarchical approach in terms of the computational efficiency. Deeks and Augarde [29] also incorporated the meshless scheme into the standard SBFEM to model the far-field problems. Although, the SBFEM has been successfully implemented in various applications, it also possesses certain drawbacks when the number of degrees of freedom resulting from the discretization becomes large, rendering the computational expenses substantial. To overcome such disadvantage, Vu and Deeks [30] integrated a *p*-adaptive scheme in the SBFEM for solving two dimensional boundary value problems. In this study, an alternative set of refinement criteria was considered to maximize the solution accuracy while minimizing the computational cost. Furthermore, He et al. [31] presented a new element-free Galerkin scaled boundary finite element method for the approximation in the

circumferential direction. This particular technique was then applied to solve a number of standard linear elasticity problems and the technique was found to offer higher and better rate of convergence than the original SBFEM. Additionally, He et al. [32] investigated the possibility of using the Fourier shape functions in the SBFEM. The developed technique was then used to solve elastostatics and steady-state heat transfer problems. It was found that the accuracy and convergence of numerical solutions were better than those of results obtained from using the polynomial shape functions and an element-free Galerkin scheme.

Applications of the SBFEM to the analysis of linear problems involving piezoelectric materials have also been recognized in the literature. Liu and Lin [4] used the SBFEM to investigate two-dimensional electrostatic problems. Li et al. [5] developed the SBFEM to solve two-dimensional, linear piezoelectric fracture problems. The stress and electric displacement intensity factors for both static and dynamic cases were calculated directly from results of the SBFEM. It was indicated that the technique requires no asymptotic solution, no local mesh refinement, and no special treatment around the crack tip. Recently, Dieringer and Becker [33] employed the SBFEM to investigate linear problems within the framework of a classical laminated plate theory. In their study, the scaled boundary finite element equations for composites were formulated in terms of the displacement, and the stress singularity at a notch was fully examined. They also demonstrated that the enhanced SBFEM can evaluate the singular stress field as an explicit function of the notch opening displacement. Li et al. [34] and Li et al. [11] also employed the SBFEM to perform two-dimensional simulations of dynamic cracks and interfacial cracks in piezoelectric composites and also study the influence of thermal loads on the fracture response. While the SBFEM has been applied successfully to solve linear piezoelectric problems, the underlying formulation and existing implemented procedures are still limited to certain scenarios and, in particular, the extension of the technique to treat other general coupled-field media has not been found in the literature.

Other recent applications of the SBFEM have been also recognized. For instance, Ooi et al. [9,10] developed an efficient procedure based on the SBFEM together with polygon elements for simulating dynamic crack propagation in elastic media; He et al. [35] presented the SBFEM for the numerical analysis of two-dimensional elastic bodies with the rotationally periodic symmetry and subjected to arbitrary loading conditions; and, most recently, Vu and Deeks [8] integrated the information of the fundamental solutions into the scaled boundary finite element method and then used the technique to investigate problems in linear elasticity with concentrated loads. As become evident from the vast amount of related publications, researches on both the novel development and the enhancement of existing SBFEM have continuously and increasingly grown. However, most of existing SBFEMs were developed specifically for problems considered and this, as a consequence, limits its generality and flexibility to treat general field problems under various scenarios when compared with standard finite element methods. A systematic generalization of the SBFEMs to treat a broader class of boundary value problems obviously requires further rigorous investigations.

The present study aims to offer the SBFEM capable of solving two-dimensional, linear, second-order, multi-field boundary value problems. The key and novel feature is that all basic field equations governing responses of interest are formulated in a general framework allowing various types of linear problems such as steady-state heat conduction and flow in porous media, Laplace's equation, linear elasticity, linear piezoelectricity and other coupled-field problems to be treated in a unified manner. In addition, the treatment of distributed body source, general boundary conditions, prescribed conditions on the side faces, and the scaled boundary approximations are integrated in the implementation.

2. Problem Formulation

Consider a two-dimensional body occupying a region Ω in $\mathbb{R}^2$ as shown schematically in Fig. 1. The region is assumed *smooth* in the sense that all involved mathematical operators (e.g., integrations and differentiations) can be performed on this region. In addition, the boundary of the body Ω , denoted by $\partial\Omega$, is assumed piecewise smooth and an outward unit normal vector at any smooth point on $\partial\Omega$ is denoted by $\mathbf{n} = \{n_1 \ n_2\}^T$. More restrictions about the geometry of the body Ω pertained in the present study will be posed later as is appropriate. A two-dimensional Cartesian coordinate system $\{\boldsymbol{\theta}; x_1, x_2\}$ with the origin located at point $\boldsymbol{\theta}$ is introduced and a symbol $f_{,\alpha}$ is used throughout to denote a partial derivative of a function f with respect to the coordinate x_α , $\alpha \in \{1, 2\}$ (i.e., $f_{,\alpha} = \partial f / \partial x_\alpha$). Here and in what follows, standard indicial notations apply for subscripts with lower-case Greek and upper-case indices.

In particular, lower-case Greek subscripts range from 1 to 2 whereas upper-case subscripts range from 1 to $\Lambda \in \{1, 2, 3, \dots\}$ and repeated subscripts imply the summation over their range unless stated otherwise.

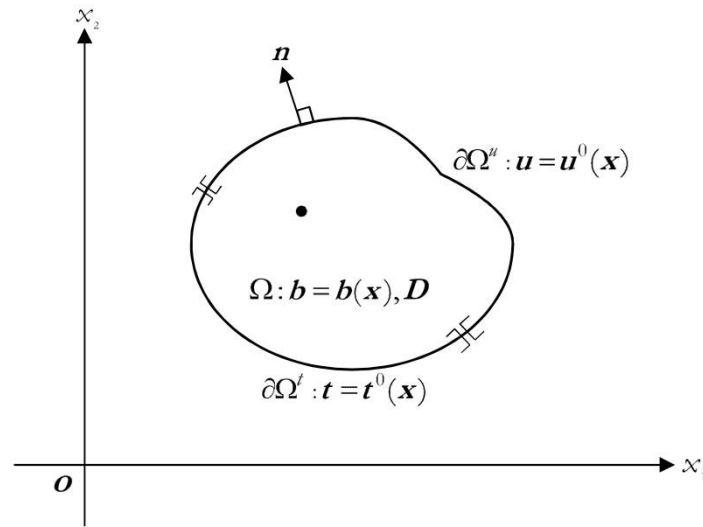


Fig. 1. Schematic of two-dimensional, multi-field body subjected to external excitations.

The body is made of a homogeneous material with its behavior completely characterized by $4\Lambda^2$ constants denoted by a set $\{E_{\alpha\beta}\}$ and subjected to a prescribed distributed body-source field, denoted by a Λ -component vector field $\mathbf{b}(\mathbf{x}) = \{b_1(\mathbf{x}) \ b_2(\mathbf{x}) \ \dots \ b_\Lambda(\mathbf{x})\}^T$. In the present study, the constants $E_{\alpha\beta}$ are assumed to satisfy the symmetry $E_{\alpha\beta} = E_{\beta\alpha}$. Responses of the body due to the distributed body-source $\mathbf{b}(\mathbf{x})$ are assumed to be completely described by following three fields: the state variable $\mathbf{u}(\mathbf{x})$, the state-variable gradient $\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$, and the body flux $\boldsymbol{\sigma}(\mathbf{x})$. The state variable $\mathbf{u}(\mathbf{x})$ contains Λ components denoted by $u_j(\mathbf{x})$ and is represented, in a vector form, by

$$\mathbf{u}(\mathbf{x}) = \{u_1(\mathbf{x}) \ u_2(\mathbf{x}) \ \dots \ u_\Lambda(\mathbf{x})\}^T \quad (1)$$

The state-variable gradient $\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ and the body flux $\boldsymbol{\sigma}(\mathbf{x})$ contain 2Λ components denoted by $\bar{\varepsilon}_{\alpha j}$ and $\sigma_{\alpha j}$, respectively, and they can be also represented in a vector form by

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}) = \{\bar{\varepsilon}_{11}(\mathbf{x}) \ \bar{\varepsilon}_{12}(\mathbf{x}) \ \dots \ \bar{\varepsilon}_{1\Lambda}(\mathbf{x}) \ \bar{\varepsilon}_{21}(\mathbf{x}) \ \bar{\varepsilon}_{22}(\mathbf{x}) \ \dots \ \bar{\varepsilon}_{2\Lambda}(\mathbf{x})\}^T \quad (2)$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \{\sigma_{11}(\mathbf{x}) \ \sigma_{12}(\mathbf{x}) \ \dots \ \sigma_{1\Lambda}(\mathbf{x}) \ \sigma_{21}(\mathbf{x}) \ \sigma_{22}(\mathbf{x}) \ \dots \ \sigma_{2\Lambda}(\mathbf{x})\}^T \quad (3)$$

In addition, the surface flux at any smooth point $\mathbf{x}$ on the boundary is denoted by a Λ -component vector $\mathbf{t}(\mathbf{x}) = \{t_1(\mathbf{x}) \ t_2(\mathbf{x}) \ \dots \ t_\Lambda(\mathbf{x})\}^T$. The boundary of the given body $\partial\Omega$ can be decomposed into two disjoint portions; one is denoted by $\partial\Omega^u$ where the state variable $\mathbf{u}$ is fully prescribed ($\mathbf{u} = \mathbf{u}^0(\mathbf{x}) \ \forall \mathbf{x} \in \partial\Omega^u$ where $\mathbf{u}^0(\mathbf{x})$ is a prescribed vector) and the other is denoted by $\partial\Omega^t$ where the surface flux $\mathbf{t}$ is fully prescribed (i.e., $\mathbf{t} = \mathbf{t}^0(\mathbf{x}) \ \forall \mathbf{x} \in \partial\Omega^t$ where $\mathbf{t}^0(\mathbf{x})$ is a prescribed vector). In the present study, the prescribed vector-value functions $\mathbf{b}(\mathbf{x})$, $\mathbf{u}^0(\mathbf{x})$ and $\mathbf{t}^0(\mathbf{x})$ are assumed sufficiently smooth to ensure the existence of the responses $\mathbf{u} = \mathbf{u}(\mathbf{x})$, $\bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$.

A set of field equations governing all field quantities described above is formulated within a general framework to allow various classes of linear, second-order, boundary value problems encountered in various areas (e.g., steady-state heat condition problems, steady-state flow in porous media, membrane problems, linear elasticity, problems associated with multi-field materials such as piezoelectric,

piezomagnetic and piezoelectromagnetic solids, etc.) to be treated in a unified manner. The integer Λ is used as a key parameter to indicate the type of problems. For instance, $\Lambda = 1$ corresponds to steady-state heat condition problems, problems of Darcy's flow in porous media, and membrane problems, and $\{\mathbf{u}(\mathbf{x}); \bar{\mathbf{e}}(\mathbf{x}); \boldsymbol{\sigma}(\mathbf{x}); \mathbf{b}(\mathbf{x}); E_{a||\beta}; \mathbf{t}\}$ represents {temperature; temperature gradient; heat flux; heat source; thermal conductivity; surface heat flux}, {fluid pressure; pressure gradient; fluid flux; source and sink; permeability; surface flux}, and {deflection; slopes; resultant shear force; distributed transverse load; membrane stiffness; end shear force}, respectively; $\Lambda = 2$ corresponds to linear elasticity problems and $\{\mathbf{u}(\mathbf{x}); \bar{\mathbf{e}}(\mathbf{x}); \boldsymbol{\sigma}(\mathbf{x}); \mathbf{b}(\mathbf{x}); E_{a||\beta}; \mathbf{t}\}$ represents {displacement; displacement gradient; stress; body force; elastic constants; traction}; $\Lambda = 3$ corresponds to linear piezoelectric and piezomagnetic problems and $\{\mathbf{u}(\mathbf{x}); \bar{\mathbf{e}}(\mathbf{x}); \boldsymbol{\sigma}(\mathbf{x}); \mathbf{b}(\mathbf{x}); E_{a||\beta}; \mathbf{t}\}$ represents {displacement and electric potential; gradients of displacement and electric potential; stress and electric induction; body force and body charge; elastic constants, piezoelectric constants, dielectric permittivities; traction and surface charge} and {displacement and magnetic potential; gradients of displacement and magnetic potential; stress and magnetic induction; body force and magnetic body source; elastic constants, piezomagnetic constants, magnetic permeabilities; traction and surface magnetic induction}, respectively; and $\Lambda = 4$ corresponds to linear piezoelectromagnetic problems and $\{\mathbf{u}(\mathbf{x}); \bar{\mathbf{e}}(\mathbf{x}); \boldsymbol{\sigma}(\mathbf{x}); \mathbf{b}(\mathbf{x}); E_{a||\beta}; \mathbf{t}\}$ represents {displacement, electric and magnetic potentials; gradients of displacement, electric and magnetic potentials; stress, electric and magnetic inductions; body force, body charge, magnetic body source; elastic, piezoelectric, piezomagnetic and electromagnetic constants, dielectric permittivities, magnetic permeabilities; traction, surface charge, surface magnetic induction}.

The fundamental laws of conservation (e.g., conservation of linear and angular momentum, conservation of mass, conservation of heat flow, etc.), the linear constitutive laws (e.g., Darcy's law, Fourier's law, Hookes' law, generalized Hookes' law, etc), and the laws of kinematics (e.g., strain-displacement relations, electric-potential-field relations, etc.) are employed to form the basic governing field equations and they are expressed in a concise and unified form as follows:

$$\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad (4)$$

$$\boldsymbol{\sigma} = \mathbf{D} \bar{\mathbf{e}} \quad (5)$$

$$\bar{\mathbf{e}} = \mathbf{L} \mathbf{u} \quad (6)$$

where a superscript "T" indicates a matrix transpose operator, $\mathbf{D}$ is a $2\Lambda \times 2\Lambda$ -matrix termed the modulus matrix, and $\mathbf{L}$ represents the linear differential operator defined, in terms of a $2\Lambda \times \Lambda$ -matrix, by

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_{\Lambda \times \Lambda} \\ \mathbf{0}_{\Lambda \times \Lambda} \end{bmatrix} \frac{\partial}{\partial x_1} + \begin{bmatrix} \mathbf{0}_{\Lambda \times \Lambda} \\ \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \frac{\partial}{\partial x_2} = \mathbf{L}_1 \frac{\partial}{\partial x_1} + \mathbf{L}_2 \frac{\partial}{\partial x_2}; \quad \mathbf{L}_1 = \begin{bmatrix} \mathbf{I}_{\Lambda \times \Lambda} \\ \mathbf{0}_{\Lambda \times \Lambda} \end{bmatrix}, \mathbf{L}_2 = \begin{bmatrix} \mathbf{0}_{\Lambda \times \Lambda} \\ \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \quad (7)$$

with $\mathbf{I}_{\Lambda \times \Lambda}$ and $\mathbf{0}_{\Lambda \times \Lambda}$ denoting a $\Lambda \times \Lambda$ -identity matrix and a $\Lambda \times \Lambda$ -zero matrix, respectively. It should be remarked that entries of the modulus matrix $\mathbf{D}$ can be simply obtained from the set $\{E_{a||\beta}\}$ by properly considering the definition of the vectors $\boldsymbol{\sigma}$ and $\bar{\mathbf{e}}$ (i.e., $E_{1JK1} = D_{JK}$, $E_{1JK2} = D_{J,K+\Lambda}$, $E_{2JK1} = D_{J+\Lambda,K}$ and $E_{2JK2} = D_{J+\Lambda,K+\Lambda}$) and, due to the symmetry of $E_{a||\beta}$, the modulus matrix $\mathbf{D}$ is obviously symmetric. By applying the law of conservation at any smooth point $\mathbf{x}$ on the boundary $\partial\Omega$, the surface flux $\mathbf{t}(\mathbf{x})$ can be related to the body flux $\boldsymbol{\sigma}(\mathbf{x})$ and the outward unit normal vector $\mathbf{n}(\mathbf{x}) = \{n_1(\mathbf{x}) \ n_2(\mathbf{x})\}^T$ by

$$\mathbf{t} = [n_1 \mathbf{I}_{\Lambda \times \Lambda} \quad n_2 \mathbf{I}_{\Lambda \times \Lambda}] \boldsymbol{\sigma} \quad (8)$$

Other useful relations such as that directly relating the body flux and the state variable and one connecting the surface flux and the state variable on the boundary can be readily obtained as

$$\boldsymbol{\sigma} = \mathbf{D}(\mathbf{L} \mathbf{u}) \quad (9)$$

$$\mathbf{t} = \begin{bmatrix} n_1 \mathbf{I}_{\Lambda \times \Lambda} & n_2 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \mathbf{D}(\mathbf{Lu}) \quad (10)$$

2.1. Weak Formulation

A standard weighted residual technique is adopted along with the integration by parts via divergence theorem to obtain a weak-form statement of the above problem. By first taking the inner product of (4) and any sufficiently smooth weight function $\mathbf{w}(\mathbf{x}) = \{w_1(\mathbf{x}) \ w_2(\mathbf{x}) \ \dots \ w_\Lambda(\mathbf{x})\}^T$, integrating the result over the body Ω , and then applying an identity $(w_j \sigma_{\alpha j})_{, \alpha} = \mathbf{w}^T \mathbf{L}^T \boldsymbol{\sigma} + (\mathbf{L}\mathbf{w})^T \boldsymbol{\sigma}$, it leads to

$$\int_{\Omega} (\mathbf{L}\mathbf{w})^T \boldsymbol{\sigma} dA = \int_{\Omega} (w_j \sigma_{\alpha j})_{, \alpha} dA + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA \quad (11)$$

By applying divergence theorem to the first integral on the right hand side (11) and then enforcing the boundary term (8), it gives rise to

$$\int_{\Omega} (\mathbf{L}\mathbf{w})^T \boldsymbol{\sigma} dA = \int_{\partial\Omega} \mathbf{w}^T \mathbf{t} dl + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA \quad (12)$$

By further replacing the body flux $\boldsymbol{\sigma}$ appearing in (12) by that associated with the state variable via the relation (9), it finally yields

$$\int_{\Omega} (\mathbf{L}\mathbf{w})^T \mathbf{D}(\mathbf{Lu}) dA = \int_{\partial\Omega} \mathbf{w}^T \mathbf{t} dl + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA \quad (13)$$

It should be apparent from the above formulation that the weak-form Eq. (13) is valid for an arbitrary choice of the weight function $\mathbf{w}$. Only restriction placed on the weight function is the smoothness requirement to ensure the integrability of all integrals appearing in (13). This can be achieved by requiring the weight function and their first partial derivatives square integrable, i.e.,

$$\int_{\Omega} [(\mathbf{L}\mathbf{w})^T (\mathbf{L}\mathbf{w}) + \mathbf{w}^T \mathbf{w}] dA < \infty \quad (14)$$

2.2. Scaled Boundary Coordinate Transformation

Let $\mathbf{x}_0 = (x_{10}, x_{20})$ be a point in $\mathbb{R}^2$ and C be a simple, piecewise smooth curve in $\mathbb{R}^2$ parameterized by a function $\mathbf{r} : s \in [a, b] \rightarrow (x_{10} + \hat{x}_1(s), x_{20} + \hat{x}_2(s)) \in \mathbb{R}^2$ as shown in Fig. 2. Let $\theta(s)$ be the circumferential angle of a point $\mathbf{r}(s)$ on the curve C measured from a straight line connecting $\mathbf{x}_0$ and $\mathbf{r}(a)$ to a straight line connecting $\mathbf{x}_0$ and $\mathbf{r}(s)$ (see Figure 2). The simple curve C considered here can be either closed (i.e., $\mathbf{r}(a) = \mathbf{r}(b)$) or opened (i.e., $\mathbf{r}(a) \neq \mathbf{r}(b)$) and, in addition, it must not contain the point $\mathbf{x}_0$ and satisfies the conditions $\theta \in [0, 2\pi]$ and $d\theta/ds > 0 \ \forall s \in (a, b)$. Now, let us introduce the following coordinate transformation

$$\mathbf{x} = \mathbf{x}_0 + \xi \hat{\mathbf{x}}(s) \quad (15)$$

where $\xi \geq 0$. It is evident from the coordinate transformation (15) that (i) any straight line $\xi = \xi_0, a \leq s \leq b$ in the $\xi-s$ plane is mapped to a curve S in the x_1-x_2 plane which is simply a scaled version of the curve C about $\mathbf{x}_0$ and (ii) any straight line $\xi \geq 0, s = s_0 \in [a, b]$ in the $\xi-s$ plane is mapped to a semi-infinite straight line L in the x_1-x_2 plane starting from $\mathbf{x}_0$ and passing through the point $\mathbf{r}(s_0)$ on the curve C (also see Figure 2). In addition, a straight line $\xi = 0, a \leq s \leq b$ in the $\xi-s$ plane is mapped to a single point $\mathbf{x}_0$ which is termed the *scaling center* and a straight line $\xi = 1, a \leq s \leq b$ in the $\xi-s$ plane is mapped to the

curve C which is termed the *defining curve*. The coordinates ξ and s are termed the *scale boundary coordinates*. Clearly, the transformation (15) maps the region $\xi \geq 0, a \leq s \leq b$ in the $\xi-s$ plane into a region in the x_1-x_2 plane bounded by the two straight lines L_a and L_b (i.e., a shaded region shown in Figure 2). From the coordinate transformation (15), a differential line $d\boldsymbol{\zeta} = \{d\xi \ ds\}^T$ at any point (ξ, s) in the $\xi-s$ plane is related to a differential line $d\mathbf{x} = \{dx_1 \ dx_2\}^T$ at any point (x_1, x_2) in the x_1-x_2 plane by

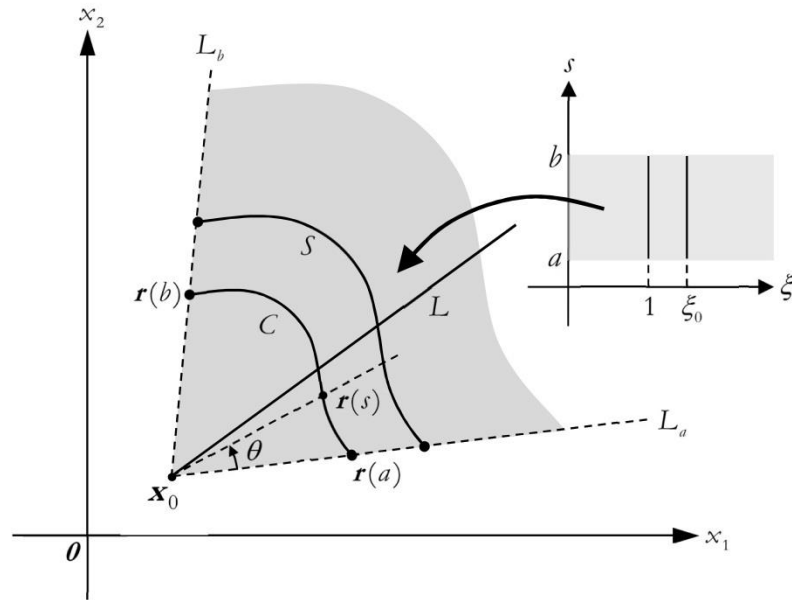


Fig. 2. Schematic of a scaling centre $\mathbf{x}_0$ and a defining curve C .

$$d\mathbf{x} = \mathbf{T} d\boldsymbol{\zeta} = \begin{bmatrix} \hat{x}_1 & \xi d\hat{x}_1/ds \\ \hat{x}_2 & \xi d\hat{x}_2/ds \end{bmatrix} d\boldsymbol{\zeta}; \quad d\boldsymbol{\zeta} = \mathbf{T}^{-1} d\mathbf{x} = \frac{1}{J} \begin{bmatrix} d\hat{x}_2/ds & -d\hat{x}_1/ds \\ -\hat{x}_2/\xi & \hat{x}_1/\xi \end{bmatrix} d\mathbf{x} \quad (16)$$

where $J = \hat{x}_1 d\hat{x}_2/ds - \hat{x}_2 d\hat{x}_1/ds$. By setting $d\boldsymbol{\zeta} = (d\lambda) \mathbf{m}$ where $\mathbf{m}$ is a unit vector and $d\lambda$ is the length of $d\boldsymbol{\zeta}$, it can be readily verified that the length of $d\mathbf{x}$, denoted by dl , is given by

$$dl = \sqrt{dx_\alpha dx_\alpha} = \sqrt{T_{\alpha\beta} T_{\alpha\gamma} m_\beta m_\gamma} d\lambda \quad (17)$$

For following two special cases: (i) $\mathbf{m} = \{1 \ 0\}^T$, $d\lambda = d\xi$ and (ii) $\mathbf{m} = \{0 \ 1\}^T$, $d\lambda = ds$, the relation (17) reduces, respectively, to

$$dl = J^\xi(s) d\xi, \quad J^\xi(s) = \sqrt{\hat{x}_1^2 + \hat{x}_2^2} \quad (18)$$

$$dl = J^s(s) ds, \quad J^s(s) = \sqrt{(d\hat{x}_1/ds)^2 + (d\hat{x}_2/ds)^2} \quad (19)$$

Similarly, the differential area $d\xi ds$ at any point (ξ, s) in the $\xi-s$ plane can be related to the differential area dA at any point (x_1, x_2) in the x_1-x_2 plane by

$$dA = J \xi d\xi ds \quad (20)$$

From the chain rule for differentiations, the partial derivative of any function with respect to the coordinate x_α can be further related to those with respect to the coordinates ξ and s via the following relation

$$\begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{Bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{d\hat{x}_2}{ds} & -\hat{x}_2 \\ -\frac{d\hat{x}_1}{ds} & \hat{x}_1 \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{1}{\xi} \frac{\partial}{\partial s} \end{Bmatrix} \quad (21)$$

The linear differential operator $\mathbf{L}$ given by (7) can be now expressed in terms of partial derivatives with respect to the coordinates ξ and s by

$$\mathbf{L} = \mathbf{b}_1 \frac{\partial}{\partial \xi} + \mathbf{b}_2 \frac{1}{\xi} \frac{\partial}{\partial s} \quad (22)$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are $2\Lambda \times \Lambda$ -matrices defined by

$$\mathbf{b}_1 = \frac{1}{J} \begin{bmatrix} \frac{d\hat{x}_2}{ds} \mathbf{I}_{\Lambda \times \Lambda} \\ -\frac{d\hat{x}_1}{ds} \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix}, \quad \mathbf{b}_2 = \frac{1}{J} \begin{bmatrix} -\hat{x}_2 \mathbf{I}_{\Lambda \times \Lambda} \\ \hat{x}_1 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \quad (23)$$

In the present study, we focus only on a body Ω whose geometry can be completely described by a single scaling center. In particular, there must exist the scaling center $\mathbf{x}_0$ and defining curve C such that there exists a region $[\xi_1, \xi_2] \times [s_1, s_2]$ in the $\xi-s$ plane that is mapped into the region Ω in the x_1-x_2 plane via the transformation (15). If the defining curve C is opened, the body is said to be *opened* and portions of the boundary $\partial\Omega$ associated with $s=s_1$ and $s=s_2$ are termed the *side faces* (see Figure 3(a), (b), (c) and (d)). If the defining curve is closed, the body is said to be closed and the boundary $\partial\Omega$ contains no side face (see Figure 3(e), (f), (g) and (h)). If ξ_2 is finite, the body Ω is said to be bounded (see Figures 3(a), (c), (e) and (g)); otherwise (i.e., $\xi_2 = \infty$), it is said to be unbounded (see Figures 3(b), (d), (f) and (h)). If $\xi_1 = 0$, the body Ω contains the scaling center $\mathbf{x}_0$ (see Figures 3(c), (d), (g) and (h)); otherwise (i.e., $\xi_1 > 0$), the body Ω does not contain the scaling center $\mathbf{x}_0$ (see Figures 3(a), (b), (e) and (f)). Portions of the boundary $\partial\Omega$ associated with $\xi = \xi_1 > 0$ and $\xi = \xi_2 < \infty$ are termed the inner and outer boundaries, respectively.

2.3. Scaled Boundary Finite Element Approximation

The defining curve C is discretized into a mesh containing n elements and m nodes. The coordinates of any point on C , denoted by $\mathbf{x} = \mathbf{x}_0 + \hat{\mathbf{x}}(s)$, is then approximated by

$$\mathbf{x}_\alpha^b(s) = \mathbf{x}_{\alpha 0} + \sum_{i=1}^m \phi_{(i)}(s) \hat{\mathbf{x}}_{\alpha(i)} = \mathbf{x}_{\alpha 0} + \mathbf{N}^G \mathbf{X}_\alpha \quad (24)$$

where the superscript “ b ” is used, here and in what follows, to designate approximate quantities, $\mathbf{N}^G = \{\phi_{(1)} \quad \phi_{(2)} \quad \cdots \quad \phi_{(m)}\}$ stands for a row-matrix containing all nodal basis functions, and $\mathbf{X}_\alpha = \{\hat{\mathbf{x}}_{\alpha(1)} \quad \hat{\mathbf{x}}_{\alpha(2)} \quad \cdots \quad \hat{\mathbf{x}}_{\alpha(m)}\}^T$ denotes a vector containing all nodal relative coordinates in which $\hat{\mathbf{x}}_{\alpha(i)} = \mathbf{x}_{\alpha(i)} - \mathbf{x}_{\alpha 0}$ represents the coordinate of the i^{th} node relative to the scaling center $\mathbf{x}_0$. The resulting discretized defining curve is denoted by C^b and the region in the x_1-x_2 plane described by the discretized defining curve C^b is then used as the approximation for the geometry of the body Ω and denoted by Ω^b . With the relation (24), the derivative $d\hat{\mathbf{x}}_\alpha/ds$ is then approximated by

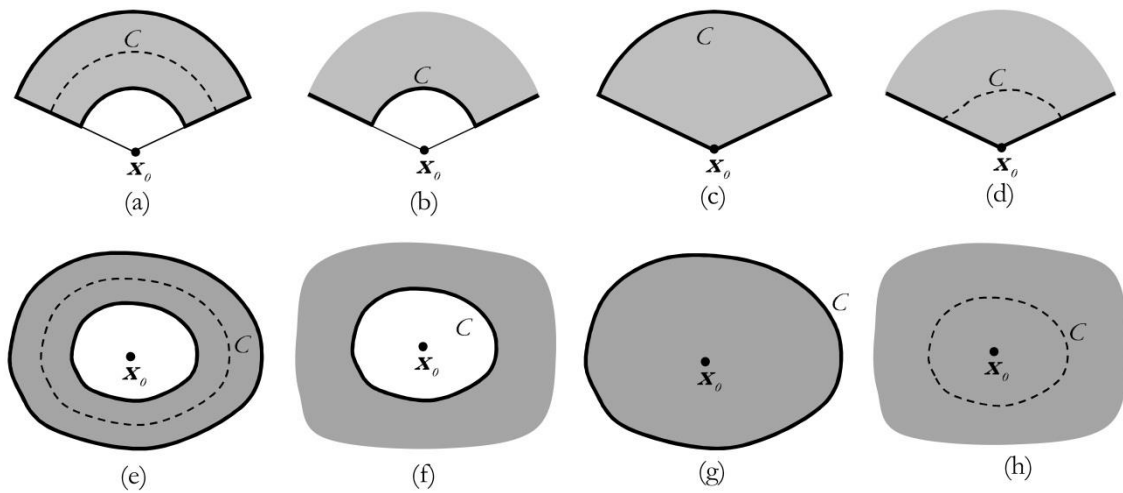


Fig. 3. Schematics of opened bodies: (a) bounded body containing no scaling center, (b) unbounded body containing no scaling center, (c) bounded body containing scaling center, and (d) unbounded body containing scaling center, and schematics of closed bodies: (e) bounded body containing a hole, (f) unbounded body containing a hole, (g) bounded body containing no hole, and (h) unbounded body containing no hole.

$$\frac{d\hat{\mathbf{x}}^b_\alpha}{ds} = \mathbf{B}^G \mathbf{X}_\alpha \quad (25)$$

where $\mathbf{B}^G = d\mathbf{N}^G/ds$. Approximations of J , J^ξ , J^s , $\mathbf{b}_1$, $\mathbf{b}_2$ and the linear operator $\mathbf{L}$ are given by

$$J^b = \mathbf{X}_1^T (\mathbf{N}^G)^T \mathbf{B}^G \mathbf{X}_2 - \mathbf{X}_2^T (\mathbf{N}^G)^T \mathbf{B}^G \mathbf{X}_1 \quad (26)$$

$$J^{\xi b} = \sqrt{\mathbf{X}_1^T (\mathbf{N}^G)^T \mathbf{N}^G \mathbf{X}_1 + \mathbf{X}_2^T (\mathbf{N}^G)^T \mathbf{N}^G \mathbf{X}_2}, \quad J^{sb} = \sqrt{\mathbf{X}_1^T (\mathbf{B}^G)^T \mathbf{B}^G \mathbf{X}_1 + \mathbf{X}_2^T (\mathbf{B}^G)^T \mathbf{B}^G \mathbf{X}_2} \quad (27)$$

$$\mathbf{b}_1^b = \frac{1}{J^b} \begin{bmatrix} \mathbf{B}^G \mathbf{X}_2 \mathbf{I}_{\Lambda \times \Lambda} \\ -\mathbf{B}^G \mathbf{X}_1 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix}, \quad \mathbf{b}_2^b = \frac{1}{J^b} \begin{bmatrix} -\mathbf{N}^G \mathbf{X}_2 \mathbf{I}_{\Lambda \times \Lambda} \\ \mathbf{N}^G \mathbf{X}_1 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix} \quad (28)$$

$$\mathbf{L}^b = \mathbf{b}_1^b \frac{\partial}{\partial \xi} + \mathbf{b}_2^b \frac{1}{\xi} \frac{\partial}{\partial s} \quad (29)$$

From the coordinate transformation (15) along with the approximation (24), the state variable $\mathbf{u}$ and the weight function $\mathbf{w}$ are now approximated, respectively, by $\mathbf{u}^b$ and $\mathbf{w}^b$ in a form

$$\mathbf{u}^b = \mathbf{u}^b(\xi, s) = \sum_{i=1}^m \varphi_{(i)}(s) \mathbf{u}_{(i)}^b(\xi) = \mathbf{N}^s \mathbf{U}^b \quad (30)$$

$$\mathbf{w}^b = \mathbf{w}^b(\xi, s) = \sum_{i=1}^m \varphi_{(i)}(s) \mathbf{w}_{(i)}^b(\xi) = \mathbf{N}^s \mathbf{W}^b \quad (31)$$

where $\mathbf{u}_{(i)}^b(\xi)$ and $\mathbf{w}_{(i)}^b(\xi)$ denote values of the state variable and arbitrary function along the line $s = s_{(i)}$, respectively; $\mathbf{N}^s = \{\varphi_{(1)} \mathbf{I}_{\Lambda \times \Lambda} \quad \varphi_{(2)} \mathbf{I}_{\Lambda \times \Lambda} \quad \cdots \quad \varphi_{(m)} \mathbf{I}_{\Lambda \times \Lambda}\}$ is a $\Lambda \times m\Lambda$ -matrix containing all nodal basis functions; and $\mathbf{U}^b = \{\mathbf{u}_{(1)}^b(\xi) \quad \mathbf{u}_{(2)}^b(\xi) \quad \cdots \quad \mathbf{u}_{(m)}^b(\xi)\}^T$ and $\mathbf{W}^b = \{\mathbf{w}_{(1)}^b(\xi) \quad \mathbf{w}_{(2)}^b(\xi) \quad \cdots \quad \mathbf{w}_{(m)}^b(\xi)\}^T$ denote vectors containing all functions $\mathbf{u}_{(i)}^b(\xi)$ and $\mathbf{w}_{(i)}^b(\xi)$, respectively. It should be remarked that the nodal basis functions $\phi_{(i)}(s)$ and $\varphi_{(i)}(s)$ are constructed using a finite element technique by simply

patching the local element shape functions associated with the i^{th} node and, as a result, it satisfies the Kronecker-delta property, i.e., $\phi_{(i)}(s_{(j)}) = \phi_{(i)}(s_{(j)}) = \delta_{ij}$ where $s_{(j)}$ is the value of the boundary coordinate s of the j^{th} node and δ_{ij} denotes the Kronecker-delta symbol. The approximation of the body flux σ and Lw at any point $\mathbf{x} \in \Omega^b$ in the $x_1 - x_2$ plane can also be obtained by

$$\sigma^b = \sigma^b(\xi, s) = D(Lu^b) = D \left[b_1^b \frac{\partial}{\partial \xi} + b_2^b \frac{1}{\xi} \frac{\partial}{\partial s} \right] (N^S U^b) = D \left[B_1 U_{,\xi}^b + \frac{1}{\xi} B_2 U^b \right] \quad (32)$$

$$Lw^b = Lw^b(\xi, s) = \left[b_1^b \frac{\partial}{\partial \xi} + b_2^b \frac{1}{\xi} \frac{\partial}{\partial s} \right] (N^S W^b) = \left[B_1 W_{,\xi}^b + \frac{1}{\xi} B_2 W^b \right] \quad (33)$$

where B_1 and B_2 are defined by

$$B_1 = b_1^b N^S, \quad B_2 = b_2^b B^S = b_2^b dN^S / ds \quad (34)$$

It is worth noting that both the matrices B_1 and B_2 are independent of the scaling coordinate ξ .

2.4. Scaled Boundary Finite Element Equations

A set of scaled boundary finite element equations is established, here, for a generic, two-dimensional body Ω as shown in Figure 4 to ensure that the resulting formulation is applicable to various cases. The boundary of the domain $\partial\Omega$ is assumed consisting of four parts resulting from the scale boundary coordinate transformation with the scaling center $\mathbf{x}_0$ and defining curve C : the inner boundary $\partial\Omega_1$, the outer boundary $\partial\Omega_2$, the side-face-1 $\partial\Omega_1^s$, and the side-face-2 $\partial\Omega_2^s$. For certain special cases such as closed bodies without the side face, bodies containing the scaling center, and unbounded bodies, it simply takes $\partial\Omega_1^s = \partial\Omega_2^s = \emptyset$, $\partial\Omega_1 = \emptyset$, and $\partial\Omega_2 = \emptyset$ in the following formulation, respectively. The approximation of the given body Ω is achieved via the discretization of the defining curve C^b along with the mapping region $[\xi_1, \xi_2] \times [s_1, s_2]$ in the $\xi - s$ plane, and the approximate body is denoted by Ω^b . In particular, the approximate inner and outer boundaries $\partial\Omega_1^b$ and $\partial\Omega_2^b$, the side-face-1 $\partial\Omega_1^s$, and the side-face-2 $\partial\Omega_2^s$ are fully described by a curve $\xi = \xi_1, s_1 \leq s \leq s_2$, a curve $\xi = \xi_2, s_1 \leq s \leq s_2$, a straight line $s = s_1, \xi_1 \leq \xi \leq \xi_2$, and a straight line $s = s_2, \xi_1 \leq \xi \leq \xi_2$, respectively.

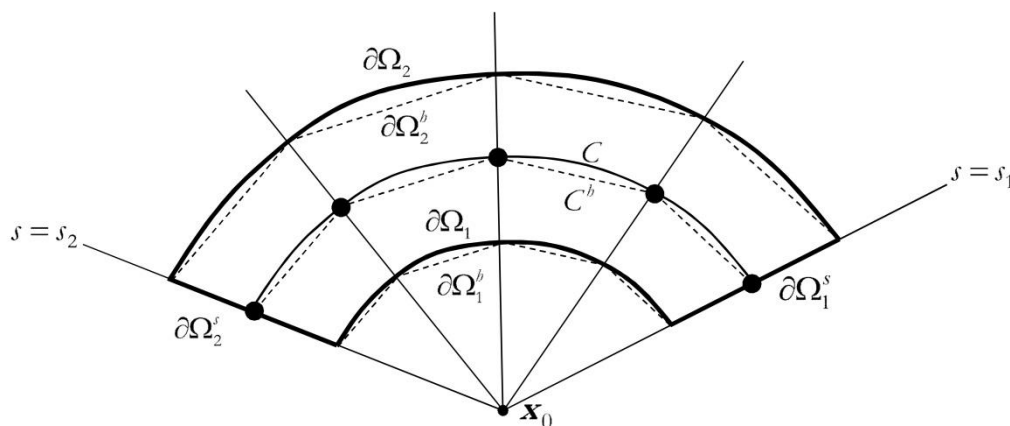


Fig. 4. Schematic of a generic body Ω and its approximation Ω^b . The dashed lines are used to represent the approximation of the defining curve, the inner boundary and the outer boundary.

From the boundary partition $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_1^s \cup \partial\Omega_2^s$ and the coordinate transformation (15), the weak-form equation (13) can be rewritten for this generic body Ω as

$$\begin{aligned} \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{L}\mathbf{w})^T \mathbf{D}(\mathbf{L}\mathbf{u}) J \xi d\xi ds &= \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} \mathbf{w}^T \mathbf{b} J \xi d\xi ds + \int_{s_1}^{s_2} \mathbf{w}_1^T \mathbf{t}_1(s) J^s(s) \xi_1 ds + \int_{s_1}^{s_2} \mathbf{w}_2^T \mathbf{t}_2(s) J^s(s) \xi_2 ds \\ &+ \int_{\xi_1}^{\xi_2} (\mathbf{w}_1^s)^T \mathbf{t}_1^s(\xi) J_1^s d\xi + \int_{\xi_1}^{\xi_2} (\mathbf{w}_2^s)^T \mathbf{t}_2^s(\xi) J_2^s d\xi \end{aligned} \quad (35)$$

where $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_1^s$, $\mathbf{w}_2^s$ are restrictions of the weight function $\mathbf{w}$ on the boundaries $\partial\Omega_1$, $\partial\Omega_2$, $\partial\Omega_1^s$, $\partial\Omega_2^s$, respectively; $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_1^s$, $\mathbf{t}_2^s$ are surface flux on the boundaries $\partial\Omega_1$, $\partial\Omega_2$, $\partial\Omega_1^s$, $\partial\Omega_2^s$, respectively; and $J_1^s = J^s(s_1)$, $J_2^s = J^s(s_2)$. Next, by introducing the approximations of the body flux (32) and the derivatives of the weight function (33) along with the domain approximation, the integral on the left-hand side of (35), denoted for convenience by $\mathcal{J}_1$, becomes

$$\begin{aligned} \mathcal{J}_1 &= \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}_{,\xi}^b)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 \mathbf{U}_{,\xi}^b J^b d\xi ds + \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}_{,\xi}^b)^T \mathbf{B}_1^T \mathbf{D} \mathbf{B}_2 \mathbf{U}^b J^b d\xi ds \\ &+ \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}^b)^T \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 \mathbf{U}_{,\xi}^b J^b d\xi ds + \int_{s_1}^{s_2} \int_{\xi_1}^{\xi_2} (\mathbf{W}^b)^T \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 \mathbf{U}^b \frac{J^b}{\xi} d\xi ds \end{aligned} \quad (36)$$

Further integrating the first two integrals by parts with respect to the coordinate ξ and recalling that the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are independent of the coordinate ξ , $\mathbf{W}^b$ and $\mathbf{U}^b$ are independent of the coordinate s , and the matrix $\mathbf{D}$ is independent of both ξ and s , the integral $\mathcal{J}_1$ is simplified to

$$\begin{aligned} \mathcal{J}_1 &= \int_{\xi_1}^{\xi_2} (\mathbf{W}^b)^T \left[-\xi \mathbf{E}_0 \mathbf{U}_{,\xi\xi}^b + (\mathbf{E}_1 - \mathbf{E}_1^T - \mathbf{E}_0) \mathbf{U}_{,\xi}^b + \frac{1}{\xi} \mathbf{E}_2 \mathbf{U}^b \right] d\xi \\ &+ (\mathbf{W}_2^b)^T \left\{ \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^b + \mathbf{E}_1^T \mathbf{U}^b \right\}_{\xi=\xi_2} - (\mathbf{W}_1^b)^T \left\{ \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^b + \mathbf{E}_1^T \mathbf{U}^b \right\}_{\xi=\xi_1} \end{aligned} \quad (37)$$

where $\mathbf{W}_1^b = \mathbf{W}^b(\xi = \xi_1)$, $\mathbf{W}_2^b = \mathbf{W}^b(\xi = \xi_2)$, and $\mathbf{E}_0$, $\mathbf{E}_1$, and $\mathbf{E}_2$ are defined by

$$\mathbf{E}_0 = \int_{s_1}^{s_2} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 J^b ds, \quad \mathbf{E}_1 = \int_{s_1}^{s_2} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 J^b ds, \quad \mathbf{E}_2 = \int_{s_1}^{s_2} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 J^b ds \quad (38)$$

It is evident that both matrices $\mathbf{E}_0$ and $\mathbf{E}_2$ are symmetric. By following a similar procedure, the boundary integrals appearing on the right-hand side of (35), denoted by $\mathcal{J}_2$, can be approximated by

$$\mathcal{J}_2 = (\mathbf{W}_1^b)^T \mathbf{P}_1 + (\mathbf{W}_2^b)^T \mathbf{P}_2 + \int_{\xi_1}^{\xi_2} (\mathbf{W}^b)^T \mathbf{F}_1^t d\xi + \int_{\xi_1}^{\xi_2} (\mathbf{W}^b)^T \mathbf{F}_2^t d\xi \quad (39)$$

where $\mathbf{N}_1^s = \mathbf{N}^s(s = s_1)$, $\mathbf{N}_2^s = \mathbf{N}^s(s = s_2)$, and

$$\mathbf{P}_1 = \int_{s_1}^{s_2} (\mathbf{N}^s)^T \mathbf{t}_1(s) \xi_1 J^{sb}(s) ds, \quad \mathbf{P}_2 = \int_{s_1}^{s_2} (\mathbf{N}^s)^T \mathbf{t}_2(s) \xi_2 J^{sb}(s) ds \quad (40)$$

$$\mathbf{F}_1^t = (\mathbf{N}_1^s)^T \mathbf{t}_1^s(\xi) J_1^{sb}, \quad \mathbf{F}_2^t = (\mathbf{N}_2^s)^T \mathbf{t}_2^s(\xi) J_2^{sb} \quad (41)$$

in which $J_1^{\xi b} = J^{\xi b}(s_1)$ and $J_2^{\xi b} = J^{\xi b}(s_2)$. Without loss of generality, the *first* and *last* nodes resulting from the discretization of opened bodies are taken as a node on the side-face-1 and a node on the side-face-2, respectively, and this applies in what follows. It should be remarked from the Kronecker property of the nodal basis functions that $\varphi_{(1)}(s_1) = 1$, $\varphi_{(j)}(s_1) = 0 \ \forall j \geq 2$ and $\varphi_{(m)}(s_2) = 1$, $\varphi_{(j)}(s_2) = 0 \ \forall j \leq m-1$. Now, both the matrices $\mathbf{N}_1^S$ and $\mathbf{N}_2^S$ clearly contain many zero entries and simply take the form

$$\mathbf{N}_1^S = \{\mathbf{I}_{\Lambda \times \Lambda} \quad \mathbf{0}_{\Lambda \times \Lambda} \quad \cdots \quad \mathbf{0}_{\Lambda \times \Lambda}\}, \quad \mathbf{N}_2^S = \{\mathbf{0}_{\Lambda \times \Lambda} \quad \mathbf{0}_{\Lambda \times \Lambda} \quad \cdots \quad \mathbf{I}_{\Lambda \times \Lambda}\} \quad (42)$$

Substituting (42) into (41) leads to

$$\mathbf{F}_1^t = \{J_1^{\xi b} \mathbf{t}_1^t(\xi) \quad \mathbf{0}_\Lambda \quad \cdots \quad \mathbf{0}_\Lambda\}^T, \quad \mathbf{F}_2^t = \{\mathbf{0}_\Lambda \quad \mathbf{0}_\Lambda \quad \cdots \quad J_2^{\xi b} \mathbf{t}_2^t(\xi)\}^T \quad (43)$$

where $\mathbf{0}_\Lambda$ is a zero Λ -component vector. Finally, the domain integral associated with the distributed body source on the right-hand side of (35), denoted by $\mathcal{J}_3$, can be approximated by

$$\mathcal{J}_3 = \int_{\xi_1}^{\xi_2} (\mathbf{W}^b)^T \xi \mathbf{F}^b d\xi \quad (44)$$

where the matrix $\mathbf{F}^b$ is defined by

$$\mathbf{F}^b = \int_{s_1}^{s_2} (\mathbf{N}^S)^T \mathbf{b} J^b ds = \left\{ \int_{s_1}^{s_2} \phi_{(1)} \mathbf{b} J^b ds \quad \int_{s_1}^{s_2} \phi_{(2)} \mathbf{b} J^b ds \quad \cdots \quad \int_{s_1}^{s_2} \phi_{(m)} \mathbf{b} J^b ds \right\}^T \quad (45)$$

By combining the results (37), (39) and (44), the approximation of the weak-form (35) becomes

$$\begin{aligned} & \int_{\xi_1}^{\xi_2} (\mathbf{W}^b)^T \left[-\xi \mathbf{E}_0 \mathbf{U}_{,\xi\xi}^b + (\mathbf{E}_1 - \mathbf{E}_1^T - \mathbf{E}_0) \mathbf{U}_{,\xi}^b + \frac{1}{\xi} \mathbf{E}_2 \mathbf{U}^b - \mathbf{F}^t - \xi \mathbf{F}^b \right] d\xi \\ & + (\mathbf{W}_2^b)^T \left[\left\{ \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^b + \mathbf{E}_1^T \mathbf{U}^b \right\}_{\xi=\xi_2} - \mathbf{P}_2 \right] - (\mathbf{W}_1^b)^T \left[\left\{ \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^b + \mathbf{E}_1^T \mathbf{U}^b \right\}_{\xi=\xi_1} + \mathbf{P}_1 \right] = 0 \end{aligned} \quad (46)$$

where $\mathbf{F}^t = \mathbf{F}_1^t + \mathbf{F}_2^t$. From the arbitrariness of the weight function $\mathbf{W}^b$, it can be deduced that

$$\xi^2 \mathbf{E}_0 \mathbf{U}_{,\xi\xi}^b + \xi (\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1) \mathbf{U}_{,\xi}^b - \mathbf{E}_2 \mathbf{U}^b + \xi \mathbf{F}^t + \xi^2 \mathbf{F}^b = \mathbf{0} \quad \forall \xi \in (\xi_1, \xi_2) \quad (47)$$

$$\mathbf{Q}^b(\xi_1) = -\mathbf{P}_1 \quad (48)$$

$$\mathbf{Q}^b(\xi_2) = \mathbf{P}_2 \quad (49)$$

where the vector $\mathbf{Q}^b = \mathbf{Q}^b(\xi)$ known as the nodal internal flux is defined by

$$\mathbf{Q}^b(\xi) = \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^b + \mathbf{E}_1^T \mathbf{U}^b \quad (50)$$

A set of Eqs. (47)-(49) is known as the scaled boundary finite element equations governing the unknown function $\mathbf{U}^b = \mathbf{U}^b(\xi)$. It can be remarked that (47) forms a system of linear, second-order, nonhomogeneous, ordinary differential equations with respect to the coordinate ξ whereas (48) and (49) pose the boundary conditions on both inner and outer boundaries of the body. As previously pointed out, the governing equations and boundary conditions (47)-(49) are formulated in a general context; as a result, for certain special cases, some terms or equations must be eliminated properly. For instances, the boundary

condition (49) is ignored for unbounded bodies; the boundary condition (48) is ignored for bounded bodies containing the scaling center; the term $\mathbf{F}^f$ vanishes for closed bodies; and the term $\mathbf{F}^b$ vanishes for bodies free of the distributed body source.

2.5. Treatment of Prescribed Conditions on Side Faces

It should be evident from (47)-(49) that the information associated with the prescribed distributed body source and the prescribed boundary conditions on both inner and outer boundaries can be integrated into the formulation via the term $\mathbf{F}^b$ and the boundary conditions (48)-(49), respectively. However, the consideration of the prescribed surface flux and state variable on the side face is still not apparent from the above formulation. The system of linear differential equations (47) can be further re-expressed in a form well-suited for the treatment of those prescribed side-face conditions.

First, bodies considered in the present study are divided into five groups, *Group-1*, *Group-2*, *Group-3*, *Group-4*, and *Group-5*, associated with closed bodies without the side-face, opened bodies with prescribed surface flux on both side faces, opened bodies with prescribed surface flux on the side-face-1 and prescribed state variable on the side-face-2, opened bodies with prescribed surface flux on the side-face-2 and prescribed state variable on the side-face-1, and opened bodies with prescribed state variable on both side faces, respectively. For bodies in *Group-1*, $\mathbf{U}^b$ contains only unknown functions and $\mathbf{t}_1^s, \mathbf{t}_2^s$ disappear; for bodies in *Group-2*, $\mathbf{U}^b$ contains only unknown functions whereas $\mathbf{t}_1^s, \mathbf{t}_2^s$ are fully prescribed; for bodies in *Group-3*, $\mathbf{t}_1^s$, and $\mathbf{u}_{(m)}^b(\xi)$ are fully prescribed and $\mathbf{t}_2^s$ and the remaining functions $\mathbf{u}_{(i)}^b(\xi) \forall i \leq m-1$ in $\mathbf{U}^b$ are unknown; for bodies in *Group-4*, $\mathbf{t}_2^s$ and $\mathbf{u}_{(1)}^b(\xi)$ are fully prescribed and $\mathbf{t}_1^s$ and the remaining functions $\mathbf{u}_{(i)}^b(\xi) \forall i \geq 2$ in $\mathbf{U}^b$ are unknown; and for bodies in *Group-5*, both $\mathbf{u}_{(1)}^b(\xi)$ and $\mathbf{u}_{(m)}^b(\xi)$ are fully prescribed whereas $\mathbf{t}_1^s, \mathbf{t}_2^s$, and the remaining functions $\mathbf{u}_{(i)}^b(\xi) \forall 2 \leq i \leq m-1$ in $\mathbf{U}^b$ are unknown. To treat the prescribed conditions on the side-face for bodies in all groups, the vector $\mathbf{U}^b$ is partitioned and rearranged into known and unknown parts as $\mathbf{U}^b = \{\mathbf{U}^{bu} \quad \mathbf{U}^{bc}\}^T$ where $\mathbf{U}^{bu} = \mathbf{U}^{bu}(\xi)$ contains only unknown functions from a collection $\mathbf{u}_{(i)}^b(\xi), i = 1, 2, \dots, m$ and $\mathbf{U}^{bc} = \mathbf{U}^{bc}(\xi)$ contains the remaining known functions associated with the prescribed state variable on the side face. By defining p as the number of known functions $\mathbf{u}_{(i)}^b(\xi)$ contained in $\mathbf{U}^{bc}$, the number of remaining unknown functions $\mathbf{u}_{(i)}^b(\xi)$ contained in $\mathbf{U}^{bu}$ is therefore equal to $m - p$. Clearly, the value of p associated with bodies in *Group-1*, *Group-2*, *Group-3*, *Group-4*, and *Group-5* are 0, 0, 1, 1, and 2 respectively. Consistent with the partition of the vector $\mathbf{U}^b$, the vector $\mathbf{F}^f$ can be also partitioned into $\mathbf{F}^f = \{\mathbf{F}^{fu} \quad \mathbf{F}^{fc}\}^T$ where $\mathbf{F}^{fu} = \mathbf{F}^{fu}(\xi)$ contains many zero functions and known functions corresponding to the prescribed surface flux on the side face and $\mathbf{F}^{fc} = \mathbf{F}^{fc}(\xi)$ contains unknown functions associated with the unknown surface flux on the side face. According to this partition, the system of differential equations (47) can be separated into

$$\xi^2 \mathbf{E}_0^{uu} \mathbf{U}_{,\xi\xi}^{bu} + \xi [\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu}] \mathbf{U}_{,\xi}^{bu} - \mathbf{E}_2^{uu} \mathbf{U}^{bu} = -\xi \mathbf{F}^{fu} - \xi^2 \mathbf{F}^{bu} - \mathbf{F}^{suu} \quad (51)$$

$$\xi \mathbf{F}^{fc} = -\xi^2 \mathbf{F}^{bc} - \mathbf{F}^{suc} - \mathbf{F}^{sc} \quad (52)$$

where $\mathbf{F}^{bu}$ and $\mathbf{F}^{bc}$ results directly from the partition of the known vector $\mathbf{F}^b = \{\mathbf{F}^{bu} \quad \mathbf{F}^{bc}\}^T$ and the vectors $\mathbf{F}^{suu}$, $\mathbf{F}^{sc}$ and $\mathbf{F}^{suc}$ are defined by

$$\mathbf{F}^{suu} = \xi^2 \mathbf{E}_0^{uc} \mathbf{U}_{,\xi\xi}^{bc} + \xi [\mathbf{E}_0^{uc} + (\mathbf{E}_1^{uc})^T - \mathbf{E}_1^{uc}] \mathbf{U}_{,\xi}^{bc} - \mathbf{E}_2^{uc} \mathbf{U}^{bc} \quad (53)$$

$$\mathbf{F}^{sc} = \xi^2 \mathbf{E}_0^{cc} \mathbf{U}_{,\xi\xi}^{bc} + \xi [\mathbf{E}_0^{cc} + (\mathbf{E}_1^{cc})^T - \mathbf{E}_1^{cc}] \mathbf{U}_{,\xi}^{bc} - \mathbf{E}_2^{cc} \mathbf{U}^{bc} \quad (54)$$

$$\mathbf{F}^{suc} = \xi^2 (\mathbf{E}_0^{uc})^T \mathbf{U}_{,\xi\xi}^{bu} + \xi [(\mathbf{E}_0^{uc})^T + (\mathbf{E}_1^{uc})^T - \mathbf{E}_1^{uc}] \mathbf{U}_{,\xi}^{bu} - (\mathbf{E}_2^{uc})^T \mathbf{U}^{bu} \quad (55)$$

in which $\mathbf{E}_i^{uu}, \mathbf{E}_i^{uc}, \mathbf{E}_i^{cu}, \mathbf{E}_i^{cc}$ for $i=0,1,2$ are sub-matrices resulting from the partition of the matrix $\mathbf{E}_i$. Note that both $\mathbf{F}^{uu}$ and $\mathbf{F}^{uc}$ are known vectors obtained from the prescribed state variable on the side face whereas $\mathbf{F}^{uc}$ is given in terms of the unknown vector $\mathbf{U}^{hu}$. By following the same procedure, the relation (50) can be also partitioned into

$$\mathbf{Q}^{hu}(\xi) = \xi \mathbf{E}_0^{uu} \mathbf{U}_{,\xi}^{hu} + (\mathbf{E}_1^{uu})^T \mathbf{U}^{hu} + \mathbf{Q}^{huc}(\xi) \quad (56)$$

$$\mathbf{Q}^{hc}(\xi) = \xi (\mathbf{E}_0^{uc})^T \mathbf{U}_{,\xi}^{hu} + (\mathbf{E}_1^{uc})^T \mathbf{U}^{hu} + \mathbf{Q}^{bcc}(\xi) \quad (57)$$

where $\mathbf{Q}^{huc}(\xi)$ and $\mathbf{Q}^{bcc}(\xi)$ are known vectors defined by

$$\mathbf{Q}^{huc}(\xi) = \xi \mathbf{E}_0^{uc} \mathbf{U}_{,\xi}^{hc} + (\mathbf{E}_1^{cu})^T \mathbf{U}^{hc}, \quad \mathbf{Q}^{bcc}(\xi) = \xi \mathbf{E}_0^{cc} \mathbf{U}_{,\xi}^{hc} + (\mathbf{E}_1^{cc})^T \mathbf{U}^{hc} \quad (58)$$

Now, a system of differential equations (51) together with the following two boundary conditions on the inner and outer boundaries (i.e., $\mathbf{Q}^{hu}(\xi_1) = -\mathbf{P}_1^u$ and $\mathbf{Q}^{hu}(\xi_2) = \mathbf{P}_2^u$) is sufficient for determining the general solution $\mathbf{U}^{hu}$ of the given boundary value problem. Note that $\mathbf{P}_1^u$ and $\mathbf{P}_2^u$ are vectors resulting from the partition of the vectors $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively (i.e., $\mathbf{P}_1 = \{\mathbf{P}_1^u \quad \mathbf{P}_1^c\}^T$ and $\mathbf{P}_2 = \{\mathbf{P}_2^u \quad \mathbf{P}_2^c\}^T$). Once $\mathbf{U}^{hu}$ is determined, the vectors $\mathbf{F}^{uc}$, $\mathbf{P}_1^c$, and $\mathbf{P}_2^c$ can be readily obtained.

3. Solution Procedure

This section presents the procedure for obtaining the analytical solution of a system of linear, second-order, nonhomogeneous, ordinary differential equations (51) along with the prescribed conditions on both inner and outer boundaries. A corresponding eigenvalue problem is solved first to determine the homogeneous solution and the method of undetermined coefficients is then utilized to construct the particular solution associated with the distributed body source and the prescribed conditions on the side faces. Once the general solution is obtained, the boundary conditions on both inner and outer boundaries are enforced to determine all involved constants. Finally, the post-process for field quantities of interest such as the state variable and body flux is briefly described.

3.1. Determination of Homogeneous Solution

Following the standard procedure in the theory of differential equations, a homogeneous solution of a system of linear, second-order, Euler-Cauchy differential equations (51), denoted by $\mathbf{U}_0^{hu}$, takes the following form

$$\mathbf{U}_0^{hu}(\xi) = \sum_{i=1}^{2(m-p)\Lambda} c_i \xi^{\lambda_i} \boldsymbol{\psi}_i^u \quad (59)$$

where λ_i is termed the modal scaling factor, $\boldsymbol{\psi}_i$ is the $(m-p)\Lambda$ -component vector representing the i^{th} mode of the state variable, and c_i are arbitrary constants denoting the contribution of the i^{th} mode. The nodal internal flux $\mathbf{Q}_0^{hu}(\xi)$ associated with $\mathbf{U}_0^{hu}$ can be obtained as

$$\mathbf{Q}_0^{hu}(\xi) = \sum_{i=1}^{2(m-p)\Lambda} c_i \xi^{\lambda_i} [\lambda_i \mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T] \boldsymbol{\psi}_i^u = \sum_{i=1}^{2(m-p)\Lambda} c_i \xi^{\lambda_i} \mathbf{q}_i^u \quad (60)$$

where $\mathbf{q}_i^u$ is termed the i^{th} modal internal flux defined in terms of $\boldsymbol{\psi}_i^u$ by

$$\mathbf{q}_i^u = [\lambda_i \mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T] \boldsymbol{\psi}_i^u \quad (61)$$

By substituting (59) into (51) and then employing arbitrariness of ϵ_i , it results in

$$\left\{ \lambda_i^2 \mathbf{E}_0''' + \lambda_i \left[(\mathbf{E}_1''')^T - \mathbf{E}_1''' \right] - \mathbf{E}_2''' \right\} \boldsymbol{\psi}_i'' = \mathbf{0} \quad \forall i \in \{1, 2, \dots, 2(m-p)\Lambda\} \quad (62)$$

By further rearranging terms in (61) such that

$$\lambda_i \boldsymbol{\psi}_i'' = -(\mathbf{E}_0''')^{-1} (\mathbf{E}_1''')^T \boldsymbol{\psi}_i'' + (\mathbf{E}_0''')^{-1} \mathbf{q}_i'' \quad (63)$$

where $(\mathbf{E}_0''')^{-1}$ denotes the inverse of $\mathbf{E}_0'''$, and then substituting this result into (62), it finally yields

$$\lambda_i \mathbf{q}_i'' = \left[\mathbf{E}_2''' - \mathbf{E}_1''' (\mathbf{E}_0''')^{-1} (\mathbf{E}_1''')^T \right] \boldsymbol{\psi}_i'' + \mathbf{E}_1''' (\mathbf{E}_0''')^{-1} \mathbf{q}_i'' \quad (64)$$

Now, by introducing a $2(m-p)\Lambda$ -component vector $\mathbf{X}_i$ such that $\mathbf{X}_i = \{\boldsymbol{\psi}_i'' \quad \mathbf{q}_i''\}^T$, equations (63) and (64) can be combined into a system of linear algebraic equations

$$\mathbf{A} \mathbf{X}_i = \lambda_i \mathbf{X}_i \quad (65)$$

where a coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} -(\mathbf{E}_0''')^{-1} (\mathbf{E}_1''')^T & (\mathbf{E}_0''')^{-1} \\ \mathbf{E}_2''' - \mathbf{E}_1''' (\mathbf{E}_0''')^{-1} (\mathbf{E}_1''')^T & \mathbf{E}_1''' (\mathbf{E}_0''')^{-1} \end{bmatrix} \quad (66)$$

Determination of $2(m-p)\Lambda$ pairs of $\{\lambda_i, \mathbf{X}_i\}$ can be achieved by solving the eigenvalue problem (65) where λ_i denotes the eigenvalue and $\mathbf{X}_i$ are the associated eigenvector. It should be remarked that since $\mathbf{A}$ is, in general, not symmetric, $\{\lambda_i, \mathbf{X}_i\}$ involves complex numbers. In fact, only a half of the eigenvalues has the positive real part whereas the other half has the negative real part. Let $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$ be $(m-p)\Lambda \times (m-p)\Lambda$ diagonal matrices containing eigenvalues with the positive and negative real parts, respectively, $\boldsymbol{\Phi}^{w+}$ and $\boldsymbol{\Phi}^{q+}$ be matrices whose columns are $\boldsymbol{\psi}_i''$ and $\mathbf{q}_i''$ of the eigenvector $\mathbf{X}_i = \{\boldsymbol{\psi}_i'' \quad \mathbf{q}_i''\}^T$ associated with λ_i in $\boldsymbol{\lambda}^+$, and $\boldsymbol{\Phi}^{w-}$ and $\boldsymbol{\Phi}^{q-}$ be matrices whose columns are $\boldsymbol{\psi}_i''$ and $\mathbf{q}_i''$ of the eigenvector $\mathbf{X}_i = \{\boldsymbol{\psi}_i'' \quad \mathbf{q}_i''\}^T$ associated with λ_i in $\boldsymbol{\lambda}^-$. Now, the homogeneous solutions $\mathbf{U}_0^{bu}$ and $\mathbf{Q}_0^{bu}(\xi)$ are given by

$$\mathbf{U}_0^{bu}(\xi) = \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- \quad (67)$$

$$\mathbf{Q}_0^{bu}(\xi) = \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- \quad (68)$$

where $\boldsymbol{\Pi}^+$ and $\boldsymbol{\Pi}^-$ are diagonal matrices obtained by replacing the diagonal entries λ_i of the matrices $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$ by a function ξ^{λ_i} , respectively; and $\mathbf{C}^+$ and $\mathbf{C}^-$ are vectors containing arbitrary constants representing the contribution of each mode. It is apparent that the diagonal entries of $\boldsymbol{\Pi}^+$ become infinite when $\xi \rightarrow \infty$ whereas those of $\boldsymbol{\Pi}^-$ is unbounded when $\xi \rightarrow 0$. As a result, $\mathbf{C}^+$ is taken to be $\mathbf{0}$ to ensure the boundedness of the solution for unbounded bodies and, similarly, the condition $\mathbf{C}^- = \mathbf{0}$ is enforced for bodies containing the scaling center.

3.2. Determination of Particular Solution

In the present study, the distributed body source $\mathbf{b}$, the prescribed surface flux $\mathbf{t}_1^s$ on the side-face-1, the prescribed surface flux $\mathbf{t}_2^s$ on the side-face-2, and the prescribed state variable $\mathbf{U}^{bc}$ on the side faces are assumed to admit the form

$$\mathbf{b}(\xi, s) = \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j} \bar{\mathbf{b}}_j(s), \mathbf{t}_1(\xi) = \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j} \bar{\mathbf{t}}_j^{s1}, \mathbf{t}_2(\xi) = \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j} \bar{\mathbf{t}}_j^{s2}, \mathbf{U}^{bc}(\xi) = \sum_{\omega_j \in \mathbb{R}^*} \xi^{\omega_j} \bar{\mathbf{U}}_j^{bc} \quad (69)$$

where $\mathbb{R}^*$ denotes a set of non-negative real numbers, $\bar{\mathbf{b}}_j(s)$ are given vector-value functions, and $\bar{\mathbf{t}}_j^{s1}$, $\bar{\mathbf{t}}_j^{s2}$, $\bar{\mathbf{U}}_j^{bc}$ are given constant vectors. Substituting (69) into (43), (45) and (53) yields

$$\mathbf{F}^b = \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j} \bar{\mathbf{F}}_j^b, \mathbf{F}_1^t = \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j} \bar{\mathbf{F}}_j^{t1}, \mathbf{F}_2^t = \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j} \bar{\mathbf{F}}_j^{t2}, \mathbf{F}^{sum} = \sum_{\omega_j \in \mathbb{R}^*} \xi^{\omega_j} \bar{\mathbf{F}}_j^{sum} \quad (70)$$

where $\bar{\mathbf{F}}_j^b$, $\bar{\mathbf{F}}_j^{t1}$, $\bar{\mathbf{F}}_j^{t2}$, and $\bar{\mathbf{F}}_j^{sum}$ are constant vectors defined, in terms of prescribed data, by

$$\bar{\mathbf{F}}_j^b = \left\{ \int_{s_1}^{s_2} \phi_{(1)} \bar{\mathbf{b}}_j(s) J^b ds, \int_{s_1}^{s_2} \phi_{(2)} \bar{\mathbf{b}}_j(s) J^b ds, \dots, \int_{s_1}^{s_2} \phi_{(m)} \bar{\mathbf{b}}_j(s) J^b ds \right\}^T \quad (71)$$

$$\bar{\mathbf{F}}_j^{t1} = \{J_1^{\xi b} \bar{\mathbf{t}}_j^{s1} \quad \mathbf{0} \quad \dots \quad \mathbf{0}\}, \bar{\mathbf{F}}_j^{t2} = \{\mathbf{0} \quad \mathbf{0} \quad \dots \quad J_2^{\xi b} \bar{\mathbf{t}}_j^{s2}\}^T \quad (72)$$

$$\bar{\mathbf{F}}_j^{sum} = \{\omega_j(\omega_j - 1)\mathbf{E}_0^{uc} + \omega_j(\mathbf{E}_0^{uc} + (\mathbf{E}_1^{uc})^T - \mathbf{E}_1^{uc}) - \mathbf{E}_2^{uc}\} \bar{\mathbf{U}}_j^{bc} \quad (73)$$

Based on this form of prescribed data and the method of undetermined coefficients, the particular solution of (51), denoted by $\mathbf{U}_1^{bu}$, takes the form

$$\mathbf{U}_1^{bu}(\xi) = \mathbf{U}_1^{hub}(\xi) + \mathbf{U}_1^{hut1}(\xi) + \mathbf{U}_1^{hut2}(\xi) + \mathbf{U}_1^{hsum}(\xi) \quad (74)$$

where

$$\mathbf{U}_1^{hub}(\xi) = \sum_{\kappa_j \in \mathbb{R}^*} \xi^{\kappa_j+2} \mathbf{c}_j^b, \mathbf{U}_1^{hut1}(\xi) = \sum_{\tau_j \in \mathbb{R}^*} \xi^{\tau_j+1} \mathbf{c}_j^{t1}, \mathbf{U}_1^{hut2}(\xi) = \sum_{\gamma_j \in \mathbb{R}^*} \xi^{\gamma_j+1} \mathbf{c}_j^{t2}, \mathbf{U}_1^{hsum}(\xi) = \sum_{\omega_j \in \mathbb{R}^*} \xi^{\omega_j} \mathbf{c}_j^{uc} \quad (75)$$

with $\mathbf{c}_j^b$, $\mathbf{c}_j^{t1}$, $\mathbf{c}_j^{t2}$, and $\mathbf{c}_j^{uc}$ being vectors of unknown constants determined from the following four systems of linear algebraic equations

$$\{(\kappa_j + 2)(\kappa_j + 1)\mathbf{E}_0^{uu} + (\kappa_j + 2)[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu}] - \mathbf{E}_2^{uu}\} \mathbf{c}_j^b + \bar{\mathbf{F}}_j^{bu} = \mathbf{0} \quad (76)$$

$$\{(\tau_j + 1)\tau_j \mathbf{E}_0^{uu} + (\tau_j + 1)[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu}] - \mathbf{E}_2^{uu}\} \mathbf{c}_j^{t1} + \bar{\mathbf{F}}_j^{tu1} = \mathbf{0} \quad (77)$$

$$\{(\gamma_j + 1)\gamma_j \mathbf{E}_0^{uu} + (\gamma_j + 1)[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu}] - \mathbf{E}_2^{uu}\} \mathbf{c}_j^{t2} + \bar{\mathbf{F}}_j^{tu2} = \mathbf{0} \quad (78)$$

$$\{\omega_j(\omega_j - 1)\mathbf{E}_0^{uu} + (\omega_j - 1)[\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu}] - \mathbf{E}_2^{uu}\} \mathbf{c}_j^{uc} + \bar{\mathbf{F}}_j^{sum} = \mathbf{0} \quad (79)$$

where $\bar{\mathbf{F}}_j^{bu}$, $\bar{\mathbf{F}}_j^{tu1}$ and $\bar{\mathbf{F}}_j^{tu2}$ result from the following partitions $\bar{\mathbf{F}}_j^b = \{\bar{\mathbf{F}}_j^{bu} \quad \bar{\mathbf{F}}_j^{bc}\}^T$, $\bar{\mathbf{F}}_j^{t1} = \{\bar{\mathbf{F}}_j^{tu1} \quad \bar{\mathbf{F}}_j^{tc1}\}^T$, and $\bar{\mathbf{F}}_j^{t2} = \{\bar{\mathbf{F}}_j^{tu2} \quad \bar{\mathbf{F}}_j^{tc2}\}^T$, respectively. Once the particular solution $\mathbf{U}_1^{bu}(\xi)$ is obtained, the corresponding particular nodal internal flux, denoted by $\mathbf{Q}_1^{bu}(\xi)$, is calculated from

$$\mathbf{Q}_1^{bu}(\xi) = \xi \mathbf{E}_0^{uu} \mathbf{U}_{1,\xi}^{bu} + (\mathbf{E}_1^{uu})^T \mathbf{U}_1^{bu} \quad (80)$$

3.3. Final General Solution

The general solution $\mathbf{U}^{bu}(\xi)$ of (51) and the corresponding nodal internal flux $\mathbf{Q}^{bu}(\xi)$ are given by

$$\mathbf{U}^{hu}(\xi) = \mathbf{U}_0^{hu}(\xi) + \mathbf{U}_1^{hu}(\xi) = \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- + \mathbf{U}_1^{hu}(\xi) \quad (81)$$

$$\mathbf{Q}^{hu}(\xi) = \mathbf{Q}_0^{hu}(\xi) + \mathbf{Q}_1^{hu}(\xi) = \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- + \mathbf{Q}_1^{hu}(\xi) \quad (82)$$

By enforcing conditions on both inner and outer boundaries (i.e., $\mathbf{Q}^{hu}(\xi_1) = -\mathbf{P}_1^u$, $\mathbf{Q}^{hu}(\xi_2) = \mathbf{P}_2^u$), it leads to a systems of linear equations governing all constants contained in $\mathbf{C}^+$ and $\mathbf{C}^-$:

$$\begin{bmatrix} \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix} \begin{Bmatrix} \mathbf{C}^+ \\ \mathbf{C}^- \end{Bmatrix} = \begin{Bmatrix} -\mathbf{P}_1^u \\ \mathbf{P}_2^u \end{Bmatrix} - \begin{Bmatrix} \mathbf{Q}_1^{hu}(\xi_1) \\ \mathbf{Q}_1^{hu}(\xi_2) \end{Bmatrix} \quad (83)$$

The system (83) can be, in principle, inverted to obtain

$$\begin{Bmatrix} \mathbf{C}^+ \\ \mathbf{C}^- \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix}^{-1} \left(\begin{Bmatrix} -\mathbf{P}_1^u \\ \mathbf{P}_2^u \end{Bmatrix} - \begin{Bmatrix} \mathbf{Q}_1^{hu}(\xi_1) \\ \mathbf{Q}_1^{hu}(\xi_2) \end{Bmatrix} \right) \quad (84)$$

By first applying (81) to obtain $\mathbf{U}^{hu}$ at $\xi = \xi_1$ and $\xi = \xi_2$, and then using the relation (84), it leads to

$$\mathbf{K} \begin{Bmatrix} \mathbf{U}^{hu}(\xi_1) \\ \mathbf{U}^{hu}(\xi_2) \end{Bmatrix} = \begin{Bmatrix} -\mathbf{P}_1^u \\ \mathbf{P}_2^u \end{Bmatrix} + \mathbf{K} \begin{Bmatrix} \mathbf{U}_1^{hu}(\xi_1) \\ \mathbf{U}_1^{hu}(\xi_2) \end{Bmatrix} - \begin{Bmatrix} \mathbf{Q}_1^{hu}(\xi_1) \\ \mathbf{Q}_1^{hu}(\xi_2) \end{Bmatrix} \quad (85)$$

where the coefficient matrix $\mathbf{K}$, commonly termed the stiffness matrix, is given by

$$\mathbf{K} = \begin{bmatrix} \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{w+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{w-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix}^{-1} \quad (86)$$

By enforcing the prescribed surface flux and the state variable on both inner and outer boundaries, a system of linear algebraic equations (85) is sufficient for determining all unknowns contained in the vectors $\{\mathbf{U}^{hu}(\xi_1) \ \mathbf{U}^{hu}(\xi_2)\}^T$ and $\{-\mathbf{P}_1^u \ \mathbf{P}_2^u\}^T$. Once the unknowns on both inner and outer boundaries are solved, the constant vectors $\mathbf{C}^+$ and $\mathbf{C}^-$ can be obtained from (84) and the general solution for both $\mathbf{U}^{hu}(\xi)$ and $\mathbf{Q}^{hu}(\xi)$ are obtained from (81) and (82), respectively.

3.4. Post-process for Field Quantities

Once the approximate general solution $\mathbf{U}^{hu}(\xi)$ is obtained, the approximate field quantities such as the state variable and the surface flux within the body can be determined. For instance, the approximate state variable at (ξ, s) can be obtained from

$$\mathbf{u}^b(\xi, s) = \mathbf{N}^s(s) \mathbf{U}^b(\xi) = \begin{bmatrix} \mathbf{N}^{Su}(s) & \mathbf{N}^{Sc}(s) \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{hu}(\xi) \\ \mathbf{U}^{hc}(\xi) \end{Bmatrix} = \mathbf{N}^{Su}(s) \mathbf{U}^{hu}(\xi) + \mathbf{N}^{Sc}(s) \mathbf{U}^{hc}(\xi) \quad (87)$$

where $\mathbf{N}^{Su}$ and $\mathbf{N}^{Sc}$ are matrices resulting from the partition of $\mathbf{N}^s$. Similarly, the approximate body flux can be computed from

$$\boldsymbol{\sigma}^b(\xi, s) = \mathbf{D} \left[\mathbf{B}_1^u(s) \mathbf{U}_{,\xi}^{hu}(\xi) + \frac{1}{\xi} \mathbf{B}_2^u(s) \mathbf{U}^{hu}(\xi) \right] + \mathbf{D} \left[\mathbf{B}_1^c(s) \mathbf{U}_{,\xi}^{hc}(\xi) + \frac{1}{\xi} \mathbf{B}_2^c(s) \mathbf{U}^{hc}(\xi) \right] \quad (88)$$

where $\mathbf{B}_1^u$, $\mathbf{B}_1^c$ and $\mathbf{B}_2^u$, $\mathbf{B}_2^c$ are matrices resulting from the partition of the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$, respectively. It is emphasized here again that the solutions (87) and (88) also apply to the special cases of

bounded and unbounded bodies. For bounded bodies containing the scaling center, $\mathbf{C}^-$ simply vanishes and, for unbounded bodies, $\mathbf{C}^+ = \mathbf{0}$.

4. Numerical Results and Discussion

Numerical results of four representative problems are presented, in this section, to verify both the formulation and the implementation of the proposed technique. To demonstrate its capability to treat a variety of boundary value problems, general boundary conditions, and prescribed data on the side faces, three different types of problems associated with steady-state heat conduction ($\Lambda = 1$), linear elasticity ($\Lambda = 2$), and linear piezoelectricity ($\Lambda = 3$), under various scenarios are considered. In the approximation, standard linear finite elements are employed to discretize both the defining curve and the trial and test functions. The accuracy and convergence of numerical solutions are confirmed by benchmarking with available analytical solutions and carrying out the analysis via a series of meshes.

4.1. Heat Conduction in Rectangular Domain

The proposed technique is first tested with a representative problem associated with the steady-state heat conduction ($\Lambda = 1$). Let us consider a two-dimensional rectangular domain of dimensions l_1 and l_2 as shown schematically in Fig. 5(a). The domain is made of a medium with an isotropic heat conductivity k_0 (or, equivalently, $D_{11} = D_{22} = k_0, D_{12} = D_{21} = 0$) and subjected to a uniform heat source $b_1 = Q_0$. A zero temperature is prescribed along the boundary AD (i.e., $u_1^{AD} = 0$) and a uniform surface heat flux q_0 is prescribed along the boundary BC (i.e., $t_1^{BC} = q_0$) whereas, along the boundaries AB and CD , the surface heat flux vanishes (i.e., $t_1^{AB} = t_1^{CD} = 0$). For this particular case, the exact solution for the temperature field (i.e., $u_1(x)$) and body heat flux (i.e., $\sigma_{11}(x)$ and $\sigma_{21}(x)$) is given by

$$u_1 = \frac{1}{k_0} \left[(q_0 + Q_0 l_1) x_1 - \frac{1}{2} Q_0 x_1^2 \right] \quad (89)$$

$$\sigma_{11} = q_0 + Q_0(l_1 - x_1), \quad \sigma_{21} = 0 \quad (90)$$

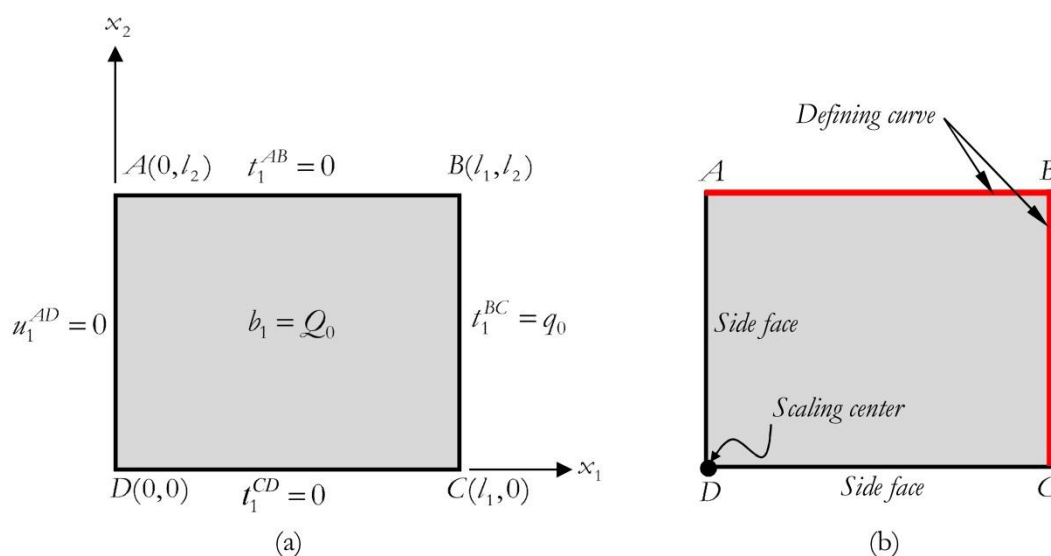


Fig. 5. Schematic of (a) rectangular domain under body heat source and mixed boundary conditions and (b) scaling center and defining curve used in scale boundary finite element analysis.

The domain geometry is described by the scaling center at point D and the defining curve ABC ; as a result, the boundaries AD and CD become the side faces (see Figure 5(b)). In the numerical study, q_0 and Q_0 are chosen such that $Q_0/l_1 = 3q_0$ and the defining curve is discretized by a series of N identical linear elements. The normalized temperature $u_1 k_0 / q_0 l_1$ and the non-zero normalized body heat flux σ_{11}/q_0 along the boundary AB are reported in Table 1 for various values of normalized coordinate $\bar{x}_1 = x_1/l_1$ and meshes. Clearly, computed numerical solutions exhibit an excellent convergence behavior as the number of elements increases and, in addition, accurate results in comparison with the benchmark solution can be obtained even when few elements are employed to discretize the defining curve and solution along the boundary direction.

4.2. Linear Elastic Hollowed Cylinder under Uniform Pressure

Consider, next, a hollowed cylinder of the inner radius R_1 and outer radius R_2 (see Figure 6(a)). The cylinder is made of a homogeneous, linearly elastic, isotropic material with Young's modulus E and Poisson's ratio ν and is subjected to a plane-strain condition and uniform pressure p_1 and p_2 on the inner and outer boundaries, respectively, (i.e., $\Lambda = 2$ and the modulus matrix $\mathbf{D}$ with non-zero entries $D_{11} = D_{44} = (1-\nu)E/(1+\nu)(1-2\nu)$, $D_{14} = \nu E/(1+\nu)(1-2\nu)$, $D_{41} = \nu E/(1+\nu)(1-2\nu)$, $D_{22} = D_{23} = D_{32} = D_{33} = E/2(1+\nu)$). Due to the symmetry, it is sufficient to model this problem using only a quarter of the cylinder (see Figure 6(b)) with appropriate conditions on both side faces (i.e., the normal displacement and tangential traction on the side faces vanish). To describe the geometry, the scaling center is chosen at the center of the cylinder whereas the inner boundary is treated as the defining curve. In a numerical study, $R_2/R_1 = 1.5$, $p_2/p_1 = 2$, $\nu = 0.3$, and meshes with N identical linear elements are employed.

Results for the normalized radial displacement ($u_r/(p_1 R_1/E)$), normalized radial stress (σ_r/p_1) and normalized hoop stress ($\sigma_{\theta\theta}/p_1$) are reported along with existing analytical solutions [36] in Tables 2, 3 and 4, respectively, for four meshes (i.e., $N = 4, 8, 16, 32$). It is seen that numerical solutions generated by the proposed technique converge and exhibit excellent agreement with the benchmark solution. It is worth noting that the discretization with only few linear elements can capture numerical solution with sufficient accuracy.

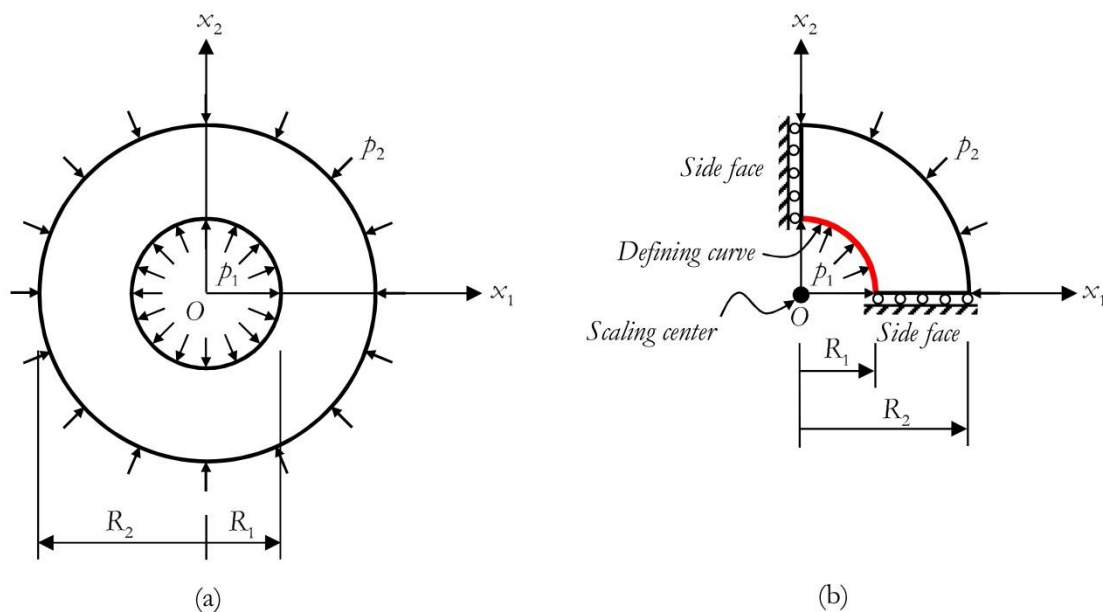


Fig. 6. Schematic of (a) hollowed cylinder under uniform internal and external pressure and (b) quarter of cylinder used in the analysis.

Table 1. Normalized temperatures and body heat flux along the boundary AB of a rectangular domain subjected to body heat source and mixed boundary conditions.

	$u_1 k_0 / q_0 l_1$					
$\bar{x}_1$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=4$	0.0000	0.7196	1.3287	1.8290	2.2175	2.4684
SBFEM, $N=8$	0.0000	0.7352	1.3520	1.8512	2.2332	2.4950
SBFEM, $N=16$	0.0000	0.7388	1.3580	1.8577	2.2382	2.4986
SBFEM, $N=32$	0.0000	0.7397	1.3595	1.8594	2.2395	2.4996
Exact solution	0.0000	0.7400	1.3600	1.8600	2.2400	2.5000

	σ_{11} / q_0					
$\bar{x}_1$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=4$	3.8735	3.3211	2.7724	2.2297	1.6359	0.8099
SBFEM, $N=8$	3.9709	3.3802	2.7890	2.2037	1.6121	1.0222
SBFEM, $N=16$	3.9929	3.3950	2.7970	2.2005	1.6034	1.0006
SBFEM, $N=32$	3.9982	3.3988	2.7992	2.2001	1.6008	1.0000
Exact solution	4.0000	3.4000	2.8000	2.2000	1.6000	1.0000

Table 2. Normalized radial displacement of hollowed cylinder under internal and external uniform pressure. Results are reported at different values of radial coordinate $r = \sqrt{x_1^2 + x_2^2}$ for four meshes.

	$u_r / (p_1 R_1 / E)$					
r / R_1	1.0	1.1	1.2	1.3	1.4	1.5
SBFEM, $N=4$	1.0958	0.9864	0.8943	0.8155	0.7473	0.6875
SBFEM, $N=8$	1.1124	1.0014	0.9080	0.8282	0.7590	0.6984
SBFEM, $N=16$	1.1166	1.0052	0.9115	0.8313	0.7619	0.7011
SBFEM, $N=32$	1.1177	1.0061	0.9123	0.8321	0.7627	0.7018
Exact solution	1.1180	1.0064	0.9126	0.8324	0.7629	0.7020

Table 3. Normalized radial stress of hollowed cylinder under internal and external uniform pressure. Results are reported at different values of radial coordinate $r = \sqrt{x_1^2 + x_2^2}$ for four meshes.

	σ_r / p_1					
r / R_1	1.0	1.1	1.2	1.3	1.4	1.5
SBFEM, $N=4$	-0.9821	-0.8289	-0.7124	-0.6217	-0.5497	-0.4917
SBFEM, $N=8$	-0.9955	-0.8401	-0.7218	-0.6298	-0.5568	-0.4979
SBFEM, $N=16$	-0.9989	-0.8429	-0.7242	-0.6319	-0.5586	-0.4995
SBFEM, $N=32$	-0.9997	-0.8436	-0.7248	-0.6324	-0.5590	-0.4999
Exact solution	-1.0000	-0.8438	-0.7250	-0.6325	-0.5592	-0.5000

Table 4. Normalized hoop stress of hollowed cylinder under internal and external uniform pressure. Results are reported at different values of radial coordinate $r = \sqrt{x_1^2 + x_2^2}$ for four meshes.

	$\sigma_{\theta\theta} / p_1$					
r / R_1	1.0	1.1	1.2	1.3	1.4	1.5
SBFEM, $N=4$	0.7833	0.6301	0.5136	0.4229	0.3510	0.2929
SBFEM, $N=8$	0.7958	0.6404	0.5222	0.4301	0.3571	0.2982
SBFEM, $N=16$	0.7990	0.6429	0.5243	0.4319	0.3587	0.2996
SBFEM, $N=32$	0.7997	0.6436	0.5248	0.4324	0.3591	0.2999
Exact solution	0.8000	0.6438	0.5250	0.4325	0.3592	0.3000

4.3. Linear Elastic Square Plate under Mixed Boundary Conditions

Another representative problem for linear elasticity ($\Lambda = 2$) is chosen to demonstrate the capability of the implemented procedure to treat problems with distributed body source and prescribed state variable and surface flux on the boundary. Consider a plane-strain, square plate $ABCD$ of dimension $l \times l$ and made of a homogeneous, isotropic, linearly elastic material of Young's modulus E and Poisson's ratio ν as shown schematically in Figure 7(a). Note that the modulus matrix $\mathbf{D}$ for this particular problem is the same as the previous case. The plate is subjected to a linear body force field $b_1 = 2b_0(1 + \bar{x}_2)$, $b_2 = 2b_0(1 - \bar{x}_1)$ with b_0 denoting a constant and $\bar{x}_1 = x_1/l$, $\bar{x}_2 = x_2/l$ denoting the normalized coordinates whereas the nonuniform traction and homogeneous displacement boundary conditions are prescribed on its four sides as follows:

Side AB : $t_1^{AB} = b_0 l(3 + 14\bar{x}_1 - \bar{x}_1^2)$ and $t_2^{AB} = 2b_0 l(-4 + 4\bar{x}_1 + 3\bar{x}_1^2)$

Side BC : $t_1^{BC} = 6b_0 l(\bar{x}_2^2 - 2)$ and $t_2^{BC} = b_0 l(1 + 14\bar{x}_2 + \bar{x}_2^2)$

Side AD : $u_1^{AD} = 0$ and $t_2^{AD} = -b_0 l(2\bar{x}_2 + \bar{x}_2^2)$

Side CD : $t_1^{CD} = b_0 l(\bar{x}_1^2 - 2\bar{x}_1)$ and $u_2^{CD} = 0$

The exact solution for this particular problem under the plane strain condition can be readily obtained from a classical theory of linear elasticity and final results are given by

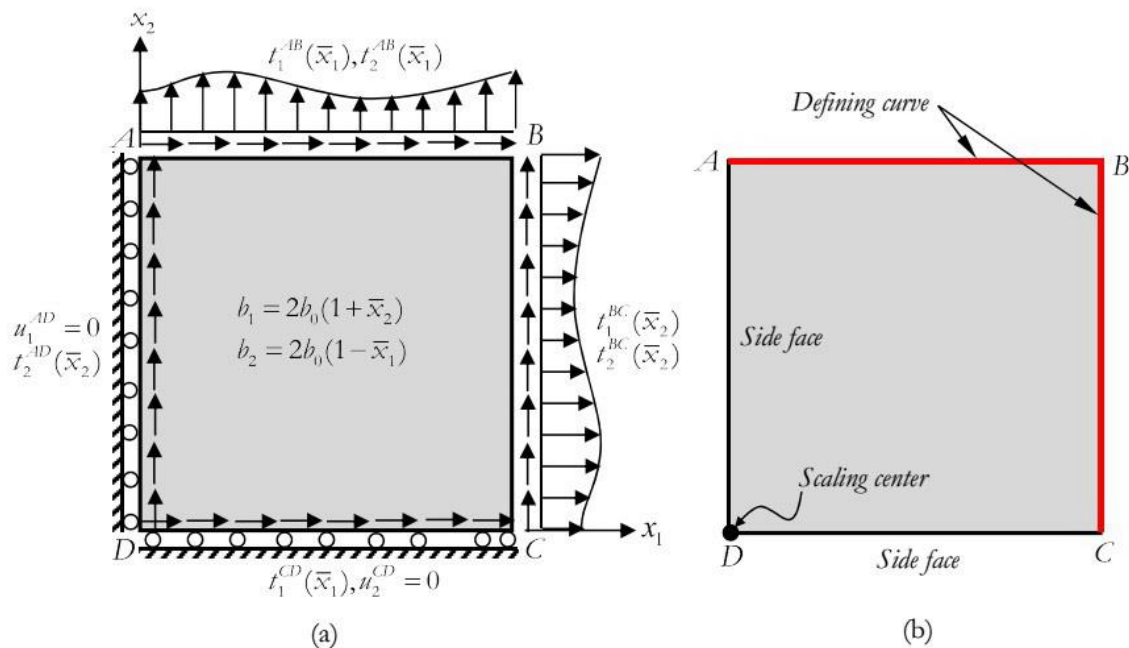


Fig 7. Schematic of (a) elastic square plate under mixed boundary conditions and (b) scaling center and defining curve used in scale boundary finite element analysis.

$$Eu_1^{exact} = -2b_0 l^2 (1 + \nu) (1 + \bar{x}_1 - 2\bar{x}_2 + \bar{x}_1^2 - 3\bar{x}_2^2 + \bar{x}_1 \bar{x}_2) \bar{x}_1 \quad (91)$$

$$Eu_2^{exact} = -2b_0 l^2 (1 + \nu) (-1 - 2\bar{x}_1 + \bar{x}_2 - 3\bar{x}_1^2 + \bar{x}_2^2 - \bar{x}_1 \bar{x}_2) \bar{x}_2 \quad (92)$$

$$\sigma_{11}^{exact} = -\sigma_{22}^{exact} = -2b_0 l (1 + 2\bar{x}_1 - 2\bar{x}_2 + 3\bar{x}_1^2 - 3\bar{x}_2^2 + 2\bar{x}_1 \bar{x}_2) \quad (93)$$

$$\sigma_{12}^{exact} = -b_0 l (-2\bar{x}_1 - 2\bar{x}_2 + \bar{x}_1^2 - \bar{x}_2^2 - 12\bar{x}_1 \bar{x}_2) \quad (94)$$

The geometry of the plate is fully described by the scaling center at point D and the defining curve ABC and, as a result, the boundaries AD and CD become the side faces (see Figure 7(b)). In the analysis, Poisson's ratio is taken as $\nu = 0.3$ and the defining curve is discretized by N identical linear elements. The

normalized displacements Eu_1/b_0l^2 and Eu_2/b_0l^2 along the boundaries AB and BC are reported in Tables 5 and 6 for various locations and meshes. It can be seen that numerical solutions converges to the exact solution as the number of elements N used to discretize the defining curve increases and, in addition, only few number of degrees of freedom is sufficient to obtain accurate displacements. The normalized normal stress components σ_{11}/b_0l and σ_{22}/b_0l along the boundaries CD and AD are also reported in Figures 8 and 9 along with the exact solution. Similar to the displacements, the proposed method also yields highly accurate stress components and the good convergent behavior; in particular, results obtained from all meshes are nearly indistinguishable from the benchmark solution.

Table 5. Normalized displacements along the boundary AB of elastic square plate subjected to mixed boundary conditions.

	Eu_1/b_0l^2					
$\bar{x}_1$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=8$	0.0000	1.8132	3.0677	3.6989	3.5973	2.6088
SBFEM, $N=16$	0.0000	1.8340	3.1482	3.7916	3.6349	2.6036
SBFEM, $N=32$	0.0000	1.8489	3.1558	3.7996	3.6578	2.6013
SBFEM, $N=64$	0.0000	1.8501	3.1608	3.8055	3.6592	2.6004
Exact solution	0.0000	1.8512	3.1616	3.8064	3.6608	2.6000

	Eu_2/b_0l^2					
$\bar{x}_1$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=8$	-2.7256	-0.7721	1.8011	4.9689	8.6766	12.8397
SBFEM, $N=16$	-2.6262	-0.7306	1.7661	4.8982	8.6579	12.9369
SBFEM, $N=32$	-2.6051	-0.7310	1.7700	4.8929	8.6363	12.9780
SBFEM, $N=64$	-2.6009	-0.7281	1.7679	4.8886	8.6333	13.0038
Exact solution	-2.6000	-0.7280	1.7680	4.8880	8.6320	13.0000

Table 6. Normalized displacements along the boundary BC of elastic square plate subjected to mixed boundary conditions.

	Eu_1/b_0l^2					
$\bar{x}_2$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=8$	-7.9407	-7.0248	-5.5067	-3.3831	-0.6694	2.6008
SBFEM, $N=16$	-7.8346	-6.9726	-5.5213	-3.4319	-0.7027	2.6012
SBFEM, $N=32$	-7.8086	-6.9716	-5.5119	-3.4297	-0.7246	2.6006
SBFEM, $N=64$	-7.8021	-6.9683	-5.5126	-3.4320	-0.7268	2.6143
Exact solution	-7.8000	-6.9680	-5.5120	-3.4320	-0.7280	2.6000

	Eu_2/b_0l^2					
$\bar{x}_2$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=8$	0.0000	3.0812	6.0130	8.7017	11.0266	12.8397
SBFEM, $N=16$	0.0000	3.0922	6.0621	8.7806	11.1096	12.9369
SBFEM, $N=32$	0.0000	3.0978	6.0699	8.7924	11.1408	12.9780
SBFEM, $N=64$	0.0000	3.0988	6.0729	8.7973	11.1465	13.0038
Exact solution	0.0000	3.0992	6.0736	8.7984	11.1488	13.0000

4.4. Linear Piezoelectric Square Plate

As a final example, a representative boundary value problem associated with the linear piezoelectricity ($\Lambda = 3$) is investigated to further highlight the capability of the proposed technique to treat linear multi-field problems. Let us consider a two-dimensional square plate of length l and made of a linear piezoelectric solid with all material constants taken from PZT-4 [5] as shown in Figure 10(a). In particular, all non-zero entries of the modulus matrix $\mathbf{D}$ are given explicitly by $D_{11} = 139 \text{ GPa}$, $D_{15} = D_{51} = 74.3 \text{ GPa}$,

$D_{16} = D_{61} = -6.98 \text{ Cm}^{-2}$, $D_{22} = D_{24} = D_{42} = D_{44} = 25.6 \text{ GPa}$, $D_{33} = 6.0 \times 10^{-9} \text{ C(Vm)}^{-1}$, $D_{34} = D_{43} = 13.44 \text{ Cm}^{-2}$, $D_{55} = 113 \text{ GPa}$, $D_{56} = D_{65} = 13.84 \text{ Cm}^{-2}$, and $D_{66} = 5.47 \times 10^{-9} \text{ C(Vm)}^{-1}$. For a purpose of verification, a set of prescribed data such as the distributed body source and boundary conditions on four sides is chosen such that the exact solution for the displacements (u_1 and u_2) and the electric potential (u_3) takes the following form

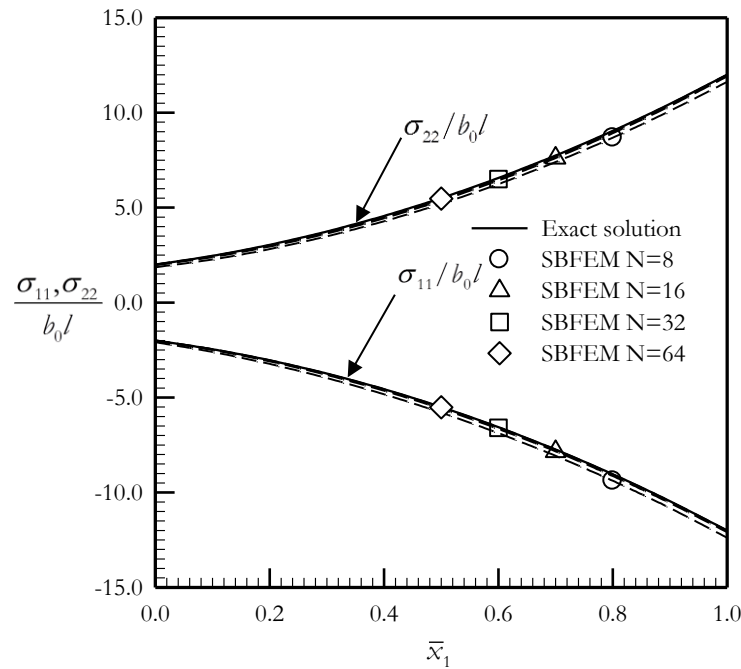


Fig. 8 Normalized normal stress components along the boundary CD of elastic square plate subjected to mixed boundary conditions.

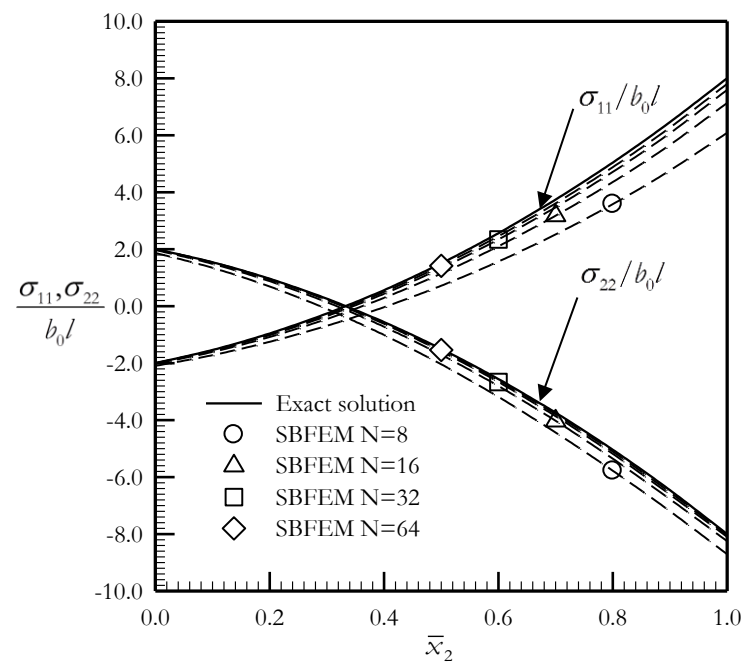


Fig. 9 Normalized normal stress components along the boundary AD of elastic square plate subjected to mixed boundary conditions.

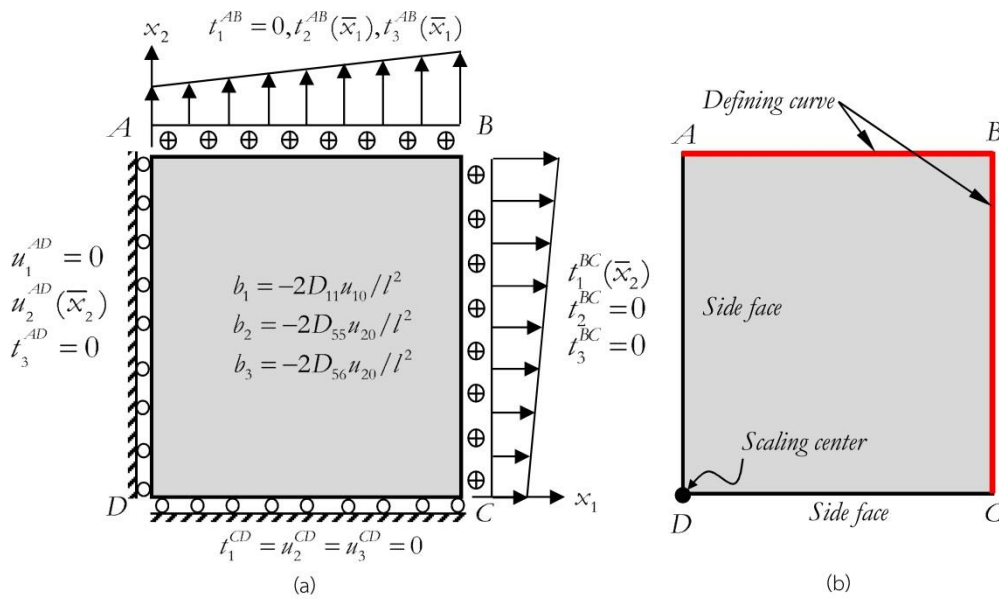


Fig. 10. Schematic of (a) linear piezoelectric square plate under mixed boundary conditions and (b) scaling center and defining curve used in scale boundary finite element analysis.

$$u_1 = u_{10}\bar{x}_1^2, \quad u_2 = u_{20}\bar{x}_2^2, \quad u_3 = \phi_0\bar{x}_2 \quad (95)$$

where u_{10} , u_{20} , ϕ_0 are given constants and $\bar{x}_1 = x_1/l$, $\bar{x}_2 = x_2/l$ are normalized coordinates. The corresponding exact stress field $(\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22})$ and electrical induction vector $(\sigma_{13}, \sigma_{23})$ are given by

$$\sigma_{11} = (2D_{11}u_{10}\bar{x}_1 + 2D_{15}u_{20}\bar{x}_2 + D_{16}\phi_0)/l, \quad \sigma_{12} = \sigma_{21} = 0, \quad (96)$$

$$\sigma_{22} = (2D_{15}u_{10}\bar{x}_1 + 2D_{55}u_{20}\bar{x}_2 + D_{56}\phi_0)/l \quad (97)$$

$$\sigma_{13} = 0, \quad \sigma_{23} = (2D_{16}u_{10}\bar{x}_1 + 2D_{56}u_{20}\bar{x}_2 - D_{66}\phi_0)/l \quad (98)$$

The distributed body source that is in equilibrium with the above stress and electric induction can readily be obtained from (4) as

$$b_1 = -2D_{11}u_{10}/l^2, \quad b_2 = -2D_{55}u_{20}/l^2, \quad b_3 = -2D_{56}u_{20}/l^2 \quad (99)$$

In the modeling, the scaling center is chosen at a point D and the prescribed conditions on the side faces AD and CD and the boundaries AB and BC are given below

Side AB: $t_1^{AB} = 0, t_2^{AB} = (2D_{15}u_{10}\bar{x}_1 + 2D_{55}u_{20} + D_{56}\phi_0)/l, t_3^{AB} = (2D_{16}u_{10}\bar{x}_1 + 2D_{56}u_{20} - D_{66}\phi_0)/l$

Side BC: $t_1^{BC} = (2D_{11}u_{10} + 2D_{15}u_{20}\bar{x}_2 + D_{16}\phi_0)/l, t_2^{BC} = 0, t_3^{BC} = 0$

Side AD: $u_1^{AD} = 0, u_2^{AD} = u_{20}\bar{x}_2^2, t_3^{AD} = 0$

Side CD: $t_1^{CD} = 0, u_2^{CD} = 0, u_3^{CD} = 0$

It is worth noting that the boundary conditions shown above are chosen to represent the general prescribed data on the side faces and boundaries of the domain. In the numerical study, a series of meshes with N identical linear elements is constructed to discretize the defining curve and solution along the scale boundary direction and $u_{20}/u_{10} = 2$, $D_{16}\phi_0/D_{11}u_{10} = 1$ are employed. Computed displacements and electric potential along the diagonal line BD are reported in Table 7 for various meshes. It is seen for this particular problem that the proposed technique yield highly accurate results even when relatively coarse meshes containing only few degrees of freedom are employed. In addition, the improvement of solutions as the

mesh is refined is clearly observed. The good quality of numerical solutions is also confirmed for the body flux as indicate in Figures 11 and 12. Computed stresses and electrical induction along the diagonal line BD show an excellent agreement with the exact solution; in particular, only slight difference between solutions can be seen for very coarse meshes while almost indistinguishable results are obtained for fine meshes.

Table 7. Normalized displacements and electrical potential along the diagonal line BD of linear piezoelectric square plate subjected to mixed boundary conditions. Results are reports as a function of normalized length $\bar{s} = s / \sqrt{2}l$ where s is the length along the line BD measured from point D .

u_1/u_{10}						
$\bar{s}$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=4$	0.0000	0.0369	0.1351	0.3278	0.6291	1.0459
SBFEM, $N=8$	0.0000	0.0393	0.1531	0.3503	0.6360	1.0252
SBFEM, $N=16$	0.0000	0.0398	0.1582	0.3574	0.6385	1.0098
SBFEM, $N=32$	0.0000	0.0400	0.1596	0.3593	0.6396	1.0033
Exact solution	0.0000	0.0400	0.1600	0.3600	0.6400	1.0000

u_2/u_{20}						
$\bar{s}$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=4$	0.0000	0.0821	0.3272	0.7222	1.2554	1.9134
SBFEM, $N=8$	0.0000	0.0803	0.3220	0.7213	1.2739	1.9679
SBFEM, $N=16$	0.0000	0.0801	0.3205	0.7204	1.2786	1.9891
SBFEM, $N=32$	0.0000	0.0800	0.3201	0.7201	1.2797	1.9965
Exact solution	0.0000	0.0800	0.3200	0.7200	1.2800	2.0000

u_3/ϕ_0						
$\bar{s}$	0.0	0.2	0.4	0.6	0.8	1.0
SBFEM, $N=4$	0.0000	0.2013	0.4019	0.6013	0.7991	0.9938
SBFEM, $N=8$	0.0000	0.2003	0.4004	0.6003	0.7998	0.9976
SBFEM, $N=16$	0.0000	0.2001	0.4001	0.6001	0.7999	0.9992
SBFEM, $N=32$	0.0000	0.2000	0.4000	0.6000	0.8000	0.9997
Exact solution	0.0000	0.2000	0.4000	0.6000	0.8000	1.0000

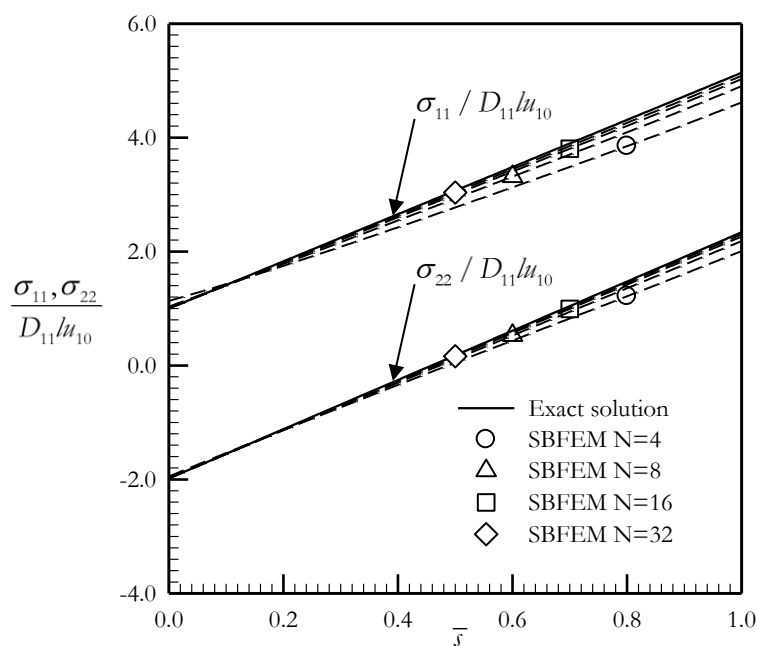


Fig. 11. Normalized non-zero stress components along the diagonal line BD of a piezoelectric square plate subjected to mixed boundary conditions. Results are reports as a function of normalized length $\bar{s} = s / \sqrt{2}l$ where s is the length along the line BD measured from point D .

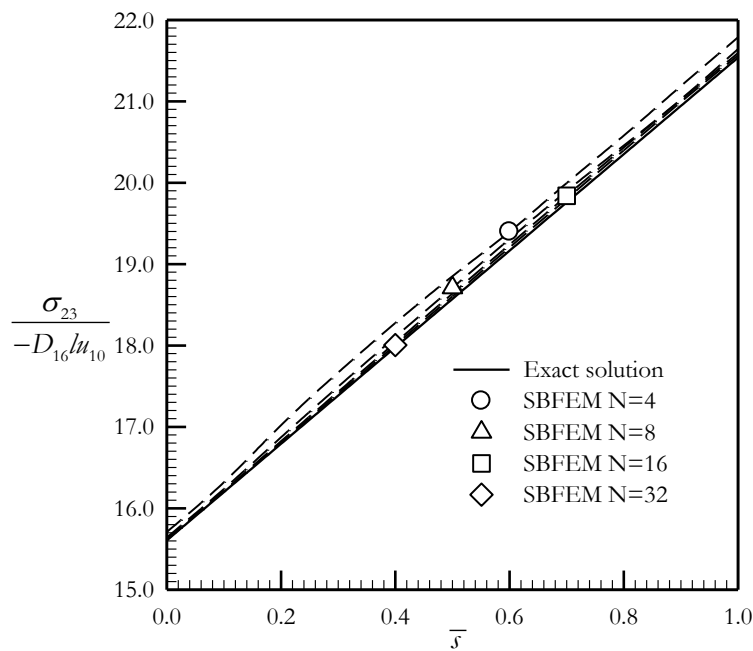


Fig. 12. Normalized a non-zero electrical induction component along the diagonal line BD of a piezoelectric square plate subjected to mixed boundary conditions. Results are reports as a function of normalized $\bar{s} = s / \sqrt{2}l$ where s is the length along the line BD measured from point D .

5. Conclusion and Remarks

A numerical technique based upon the scaled boundary finite element method (SBFEM) has been successfully developed for solving two-dimensional, multi-field boundary value problems. Both the formulation and implementations have been established in a general framework allowing various classes of linear boundary value problems (e.g., steady-state heat conduction problems, Laplace's equation, linear elasticity, linear piezoelectricity, etc.) and a set of general data such as the domain geometry, the prescribed distributed body source, the prescribed boundary conditions, and the contribution of the side-face conditions to be treated in a single, unified fashion. Results from an extensive numerical study for various scenarios have revealed that the proposed SBFEM yields highly accurate numerical solutions with the good convergence behavior. Relatively coarse meshes containing only few degrees of freedom have been found to accurately capture both the state variable and the body flux. In particular, both prescribed state variables and prescribed surface flux along the side faces and the general mixed boundary conditions along the scale boundary direction have been implemented into the proposed procedure and this, therefore, provides the flexibility in the selection of the scaling center and defining curve for describing the domain geometry.

Although the proposed technique has been implemented within the context of two-dimensional boundary value problems with domains described by a single scaling center, its underlying formulation and computational procedure are sufficiently general and should provide an essential basis for the extension to treat more general bodies such as those requiring multiple scaling centers to fully describe their geometry. Another potential extension is to generalize the present unified framework to three-dimensional multi-field problems.

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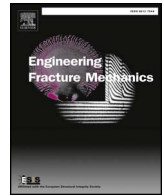
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Generalized T-stress solutions for penny-shaped cracks in transversely isotropic piezoelectric media

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ABSTRACT

A novel set of analytical and semi-analytical solutions is obtained for the generalized T-stress for both electrically impermeable and permeable penny-shaped cracks embedded in a transversely isotropic, linear piezoelectric, infinite medium under uniform far-field loading and general mechanical and electrical crack-face loading. The generalized stress field of the crack for two basic loading cases, viz. uniformly distributed self-equilibrated crack-face loading and a pair of crack-face self-equilibrated unit normal point forces and unit point electrical charges, are employed together with the asymptotic near-front expansion to establish fundamental solutions for the generalized T-stress. Such fundamental solutions are explicit and involve only elementary functions, and are then utilized to form an explicit integral formula for computing the generalized T-stress of a penny-shaped crack under general crack-face loading. The exact integration of the integral formula is also established for specific loading conditions, whereas its evaluation via an efficient quadrature for general loading conditions is also addressed. A selected set of numerical results is presented to verify the proposed quadrature and significant and useful findings are also reported.

1. Introduction

The modeling of fracture in piezoelectric media has received attention from many researchers due to the increasing applications of piezoelectric materials in different fields and the need for fundamental understanding of fracture characteristics in the safe design of electro-mechanical systems [1–3]. The theory of linear piezoelectricity has been utilized in the modeling of cracks in piezoelectric media and predicted responses such as the localized generalized stress field, relative crack-face data, intensity factors and generalized T-stress are crucial quantities often used in linear piezoelectric fracture modeling (e.g., [4,5]).

Similar to the linear elastic case (e.g., [6–12]), in linear piezoelectric media the generalized T-stress components have been found to play a crucial role in fracture-related response prediction, such as the size and shape of the inelastic zone embedding the crack front, tri-axiality and stress confinement ahead of the crack front, and the characteristics of crack initiation and crack growth (e.g., [13–17]). These previous studies have confirmed that the use of a single-parameter fracture model based solely on the intensity factors is insufficient for many scenarios and, as a result, the necessity to integrate the information of the non-singular terms to

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enhance the modeling capability is obligatory (e.g., [18–21]). To facilitate the use of such two-parameter fracture models, the accurate determination of the generalized T-stress data along the crack front, in addition to the intensity factors, is found to be an essential step.

Based on an extensive literature survey, it can be said that studies related to the analysis of the generalized T-stress and non-singular terms for cracks in piezoelectric media are still very limited [13–17], in comparison with studies carried out on cracks in elastic media. For the latter case, various solution schemes including analytical, semi-analytical, and numerical techniques have been proposed extensively, to determine the T-stress components along the crack boundary for different scenarios (e.g., [22–28]). Although problems of cracks in linear piezoelectric solids have also been investigated in the past three decades, most of the work focused primarily on the determination of the singular terms or the generalized intensity factors (e.g., [29–37]). Previous studies on the generalized T-stress have been limited to simple boundary value problems involving two-dimensional bodies and/or simple crack geometries [13–17]. This may result not only from the complexity of electro-mechanical coupling effect and the proper treatment of the crack-face conditions (e.g., [37–39]) but also from the lack of recognition of the generalized T-stress and its role. The brief overview of relevant literature concerning the determination of the generalized T-stress solution in linear piezoelectric media presented here to not only review the problems solved previously but also to indicate the gap in knowledge and the merits of the present study. Zhu and Yang [13] applied the continuous dislocation theory and electric dipoles together with the Stroh formalism and a boundary integral equation method to determine the generalized T-stress for an impermeable kinked crack in a two-dimensional, linear piezoelectric, infinite body. In addition, they also confirmed the strong contribution of the generalized T-stress on the characteristics of crack propagation. Later, Hao and Biao [14] derived the exact solution of the generalized T-stress of an impermeable straight crack embedded in a transversely isotropic, linear piezoelectric infinite plane under uniform mechanical and electrical far-field loading. The principle of superposition and Plemelj formulation were adopted in the analysis and the non-singular terms were solved from the corresponding Riemann-Hilbert problem. Results from their study also indicated that the value of the generalized T-stress depends strongly on both the elastic and electric material constants. In the same year, Li and Lee [15] investigated a semi-permeable Griffith crack oriented normal to the poling direction and subjected to uniform far-field loading. In their study, Fourier integral transforms together with the standard procedure for solving dual integral equations were employed to establish analytical expressions for the electro-elastic field, the intensity factors and the generalized T-stress. Later, Viola et al. [16] investigated the influence of the non-singular terms on the near-tip generalized stresses and displacements of a Griffith crack embedded in a transversely isotropic, linear piezoelectric, infinite plane under uniform biaxial far-field loading and various crack-face conditions. In their work, an analytical method along with the similarity transformation and complex potential theory were adopted to obtain mechanical and electrical fields and the non-singular part. Recently, Liu et al. [17] studied an elliptical hole and a straight crack in a two-dimensional, transversely isotropic, linear piezoelectric, infinite medium subjected to both uniform crack-face pressure and uniform far-field electro-mechanical loading. The complete analytical solution of both mechanical and electrical fields and non-singular terms for an electrically permeable crack-face condition was obtained by a complex-variable representation method. They also pointed out, that the non-singular terms significantly influence the electro-mechanical field in the neighborhood of the crack front. It should be evident from these earlier studies [13–17], that analytical and numerical solutions for the generalized T-stress have been very limited and restricted mostly to two-dimensional settings. Lack of generalized T-stress solutions for the three-dimensional case clearly limits the modeling capability to simulate practical and complex scenarios.

In the present study, the solution for the generalized T-stress components has been obtained for electrically impermeable and electrically permeable penny-shaped cracks embedded in a transversely isotropic, linear piezoelectric, infinite medium under uniform far-field loading and general mechanical and electrical crack-face loading. This novel set of solutions is presented in an analytical and semi-analytical form suitable for further use as reliable benchmark solutions in the verification procedure of numerical techniques developed for modeling more complex crack problems. In addition to such merit, results from the present study can be used as a basis for the investigation of the role of the generalized T-stress in fracture-response prediction within a three-dimensional context.

2. Problem description and basic equations

Consider a three-dimensional infinite medium containing a penny-shaped crack of radius a , as shown schematically in Fig. 1. The reference Cartesian coordinate system $\{O; x_1, x_2, x_3\}$ and the corresponding cylindrical coordinate system $\{O; r, \theta, z\}$ are chosen such that the origin O is located at the center of the crack, the x_3 -axis is directed upward and normal to the crack surface, and the remaining axes follow the right-hand rule. The body is made of a homogeneous, transversely isotropic, linear piezoelectric material with the axis of material symmetry and the poling direction perpendicular to the crack surface.

The basic field equations for a linear piezoelectric body can be written in a concise form as,

$$\sigma_{ij,i} = 0, \quad (1)$$

$$\sigma_{ij} = E_{ijkl} u_{k,m}, \quad (2)$$

(see [35,37] for details of this concise notation), where lower-case and upper-case indices range from 1 to 3 and 1 to 4, respectively, and repeated indices imply summation over their range; a standard comma symbol $(\cdot)_{,i}$ is used to represent a partial derivative with respect to the spatial coordinate x_i ; σ_{ij} are the generalized stress components which contain the components of the stress tensor σ_{ij} and the components of the electric induction vector $\sigma_{i4} \equiv D_i$; u_K are the generalized displacement components which contain the components of the displacement u_k and the electric potential $u_4 \equiv \phi$; and E_{ijkl} are the generalized moduli which contain the elastic moduli E_{ijkl} , the piezoelectric constants $E_{ij4l} \equiv E_{l4ji}$, and the dielectric permittivities $-E_{i44l} \equiv -E_{l44i}$. For the transversely isotropic case, only

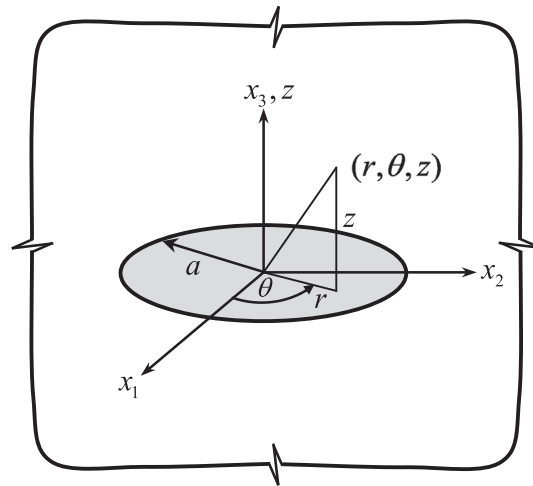


Fig. 1. Schematic of a penny-shaped crack of radius a embedded in a transversely isotropic, linear piezoelectric, infinite medium.

ten independent material constants (i.e., five elastic constants $E_{1111}, E_{1122}, E_{1133}, E_{3333}, E_{2323}$, three piezoelectric constants $E_{1341}, E_{1143}, E_{3343}$, and two dielectric permittivities $-E_{1441}, -E_{3443}$) are required to completely describe the generalized moduli E_{ijkl} . The components of the mechanical tractions t_j and the surface electric charge t_4 at any point on the smooth boundary can be related to the generalized stress components σ_{ij} by $t_j = \sigma_{ij} n_i$ where t_j are the generalized traction components and n_i are the components of the outward unit normal vector to the boundary.

On the upper and lower crack surfaces, the generalized displacement components $\{u_j^+, u_j^-\}$ and the generalized traction components $\{t_j^+, t_j^-\}$ must be properly prescribed. In the present study, two crack-face models, viz. electrically impermeable cracks and electrically permeable cracks, are considered. For the electrically impermeable cracks (e.g., [40]), the crack-face generalized traction components $\{t_j^+, t_j^-\}$ are prescribed, whereas the jump in the generalized displacement components across the crack surface $\Delta u_j = u_j^+ - u_j^-$ are unknown a priori. For this case, four types of loading conditions, (i) uniform far-field triaxial stress $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ (Fig. 2(a)), (ii) uniform far-field electrical induction σ_{i4}^∞ (Fig. 2(a)), (iii) general self-equilibrated, crack-face normal traction t_3^0 (Fig. 2(b)), and (iv) general self-equilibrated, crack-face surface electric charge t_4^0 (Fig. 2(c)), are considered. For the electrically permeable cracks (e.g., [41]), the crack-face mechanical traction components t_j^+ and t_j^- are prescribed and the jump in the crack-face electric potential and the sum of the crack surface electric charge vanishes (i.e., $\Delta u_4 \equiv u_4^+ - u_4^- = 0$ and $\Sigma t_4 \equiv t_4^+ + t_4^- = 0$). Here, the sum of the crack-face generalized displacement components $\Sigma u_j \equiv u_j^+ + u_j^-$, the jump in the crack-face displacement components $\Delta u_j \equiv u_j^+ - u_j^-$, and the jump in the crack-face electric charge $\Delta t_4 \equiv t_4^+ - t_4^-$ are unknown a priori. For this case, three loading conditions, (i) uniform far-field triaxial stress $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ (Fig. 2(a)), (ii) uniform far-field electrical induction σ_{i4}^∞ (Fig. 2(a)), and (iii) general self-equilibrated, crack-face normal traction t_3^0 (Fig. 2(b)) are treated.

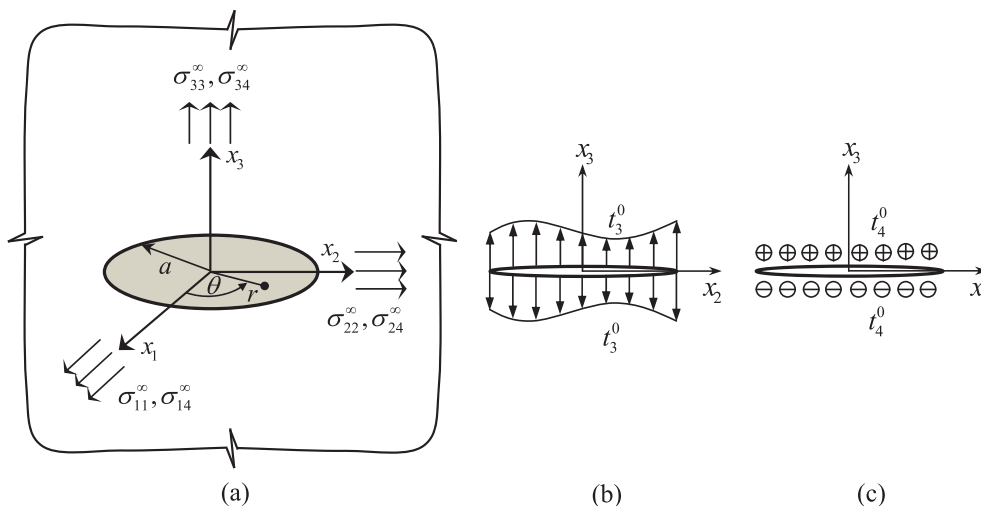


Fig. 2. Schematic of penny-shaped crack under (a) uniform far-field triaxial stress $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and uniform far-field electrical induction $\{\sigma_{14}^\infty, \sigma_{24}^\infty, \sigma_{34}^\infty\}$; (b) general, self-equilibrated, crack-face, normal traction t_3^0 and (c) general, self-equilibrated, crack-face, surface electric charge t_4^0 .

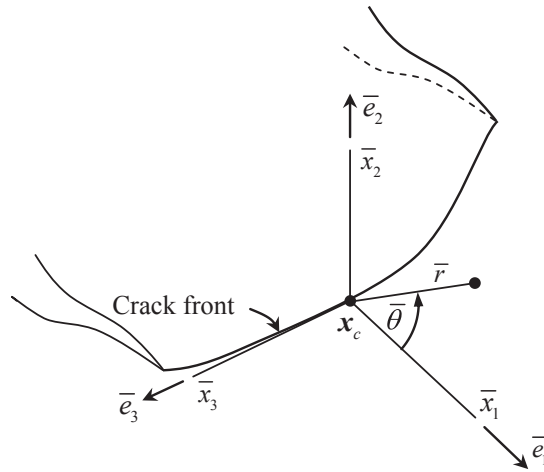


Fig. 3. Schematic of crack front and local reference coordinate system $\{x_c, \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ with origin at x_c used for defining generalized T-stress components.

3. Near-front generalized stress field

By adopting the same strategy employed by Williams [42], the local generalized stress field in a region surrounding the point x_c on the crack front is represented by,

$$\bar{\sigma}_{ij}(x_c; \bar{r}, \bar{\theta}) = \frac{1}{\sqrt{\bar{r}}} \bar{\sigma}_{ij}^K(x_c; \bar{\theta}) + \bar{\sigma}_{ij}^T(x_c; \bar{\theta}) + \sum_{m=1}^{\infty} \bar{r}^{m/2} \bar{\sigma}_{ij}^{(m)}(x_c; \bar{\theta}), \quad (3)$$

where the “bar” is used to designate quantities referring to the local reference coordinate system $\{x_c, \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ with origin at x_c and corresponding orthonormal basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$, such that $\bar{x}_1$ – $\bar{x}_3$ is a tangent plane to the crack surface at x_c , whereas the $\bar{x}_1$ – $\bar{x}_2$ plane is normal to the crack front at x_c as illustrated in Fig. 3; $(\bar{r}, \bar{\theta})$ denotes the polar coordinates of a point in the $\bar{x}_1$ – $\bar{x}_2$ plane; $\bar{\sigma}_{ij}(x_c; \bar{r}, \bar{\theta})$ denotes the generalized stress components at any point in the $\bar{x}_1$ – $\bar{x}_2$ plane in the neighborhood of the point x_c ; and $\bar{\sigma}_{ij}^K$, $\bar{\sigma}_{ij}^T$, and $\bar{\sigma}_{ij}^{(m)}$ are functions independent of the local radial coordinate $\bar{r}$. This asymptotic expansion of the near-front field is expressed in a form analogous to that appearing in the work of [28]. Regarding the near-front expansion given in Eq. (3), it is evident that the first term represents the dominant part of the field which is singular of order $1/\sqrt{\bar{r}}$ at x_c ; the second term represents the part that is independent of the local radial coordinate $\bar{r}$, and the third term represents the non-singular part that vanishes identically at x_c . It is also noted that the function $\bar{\sigma}_{ij}^T$ must also be independent of the angular coordinate $\bar{\theta}$ in order to eliminate the essential singularity and ensure that it represents the finite part of the generalized stress field along the crack front in the local Cartesian coordinate system. Introduce the concise notation T_{ij} such that,

$$T_{ij} = T_{ij}(x_c) = \bar{\sigma}_{ij}^T(x_c). \quad (4)$$

From Eq. (4) and the definition of the generalized stress Eq. (2), it is apparent that $T_{ij} = T_{ji}$ and as a result, only nine components out of twelve are independent. In addition, not all of these components are unknown a priori. From the continuity of the finite part of the generalized stress at x_c , it can be readily verified that the four components T_{21} , T_{22} , T_{23} and T_{24} can be obtained in terms of the generalized crack-face traction at a limiting point x_c on the surface of the crack. As a result, $\{T_{21}, T_{22}, T_{23}, T_{24}\}$ and $\{T_{21}, T_{22}, T_{23}\}$ are known a priori for electrically impermeable and electrically permeable cracks, respectively. The components T_{22} and T_{24} can be readily obtained once the unknown crack-face generalized tractions are determined. The five independent components $\{T_{11}, T_{13}, T_{33}, T_{14}, T_{34}\}$ are always unknown and they are commonly termed the generalized T-stress components. The first three components $\{T_{11}, T_{13}, T_{33}\}$ are the mechanical T-stress components associated with the stress field similar to the elastic case (see [28]), whereas the last two components $\{T_{14}, T_{34}\}$ are the electrical T-stress components associated with the electric induction field. It can be seen that the generalized T-stress components $\{T_{11}, T_{13}, T_{33}, T_{14}, T_{34}\}$ can be obtained if the generalized stress field at least in the region embedding the crack front is known. From the near-front asymptotic expansion of the generalized stress field Eq. (3) together with Eq. (4), the generalized T-stress components can be expressed explicitly in terms of the generalized stress field by,

$$T_{\tau}(x_c) = 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} [\sqrt{\bar{r}} \bar{\sigma}_{\tau}(x_c; \bar{r}, \bar{\theta} = 0)] \right\}, \quad \tau \in \{11, 13, 33, 14, 34\}. \quad (5)$$

From Eq. (5), it becomes evident that the calculation of the generalized T-stress components from the existing generalized stress field of a cracked body, only requires standard differentiations and taking the limits.

4. Integral formula for generalized T-stress components

Here an explicit formula is presented to compute the generalized T-stress components for both electrically impermeable and

electrically permeable penny-shaped cracks, subjected to uniform far-field loading and general mechanical and electrical crack-face loading indicated in Section 2. Instead of directly solving for the generalized stress field and then using Eq. (5) to calculate the generalized T-stress components $\{T_{11}, T_{13}, T_{33}, T_{14}, T_{34}\}$ for general mechanical and electrical crack-face loading, it is more appealing to consider the linearity of the problem and use the method of superposition and available fundamental solutions to obtain the desired results. This particular technique is similar to that employed by Jin and Wang [26]. By adopting this strategy, only solutions for two fundamental problems associated with the penny-shaped crack subjected to a pair of concentrated mechanical and electrical crack-face loading and a uniformly distributed mechanical and electrical crack-face loading are required. A single-surface integral formula for general mechanical and electrical crack-face loading can be readily established from these basic results via the superposition technique.

4.1. Generalized T-stress components for uniform far-field loading and general mechanical and electrical crack-face loading

Consider a generic point $\mathbf{x}_c$ along the boundary of the penny-shaped crack of radius a with the global cylindrical coordinates $(a, \theta, 0)$. The local coordinate system $\{\mathbf{x}_c, \bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3\}$ and the corresponding polar coordinates $(\bar{r}, \bar{\theta})$ shown in Fig. 3 are employed throughout, to describe quantities in a region surrounding the point $\mathbf{x}_c$. From the principle of superposition, the generalized stress field for electrically impermeable cracks and electrically permeable cracks in the neighborhood of the point $\mathbf{x}_c$ admits the following integral representation,

$$\bar{\sigma}_{ij}(a, \theta, \bar{r}, \bar{\theta}) = \bar{\sigma}_{ij}^{\infty} + \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{ij}^{GP}(a, \theta, \bar{r}, \bar{\theta}; r_1, \theta_1) \{t_3^0(r_1, \theta_1) + \sigma_{33}^{\infty}\} r_1 dr_1 d\theta_1 + \Lambda \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{ij}^{GQ}(a, \theta, \bar{r}, \bar{\theta}; r_2, \theta_2) \{t_4^0(r_2, \theta_2) + \sigma_{34}^{\infty}\} r_2 dr_2 d\theta_2, \quad (6)$$

where $\bar{\sigma}_{ij}$ are local components of the generalized stress field; $\bar{\sigma}_{ij}^{\infty}$ are local components of the uniform far-field generalized stress (directly obtained from the prescribed global far-field loading $\{\sigma_{11}^{\infty}, \sigma_{22}^{\infty}, \sigma_{33}^{\infty}\}$ and σ_{i4}^{∞} via coordinate transformation); $\bar{\sigma}_{ij}^{GP}$ are local components of the generalized stress Green's function due to a pair of self-equilibrated, unit normal point forces P acting on the crack surfaces at $(r_1, \theta_1, 0^{\pm})$ (see Fig. 4(a)); $\bar{\sigma}_{ij}^{GQ}$ are local components of the generalized stress Green's function due to a pair of self-equilibrated, unit point charges Q acting on the crack surfaces at $(r_2, \theta_2, 0^{\pm})$ (see Fig. 4(b)); and $\Lambda = 1, 0$ for electrically impermeable cracks and electrically permeable cracks, respectively. By following the same procedure as that utilized by Rungamornrat and Pinitpanich [28] (i.e., using Eq. (5) together with the integral relation Eq. (6)), the generalized T-stress components at any point $\mathbf{x}_c$ along the crack front can be expressed as,

$$T_{\tau}(\theta) = T_{\tau}^{\infty} + T_{\tau}^0(\theta) + T_{\tau}^1(\theta), \quad \tau \in \{11, 13, 33, 14, 34\}, \quad (7)$$

where T_{τ}^{∞} , $T_{\tau}^0(\theta)$, and $T_{\tau}^1(\theta)$ are defined by

$$T_{\tau}^{\infty} = \bar{\sigma}_{\tau}^{\infty} + 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{\tau}^{GP}(a, \theta, \bar{r}, \bar{\theta} = 0; r_1, \theta_1) \sigma_{33}^{\infty} r_1 dr_1 d\theta_1 \right] \right. \\ \left. + 2\Lambda \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{\tau}^{GQ}(a, \theta, \bar{r}, \bar{\theta} = 0; r_2, \theta_2) \sigma_{34}^{\infty} r_2 dr_2 d\theta_2 \right] \right\} \right\}, \quad (8)$$

$$T_{\tau}^0(\theta) = 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{\tau}^{GP}(a, \theta, \bar{r}, \bar{\theta} = 0; r_1, \theta_1) t_3^0(a, \theta) r_1 dr_1 d\theta_1 \right] \right. \\ \left. + 2\Lambda \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_{\tau}^{GQ}(a, \theta, \bar{r}, \bar{\theta} = 0; r_2, \theta_2) t_4^0(a, \theta) r_2 dr_2 d\theta_2 \right] \right\} \right\}, \quad (9)$$

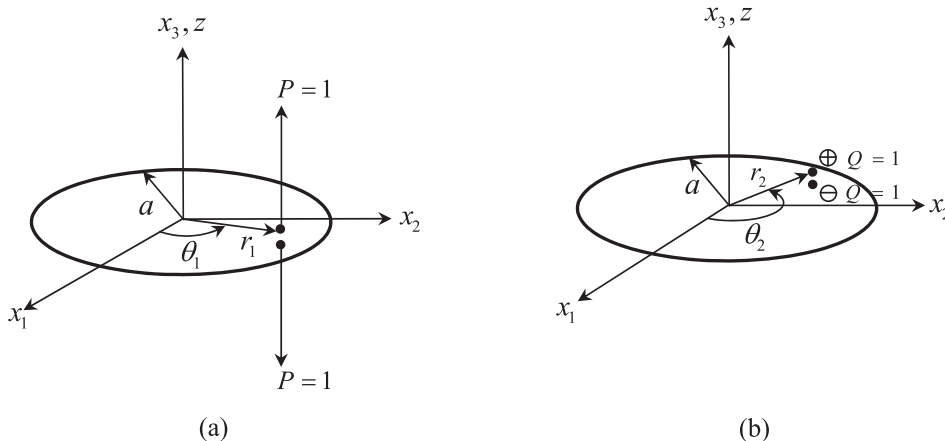


Fig. 4. Schematic of penny-shaped crack under (a) pair of self-equilibrated, crack-face, unit normal point forces P and (b) pair of self-equilibrated, crack-face, unit point electric charges Q .

$$T_r^1(\theta) = 2 \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_r^{GP}(a, \theta, \bar{r}, \bar{\theta} = 0; r_1, \theta_1) [t_3^0(r_1, \theta_1) - t_3^0(a, \theta)] r_1 dr_1 d\theta_1 \right] \right\} \\ + 2\Lambda \lim_{\bar{r} \rightarrow 0} \left\{ \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \int_{-\pi}^{\pi} \int_0^a \bar{\sigma}_r^{GQ}(a, \theta, \bar{r}, \bar{\theta} = 0; r_2, \theta_2) [t_4^0(r_2, \theta_2) - t_4^0(a, \theta)] r_2 dr_2 d\theta_2 \right] \right\}. \quad (10)$$

From above expressions, T_r^∞ denotes the generalized T-stress components due to the uniform far-field loading $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and σ_{i4}^∞ ; $T_r^0(\theta)$ denotes the generalized T-stress components due to the uniformly distributed crack-face normal traction and electric surface charge of magnitude $t_3^0(a, \theta)$ and $t_4^0(a, \theta)$, respectively; and $T_r^1(\theta)$ denotes the generalized T-stress components associated with the remaining (non-uniform) part of the general applied crack-face loading $t_3^0(r_1, \theta_1)$ and $t_4^0(r_2, \theta_2)$. It is worth noting that T_r^∞ is independent of angle θ due to axisymmetry and the non-uniform part of t_3^0 and t_4^0 vanishes at the point $\mathbf{x}_c \equiv (a, \theta, 0)$.

To avoid carrying out the direct integration of double line integrals containing the generalized stress Green's function $\bar{\sigma}_r^{GP}$ and $\bar{\sigma}_r^{GQ}$ in Eqs. (8) and (9), it is more appealing to utilize existing closed-form solutions of the generalized stress field of a penny-shaped crack under uniformly distributed crack-face loading (e.g., [44,45]), to establish the closed-form expressions for T_r^∞ and $T_r^0(\theta)$. With this strategy, Eqs. (8) and (9) can be replaced by,

$$T_r^\infty = 2 \lim_{\bar{r} \rightarrow 0} \left[\sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\sqrt{\bar{r}} \bar{\sigma}_r^{\infty M} \right) \right] + 2 \lim_{\bar{r} \rightarrow 0} \left[\sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\sqrt{\bar{r}} \bar{\sigma}_r^{\infty E} \right) \right], \quad (11)$$

$$T_r^0(\theta) = 2 \lim_{\bar{r} \rightarrow 0} \left[\sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\sqrt{\bar{r}} \bar{\sigma}_r^{uM} \right) \right] + 2\Lambda \lim_{\bar{r} \rightarrow 0} \left[\sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\sqrt{\bar{r}} \bar{\sigma}_r^{uE} \right) \right], \quad (12)$$

where $\bar{\sigma}_r^{\infty M}$, $\bar{\sigma}_r^{\infty E}$ are local components of the generalized stress field due to the uniform far-field loading $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and σ_{i4}^∞ , respectively, and $\bar{\sigma}_r^{uM}$, $\bar{\sigma}_r^{uE}$ are local components of the generalized stress field due to the uniformly distributed, self-equilibrated, crack-face loading $t_3^0(a, \theta)$ and $t_4^0(a, \theta)$, respectively. It is important to note that the generalized stress fields $\{\bar{\sigma}_r^{\infty M}, \bar{\sigma}_r^{\infty E}\}$ can be directly related to $\{\bar{\sigma}_r^{\infty M}, \bar{\sigma}_r^{\infty E}\}$ via the superposition of cracked and un-cracked cases and, as a direct consequence, only one set of this information is sufficient to obtain the expressions for T_r^∞ and $T_r^0(\theta)$ given by Eqs. (11) and (12) as shown in Section 4.2.

By recognizing the conditions $t_3^0(r_1, \theta_1) = t_3^0(a, \theta)$ and $t_4^0(r_2, \theta_2) = t_4^0(a, \theta)$ at the point $\mathbf{x}_c \equiv (a, \theta, 0)$ and the fact that the singularity of the generalized stress Green's functions $\bar{\sigma}_r^{GP}$ and $\bar{\sigma}_r^{GQ}$ is present only at the location where the concentrated load is applied, the order of limits, differentiations and integrations in Eq. (10) can be interchanged and the expression for $T_r^1(\theta)$ can be rewritten as,

$$T_r^1(\theta) = \int_{-\pi}^{\pi} \int_0^a 2 \left\{ \lim_{\bar{r} \rightarrow 0} \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \bar{\sigma}_r^{GP}(a, \theta, \bar{r}, \bar{\theta} = 0; r_1, \theta_1) \right] \right\} [t_3^0(r_1, \theta_1) - t_3^0(a, \theta)] r_1 dr_1 d\theta_1 \\ + \Lambda \int_{-\pi}^{\pi} \int_0^a 2 \left\{ \lim_{\bar{r} \rightarrow 0} \sqrt{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[\sqrt{\bar{r}} \bar{\sigma}_r^{GQ}(a, \theta, \bar{r}, \bar{\theta} = 0; r_2, \theta_2) \right] \right\} [t_4^0(r_2, \theta_2) - t_4^0(a, \theta)] r_2 dr_2 d\theta_2. \quad (13)$$

By using the definition of generalized T-stress given by Eq. (5), $T_r^1(\theta)$ in Eq. (13) can be reduced to,

$$T_r^1(\theta) = \int_{-\pi}^{\pi} \int_0^a T_r^{GP}(\theta; r_1, \theta_1) P(r_1, \theta_1) r_1 dr_1 d\theta_1 + \Lambda \int_{-\pi}^{\pi} \int_0^a T_r^{GQ}(\theta; r_2, \theta_2) Q(r_2, \theta_2) r_2 dr_2 d\theta_2, \quad (14)$$

where $P(r_1, \theta_1) = t_3^0(r_1, \theta_1) - t_3^0(a, \theta)$ and $Q(r_2, \theta_2) = t_4^0(r_2, \theta_2) - t_4^0(a, \theta)$; T_r^{GP} is the generalized T-stress Green's function for a pair of self-equilibrated, unit normal point forces P ; and T_r^{GQ} is the generalized T-stress Green's function for a pair of self-equilibrated, unit point charges Q . The closed-form expressions for T_r^{GP} and T_r^{GQ} are presented in Section 4.3.

4.2. Explicit expressions for T_r^∞ and $T_r^0(\theta)$

Here, the closed-form expressions for T_r^∞ and $T_r^0(\theta)$ are derived for electrically impermeable and electrically permeable cracks, using available solutions for the generalized stress field for a penny-shaped crack in a piezoelectric medium subjected to uniform far-field loading (e.g., [44,45]), along with Eqs. (11) and (12).

4.2.1. Electrically impermeable crack

For the electrically impermeable case ($\Lambda = 1$), the solutions for the generalized stress fields $\bar{\sigma}_r^{\infty M}$ and $\bar{\sigma}_r^{\infty E}$ due to uniform far-field mechanical and electrical loading obtained by Chen et al. [44] are employed, to determine expressions for T_r^∞ and $T_r^0(\theta)$. By carrying out the differentiation and taking the limits appearing in Eq. (11), the generalized T-stress components T_r^∞ due to uniform far-field triaxial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and uniform far-field electrical induction σ_{i4}^∞ are given in closed-form by,

$$T_{11}^\infty = \bar{\sigma}_{11}^\infty + \Upsilon_{ic}^M \sigma_{33}^\infty + \Upsilon_{ic}^E \sigma_{34}^\infty, \quad T_{13}^\infty = \bar{\sigma}_{13}^\infty, \quad T_{33}^\infty = \bar{\sigma}_{33}^\infty + \Upsilon_{ic}^M \sigma_{33}^\infty + \Upsilon_{ic}^E \sigma_{34}^\infty, \quad T_{14}^\infty = \bar{\sigma}_{14}^\infty, \quad T_{34}^\infty = \bar{\sigma}_{34}^\infty, \quad (15)$$

where $\{\bar{\sigma}_{11}^\infty, \bar{\sigma}_{13}^\infty, \bar{\sigma}_{33}^\infty\}$ and $\{\bar{\sigma}_{14}^\infty, \bar{\sigma}_{34}^\infty\}$ are components of the far-field triaxial stresses and far-field electrical induction with respect to the local coordinate system $\{\mathbf{x}_c, \bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3\}$,

$$\Upsilon_{ic}^M = \sum_{j=1}^3 [(c_{66} - c_{11}) + c_{13} s_j \alpha_{j1} + e_{31} s_j \alpha_{j2}] (c_j g_4 - d_j g_3) / (g_1 g_4 - g_2 g_3), \quad (16)$$

$$\Upsilon_{ic}^E = \sum_{j=1}^3 [(c_{66} - c_{11}) + c_{13} s_j \alpha_{j1} + e_{31} s_j \alpha_{j2}] (d_j g_1 - c_j g_2) / (g_1 g_4 - g_2 g_3), \quad (17)$$

where,

$$g_1 = - \sum_{j=1}^3 c_j \gamma_{j1}, \quad g_2 = - \sum_{j=1}^3 d_j \gamma_{j1}, \quad g_3 = - \sum_{j=1}^3 c_j \gamma_{j2}, \quad g_4 = - \sum_{j=1}^3 d_j \gamma_{j2}, \quad (18)$$

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{1}{2\pi} \begin{bmatrix} s_1 & s_2 & s_3 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \frac{1}{2\pi} \begin{bmatrix} s_1 & s_2 & s_3 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{bmatrix}^{-1} \begin{Bmatrix} e_{15}/c_{44} \\ 0 \\ -1 \end{Bmatrix}, \quad (19)$$

$$\gamma_{j1} = -c_{13} + c_{33}s_j\alpha_{j1} + e_{33}s_j\alpha_{j2}, \quad \gamma_{j2} = -e_{31} + e_{33}s_j\alpha_{j1} - e_{33}s_j\alpha_{j2}, \quad (20)$$

$$\alpha_{j1} = \frac{c_{11}\varepsilon_{11} - m_3 s_j^2 + c_{44}\varepsilon_{33}s_j^4}{(m_1 - m_2 s_j^2)s_j}, \quad \alpha_{j2} = \frac{c_{11}e_{15} - m_4 s_j^2 + c_{44}e_{33}s_j^4}{(m_1 - m_2 s_j^2)s_j}, \quad (21)$$

$$m_1 = \varepsilon_{11}(c_{13} + c_{44}) + e_{15}(e_{15} + e_{31}), \quad m_2 = \varepsilon_{33}(c_{13} + c_{44}) + e_{33}(e_{15} + e_{31}), \\ m_3 = c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2, \quad m_4 = c_{11}e_{33} + c_{44}e_{15} - (c_{13} + c_{44})(e_{15} + e_{31}). \quad (22)$$

The relevant material constants are defined by $c_{11} = E_{1111}$, $c_{12} = E_{1122}$, $c_{13} = E_{1133}$, $c_{33} = E_{3333}$, $c_{44} = E_{2323}$, $c_{66} = \frac{1}{2}(c_{11} - c_{12})$, $e_{15} = E_{1341}$, $e_{31} = E_{1143}$, $e_{33} = E_{3343}$, $\varepsilon_{11} = -E_{1441}$, $\varepsilon_{33} = -E_{3443}$ and the three parameters $\{s_1^2, s_2^2, s_3^2\}$ are solutions of the equation,

$$as^6 - bs^4 + cs^2 - d = 0, \quad (23)$$

where constants a , b , c , d are defined by,

$$a = c_{44}(e_{33}^2 + c_{33}\varepsilon_{33}), \quad b = c_{33}m_3 + \varepsilon_{33}[c_{44}^2 - (c_{13} + c_{44})^2] + e_{33}(2m_4 - c_{11}e_{33}), \\ c = c_{44}m_3 + \varepsilon_{11}[c_{11}c_{33} - (c_{13} + c_{44})^2] + e_{15}(2m_4 - c_{44}e_{15}), \quad d = c_{11}(e_{15}^2 + c_{44}\varepsilon_{11}). \quad (24)$$

By using the fact that the problem of a cracked medium under uniform far-field loading can be decomposed into two subproblems, one subproblem associated with the un-cracked medium under uniform far-field loading and the other subproblem corresponding to the cracked medium subjected to the equivalent crack-face generalized tractions, the solutions $\{\bar{\sigma}_r^{uM}, \bar{\sigma}_r^{uE}\}$ can then be readily obtained from $\{\bar{\sigma}_r^{\infty M}, \bar{\sigma}_r^{\infty E}\}$. The closed-form solution for $T_r^0(\theta)$ due to the uniformly distributed, crack-face loads $t_3^0(a, \theta)$ and $t_4^0(a, \theta)$ can then be established using Eq. (12) and the final results are given by,

$$T_{11}^0(\theta) = T_{33}^0(\theta) = \Upsilon_{ic}^M t_3^0(a, \theta) + \Upsilon_{ic}^E t_4^0(a, \theta), \quad T_{13}^0 = 0, \quad T_{14}^0 = 0, \quad T_{34}^0 = 0. \quad (25)$$

4.2.2. Electrically permeable crack

For the electrically permeable case ($\Lambda = 0$), the solutions for the generalized stress fields $\bar{\sigma}_r^{\infty M}$ and $\bar{\sigma}_r^{\infty E}$ due to uniform far-field mechanical and electrical loading reported by Chen and Lim [45] are utilized, to determine expressions for T_r^{∞} and $T_r^0(\theta)$. By following the same procedure as that employed in the electrically impermeable case, the closed-form expressions for the generalized T-stress components T_r^{∞} and $T_r^0(\theta)$ can be obtained as,

$$T_{11}^{\infty} = \bar{\sigma}_{11}^{\infty} + \Upsilon_{pc}^M \sigma_{33}^{\infty}, \quad T_{13}^{\infty} = \bar{\sigma}_{13}^{\infty}, \quad T_{33}^{\infty} = \bar{\sigma}_{33}^{\infty} + \Upsilon_{pc}^M \sigma_{33}^{\infty}, \quad T_{14}^{\infty} = \bar{\sigma}_{14}^{\infty}, \quad T_{34}^{\infty} = \bar{\sigma}_{34}^{\infty}, \quad (26)$$

$$T_{11}^0(\theta) = T_{33}^0(\theta) = \Upsilon_{pc}^M t_3^0(a, \theta), \quad T_{13}^0 = 0, \quad T_{14}^0 = 0, \quad T_{34}^0 = 0, \quad (27)$$

where the constant Υ_{pc}^M is defined by,

$$\Upsilon_{pc}^M = \sum_{j=1}^3 [(c_{66} - c_{11}) + c_{13}s_j\alpha_{j1} + e_{31}s_j\alpha_{j2}]c_j/g_1, \quad (28)$$

the constants c_j ($j = 1, 2, 3$) for the electrically permeable case, are obtained from,

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = -\frac{1}{2\pi} \begin{bmatrix} \gamma_{11}s_1 & \gamma_{21}s_2 & \gamma_{31}s_3 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad (29)$$

and all other constants are defined in the same manner as in Section 4.2.1 for the case of electrically impermeable cracks (the definition of all parameters can also be found in the work of Chen and Lim [45]). Unlike in the impermeable case, the component of the far-field electric induction σ_{34}^{∞} does not appear in the generalized T-stress components T_r^{∞} and $T_r^0(\theta)$.

4.3. Closed-form expressions for the generalized T-stress Green's functions

Here, closed-form expressions for the generalized T-stress Green's functions T_r^{GP} and T_r^{GQ} for a pair of self-equilibrated, crack-face, unit normal point forces P and for a pair of self-equilibrated, crack-face, unit point electric charges Q , respectively, are obtained for electrically impermeable and electrically permeable cracks. Available closed-form expressions for the generalized stress field (e.g., [43,45]) are used as the basis for such development.

4.3.1. Electrically impermeable crack

Consider the fundamental problem of an impermeable penny-shaped crack subjected to a pair of self-equilibrated, unit normal point forces P at $(r_1, \theta_1, 0^\pm)$ (Fig. 4(a)) and a pair of self-equilibrated, unit point charges Q at $(r_2, \theta_2, 0^\pm)$ (Fig. 4(b)), similar to the problems considered by Chen and Shioya [43]. The closed-form solution for the generalized stress Green's function, referring to the global cylindrical coordinate system $\{O; r, \theta, z\}$ defined in Section 2, is given by,

$$\sigma_1^G(r, \theta, z; r_\xi, \theta_\xi) = 8A \sum_{j=1}^3 [(c_{66} - c_{11}) + c_{13}s_j\alpha_{j1} + e_{31}s_j\alpha_{j2}][\beta_{j1}f_{31}(z_j) + \beta_{j2}f_{32}(z_j)], \quad (30)$$

$$\sigma_2^G(r, \theta, z; r_\xi, \theta_\xi) = 8Ac_{66} \sum_{j=1}^3 [\beta_{j1}f_{41}(z_j) + \beta_{j2}f_{42}(z_j)], \quad (31)$$

$$\tau_z^G(r, \theta, z; r_\xi, \theta_\xi) = 4A \sum_{j=1}^3 [c_{44}(s_j + \alpha_{j1}) + e_{15}\alpha_{j2}][\beta_{j1}f_{51}(z_j) + \beta_{j2}f_{52}(z_j)], \quad (32)$$

$$D^G(r, \theta, z; r_\xi, \theta_\xi) = 4A \sum_{j=1}^3 [e_{15}(s_j + \alpha_{j1}) - e_{11}\alpha_{j2}][\beta_{j1}f_{51}(z_j) + \beta_{j2}f_{52}(z_j)], \quad (\xi = 1, 2) \quad (33)$$

where the superscript “G” is used to designate the Green's function; $\sigma_1^G = \sigma_{rr}^G + \sigma_{\theta\theta}^G$, $\sigma_2^G = (\sigma_{rr}^G - \sigma_{\theta\theta}^G + 2i\sigma_{r\theta}^G)e^{2i\theta}$, $\tau_z^G = (\sigma_{rz}^G + i\sigma_{\theta z}^G)e^{i\theta}$, $D^G = (D_r^G + iD_\theta^G)e^{i\theta}$, $\beta_{j1} = c_j g_4 - d_j g_3$, $\beta_{j2} = d_j g_1 - c_j g_2$, $i = \sqrt{-1}$, $z_j = s_j z$ and $A = 1/4\pi^2(g_1 g_4 - g_2 g_3)$. All other parameters defined in Section 4.2.1 apply here and the complex functions $f_{3\xi}$, $f_{4\xi}$ and $f_{5\xi}$ are defined by,

$$f_{3\xi}(z) = \frac{z}{R_\xi^2} \tan^{-1} \left(\frac{h_\xi}{R_\xi} \right) - \frac{h_\xi}{z(R_\xi^2 + h_\xi^2)} \left[\frac{r^2 - \delta_1^2}{\delta_2^2 - \delta_1^2} - \frac{z^2}{R_\xi^2} \right], \quad (34)$$

$$f_{4\xi}(z) = \frac{\sqrt{a^2 - r_\xi^2}}{\tilde{t}_\xi \tilde{s}_\xi} \left(\frac{2}{\tilde{t}_\xi} - \frac{r_\xi e^{i\theta_\xi}}{\tilde{s}_\xi^2} \right) \tan^{-1} \left[\frac{\tilde{s}_\xi}{\sqrt{\delta_2^2 - a^2}} \right] - \frac{z(3R_\xi^2 - z^2)}{\tilde{t}_\xi^2 R_\xi^2} \tan^{-1} \left(\frac{h_\xi}{R_\xi} \right) + \frac{\sqrt{a^2 - r_\xi^2} \sqrt{\delta_2^2 - a^2} r_\xi e^{i\theta_\xi}}{\tilde{t}_\xi \tilde{s}_\xi^2 [\delta_2^2 - r r_\xi e^{-i(\theta - \theta_\xi)}]} \\ - \frac{zh_\xi}{R_\xi^2 + h_\xi^2} \left[\frac{t_\xi}{\tilde{t}_\xi R_\xi^2} - \frac{r^2 e^{2i\theta}}{(\delta_2^2 - \delta_1^2)(\delta_2^2 - r^2)} \right], \quad (35)$$

$$f_{5\xi}(z) = \frac{t_\xi}{R_\xi^2} \tan^{-1} \left(\frac{h_\xi}{R_\xi} \right) + \frac{h_\xi}{R_\xi^2 + h_\xi^2} \left[\frac{r e^{i\theta}}{\delta_2^2 - \delta_1^2} + \frac{t_\xi}{R_\xi^2} \right], \quad (\xi = 1, 2) \quad (36)$$

with $2\delta_1 = \sqrt{(r+a)^2 + z^2} - \sqrt{(r-a)^2 + z^2}$, $2\delta_2 = \sqrt{(r+a)^2 + z^2} + \sqrt{(r-a)^2 + z^2}$, $t_\xi = r e^{i\theta} - r_\xi e^{i\theta_\xi}$, $\tilde{t}_\xi = r e^{-i\theta} - r_\xi e^{-i\theta_\xi}$, $R_\xi = \sqrt{r^2 + r_\xi^2 - 2rr_\xi \cos(\theta - \theta_\xi) + z^2}$, $h_\xi a = \sqrt{a^2 - \delta_1^2} \sqrt{a^2 - r_\xi^2}$ and $\tilde{s}_\xi = \sqrt{a^2 - r r_\xi e^{-i(\theta - \theta_\xi)}}$.

By applying the following results,

$$\lim_{z \rightarrow 0} \frac{\sqrt{a^2 - \delta_1^2}}{z} = \frac{\sqrt{a}}{\sqrt{2(r^2 - a^2)}}, \quad \lim_{z \rightarrow 0} \frac{zh_\xi}{\delta_2 - r} = \frac{2\sqrt{r^2 - a^2} \sqrt{a^2 - r_\xi^2}}{r}, \quad (37)$$

to Eqs. (30)–(36), the Green's function for the generalized stress at any point on the plane $z = 0$ can be obtained explicitly by,

$$\sigma_1^G(r, \theta, 0; r_\xi, \theta_\xi) = 8A \sum_{j=1}^3 [(c_{66} - c_{11}) + c_{13}s_j\alpha_{j1} + e_{31}s_j\alpha_{j2}][\beta_{j1}f_{31}(0) + \beta_{j2}f_{32}(0)], \quad (38)$$

$$\sigma_2^G(r, \theta, 0; r_\xi, \theta_\xi) = 8Ac_{66} \sum_{j=1}^3 [\beta_{j1}f_{41}(0) + \beta_{j2}f_{42}(0)], \quad (39)$$

$$\tau_z^G(r, \theta, 0; r_\xi, \theta_\xi) = 0, \quad (40)$$

$$D^G(r, \theta, 0; r_\xi, \theta_\xi) = 0, \quad (41)$$

where

$$f_{3\xi}(0) = -\frac{\sqrt{a^2 - r_\xi^2}}{\sqrt{r^2 - a^2} R_{0\xi}^2}, \quad (42)$$

$$f_{4\xi}(0) = \frac{\sqrt{a^2 - r_\xi^2}}{\tilde{t}_\xi \tilde{s}_\xi} \left(\frac{2}{\tilde{t}_\xi} - \frac{r_\xi e^{i\theta_\xi}}{\tilde{s}_\xi^2} \right) \tan^{-1} \left(\frac{\tilde{s}_\xi}{\sqrt{r^2 - a^2}} \right) + \frac{\sqrt{r^2 - a^2} \sqrt{a^2 - r_\xi^2} r_\xi e^{i\theta_\xi}}{\tilde{t}_\xi \tilde{s}_\xi^2 (r^2 - r r_\xi e^{-i(\theta - \theta_\xi)})} + \frac{e^{2i\theta} \sqrt{a^2 - r_\xi^2}}{R_{0\xi}^2 \sqrt{r^2 - a^2}}, \quad (43)$$

$$f_{5\xi}(0) = 0, \quad (\xi = 1, 2), \quad (44)$$

with the parameter $R_{0\xi}$ defined by,

$$R_{0\xi} = R_\xi(z=0) = \sqrt{r^2 + r_\xi^2 - 2rr_\xi \cos(\theta - \theta_\xi)}. \quad (45)$$

By introducing the variable transformation $\hat{r} = r - a$ together with the proper limiting process, for $(r_\xi, \theta_\xi) \neq (a, \theta)$, it can readily be verified that,

$$\begin{aligned} \lim_{\hat{r} \rightarrow 0} \left\{ \sqrt{\hat{r}} \frac{\partial}{\partial \hat{r}} \left[\sqrt{\hat{r}} \left(\frac{\sqrt{a^2 - r_\xi^2}}{\sqrt{r^2 - a^2}} \right) \right] \right\} &= 0, \\ \lim_{\hat{r} \rightarrow 0} \left\{ \sqrt{\hat{r}} \frac{\partial}{\partial \hat{r}} \left[\sqrt{\hat{r}} \left(\frac{\sqrt{r^2 - a^2} \sqrt{a^2 - r_\xi^2}}{\bar{l}_\xi \bar{s}_\xi^2 (r^2 - rr_\xi e^{-i(\theta - \theta_\xi)})} \right) \right] \right\} &= 0, \\ \lim_{\hat{r} \rightarrow 0} \left\{ \sqrt{\hat{r}} \frac{\partial}{\partial \hat{r}} \left[\sqrt{\hat{r}} \left(\frac{\sqrt{a^2 - r_\xi^2}}{\bar{l}_\xi \bar{s}_\xi} \left(\frac{2}{\bar{l}_\xi} - \frac{r_\xi e^{i\theta_\xi}}{\bar{s}_\xi^2} \right) \tan^{-1} \left(\frac{\bar{s}_\xi}{\sqrt{r^2 - a^2}} \right) \right) \right] \right\} &= \pi \sqrt{a^2 - r_\xi^2} \left(\frac{1}{2\bar{l}_\xi^2 \bar{s}_{0\xi}} - \frac{r_\xi e^{i\theta_\xi}}{4\bar{l}_0 \bar{s}_0^3} \right), \end{aligned} \quad (46)$$

where $\bar{l}_{0\xi} = ae^{-i\theta} - r_\xi e^{-i\theta_\xi}$ and $\bar{s}_{0\xi} = \sqrt{a^2 - ar_\xi e^{-i(\theta - \theta_\xi)}}$. By employing Eq. (5) together with Eqs. (38)–(46), the generalized T-stress Green's function for the impermeable case, resulting from both pairs of self-equilibrated unit normal point forces P and unit point charges Q , are obtained as,

$$T_{11}^G(\theta; r_\xi, \theta_\xi) = -T_{33}^G(\theta; r_\xi, \theta_\xi) = 4\pi Ac_{66} \sum_{j=1}^3 \sum_{\xi=1}^2 \operatorname{Re}(\beta_{j\xi} \Phi), \quad T_{13}^G(\theta; r_\xi, \theta_\xi) = -4\pi Ac_{66} \sum_{j=1}^3 \sum_{\xi=1}^2 \operatorname{Im}(\beta_{j\xi} \Phi), \quad T_{14}^G = T_{34}^G = 0, \quad (47)$$

where $T_\tau^G(\theta; r_\xi, \theta_\xi) \equiv T_\tau^{GP}(\theta; r_1, \theta_1) + T_\tau^{GQ}(\theta; r_2, \theta_2)$, $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ stand for the real and imaginary parts of a complex function f , respectively, and the function Φ is defined by,

$$\Phi(\theta; r_\xi, \theta_\xi) = \left(\frac{\sqrt{a^2 - r_\xi^2}}{\sqrt{a} (ae^{-i\theta} - r_\xi e^{-i\theta_\xi}) \sqrt{a - r_\xi e^{-i(\theta - \theta_\xi)}}} \right) \times \left(\frac{1}{a - r_\xi e^{-i(\theta_\xi - \theta)}} - \frac{r_\xi e^{i(\theta_\xi - \theta)}}{2a(a - r_\xi e^{-i(\theta - \theta_\xi)})} \right). \quad (48)$$

By expressing $\beta_{j\xi} = A_{j\xi} + iB_{j\xi}$ with $A_{j\xi}$ and $B_{j\xi}$ denoting the real and imaginary parts of $\beta_{j\xi}$, the generalized T-stress Green's function T_τ^{GP} and T_τ^{GQ} can be obtained explicitly as,

$$\begin{aligned} T_{11}^{GP}(\theta; r_1, \theta_1) &= -T_{33}^{GP}(\theta; r_1, \theta_1) = -4\pi Ac_{66} \left(\Psi_1 \sum_{j=1}^3 A_{j1} - \Gamma_1 \sum_{j=1}^3 B_{j1} \right), \\ T_{13}^{GP}(\theta; r_1, \theta_1) &= 4\pi Ac_{66} \left(\Psi_1 \sum_{j=1}^3 B_{j1} + \Gamma_1 \sum_{j=1}^3 A_{j1} \right), \\ T_{14}^{GP}(\theta; r_1, \theta_1) &= T_{34}^{GP}(\theta; r_1, \theta_1) = 0, \end{aligned} \quad (49)$$

$$\begin{aligned} T_{11}^{GQ}(\theta; r_2, \theta_2) &= -T_{33}^{GQ}(\theta; r_2, \theta_2) = -4\pi Ac_{66} \left(\Psi_2 \sum_{j=1}^3 A_{j2} - \Gamma_2 \sum_{j=1}^3 B_{j2} \right), \\ T_{13}^{GQ}(\theta; r_2, \theta_2) &= 4\pi Ac_{66} \left(\Psi_2 \sum_{j=1}^3 B_{j2} + \Gamma_2 \sum_{j=1}^3 A_{j2} \right), \\ T_{14}^{GQ}(\theta; r_2, \theta_2) &= T_{34}^{GQ}(\theta; r_2, \theta_2) = 0, \end{aligned} \quad (50)$$

where Ψ_ξ and Γ_ξ ($\xi = 1, 2$) are defined by,

$$\Psi_\xi = \frac{1}{l_\xi^2} \sqrt{2\cos\phi_\xi - \frac{l_\xi}{a}} \left[\frac{1}{2} \left(1 - \frac{l_\xi}{a} \right) \cos \frac{\phi_\xi}{2} - \cos \frac{3\phi_\xi}{2} \right], \quad (51)$$

$$\Gamma_\xi = \frac{1}{l_\xi^2} \sqrt{2\cos\phi_\xi - \frac{l_\xi}{a}} \left[\frac{1}{2} \left(1 + \frac{l_\xi}{a} \right) \sin \frac{\phi_\xi}{2} + \sin \frac{3\phi_\xi}{2} \right], \quad (52)$$

with l_1, l_2, ϕ_1, ϕ_2 being related to $r_1, r_2, \theta_1, \theta_2$ via the following variable transformations (see Fig. 5 for the graphical interpretation),

$$\begin{aligned} r_1 \sin(\theta_1 - \theta) &= l_1 \sin\phi_1, \quad r_1 \cos(\theta_1 - \theta) + l_1 \cos\phi_1 = a, \\ r_2 \sin(\theta_2 - \theta) &= l_2 \sin\phi_2, \quad r_2 \cos(\theta_2 - \theta) + l_2 \cos\phi_2 = a. \end{aligned} \quad (53)$$

It is apparent from Eqs. (49) and (50) that the mechanical T-stress Green's functions (i.e., T_τ^{GP} and T_τ^{GQ} with $\tau \in \{11, 13, 33\}$) for the electrically impermeable crack depend on the material properties and are singular only at $l_\xi = 0$ of $\mathcal{O}(1/l_\xi^2)$, whereas the electrical T-

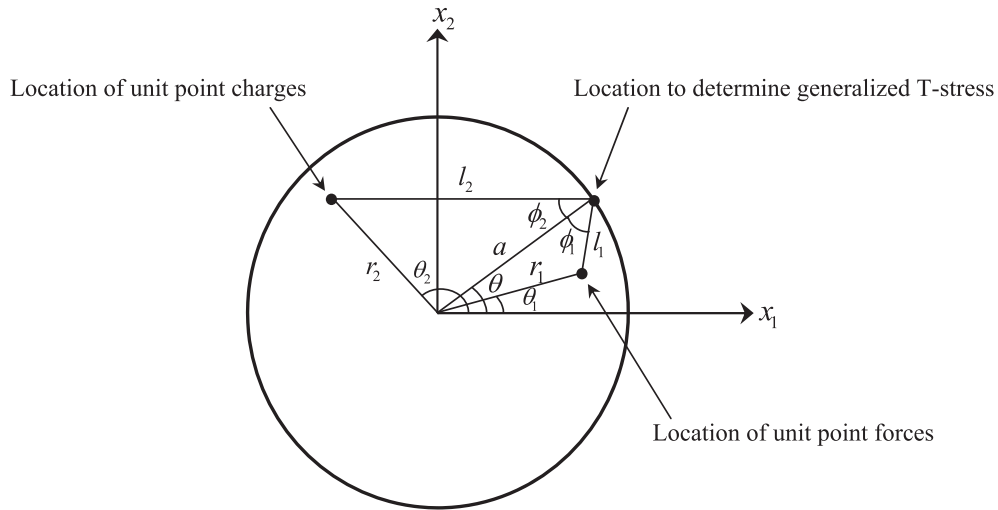


Fig. 5. Graphical interpretation of variables r_1 , r_2 , l_1 , l_2 , θ_1 , θ_2 , ϕ_1 , and ϕ_2 .

stress Green's functions (i.e., T_τ^{GP} and T_τ^{GQ} with $\tau \in \{14, 34\}$) vanish identically.

4.3.2. Electrically permeable crack

Consider, next, the fundamental problem of a permeable penny-shaped crack subjected to a pair of self-equilibrated, unit normal point forces P at $(r_1, \theta_1, 0^\pm)$ (Fig. 4(a)) similar to that reported by Chen and Lim [45]. The closed-form solution of the generalized stress Green's function, referring to the global cylindrical coordinate system $\{O; r, \theta, z\}$, is given by

$$\sigma_1^G(r, \theta, z; r_1, \theta_1) = \frac{1}{g_1 \pi^2} \sum_{j=1}^3 \gamma_{j3} c_j f_{31}(z_j), \quad (54)$$

$$\sigma_2^G(r, \theta, z; r_1, \theta_1) = \frac{2c_{66}}{g_1 \pi^2} \sum_{j=1}^3 c_j f_{41}(z_j), \quad (55)$$

$$\tau_z^G(r, \theta, z; r_1, \theta_1) = \frac{1}{g_1 \pi^2} \sum_{j=1}^3 \gamma_{j1} s_j c_j f_{51}(z_j), \quad (56)$$

$$D^G(r, \theta, z; r_1, \theta_1) = \frac{1}{g_1 \pi^2} \sum_{j=1}^3 \gamma_{j2} s_j c_j f_{51}(z_j), \quad (57)$$

where the constants c_j are defined by Eq. (29), the constant g_1 is defined by Eq. (18), the constants γ_{j1} and γ_{j2} are defined by Eq. (20), $\gamma_{j3} = 2(c_{66} - c_{11}) + 2c_{13}s_j\alpha_{j1} + 2e_{31}s_j\alpha_{j2}$, and all other related constants are defined in the same way as those indicated in Section 4.3.1. Using Eq. (37) along with Eqs. (34)–(36) and Eqs. (54)–(57), the generalized stress Green's function at any point on the plane $z = 0$ is given by,

$$\sigma_1^G(r, \theta, 0; r_1, \theta_1) = \frac{1}{g_1 \pi^2} \sum_{j=1}^3 \gamma_{j3} c_j f_{31}(0), \quad (58)$$

$$\sigma_2^G(r, \theta, 0; r_1, \theta_1) = \frac{2c_{66}}{g_1 \pi^2} \sum_{j=1}^3 c_j f_{41}(0), \quad (59)$$

$$\tau_z^G(r, \theta, 0; r_1, \theta_1) = 0, \quad (60)$$

$$D^G(r, \theta, 0; r_1, \theta_1) = 0, \quad (61)$$

where $f_{31}(0)$ and $f_{41}(0)$ are defined by Eqs. (42) and (43) with $\xi = 1$, respectively. Using Eq. (5) together with Eq. (46) and Eqs. (58)–(61) finally yields the explicit expression for the generalized T-stress Green's function T_τ^{GP} for the electrically permeable case,

$$\begin{aligned}
T_{11}^{GP}(\theta; r_1, \theta_1) &= -T_{33}^{GP}(\theta; r_1, \theta_1) = \frac{c_{66}}{g_1 \pi} \sum_{j=1}^3 \operatorname{Re}(c_j \Phi), \\
T_{13}^{GP}(\theta; r_1, \theta_1) &= -\frac{c_{66}}{g_1 \pi} \sum_{j=1}^3 \operatorname{Im}(c_j \Phi), \\
T_{14}^{GP}(\theta; r_1, \theta_1) &= T_{34}^{GP}(\theta; r_1, \theta_1) = 0,
\end{aligned} \tag{62}$$

where the function Φ is defined in the same fashion as for the impermeable case, i.e., by Eq. (48). Eq. (62) can be reduced further by using the variable transformations given in Eq. (53) and the representation $c_j = A_{j1} + iB_{j1}$ where A_{j1} and B_{j1} are the real and imaginary parts of the complex constant c_j . It is crucial to emphasize that the constants A_{j1}, B_{j1} defined here are different from the electrically impermeable case. The final results for T_τ^{GP} are given by,

$$\begin{aligned}
T_{11}^{GP}(\theta; r_1, \theta_1) &= -T_{33}^{GP}(\theta; r_1, \theta_1) = -\frac{c_{66}}{g_1 \pi} \left(\Psi_1 \sum_{j=1}^3 A_{j1} - \Gamma_1 \sum_{j=1}^3 B_{j1} \right), \\
T_{13}^{GP}(\theta; r_1, \theta_1) &= \frac{c_{66}}{g_1 \pi} \left(\Psi_1 \sum_{j=1}^3 B_{j1} + \Gamma_1 \sum_{j=1}^3 A_{j1} \right), \\
T_{14}^{GP}(\theta; r_1, \theta_1) &= T_{34}^{GP}(\theta; r_1, \theta_1) = 0,
\end{aligned} \tag{63}$$

where Ψ_1 and Γ_1 are defined by Eqs. (51) and (52) with $\xi = 1$, respectively. From Eq. (63), it is apparent that the mechanical T-stress Green's function for the electrically permeable case depends on the material properties and are singular only at $l_1 = 0$ of $\mathcal{O}(1/l_1^2)$ whereas the electrical T-stress Green's function vanishes identically.

5. Generalized T-stress for particular loading cases

Solutions for the general loading case developed in Section 4 are employed as the basis to derive closed-form expressions for the generalized T-stress components for electrically impermeable and electrically permeable penny-shaped cracks, subjected to some particular loading conditions. Some of those results are further reduced to the elastic case and then compared with available benchmark solutions; as a result, this serves as a means for verifying not only the closed-form solutions presented in this section but also the integral formula and Green's functions used to generate such results. In the following sub-sections $\bar{\Upsilon}^M = \Upsilon_{ic}^M, \Upsilon_{pc}^M$ for the electrically impermeable crack and electrically permeable crack, respectively. For the special case of transversely isotropic elastic media, by setting the piezoelectric constants $e_{15} = e_{31} = e_{33} = 0$, the constant $\bar{\Upsilon}^M$ is obtained as,

$$\bar{\Upsilon}^M = \frac{c_{66}}{c_{13} + \sqrt{c_{11}c_{33}}} - \frac{c_{11}}{\sqrt{c_{11}c_{33}}}. \tag{64}$$

5.1. Uniform far-field loading

For either an electrically impermeable or electrically permeable penny-shaped crack, subjected only to uniform far-field triaxial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$ and uniform far-field electrical induction σ_{i4}^∞ , both T_τ^0 and T_τ^1 identically vanish. The generalized T-stress components for this case are given by,

$$T_{11} = \bar{\sigma}_{11}^\infty + \bar{\Upsilon}^M \sigma_{33}^\infty + \Lambda \Upsilon^E \sigma_{34}^\infty, \quad T_{13} = \bar{\sigma}_{13}^\infty, \quad T_{33} = \bar{\sigma}_{33}^\infty + \bar{\Upsilon}^M \sigma_{33}^\infty + \Lambda \Upsilon^E \sigma_{34}^\infty, \quad T_{14} = \bar{\sigma}_{14}^\infty, \quad T_{34} = \bar{\sigma}_{34}^\infty, \tag{65}$$

Eq. (65), when specialized to a penny-shaped crack embedded in elastic media subjected to uniform far-field triaxial stresses $\{\sigma_{11}^\infty, \sigma_{22}^\infty, \sigma_{33}^\infty\}$, agrees with the results reported by [28] for transversely isotropic elastic media and [46] for isotropic elastic media.

5.2. Uniform crack-face loading

For either an electrically impermeable penny-shaped crack subjected only to self-equilibrated, uniform crack-face normal traction $t_3^0(r_1, \theta_1) = p_0$ and self-equilibrated, uniform surface electric charge $t_4^0(r_2, \theta_2) = q_0$ or an electrically permeable penny-shaped crack subjected only to self-equilibrated, uniform crack-face normal traction $t_3^0(r_1, \theta_1) = p_0$, both T_τ^∞ and T_τ^1 identically vanish. The generalized T-stress components are thus given by,

$$T_{11}(\theta) = T_{33}(\theta) = \bar{\Upsilon}^M p_0 + \Lambda \Upsilon^E q_0, \quad T_{13} = 0, \quad T_{14} = 0, \quad T_{34} = 0. \tag{66}$$

The closed-form expressions in Eq. (66) for this loading condition, when specialized to the elastic case agree with the expressions proposed by [28] and [46].

5.3. Linear crack-face loading proportional to x_1

Consider, next, the following two cases: (i) an electrically impermeable penny-shaped crack subjected to a linear, self-equilibrated, crack-face normal traction proportional to x_1 , $t_3^0(r_1, \theta_1) = (r_1 \cos \theta_1) p_0 / a$ and a linear, self-equilibrated, surface electric charge

proportional to x_1 , $t_4^0(r_2, \theta_2) = (r_2 \cos \theta_2) q_0 / a$ and (ii) an electrically permeable penny-shaped crack subjected to a linear, self-equilibrated, crack-face normal traction proportional to x_1 , $t_3^0(r_1, \theta_1) = (r_1 \cos \theta_1) p_0 / a$ where p_0 and q_0 are constants. It is evident that for these two cases, T_r^∞ vanishes and T_r^0 can be readily obtained from Eqs. (25) and (27) with $t_3^0(a, \theta) = p_0 \cos \theta$ and $t_4^0(a, \theta) = q_0 \cos \theta$,

$$T_{11}^0(\theta) = T_{33}^0(\theta) = \bar{\gamma}^M p_0 \cos \theta + \Lambda \gamma^E q_0 \cos \theta, \quad T_{13}^0 = 0, \quad T_{14}^0 = 0, \quad T_{34}^0 = 0. \quad (67)$$

Performing the direct integration of Eq. (14) with $P(r_1, \theta_1) = p_0 [(r_1/a) \cos \theta_1 - \cos \theta]$, $Q(r_2, \theta_2) = q_0 [(r_2/a) \cos \theta_2 - \cos \theta]$ and the Green's functions T_τ^{GP} and T_τ^{GQ} obtained in Section 4.3, finally yields the explicit results for T_τ^1 as,

$$\begin{aligned} T_{11}^1(\theta) &= -T_{33}^1(\theta) = \mathcal{C} \left[\sum_{j=1}^3 A_{j1} \cos \theta + \sum_{j=1}^3 B_{j1} \sin \theta \right] p_0 + \mathcal{D} \left[\sum_{j=1}^3 A_{j2} \cos \theta + \sum_{j=1}^3 B_{j2} \sin \theta \right] q_0, \\ T_{13}^1(\theta) &= \mathcal{C} \left[\sum_{j=1}^3 A_{j1} \sin \theta - \sum_{j=1}^3 B_{j1} \cos \theta \right] p_0 + \mathcal{D} \left[\sum_{j=1}^3 A_{j2} \sin \theta - \sum_{j=1}^3 B_{j2} \cos \theta \right] q_0, \\ T_{14}^1(\theta) &= T_{34}^1(\theta) = 0, \end{aligned} \quad (68)$$

where $\mathcal{C} = \mathcal{D} = -2\pi^2 \mathcal{A} c_{66}$ for the impermeable case and $\mathcal{C} = -c_{66}/2g_1$, $\mathcal{D} = 0$ for the permeable case. The generalized T-stress components for this particular loading condition finally take the form,

$$\begin{aligned} T_{11}(\theta) &= \left[(\bar{\gamma}^M + \mathcal{C} \sum_{j=1}^3 A_{j1}) \cos \theta + (\mathcal{C} \sum_{j=1}^3 B_{j1}) \sin \theta \right] p_0 + \left[(\Lambda \gamma^E + \mathcal{D} \sum_{j=1}^3 A_{j2}) \cos \theta + (\mathcal{D} \sum_{j=1}^3 B_{j2}) \sin \theta \right] q_0, \\ T_{33}(\theta) &= \left[(\bar{\gamma}^M - \mathcal{C} \sum_{j=1}^3 A_{j1}) \cos \theta - (\mathcal{C} \sum_{j=1}^3 B_{j1}) \sin \theta \right] p_0 + \left[(\Lambda \gamma^E - \mathcal{D} \sum_{j=1}^3 A_{j2}) \cos \theta - (\mathcal{D} \sum_{j=1}^3 B_{j2}) \sin \theta \right] q_0, \\ T_{13}(\theta) &= \left[(\mathcal{C} \sum_{j=1}^3 A_{j1}) \sin \theta - (\mathcal{C} \sum_{j=1}^3 B_{j1}) \cos \theta \right] p_0 + \left[(\mathcal{D} \sum_{j=1}^3 A_{j2}) \sin \theta - (\mathcal{D} \sum_{j=1}^3 B_{j2}) \cos \theta \right] q_0, \\ T_{14}(\theta) &= T_{34}(\theta) = 0. \end{aligned} \quad (69)$$

Here, when the piezoelectric constants are set to zero,

$$2\mathcal{C} \sum_{j=1}^3 A_{j1} = -\frac{c_{66}}{c_{13} + \sqrt{c_{11}c_{33}}}, \quad \sum_{j=1}^3 B_{j1} = 0. \quad (70)$$

Eq. (69) when specialized to a penny-shaped crack in an elastic medium and ignoring the electrical loading agree with the results given by [28] and [46].

5.4. Axisymmetric crack-face loading

Finally, consider (i) an electrically impermeable penny-shaped crack subjected to axisymmetric, self-equilibrated, crack-face normal traction $t_3^0(r_1, \theta_1) = f(r_1)$ and axisymmetric, self-equilibrated, surface electric charge $t_4^0(r_2, \theta_2) = g(r_2)$, and (ii) an electrically permeable penny-shaped crack subjected to axisymmetric, self-equilibrated, crack-face normal traction $t_3^0(r_1, \theta_1) = f(r_1)$ where f and g are any prescribed functions. For these two cases, T_r^∞ identically vanishes, whereas T_r^0 and T_r^1 can be given by,

$$T_{11}^0(\theta) = T_{33}^0(\theta) = \bar{\gamma}^M f(a) + \Lambda \gamma^E g(a), \quad T_{13}^0(\theta) = T_{14}^0(\theta) = T_{34}^0(\theta) = 0, \quad (71)$$

$$T_\tau^1(\theta) = \int_{-\pi}^{\pi} \int_0^a T_\tau^{GP}(\theta; r_1, \theta_1) [f(r_1) - f(a)] r_1 dr_1 d\theta_1 + \Lambda \int_{-\pi}^{\pi} \int_0^a T_\tau^{GQ}(\theta; r_2, \theta_2) [g(r_2) - g(a)] r_2 dr_2 d\theta_2. \quad (72)$$

By employing the variable transformations given in Eq. (53) to obtain the relations $r_1(l_1, \phi_1) = \sqrt{l_1^2 - 2l_1 a \cos \phi_1 + a^2}$ and $r_2(l_2, \phi_2) = \sqrt{l_2^2 - 2l_2 a \cos \phi_2 + a^2}$, together with the generalized T-stress Green's functions T_τ^{GP} and T_τ^{GQ} obtained in Section 4.3, the integrals appearing in Eq. (72) can be integrated analytically and $T_\tau^1(\theta)$ is given by,

$$T_{11}^1 = -T_{33}^1 = -\frac{2}{\pi a^2} (R_p - R_p^a) \mathcal{C} \sum_{j=1}^3 A_{j1} - \frac{2}{\pi a^2} (R_Q - R_Q^a) \mathcal{D} \sum_{j=1}^3 A_{j2}, \quad T_{13}^1 = T_{14}^1 = T_{34}^1 = 0, \quad (73)$$

where $R_p = 2\pi \int_0^a f(r_1) r_1 dr_1$ denotes the resultant of the applied normal traction $t_3^0(r_1, \theta_1) = f(r_1)$; $R_p^a = \pi a^2 f(a)$; $R_Q = 2\pi \int_0^a g(r_2) r_2 dr_2$ denotes the resultant of the applied surface electric charge $t_4^0(r_2, \theta_2) = g(r_2)$; and $R_Q^a = \pi a^2 g(a)$. Eqs. (71) and (73) yield the following closed-form expressions for the generalized T-stress components, for both impermeable and permeable penny-shaped cracks under axisymmetric crack-face loading,

$$\begin{aligned} T_{11} &= \frac{1}{\pi a^2} \left[\bar{\gamma}^M R_p^a - 2(R_p - R_p^a) \mathcal{C} \sum_{j=1}^3 A_{j1} \right] + \frac{1}{\pi a^2} \left[\Lambda \gamma^E R_Q^a - 2(R_Q - R_Q^a) \mathcal{D} \sum_{j=1}^3 A_{j2} \right], \\ T_{33} &= \frac{1}{\pi a^2} \left[\bar{\gamma}^M R_p^a + 2(R_p - R_p^a) \mathcal{C} \sum_{j=1}^3 A_{j1} \right] + \frac{1}{\pi a^2} \left[\Lambda \gamma^E R_Q^a + 2(R_Q - R_Q^a) \mathcal{D} \sum_{j=1}^3 A_{j2} \right], \\ T_{13} &= T_{14} = T_{34} = 0. \end{aligned} \quad (74)$$

For the elastic case, it can be readily verified by using Eqs. (64) and (70) that the closed-form expressions given by Eq. (74) simply reduce to that obtained by [28] for a transversely isotropic linearly elastic medium.

6. Numerical quadrature for T_τ^1

By applying the variable transformations given in Eq. (53), the integral formula (14) for T_τ^1 , $\tau \in \{11, 33, 13, 14, 34\}$ can be transformed into,

$$T_\tau^1(\theta) = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\phi_1} [l_1 \hat{T}_\tau^{GP}(\theta; l_1, \phi_1)] \hat{P}(l_1, \phi_1) dl_1 d\phi_1 + \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\phi_2} [l_2 \hat{T}_\tau^{GQ}(\theta; l_2, \phi_2)] \hat{Q}(l_2, \phi_2) dl_2 d\phi_2, \quad (75)$$

where $\hat{T}_\tau^{GP}(\theta; l_1, \phi_1) = T_\tau^{GP}(\theta; r_1(l_1, \phi_1), \theta_1(l_1, \phi_1))$, $\hat{T}_\tau^{GQ}(\theta; l_2, \phi_2) = T_\tau^{GQ}(\theta; r_2(l_2, \phi_2), \theta_2(l_2, \phi_2))$, $\hat{P}(l_1, \phi_1) = P(r_1(l_1, \phi_1), \theta_1(l_1, \phi_1))$ and $\hat{Q}(l_2, \phi_2) = Q(r_2(l_2, \phi_2), \theta_2(l_2, \phi_2))$. Since the generalized T-stress Green's functions $\hat{T}_\tau^{GP}$ and $\hat{T}_\tau^{GQ}$ have singularities of $\mathcal{O}(1/l_1^2)$ at $l_1 = 0$ and of $\mathcal{O}(1/l_2^2)$ at $l_2 = 0$, respectively, the functions $l_1 \hat{T}_\tau^{GP}$ and $l_2 \hat{T}_\tau^{GQ}$ appearing in Eq. (75) have singularities of $\mathcal{O}(1/l_1)$ at $l_1 = 0$ and of $\mathcal{O}(1/l_2)$ at $l_2 = 0$, respectively. If the prescribed crack-face loads $t_3^0(r_1, \theta_1)$ and $t_4^0(r_2, \theta_2)$ are sufficiently smooth at the point $\mathbf{x}_c \equiv (a, \theta, 0)$ such that $\hat{P}(l_1, \phi_1) = \mathcal{O}(l_1^{\lambda_1})$ and $\hat{Q}(l_2, \phi_2) = \mathcal{O}(l_2^{\lambda_2})$ with $\lambda_1, \lambda_2 > 0$, the integrands of the first and second integrals in Eq. (75) are obviously of $\mathcal{O}(l_1^{\lambda_1-1})$ and $\mathcal{O}(l_2^{\lambda_2-1})$, respectively. Hence, the existence of both integrals in the sense of Riemann is confirmed. For $\lambda_1, \lambda_2 \geq 1$, the integrands are not singular and both integrals can be integrated efficiently by using standard Gaussian quadrature. For $0 < \lambda_1, \lambda_2 < 1$, the integrands are weakly singular and the following regularization procedure is employed to remove such weak singularities. By introducing the variable transformations $l_1 = l_1(\bar{l}_1) = \bar{l}_1^{\gamma_1}$ and $l_2 = l_2(\bar{l}_2) = \bar{l}_2^{\gamma_2}$ where γ_1 and γ_2 denote pre-selected constants, Eq. (75) can be written as,

$$T_\tau^1(\theta) = \gamma_1 \int_{-\pi/2}^{\pi/2} \int_0^{(2a\cos\phi_1)^{1/\gamma_1}} [l_1(\bar{l}_1) \hat{T}_\tau^{GP}(\theta; l_1(\bar{l}_1), \phi_1)] \hat{P}(l_1(\bar{l}_1), \phi_1) [l_1(\bar{l}_1)]^{(\gamma_1-1)/\gamma_1} d\bar{l}_1 d\phi_1 \\ + \gamma_2 \int_{-\pi/2}^{\pi/2} \int_0^{(2a\cos\phi_2)^{1/\gamma_2}} [l_2(\bar{l}_2) \hat{T}_\tau^{GQ}(\theta; l_2(\bar{l}_2), \phi_2)] \hat{Q}(l_2(\bar{l}_2), \phi_2) [l_2(\bar{l}_2)]^{(\gamma_2-1)/\gamma_2} d\bar{l}_2 d\phi_2. \quad (76)$$

Setting $\gamma_1 = 1/\lambda_1$ and $\gamma_2 = 1/\lambda_2$ in Eq. (76), results in,

$$T_\tau^1(\theta) = \frac{1}{\lambda_1} \int_{-\pi/2}^{\pi/2} \int_0^{(2a\cos\phi_1)^{\lambda_1}} [l_1(\bar{l}_1) \hat{T}_\tau^{GP}(\theta; l_1(\bar{l}_1), \phi_1)] \hat{P}(l_1(\bar{l}_1), \phi_1) [l_1(\bar{l}_1)]^{(1-\lambda_1)} d\bar{l}_1 d\phi_1 \\ + \frac{1}{\lambda_2} \int_{-\pi/2}^{\pi/2} \int_0^{(2a\cos\phi_2)^{\lambda_2}} [l_2(\bar{l}_2) \hat{T}_\tau^{GQ}(\theta; l_2(\bar{l}_2), \phi_2)] \hat{Q}(l_2(\bar{l}_2), \phi_2) [l_2(\bar{l}_2)]^{(1-\lambda_2)} d\bar{l}_2 d\phi_2. \quad (77)$$

It is apparent that the integrals appearing in Eq. (77) can now be efficiently integrated by standard Gaussian quadrature, since the weak singularities at $l_1 = 0$ and $l_2 = 0$ are completely removed by the specific choices of γ_1 and γ_2 . To validate the proposed numerical quadrature, Eq. (77) is employed to compute the generalized T-stress components of a penny-shaped crack under linear crack-face loading and axisymmetric crack-face loading and the results are then compared with the analytical solutions obtained in Section 5. The cracked medium is a transversely isotropic, piezoelectric material PZT-5H with the material properties given in Table 1 ([15]).

First, consider a penny-shaped crack subjected to a linearly distributed, self-equilibrated, crack-face normal traction defined by $t_3^0(r_1, \theta_1) = [1 + (r_1 \cos \theta_1)/a] \sigma_0/2$ where σ_0 is a given constant representing the maximum value of the traction along the crack front. The generalized T-stress components are computed numerically using Eq. (7) together with Eqs. (25), (27) and (77) and then plotted along with the analytical solution derived in Sections 5.2 and 5.3, in Figs. 6 and 7. As evident from this set of results, the numerical solutions obtained from the proposed quadrature are in excellent agreement with the benchmark solutions for both electrically impermeable and electrically permeable cracks. It can also be pointed out that for this particular loading condition, the generalized T-stress components exhibit only weak dependence on the crack-face condition.

Next, consider a penny-shaped crack of radius a subjected to axisymmetric, self-equilibrated, crack-face normal traction $t_3^0(r, \theta) = (r/a)^n \sigma_0$, where σ_0 is the maximum value of the traction along the crack front and n is an exponent indicating the distribution of traction across the crack surface (Fig. 8). Numerical results for the generalized T-stress components are obtained from Eq. (7) together with Eqs. (25), (27) and (77). Due to the axisymmetric nature of the problem, only the generalized T-stress components T_{11} and T_{33} are non-zero and independent of the position along the crack front for both impermeable and permeable cases. The computed generalized T-stress components T_{11} and T_{33} , normalized either by σ_0 or $T_0/\pi a^2$ are plotted as a function of the exponent n in Fig. 9 where $T_0 = 2\pi a^2 \sigma_0/(n+2)$ is the resultant of the applied traction. It is seen that computed numerical results from the proposed integral formula (7) are highly accurate and in excellent agreement with the exact solution given by Eq. (74). As indicated by results in Fig. 9(a) and (b), for both crack-face conditions the normalized generalized T-stress component T_{11} increases monotonically in

Table 1
Material properties of transversely isotropic, piezoelectric solid PZT-5H [15].

Elastic constants ($\times 10^9$ Pa)	c_{11} 126.0	c_{12} 55.0	c_{13} 53.0	c_{33} 117.0	c_{44} 35.3
Piezoelectric constants (C/m ²)	e_{15} 17.0	e_{31} −6.5	e_{33} 23.3		
Dielectric permittivities ($\times 10^{-9}$ C/(Vm))	ϵ_{11} 15.1	ϵ_{33} 13.0			

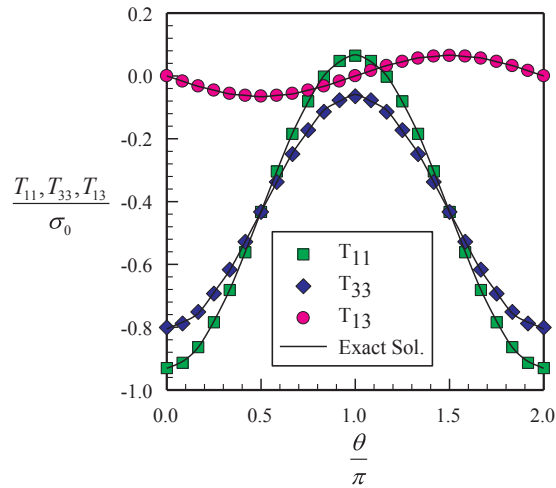


Fig. 6. Generalized T-stress components for an electrically impermeable penny-shaped crack of radius a embedded in a transversely isotropic, piezoelectric material PZT-5H under linearly distributed, self-equilibrated, crack-face, normal traction $t_3^0(r_1, \theta_1) = [1 + (r_1 \cos \theta_1)/a] \sigma_0/2$. The exact solution is taken from results in Sections 5.2 and 5.3.

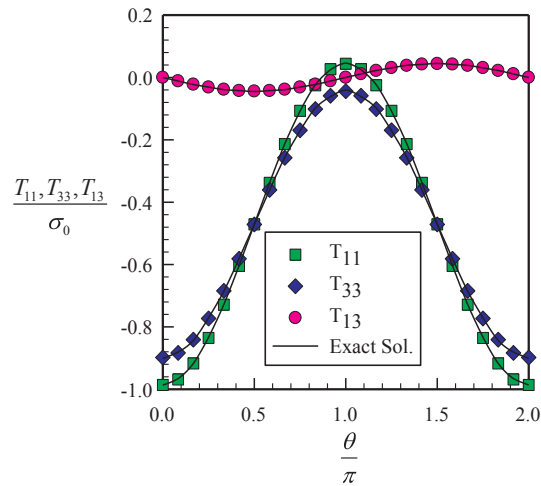


Fig. 7. Generalized T-stress components for an electrically permeable penny-shaped crack of radius a embedded in a transversely isotropic, piezoelectric material PZT-5H under linearly distributed, self-equilibrated, crack-face, normal traction $t_3^0(r_1, \theta_1) = [1 + (r_1 \cos \theta_1)/a] \sigma_0/2$. The exact solution is taken from results in Sections 5.2 and 5.3.

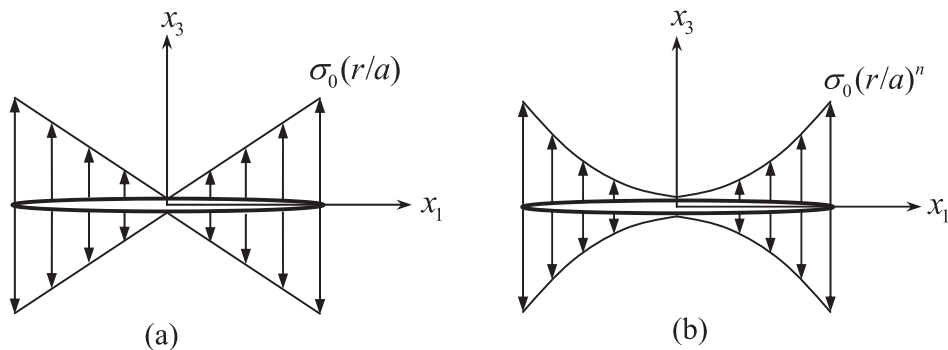


Fig. 8. Penny-shaped crack of radius a under axisymmetric, self-equilibrated, crack-face, normal traction $(r/a)^n \sigma_0$: (a) $n = 1$ and (b) $n > 1$.

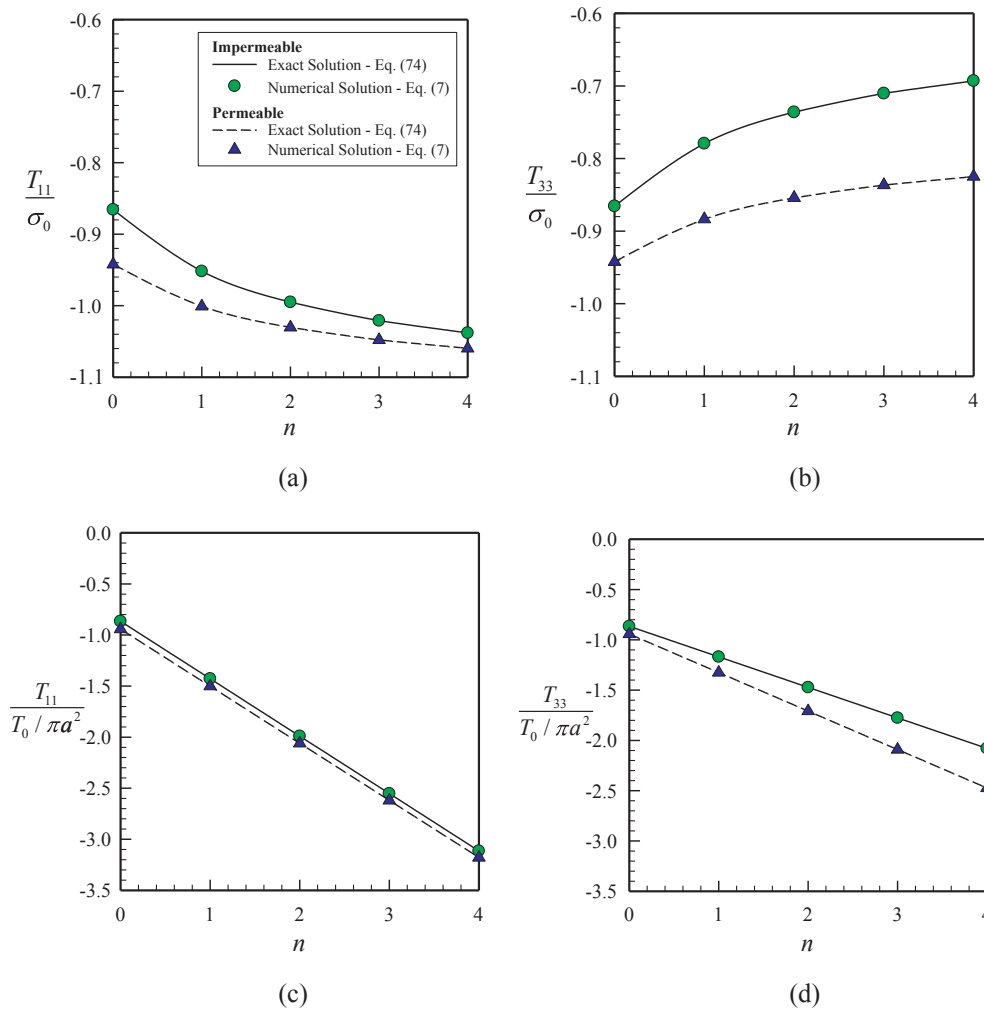


Fig. 9. Generalized T-stress components for a penny-shaped crack of radius a embedded in a transversely isotropic, piezoelectric material PZT-5H under axisymmetric, self-equilibrated, crack-face, normal traction $t_3^0 = (r/a)^n \sigma_0$: (a) T_{11}/σ_0 , (b) T_{33}/σ_0 , (c) $T_{11}/(T_0/\pi a^2)$ and (d) $T_{33}/(T_0/\pi a^2)$, where σ_0 is the maximum value of the traction along the crack front and $T_0 = 2\pi a^2 \sigma_0/(n+2)$ is the resultant of traction.

magnitude as the exponent n increases, whereas the reverse effect is observed for the component T_{33} . Note also that both T_{11} and T_{33} are negative and for a fixed value of the exponent n , the magnitude of T_{11} and T_{33} for the electrically permeable model is greater than those generated by the electrically impermeable model. By changing the means of normalization from σ_0 to $T_0/\pi a^2$, it is evident from Fig. 9(c) and (d) that the normalized generalized T-stresses T_{11} and T_{33} depend linearly on the exponent n .

7. Conclusions and remarks

Analytical and semi-analytical solutions for the generalized T-stress have been obtained for electrically impermeable and electrically permeable penny-shaped cracks embedded in a transversely isotropic, linear piezoelectric, infinite medium under uniform far-field loading, and general mechanical and electrical crack-face loading. The generalized T-stresses for two fundamental loading cases, one associated with uniformly distributed, self-equilibrated, crack-face loading (Section 4.2) and the other corresponding to a pair of crack-face, self-equilibrated, unit normal point forces and unit point electrical charges (Section 4.3), have been successfully derived in closed-form involving only elementary functions and such results have been directly utilized to establish the simple integral formula for cracks under general crack-face loading. In the derivation of the fundamental generalized T-stresses, the near-front asymptotic expansion of the generalized stress together with standard differentiation and limiting process have been employed. The proposed formula for uniform far-field loading and general mechanical and electrical crack-face loading has been directly integrated to obtain the closed-form solutions for the generalized T-stress components for some particular loading cases such as the uniform far-field loading, uniform crack-face loading, linear crack-face loading proportional to x_1 , and axisymmetric crack-face loading. This set of solutions is explicit and involves only elementary functions and as a result, is suitable for use as reliable benchmark solutions in the verification procedure of newly developed, boundary element and finite element numerical schemes, for

analysis of complex crack problems. The validity of the obtained solutions has been confirmed by comparing with available solutions reported in the literature. In addition, it is noted that the generalized T-stress components T_{14} and T_{34} (representing the finite part components of the electric induction vector along the crack front) identically vanish for the penny-shaped cracks treated in the present investigation because of the relatively basic settings of the problem considered, including the crack geometry, the orientation relative to the material symmetry, and the crack-face conditions. However, these two terms will not necessarily be zero for a more general crack geometry and loading conditions.

The integral formula that is obtained for computing the generalized T-stress components of a penny-shaped crack under general loading conditions is explicit and free of discretization. After integrand regularization, all integrals in the formula are regular and can be integrated efficiently by using standard Gaussian quadrature. The proposed algorithm has been fully tested and the results obtained have confirmed its accuracy and robustness. In addition, results from the parametric study have shown that both the crack-face boundary conditions and crack-face loading play an important role on the value and distribution of the generalized T-stress along the crack front. To gain an in-depth understanding of the influence of various parameters on the generalized T-stress, extensive parametric studies can be carried out and solutions and expressions obtained from the present study should provide a useful basis for such investigations.

As a final remark, it is noted that the theoretical modeling and simulation of fracture in piezoelectric solids are quite challenging tasks and generally require sophisticated governing physics, to be capable of sufficiently capturing the phenomena to the level of complexity involved. Attempts to properly treat the complex nature of the strong electro-mechanical coupling, the conditions on the crack surface, and the fracture process induced within a local region surrounding the crack, has been of central interest to many researchers during the past three decades. Various models and fracture criteria have been continuously proposed and implemented, to simulate the complex fracture response in piezoelectric media (e.g., [47–53]), where the capabilities, limitations, pros and cons of the developed models have been fully discussed, and potential extensions to further enhance the modeling technology have also been recommended. Although the results of the present study are based mainly upon the simplified theory of linear piezoelectricity and certain geometric configurations and loading conditions; these results can be considered as a first or preliminary approximation but would contribute to the fundamental understanding of the actual complex process of fracture in piezoelectric solids.

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Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at <http://dx.doi.org/10.1016/j.engfracmech.2018.01.020>.

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Weakly singular BIEM for analysis of cracks embedded in symmetric elastic whole space

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ABSTRACT

In this paper, a weakly singular boundary integral equation method is developed for the stress analysis of an anisotropic, linearly elastic, cracked whole space possessing a plane of symmetry. This study should offer an alternative powerful tool essential for the modeling of both near-surface and deeply embedded defects in a rock/soil medium. A system of governing equations is established using a pair of weakly singular, weak-form, displacement and traction integral equations for the cracked whole space along with the symmetric condition. The final equations contain only unknown crack-face data in a lower-half of the whole space. In addition to their capability to treat cracks of arbitrary shape, material anisotropy and general loading conditions, all involved kernels are only weakly singular allowing all integrals to be interpreted in the sense of Riemann sum. A symmetric Galerkin boundary element method together with the Galerkin approximation is implemented to solve the governing integral equations for the unknown crack-face data. To further enhance the accuracy and efficiency of the proposed scheme, special basis functions are introduced to approximate the near-front field and the interpolation technique is adopted to evaluate all kernels for generally anisotropic materials. The solved crack-face displacement data is then utilized to post-process for the essential fracture information along the crack front. Various scenarios are employed to verify the proposed technique and a selected set of results is presented to demonstrate its accuracy and computational robustness.

1. Introduction

Modeling and analysis of a medium containing pre-existing damages and flaws (e.g., dislocations and cracks) has been found an essential procedure in the fracture-based failure/fatigue assessment. There are

various situations encountered in practices when damages and flaws are located in a region relatively near the boundary or deeply inside the body and, in addition, their sizes are comparatively small in comparison with the characteristic length scale of the body; for instance, the modeling of near-surface/deep fractures in a large scale

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rock and soil medium and simulations of hydraulically induced cracks deeply beneath the ground used in the production enhancement of natural oil and gas (e.g., Mendelsohn, 1984a; Mendelsohn, 1984b; Yew, 1997). To simplify the modeling of those physical problems, a half space or whole space together with a selected set of governing physics (e.g., a theory of linear elasticity and linear elastic fracture mechanics) is commonly used to describe the response of such large scale cracked body. The influence of the remote boundary on the local response of interest is generally insignificant and can, therefore, be discarded in the modeling without loss of accuracy of predicted results. To perform the full analysis of such simplified mathematical model, the solution procedure plays a crucial role in both the accuracy and computational efficiency and it must be properly chosen to suit each involved scenario.

Various analytical techniques such as methods of integral transform, representation theories, and potential functions have been proposed and used extensively to solve various types of boundary value problems with semi-infinite and infinite domains (e.g., Srivastava and Singh, 1969; Mayrhofer and Fischer, 1989; Wang, 2004; Chen and Shioya, 2000). However, their applications are quite limited to either two-dimensional problems or three-dimensional cases with extremely idealized settings. Such limitation becomes more apparent when the complexity of involved physical phenomena increases (e.g., material constitutive laws, presence of singularities, and general boundary and loading conditions). To enhance the modeling capability, a variety of numerical procedures have been continuously developed to solve half-space and whole-space problems. Standard finite element methods (FEMs) have been well-established in the past several decades (Oden and Carey, 1984; Zienkiewicz and Taylor, 2000; Hughes, 2000) and successfully applied to solve numerous problems in various disciplines. It is worth noting, however, that when applied to treat problems involving unbounded domains, the computational efficiency of FEMs can be significantly degraded. For instance, a standard discretization procedure cannot be directly applied to infinite and semi-infinite domains. A domain truncation together with a set of proper remote boundary conditions is commonly employed to establish an approximate domain of finite dimensions prior to the discretization. Another limited capability of standard FEMs is apparent when they are applied to solve fracture-related problems. In the analysis, substantially fine meshes are required in a region surrounding the discontinuities in order to accurately capture the complex field and extract essential local fracture information along the crack front (Swenson and Ingraffea, 1988; Martha et al., 1993; Ayhan et al., 2003).

Boundary integral equation methods (BIEMs) have been also well-recognized as one of the most efficient numerical techniques for modeling linear boundary value problems (e.g., Brebbia and Dominguez, 1989; Cruse, 1988; Liggett and Liu, 1983). Efficiency of BIEMs over standard FEMs becomes more apparent when a domain to be treated is unbounded (e.g., Gu and Yew, 1988; Xu and Ortiz, 1993; Xu, 2000; Rungamornrat and Wheeler, 2006; Rungamornrat and Mear, 2008b; Rungamornrat et al., 2015), the ratio between the domain-measure and the boundary-measure is large, and the domain contains an embedded singularity such as dislocations and cracks (Rungamornrat and Mear, 2008b; Rungamornrat et al., 2015; Saez et al., 1997; Li et al., 1998; Frangi et al., 2002; Ariza and Dominguez, 2004; Rungamornrat, 2006). This is due to the key nature of the governing integral equation underlying the methods; for a domain that is homogeneous and free of distributed source, the governing equation involves only integrals over the boundary of the domain and the surface of discontinuity. Besides those desirable features, BIEMs still possess a major drawback associated with the treatment of singularity induced by kernels present in the governing integral equations. For conventional BIEMs, involved strongly singular and hyper-singular kernels often pose theoretical and numerical difficulties including the existence and interpretation of singular integrals (e.g., Martin and Rizzo, 1996; Chen, 2003a), issues associated with the smoothness requirement of boundary data (e.g., Martin and Rizzo, 1996), and the requirement of significant computational cost and special numerical quadrature to evaluate involved integrals (e.g., Gray et al., 1990; Martha et al., 1992; Ariza et al., 1997; Chen, 2003b; Qin and Noda, 2004; Zhao et al., 2004). To circumvent such drawback, the BIEMs based on a set of singularity-reduced integral equations are commonly employed.

Regularized BIEMs have been continuously developed for past several decades and the historical background and current advances relevant to the present study can be briefly summarized below. Bui (1977) and Weaver (1977) independently developed the singularity-reduced traction integral equations for isolated planar cracks in an isotropic, elastic whole space under pure mode-I loading conditions. The extension to treat material anisotropy was carried out later by Sladek and Sladek (1982). While the strength of singularity was reduced from hypersingular to strongly singular, the regularization was still incomplete in the sense that the validity of involved integrals still requires the derivative of the relative crack-face displacement to be continuous. Within the context of fracture analysis in linear elastic media, development of the weakly singular boundary integral equations has

been well-established. Gu and Yew (1988) was able to develop the first weakly singular boundary integral equation for the traction. While the derived integral equation is completely regularized, the development is restricted to an isotropic elastic whole space and a planar crack under the pure mode-I loading condition. Xu and Ortiz (1993) applied results from the dislocation theory to develop the weakly-singular, weak-form traction boundary integral equation for modeling isolated cracks of arbitrary shapes in an isotropic elastic, whole space. Later, Bonnet (1995) exploited the integration-by-parts technique together with certain representations of fundamental solutions to establish a pair of weakly singular displacement and traction integral equations for modeling isotropic, linearly elastic, uncracked finite bodies. Li (1996) developed a regularization procedure to derive the weakly singular integral equations for modeling cracks of arbitrary shapes in both isotropic, linearly elastic, whole space and half-space. Li et al. (1998) extended the work of Li (1996) to derive the weakly singular, weak-form integral equations for both the displacement and traction and successfully implemented the symmetric Galerkin boundary element method to solve cracks in isotropic, finite bodies. Later, Rungamornrat and Mear (2008a) proposed a systematic regularization technique to develop a complete set of singularity-reduced integral relations for both dislocations and cracks in generally anisotropic elastic media. While a vast amount of researches concerning the development of weakly singular BIEMs for cracks in unbounded elastic media has been well established, most of existing investigations are restricted only to the derivation of traction integral equation for cracks and the determination of stress intensity factors. The analysis for the nonsingular terms such as the T-stress along the crack front has not been recognized. In addition, the integration of existing symmetry to enhance the computational efficiency and the treatment of material

anisotropy for cracks in an elastic half-space under symmetric boundary conditions has not been found.

In the present study, a systematic regularization technique proposed by Rungamornrat and Mear (2008a) is extended to establish a set of singularity-reduced boundary integral equations for either a cracked elastic whole space containing a plane of symmetry or a cracked elastic half-space subjected to a symmetric condition on the free surface. The development is carried out in a general framework allowing the treatment of material anisotropy, cracks of arbitrary shapes, and general crack-face loading. The resulting weakly singular, weak-form boundary integral equations for both the displacement and traction are then employed as the basis in the implementation of the weakly singular, symmetric Galerkin boundary element method for solving unknown crack-face data and post-processing for essential fracture information. Remaining sections of this paper are organized to incorporate the clear problem description, the development of regularized boundary integral equations, the solution procedure, results and discussion, and conclusions and remarks.

2. Problem description

Consider an elastic cracked whole space Ω that possesses a plane of symmetry as shown schematically in **Fig. 1(a)**. A reference Cartesian coordinate system $\{O; x_1, x_2, x_3\}$ with a set of orthonormal base vectors $\{e_1, e_2, e_3\}$ used in the present development is chosen such that the origin O is located on the plane of symmetry; the x_3 -axis directs downward normal to the plane of symmetry; and the x_1 - and x_2 - axes follow the right hand rule. The medium is made of a homogeneous, anisotropic, linearly elastic material with $x_3 = 0$ being the plane of material symmetry. The cracks in the undeformed state are represented by two geometrically identical, piecewise smooth surfaces S_c^+ and S_c^- with the

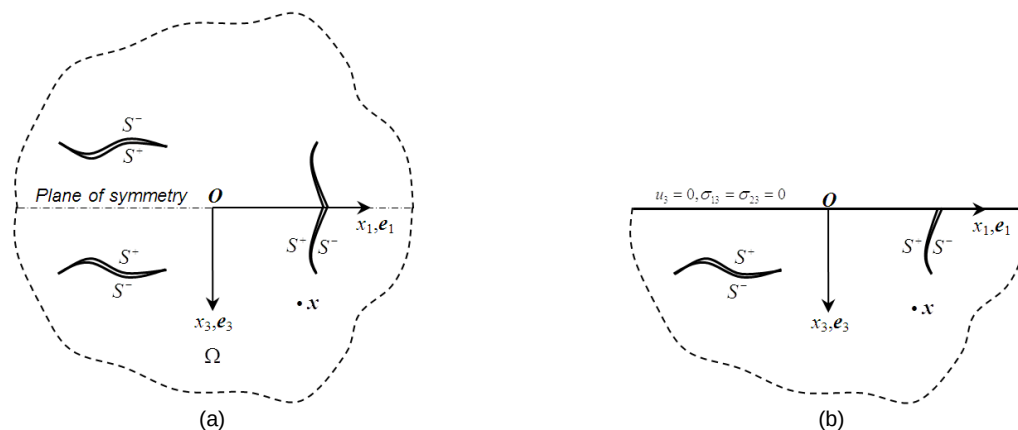


Fig. 1. Schematic of **(a)** cracked whole space with $x_3 = 0$ as plane of symmetry and **(b)** equivalent cracked half-space with symmetric conditions on free surface.

outward unit normal vectors $\mathbf{n}^+$ and $\mathbf{n}^-$, respectively. In the present study, it is assumed that the body force and remote loading are absent and the medium is loaded by self-equilibrated tractions on the crack surfaces. It is worth noting that the influence of the body force and remote loading condition can be readily treated via the superposition technique together with an uncracked state. In addition to an elastic field induced within the cracked medium due to applied loadings, the essential fracture data such as the relative crack-face displacement, the stress intensity factors, and the T-stress components along the crack front are of primary interest.

From the symmetry of all prescribed data, it can be verified that an elastic field of a lower-half of the cracked whole space shown in **Fig. 1(a)** is identical to that of a half-space containing the same crack and subjected to symmetric boundary conditions on the free surface (i.e., the displacement normal to the free surface u_3 and shear tractions σ_{13}, σ_{23} vanish) as indicated in **Fig. 1(b)**. As a result, it is sufficient to solve the equivalent cracked half-space and then exploit the symmetry to obtain the complete solution of the cracked whole space. Using such reduced domain in the simulations clearly reduces the number of degrees of freedom approximately by half.

3. Formulation of governing equations

This section presents the development of a set of completely regularized boundary integral equations for the cracked whole space possessing a plane of symmetry shown in **Fig. 1(a)**. The derivation is carried out only for the lower-half of the domain or, equivalently, for the equivalent cracked half-space shown in **Fig. 1(b)**. First, the displacement and stress fundamental solutions for an uncracked half-space subjected to the symmetric condition on the free surface are constructed and such results are then utilized to form a pair of boundary

integral relations for the displacement and stress of the equivalent cracked half-space. Finally, a systematic regularization technique based on the integration by parts via Stokes' theorem is adopted to establish a set of singularity-reduced boundary integral equations. Special representations of involved kernels essential for assisting such regularization procedure are also provided in details.

3.1 Fundamental solutions for uncracked half-space under symmetric conditions

To construct the fundamental solutions for the displacement and stress of an uncracked half-space subjected to symmetric boundary conditions on the surface $x_3 = 0$, the existing fundamental solutions of an elastic whole space (Ting and Lee, 1997; Wang, 1997) can be employed together with the symmetry as described below.

Now, consider the equivalent uncracked half-space subjected to symmetric boundary conditions and a unit concentrated force $\delta_{ip} \mathbf{e}_i$ at a source point $\mathbf{x} = x_k \mathbf{e}_k$ as illustrated in **Fig. 2(b)** where δ_{ip} denotes the Kronecker-delta symbol. Here and in what follows, standard indicial notation and Einstein summation convention apply. It is apparent that the considered half-space is identical to the lower half of a uncracked whole space subjected to a unit concentrated force $\delta_{ip} \mathbf{e}_i$ at a point $\mathbf{x} = x_k \mathbf{e}_k$ and a unit concentrated force $\bar{\delta}_{ip} \mathbf{e}_i$ at the image point $\bar{\mathbf{x}} = \bar{x}_k \mathbf{e}_k$ as illustrated in **Fig. 2(a)** where $\bar{x}_k = \delta_{kp} x_p$ and $\bar{\delta}_{ip}$ are defined by $\bar{\delta}_{11} = \bar{\delta}_{22} = -\bar{\delta}_{33} = 1$ and $\bar{\delta}_{ij} = 0$ for $i \neq j$. The symmetric boundary conditions (i.e., $u_3 = 0$ and $\sigma_{13} = \sigma_{23} = 0$) are automatically satisfied along the plane $x_3 = 0$ of the whole space for the given applied loads. Upon employing this correspondence together with the superposition of the whole space fundamental solutions, the fundamental solutions for the displacement and stress of the uncracked half-space under the symmetric boundary conditions, denoted respectively by $\bar{U}_j^p(\xi, \mathbf{x})$ and $\bar{S}_{ij}^p(\xi, \mathbf{x})$,

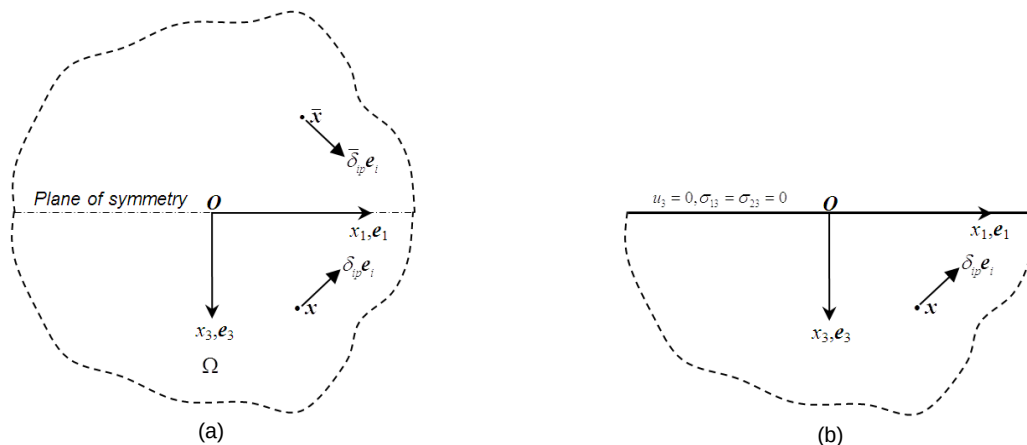


Fig. 2. Schematics of (a) uncracked whole space under unit concentrated force $\delta_{ip} \mathbf{e}_i$ at $\mathbf{x}$ and unit concentrated force $\bar{\delta}_{ip} \mathbf{e}_i$ at image point $\bar{\mathbf{x}}$ and (b) uncracked half-space under symmetric boundary conditions on $x_3 = 0$ and unit concentrated force $\delta_{ip} \mathbf{e}_i$ at $\mathbf{x}$.

can be obtained as

$$\bar{U}_j^p(\xi, \mathbf{x}) = U_j^p(\xi - \mathbf{x}) + \bar{\delta}_{pk} U_j^k(\xi - \bar{\mathbf{x}}) \quad [1]$$

$$\bar{S}_{ij}^p(\xi, \mathbf{x}) = S_{ij}^p(\xi - \mathbf{x}) + \bar{\delta}_{pk} S_{ij}^k(\xi - \bar{\mathbf{x}}) \quad [2]$$

where $U_j^p(\xi - \mathbf{x})$ and $S_{ij}^p(\xi - \mathbf{x})$ are the displacement and stress fundamental solutions of an uncracked whole space under a unit concentrated force $\delta_{ij}e_i$ at a source point $\mathbf{x} = x_k e_k$. The explicit form of $U_j^p(\xi - \mathbf{x})$ and $S_{ij}^p(\xi - \mathbf{x})$ can be found in the work of Rungamornrat and Mear (2008a), Ting and Lee (1997), and Wang (1997). From the structure of $U_j^p(\xi - \mathbf{x})$ and $S_{ij}^p(\xi - \mathbf{x})$ together with [1] and [2], it can be concluded that $\bar{U}_j^p(\xi, \mathbf{x})$ and $\bar{S}_{ij}^p(\xi, \mathbf{x})$ are singular only at a single point $\xi = \mathbf{x}$ within the half-space of $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r^2)$, respectively, where $r = \|\xi - \mathbf{x}\|$.

3.2 Standard integral relations for equivalent cracked half-space

By applying the reciprocal theorem to the equivalent cracked half-space shown in **Fig. 1(b)** along with the elastic state associated with a fundamental problem of an uncracked half-space shown in **Fig. 2(b)**, it yields an integral relation for the displacement at any interior point $\mathbf{x}$ of the equivalent cracked half-space as

$$u_p(\mathbf{x}) = \int_{S_c^+} \bar{U}_j^p(\xi, \mathbf{x}) \Sigma t_j(\xi) dA(\xi) - \int_{S_c^-} \bar{S}_{ij}^p(\xi, \mathbf{x}) n_i^+(\xi) \Delta u_j(\xi) dA(\xi) \quad [3]$$

where $\Sigma t_j(\xi) = t_j(\xi^+) + t_j(\xi^-)$ denotes the sum of the prescribed crack-face traction; $\Delta u_j(\xi) = u_j(\xi^+) - u_j(\xi^-)$ denotes the jump in the unknown crack-face displacement; and $\xi^+ \in S_c^+$ and $\xi^- \in S_c^-$ are two coincident points on the crack surface. It should be remarked that the reduction to integrals over a single crack surface S_c^+ stems directly from the continuity of the fundamental solutions $\bar{U}_j^p(\xi, \mathbf{x})$ and $\bar{S}_{ij}^p(\xi, \mathbf{x})$ at any field point ξ . By first taking the derivative of [3] to obtain the spatial gradient of $u_p(\mathbf{x})$ with respect to x_i and then employing the constitutive law for linear elastic materials, it finally yields a boundary integral relation for the stress at any interior point $\mathbf{x}$ of the equivalent cracked half-space:

$$\sigma_{ik}(\mathbf{x}) = \int_{S_c^+} E_{ikpq} \frac{\partial \bar{U}_j^p(\xi, \mathbf{x})}{\partial x_q} \Sigma t_j(\xi) dA(\xi) - \int_{S_c^+} E_{ikpq} \frac{\partial \bar{S}_{ij}^p(\xi, \mathbf{x})}{\partial x_q} n_i^+(\xi) \Delta u_j(\xi) dA(\xi) \quad [4]$$

From the relations [1] and [2] along with the properties of the fundamental solutions $U_j^p(\xi - \mathbf{x})$ and $S_{ij}^p(\xi - \mathbf{x})$ for the uncracked whole space, the kernels $E_{ikpq} \partial \bar{U}_j^p(\xi, \mathbf{x}) / \partial x_q$ and $E_{ikpq} \partial \bar{S}_{ij}^p(\xi, \mathbf{x}) / \partial x_q$ contained in the boundary integral relation [4] can be further expressed as

$$E_{ikpq} \frac{\partial \bar{U}_j^p(\xi, \mathbf{x})}{\partial x_q} = -S_{ik}^j(\xi - \mathbf{x}) + \bar{\delta}_{jp} S_{ik}^p(\mathbf{x} - \bar{\xi}) \quad [5]$$

$$E_{ikpq} \frac{\partial \bar{S}_{ij}^p(\xi, \mathbf{x})}{\partial x_q} = -\Sigma_{ij}^{lk}(\xi - \mathbf{x}) - \bar{\delta}_{ia} \bar{\delta}_{jb} \Sigma_{ab}^{lk}(\mathbf{x} - \bar{\xi}) \quad [6]$$

where $\Sigma_{ij}^{lk}(\xi - \mathbf{x}) = E_{ikpq} \partial S_{ij}^p(\xi - \mathbf{x}) / \partial x_q$ and $\bar{\xi}$ is an image point of ξ with respect to the plane $x_3 = 0$. From the property of the stress fundamental solution for the uncracked whole space $S_{ij}^p(\xi - \mathbf{x})$, the kernels $E_{ikpq} \partial \bar{U}_j^p(\xi, \mathbf{x}) / \partial x_q$ and $E_{ikpq} \partial \bar{S}_{ij}^p(\xi, \mathbf{x}) / \partial x_q$ are clearly singular at $\xi = \mathbf{x}$ of $\mathcal{O}(1/r^2)$ and $\mathcal{O}(1/r^3)$, respectively. The two integral relations [3] and [4] form an essential basis for the development of governing integral equations for determining unknown crack-face data such as the sum of and the jump in the crack-face displacements. However, the direct use of those integral relations poses certain difficulties including the interpretation and numerical treatment of all involved strongly singular and hyper-singular integrals.

3.3 Regularized displacement and stress boundary integral relations

To regularize the boundary integral relations [3] and [4], special decompositions of involved strongly singular and hypersingular kernels are first established to aid the integration-by-part procedure. The essential component for establishing such decompositions is the special representations of the stress fundamental solution $S_{ij}^p(\xi - \mathbf{x})$ and the hypersingular kernel $\Sigma_{ij}^{lk}(\xi - \mathbf{x})$ for the uncracked whole space proposed by Rungamornrat and Mear (2008a). Such results are valid for generally anisotropic, linearly elastic materials and given explicitly by

$$S_{ij}^p(\xi - \mathbf{x}) = H_{ij}^p(\xi - \mathbf{x}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} G_{mj}^p(\xi - \mathbf{x}) \quad [7]$$

$$\Sigma_{ij}^{lk}(\xi - \mathbf{x}) = -E_{ijkl} \delta(\xi - \mathbf{x}) + \varepsilon_{ism} \varepsilon_{lrs} \frac{\partial}{\partial \xi_s} \frac{\partial}{\partial \xi_r} C_{mj}^{tk}(\xi - \mathbf{x}) \quad [8]$$

where ε_{ism} denotes an alternating symbol; $\delta(\xi - \mathbf{x})$ is a Dirac-delta distribution centered at $\mathbf{x}$; and the functions $H_{ij}^p(\xi - \mathbf{x})$, $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{tk}(\xi - \mathbf{x})$ are defined by

$$H_{ij}^p(\xi - \mathbf{x}) = -\delta_{jp} \frac{\xi_i - x_i}{4\pi r^3} \quad [9]$$

$$G_{mj}^p(\xi - \mathbf{x}) = \frac{\varepsilon_{mqa} E_{qkl}}{8\pi^2 r} \oint_{z=r=0} (\mathbf{z}, \mathbf{z})^{-1} z_a z_l ds(\mathbf{z}) \quad [10]$$

$$C_{mj}^{tk}(\xi - \mathbf{x}) = \frac{A_{mtsl}^{kjap}}{8\pi^2 r} \oint_{z=r=0} (\mathbf{z}, \mathbf{z})^{-1} z_a z_s z_l ds(\mathbf{z}) \quad [11]$$

where $A_{mtsl}^{kjap} = \varepsilon_{ptd} \varepsilon_{pmq} (E_{djpl} E_{qlms} - E_{lmps} E_{djkl} / 3)$; $\mathbf{r} = \xi - \mathbf{x}$; $\mathbf{z}$ is a unit vector; $(\mathbf{z}, \mathbf{z})$ is a tensor whose components are defined by $(\mathbf{z}, \mathbf{z})_{kp} = z_k E_{mkpn} z_n$; and $(\mathbf{z}, \mathbf{z})^{-1}$ is the inverse of $(\mathbf{z}, \mathbf{z})$. Note that the line integrals appearing in [10] and

[11] are traced along a unit circle $\|z\|=1$ on the plane $z \cdot r = 0$. It is evident from [9]-[11] that the functions $H_{ij}^p(\xi - x)$, $G_{mj}^p(\xi - x)$ and $C_{mj}^{ik}(\xi - x)$ are singular at $\xi = x$ of $\mathcal{O}(1/r^2)$, $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r)$, respectively. By employing the relations [2] and [5] together with the decomposition [7], the stress fundamental solution $\bar{S}_{ij}^p(\xi, x)$ and the function $E_{lkpq} \partial \bar{U}_j^p(\xi, x) / \partial x_q$ for the uncracked half-space under the symmetric boundary conditions admits the representation

$$\bar{S}_{ij}^p(\xi, x) = \bar{H}_{ij}^p(\xi, x) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} \bar{G}_{mj}^p(\xi, x) \quad [12]$$

$$E_{lkpq} \frac{\partial \bar{U}_j^p(\xi, x)}{\partial x_q} = \bar{H}_{lk}^j(x, \xi) + \varepsilon_{lrt} \frac{\partial}{\partial x_r} \bar{G}_{tk}^j(x, \xi) \quad [13]$$

where the functions $\bar{H}_{ij}^p(\xi, x)$ and $\bar{G}_{mj}^p(\xi, x)$ are defined by

$$\bar{H}_{ij}^p(\xi, x) = H_{ij}^p(\xi - x) + \bar{\delta}_{pk} H_{ij}^k(\xi - \bar{x}) \quad [14]$$

$$\bar{G}_{mj}^p(\xi, x) = G_{mj}^p(\xi - x) + \bar{\delta}_{pk} G_{mj}^k(\xi - \bar{x}) \quad [15]$$

Finally, by applying the decomposition [8] to [6], it yields the representation of the kernel $E_{lkpq} \partial \bar{S}_{ij}^p(\xi, x) / \partial x_q$ as

$$E_{lkpq} \frac{\partial \bar{S}_{ij}^p(\xi, x)}{\partial x_q} = E_{ijkl} \delta(\xi - x) - \bar{\delta}_{ia} \bar{\delta}_{jb} E_{abkl} \delta(x - \bar{\xi}) + \varepsilon_{ism} \varepsilon_{lrt} \frac{\partial}{\partial \xi_s} \frac{\partial}{\partial x_r} \bar{C}_{mj}^{ik}(\xi, x) \quad [16]$$

where the function $\bar{C}_{mj}^{ik}(\xi, x)$ is given by

$$\bar{C}_{mj}^{ik}(\xi, x) = C_{mj}^{ik}(\xi - x) - \bar{\delta}_{am} \bar{\delta}_{bj} C_{ab}^{ik}(x - \bar{\xi}) \quad [17]$$

From the singularity behavior of the functions $H_{ij}^p(\xi - x)$, $G_{mj}^p(\xi - x)$ and $C_{mj}^{ik}(\xi - x)$ indicated by [9]-[11], it is apparent from [14]-[15] and [17] that the functions $\bar{H}_{ij}^p(\xi, x)$, $\bar{G}_{mj}^p(\xi, x)$ and $\bar{C}_{mj}^{ik}(\xi, x)$ are singular only at a point $\xi = x$ of order $\mathcal{O}(1/r^2)$, $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r)$, respectively.

By applying the decomposition [7] to the displacement boundary integral relation [3] and then integrating a term containing the function $\bar{G}_{mj}^p(\xi, x)$ by parts via Stokes' theorem, it finally results in

$$u_p(x) = \int_{S_c^+} \bar{U}_j^p(\xi, x) \Sigma t_j(\xi) dA(\xi) - \int_{S_c^+} \bar{H}_{ij}^p(\xi, x) n_i^+(\xi) \Delta u_j(\xi) dA(\xi) + \int_{S_c^+} \bar{G}_{mj}^p(\xi, x) D_m \Delta u_j(\xi) dA(\xi) \quad [18]$$

where $D_m(\cdot) = n_i \varepsilon_{ism} \partial(\cdot) / \partial \xi_s$ denotes a surface differential operator. The boundary term resulting from the integration by part vanishes due to the enforcement of the closure condition along the crack front (i.e., $\Delta u_j(\xi) = 0 \forall \xi \in \partial S_c^+$). It is worth noting that the boundary

integral relation [18] contains only weakly singular kernels of $\mathcal{O}(1/r)$. The weak singularity of the product $\bar{H}_{ij}^p(\xi, x) n_i^+(\xi)$ results directly from the relation [14] and the property of the function $H_{ij}^p(\xi - x)$, as pointed out before by Xiao (1998). Similarly, by applying the decompositions [13] and [16] to the boundary integral relation for the stress [4] and then integrating the term associated with the kernel $\bar{C}_{mj}^{ik}(\xi, x)$ by parts via Stokes' theorem, it finally leads to

$$\sigma_{ik}(x) = \varepsilon_{lrt} \frac{\partial}{\partial x_r} \left\{ \int_{S_c^+} \bar{C}_{mj}^{ik}(\xi, x) D_m \Delta u_j(\xi) dA(\xi) + \int_{S_c^+} \bar{G}_{tk}^j(x, \xi) \Sigma t_j(\xi) dA(\xi) \right\} + \int_{S_c^+} \bar{H}_{lk}^j(x, \xi) \Sigma t_j(\xi) dA(\xi) \quad [19]$$

where the closure condition has been enforced again and the contribution of the Dirac-delta distributions disappear at any interior point x . A pair of singularity-reduced boundary integral relations [18] and [19] forms the basis for the post-process of the displacement and stress at any interior point x once the unknown jump in the crack-face displacement (i.e., Δu_j) is fully determined.

3.4 Weakly singular, weak-form integral equations for equivalent cracked half-space

By properly forming the limit $x \rightarrow y \in S_c^+$ of the regularized boundary integral relation [18], it yields the boundary integral equation for the sum of the crack-face displacement as

$$c(y) \Sigma u_p(y) = \int_{S_c^+} \bar{U}_j^p(\xi, y) \Sigma t_j(\xi) dA(\xi) + \int_{S_c^+} \bar{G}_{mj}^p(\xi, y) D_m \Delta u_j(\xi) dA(\xi) - \int_{S_c^+} \bar{H}_{ij}^p(\xi, y) n_i^+(\xi) \Delta u_j(\xi) dA(\xi) \quad [20]$$

where $\Sigma u_p(y) = u_p(y^+) + u_p(y^-)$ denotes the sum of the crack-face displacement and the function $c = c(y)$ is defined such that $0 < c(y) < 1$ and $c(y) = 1/2$ if the surface is sufficiently smooth at y . By multiplying [20] by a sufficiently smooth test function $\tilde{t}_p(y)$ and then integrating the result over S_c^+ , it leads to a weakly-singular, weak-form boundary integral equation for the sum of the crack-face displacement:

$$\frac{1}{2} \int_{S_c^+} \tilde{t}_p(y) \Sigma u_p(y) dA(y) = \int_{S_c^+} \tilde{t}_p(y) \int_{S_c^+} \bar{U}_j^p(\xi, y) \Sigma t_j(\xi) dA(\xi) dA(y) + \int_{S_c^+} \tilde{t}_p(y) \int_{S_c^+} \bar{G}_{mj}^p(\xi, y) D_m \Delta u_j(\xi) dA(\xi) dA(y) - \int_{S_c^+} \tilde{t}_p(y) \int_{S_c^+} \bar{H}_{ij}^p(\xi, y) n_i^+(\xi) \Delta u_j(\xi) dA(\xi) dA(y) \quad [21]$$

It is important to emphasize that the function $c = c(y)$ simply reduces to $1/2$ since the crack surface is assumed

piecewise smooth (i.e., a set of points y whose unit normal vector is not well-defined is of measure zero). Clearly, the weak-form boundary integral equation [21] contains only weakly singular kernels of $\mathcal{O}(1/r)$.

To establish the weakly singular, weak-form boundary integral equation for the crack-face traction, we first form the product $n_i^+(y)\sigma_{ik}(x)$, $y \in S_c^+$ using the boundary integral relation [19] and then take the limit $x \rightarrow y$. This process yields the following singularity-reduced boundary integral equation for the crack-face traction

$$\rho(y)\Delta t_k(y) = D_i \left\{ \int_{S_c^+} \bar{C}_{mj}^{ik}(\xi, y) D_m \Delta u_j(\xi) dA(\xi) + \int_{S_c^+} \bar{G}_{ik}^j(y, \xi) \Sigma t_j(\xi) dA(\xi) \right\} + n_i^+(y) \int_{S_c^+} \bar{H}_{ik}^j(y, \xi) \Sigma t_j(\xi) dA(\xi) \quad [22]$$

where $\Delta t_k(y) = t_k(y^+) - t_k(y^-)$ denotes the jump in the crack-face traction and $\rho = \rho(y)$ is defined such that $0 < \rho(y) < 1$ and $\rho(y) = 1/2$ if the surface is sufficiently smooth at y . By multiplying the boundary integral equation [22] by a sufficiently smooth test function $\tilde{u}_k(y)$, integrating the result over the crack surface S_c^+ , then performing an integration by parts via Stokes' theorem of integrals containing the kernels $\bar{C}_{mj}^{ik}(\xi, x)$ and $\bar{G}_{ik}^j(y, \xi)$, and finally choosing the test function $\tilde{u}_k(y)$ to satisfy the closure condition along the crack front (i.e., $\tilde{u}_k(y) = 0 \forall y \in \partial S_c^+$), it gives rise to

$$\begin{aligned} \frac{1}{2} \int_{S_c^+} \tilde{u}_k(y) \Delta t_k(y) dA(y) &= \int_{S_c^+} \tilde{u}_k(y) \int_{S_c^+} \bar{H}_{ik}^j(y, \xi) n_i^+(y) \Sigma t_j(\xi) dA(\xi) dA(y) \\ &\quad - \int_{S_c^+} D_i \tilde{u}_k(y) \int_{S_c^+} \bar{G}_{ik}^j(y, \xi) \Sigma t_j(\xi) dA(\xi) dA(y) \\ &\quad - \int_{S_c^+} D_i \tilde{u}_k(y) \int_{S_c^+} \bar{C}_{mj}^{ik}(\xi, y) D_m \Delta u_j(\xi) dA(\xi) dA(y) \end{aligned} \quad [23]$$

Again, from the piecewise smoothness of the crack surface, the function $\rho = \rho(y)$ simply reduces to $1/2$. It is also important to point out that the weak-form, traction boundary integral equation [23] contains only weakly singular kernels of $\mathcal{O}(1/r)$.

4. Solution methodology

A pair of weakly singular, weak-form boundary integral equations [21] and [23] provides the essential basis for the formulation of the boundary value problem of the equivalent cracked half-space shown in **Fig. 1(b)**. The final governing equations contain only unknowns on the crack surface including the sum of the crack-face displacement Σu_p and the jump in the crack-face displacement Δu_p . It is apparent that the weakly-singular, weak-form traction boundary integral equation [23] contains the complete information of the prescribed crack-face tractions (i.e., Δt_i and Σt_i) and is independent

of the unknown Σu_p . The solution of the jump in the crack-face displacement Δu_p can be, therefore, determined by solving this weak-form equation. Once the data Δu_p is fully obtained, it is then employed together with the weak-form integral equation [21] to solve for the unknown sum of the crack-face displacement Σu_p .

In the present study, a symmetric Galerkin boundary element method (SGBEM) is adopted to construct the numerical solution of the weak-form equation [23] (also see the work of Rungamornrat and Mear (2008b), Li et al. (1998), Frangi et al. (2002)). Due to the weakly singular nature of all involved integrals, continuous finite element basis functions are employed everywhere in the approximation; in particular, 6-node, 8-node, and 9-node quadratic elements are employed in the discretization of the crack surface. To further enhance the accuracy of the approximation, special 9-node crack-tip elements proposed by Rungamornrat and Mear (2008b) and Li et al. (1998) are adopted to discretize the jump in the crack-face displacement in a region adjacent to the crack front. Such special crack-tip elements possess two positive features including their shape functions properly enriched by a square-root function and extra degrees of freedom introduced along the crack front to represent the gradient of the jump in the crack-face displacement. The former feature allows the near-front solution to be captured accurately using relatively coarse meshes whereas the latter renders the stress intensity factors to be extracted directly from those extra degrees of freedom without carrying the extrapolation. To form a system of linear algebraic equations resulting from the discretization, two essential tasks are accomplished with a special care. One is associated with the numerical evaluation of regular, weakly singular and nearly singular integrals over elements or pairs of elements resulting from the discretization procedure. The regular integrals including both single surface and double surface integrals over pairs of remote elements are integrated efficiently and accurately by standard Gaussian quadrature. For the last two types of integrals, the weak singularity and rapid variation of integrands induced when pairs of coincident or adjacent elements are involved renders the integrals very difficult to be evaluated by standard Gaussian quadrature (e.g., Xiao, 1998). In the present study, an integrand-regularization technique similar to that proposed by Xiao (1998) is employed to efficiently integrate those double surface integrals. Specifically, a family of variable transformations such as triangular-polar transformation and logarithmic transformations is applied to either remove the weak singularity or alleviate the fast variation of the integrand before they are integrated by standard Gaussian quadrature. The other important task corresponds directly to the evaluation of all involved

kernels $\bar{H}_{ij}^p(\xi, \mathbf{x})$, $\bar{G}_{mj}^p(\xi, \mathbf{x})$ and $\bar{C}_{mj}^{ik}(\xi, \mathbf{x})$ for every pair of points $(\xi, \mathbf{x})$ resulting from the numerical quadrature. It is evident from the relations [14], [15] and [17] that such task only requires calculations of the kernels $H_{ij}^p(\xi - \mathbf{x})$, $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{ik}(\xi - \mathbf{x})$ for the uncracked whole space. Since $H_{ij}^p(\xi - \mathbf{x})$ is independent of material property and involves only elementary functions, its evaluation can be readily achieved via the direct substitution. On the contrary, the kernels $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{ik}(\xi - \mathbf{x})$ are material dependent and the direct evaluation by performing all involved contour integrals for every pair of points $(\xi, \mathbf{x})$ is computationally expensive. To avoid such massive calculations, an interpolation scheme similar to that used by Rungamornrat and Mear (2008b) is adopted. In this technique, the kernels $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{ik}(\xi - \mathbf{x})$ given by [10] and [11] are first rewritten as a product of two parts:

$$G_{mj}^p(\xi - \mathbf{x}) = \frac{\varepsilon_{mqa} E_{ijkl}}{8\pi^2 r} \times I_{ij}^{ql}(\theta, \phi) \quad [24]$$

$$C_{mj}^{ik}(\xi - \mathbf{x}) = \frac{A_{msl}^{kijp}}{8\pi^2 r} \times I_{ap}^{sl}(\theta, \phi) \quad [25]$$

where $r = \|\xi - \mathbf{x}\|$, $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$ denote spherical coordinates of point ξ with the origin at point $\mathbf{x}$ and the angular dependent function $I_{ij}^{kl}(\theta, \phi)$ is defined by

$$I_{ij}^{kl}(\theta, \phi) = \oint_{z=r=0} (\mathbf{z}, \mathbf{z})_{ij}^{-1} z_k z_l ds(\mathbf{z}) \quad [26]$$

It is evident that the first part of both kernels involves mainly elementary functions and can, therefore, be calculated efficiently via the direct substitution. For the second part, the function $I_{ij}^{kl}(\theta, \phi)$ is approximated over a domain $[0, 2\pi] \times [0, \pi]$ using finite element interpolations. In such procedure, the direct evaluation of the line integral [26] is only required at nodal points and it can be achieved efficiently via Gaussian quadrature. The accuracy of the approximation can be readily controlled and enhanced by refining the interpolation grid. The final, symmetric, dense system of linear algebraic equations is then solved by a selected efficient indirect linear solver such as a preconditioning conjugate gradient method.

To determine the sum of the crack-face displacement Σu_p , the weak-form boundary integral equation [21] is subsequently solved by a standard Galerkin technique. Note that once the jump in the crack-face displacement Δu_p is obtained, all double surface integrals appearing on the right hand side of [21] serve only as the prescribed data and the single surface integral containing the unknown Σu_p is relatively simple. Due to the regularity behavior of Σu_p over the entire crack surface, standard two-dimensional isoparametric elements are adopted everywhere in the discretization of trial and test functions. Essential ingredients described above also apply to treat

all involved single surface and double surface integrals and the calculations of all kernels. The resulting, symmetric, sparse system of linear algebraic equations is then efficiently solved by the same indirect linear solver.

5. Post-process

Once the unknown crack-face data $\{\Sigma u_p, \Delta u_p\}$ are obtained, other related quantities such as the displacements and stresses at any point within the cracked medium and the stress intensity factors and T-stress components along the crack front can be determined. The former can be readily computed using the singularity-reduced boundary integral relations [18] and [19] whereas the post-process of the latter from the local stress field in the neighborhood of the crack front still requires non-trivial treatment of nearly singular integrals and limiting procedure. To avoid difficulties posed in such task, the solved crack-face displacement data (i.e., Σu_p and Δu_p) is utilized, instead, to extract the stress intensity factors and the T-stress components. Such procedure is briefly summarized as follows (see also the work of Rungamornrat and Mear (2008b), Li et al. (1998), Pham (2015) and Rungamornrat et al. (2018)).

Let $\mathbf{x}_c$ be a point on the crack front and $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ be a local Cartesian coordinate system with the origin at $\mathbf{x}_c$ and the orthonormal base vectors $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$. In particular, $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ are chosen such that $\bar{\mathbf{e}}_3$ is tangent to the crack front at $\mathbf{x}_c$; $\bar{\mathbf{e}}_2$ is normal to the crack surface S_c^+ at $\mathbf{x}_c$ and $\bar{\mathbf{e}}_2 = -\mathbf{n}^+$; and $\bar{\mathbf{e}}_1 = \bar{\mathbf{e}}_2 \times \bar{\mathbf{e}}_3$ is normal to the crack front and directs inside the medium as clearly indicated in Fig. 3. By using the property of special crack-tip elements, Li et al. (1998) and Rungamornrat and Mear (2008b) proposed an explicit formula for determining the mixed-mode stress intensity factors (K_I, K_{II}, K_{III}) in terms of the extra nodal degrees of freedom placed along the crack front as

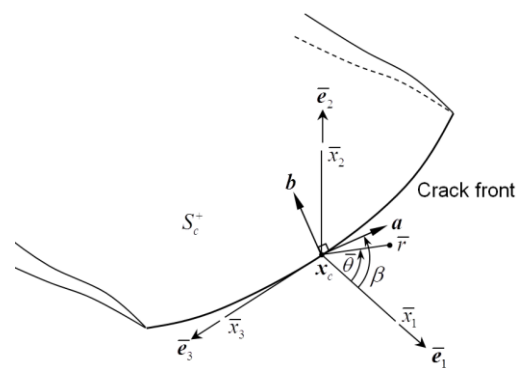


Fig. 3. Local Cartesian coordinate system and all involved parameters for determining stress intensity factors and T-stress components.

$$k_i(\mathbf{x}_c) = \sqrt{\frac{\pi}{2\gamma}} B_{ij} [\hat{\mathbf{u}}(\mathbf{x}_c) \cdot \bar{\mathbf{e}}_j] \quad [27]$$

where $k_1 \equiv K_{II}$, $k_2 \equiv K_I$, $k_3 \equiv K_{III}$;

$$\gamma = -\bar{\mathbf{e}}_1 \cdot \frac{\partial \mathbf{r}_c}{\partial \eta}(\xi_c, -1), \quad [28]$$

$$\mathbf{r}_c = \sum_{i=1}^9 \mathbf{x}^{(i)} \psi^{(i)}(\xi, \eta) - \mathbf{x}_c, \quad [29]$$

$$\hat{\mathbf{u}}(\mathbf{x}_c) = \sum_i \Delta \mathbf{u}^{(i)} \psi^{(i)}(\xi_c, -1); \quad [30]$$

$\mathbf{x}^{(i)}$ is the i^{th} node of the crack-tip element; $(\xi, \eta) \in [-1, 1] \times [-1, 1]$ are master coordinates of any point within the crack-tip element and $(\xi_c, -1)$ are master coordinates of the point $\mathbf{x}_c$; $\psi^{(i)}(\xi, \eta)$ is a standard quadratic shape function of the i^{th} node defined on a 9-node master element; $\Delta \mathbf{u}^{(i)}$ represents the extra degree of freedom of the i^{th} node of the crack-tip element along the crack front; and B_{ij} is given by

$$B_{ij} = \frac{1}{2\pi} \int_0^{2\pi} [(\mathbf{a}, \mathbf{a})_{ij} - (\mathbf{a}, \mathbf{b})_{im} (\mathbf{b}, \mathbf{b})_{mn}^{-1} (\mathbf{b}, \mathbf{a})_{nj}] d\beta \quad [31]$$

with $\mathbf{a}$ and $\mathbf{b}$ denoting orthonormal vectors in the plane $\bar{x}_3 = 0$ and β denoting the angle between $\mathbf{a}$ and the unit vector $\bar{\mathbf{e}}_1$ as indicated in **Fig. 3**. It is worth noting that the summation in [30] is taken only over nodes of the crack-tip element located along the crack front.

The T-stress components along the boundary of the crack can be also obtained directly from the sum of the crack-face displacement as described below. First, the finite part of the stress tensor at the point $\mathbf{x}_c$ along the crack front, with $T_{ij}(\mathbf{x}_c)$ denoting its local components, is related to the finite part of the strain tensor at the same point, with $\bar{\varepsilon}_{kl}(\mathbf{x}_c)$ denoting its local components, by

$$T_{ij}(\mathbf{x}_c) = \bar{E}_{ijkl}(\mathbf{x}_c) \bar{\varepsilon}_{kl}(\mathbf{x}_c) \quad [32]$$

where $\bar{E}_{ijkl}(\mathbf{x}_c)$ are local components of the elastic moduli at $\mathbf{x}_c$. From the continuity of the finite part of the stress at $\mathbf{x}_c$, three local components T_{12} , T_{22} , and T_{23} are, therefore, known a priori and can be obtained from the prescribed crack-face traction at $\mathbf{x}_c$. In addition, three in-plane components of $\bar{\varepsilon}_{kl}(\mathbf{x}_c)$ at $\mathbf{x}_c$ can be computed from the solved sum of the crack-face displacement as

$$\begin{aligned} \bar{\varepsilon}_{11}(\mathbf{x}_c) &= \frac{1}{2} \frac{\partial \Sigma \bar{u}_1(\mathbf{x}_c)}{\partial \bar{x}_1}; \bar{\varepsilon}_{33}(\mathbf{x}_c) = \frac{1}{2} \frac{\partial \Sigma \bar{u}_3(\mathbf{x}_c)}{\partial \bar{x}_3}; \\ \bar{\varepsilon}_{13}(\mathbf{x}_c) &= \frac{1}{4} \frac{\partial \Sigma \bar{u}_1(\mathbf{x}_c)}{\partial \bar{x}_3} + \frac{1}{4} \frac{\partial \Sigma \bar{u}_3(\mathbf{x}_c)}{\partial \bar{x}_1} \end{aligned} \quad [33]$$

The derivatives involved in the expressions [33] can be carried out directly within elements along the crack front. From the prescribed information of T_{12} , T_{22} , and T_{23} and

the computed components $\bar{\varepsilon}_{11}$, $\bar{\varepsilon}_{13}$ and $\bar{\varepsilon}_{33}$, the unknown components T_{11} , T_{13} , and T_{33} at any point $\mathbf{x}_c$ along the crack front, commonly termed the T-stress components, can be obtained by solving a system of six independent linear algebraic equations [32].

6. Numerical results

In this section, the boundary integral formulation and the implemented solution procedure for a cracked whole space possessing a plane of symmetry is fully tested. Three different boundary value problems are considered to demonstrate the computational performance and robustness of the proposed technique. In numerical simulations, two representative, linearly elastic materials, one for isotropic and the other for transversely isotropic solids, with elastic constants shown in **Table 1** are employed. For the transversely isotropic material, the axis of material symmetry is taken to direct normal to the plane of symmetry, i.e., the $x_1 - x_2$ plane. Special 9-node crack-tip elements are used to discretize the region adjacent to the crack front whereas standard 8-node quadrilateral elements and 6-node triangular elements are adopted to discretize the majority of the crack surface.

6.1 Pair of identical horizontal penny-shaped cracks

Consider, first, a pair of identical horizontal penny-shaped cracks embedded in a whole space and the distance from their surfaces to the symmetric plane is h as shown in **Fig. 4(a)**. The crack radius is denoted by a and the lower crack front is parameterized by $x_1 = a \cos \beta$, $x_2 = -a \sin \beta$, $x_3 = h$ for $\beta \in [0, 2\pi]$. The equivalent cracked half-space under the symmetric boundary conditions is indicated in **Fig. 4(b)** and each crack is subjected to a uniform traction $t_1^+ = -t_1^- = -\sigma_0$, $t_2^+ = -t_2^- = 0$, $t_3^+ = -t_3^- = -\sigma_0$, (see **Fig. 4(c)**). In the analysis, four meshes are adopted as indicated in **Fig. 4(d)** and the normalized depth $h/a = 0.5$ is considered.

Computed stress intensity factors and T-stress components at $\beta = 0^\circ, 90^\circ$ and 180° are normalized by the

Table 1. Elastic constants for isotropic material (associated with Poisson's ratio $\nu = 0.3$ and $E = 2.6$ GPa) and transversely isotropic material with the axis of material symmetry normal to half-space surface.

Materials	Elastic constants (GPa)				
	E_{1111}	E_{1122}	E_{1133}	E_{2222}	E_{1313}
Isotropic	3.500	1.500	1.500	3.500	1.000
Transversely isotropic	16.090	3.350	5.010	6.100	3.830

reference solution and then reported in **Tables 2** and **3**. It is remarked that the reference results are taken as the solution of a cracked whole space shown in **Fig. 4(a)**

without exploiting the symmetry by the SGBEM proposed by Rungamornrat and Mear (2008b) with use of the Mesh 4. As can be seen from results in **Table 2**, numerical

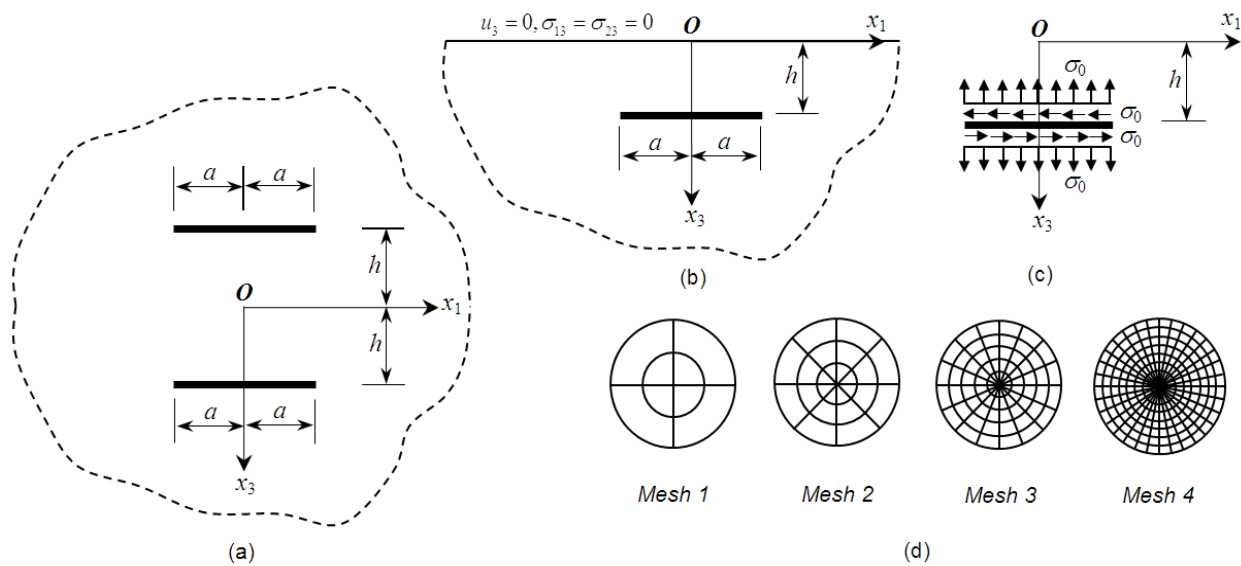


Fig. 4. Schematics of **(a)** pair of identical, horizontal penny-shaped cracks in whole space, **(b)** equivalent cracked half-space, **(c)** tractions acting on crack surfaces, and **(d)** four meshes used in analysis.

Table 2. Normalized stress intensity factors at $\beta = 0^\circ, 90^\circ$ and 180° for lower penny-shaped crack shown in **Fig. 4(a)** with $h/a = 0.5$.

β	Mesh	Isotropic			Transversely isotropic		
		K_I / K_I^{ref}	K_{II} / K_{II}^{ref}	K_{III} / K_{III}^{ref}	K_I / K_I^{ref}	K_{II} / K_{II}^{ref}	K_{III} / K_{III}^{ref}
0°	1	0.9949	0.9896	-	0.9928	0.9907	-
	2	1.0004	1.0001	-	1.0000	1.0004	-
	3	1.0003	1.0002	-	1.0002	1.0004	-
90°	1	0.9935	1.0011	0.9817	0.9927	0.9903	0.9835
	2	1.0002	1.0016	0.9990	1.0000	1.0000	0.9994
	3	1.0003	1.0004	1.0000	1.0002	1.0000	1.0002
180°	1	0.9916	0.9911	-	0.9927	0.9907	-
	2	1.0000	1.0003	-	1.0000	1.0003	-
	3	1.0002	1.0002	-	1.0002	1.0003	-

Table 3. Normalized T-stress components at $\beta = 0^\circ, 90^\circ$ and 180° for lower penny-shaped crack shown in **Fig. 4(a)** with $h/a = 0.5$.

β	Mesh	Isotropic			Transversely isotropic		
		T_{11} / T_{11}^{ref}	T_{33} / T_{33}^{ref}	T_{13} / T_{13}^{ref}	T_{11} / T_{11}^{ref}	T_{33} / T_{33}^{ref}	T_{13} / T_{13}^{ref}
0°	1	1.0359	1.0252	-	1.0184	1.0055	-
	2	1.0186	1.0166	-	1.0075	1.0057	-
	3	1.0055	1.0049	-	1.0023	1.0017	-
90°	1	1.0037	0.9988	0.9626	1.0063	0.9969	1.0787
	2	1.0042	1.0013	1.1549	1.0034	0.9995	1.0825
	3	1.0021	1.0008	1.0662	1.0015	0.9999	1.0297
180°	1	0.9658	0.9603	-	0.9930	0.9868	-
	2	0.9873	0.9790	-	0.9989	0.9921	-
	3	0.9980	0.9947	-	1.0006	0.9977	-

solutions exhibit excellent agreement with the reference solutions for the first three meshes and they are weakly dependent on the level of mesh refinement. In particular, the discrepancy between the stress intensity factors generated by the coarsest and intermediate meshes and the reference solutions is less than 1.9% and 0.2% for the isotropic case and 1.7% and 0.1% for the transversely isotropic case, respectively, whereas results obtained from the Mesh 3 are nearly identical to the reference solution. The high quality of the numerical solutions, while employing relatively coarse meshes, is the direct consequence of the use of special crack-tip elements to enhance the near-front approximation of the relative crack-face displacement.

Similar convergence behavior can be also observed for results of the T-stress components shown in **Table 3**; however, it is apparent that the difference between the computed solutions T_{13} from the Mesh 1 and Mesh 2 and the reference solution is larger than the case of the stress intensity factors. The reduction in the accuracy results directly from the fact that the derivatives of the sum of the crack-face displacement are required in the calculation of the T-stress components.

6.2 Pair of identical vertical penny-shaped cracks

Consider, next, a pair of identical, vertical penny-shaped cracks of radius a embedded in a whole space with a depth h (measured from the center of each crack to the plane of symmetry) as illustrated in **Fig. 5(a)**. The lower crack front is parameterized by $x_1 = 0$, $x_2 = a \sin \beta$, $x_3 = h - a \cos \beta$ for $\beta \in [0, 2\pi]$. The equivalent cracked half-space subjected to symmetric boundary conditions is

shown in **Fig. 5(b)** and each crack is subjected to the uniformly distributed, self-equilibrated, normal tractions $t_1^+ = -t_1^- = -\sigma_0$, $t_2^+ = -t_2^- = 0$, $t_3^+ = -t_3^- = 0$ (see **Fig. 5(c)**). It is apparent from the symmetry of the crack-face loading about the plane $x_1 = 0$ that the mode-II and mode-III stress intensity factors identically vanish along the crack front. In the numerical study, four meshes of the penny-shaped crack are adopted as illustrated in **Fig. 5(d)** and the aspect ratio with $h/a = 1.25$ is considered.

The computed mode-I stress intensity factor and T-stress components for both isotropic and transversely isotropic materials are normalized by the reference solution (taken from the converged SGBEM solution of a cracked whole space shown in **Fig. 5(a)** without exploiting the symmetry) and then reported in **Figs. 6** and **7**. As can be seen from results in **Fig. 6**, the computed mode-I stress intensity factor shows very good agreement with the reference solution for all three meshes and both types of materials. Similarly, numerical solutions for the T-stress components shown in **Fig. 7** for both materials also indicate the good convergence behavior and the weak dependence on the level of mesh refinement. Evidently, results generated by the coarsest mesh, intermediate mesh and finest mesh are nearly indistinguishable from the benchmark solution. The high quality of the numerical solutions results directly, again, from the use of special crack-tip elements to capture the near-front relative crack-face displacement.

6.3 Vertical penny-shaped crack under non-uniform tractions

Consider, as a final example, a circular crack of radius a

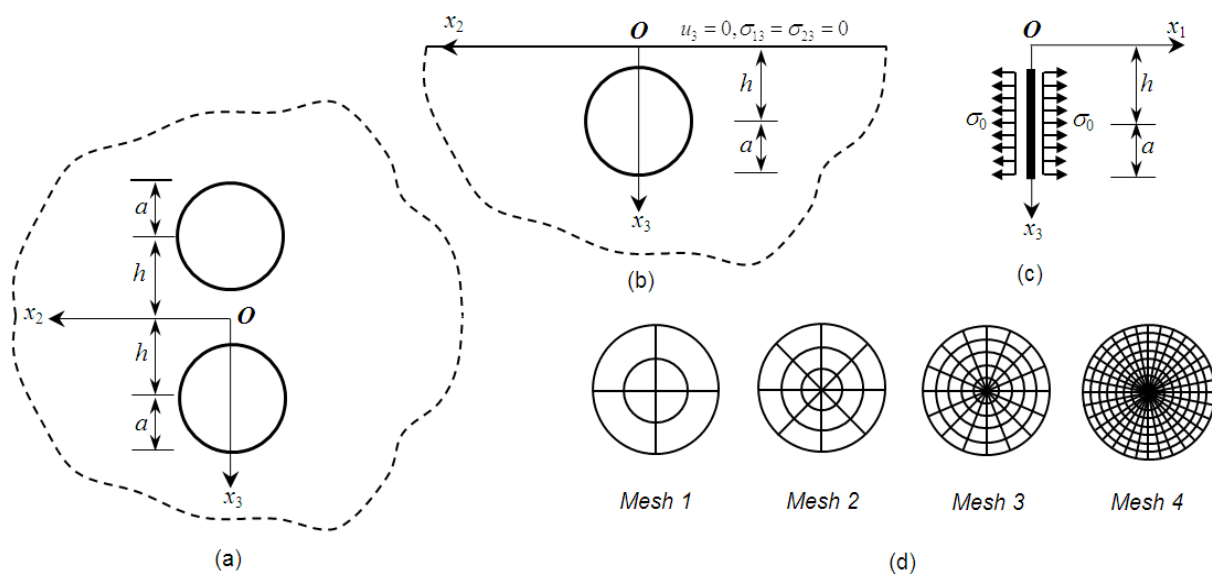


Fig. 5. Schematics of **(a)** pair of identical, vertical penny-shaped cracks in whole space, **(b)** equivalent cracked half-space, **(c)** tractions acting on crack surfaces, and **(d)** four meshes used in analysis.

embedded in a whole space and oriented perpendicular to the plane of symmetry, i.e., the plane $x_3 = 0$, as shown schematically in **Fig. 8(a)**. The equivalent cracked half-space under the symmetric boundary conditions used in the analysis is also illustrated in **Fig. 8(b)**. The corresponding semi-circular crack front is parameterized by $x_1 = -a \cos \beta$, $x_2 = 0$, $x_3 = a \sin \beta$ for $\beta \in [0, 2\pi]$. To demonstrate the capability of the proposed technique to treat general loading conditions, three types of symmetric, self-equilibrated, crack-face tractions including the non-uniform normal traction $t_1^+ = t_3^+ = 0, t_2^+ = -\sigma_0(x_3/a)^2$, the non-uniform horizontal shear traction $t_2^+ = t_3^+ = 0, t_1^+ = \sigma_0(x_3/a)^2$, and the non-uniform vertical shear traction $t_1^+ = t_2^+ = 0$,

$t_3^+ = \sigma_0(x_3/a)^2$ are treated (as indicated in **Fig. 9**). Three meshes as indicated in **Fig. 8(c)** are adopted in the analysis and the reference solution associated with the cracked whole-space shown in **Fig. 8(a)** without using the symmetry is obtained via the weakly singular SGBEM (Rungamornrat and Mear, 2008b).

For the first loading condition (see **Fig. 9(a)**), the mode-II and mode-III stress intensity factors vanish whereas the mode-I stress intensity factor and all T-stress components vary along the crack front. The computed mode-I stress intensity factor and T-stress components are reported in **Figs. 10** and **11**, respectively, for both isotropic and transversely isotropic materials. In

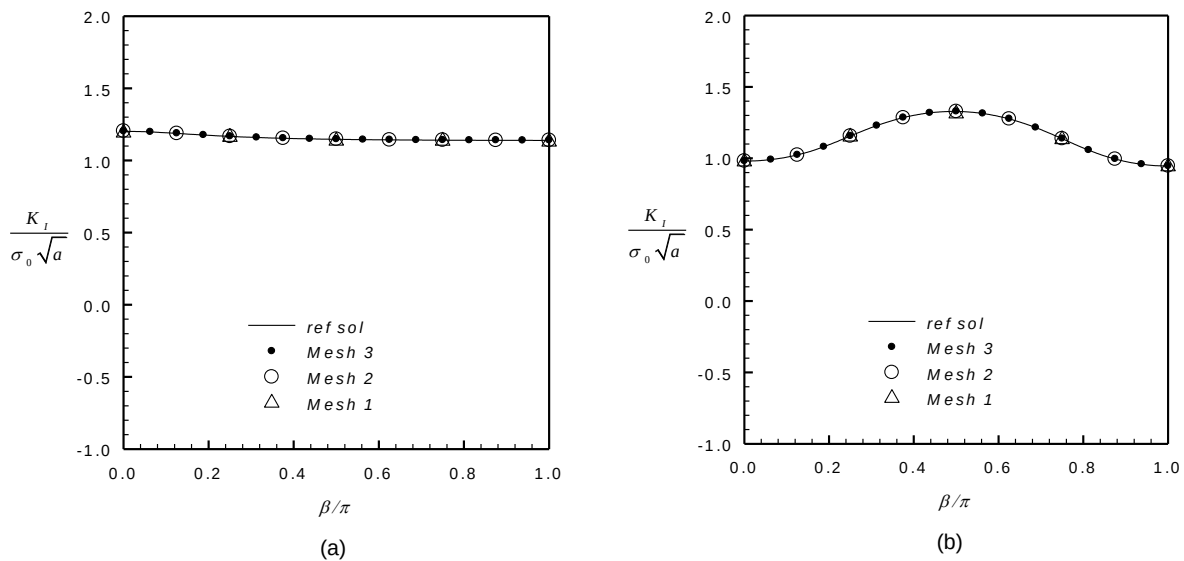


Fig. 6. Normalized mode-I stress intensity factors of lower penny-shaped crack under uniform normal traction for (a) isotropic case and (b) transversely isotropic case.

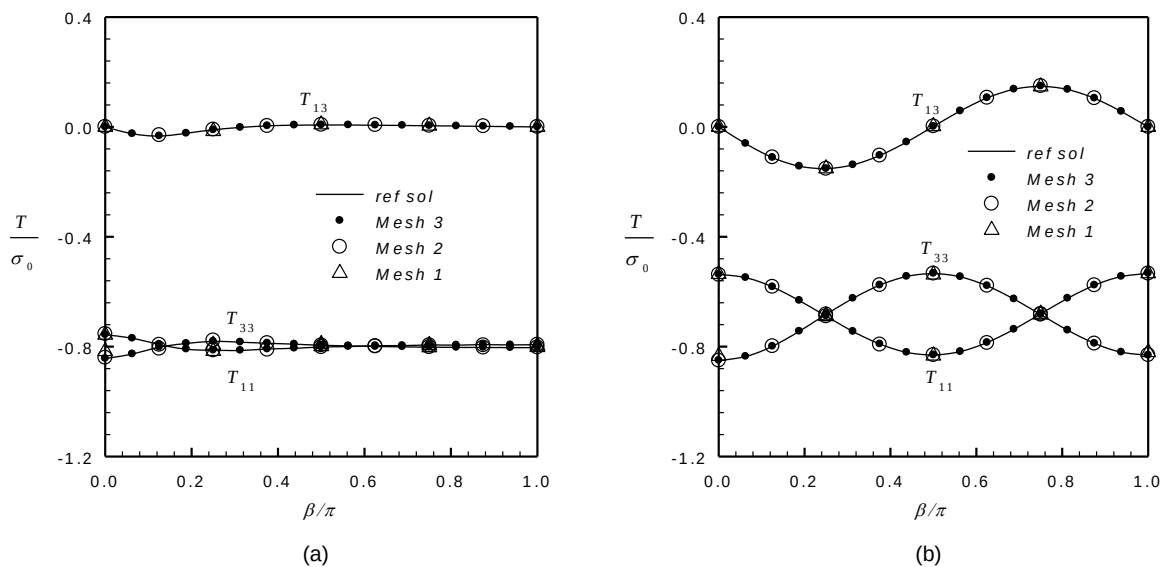


Fig. 7. Normalized T-stress components of lower penny-shaped crack under uniform normal traction for (a) isotropic case and (b) transversely isotropic case.

addition to the good agreement between the computed results and the reference solution and the weak dependence on the level of discretization, it can be also seen that the influence of the material anisotropy on the value and distribution of the mode-I stress intensity factor and all T-stress components along the crack front is not significant. For the second and third loading conditions, the applied shear traction in either x_1 – direction or x_3 – direction yields zero mode-I stress intensity factor and all zero T-stress components along the entire crack front. Computed mode-II and mode-III stress intensity factors of such shear loading conditions are reported in **Figs. 12** and **13** for all three meshes and both materials together with the reference solution. Similar to the previous case, this set of results indicates that the proposed technique with use of special crack-tip elements yields highly accurate stress intensity factors comparable to the reference solution for all meshes employed. In addition, use of the symmetry or the equivalent cracked half-space in the analysis can

significantly reduce the number of degrees of freedom in comparison with the treatment of the whole cracked space. In particular, a semi-circular crack is discretized instead of the entire penny-shaped crack.

7. Conclusions and remarks

A set of singularity-reduced integral representations has been established for the analysis of cracks in an anisotropic, linearly elastic, whole space possessing a plane of symmetry. The systematic regularization procedure based on the derivative transferring technique via the integration by parts and Stokes' theorem has been utilized along with special decompositions of strongly singular and hypersingular kernels to derive a pair of weakly singular, weak-form boundary integral equations governing the unknown crack-face data for the equivalent cracked half-spaces. Another key feature of the developed integral equations, in addition to the

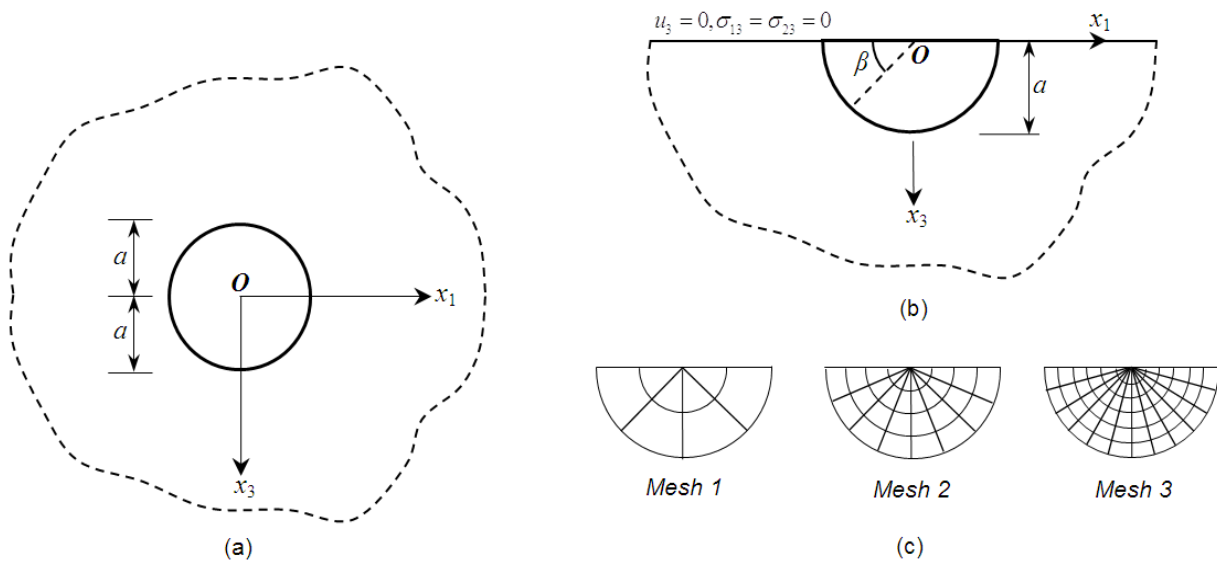


Fig. 8. Schematics of (a) vertical penny-shaped crack in elastic whole space, (b) equivalent cracked half-space, and (c) three meshes used in analysis.

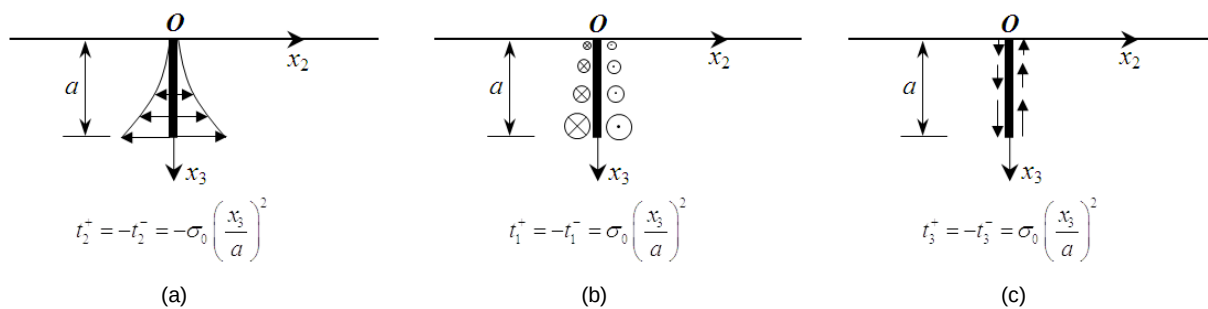


Fig. 9. Schematics of crack under (a) non-uniform normal traction $t_2^+ = t_2^- = -\sigma_0 (x_3/a)^2$, (b) non-uniform horizontal shear traction $t_1^+ = t_1^- = \sigma_0 (x_3/a)^2$, and (c) non-uniform vertical shear traction $t_3^+ = t_3^- = \sigma_0 (x_3/a)^2$.

weakly singular nature, is the automatic treatment of the existing symmetry via the use of fundamental solutions of the half-space under symmetric boundary conditions on the free surface. This therefore avoids the discretization of the free surface in the solution procedure.

The weakly singular symmetric Galerkin boundary element method (SGBEM) and standard Galerkin finite element technique have been successfully implemented to solve a pair of weakly-singular, weak-form traction boundary integral equations for the sum of and the jump in the crack-face displacement. Special crack-tip elements have been also employed in the solution discretization to enhance the approximation of the near-

front relative crack-face displacement. Essential fracture data along the crack front such as the stress intensity factors and the T-stress components has been directly post-processed from the solved data of the crack-face displacements using the explicit formula. Results from extensive numerical experiments and the comparison with several benchmarked cases have revealed that the proposed numerical procedure is highly accurate and computationally robust for the analysis of a cracked half-space under the symmetric boundary conditions. Use of the special crack-tip elements along the crack front has indicated that the stress intensity factors can be accurately captured using relatively very coarse meshes

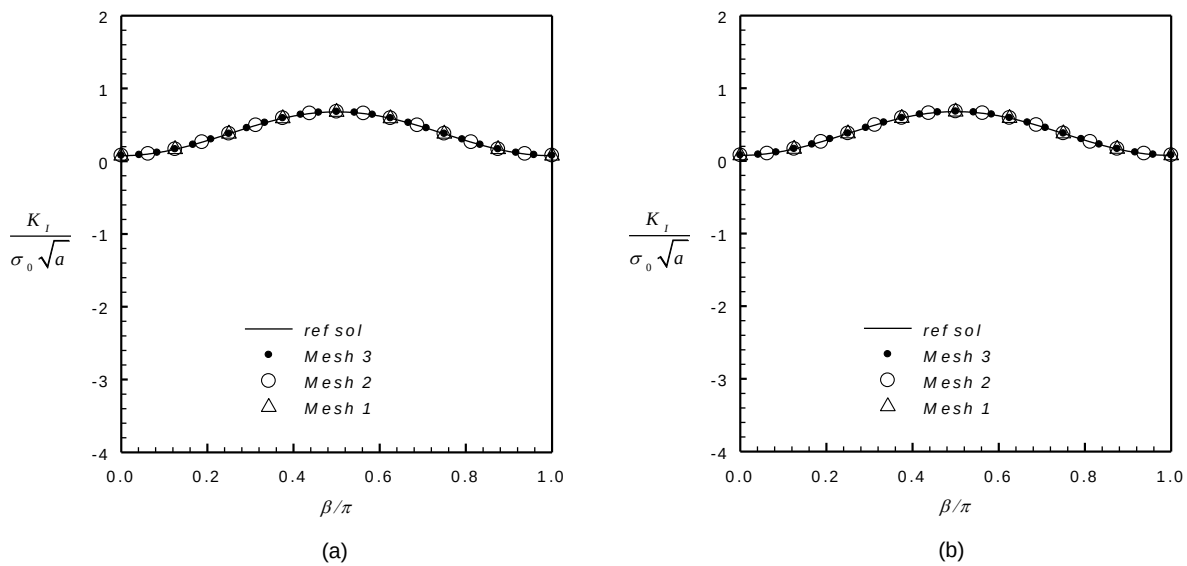


Fig. 10. Normalized mode-I stress intensity factor for vertical crack in whole space subjected to self-equilibrated, non-uniform normal traction $t_1^+ = t_3^+ = 0$, $t_2^+ = -\sigma_0(x_3/a)^2$ for (a) isotropic material and (b) transversely isotropic material.

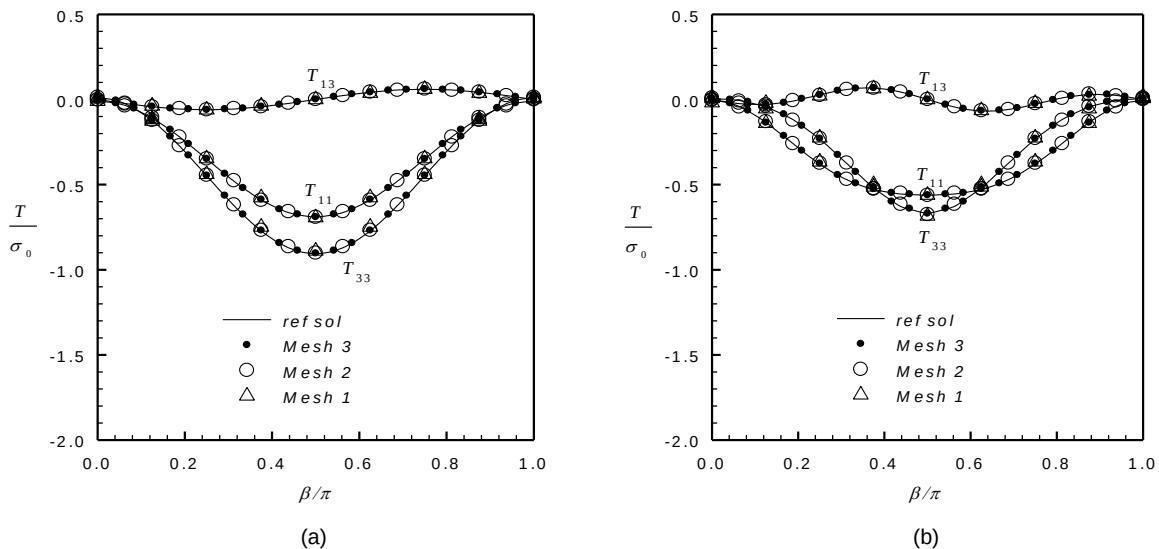


Fig. 11. Normalized T-stress components for vertical crack in whole space subjected to self-equilibrated, non-uniform normal traction $t_1^+ = t_3^+ = 0$, $t_2^+ = -\sigma_0(x_3/a)^2$ for (a) isotropic material and (b) transversely isotropic material.

and this therefore renders the technique more suitable for linear fracture analysis than the standard finite element method which generally requires sufficiently fine meshes to capture the near-front field and experiences difficulty in the treatment of an unbounded domain. While the proposed technique has been successfully implemented, it is still restricted to either a cracked whole-space possessing the plane of symmetry or a cracked half-space under symmetric boundary conditions. The potential extension of the current work to treat other types of boundary conditions such as the anti-symmetric,

traction-free and rigid boundary conditions and other types of materials such as multi-field and smart solids is considered essential. It is important to emphasize that besides the reduction of the computational cost directly gained from using the equivalent half-space model instead of the full treatment of a cracked whole space under a symmetric condition, the key ingredients and results established in the present study also form the useful and essential basis for further nontrivial generalization.

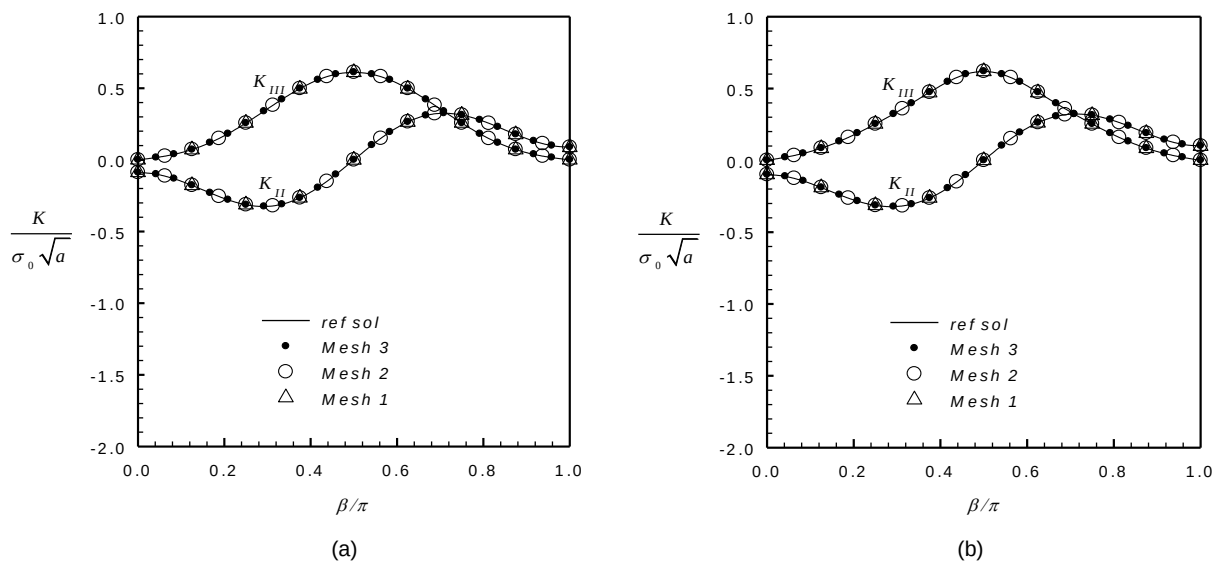


Fig. 12. Normalized mode-II and mode-III stress intensity factors for vertical crack in whole space subjected to self-equilibrated, non-uniform shear traction $t_2^+ = t_3^+ = 0, t_1^+ = \sigma_0(x_3/a)^2$ for **(a)** isotropic material and **(b)** transversely isotropic material.

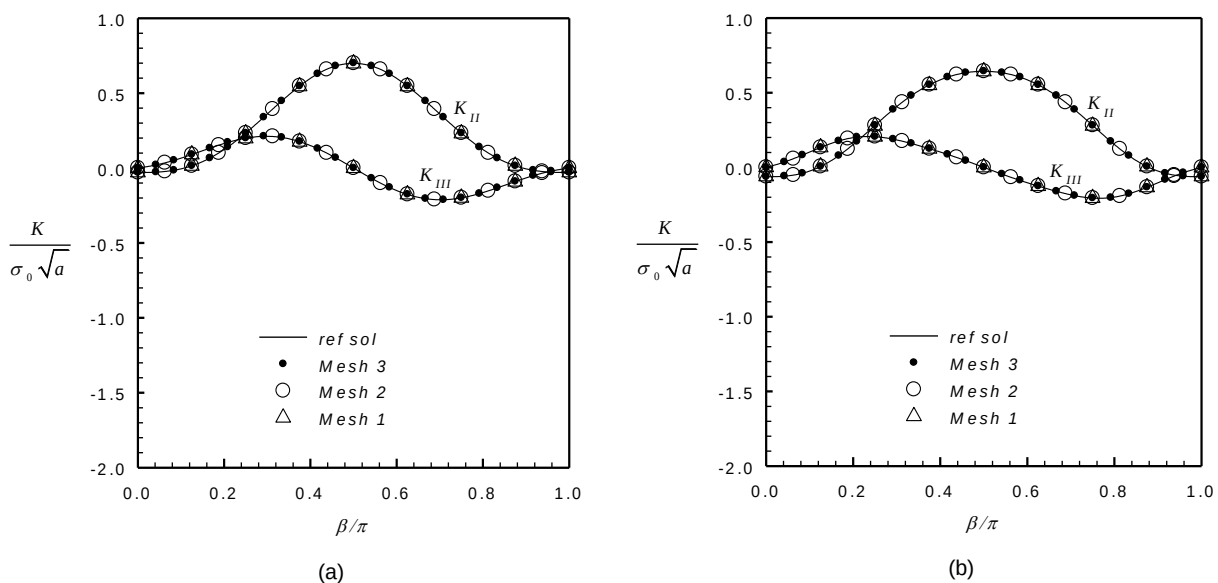


Fig. 13. Normalized mode-II and mode-III stress intensity factors for vertical crack in whole space subjected to self-equilibrated, non-uniform shear traction $t_1^+ = t_2^+ = 0, t_3^+ = \sigma_0(x_3/a)^2$ for **(a)** isotropic material and **(b)** transversely isotropic material.

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Symbols and abbreviations

a	Crack radius
BIEM	Boundary integral equation method
$\mathbf{a}, \mathbf{b}$	Orthonormal vectors
C_{mj}^{tk}, G_{mj}^p	Weakly singular kernels for whole space
$\bar{C}_{mj}^{tk}, \bar{G}_{mj}^p$	Weakly singular kernels for whole space under symmetric conditions
$D_m(\cdot)$	Surface differential operator
E_{ijkl}	Global components of elastic moduli
$\bar{E}_{ijkl}$	Local components of elastic moduli
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Orthonormal base vectors of global Cartesian coordinate system

$\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3$	Orthonormal base vectors of local Cartesian coordinate system	$\mathbf{x}$	Source point
FEM	Finite element method	$\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$	Local Cartesian coordinate system at $\mathbf{x}_c$
H_{mj}^p	Singular function of $\mathcal{O}(1/r^2)$	$\bar{\mathbf{x}}$	Image point of source point $\mathbf{x}$
$\bar{H}_{mj}^p$	Singular function of $\mathcal{O}(1/r^2)$	$\mathbf{z}$	Unit vector
K_I, K_{II}, K_{III}	Mode-I, Mode-II, Mode-III stress intensity factors	$\delta(\boldsymbol{\xi} - \mathbf{x})$	Dirac-delta distribution centered at $\mathbf{x}$
$\mathbf{n}^+, \mathbf{n}^-$	Outward unit normal vectors to crack surfaces	δ_{ij}	Standard Kronecker-delta symbol
$\{\mathbf{O}; x_1, x_2, x_3\}$	Global Cartesian coordinate system	$\bar{\delta}_{ij}$	Modified Kronecker-delta symbol
r	Distance between source and field points	ε_{ijk}	Standard alternating symbol
S_c^+, S_c^-	Geometrically identical crack surfaces	$\bar{\varepsilon}_{ij}$	Local components of finite part of strain tensor
S_{ij}^p	Stress fundamental solution for whole space	(ζ, η)	Master coordinates of point within the crack-tip element
T_{ij}	T-stress tensor	$\boldsymbol{\zeta}$	Field point
t_j^+, t_j^-	Tractions applied to crack surfaces	$\bar{\boldsymbol{\zeta}}$	Image point of field point $\boldsymbol{\zeta}$
$\tilde{t}_p, \tilde{u}_p$	Test functions	$\boldsymbol{\sigma}$	Stress field
U_j^p	Displacement fundamental solution for whole space	Σ_{ij}^{lk}	Hyper singular kernel of whole space
$\mathbf{u}$	Displacement field	Ω	Cracked whole space under symmetric condition

Journal Article No. 4

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Scaled boundary finite element method with exact defining curves for two-dimensional linear multi-field media

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ABSTRACT This paper presents an efficient and accurate numerical technique based upon the scaled boundary finite element method for the analysis of two-dimensional, linear, second-order, boundary value problems with a domain completely described by a circular defining curve. The scaled boundary finite element formulation is established in a general framework allowing single-field and multi-field problems, bounded and unbounded bodies, distributed body source, and general boundary conditions to be treated in a unified fashion. The conventional polar coordinates together with a properly selected scaling center are utilized to achieve the exact description of the circular defining curve, exact geometry of the domain, and exact spatial differential operators. Standard finite element shape functions are employed in the discretization of both trial and test functions in the circumferential direction and the resulting eigenproblem is solved by a selected efficient algorithm. The computational performance of the implemented procedure is then fully investigated for various scenarios to demonstrate the accuracy in comparison with standard linear elements.

KEYWORDS multi-field problems, defining curve, exact geometry, general boundary conditions, SBFEM

1 Introduction

A scaled boundary finite element method (SBFEM) has been well recognized, in the past three decades, as one of accurate and powerful solution procedures for analysis of various problems in applied mechanics, especially when involving domains are unbounded [1,2]. In this particular technique, the solution is mainly assumed in a separable form in terms of functions described by two sets of spatial coordinates commonly termed the scaling and boundary coordinates. A part of the solution depending on the boundary coordinate is approximated using available standard discretization procedures whereas that depending on the scaling coordinate is determined directly by solving a certain set of ordinary differential equations [1,3]. Due to the reduction of one spatial dimension in the discretization,

degrees of freedom are equivalently located at the domain boundary, therefore the method can be categorized as a particular type of boundary element methods. It is worth noting, however, that the formulation of the SBFEM contains no fundamental solution, and as a direct consequence, the non-trivial treatment of singular integrals is not required in comparison with boundary integral equation methods [4–6]. Such positive and attractive features render this particular technique suitable and computationally efficient when applied to various scenarios including problems related to potential fields, linear elasticity, electrostatics, fractures, elasto-plastic, wave propagation, and geo-mechanics applications [1,3,7–16]. The collection of earlier investigations indicating the history and background, the current advances, and the enhanced capability of the SBFEM is briefly summarized below.

The primitive formulation of the SBFEM was initially

proposed by Wolf and Song [2] in the dynamic analysis of linear elastic unbounded media. The complexity of their original formulation based upon the mechanical-based approach was later improved by applying either the standard weighted residual technique [17,18] or the well-known principle of virtual work [3,19]. While the SBFEM, in its early age, was developed specifically to treat special classes of problems involving unbounded media, its capability has been continuously enhanced and is recently far superior to its original version [7–11,16,20–25]. For instance, use of various families of interpolation functions, including both element-based and element-free bases, in the solution discretization and the corresponding computational performance was fully investigated [21–23,26]. Both h - and p -hierarchical adaptive procedures together with various refinement criteria were also integrated into the SBFEM to enhance the solution accuracy with optimal computational costs [1,24,25]. The capability of the SBFEM to solve linear elasticity problems associated with applied concentrated forces was achieved by directly incorporating the known fundamental solutions into the solution procedure [9]. Extension of the technique to investigate linear boundary value problems involving coupled-field materials such as piezoelectric solids has been also successfully developed [7,8,16,27,28]. In addition, applications of the SBFEM to solve problems involving embedded static and moving singularities such as the displacement discontinuities and their propagation has been increasingly found and its success and attractiveness stem directly from the fundamental characteristic of the technique to directly capture the order of singularity and behavior of the surrounding fields [8,16,29–31]. Besides linear boundary value problems, the SBFEM along with the polygon formulation was also applied successfully in the stress analysis of bodies made of elastoplastic materials [10]. It is known that the accuracy of the SBFEM relies mainly upon the discretization of both the geometry and solution. For relatively complex domains, it is generally required sufficiently fine meshes to achieve the accurate representation of the geometry and this is a non-trivial issue in the computer-aided designs. The concept of isogeometric analysis (IGA) has been recently employed to enhance the approximation of the geometry in the SBFEM (e.g., Ref. [32]). In addition, IGA has also been successfully integrated in the boundary integral equation methods for the analysis of both two- and three-dimensional problems in linear elasticity (e.g., Refs. [33,34]).

All aforementioned studies have demonstrated the significant progress of the SBFEM in the analysis of numerous engineering problems. However, the key formulation and implementations of the solution procedure were mainly established in a problem-dependent style. Treatment of various types of boundary value problems and all essential ingredients such as body loads, side-faces, and prescribed conditions based upon a unified or single formulation has not been well established. In addition, use

of the exact description of the defining curve to further improve the approximation of the domain geometry and its corresponding boundary has not been recognized. Standard C^0 linear finite elements have been commonly employed in the SBFEM and found to provide reasonably accurate results with relatively low number of degrees of freedom in the discretization when the domain boundary consists of piecewise linear segments. However, such low-order elements can be inefficient, in terms of degrees of freedom required, when applied to problems with circular or curved boundaries. Relatively fine meshes are generally required to accurately capture the domain geometry and the prescribed boundary conditions. In the present study, the SBFEM is formulated in a general framework allowing two-dimensional, linear, second-order, multi-field boundary value problems to be treated in a unified fashion. The exact defining curve for a circular arc is also established and integrated into the current implementation to investigate the quality of approximate solutions in comparison with those generated by simple linear elements. Results from the present study should, at least, provide some useful information and guidelines for choosing either exact or approximate defining curves in the SBFEM. Remaining sections of this paper are organized to cover the problem formulation, scaled boundary formulation, solution procedure, numerical results and discussion, and conclusion and remarks.

2 Problem formulation

Consider a two-dimensional body occupying a region $\Omega \in \mathbb{R}^2$ as shown schematically in Fig. 1. The region is assumed sufficiently smooth such that all involved mathematical operators such as differentiations and integrations are well defined on this region. In addition, the boundary of the body Ω , denoted by $\partial\Omega$, is assumed piecewise smooth and an outward unit normal vector at any smooth point on $\partial\Omega$

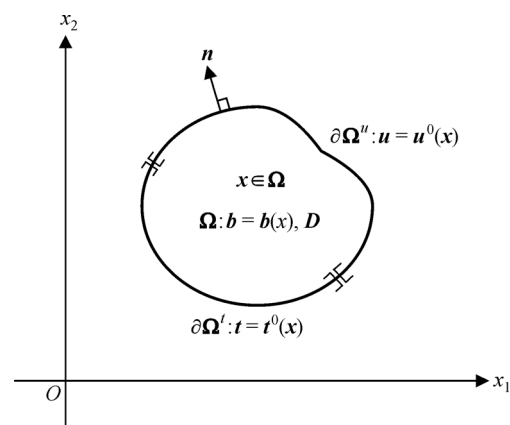


Fig. 1 Schematic of two-dimensional, multi-field body subjected to external excitations

is denoted by $\mathbf{n} = \{n_1 \ n_2\}^T$. The state of the body Ω is assumed to be completely described by the following three basic field quantities: The state variable denoted by a Λ -component vector $\mathbf{u} = \mathbf{u}(x)$, the state-variable gradient denoted by a 2Λ -component vector $\bar{\mathbf{e}} = \bar{\mathbf{e}}(x)$ and the body flux denoted by a 2Λ -component vector $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x)$ where Λ is any positive integer. The three field quantities $\mathbf{u}(x)$, $\bar{\mathbf{e}}(x)$ and $\boldsymbol{\sigma}(x)$ are governed by three linear field equations including the fundamental law of conservation, the constitutive law of materials, and the relation between the state variable and its measure of variation:

$$\mathbf{L}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}, \quad (1)$$

$$\boldsymbol{\sigma} = \mathbf{D} \bar{\mathbf{e}}, \quad (2)$$

$$\bar{\mathbf{e}} = \mathbf{L} \mathbf{u}, \quad (3)$$

where $\mathbf{L}$ represents the linear differential operator defined by

$$\mathbf{L} = \mathbf{L}_1 \frac{\partial}{\partial x_1} + \mathbf{L}_2 \frac{\partial}{\partial x_2}, \mathbf{L}_1 = \begin{bmatrix} \mathbf{I}_{\Lambda \times \Lambda} \\ \mathbf{0}_{\Lambda \times \Lambda} \end{bmatrix}, \mathbf{L}_2 = \begin{bmatrix} \mathbf{0}_{\Lambda \times \Lambda} \\ \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix}, \quad (4)$$

with $\partial/\partial x_1$ and $\partial/\partial x_2$ denoting the partial derivatives with respect to the coordinates x_1 and x_2 , respectively, and $\mathbf{I}_{\Lambda \times \Lambda}$ and $\mathbf{0}_{\Lambda \times \Lambda}$ standing for $\Lambda \times \Lambda$ identity- and zero-matrices, respectively, the superscript “T” denotes the matrix transpose operator, $\mathbf{b}$ is a Λ -component vector denoting the prescribed distributed body source, and $\mathbf{D}$ is a $2\Lambda \times 2\Lambda$ modulus matrix containing prescribed constants used for completely characterizing the behaviour of a constituting material of the body. At any smooth point $\mathbf{x} \in \partial\Omega$, the surface flux, denoted by a Λ -component vector $\mathbf{t} = \mathbf{t}(\mathbf{x})$, is linearly related to the body flux $\boldsymbol{\sigma}(\mathbf{x})$ by

$$\mathbf{t} = [n_1 \mathbf{I}_{\Lambda \times \Lambda} \quad n_2 \mathbf{I}_{\Lambda \times \Lambda}] \boldsymbol{\sigma}. \quad (5)$$

Based on the prescribed information on the boundary of the body, $\partial\Omega$ can be decomposed into a portion $\partial\Omega^u$ on which the state variable is fully prescribed (i.e., $\mathbf{u} = \mathbf{u}^0(x)$ where $\mathbf{u}^0$ is a given function) and a (complementary) portion $\partial\Omega^t$ on which the surface flux is prescribed (i.e., $\mathbf{t} = \mathbf{t}^0(x)$ where $\mathbf{t}^0$ is a given function). It should be clear from the form of the basic field Eqs. (1)–(3) and the body-surface-flux relation Eq. (5) that various types of boundary value problems can be readily treated as a special case by appropriately setting the value of the integer Λ and defining the modulus matrix $\mathbf{D}$, for instance, Laplace’s and Poisson’s equation ($\Lambda = 1$), linear elasticity ($\Lambda = 2$), linear piezoelectric and linear piezomagnetic problems ($\Lambda = 3$), and linear piezoelectromagnetic problems ($\Lambda = 4$) (see the work of Chung [27]).

The weak formulation of the above governing field equations can be readily established via the following standard procedure: The standard weighted residual

technique is first applied to the law of conservation Eq. (1); the term involving the operator $\mathbf{L}$ is then integrated by parts via Gauss-divergence theorem; and finally, the constitutive law Eq. (2) and the relation Eq. (3) are directly employed upon substitution. The final weak-form equation in terms of the state variable is given by

$$\int_{\Omega} (\mathbf{L}\mathbf{w})^T \mathbf{D}(\mathbf{L}\mathbf{u}) dA = \int_{\partial\Omega} \mathbf{w}^T \mathbf{t} dl + \int_{\Omega} \mathbf{w}^T \mathbf{b} dA, \quad (6)$$

where $\mathbf{w}$ is a Λ -component vector of sufficiently smooth weight or test functions. To ensure the integrability of all involved integrals in Eq. (6), the test functions $\mathbf{w}$ must satisfy the following condition:

$$\int_{\Omega} [(\mathbf{L}\mathbf{w})^T (\mathbf{L}\mathbf{w}) + \mathbf{w}^T \mathbf{w}] dA < \infty. \quad (7)$$

3 Scaled boundary formulation

To introduce the scaled boundary coordinate transformation in $\mathbb{R}^2$, a pair $(\mathbf{x}_0, C)$ is first selected where $\mathbf{x}_0 = (x_{10}, x_{20}) \in \mathbb{R}^2$ is termed a scaling center and $C \in \mathbb{R}^2$ is termed a defining curve. The defining curve can be parameterized by $(x_{10} + \hat{x}_1(s), x_{20} + \hat{x}_2(s))$, $s \in [s_1, s_2]$ where s is termed the boundary coordinate and $\hat{x}_1$ and $\hat{x}_2$ are prescribed functions of s . Any point $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ can be now related to the scaling center $\mathbf{x}_0$ and the defining curve C via the following coordinate transformation:

$$x_{\alpha} = x_{\alpha 0} + \xi \hat{x}_{\alpha}(s), \quad \alpha = 1, 2. \quad (8)$$

A choice of the scaling center and the defining curve must ensure that the transformation Eq. (8) is one-to-one. By applying the coordinate transformation Eq. (8), the first-order linear differential operator $\mathbf{L}$ given by Eq. (4) can be expressed in terms of partial derivatives with respect to the scaled boundary coordinates ξ and s by

$$\mathbf{L} = \mathbf{b}_1 \frac{\partial}{\partial \xi} + \mathbf{b}_2 \frac{1}{\xi} \frac{\partial}{\partial s}, \quad (9)$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are $2\Lambda \times \Lambda$ -matrices defined by

$$\mathbf{b}_1 = \frac{1}{J} \begin{bmatrix} \frac{d\hat{x}_2}{ds} \mathbf{I}_{\Lambda \times \Lambda} \\ -\frac{d\hat{x}_1}{ds} \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix}, \quad \mathbf{b}_2 = \frac{1}{J} \begin{bmatrix} -\hat{x}_2 \mathbf{I}_{\Lambda \times \Lambda} \\ \hat{x}_1 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix}, \quad (10)$$

with $J = \hat{x}_1(d\hat{x}_2/ds) - \hat{x}_2(d\hat{x}_1/ds)$. To provide the complete description of the body Ω in terms of the scaled boundary coordinates ξ and s , pairs of scaling center and defining curve must be properly chosen and such choices are generally not unique and depends primarily on the geometry of the body itself. In the present study, we focus

mainly on the body Ω that can be fully described by a single pair of $(\mathbf{x}_0, C)$. The treatment of bodies with multiple pairs of $(\mathbf{x}_0, C)$ can be readily extended using the sub-domain technique (e.g., Refs. [35,36]). Now, let any point $\mathbf{x} \in \Omega$ be fully described by the coordinate transformation Eq. (8) for $(\zeta, s) \in [\zeta_1, \zeta_2] \times [s_1, s_2]$. Portions of the boundaries associated with $\zeta = \zeta_1$ and $\zeta = \zeta_2$ are termed the inner and outer boundaries, respectively. If the defining curve C is not a closed curve (i.e., $\hat{\mathbf{x}}(s_1) \neq \hat{\mathbf{x}}(s_2)$), the body is said to be opened and portions of its boundary associated with $s = s_1$ and $s = s_2$ are termed the side-faces; otherwise, the body is closed and contains no side-face. If ζ_2 is finite, the body is bounded; otherwise, it is unbounded (i.e., $\zeta_2 \rightarrow \infty$). The body contains the scaling center if and only if $\zeta_1 = 0$.

3.1 Solution discretization

The defining curve C is first discretized into n finite elements and m nodes. The state variable $\mathbf{u}$ and the test function $\mathbf{w}$ can be approximated by

$$\mathbf{u}^h = \mathbf{u}^h(\zeta, s) = \sum_{i=1}^m \phi_{(i)}(s) \mathbf{u}_{(i)}^h(\zeta) = \mathbf{N}(s) \mathbf{U}^h(\zeta), \quad (11)$$

$$\mathbf{w}^h = \mathbf{w}^h(\zeta, s) = \sum_{i=1}^m \phi_{(i)}(s) \mathbf{w}_{(i)}^h(\zeta) = \mathbf{N}(s) \mathbf{W}^h(\zeta), \quad (12)$$

where the superscript “ h ” is employed, here and in what follows, to designate approximate quantities, $\phi_{(i)}(s)$ denotes the i th nodal basis function, $\mathbf{u}_{(i)}^h(\zeta)$ denotes the i th unknown nodal function of the state variable, $\mathbf{w}_{(i)}^h(\zeta)$ denotes the i th arbitrary nodal function, $\mathbf{N}(s)$ denotes a matrix containing $\phi_{(i)}(s)$, and $\mathbf{U}^h(\zeta)$ and $\mathbf{W}^h(\zeta)$ denote vector containing nodal functions $\mathbf{u}_{(i)}^h(\zeta)$ and $\mathbf{w}_{(i)}^h(\zeta)$, respectively. By employing Eqs. (2), (3), (9), (11) and (12), the body flux σ and $\mathbf{L}\mathbf{w}$ can be approximated by

$$\sigma^h = \sigma^h(\zeta, s) = \mathbf{D}(\mathbf{L}^h \mathbf{u}^h) = \mathbf{D} \left(\mathbf{B}_1 \mathbf{U}_{,\zeta}^h + \frac{1}{\zeta} \mathbf{B}_2 \mathbf{U}^h \right), \quad (13)$$

$$\mathbf{L}^h \mathbf{w}^h = \mathbf{L}^h \mathbf{w}^h(\zeta, s) = \left(\mathbf{B}_1 \mathbf{W}_{,\zeta}^h + \frac{1}{\zeta} \mathbf{B}_2 \mathbf{W}^h \right), \quad (14)$$

where $(\cdot)_{,\zeta}$ denotes the derivative with respect to the scaling coordinate ζ and the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are given by

$$\mathbf{B}_1 = \mathbf{b}_1^h \mathbf{N}, \quad (15)$$

$$\mathbf{B}_2 = \mathbf{b}_2^h \mathbf{B} = \mathbf{b}_2^h d\mathbf{N}/ds. \quad (16)$$

3.2 Scaled boundary finite element equations

To establish a set of scaled boundary finite element equations for a general, two-dimensional body Ω as shown schematically in Fig. 2, let us assume that the boundary of the domain $\partial\Omega$ consists of four parts resulting from the scaled boundary coordinate transformation with the scaling center $\mathbf{x}_0$, the defining curve C , and $(\zeta, s) \in [\zeta_1, \zeta_2] \times [s_1, s_2]$: The inner boundary $\partial\Omega_1$ (associated with $\zeta = \zeta_1$), the outer boundary $\partial\Omega_2$ (associated with $\zeta = \zeta_2$), the side-face-1 $\partial\Omega_{s1}$ (associated with $s = s_1$), and the side-face-2 $\partial\Omega_{s2}$ (associated with $s = s_2$). As a result of such boundary partition $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_{s1} \cup \partial\Omega_{s2}$ and the coordinate transformation Eq. (8), the weak-form Eq. (6) becomes

$$\begin{aligned} & \int_{s_1}^{s_2} \int_{\zeta_1}^{\zeta_2} (\mathbf{L}\mathbf{w})^T \mathbf{D}(\mathbf{L}\mathbf{u}) J \zeta d\zeta ds \\ &= \int_{s_1}^{s_2} \mathbf{w}_1^T \mathbf{t}_1(s) J_s(s) \zeta_1 ds + \int_{s_1}^{s_2} \mathbf{w}_2^T \mathbf{t}_2(s) J_s(s) \zeta_2 ds \\ &+ \int_{\zeta_1}^{\zeta_2} (\mathbf{w}_{s1})^T \mathbf{t}_{s1}(\zeta) J_{\zeta 1} d\zeta + \int_{\zeta_1}^{\zeta_2} (\mathbf{w}_{s2})^T \mathbf{t}_{s2}(\zeta) J_{\zeta 2} d\zeta \\ &+ \int_{s_1}^{s_2} \int_{\zeta_1}^{\zeta_2} \mathbf{w}^T \mathbf{b} J \zeta d\zeta ds, \end{aligned} \quad (17)$$

where $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_{s1}$, $\mathbf{w}_{s2}$ are restrictions of the test function on the boundaries $\partial\Omega_1$, $\partial\Omega_2$, $\partial\Omega_{s1}$, $\partial\Omega_{s2}$, respectively, $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_{s1}$, $\mathbf{t}_{s2}$ are surface fluxes on the boundaries $\partial\Omega_1$, $\partial\Omega_2$, $\partial\Omega_{s1}$, $\partial\Omega_{s2}$, respectively, and $J_{\zeta 1} = \sqrt{\hat{x}_1^2(s_1) + \hat{x}_2^2(s_1)}$, $J_{\zeta 2} = \sqrt{\hat{x}_1^2(s_2) + \hat{x}_2^2(s_2)}$ and $J_s(s) = \sqrt{(d\hat{x}_1/ds)^2 + (d\hat{x}_2/ds)^2}$. By substituting the approximations Eqs. (11)–(14) into Eq. (17), then integrating the integral on the left hand side

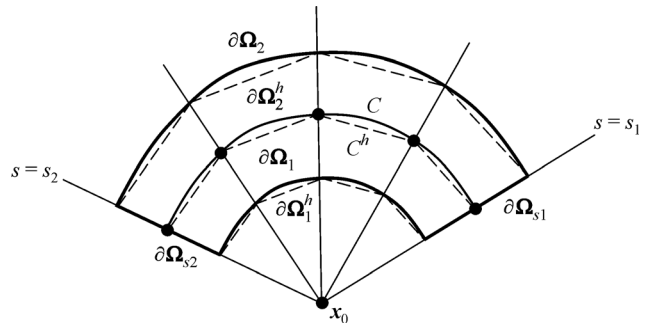


Fig. 2 Schematic of generic body Ω , corresponding boundary, and its approximation Ω^h

by parts, and finally carrying out all involved matrix algebra, it leads to

$$\begin{aligned} & \int_{\xi_1}^{\xi_2} (\mathbf{W}^h)^T \left[-\xi \mathbf{E}_0 \mathbf{U}_{,\xi\xi}^h + (\mathbf{E}_1 - \mathbf{E}_1^T - \mathbf{E}_0) \mathbf{U}_{,\xi}^h \right. \\ & \left. + \frac{1}{\xi} \mathbf{E}_2 \mathbf{U}^h - \mathbf{F}^t - \xi \mathbf{F}^b \right] d\xi \\ & + \left(\mathbf{W}^h(\xi_2) \right)^T [(\xi \mathbf{E}_0 \mathbf{U}_{,\xi}^h + \mathbf{E}_1^T \mathbf{U}^h)_{\xi=\xi_2} - \mathbf{P}_2] \\ & - \left(\mathbf{W}^h(\xi_1) \right)^T [(\xi \mathbf{E}_0 \mathbf{U}_{,\xi}^h + \mathbf{E}_1^T \mathbf{U}^h)_{\xi=\xi_1} + \mathbf{P}_1] = 0, \end{aligned} \quad (18)$$

where the matrices $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, and all involved quantities are defined by

$$\begin{aligned} \mathbf{E}_0 &= \int_{s_1}^{s_2} \mathbf{B}_1^T \mathbf{D} \mathbf{B}_1 J^h ds, \quad \mathbf{E}_1 = \int_{s_1}^{s_2} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_1 J^h ds, \\ \mathbf{E}_2 &= \int_{s_1}^{s_2} \mathbf{B}_2^T \mathbf{D} \mathbf{B}_2 J^h ds, \end{aligned} \quad (19)$$

$$\begin{aligned} \mathbf{P}_1 &= \xi_1 \int_{s_1}^{s_2} (\mathbf{N}^S)^T \mathbf{t}_1(s) J_s^h(s) ds, \\ \mathbf{P}_2 &= \xi_2 \int_{s_1}^{s_2} (\mathbf{N}^S)^T \mathbf{t}_2(s) J_s^h(s) ds, \end{aligned} \quad (20)$$

$$\mathbf{F}_1^t = (\mathbf{N}_1^S)^T \mathbf{t}_{s1}(\xi) J_{\xi_1}^h, \quad \mathbf{F}_2^t = (\mathbf{N}_2^S)^T \mathbf{t}_{s2}(\xi) J_{\xi_2}^h, \quad (21)$$

$$\mathbf{F}^b = \mathbf{F}^b(\xi) = \int_{s_1}^{s_2} \mathbf{N}^T(s) \mathbf{b}(\xi, s) J^h ds. \quad (22)$$

From the arbitrariness of the nodal functions $\mathbf{W}^h$, it can be deduced from Eq. (18) that

$$\begin{aligned} & \xi^2 \mathbf{E}_0 \mathbf{U}_{,\xi\xi}^h + \xi (\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1) \mathbf{U}_{,\xi}^h - \mathbf{E}_2 \mathbf{U}^h + \xi \mathbf{F}^t + \xi^2 \mathbf{F}^b = \mathbf{0}, \\ & \forall \xi \in (\xi_1, \xi_2), \end{aligned} \quad (23)$$

$$\mathbf{Q}^h(\xi_1) = -\mathbf{P}_1, \quad (24)$$

$$\mathbf{Q}^h(\xi_2) = \mathbf{P}_2, \quad (25)$$

where the vector $\mathbf{Q}^h = \mathbf{Q}^h(\xi)$ known as the nodal internal flux is defined by

$$\mathbf{Q}^h(\xi) = \xi \mathbf{E}_0 \mathbf{U}_{,\xi}^h + \mathbf{E}_1^T \mathbf{U}^h. \quad (26)$$

Equations (23)–(25) form a set of the so-called scaled boundary finite element equations governing the function $\mathbf{U}^h = \mathbf{U}^h(\xi)$. In particular, Eq. (23) is a system of linear, second-order, nonhomogeneous, ordinary differential equations with respect to the scaling coordinate ξ whereas Eqs. (24) and (25) correspond to the boundary conditions prescribed on the inner and outer boundaries of the body.

3.3 Treatment of prescribed conditions on side-faces

For bodies containing side-faces, the treatment of prescribed conditions on those boundaries can be achieved by partitioning the vector $\mathbf{U}^h$ into $\{\mathbf{U}^{hu} \quad \mathbf{U}^{hc}\}^T$ where $\mathbf{U}^{hu}$ denotes a vector containing only the unknown nodal functions and $\mathbf{U}^{hc}$ denotes a vector containing known nodal functions associated with the prescribed state variable on the side-faces. By partitioning all other involved quantities in a consistent fashion and then partitioning the system of Eq. (23) and the boundary conditions Eqs. (24) and (25), it finally leads to the scaled boundary finite element equations for the unknown nodal functions $\mathbf{U}^{hu}$:

$$\begin{aligned} & \xi^2 \mathbf{E}_0^{uu} \mathbf{U}_{,\xi\xi}^{hu} + \xi [\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu}] \mathbf{U}_{,\xi}^{hu} - \mathbf{E}_2^{uu} \mathbf{U}^{hu} \\ & = -\xi \mathbf{F}^{tu} - \xi^2 \mathbf{F}^{bu} - \mathbf{F}^{suu}, \end{aligned} \quad (27)$$

$$\mathbf{Q}^{hu}(\xi_1) = -\mathbf{P}_1^u, \quad (28)$$

$$\mathbf{Q}^{hu}(\xi_2) = \mathbf{P}_2^u, \quad (29)$$

where the function $\mathbf{F}^{suu}$ is defined by

$$\mathbf{F}^{suu} = \xi^2 \mathbf{E}_0^{uc} \mathbf{U}_{,\xi\xi}^{hc} + \xi [\mathbf{E}_0^{uc} + (\mathbf{E}_1^{cu})^T - \mathbf{E}_1^{uc}] \mathbf{U}_{,\xi}^{hc} - \mathbf{E}_2^{uc} \mathbf{U}^{hc}, \quad (30)$$

and $\{\mathbf{E}_0^{uu}, \mathbf{E}_0^{uc}\}$, $\{\mathbf{E}_1^{uu}, \mathbf{E}_1^{uc}\}$, $\{\mathbf{E}_2^{uu}, \mathbf{E}_2^{uc}\}$, $\mathbf{F}^{tu}$, $\mathbf{F}^{bu}$, $\mathbf{Q}^{hu}$, $\mathbf{P}_1^u$, $\mathbf{P}_2^u$ result from the partition of $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{F}^t$, $\mathbf{F}^b$, $\mathbf{Q}^h$, $\mathbf{P}_1$, $\mathbf{P}_2$, respectively (also see the work of Chung [27]). It should be remarked that the right-hand side of Eq. (27) now involves only the prescribed conditions on the side-faces and the distributed body source.

4 Solution procedure

A homogeneous solution of Eq. (27), denoted by $\mathbf{U}_0^{hu}$, can be readily obtained by solving a system of linear, second-order, Euler-Cauchy differential equations:

$$\xi^2 \mathbf{E}_0^{uu} \mathbf{U}_{0,\xi\xi}^{hu} + \xi [\mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T - \mathbf{E}_1^{uu}] \mathbf{U}_{0,\xi}^{hu} - \mathbf{E}_2^{uu} \mathbf{U}_0^{hu} = \mathbf{0}, \quad (31)$$

and it takes the following form:

$$\mathbf{U}_0^{hu}(\xi) = \sum_{i=1}^{2N} c_i \xi^{\lambda_i} \boldsymbol{\psi}_i, \quad (32)$$

where N is the number of equations in the system Eq. (31), $(\lambda_i, \boldsymbol{\psi}_i)$ denote a scaling factor and a vector representing the state variable of the i th mode, and c_i is an unknown constant denoting the contribution of the i th mode to the solution. The nodal internal flux generated by $\mathbf{U}_0^{hu}$, denoted by $\mathbf{Q}_0^{hu}$, is obtained by substituting Eq. (32) into $\mathbf{Q}^{hu}$ as

$$\mathbf{Q}_0^{hu}(\xi) = \sum_{i=1}^{2N} c_i \xi^{\lambda_i} \mathbf{q}_i, \quad (33)$$

where $\mathbf{q}_i = [\lambda_i \mathbf{E}_0^{uu} + (\mathbf{E}_1^{uu})^T] \boldsymbol{\psi}_i$ is termed the i th modal internal flux. A pair $(\lambda_i, \mathbf{X}_i)$ with $\mathbf{X}_i = \{\boldsymbol{\psi}_i \quad \mathbf{q}_i\}^T$ can be obtained by solving the corresponding linear eigenvalue problem:

$$\mathbf{A} \mathbf{X}_i = \lambda_i \mathbf{X}_i, \quad (34)$$

where the coefficient matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{bmatrix} -(\mathbf{E}_0^{uu})^{-1}(\mathbf{E}_1^{uu})^T & (\mathbf{E}_0^{uu})^{-1} \\ \mathbf{E}_2^{uu} - \mathbf{E}_1^{uu}(\mathbf{E}_0^{uu})^{-1}(\mathbf{E}_1^{uu})^T & \mathbf{E}_1^{uu}(\mathbf{E}_0^{uu})^{-1} \end{bmatrix}. \quad (35)$$

For further use in the solution procedure of different types of bodies, the homogeneous solution for both $\mathbf{U}_0^{hu}$ and $\mathbf{Q}_0^{hu}$ can be re-expressed as

$$\mathbf{U}_0^{hu}(\xi) = \boldsymbol{\Phi}^{\psi+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{\psi-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^-, \quad (36)$$

$$\mathbf{Q}_0^{hu}(\xi) = \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^-, \quad (37)$$

where the superscripts “+” and “−” are used to designate quantities associated with λ_i possessing positive and negative real parts, respectively, $\boldsymbol{\Phi}^{\psi+}$ and $\boldsymbol{\Phi}^{\psi-}$ are matrices with each column containing a vector $\boldsymbol{\psi}_i$, $\boldsymbol{\Phi}^{q+}$ and $\boldsymbol{\Phi}^{q-}$ are matrices with each column containing a vector $\mathbf{q}_i$, $\boldsymbol{\Pi}^+$ and $\boldsymbol{\Pi}^-$ are diagonal matrices with each diagonal entries containing a function ξ^{λ_i} , and $\mathbf{C}^+$ and $\mathbf{C}^-$ are vectors containing unknown constants representing the contribution of each mode. Note in particular that the order of columns and entries of involved matrices and vectors in Eqs. (36) and (37) follows consistently the eigenvalues λ_i . For bodies containing the scaling center, it should be apparent that $\boldsymbol{\Pi}^-$ becomes infinite at $\xi = 0$ and the vector $\mathbf{C}^-$ must vanish to maintain the boundedness of the solution at the scaling center. Similarly, for unbounded bodies, the vector $\mathbf{C}^+$ must be chosen equal to zero if the solution is expected to be bounded at infinity.

A particular solution of the system Eq. (27) associated with the distributed body source and the prescribed boundary conditions on the side-faces, denoted by $\mathbf{U}_1^{hu}$, can be readily established from existing standard proce-

dures such as the method of undetermined coefficients. Once the particular solution $\mathbf{U}_1^{hu}$ is obtained, the corresponding particular nodal internal flux $\mathbf{Q}_1^{hu}$ can be directly calculated in the same fashion as $\mathbf{Q}_0^{hu}$. Finally, the general solution of Eq. (27) and the corresponding nodal internal flux are given by

$$\begin{aligned} \mathbf{U}^{hu}(\xi) &= \mathbf{U}_0^{hu}(\xi) + \mathbf{U}_1^{hu}(\xi) \\ &= \boldsymbol{\Phi}^{\psi+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{\psi-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- \\ &\quad + \mathbf{U}_1^{hu}(\xi), \end{aligned} \quad (38)$$

$$\begin{aligned} \mathbf{Q}^{hu}(\xi) &= \mathbf{Q}_0^{hu}(\xi) + \mathbf{Q}_1^{hu}(\xi) \\ &= \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi) \mathbf{C}^+ + \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi) \mathbf{C}^- \\ &\quad + \mathbf{Q}_1^{hu}(\xi). \end{aligned} \quad (39)$$

By enforcing the boundary conditions on both inner and outer boundaries along with Eqs. (28), (29) and (39), the unknown constants $\mathbf{C}^+$ and $\mathbf{C}^-$ can be obtained as

$$\begin{aligned} \begin{Bmatrix} \mathbf{C}^+ \\ \mathbf{C}^- \end{Bmatrix} &= \begin{bmatrix} \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix}^{-1} \\ &\quad \cdot \left(\begin{Bmatrix} -\mathbf{P}_1^u \\ \mathbf{P}_2^u \end{Bmatrix} - \begin{Bmatrix} \mathbf{Q}_1^{hu}(\xi_1) \\ \mathbf{Q}_1^{hu}(\xi_2) \end{Bmatrix} \right). \end{aligned} \quad (40)$$

By substituting Eq. (40) into Eq. (38) and then evaluating the result at $\xi = \xi_1$ and $\xi = \xi_2$, it finally yields

$$\begin{aligned} \mathbf{K} \begin{Bmatrix} \mathbf{U}^{hu}(\xi_1) \\ \mathbf{U}^{hu}(\xi_2) \end{Bmatrix} &= \begin{Bmatrix} -\mathbf{P}_1^u \\ \mathbf{P}_2^u \end{Bmatrix} + \mathbf{K} \begin{Bmatrix} \mathbf{U}_1^{hu}(\xi_1) \\ \mathbf{U}_1^{hu}(\xi_2) \end{Bmatrix} - \begin{Bmatrix} \mathbf{Q}_1^{hu}(\xi_1) \\ \mathbf{Q}_1^{hu}(\xi_2) \end{Bmatrix}, \end{aligned} \quad (41)$$

where the coefficient matrix $\mathbf{K}$ is given by

$$\begin{aligned} \mathbf{K} &= \begin{bmatrix} \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{q+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{q-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \boldsymbol{\Phi}^{\psi+} \boldsymbol{\Pi}^+(\xi_1) & \boldsymbol{\Phi}^{\psi-} \boldsymbol{\Pi}^-(\xi_1) \\ \boldsymbol{\Phi}^{\psi+} \boldsymbol{\Pi}^+(\xi_2) & \boldsymbol{\Phi}^{\psi-} \boldsymbol{\Pi}^-(\xi_2) \end{bmatrix}^{-1}. \end{aligned} \quad (42)$$

By applying the prescribed conditions on both inner and outer boundaries to Eq. (41), all involved unknowns on those boundaries can be determined. Once such unknowns are solved, the constants $\mathbf{C}^+$ and $\mathbf{C}^-$ are computed from Eq. (40), the unknown nodal functions contained in $\mathbf{U}^{hu}$ can be determined from Eq. (38), and the state variable and

the body flux at any point within the body can be obtained from Eqs. (11) and (13), respectively. The relative error of the scaled boundary finite element solution is defined by

$$\text{Error} = \frac{\|\mathbf{e}\|_{L^2}}{\|\mathbf{u}^{\text{exact}}\|_{L^2}} \times 100\%, \quad (43)$$

where $\mathbf{e} = \mathbf{u}^{\text{exact}} - \mathbf{u}^h$ denotes the error function, $\mathbf{u}^{\text{exact}}$ is an exact or reference solution, and $\|\mathbf{f}\|_{L^2}$ denotes the L_2 -norm of a vector-value function $\mathbf{f}$ given by

$$\|\mathbf{f}\|_{L^2} = \sqrt{\int_{\Omega} \mathbf{f}^T \mathbf{f} dA}. \quad (44)$$

5 Description of linear straight and circular arc elements

In this section, the description of two types of elements used in the present study to describe the domain geometry and to discretize the solution in the boundary direction, one associated with a 2-node isoparametric linear element and the other corresponding to a 2-node circular-arc element (Fig. 3), is briefly presented. For the 2-node isoparametric linear element, the shape functions used in the discretization of the defining curve and the solution are given explicitly by

$$\varphi_1(s) = (1-s)/2, \quad \varphi_2(s) = (1+s)/2, \quad (45)$$

where $s \in [-1, 1]$ denotes the local or element boundary coordinate. The approximate defining curve of this particular element is parameterized by

$$\mathbf{x}_{\alpha}^h(s) = \mathbf{x}_{\alpha 0} + \hat{x}_{\alpha 1} \varphi_1(s) + \hat{x}_{\alpha 2} \varphi_2(s), \quad \alpha = 1, 2, \quad (46)$$

where $\mathbf{x}_1 = (x_{10} + \hat{x}_{11}, x_{20} + \hat{x}_{21})$ and $\mathbf{x}_2 = (x_{10} + \hat{x}_{12}, x_{20} + \hat{x}_{22})$ are physical coordinates of the two nodes of the element on the defining curve. By using the parameterization Eq. (46), the approximation of the matrices $\mathbf{b}_1$ and $\mathbf{b}_2$ and J in Eq. (10) can be obtained explicitly by

$$\mathbf{b}_1^h = \frac{1}{J^h} \begin{bmatrix} \Delta_2 \mathbf{I}_{\Lambda \times \Lambda} \\ -\Delta_1 \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix}, \quad \mathbf{b}_2^h = \frac{1}{J^h} \begin{bmatrix} -(\Sigma_2 + \Delta_2 s) \mathbf{I}_{\Lambda \times \Lambda} \\ (\Sigma_1 + \Delta_1 s) \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix},$$

$$J^h = \Sigma_1 \Delta_2 - \Sigma_2 \Delta_1, \quad (47)$$

where $\Sigma_1 = (\hat{x}_{11} + \hat{x}_{12})/2$, $\Sigma_2 = (\hat{x}_{21} + \hat{x}_{22})/2$, $\Delta_1 = (\hat{x}_{12} - \hat{x}_{11})/2$ and $\Delta_2 = (\hat{x}_{22} - \hat{x}_{21})/2$. It is apparent from Eq. (47) that both J^h and $\mathbf{b}_1^h$ are constant whereas $\mathbf{b}_2^h$ is a linear function of s . Using Eq. (47) along with Eqs. (15) and (16) and the fact that the solution on the element is also discretized by the same linear shape functions Eq. (45), all entries of the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are linear functions of s and, as a result, the matrices $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$ for the 2-node

linear element can be readily obtained in a closed form. It is noted that the representation of the geometry of the body and its associated boundaries using the 2-node linear elements can be exact if the defining curve is piecewise linear. However, when the defining curve used in describing the body becomes more complex or non-straight, a significant number of linear elements is required, in general, to achieve the accurate description of the geometry.

For the 2-node circular-arc element, the defining curve on this element is parameterized in terms of the local boundary coordinate $s \in [-1, 1]$ by

$$x_1 = x_{10} + R \cos[\theta(s)], \quad x_2 = x_{20} + R \sin[\theta(s)], \quad (48)$$

where R denotes the radius of the circular arc, $\theta(s) = \theta_1 \varphi_1(s) + \theta_2 \varphi_2(s)$, φ_1 and φ_2 are given by Eq. (45), and θ_1 and θ_2 are angles between the x_1 -axis and a straight line passing through the scaling center and Nodes 1 and 2, respectively (Fig. 3(b)). The approximate solution along the boundary direction for this particular element is obtained by interpolating the nodal functions at the two nodes using the linear shape functions given by Eq. (45). It should be evident that the 2-node circular-arc elements can be used to describe the domain geometry exactly when the defining curve consists of only circular arcs. By employing the parameterization Eq. (48) along with choosing the origin of the x_1 - x_2 coordinate system coincident with the scaling center, the matrices $\mathbf{b}_1^h$ and $\mathbf{b}_2^h$ and J^h for this case become

$$\mathbf{b}_1^h = \frac{1}{R} \begin{bmatrix} \cos \theta \mathbf{I}_{\Lambda \times \Lambda} \\ \sin \theta \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix}, \quad \mathbf{b}_2^h = \frac{2}{R(\theta_2 - \theta_1)} \begin{bmatrix} -\sin \theta \mathbf{I}_{\Lambda \times \Lambda} \\ \cos \theta \mathbf{I}_{\Lambda \times \Lambda} \end{bmatrix},$$

$$J^h = \frac{R^2(\theta_2 - \theta_1)}{2}. \quad (49)$$

As evident from Eq. (49), J^h is constant whereas $\mathbf{b}_1^h$ and $\mathbf{b}_2^h$ involve simple elementary functions. This together with the linear approximation along the boundary direction allows the matrices $\mathbf{E}_0$, $\mathbf{E}_1$ and $\mathbf{E}_2$ to be constructed analytically. To further enhance the approximation of the solution along the boundary direction, higher-order Lagrange basis functions can be employed by simply adding interior nodes while the geometry is still described by the exact representation Eq. (48).

6 Numerical results and discussion

In this section, results of three examples are reported not only to verify the proposed technique but also to demonstrate its computational performance when the discretization of the defining curve is exact. Representative problems of heat conduction ($\Lambda = 1$), linear elasticity ($\Lambda = 2$) and linear piezoelectricity ($\Lambda = 3$) with bodies

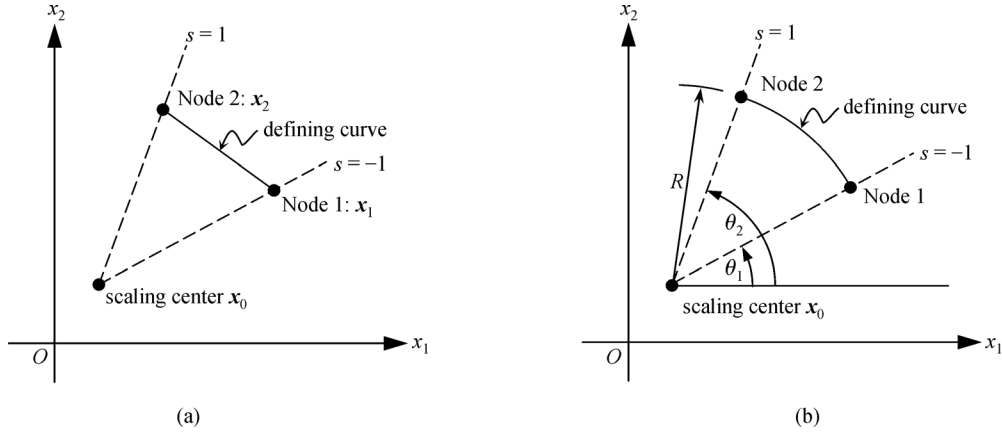


Fig. 3 Schematic of scaling center and defining curve of (a) 2-node isoparametric linear element and (b) 2-node circular-arc element

possessing circular boundaries and subjected to various scenarios are chosen in the numerical study to mainly serve the latter purpose and to additionally exhibit the capability of the technique to treat general boundary conditions and prescribed data on the side-faces. Two types of elements described in the previous section are employed in the discretization and results are then compared to investigate the quality of approximate solutions when the exact description of the defining curve is utilized. For brevity, the symbols “Type-1” and “Type-2” are later used to indicate 2-node linear elements and 2-node circular-arc elements, respectively. Accuracy and convergence of computed numerical solutions are confirmed by benchmarking with available reference solutions and carrying out the analysis via a series of meshes with different levels of refinement, respectively.

6.1 Heat conduction problem

First, a representative problem associated with the steady-state heat conduction ($\Lambda = 1$) is investigated. Consider a two-dimensional domain which is a quarter of a ring of inner radius R_1 and outer radius R_2 and occupies a region in $\mathbb{R}^2$ as shown schematically in Fig. 4. The domain is made of a medium with an isotropic heat conductivity k_0 (or, equivalently, $D_{11} = D_{22} = k_0$, $D_{12} = D_{21} = 0$) and subjected to a non-uniform heat source $b_1 = Q_0(x_1^2 + x_2^2)/R_1^2$, where Q_0 is a constant, and mixed boundary conditions as indicated below.

Side AB : $u_1^{AB} = -Q_0 x_1^4 / (16k_0 R_1^2)$;

Side BC : $u_1^{BC} = -Q_0 R_2^4 / (16k_0 R_1^2)$;

Side CD : $t_1^{CD} = 0$;

Side DA : $t_1^{DA} = Q_0 R_1 / 4$.

For this particular case, the exact solutions for the temperature field $u_1(x_1, x_2)$ and the body heat flux $\sigma_{11}(x_1, x_2)$ and $\sigma_{21}(x_1, x_2)$ are given by

$$u_1 = -Q_0(x_1^2 + x_2^2)^2 / (16k_0 R_1^2), \quad (50)$$

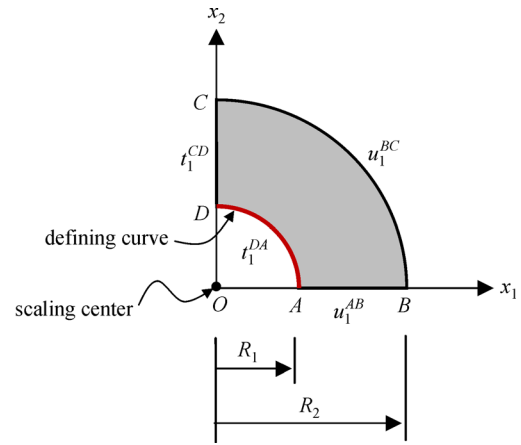


Fig. 4 Schematic of a quarter of a ring subjected to non-uniform heat source and mixed boundary conditions

$$\sigma_{11} = -Q_0(x_1^2 + x_2^2)x_1 / (4R_1^2),$$

$$\sigma_{21} = -Q_0(x_1^2 + x_2^2)x_2 / (4R_1^2). \quad (51)$$

The domain geometry is described by the scaling center at Point O and the defining curve AD ; as a result, the boundaries AB and CD become the side faces (Fig. 4). In the numerical study, $R_2/R_1 = 2$ is chosen and the defining curve is discretized by a uniform mesh containing n identical elements of either Type-1 or Type-2. Results for the normalized temperature along the circular arc between DA and BC (i.e., along $\sqrt{x_1^2 + x_2^2} = (R_1 + R_2)/2$) are normalized by the existing analytical solution and then reported in Table 1 at different values of normalized angle $\theta/\pi = \tan^{-1}(x_2/x_1)/\pi$ and for four different meshes and both types of elements. It is seen that numerical solutions generated by the proposed technique with Type-1 elements converge to the reference solution as the mesh is refined whereas those obtained from the Type-2 elements exhibit excellent agreement with the exact solution for all levels of

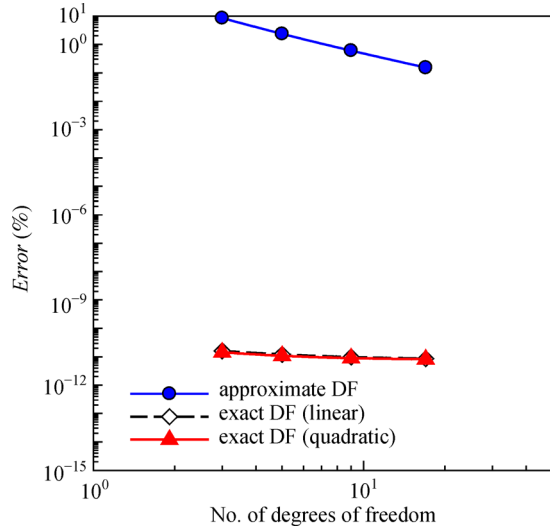


Fig. 5 Relative errors of SBFEM solutions versus number of degrees of freedom used in discretization of circular defining curve. DF: Defining curve

refinement considered. It is worth noting that numerical solutions generated from the Type-2 elements are much more accurate than those from the Type-1 elements for the same level of refinement. To additionally confirm this argument, the relative errors of scaled boundary finite element solutions obtained for both types of elements are computed for various levels of discretization and results are shown in Fig. 5. As indicated by this set of results, using the exact defining curve in the solution procedure yields significantly less solution error for all levels of refinement and obtained approximate solutions are therefore nearly identical to the exact solution. This is due mainly to that the geometry of the body, prescribed data and all involved differential operators are treated exactly. As the direct result, using circular-arc elements with quadratic element shape functions in the discretization of

solution along the boundary direction yields nearly identical relative error to that for the case of Type-2 elements.

6.2 Linear elasticity problem

Consider, next, a representative problem in linear elasticity ($\Lambda = 2$) corresponding to a plane-strain, elastic, hollowed disk with the inner radius and outer radius denoted by R_1 and R_2 , respectively, as shown in Fig. 6(a). The body is made of a homogeneous, isotropic, linearly elastic material with Young's modulus E and Poisson's ratio ν and free of the body force field. On the inner boundary, the normal traction vanishes and the uniform shear traction τ_0 is fully prescribed whereas on the outer boundary, the body is fully restrained against the movement. For this particular problem, the exact solution for the displacement and stress fields can be readily obtained from the theory of linear elasticity and they are given explicitly by

$$\begin{aligned} u_1 &= -\frac{\tau_0}{2\mu} \bar{R}_1^2 R_2 \left(\bar{x}_2 - \frac{\bar{x}_2}{\bar{x}_1^2 + \bar{x}_2^2} \right), \\ u_2 &= \frac{\tau_0}{2\mu} \bar{R}_1^2 R_2 \left(\bar{x}_1 - \frac{\bar{x}_1}{\bar{x}_1^2 + \bar{x}_2^2} \right), \end{aligned} \quad (52)$$

$$\begin{aligned} \sigma_{11} &= -\sigma_{22} = -2\tau_0 \bar{R}_1^2 \frac{\bar{x}_1 \bar{x}_2}{(\bar{x}_1^2 + \bar{x}_2^2)^2}, \\ \sigma_{12} &= \sigma_{21} = \tau_0 \bar{R}_1^2 \frac{\bar{x}_1^2 - \bar{x}_2^2}{(\bar{x}_1^2 + \bar{x}_2^2)^2}, \end{aligned} \quad (53)$$

where μ is the shear modulus, $\bar{R}_1 = R_1/R_2$, $\bar{x}_1 = x_1/R_2$, and $\bar{x}_2 = x_2/R_2$. Due to the rotational symmetry and to demonstrate the capability of the present method to treat mixed boundary conditions and prescribed non-zero data

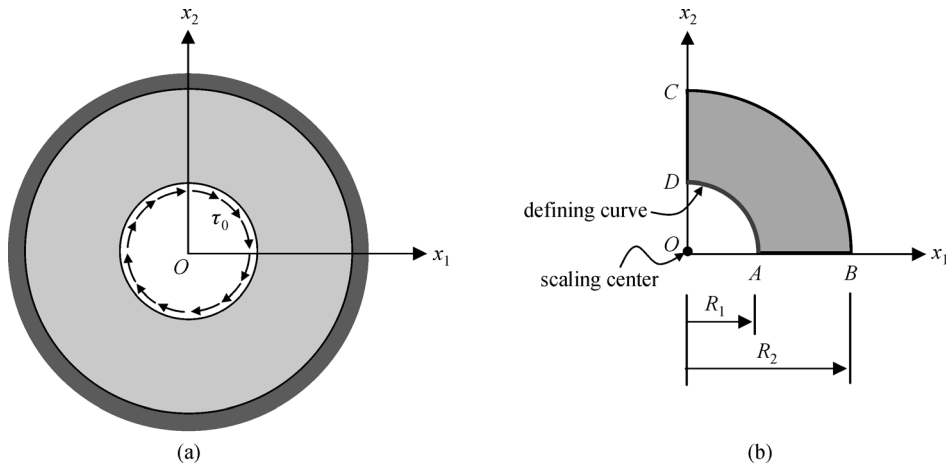


Fig. 6 Schematics of (a) plane-strain hollowed disk fully restrained against the movement on its outer boundary and subjected to uniform shear traction on its inner boundary and (b) equivalent quarter of body used in analysis

on the side-faces, a quarter of the body, as illustrated in Fig. 6(b), is considered. On four sides of the reduced domain, following non-uniform mixed boundary conditions are prescribed:

$$\text{Side } AB: t_1^{AB} = -\tau_0 \frac{\bar{R}_1^2}{\bar{x}_1^2}, u_2^{AB} = \frac{\tau_0 \bar{R}_1^2 R_2}{2\mu} \left(\bar{x}_1 - \frac{1}{\bar{x}_1} \right);$$

$$\text{Side } BC: u_1^{BC} = u_2^{BC} = 0;$$

$$\text{Side } CD: u_1^{CD} = -\frac{\tau_0}{2\mu} \bar{R}_1^2 R_2 \left(\bar{x}_2 - \frac{1}{\bar{x}_2} \right), t_2^{CD} = \tau_0 \frac{\bar{R}_1^2}{\bar{x}_2^2};$$

$$\text{Side } DA: t_1^{DA} = \tau_0 \frac{x_2}{R_1}, t_2^{AD} = -\tau_0 \frac{x_1}{R_1}.$$

To describe the geometry, the scaling center is chosen at the center of the hollowed disk whereas its inner boundary is treated as the defining curve. In numerical simulations, $R_2/R_1 = 2$ is chosen and uniform meshes containing n elements of either Type-1 or Type-2 are employed to discretize the defining curve of the reduced domain. Numerical solutions for the displacement components along the circular arc between DA and BC (i.e., along

$\sqrt{x_1^2 + x_2^2} = (R_1 + R_2)/2$) are shown in Tables 2 and 3 at different values of normalized angle $\theta/\pi = \tan^{-1}(x_2/x_1)/\pi$ and for different levels of refinements and two types of elements used in the discretization of the defining curve. Besides the good agreement between the scaled boundary finite element and benchmark solutions and the good convergence behaviour upon the mesh refinement, using the 2-node circular-arc elements in the discretization of the defining curve yields more accurate solutions. The improvement of the solution accuracy can also be confirmed from the relationship between the relative error and the degrees of freedom used in the discretization shown in Fig. 7. Similar to the previous case, the use of the exact defining curve does not alter the rate of convergence as the mesh is refined but significantly reducing the solution error. Note also that if the quadratic circular-arc elements are employed in the solution discretization, the rate of convergence is apparently enhanced and, for the same number of degrees of freedom, the relative error is less than that for the case of Type-2 elements.

Table 1 Normalized temperatures u_1/u_1^{exact} along circular arc between DA and BC of two-dimensional domain associated with quarter of ring

n	u_1/u_1^{exact}							
	$\theta/\pi = 0.1$		$\theta/\pi = 0.2$		$\theta/\pi = 0.3$		$\theta/\pi = 0.4$	
	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2
2	1.0609	1.0000	1.1218	1.0000	1.1483	1.0000	1.1405	1.0000
4	1.0252	1.0000	1.0336	1.0000	1.0352	1.0000	1.0354	1.0000
8	1.0067	1.0000	1.0085	1.0000	1.0088	1.0000	1.0089	1.0000
16	1.0017	1.0000	1.0021	1.0000	1.0022	1.0000	1.0022	1.0000

Table 2 Normalized displacements u_1/u_1^{exact} along circular arc between DA and BC of plane-strain hollowed disk

n	u_1/u_1^{exact}							
	$\theta/\pi = 0.1$		$\theta/\pi = 0.2$		$\theta/\pi = 0.3$		$\theta/\pi = 0.4$	
	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2
2	0.7449	0.9529	0.7832	1.0019	0.7702	0.9853	0.7554	0.9664
4	0.9411	1.0009	0.9330	0.9923	0.9320	0.9913	0.9379	0.9975
8	0.9829	0.9981	0.9840	0.9993	0.9843	0.9996	0.9826	0.9979
16	0.9960	0.9998	0.9956	0.9995	0.9956	0.9995	0.9960	0.9999

Table 3 Normalized displacements u_2/u_2^{exact} along circular arc between DA and BC of plane-strain hollowed disk

n	u_2/u_2^{exact}							
	$\theta/\pi = 0.1$		$\theta/\pi = 0.2$		$\theta/\pi = 0.3$		$\theta/\pi = 0.4$	
	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2
2	0.7554	0.9664	0.7702	0.9853	0.7832	1.0019	0.7449	0.9529
4	0.9379	0.9975	0.9320	0.9913	0.9330	0.9923	0.9411	1.0009
8	0.9826	0.9979	0.9843	0.9996	0.9840	0.9993	0.9829	0.9981
16	0.9960	0.9999	0.9956	0.9995	0.9956	0.9995	0.9960	0.9998

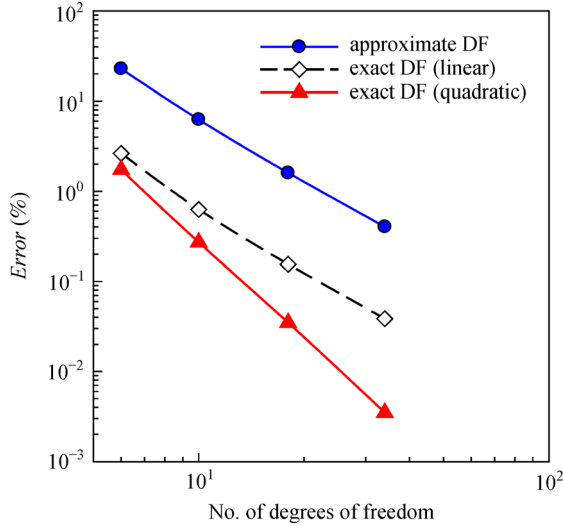


Fig. 7 Relative errors of SBFEM solutions versus number of degrees of freedom used in discretization of circular defining curve. DF: Defining curve

6.3 Linear piezoelectricity problem

As a final example, a representative boundary value problem associated with the linear piezoelectricity ($\Lambda = 3$) is investigated to further emphasize the capability of the proposed technique to treat linear multi-field problems. Let us consider a quarter of hollowed circular plate with the inner radius and outer radius denoted by R_1 and R_2 , respectively, and made of a linear piezoelectric solid with all material constants taken from PZT-4 [8] as shown in Fig. 6(b). In particular, all non-zero entries of the modulus matrix $\mathbf{D}$ are given explicitly by $D_{11} = 139$ GPa, $D_{15} = D_{51} = 74.3$ GPa, $D_{16} = D_{61} = -6.98$ C/m², $D_{22} = D_{24} = 25.6$ GPa, $D_{42} = D_{44} = 25.6$ GPa, $D_{55} = 113$ GPa, $D_{33} = 6.0 \times 10^{-9}$ C/(V·m), $D_{34} = D_{43} = 13.44$ C/m², $D_{56} = D_{65} = 13.84$ C/m², and $D_{66} = -5.47 \times 10^{-9}$ C/(V·m). For the purpose of verification, a set of prescribed data such as the distributed body source and boundary conditions on four sides is chosen such that the exact solution for the displacements (u_1, u_2) and the electric potential (u_3) takes the following form:

$$u_1 = u_{10}\bar{x}_1^2, \quad u_2 = u_{20}\bar{x}_2^2, \quad u_3 = \phi_0\bar{x}_2, \quad (54)$$

where u_{10}, u_{20}, ϕ_0 are given constants, $\bar{x}_1 = x_1/R$ and $\bar{x}_2 = x_2/R$ with R taken as R_1 . The corresponding exact stress field ($\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$) and electrical induction vector (σ_{13}, σ_{23}) are given by

$$\begin{aligned} \sigma_{11} &= (2D_{11}u_{10}\bar{x}_1 + 2D_{15}u_{20}\bar{x}_2 + D_{16}\phi_0)/R, \\ \sigma_{22} &= (2D_{15}u_{10}\bar{x}_1 + 2D_{55}u_{20}\bar{x}_2 + D_{56}\phi_0)/R, \\ \sigma_{12} &= \sigma_{21} = 0, \end{aligned} \quad (55)$$

$$\sigma_{13} = 0, \quad \sigma_{23} = (2D_{16}u_{10}\bar{x}_1 + 2D_{56}u_{20}\bar{x}_2 + D_{66}\phi_0)/R. \quad (56)$$

The distributed body source required to ensure the equilibrium state is given by

$$\begin{aligned} b_1 &= -2D_{11}u_{10}/R^2, \quad b_2 = -2D_{55}u_{20}/R^2, \\ b_3 &= -2D_{56}u_{20}/R^2. \end{aligned} \quad (57)$$

In the analysis, the scaling center is chosen at a Point O and the prescribed conditions on the side faces AB and DC and the boundaries AD and BC are given below:

$$\begin{aligned} \text{Side } AB: & \quad t_1^{AB} = 0, \quad u_2^{AB} = 0, \quad u_3^{AB} = 0; \\ \text{Side } CD: & \quad u_1^{CD} = 0, \quad u_2^{CD} = u_{20}\bar{x}_2^2, \quad t_3^{CD} = 0; \\ \text{Side } DA: & \end{aligned}$$

$$\begin{aligned} t_1^{DA} &= -(2D_{11}u_{10}\bar{x}_1 + 2D_{15}u_{20}\bar{x}_2 + D_{16}\phi_0)\bar{x}_1/R, \\ t_2^{DA} &= -(2D_{15}u_{10}\bar{x}_1 + 2D_{55}u_{20}\bar{x}_2 + D_{56}\phi_0)\bar{x}_2/R, \\ t_3^{DA} &= -(2D_{16}u_{10}\bar{x}_1 + 2D_{56}u_{20}\bar{x}_2 + D_{66}\phi_0)\bar{x}_2/R; \end{aligned}$$

$$\text{Side } BC: \quad u_1^{BC} = u_{10}\bar{x}_1^2, \quad u_2^{BC} = u_{20}\bar{x}_2^2, \quad u_3^{BC} = \phi_0\bar{x}_2.$$

The prescribed boundary conditions indicated above are chosen mainly to demonstrate the capability of the present technique to deal with general prescribed data on the side faces and the boundary of the domain. In numerical simulations, uniform meshes containing n elements of either Type-1 or Type-2 are employed to discretize the defining curve (i.e., the boundary DA) and $R_2/R_1 = 2$, $u_{20}/u_{10} = 2$ and $D_{16}\phi_0/(D_{11}u_{10}) = 1$ are chosen. Computed displacements, electric potentials along the line $x_1 = x_2$, normalized by the analytical solution, are reported at different values of $\bar{x}_1 = x_1/R$ in Tables 4–6 for various levels of discretization and both types of elements. It is seen, for this particular problem, that the proposed technique yields highly accurate results even when relatively coarse meshes containing only few degrees of freedom are employed. Again, use of the exact defining curve to describe the domain geometry and its boundary can significantly improve the accuracy of the numerical solutions for the entire domain and allows coarser meshes to be utilized to obtain the same level of accuracy. This is confirmed by the plot of the relative error versus the number of degrees of freedom used in the discretization of the defining curve shown in Fig. 8. Similar to the previous problem, the rate of convergence of the numerical solution can be improved and the relative error can be significantly reduced by using the quadratic circular-arc elements.

7 Conclusions and remarks

An efficient numerical technique based on the scaled boundary finite element method has been successfully implemented for the analysis of two-dimensional, linear,

Table 4 Normalized displacements u_1/u_1^{exact} along the line $x_1 = x_2$ of quarter of hollowed circular plate

n	u_1/u_1^{exact}							
	$\bar{x}_1 = 1.25$		$\bar{x}_1 = 1.50$		$\bar{x}_1 = 1.75$		$\bar{x}_1 = 2.00$	
	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2
2	0.7368	0.8935	0.7228	0.9262	0.7799	1.0043	0.8605	1.1004
4	0.9361	0.9806	0.9309	0.9936	0.9385	1.0115	0.9413	1.0199
8	0.9830	0.9946	0.9812	0.9974	0.9830	1.0014	0.9852	1.0046
16	0.9957	0.9986	0.9952	0.9993	0.9957	1.0003	0.9963	1.0011

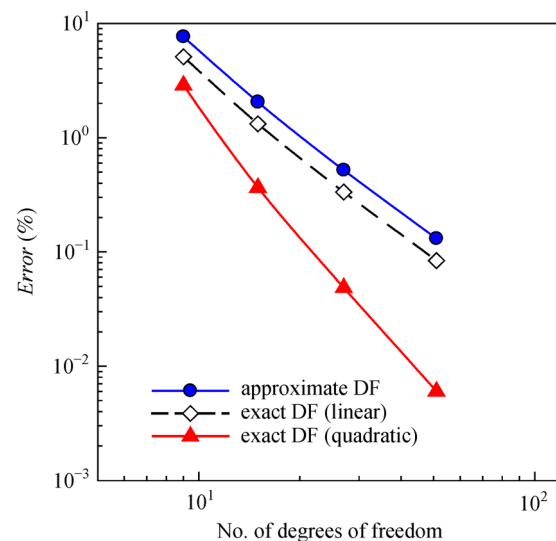
Table 5 Normalized displacements u_2/u_2^{exact} along the line $x_1 = x_2$ of quarter of hollowed circular plate

n	u_2/u_2^{exact}							
	$\bar{x}_1 = 1.25$		$\bar{x}_1 = 1.50$		$\bar{x}_1 = 1.75$		$\bar{x}_1 = 2.00$	
	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2
2	1.0289	0.9705	1.0421	0.9801	1.0493	0.9917	1.0525	0.9981
4	1.0072	0.9984	1.0088	0.9987	1.0097	0.9998	1.0111	1.0019
8	1.0016	0.9997	1.0021	0.9998	1.0023	0.9999	1.0022	1.0000
16	1.0004	0.9999	1.0005	1.0000	1.0006	1.0000	1.0005	1.0000

Table 6 Normalized electric potentials u_3/u_3^{exact} along the line $x_1 = x_2$ of quarter of hollowed circular plate

n	u_3/u_3^{exact}							
	$\bar{x}_1 = 1.25$		$\bar{x}_1 = 1.50$		$\bar{x}_1 = 1.75$		$\bar{x}_1 = 2.00$	
	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2	Type-1	Type-2
2	0.9864	1.0111	0.9699	1.0088	0.9500	0.9992	0.9268	0.9844
4	0.9973	1.0018	0.9944	1.0019	0.9914	1.0014	0.9892	1.0010
8	0.9993	1.0004	0.9986	1.0004	0.9978	1.0003	0.9970	1.0000
16	0.9998	1.0001	0.9996	1.0001	0.9994	1.0001	0.9992	1.0000

boundary value problems. Both the formulation and numerical implementations have been established in a general framework allowing a variety of linear field problems (e.g., potential and Laplace-type problems, linear elasticity, linear coupled-field problems) and the general associated data (e.g., domain geometry, prescribed distributed body source, boundary conditions, contribution of prescribed conditions on the side-faces) to be treated in a single, unified fashion. The exact treatment of the defining curve used for describing the geometry of the domain and the corresponding boundaries and its influence on the improvement of solution accuracy has also been fully investigated via a special class of boundary value problems concerning domains with a circular boundary. A 2-node circular arc element has been developed and implemented to serve that purpose and obtained results have been benchmarked with those generated by standard 2-node linear elements. Numerical results obtained from extensive numerical experiments have indicated that the proposed SBFEM yields highly accurate solutions for various types

**Fig. 8** Relative errors of SBFEM solutions versus number of degrees of freedom used in discretization of circular defining curve. DF: Defining curve

of boundary value problems considered and, in particular, only few degrees of freedom are required along the defining curve to attain such high-quality results. In addition, it has been found that use of the exact description of the defining curve in the solution procedure can significantly reduce the solution error and, as a result, reduce the number of degrees of freedom required to achieve the target accuracy in comparison with standard linear elements.

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Analysis for T-stress of cracks in 3D anisotropic elastic media by weakly singular integral equation method

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Highlights

- A numerical method is established to determine T-stress for cracks in 3D media.
- The formulation is based on a pair of weakly singular weak-form integral equations.
- A special element is used for the near-front relative crack-face displacement.
- Explicit formulae are used to extract the T-stress from the crack-face data.
- Highly accurate T-stress results are obtained using relatively coarse meshes.

Abstract

This paper proposes an efficient numerical procedure for computing T-stresses of cracks in three-dimensional, linearly elastic, infinite media. The technique is established in a broad framework allowing a medium made of generally anisotropic materials and cracks of arbitrary shape and under general loading conditions to be treated. A pair of weakly singular, weak-form, displacement and traction boundary integral equations is utilized to formulate the key equations governing the unknown crack-face fields. Besides the basic benefits such as the reduction of spatial dimensions of the solution space and the efficient treatment of unbounded domains and remote boundary data, use of such integral equations in the formulation offers additional positive features including the computational simplicity resulting from the weakly singular nature of all involved integrals and the involvement of a complete set of crack-face displacement fields. A weakly singular symmetric Galerkin boundary element method together with the special near-front approximation is utilized to solve for the unknown relative crack-face displacement whereas the sum of the crack-face displacement is obtained from the displacement boundary integral equation for cracks via the Galerkin technique. The latter step is one of the essential aspects of the present study that provides the direct means for determining the T-stress data in terms of the sum of the crack-face displacement in the neighborhood of the crack-front. An extensive numerical study is conducted for various scenarios and a selected set of results is reported to demonstrate both the accuracy and capability of the proposed technique.

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1. Introduction

Mathematical models based upon a classical theory of linear elasticity have been widely used and found well-suited for simulating the response of cracked bodies provided that the material nonlinear process zone is only confined in

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a region of small size in comparison with the crack dimensions. In view of these linear models, cracks contained in a body serve as a local stress riser and the predicted stress field becomes singular at points along the crack front. The dominant behavior of the near-front stress field is completely characterized by a singular term, called the K -field. The spatial dependence of this K -field can be obtained from the near-front eigenanalysis (e.g., [1–3]) whereas its dependence on the geometry, material properties, and loading conditions is lumped into a single set of parameters, called the stress intensity factors. This essential fracture data can be found from a stress analysis of the whole body and plays a vital role in classical linear elastic fracture mechanics. In particular, knowledge of the stress intensity factors not only provides a complete measure of the dominant stress field around the crack front but also forms a basis in the investigation of crack growth and failure analysis. It should be noted, however, that the fracture models based solely on the information of the K -field can still yield inaccurate results if the influence of the nonsingular part of the near-front stress field is not negligible. As pointed out by various researchers, the first nonsingular constant resulting from the near-front stress expansion, called the T -field or the T -stress, can also play a crucial role in the estimation of crack-tip plastic-zone size and shape, the investigation of stress tri-axiality and crack-front stress confinement, and the directional stability of advancing cracks (e.g., [4–10]). Similar to the stress intensity factors, determination of the T -stress components along the crack front still requires solving a complete boundary value problem associated with the whole cracked body and this analysis step is essential if $K - T$ fracture models are employed in the fracture simulations.

A large number of research studies directed towards the development of solution techniques for determining the T -stress of cracks in linearly elastic media have been conducted in the past four decades (see extensive review in [11]). Some of those investigations are briefly summarized here not only to reveal the historical background and the current advances in this specific discipline but also to pinpoint the existing gap of knowledge to be emphasized in the present study. For two-dimensional crack problems, many solution procedures have been successfully implemented and solutions of the T -stress for various scenarios were reported for both isotropic and anisotropic solids. A technique based upon path-independent contour integrals and Eshelby's method was commonly employed along with numerical solutions from either the finite element analysis (e.g., [12–17]) or the boundary integral equation methods (e.g., [18–20]) to compute the T -stress at the crack tip. Another method using the weight functions, Green's functions, and reciprocal integrals was also introduced to extract the T -stress from the finite element solutions (e.g., [21–25]). While this group of techniques does not directly utilize the information of the near-tip field and generally yield accurate T -stress solutions, they require non-trivial implementations and computational effort associated with the post-process. Other techniques based directly upon the near-tip stress and displacement solution were also developed; for instance, a boundary-layer method via the subtraction of tangential stress components (e.g., [4,26]), a stress-difference method (e.g., [27]), a technique based on the near-front crack-face displacement (e.g., [28,29]), Green's function technique along with the collocation boundary element method (e.g., [30–33]), and methods based on a complex potential theory (e.g., [34]) and complex variable representations along with crack-back and crack-front position techniques [35]. It should be remarked that most of the techniques indicated above are restricted mainly to cracks in isotropic media. The integration of material anisotropy in the calculation of T -stress within the context of two-dimensional crack problems has also been addressed in the literature; however, the number of investigations is significantly smaller in comparison with the isotropic case. Results from those studies include the analytical solutions of the T -stress for curved and kinked cracks in orthotropic media via a perturbation technique [36], the explicit expressions of the T -stress and near-tip stress and displacement of cracks in anisotropic media based on the Stroh formalism [37], the T -stress formulated in terms of either J-integral or reciprocity techniques [38], the numerical solution of T -stress for cracks in orthotropic solids via a fractal finite element method and the representation of near-tip displacement field [39,40], and the T -stress solution for cracks in orthotropic materials via the M-integral technique along with the finite element method [41] and the boundary integral equation method [42]. Recently, Tran and Mear [43] proposed an efficient and accurate technique, based upon a weakly singular symmetric Galerkin boundary element method, to compute the T -stress of cracks in anisotropic two-dimensional media. Special crack-tip elements were adopted to enhance the near-tip approximation and explicit formulae were proposed to extract the T -stress for finite domains containing an arbitrary number of curvilinear cracks under general loading conditions. As indicated by the significant amount of existing literature, solution procedures for calculating the T -stress have been fairly well established for two-dimensional crack problems.

However, on the basis of an extensive literature survey, work towards the development of solution procedures for calculating the T -stress of cracks in three-dimensional elastic media appears to be relatively limited and most of the

results are applicable only to simple material models and specific crack geometries, loadings, and boundary conditions. For instance, Wang [44] proposed a technique based upon potential theory and the Hankel integral transform to derive the exact solution of the T-stress for a penny-shaped crack in an isotropic infinite medium under remote tension and bending. The generalization of the work of Wang [44] to an elliptical crack was later achieved by Kirilyuk and Levchuk [45] via the method of potential theory and representations in terms of harmonic functions. Recently, Rungamornrat and Pinitpanich [46] developed analytical solutions of the T-stress for a penny-shaped crack in transversely isotropic, linearly elastic media under remote loading and general crack-face pressure. In their development, the near-front stress expansion was utilized along with an existing stress field reported by Fabrikant [47] to derive the T-stress Green's function and the obtained results were then utilized to form the integral formula for computing the T-stress of cracks under a general crack-face pressure. Closed-form solutions of the T-stress for remote loading and uniform, linear and axisymmetric crack-face pressures were also reported. While those existing analytical techniques yield closed-form solutions, their applications are restricted to very simple crack configurations and, apparently, no other analytical T-stress solution of cracks in three-dimensional elastic media has been reported in the literature. Solution methodologies based upon numerical procedures have been considered more robust due to their vast features and computational capability and the continuous growth of this group of techniques in the analysis of T-stress has been apparent. For instance, Henry and Luxmoore [48] proposed a three-dimensional finite element model to analyze a center cracked plate under uniform displacement. Both the boundary layer and displacement field methods were adopted to extract the T-stress along the crack front. Later, Wang [49] applied the interaction integral formula together with the finite element analysis to determine the T-stress of a semi-elliptical, surface-breaking crack in an isotropic, linear elastic, finite plate under both tensile and bending loads. This technique was later utilized by Wang and Bell [50] to obtain the T-stress of the same cracked plate but subjected to more general applied loads with uniform, linear, parabolic and cubic distributions, by Qu and Wang [51] to determine the T-stress of a quarter-elliptical, corner crack in a thick plate under tensile and bending loads, and by Lewis and Wang [52] to obtain the T-stress of a circumferential through-wall crack in a finite cylinder subjected to remote tension, remote bending, and the crack-face pressure with uniform, linear, parabolic, and cubic distributions. In addition, Schütte and Molla-Abbasi [53] and Molla-Abbasi and Schütte [54] applied various techniques including the method of potential theory, Hankel integral transform, Green's function technique, and finite element method to obtain the full set of T-stress solutions for both penny-shaped and elliptical cracks contained in an isotropic infinite body under mixed-mode remote loadings. More recently, Meshii et al. [55] also employed the finite element method along with an interaction integral scheme to determine the T-stress of a semi-elliptical crack embedded in a three-dimensional, isotropic, linearly elastic, finite cylinder under mode-I crack-face pressure with uniform, linear, parabolic, or cubic distributions. As is evident, the bulk of these existing studies involve cracks in isotropic media and are very specific to crack configurations and domain geometries. To the best of the authors' knowledge, the integration of material anisotropy and boundary integral equation methods, which are considered efficient for fracture modeling, into the three-dimensional numerical analysis of the T-stress has not been well established.

This paper aims to enhance the modeling capacity of existing techniques for calculating the T-stress of cracks in three-dimensional elastic media. The proposed numerical procedure is based upon a weakly singular boundary integral equation method and implemented in a general framework allowing the treatment of material anisotropy and cracks of arbitrary shapes and under general loading conditions. One of the essential features of the present study is the use of the weakly singular displacement boundary integral equation to solve for the sum of the crack-face displacement and this set of information provides an efficient and simple means to extract the T-stress data along the crack front. The following sections of this paper are organized to include the problem description, formulation of key governing boundary integral equations, solution procedure, numerical results and discussion, and conclusions and remarks.

2. Problem statement

Consider a three-dimensional infinite medium Ω containing isolated cracks of arbitrary shapes as shown schematically in Fig. 1. The body is made of a homogeneous, generally anisotropic, linearly elastic material with prescribed elastic moduli E_{ijkl} . The geometry of the crack in its reference state is represented by two identical surfaces, denoted by the upper crack surface S^+ and the lower crack surface S^- , with the outward unit normal vector $\mathbf{n}^+$ at any point $\xi^+ \in S^+$ opposite to the outward unit normal vector $\mathbf{n}^-$ at the geometrically coincident point $\xi^- \in S^-$ (i.e., $\mathbf{n}^+(\xi^+) = -\mathbf{n}^-(\xi^-)$). For the case of multiple cracks, S^+ and S^- are taken as the union of all upper crack

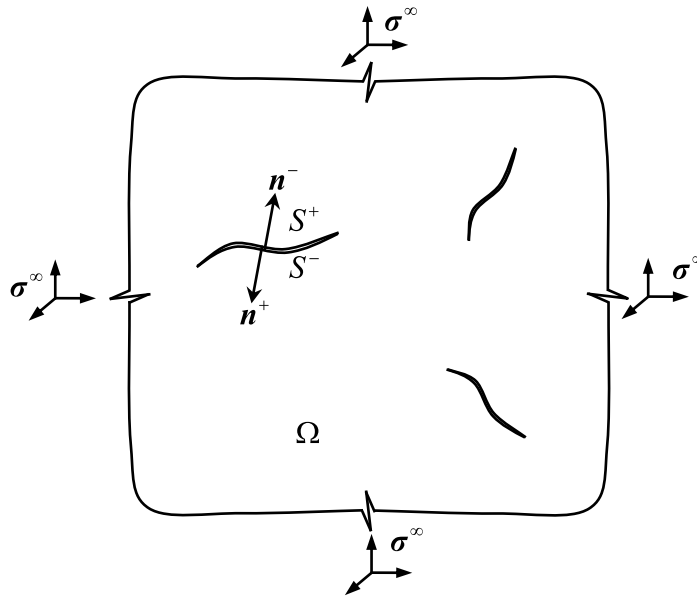


Fig. 1. Schematic of three-dimensional, homogeneous, anisotropic, linearly elastic infinite medium containing isolated cracks under uniform remote stress and general crack-face loading.

surfaces and that of the lower crack surfaces, respectively. The medium is loaded by the uniform far field σ^∞ , the traction $t^{0+}(\xi^+)$ on the upper crack surface S^+ , and the traction $t^{0-}(\xi^-)$ on the lower crack surface S^- whereas there is an absence of body force. In the present study, it is assumed that the crack surface is piecewise smooth (i.e., a set of points whose outward unit normal vector is not well defined is of measure zero) and the response of the medium obeys the classical theory of linear elasticity. The primary objective of the current investigation is to establish a numerical procedure to efficiently and accurately calculate the T-stress components at any point along the crack front (i.e., points along the boundary of S^+ and S^-).

3. Formulation of integral equations

In this section, a set of governing boundary integral equations for cracks in an infinite elastic medium is presented. The formulation is based upon a pair of weakly singular, weak-form boundary integral equations for the jump in the crack-face traction and the sum of the crack-face displacement. Standard integral relations for the displacement and stress at any interior point are first presented and the regularization procedure to arrive at the final equations is then briefly discussed.

From the linearity of the problem, the original cracked medium illustrated in Fig. 1 can be decomposed into two sub-problems: (i) an un-cracked medium subjected only to the remote far field σ^∞ and (ii) a cracked medium subjected to the prescribed crack-face tractions t^{0+} , t^{0-} and the opposite tractions generated at the crack surfaces S^+ and S^- by the far field, i.e., $-\sigma^\infty n^+$ and $-\sigma^\infty n^-$, respectively. By applying a linear superposition together with Somigliana's identity for a cracked medium, the displacement at any interior point $\mathbf{x} \in \Omega$ of the original cracked body is given by

$$u_p(\mathbf{x}) = u_p^\infty(\mathbf{x}) + \int_{S^+} U_j^p(\xi - \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) - \int_{S^+} S_{ij}^p(\xi - \mathbf{x}) n_i^+(\xi) \Delta u_j(\xi) dA(\xi) \quad (1)$$

where $u_p^\infty(\mathbf{x})$ denotes the known displacement of the un-cracked medium in the sub-problem (i); $\Sigma t_j^0 = t_j^{0+} + t_j^{0-}$ denotes the sum of the prescribed crack-face traction in the original cracked medium; $\Delta u_j = u_j^+ - u_j^-$ denotes the jump in the crack-face displacement in the sub-problem (ii); and $U_j^p(\xi - \mathbf{x})$ and $S_{ij}^p(\xi - \mathbf{x})$ are the displacement and stress fundamental solutions for an infinite, un-cracked body given by (see [56] and the references therein)

$$U_j^p(\xi - \mathbf{x}) = \frac{1}{8\pi^2 r} \oint_{z,r=0} (z, z)_{jp}^{-1} ds(z) \quad (2)$$

$$S_{ij}^p(\xi - \mathbf{x}) = E_{ijkl} \frac{\partial U_k^p(\xi - \mathbf{x})}{\partial \xi_l} \quad (3)$$

in which $\mathbf{r} = \xi - \mathbf{x}$, $r = \|\mathbf{r}\|$, $(z, z)_{jp} = z_m E_{mjpn} z_n$, and the closed contour integral is to be evaluated over a unit circle $\|\mathbf{z}\| = 1$ on a plane defined by $\mathbf{z} \cdot \mathbf{r} = 0$. Since there is no crack in the sub-problem (i), Δu_j also represents the jump in the crack-face displacement of the original cracked medium. By forming the displacement gradient of (1) and then utilizing Hooke's law, it leads to an integral relation for the stress at any interior point $\mathbf{x} \in \Omega$:

$$\sigma_{lk}(\mathbf{x}) = \sigma_{lk}^\infty - \int_{S^+} S_{lk}^j(\xi - \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) + \int_{S^+} \Sigma_{ij}^{lk}(\xi - \mathbf{x}) n_i^+(\xi) \Delta u_j(\xi) dA(\xi) \quad (4)$$

where the kernel $\Sigma_{ij}^{lk}(\xi - \mathbf{x})$ is defined by

$$\Sigma_{ij}^{lk}(\xi - \mathbf{x}) = E_{lkpq} \frac{\partial S_{ij}^p(\xi - \mathbf{x})}{\partial \xi_q} \quad (5)$$

While both standard integral relations (1) and (4) can be used to form a pair of displacement and traction integral equations sufficient for determining all crack-face unknowns (i.e., the crack-face displacements), the numerical treatment of the strongly singular and hypersingular integrals associated with the kernels $S_{ij}^p(\xi - \mathbf{x})$ and $\Sigma_{ij}^{lk}(\xi - \mathbf{x})$ is non-trivial and requires special care. To avoid such difficulty, an alternative form of the displacement and stress integral relations is established first, following the regularization procedure proposed by Rungamornrat and Mear [56], before the boundary integral equations are formed (also see [57–61]).

In the regularization procedure, the stress fundamental solution $S_{ij}^p(\xi - \mathbf{x})$ and the hypersingular kernel $\Sigma_{ij}^{lk}(\xi - \mathbf{x})$ are represented by

$$S_{ij}^p(\xi - \mathbf{x}) = H_{ij}^p(\xi - \mathbf{x}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} G_{mj}^p(\xi - \mathbf{x}) \quad (6)$$

$$\Sigma_{ij}^{lk}(\xi - \mathbf{x}) = -E_{ijkl} \delta(\xi - \mathbf{x}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} \varepsilon_{lrt} \frac{\partial}{\partial \xi_r} C_{mj}^{tk}(\xi - \mathbf{x}) \quad (7)$$

where ε_{ism} denotes the alternating symbol; $\delta(\xi - \mathbf{x})$ denotes the Dirac-delta centered at $\mathbf{x}$; and $H_{ij}^p(\xi - \mathbf{x})$, $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{tk}(\xi - \mathbf{x})$ are functions defined by

$$H_{ij}^p(\xi - \mathbf{x}) = -\frac{\xi_i - x_i}{4\pi r^3} \delta_{jp} \quad (8)$$

$$G_{mj}^p(\xi - \mathbf{x}) = \frac{\varepsilon_{abm} E_{ajdc}}{8\pi^2 r} \oint_{\mathbf{z} \cdot \mathbf{r} = 0} (\mathbf{z}, \mathbf{z})_{dp}^{-1} z_b z_c ds(\mathbf{z}) \quad (9)$$

$$C_{mj}^{tk}(\xi - \mathbf{x}) = \frac{A_{mjdn}^{tkoe}}{8\pi^2 r} \oint_{\mathbf{z} \cdot \mathbf{r} = 0} (\mathbf{z}, \mathbf{z})_{en}^{-1} z_o z_d ds(\mathbf{z}), \quad A_{mjdn}^{tkoe} = \varepsilon_{pam} \varepsilon_{pbt} \left\{ E_{ajeo} E_{bknd} - \frac{1}{3} E_{ajkb} E_{dneo} \right\} \quad (10)$$

It is worth noting that the function $H_{ij}^p(\xi - \mathbf{x})$ is independent of material properties and singular at a point $\xi = \mathbf{x}$ of $\mathcal{O}(1/r^2)$ whereas the functions $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{tk}(\xi - \mathbf{x})$ depend on the elastic moduli E_{ijkl} and are singular at a point $\xi = \mathbf{x}$ of $\mathcal{O}(1/r)$.

By substituting (6) into (1) and (6)–(7) into (4), and then employing Stokes' theorem to carry out an integration by parts of terms involving the functions $G_{mj}^p(\xi - \mathbf{x})$ and $C_{mj}^{tk}(\xi - \mathbf{x})$, we obtain

$$u_p(\mathbf{x}) = u_p^\infty(\mathbf{x}) + \int_{S^+} U_j^p(\xi - \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) - \int_{S^+} n_i^+(\xi) H_{ij}^p(\xi - \mathbf{x}) \Delta u_j(\xi) dA(\xi) + \int_{S^+} G_{mj}^p(\xi - \mathbf{x}) D_m \Delta u_j(\xi) dA(\xi) \quad (11)$$

$$\sigma_{lk}(\mathbf{x}) = \sigma_{lk}^\infty + \varepsilon_{lrt} \frac{\partial}{\partial x_r} \left\{ \int_{S^+} G_{tk}^j(\xi - \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) + \int_{S^+} C_{mj}^{tk}(\xi - \mathbf{x}) D_m \Delta u_j(\xi) dA(\xi) \right\} - \int_{S^+} H_{lk}^j(\xi - \mathbf{x}) \Sigma t_j^0(\xi) dA(\xi) \quad (12)$$

where $D_m(\cdot) = \varepsilon_{mps} n_p \partial(\cdot)/\partial \xi_s$ is a surface differential operator. It should be noted that the crack closure condition $\Delta u_j(\xi) = 0$, $\forall \xi \in \partial S^+$ has been employed to eliminate line integrals resulting from the integration by parts. By taking the limit $\mathbf{x} \rightarrow \mathbf{y} \in S^+$ of (11), then performing the inner product of the obtained result with a sufficiently smooth test

function $\tilde{t}_p(\mathbf{y})$, and finally integrating the equation over the crack surface S^+ , we obtain a weakly-singular, weak-form boundary integral equation for the displacement as

$$\begin{aligned} \int_{S^+} \tilde{t}_p(\mathbf{y}) \left\{ \frac{1}{2} \Sigma u_p(\mathbf{y}) - u_p^\infty(\mathbf{y}) \right\} dA(\mathbf{y}) = & \int_{S^+} \tilde{t}_p(\mathbf{y}) \int_{S^+} U_j^p(\boldsymbol{\xi} - \mathbf{y}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \\ & - \int_{S^+} \tilde{t}_p(\mathbf{y}) \int_{S^+} n_i^+(\boldsymbol{\xi}) H_{ij}^p(\boldsymbol{\xi} - \mathbf{y}) \Delta u_j(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \\ & + \int_{S^+} \tilde{t}_p(\mathbf{y}) \int_{S^+} G_{mj}^p(\boldsymbol{\xi} - \mathbf{y}) D_m \Delta u_j(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \end{aligned} \quad (13)$$

where $\Sigma u_p = u_p^+ + u_p^-$ denotes the sum of the crack-face displacement of the original cracked medium. By forming the product $\sigma_{lk}(\mathbf{x}) n_l^+(\mathbf{y})$ of (12) for $\mathbf{y} \in S^+$ and taking the proper limit $\mathbf{x} \rightarrow \mathbf{y} \in S^+$, then taking the inner product of the result with a sufficiently smooth test function $\tilde{u}_k(\mathbf{y})$, and finally integrating the equation over the crack surface S^+ and carrying out an integration by parts for the term involving the function $C_{mj}^{tk}(\boldsymbol{\xi} - \mathbf{y})$, there arises a weak-form boundary integral equation for the jump in the crack-face traction:

$$\begin{aligned} \int_{S^+} \tilde{u}_k(\mathbf{y}) \left\{ \frac{1}{2} \Delta t_k^0(\mathbf{y}) - \sigma_{lk}^\infty n_l^+(\mathbf{y}) \right\} dA(\mathbf{y}) = & - \int_{S^+} \tilde{u}_k(\mathbf{y}) \int_{S^+} n_l^+(\mathbf{y}) H_{lk}^j(\boldsymbol{\xi} - \mathbf{y}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \\ & - \int_{S^+} D_t \tilde{u}_k(\mathbf{y}) \int_{S^+} G_{tk}^j(\boldsymbol{\xi} - \mathbf{y}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \\ & - \int_{S^+} D_t \tilde{u}_k(\mathbf{y}) \int_{S^+} C_{mj}^{tk}(\boldsymbol{\xi} - \mathbf{y}) D_m \Delta u_j(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \end{aligned} \quad (14)$$

where $\Delta t_k^0 = t_k^{0+} - t_k^{0-}$ denotes the jump in the prescribed crack-face traction of the original cracked medium. Again, the line integral resulting from the integration by parts is eliminated from (14) due to the fact that the test function $\tilde{u}_k$ is chosen to vanish along the crack front, $\tilde{u}_k(\mathbf{y}) = 0, \forall \mathbf{y} \in \partial S^+$. The pair of weak-form, boundary integral equations (13) and (14) forms a system of governing equations sufficient for determining the unknown crack-face displacement data (i.e., the unknown sum of the crack-face displacement Σu_p and the unknown jump in the crack-face displacement Δu_j). It is important to note that this pair of integral equations involves only weakly singular kernels of $\mathcal{O}(1/r)$ (the weakly singular nature of the functions $n_i(\mathbf{x}) H_{ij}^p(\boldsymbol{\xi} - \mathbf{x})$ and $n_i(\boldsymbol{\xi}) H_{ij}^p(\boldsymbol{\xi} - \mathbf{x})$ can be found in the work of Xiao [62]).

4. Solution procedure

To determine the unknown crack-face displacement Σu_p and Δu_j and the T-stress data along the crack front, the following solution methodology is proposed. First, the weakly singular, weak-form boundary integral equation (14) is solved for the jump in the crack-face displacement Δu_j , and then the sum of the crack-face displacement Σu_p is obtained by solving the weakly singular, weak-form boundary integral equation (13). The near-front data of Σu_p is then utilized along with the post-process formulas described below to extract the T-stress components along the crack front.

4.1. Numerical solution for Δu_j and Σu_p

To numerically solve the integral equation (14) for Δu_j , a weakly singular symmetric Galerkin boundary element method (SGBEM) is implemented. First, the trial function Δu_j and test function $\tilde{u}_k$ are discretized, using Galerkin approximation together with the finite element procedure, by

$$\Delta u_j(\boldsymbol{\xi}) = \sum_{p=1}^N u_j^{\Delta(p)} \phi^{(p)}(\boldsymbol{\xi}); \quad \tilde{u}_k(\mathbf{y}) = \sum_{p=1}^N \hat{u}_k^{(p)} \phi^{(p)}(\mathbf{y}) \quad (15)$$

where $\phi^{(p)}$ is the nodal basis function associated with the p th node; $u_j^{\Delta(p)}$ are nodal degrees of freedom at the p th node; $\hat{u}_k^{(p)}$ are arbitrary constants at the p th node; and N denotes the number of nodes resulting from the discretization of the crack surface into two-dimensional finite elements. The nodal basis functions $\phi^{(p)}$ are constructed locally using the concept of finite elements and, in the present study, standard C^0 -elements are utilized everywhere except in the near-front region where special crack-tip elements, developed by Li et al. [63] and Rungamornrat and Mear [64], are used. The key purpose of utilizing these special elements is to enhance the approximation of the near-front relative

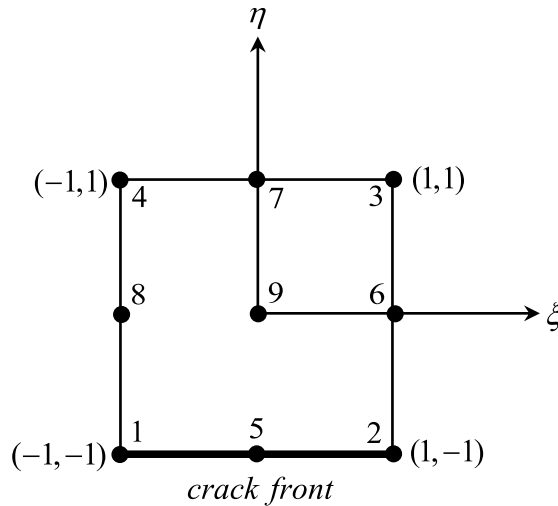


Fig. 2. Schematic of master element associated with 9-node crack-tip element with the side $\eta + 1 = 0$ corresponding to the crack front.

crack-face displacement whose asymptotic behavior is known a priori (see the [Appendix](#) and Section 4.2). To capture such near-front square-root behavior, extra degrees of freedom are placed along the crack front and they are directly related to the gradient of the relative crack-face displacement. The explicit element shape functions for a 9-node crack-tip element adopted in the present study, defined in terms of master coordinates $(\xi, \eta) \in [-1, 1] \times [-1, 1]$ shown in [Fig. 2](#), are given by

$$\bar{\psi}_{(i)}(\xi, \eta) = \frac{\sqrt{1+\eta}}{A_i} \psi_{(i)}(\xi, \eta) \quad (\text{no sum on } i) \quad (16)$$

where $\psi_{(i)}(\xi, \eta)$ are standard element shape functions for a 9-node quadratic element; the side $\eta + 1 = 0$ of the element corresponds to the crack-front; and

$$A_i = \begin{cases} 1/2 & \eta_{(i)} = -1 \\ \sqrt{1+\eta_{(i)}} & \eta_{(i)} \neq -1 \end{cases} \quad (17)$$

with $(\xi_{(i)}, \eta_{(i)})$ denoting master coordinates of the i th node.

By substituting (15) into (14) and then exploiting the arbitrariness of $\hat{u}_k^{(p)}$, there results a system of linear algebraic equations:

$$\mathbf{C} \Delta \mathbf{U} = \mathbf{T}^0 + \mathbf{T}^\infty \quad (18)$$

where $\Delta \mathbf{U}$ is a vector containing nodal degrees of freedom defined by $[\Delta \mathbf{U}]_{3(p-1)+j} = u_j^{\Delta(q)}$ and the coefficient matrix $\mathbf{C}$ and the known vectors $\mathbf{T}^0$ and $\mathbf{T}^\infty$ are defined by

$$[\mathbf{C}]_{3(p-1)+i, 3(q-1)+j} = \int_{S^+} D_t \phi^{(p)}(\mathbf{y}) \int_{S^+} C_{mj}^{ti}(\xi - \mathbf{y}) D_m \phi^{(q)}(\xi) dA(\xi) dA(\mathbf{y}) \quad (19)$$

$$[\mathbf{T}^0]_{3(p-1)+i} = -\frac{1}{2} \int_{S^+} \phi^{(p)}(\mathbf{y}) \Delta t_i^0(\mathbf{y}) dA(\mathbf{y}) - \int_{S^+} D_t \phi^{(p)}(\mathbf{y}) \int_{S^+} G_{ti}^j(\xi - \mathbf{y}) \Sigma t_j^0(\xi) dA(\xi) dA(\mathbf{y}) \\ - \int_{S^+} \phi^{(p)}(\mathbf{y}) \int_{S^+} n_l^+(\mathbf{y}) H_{li}^j(\xi - \mathbf{y}) \Sigma t_j^0(\xi) dA(\xi) dA(\mathbf{y}) \quad (20)$$

$$[\mathbf{T}^\infty]_{3(p-1)+i} = \int_{S^+} \phi^{(p)}(\mathbf{y}) \sigma_{li}^\infty n_l^+(\mathbf{y}) dA(\mathbf{y}) \quad (21)$$

It should be evident from (19) that the coefficient matrix $\mathbf{C}$ is symmetric. To efficiently construct the coefficient matrix $\mathbf{C}$ and the vectors $\mathbf{T}^0$ and $\mathbf{T}^\infty$, the integrals (19)–(21) are computed at the element level and their contribution to the global matrix and vectors are then obtained via a standard assembly procedure. In the numerical evaluation of double surface integrals over a pair of finite elements, special quadrature rules proposed by Xiao [62] are utilized to

efficiently and accurately integrate all involved weakly singular and nearly singular integrals. It should be remarked that the final coefficient matrix $\mathbf{C}$ is dense and negative definite due to the property of the kernel C_{mj}^{ii} . To solve the definite system of linear algebraic equations (18), a standard indirect solver based upon a preconditioning conjugate gradient scheme is utilized with a specified relative tolerance of 10^{-6} . It has been found in the numerical study that the dense nature of $\mathbf{C}$ does not significantly alter the convergence behavior of the linear solver.

Once the jump in the crack-face displacement Δu_j has been determined, the terms on the right hand side of the weak-form boundary integral equation (13) are known, and the as yet unknown sum of the crack-face displacement Σu_p is involved in only a single surface integral. To solve the integral equation for the unknown Σu_p , a standard Galerkin finite element technique is applied. First, the trial and test functions Σu_p and $\tilde{t}_p$ are approximated by

$$\Sigma u_p(\mathbf{y}) - 2u_p^\infty(\mathbf{y}) = \Sigma u_p^*(\mathbf{y}) = \sum_{q=1}^N u_p^{\Sigma(q)} \psi^{(q)}(\mathbf{y}); \quad \tilde{t}_p(\mathbf{y}) = \sum_{q=1}^N \hat{t}_p^{(q)} \psi^{(q)}(\mathbf{y}) \quad (22)$$

where $\Sigma u_p^*(\mathbf{y})$ denotes the sum of the crack-face displacement associated with the sub-problem (ii); $u_p^{\Sigma(q)}$ denotes the sum of the crack-face displacement associated with the sub-problem (ii) at the q th node; $\hat{t}_p^{(q)}$ are arbitrary constants at the q th node; $\psi^{(q)}$ is the nodal basis function associated with the q th node; and N is the number of all nodes. Due to the regularity of the sum of the crack-face displacement (see the Appendix and Section 4.2), the nodal basis functions $\psi^{(q)}$ are constructed based on standard C^0 -elements over the entire crack surface using the same mesh as that employed in the discretization of (14). By substituting (22) and Δu_j from (15) into (13) and then using the arbitrariness of $\hat{t}_p^{(q)}$, there results the system of linear equations

$$\mathbf{L}\Sigma\mathbf{U} = \mathbf{D}\Delta\mathbf{U} + \mathbf{T}^{\Sigma 0} \quad (23)$$

where $\Sigma\mathbf{U}$ is a vector containing nodal degrees of freedom defined by $[\Sigma\mathbf{U}]_{3(p-1)+j} = u_j^{\Sigma(q)}$, and the coefficient matrices $\mathbf{L}$ and $\mathbf{D}$ and the known vector $\mathbf{T}^{\Sigma 0}$ are defined by

$$[\mathbf{L}]_{3(p-1)+i, 3(q-1)+j} = \frac{1}{2} \int_{S^+} \psi^{(p)}(\mathbf{y}) \psi^{(q)}(\mathbf{y}) \delta_{ij} dA(\mathbf{y}) \quad (24)$$

$$[\mathbf{D}]_{3(p-1)+i, 3(q-1)+j} = \int_{S^+} \psi^{(p)}(\mathbf{y}) \int_{S^+} G_{mj}^i(\boldsymbol{\xi} - \mathbf{y}) D_m \phi^{(q)}(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \\ - \int_{S^+} \psi^{(p)}(\mathbf{y}) \int_{S^+} n_m^+(\boldsymbol{\xi}) H_{mj}^i(\boldsymbol{\xi} - \mathbf{y}) \phi^{(q)}(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \quad (25)$$

$$[\mathbf{T}^{\Sigma 0}]_{3(p-1)+i} = \int_{S^+} \psi^{(p)}(\mathbf{y}) \int_{S^+} U_j^i(\boldsymbol{\xi} - \mathbf{y}) \Sigma t_j^0(\boldsymbol{\xi}) dA(\boldsymbol{\xi}) dA(\mathbf{y}) \quad (26)$$

It is evident from (24) that the coefficient matrix $\mathbf{L}$ is symmetric. To construct the matrices $\mathbf{L}$ and $\mathbf{D}$ and the vector $\mathbf{T}^{\Sigma 0}$, a procedure similar to that used to construct $\mathbf{C}$, $\mathbf{T}^0$ and $\mathbf{T}^\infty$ is adopted. The final coefficient matrix $\mathbf{L}$ is positive definite, and it is sparse due to the local feature of the basis functions used in the discretization. The resulting system of linear algebraic equations (23) is then solved by the preconditioning conjugate gradient scheme for a specified relative tolerance 10^{-6} .

4.2. Post-process for T -stress components

Let $\mathbf{x}_c$ denote a point along the crack-front and $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ be a local Cartesian coordinate system with the origin at $\mathbf{x}_c$, the $\bar{x}_1$ – $\bar{x}_2$ plane being normal to the crack front at $\mathbf{x}_c$, and the $\bar{x}_1$ – $\bar{x}_3$ plane being tangent to the crack surface at $\mathbf{x}_c$ as indicated in Fig. 3. As suggested by the original work of Williams [1] and Hoenig [65], the near-front strain and stress fields in the local $\bar{x}_1$ – $\bar{x}_2$ plane admit the following series representations (also see the Appendix)

$$\bar{\varepsilon}_{ij}(\mathbf{x}_c; \bar{r}, \bar{\theta}) = \frac{1}{\sqrt{\bar{r}}} \bar{\varepsilon}_{ij}^K(\mathbf{x}_c; \bar{\theta}) + \bar{\varepsilon}_{ij}^{finite}(\mathbf{x}_c) + \sum_{m=1}^{\infty} \bar{r}^{m/2} \bar{\varepsilon}_{ij}^m(\mathbf{x}_c; \bar{\theta}) \quad (27)$$

$$\bar{\sigma}_{ij}(\mathbf{x}_c; \bar{r}, \bar{\theta}) = \frac{1}{\sqrt{\bar{r}}} \bar{\sigma}_{ij}^K(\mathbf{x}_c; \bar{\theta}) + \bar{\sigma}_{ij}^{finite}(\mathbf{x}_c) + \sum_{m=1}^{\infty} \bar{r}^{m/2} \bar{\sigma}_{ij}^m(\mathbf{x}_c; \bar{\theta}) \quad (28)$$

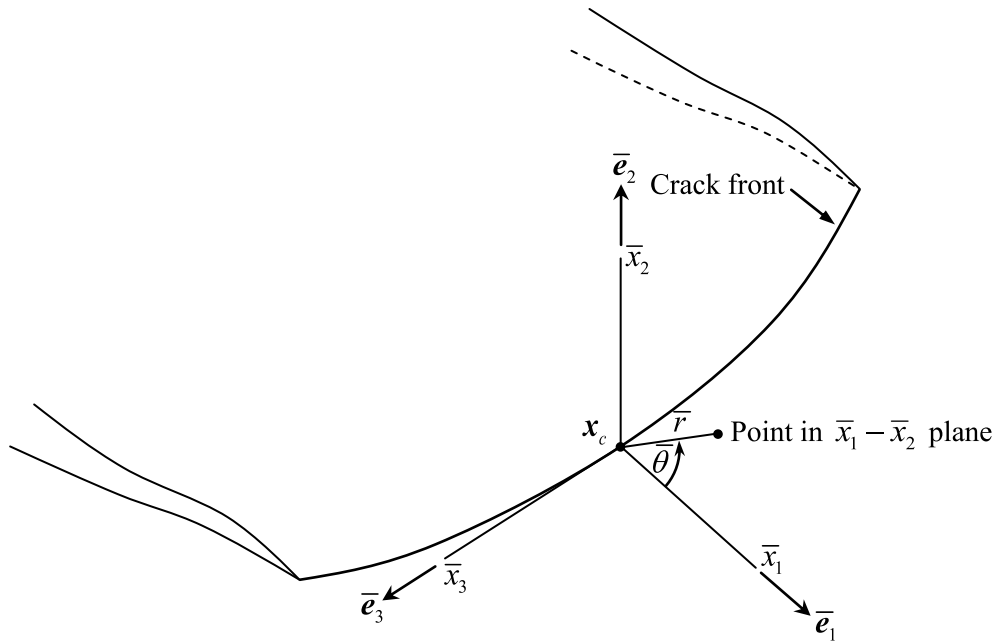


Fig. 3. Schematic of crack front, local Cartesian coordinate system $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ at point $\mathbf{x}_c$, and corresponding polar coordinates $(\bar{r}, \bar{\theta})$ of any point in the $\bar{x}_1$ - $\bar{x}_2$ plane.

where $(\bar{r}, \bar{\theta})$ denotes the local polar coordinates of a point on the local $\bar{x}_1$ - $\bar{x}_2$ plane with the origin at $\mathbf{x}_c$; $\bar{\varepsilon}_{ij}$ and $\bar{\sigma}_{ij}$ are components of the strain and stress tensors with respect to the local coordinate system $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$; $\bar{\varepsilon}_{ij}^K$, $\bar{\varepsilon}_{ij}^m$, $\bar{\sigma}_{ij}^K$ and $\bar{\sigma}_{ij}^m$ are functions independent of the local radial coordinate $\bar{r}$; and $\bar{\varepsilon}_{ij}^{finite}$ and $\bar{\sigma}_{ij}^{finite}$ are functions independent of the local radial and angular coordinates $\bar{r}, \bar{\theta}$. It is evident from (27) and (28) that all terms in the summation vanish identically at the crack front $\mathbf{x}_c$ and, as a result, the strain and stress at the crack front $\mathbf{x}_c$ can be divided into two parts, a singular part of order $1/\sqrt{\bar{r}}$ and a finite part denoted by $\bar{\varepsilon}_{ij}^{finite}$ and $\bar{\sigma}_{ij}^{finite}$, respectively. By recalling that $\bar{\varepsilon}_{ij}$ and $\bar{\sigma}_{ij}$ must satisfy the constitutive law, it is readily shown that the finite-part strain $\bar{\varepsilon}_{ij}^{finite}$ and the finite-part stress $\bar{\sigma}_{ij}^{finite}$ are linearly related by

$$\bar{\sigma}_{ij}^{finite}(\mathbf{x}_c) = \bar{E}_{ijkl}(\mathbf{x}_c) \bar{\varepsilon}_{kl}^{finite}(\mathbf{x}_c) \quad (29)$$

where $\bar{E}_{ijkl}(\mathbf{x}_c)$ denotes components of the elastic moduli with respect to the local coordinate system $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$.

By forming the near-front traction on the crack surface from the stress field $\bar{\sigma}_{ij}$ given by (28) and then using the fact that the crack-face traction is fully prescribed and finite, the components $\bar{\sigma}_{12}^{finite}$, $\bar{\sigma}_{22}^{finite}$ and $\bar{\sigma}_{23}^{finite}$ can then be obtained in terms of the prescribed crack-face tractions at the limiting point of $\mathbf{x}_c$ on the crack surface (i.e., $\bar{\theta} = \pm\pi$, $\bar{r} = -\bar{x}_1 \rightarrow 0$). The remaining independent components of the finite-part stress $\bar{\sigma}_{ij}^{finite}$ (i.e., $\bar{\sigma}_{11}^{finite}$, $\bar{\sigma}_{13}^{finite}$, $\bar{\sigma}_{33}^{finite}$) are unknown a priori, and here they are termed the T-stress components defined by $T_{11} \equiv \bar{\sigma}_{11}^{finite}$, $T_{13} \equiv \bar{\sigma}_{13}^{finite}$ and $T_{33} \equiv \bar{\sigma}_{33}^{finite}$. By integrating the strain–displacement relation together with the form of the strain field (27), the displacement at any point in the local $\bar{x}_1$ - $\bar{x}_2$ plane takes the following structure

$$\begin{aligned} \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}, \bar{\theta}) = & \bar{\mathbf{u}}^o(\mathbf{x}_c) + \sqrt{\bar{r}} \bar{\mathbf{u}}^K(\mathbf{x}_c; \bar{\theta}) + \bar{r} \left\{ \begin{aligned} & \bar{\varepsilon}_{11}^{finite}(\mathbf{x}_c) \cos \bar{\theta} + [\bar{\varepsilon}_{12}^{finite}(\mathbf{x}_c) - \bar{\omega}_3^o(\mathbf{x}_c)] \sin \bar{\theta} \\ & [\bar{\varepsilon}_{21}^{finite}(\mathbf{x}_c) + \bar{\omega}_3^o(\mathbf{x}_c)] \cos \bar{\theta} + \bar{\varepsilon}_{22}^{finite}(\mathbf{x}_c) \sin \bar{\theta} \\ & [\bar{\varepsilon}_{31}^{finite}(\mathbf{x}_c) - \bar{\omega}_2^o(\mathbf{x}_c)] \cos \bar{\theta} + [\bar{\omega}_1^o(\mathbf{x}_c) + \bar{\varepsilon}_{32}^{finite}(\mathbf{x}_c)] \sin \bar{\theta} \end{aligned} \right\} \\ & + \sum_{m=1}^{\infty} \bar{r}^{m/2+1} \bar{\mathbf{u}}^m(\mathbf{x}_c; \bar{\theta}) \end{aligned} \quad (30)$$

where $\bar{\mathbf{u}}^K$ and $\bar{\mathbf{u}}^m$ are functions independent of the local radial coordinate $\bar{r}$; and $\bar{\mathbf{u}}^o$ and $\bar{\omega}^o$ represent the rigid displacement and rotation at any point $\mathbf{x}_c$, respectively, and satisfy the following relations

$$\frac{\partial \bar{u}_1^0(\mathbf{x}_c)}{\partial \bar{x}_3} = \bar{\varepsilon}_{13}^{finite}(\mathbf{x}_c) + \bar{\omega}_2^o(\mathbf{x}_c); \quad \frac{\partial \bar{u}_2^0(\mathbf{x}_c)}{\partial \bar{x}_3} = \bar{\varepsilon}_{23}^{finite}(\mathbf{x}_c) - \bar{\omega}_1^o(\mathbf{x}_c); \quad \frac{\partial \bar{u}_3^0(\mathbf{x}_c)}{\partial \bar{x}_3} = \bar{\varepsilon}_{33}^{finite}(\mathbf{x}_c) \quad (31)$$

Next, let us define the sum of the crack-face displacements $\Sigma \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}) = \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}, \pi) + \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}, -\pi)$ and the jump in the crack-face displacements $\Delta \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}) = \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}, \pi) - \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}, -\pi)$. For two-dimensional, linear elastic media, results shown in the [Appendix](#) and those reported in [65] for the two-dimensional setting show that $\Sigma \bar{\mathbf{u}}$ is only in terms of integral powers of $\bar{r}$ while $\Delta \bar{\mathbf{u}}$ involves only fractional powers of $\bar{r}$. This feature should also pertain in the three-dimensional case and, as a result, the sum of and jump in the crack-face displacements take the following form

$$\Sigma \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}) = 2\bar{\mathbf{u}}^o(\mathbf{x}_c) - 2\bar{r} \left\{ \begin{array}{l} \bar{\varepsilon}_{11}^{finite}(\mathbf{x}_c) \\ \bar{\varepsilon}_{21}^{finite}(\mathbf{x}_c) + \bar{\omega}_3^o(\mathbf{x}_c) \\ \bar{\varepsilon}_{31}^{finite}(\mathbf{x}_c) - \bar{\omega}_2^o(\mathbf{x}_c) \end{array} \right\} + \sum_{k=1}^{\infty} \bar{r}^{k+1} [\bar{\mathbf{u}}^{2k}(\mathbf{x}_c; \pi) + \bar{\mathbf{u}}^{2k}(\mathbf{x}_c; -\pi)] \quad (32)$$

$$\Delta \bar{\mathbf{u}}(\mathbf{x}_c; \bar{r}) = \sqrt{\bar{r}} \left[\bar{\mathbf{u}}^K(\mathbf{x}_c; \pi) - \bar{\mathbf{u}}^K(\mathbf{x}_c; -\pi) + \sum_{k=1}^{\infty} \bar{r}^k \{ \bar{\mathbf{u}}^{2k-1}(\mathbf{x}_c; \pi) - \bar{\mathbf{u}}^{2k-1}(\mathbf{x}_c; -\pi) \} \right] \quad (33)$$

It is evident from (31) and (32) that

$$\left\{ \begin{array}{l} \bar{\varepsilon}_{11}^{finite}(\mathbf{x}_c) \\ \bar{\varepsilon}_{33}^{finite}(\mathbf{x}_c) \\ \bar{\varepsilon}_{13}^{finite}(\mathbf{x}_c) \end{array} \right\} = \lim_{\bar{r} \rightarrow 0} \left\{ \begin{array}{l} -\frac{1}{2} \frac{\partial \Sigma \bar{u}_1(\mathbf{x}_c; \bar{r})}{\partial \bar{r}} \\ \frac{1}{2} \frac{\partial \Sigma \bar{u}_3(\mathbf{x}_c; \bar{r})}{\partial \bar{x}_3} \\ \frac{1}{4} \frac{\partial \Sigma \bar{u}_1(\mathbf{x}_c; \bar{r})}{\partial \bar{x}_3} - \frac{1}{4} \frac{\partial \Sigma \bar{u}_3(\mathbf{x}_c; \bar{r})}{\partial \bar{r}} \end{array} \right\} \quad (34)$$

The relation (34) allows three components of the finite-part strain $\bar{\varepsilon}_{ij}^{finite}$ to be computed from the sum of the crack-face displacement data. From the method of superposition, the finite-part strain components $\bar{\varepsilon}_{kl}^{finite}(\mathbf{x}_c)$ of the given original problem can be further decomposed into

$$\bar{\varepsilon}_{kl}^{finite}(\mathbf{x}_c) = \bar{\varepsilon}_{kl}^{\infty} + \bar{\varepsilon}_{kl}^*(\mathbf{x}_c) \quad (35)$$

where $\bar{\varepsilon}_{kl}^{\infty}$ are local strain components associated with the far field σ^{∞} in the sub-problem (i) and $\bar{\varepsilon}_{kl}^*(\mathbf{x}_c)$ denotes the finite-part strain components at $\mathbf{x}_c$ of a cracked medium in the sub-problem (ii) which can be obtained from the formula (34) by simply replacing $\Sigma \bar{\mathbf{u}}$ by $\Sigma \bar{\mathbf{u}}^*$. Substituting (35) into (29) leads to

$$\bar{\sigma}_{ij}^{finite}(\mathbf{x}_c) = \bar{\sigma}_{ij}^{\infty}(\mathbf{x}_c) + \bar{E}_{ijkl}(\mathbf{x}_c) \bar{\varepsilon}_{kl}^*(\mathbf{x}_c) \quad (36)$$

where $\bar{\sigma}_{ij}^{\infty}(\mathbf{x}_c)$ are local components of the far field σ^{∞} at the point $\mathbf{x}_c$. From the fact that $\bar{\sigma}_{12}^{finite}$, $\bar{\sigma}_{22}^{finite}$ and $\bar{\sigma}_{23}^{finite}$ are known from the applied crack-face traction data, and the fact that $\bar{\varepsilon}_{11}^*(\mathbf{x}_c)$, $\bar{\varepsilon}_{13}^*(\mathbf{x}_c)$ and $\bar{\varepsilon}_{33}^*(\mathbf{x}_c)$ are given in terms of the sum of the crack-face data $\Sigma \bar{\mathbf{u}}^*$ by (34), the unknown T-stress components T_{11} , T_{13} , T_{33} can now be obtained by solving a system of six linear algebraic equations (36).

5. Numerical results

To demonstrate the capabilities and accuracy of the proposed technique, numerical studies are carried out for various boundary value problems involving planar, non-planar, and multiple cracks. First, two representative problems, one associated with a penny-shaped crack under normal crack-face tractions and remote stresses and the other associated with an elliptical crack under uniform and linear crack face tractions, are considered and the accuracy of computed results is confirmed by benchmarking with existing analytical solutions proposed by Rungamornrat and Pinitpanich [46] and Kirilyuk and Levchuk [45]. After the technique is fully tested, two problems concerning a spherical cap crack and a pair of identical penny-shaped cracks are investigated to further demonstrate the capability of the technique to treat mixed-mode conditions, non-planar crack geometry, and multiple cracks. In the numerical study, a series of meshes is adopted to explore the convergence behavior of approximate solutions and their dependency on the level of mesh refinement. In particular, standard 6-node triangular elements and 8-node quadrilateral elements are

Table 1

Elastic constants of a representative transversely isotropic elastic material used in the numerical study. Constants shown below are specific for a material with the axis of material symmetry along the x_3 -coordinate direction.

Elastic constants ($\times 10^9$ Pa)	
E_{1111}	160
E_{1122}	35
E_{1133}	50
E_{3333}	60
E_{1313}	40

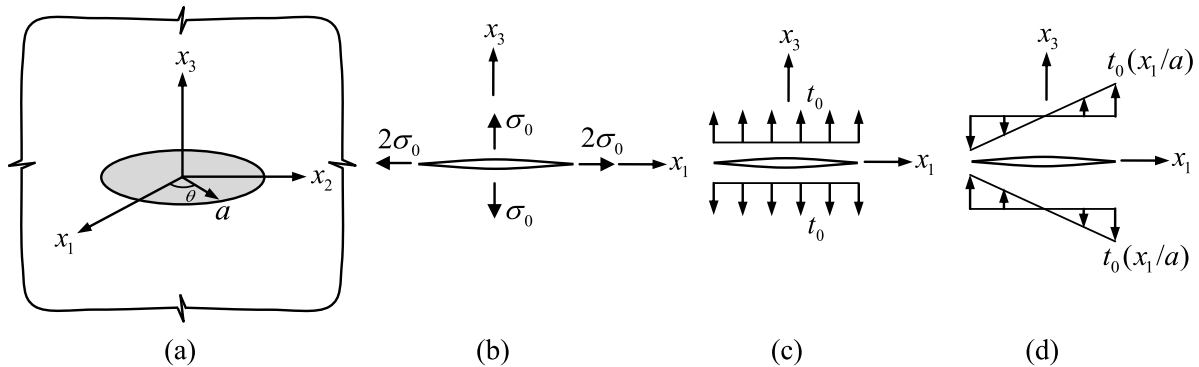


Fig. 4. Schematic of (a) a penny-shaped crack of radius a in an infinite elastic medium, (b) crack under remote triaxial stress field $\sigma_{11}^\infty = 2\sigma_0$, $\sigma_{22}^\infty = 0$, $\sigma_{33}^\infty = \sigma_0$, (c) crack under uniformly distributed normal traction $t_3^+ = -t_3^- = t_0$, and (d) crack under linearly distributed normal traction $t_3^+ = -t_3^- = t_0(x_1/a)$.

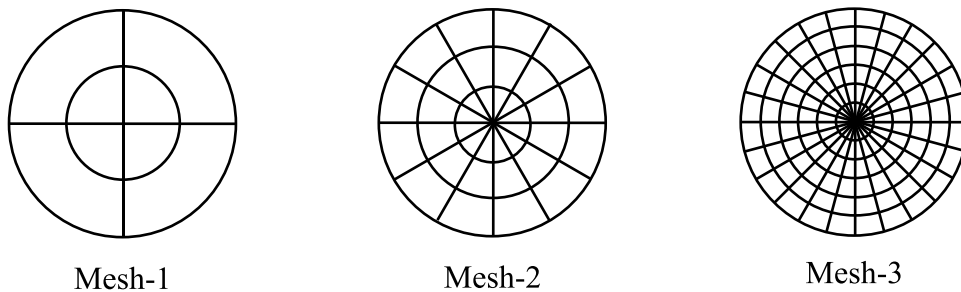


Fig. 5. Three meshes for crack surface used in the analysis.

employed everywhere in the discretization except along the crack front where special 9-node quadrilateral crack-tip elements are adopted to approximate the near-front relative crack-face displacement. An isotropic, linearly elastic material (with Poisson's ratio $\nu = 0.3$) and a transversely isotropic solid with the axis of material symmetry directing along the x_3 -axis (with elastic constants given in Table 1) are chosen as representative materials.

5.1. Penny-shaped crack

Consider a penny-shaped crack of radius a for which the crack surface is perpendicular to the axis of material symmetry (i.e., the x_3 -axis) as shown in Fig. 4(a). Three loading conditions, indicated in Fig. 4(b)–(d), are investigated and three meshes for the crack surface, shown in Fig. 5, are primarily adopted in the analysis. Note that the coarsest mesh (Mesh-1), contains only 8 elements with 4 crack-tip elements and the finest mesh (Mesh-3) contains 144 elements with 24 crack-tip elements.

Table 2

Normalized T-stress components T_{11} and T_{33} for a penny-shaped crack under uniformly distributed normal traction. Results are normalized by the exact solutions reported by Rungamornrat and Pinitpanich [46].

Mesh	Isotropic material		Transversely isotropic material	
	T_{11}/T_{11}^{exact}	T_{33}/T_{33}^{exact}	T_{11}/T_{11}^{exact}	T_{33}/T_{33}^{exact}
1	1.0004	0.9965	1.0026	0.9962
2	0.9975	0.9992	0.9983	1.0001
3	0.9954	0.9987	0.9966	1.0002
2a	0.9990	0.9998	0.9997	1.0000
3a	0.9981	0.9996	0.9991	1.0001

5.1.1. Remote triaxial stresses

For a penny-shaped crack under a remote triaxial stress $\sigma_{11}^{\infty} = 2\sigma_0$, $\sigma_{22}^{\infty} = 0$, $\sigma_{33}^{\infty} = \sigma_0$ (see Fig. 4(b)), all three T-stress components T_{11} , T_{33} and T_{13} are non-trivial along the crack front and the analytical solution for this particular case was obtained by Rungamornrat and Pinitpanich [46]. Computed T-stresses for all three meshes and two representative materials are normalized and reported in Fig. 6(a)–(c) along with the analytical solution. It is seen from these results that the proposed technique yields highly accurate numerical solutions. In particular, T-stress components generated by the finest mesh (i.e., Mesh-3) are nearly indistinguishable from the benchmark solution whereas the coarsest mesh (i.e., Mesh-1) can also capture the variation of the solution along the crack front with very good accuracy. In addition, the independence of the material properties of the T-stress component T_{13} can also be accurately captured by the present technique.

5.1.2. Uniformly distributed normal traction

For a penny-shaped crack under a uniformly distributed normal traction $t_3^+ = -t_3^- = t_0$ (see Fig. 4(c)), T_{13} vanishes identically and T_{11} and T_{33} are constant along the crack front due to the axisymmetry of the problem (also see the exact expressions derived by Rungamornrat and Pinitpanich [46]). Non-zero T-stress components at the point $(a, 0, 0)$, normalized by the exact solution, are reported in Table 2 for both the isotropic and transversely isotropic cases. As is apparent from this set of results, the computed T-stress components T_{11} and T_{33} for all three meshes (i.e., Mesh-1, Mesh-2 and Mesh-3) show excellent agreement with the reference solution; specifically, the discrepancy is less than 0.5% for all cases. We also note that while the T-stress component T_{13} is not reported here (given that its exact value is zero), the technique yields accurate numerical results for T_{13}/t_0 in that the computed values are (at most) on the order of 10^{-4} .

Now, while all three meshes yield solutions for which the error is but a fraction of one percent, it can be observed that the T-stress components generated from a finer mesh (resulting from a uniform refinement across the crack surface) are actually slightly less accurate than those from a coarser mesh, especially for the T-stress component T_{11} . Toward understanding this behavior, we first recall that a special crack-tip element is utilized and that this element is capable of capturing the first three terms of the asymptotic expansion for the relative crack-face displacement Δu . As the mesh is uniformly refined, the region modeled by the special crack-tip elements becomes smaller. At the same time, additional regular elements are added behind the crack-tip elements, but these elements are standard quadratic elements which are less capable of modeling the near crack-tip field in the radial direction. We also note that Δu serves as data in the displacement integral equation (13) and its accurate determination clearly affects the quality of the sum of the crack-face displacement and, in turn, the T-stress components.

To improve the solution accuracy upon refinement (and, specifically, a refinement that leads to a decrease in the size of the crack-tip elements), one might reasonably suspect that the mesh behind the crack-tip elements must be sufficiently refined in the direction normal to the crack front. To support this conjecture, results are presented for the two additional meshes shown in Fig. 7. These meshes, denoted Mesh-2a and Mesh-3a, are obtained by refining Mesh-2 and Mesh-3 in the radial direction behind the crack-tip elements (with the original crack-tip elements being retained).

Computed T-stress components corresponding to Mesh-2a and Mesh-3a are reported in Table 2. It is evident that as the region behind the crack-tip elements of Mesh-2 and Mesh-3 is refined in the radial direction (i.e., the direction normal to the crack front) while maintaining the size of the crack-tip elements, the accuracy of both T-stress

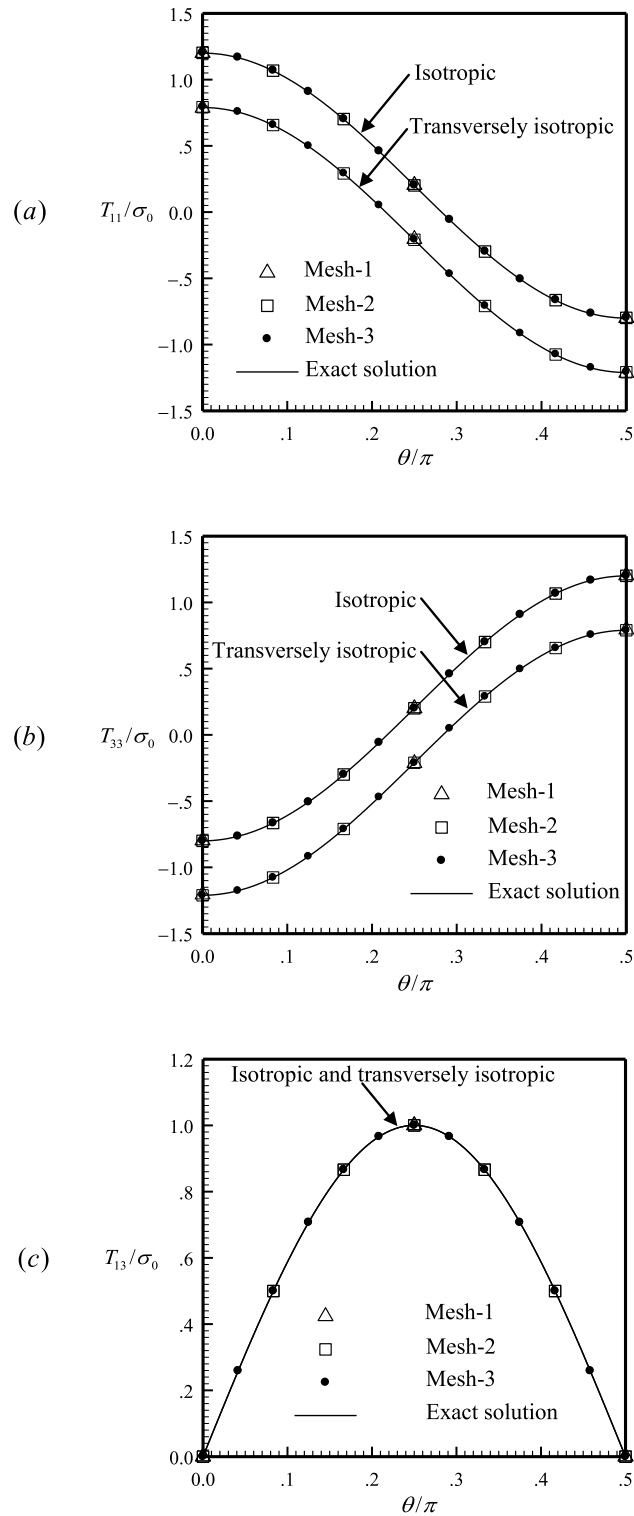


Fig. 6. Normalized T-stress components: (a) T_{11} , (b) T_{33} and (c) T_{13} for a penny-shaped crack under remote triaxial stress $\sigma_{11}^{\infty} = 2\sigma_0$, $\sigma_{22}^{\infty} = 0$, $\sigma_{33}^{\infty} = \sigma_0$. Exact solutions are taken from [46].

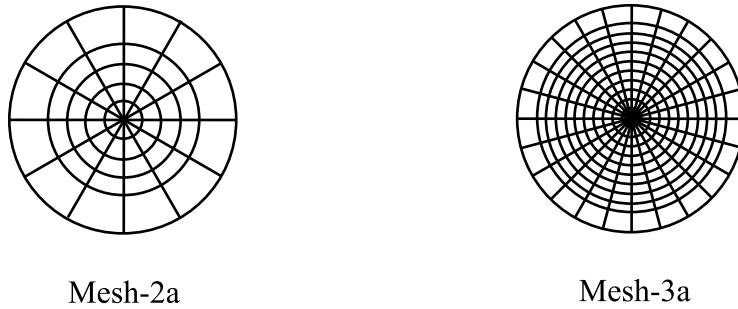


Fig. 7. Two additional meshes resulting from refining Mesh-2 and Mesh-3 shown in Fig. 5.

components is in fact significantly improved. It is worth noting that the exact sum of the crack-face displacement for this particular problem varies linearly along the radial direction; as a result, the refinement does not affect the ability to represent/interpolate this quantity, but it does nonetheless affect the quality of the computed solution since, as already emphasized, the jump in the displacements $\Delta \mathbf{u}$ is the primary data involved in the integral equation (13). It should also be noted that even without this additional refinement, the computed T-stress components are quite accurate.

5.1.3. Linearly distributed normal traction

For a penny-shaped crack under a linearly distributed normal traction $t_3^+ = -t_3^- = t_0(x_1/a)$ as indicated in Fig. 4(c), all T-stress components exist and vary along the crack front (see the analytical solutions reported by Rungamornrat and Pinitpanich [46]). That is, even though the loading is pure mode-I, the nonuniform traction induces all T-stress components along the crack front. The T-stress components computed based upon the three original meshes (see Fig. 5) are shown in Fig. 8(a)–(c); the numerical results agree very well with the benchmark solution for the entire crack front. Indeed, using only a few degrees of freedom in the solution discretization (associated with relatively large elements along the crack front) yields results of high accuracy; clearly, the T-stress components generated by the Mesh-1 are only slightly different from the exact solution. We remark again that this is due mainly to the use of special crack-tip elements in the approximation of the near-front relative crack-face displacement.

5.2. Elliptical crack

Consider next an elastic infinite medium containing an elliptical crack for which the major semi-axis is a and the minor semi-axis is b as shown in Fig. 9(a). The crack is oriented normal to the x_3 -axis with the crack front fully described by

$$x_1 = a \cos \theta, \quad x_2 = b \sin \theta, \quad x_3 = 0, \quad \theta \in [0, 2\pi] \quad (37)$$

The medium is made of an isotropic material with Poisson ratio $\nu = 0.3$ and loaded by either a uniformly distributed normal traction $t_3^+ = -t_3^- = t_0$ (Fig. 9(b)) or a linearly distributed normal traction $t_3^+ = -t_3^- = t_0(x_1/a)$ (Fig. 8(c)). This particular problem is considered here since the closed-form solutions of the T-stress component T_{11} for both loading cases were reported by Kirilyuk and Levchuk [45] and, as a result, it serves as additional verification for the proposed technique. In the numerical study, the crack aspect ratio is taken as $a/b = 2$ and the three meshes shown in Fig. 10 are utilized.

Computed T-stress components for both loading conditions are reported in Figs. 11 and 12. As is evident from these plots, the results exhibit only a slight mesh-dependence and the numerical solutions for T_{11} are in excellent agreement with the reported exact solution. In particular, Mesh-1 can accurately capture the distribution of the T-stress along the crack front and solutions generated by this very coarse mesh cannot be easily distinguished from those generated by the finer meshes. This reaffirms the validity of the present technique and its capability to obtain high quality solutions without the need for very fine meshes (especially in a local region along the crack front).

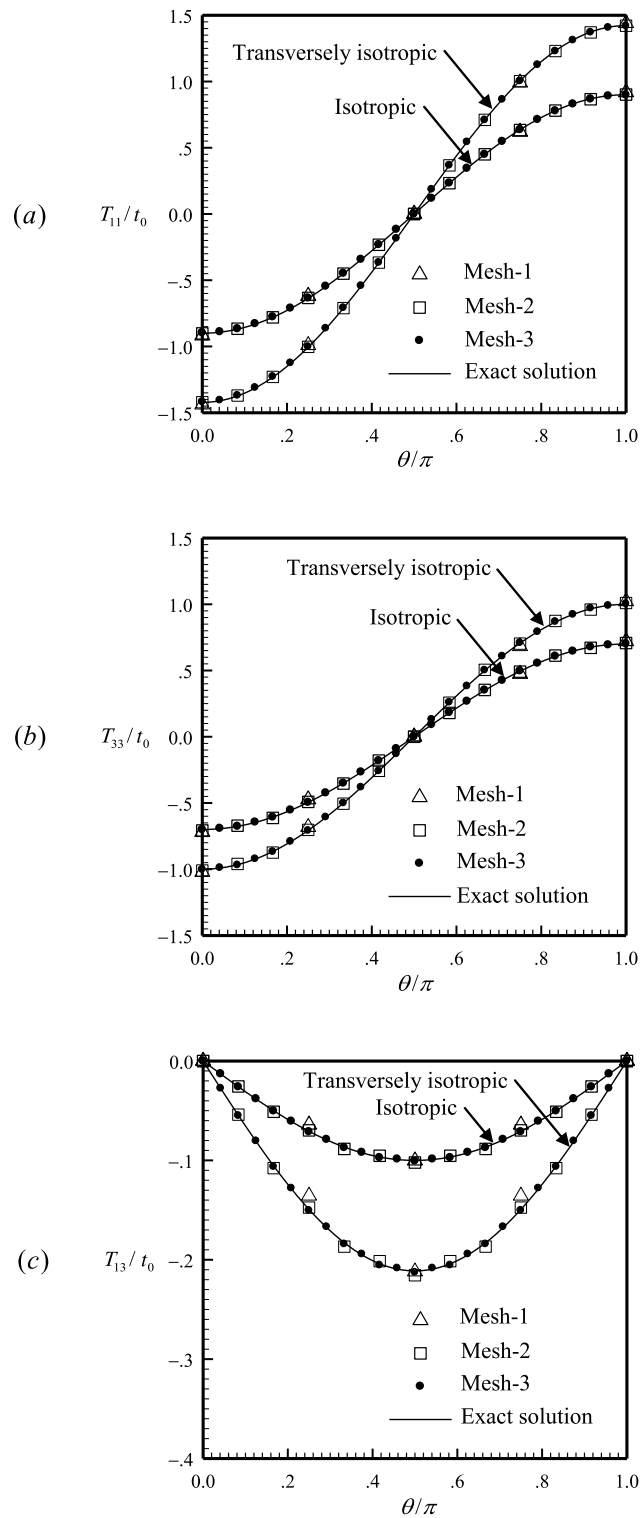


Fig. 8. Normalized T-stress components: (a) T_{11} , (b) T_{33} and (c) T_{13} for a penny-shaped crack under linear normal traction $t_3^+ = -t_3^- = t_0(x_1/a)$. Exact solutions are taken from [46].

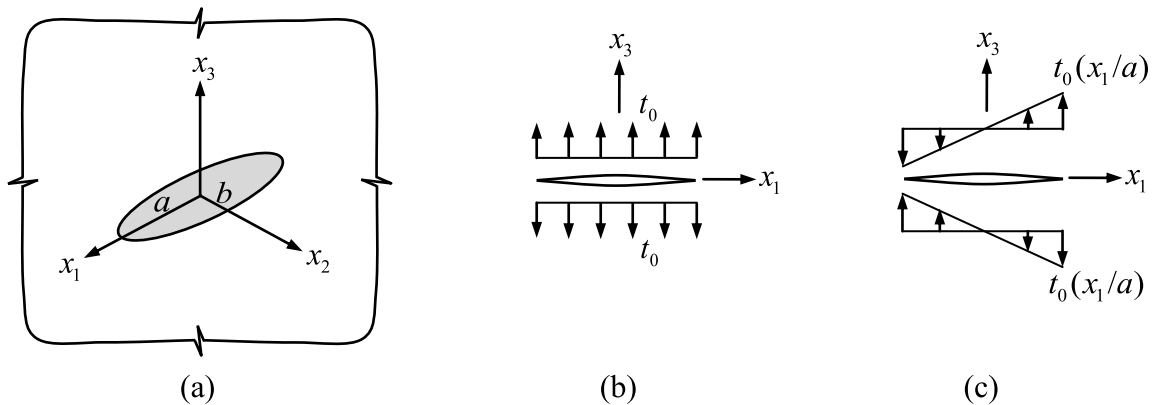


Fig. 9. Schematic of (a) an elliptical crack in an infinite elastic medium, (b) crack under uniformly distributed normal traction $t_3^+ = -t_3^- = t_0$, and (c) crack under linearly distributed normal traction $t_3^+ = -t_3^- = t_0(x_1/a)$.

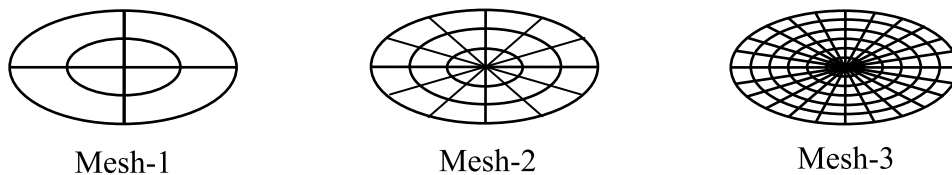


Fig. 10. Three meshes for the crack surface used in the analysis.

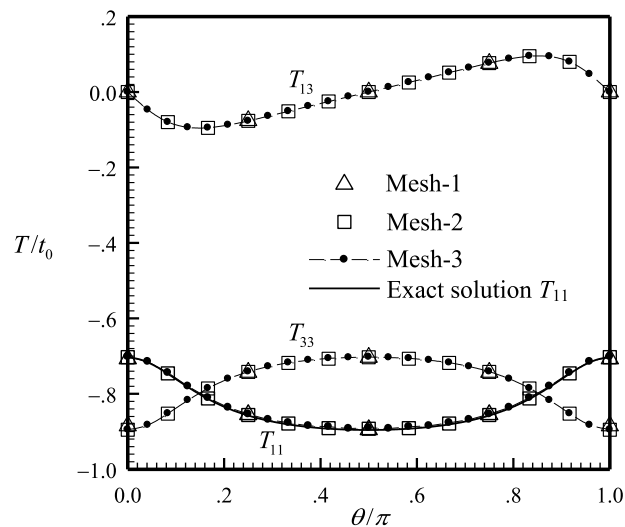


Fig. 11. Normalized T-stress components for an elliptical crack under uniformly distributed normal traction $t_3^+ = -t_3^- = t_0$. Exact solution for T_{11} is taken from [45].

5.3. Spherical cap crack

Now that the implemented technique has been fully tested against existing benchmark cases, a boundary value problem involving a nonplanar crack (specifically, a spherical cap crack) is considered to further demonstrate the capability of the proposed technique to treat more general crack geometries. We note that, for a nonplanar crack, the mesh also plays a role in representing (or approximating) the geometry of the crack, so having too coarse of a

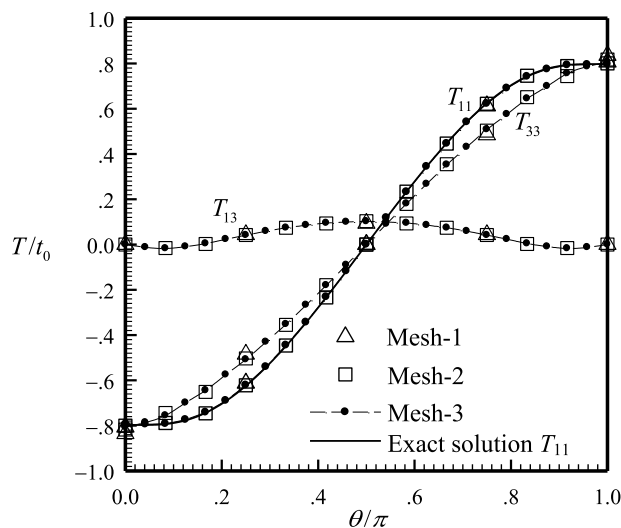


Fig. 12. Normalized T-stress components for an elliptical crack under linearly distributed normal traction $t_3^+ = -t_3^- = t_0(x_1/a)$. Exact solution for T_{11} is taken from [45].

mesh can impact the solution quality for this reason. We also note that the traction integral equation described here has previously been validated in terms of its use in evaluating the stress intensity factors for a spherical cap crack (see [60,64]), and those studies indicate that accurate data for Δu is in fact obtained from the technique.

Consider a representative spherical cap crack embedded in an infinite elastic medium as shown schematically in Fig. 13(a). The crack surface is fully described by

$$x_1 = a \sin \gamma \cos \beta, \quad x_2 = a \sin \gamma \sin \beta, \quad x_3 = a \cos \gamma, \quad \beta \in [0, 2\pi], \quad \gamma \in [0, \theta] \quad (38)$$

where a denotes the crack radius and θ represents the half subtended angle of the spherical surface. The axis of material symmetry, for the medium made of a transversely isotropic solid, is chosen to coincide with the x_3 -axis. The following three loading conditions are investigated: (i) a remote uniaxial tension σ_0 in the x_3 -direction, (ii) a remote uniaxial tension σ_0 in the x_1 -direction, and (iii) a self-equilibrated, uniformly distributed traction t_0 applied to the crack surface in the x_3 -direction (see Fig. 13(b)–(d)). To demonstrate the convergence behavior of numerical solutions generated by the proposed technique, a series of meshes with different levels of refinement is employed. In the analysis, the half-subtended angle is taken to be $\theta = 30^\circ$ and three meshes are adopted as shown in Fig. 14. It should be noted that the meshes shown should be viewed as those for a penny-shaped crack of radius $a\theta$ that have been ‘projected’ to properly fit the spherical crack surface.

For the first loading condition (see Fig. 13(b)), the problem is obviously axisymmetric and, as a consequence, the shear T-stress component T_{13} vanishes identically along the crack front whereas T_{11} and T_{33} are independent of the angle β . The computed T-stress components T_{11} and T_{33} (at $\beta = 0$ and $\gamma = 30^\circ$), normalized by that generated from Mesh-3, are reported in Table 3 for the two material models. The numerical results show a good convergence behavior in that only a modest dependence on the level of discretization is observed. In particular, the discrepancy between computed results from Mesh-1 and Mesh-2 and the reference solution is less than {2%, 0.2%} and {0.2%, 0.02%} for T_{11} and T_{33} , respectively. These results also indicate that, at least for this problem, the technique yields a better rate of convergence for T_{33} than for T_{11} .

For the second loading condition (see Fig. 13(c)), all T-stress components exist and vary along the crack front. Computed T-stress components are reported in Fig. 15(a)–(c) for both material models. It is apparent that results obtained from the three meshes are in good agreement and that even the coarsest mesh (which has only a few degrees of freedom) can capture, to reasonable accuracy, the variation of the T-stress along the crack front. Again, we mention that this high quality and good convergence of numerical solutions stem directly in large part from the use of special crack-tip elements along the crack front to accurately capture the near-front singular field.

Similarly to the first loading case, the boundary value problem is axisymmetric for the third loading case in which the crack is subjected to a uniform traction t_0 in the x_3 -direction (see Fig. 13(d)). Values of the non-zero T-stress

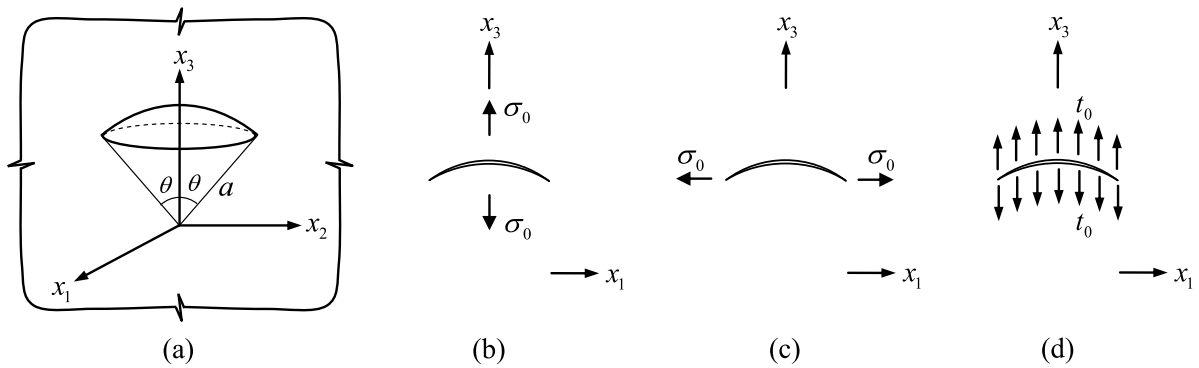


Fig. 13. Schematic of (a) a spherical cap crack of radius a in an infinite elastic medium, (b) crack under remote uniaxial stress field $\sigma_{11}^{\infty} = \sigma_{22}^{\infty} = 0$, $\sigma_{33}^{\infty} = \sigma_0$, (c) crack under remote uniaxial stress field $\sigma_{22}^{\infty} = \sigma_{33}^{\infty} = 0$, $\sigma_{11}^{\infty} = \sigma_0$, and (d) crack under uniformly distributed crack-face traction $t_3^+ = -t_3^- = t_0$.

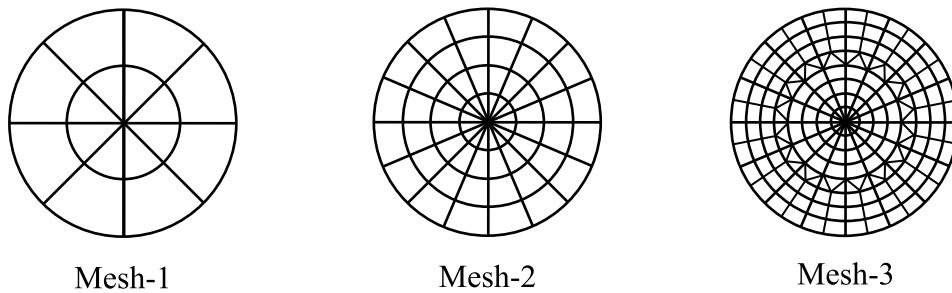


Fig. 14. Three meshes of a spherical cap crack used in analysis.

Table 3

Normalized T-stress components T_{11} and T_{33} for a spherical cap crack under remote uniaxial tension σ_0 acting in the x_3 -direction. Results are reported at the point for which $\beta = 0$, $\gamma = 30^\circ$ and are normalized by the reference solution corresponding to Mesh-3.

Mesh	Isotropic material		Transversely isotropic material	
	T_{11}/T_{11}^{ref}	T_{33}/T_{33}^{ref}	T_{11}/T_{11}^{ref}	T_{33}/T_{33}^{ref}
1	0.9971	0.9991	0.9821	0.9985
2	1.0013	1.0002	0.9992	0.9999
3	1.0000	1.0000	1.0000	1.0000
T^{ref}/σ_0	−0.3501	−0.6030	−0.3318	−1.2058

components T_{11} and T_{33} are reported in Table 4 for a representative point (specifically, the one corresponding to $\beta = 0$ and $\gamma = 30^\circ$) for both isotropic and transversely isotropic materials. To explore the convergence of the numerical solutions, all components of the T-stress are normalized by the reference solution obtained from the finest mesh (i.e., Mesh-3); as the level of refinement increases, a good convergence behavior is observed for all cases. The discrepancy between results generated by the Mesh-1 and Mesh-2 and that of the Mesh-3 is less than {2%, 0.3%} and {0.3%, 0.05%} for T-stress components T_{11} and T_{33} , respectively.

5.4. A pair of identical penny-shaped cracks

As a final example, consider an infinite elastic medium containing a pair of identical penny-shaped cracks of radius a , denoted by crack-A and crack-B, as shown in Fig. 16. The two cracks are oriented perpendicular to the x_3 -axis, and the center of crack-A is at the point $(0, 0, -d)$ whereas that of crack-B is at the point $(0, 0, d)$. The medium is made of

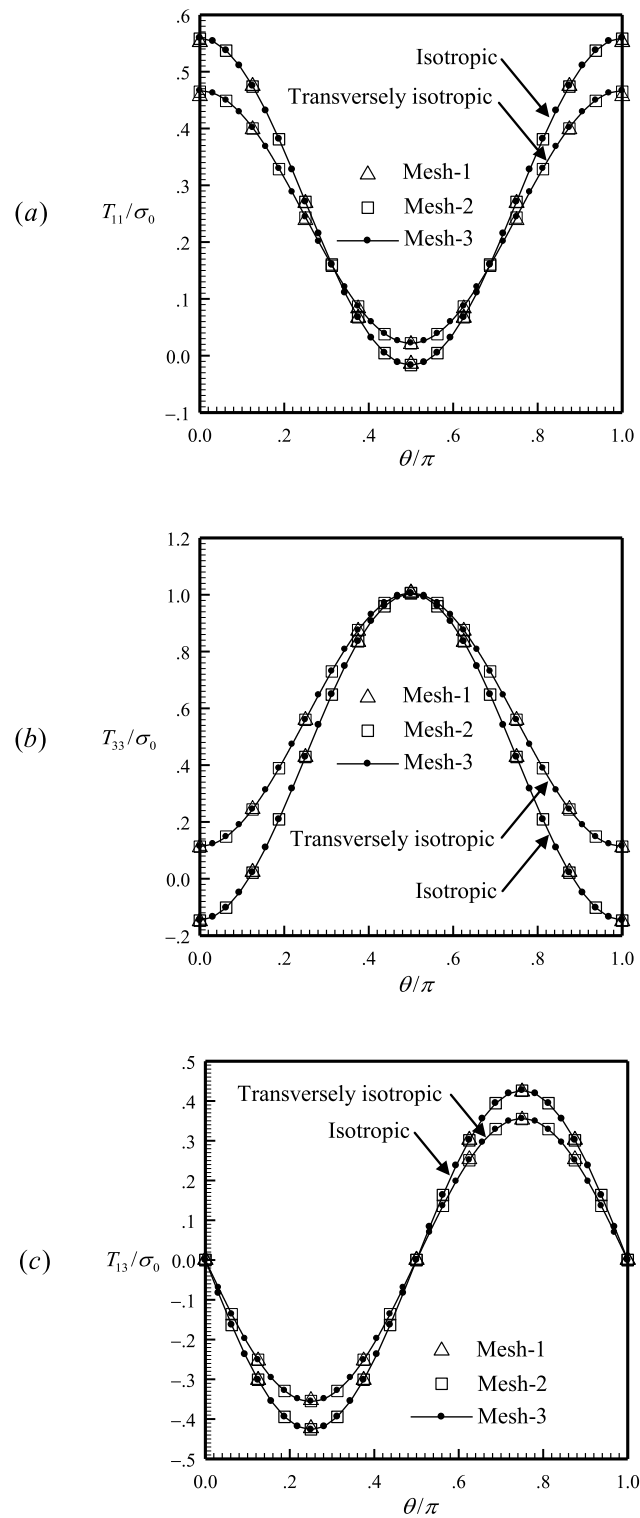
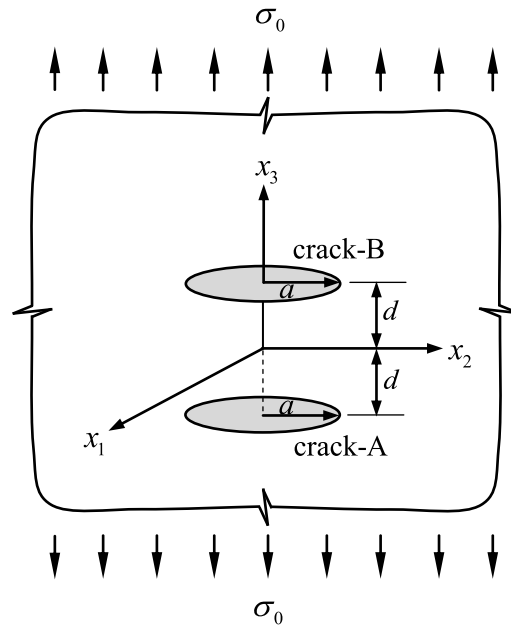


Fig. 15. Normalized T-stress components: (a) T_{11} , (b) T_{33} and (c) T_{13} for a spherical cap crack under remote uniaxial tension σ_0 in the x_1 -direction.

Table 4

Normalized T-stress components T_{11} and T_{33} for a spherical cap crack under uniformly distributed traction t_0 in the x_3 -direction. Results are reported at the point for which $\beta = 0$, $\gamma = 30^\circ$, and the reference solution is taken as that generated from Mesh-3.

Mesh	Isotropic material		Transversely isotropic material	
	T_{11}/T_{11}^{ref}	T_{33}/T_{33}^{ref}	T_{11}/T_{11}^{ref}	T_{33}/T_{33}^{ref}
1	0.9940	0.9976	0.9838	0.9978
2	0.9996	0.9998	0.9978	0.9997
3	1.0000	1.0000	1.0000	1.0000
T^{ref}/t_0	−0.7053	−0.6846	−0.6922	−1.3761

**Fig. 16.** Schematic of a pair of identical penny-shaped cracks under uniform remote uniaxial stress σ_0 in x_3 -direction.

either an isotropic material (with $\nu = 0.3$) or a transversely isotropic solid (for which the axis of material symmetry is directing along the x_3 -axis and with the material properties being those indicated in Table 1), and the loading is uniform remote uniaxial stress σ_0 acting in the x_3 -direction. In the analysis, three levels of refinement corresponding to the meshes shown in Fig. 5 are adopted for both crack surfaces in order to investigate the convergence of the numerical solutions.

Computed T-stress components are reported in Table 5 for the case $d/a = 0.25$. It can be observed that the numerical solutions tend to converge as the level of discretization is refined. In contrast with the case of a single penny-shaped crack discussed in Section 5.1, the coarsest mesh yields results with a significant difference relative to the finest mesh; this is due mainly to the complexity of the elastic field induced by the interaction between the two relatively closely-spaced cracks. However, use of the special near-front approximation still appears to alleviate the requirement of utilizing very fine mesh to capture the complex near-front elastic field.

To further investigate the interaction between the two cracks, results of the T-stress for different values of d/a are obtained using Mesh-3 and reported along with the solution of a single crack in Fig. 17(a) and (b) for the isotropic and transversely isotropic cases, respectively. It is seen that the interaction between the two cracks becomes more significant as the ratio d/a decreases and the solution of the two cracks converges quite rapidly to the single-crack solution as d/a is sufficiently large. It is worth noting that for both material models, as the ratio d/a increases, the magnitude of T_{33} increases monotonically and converges to the single-crack solution whereas that of T_{11} decreases rapidly to reach a minimum value and then increases monotonically to the single-crack solution.

Table 5

Normalized T-stress components for a pair of identical penny-shaped cracks under remote uniaxial stress. Results are normalized by the exact solution of a single penny-shaped crack under the same loading condition obtained from [46].

Mesh	Isotropic material		Transversely isotropic material	
	T_{11}/T_{11}^{ref}	T_{33}/T_{33}^{ref}	T_{11}/T_{11}^{ref}	T_{33}/T_{33}^{ref}
1	1.0454	0.8344	1.1128	0.8021
2	1.0912	0.8520	1.1197	0.8038
3	1.1094	0.8582	1.1182	0.8040

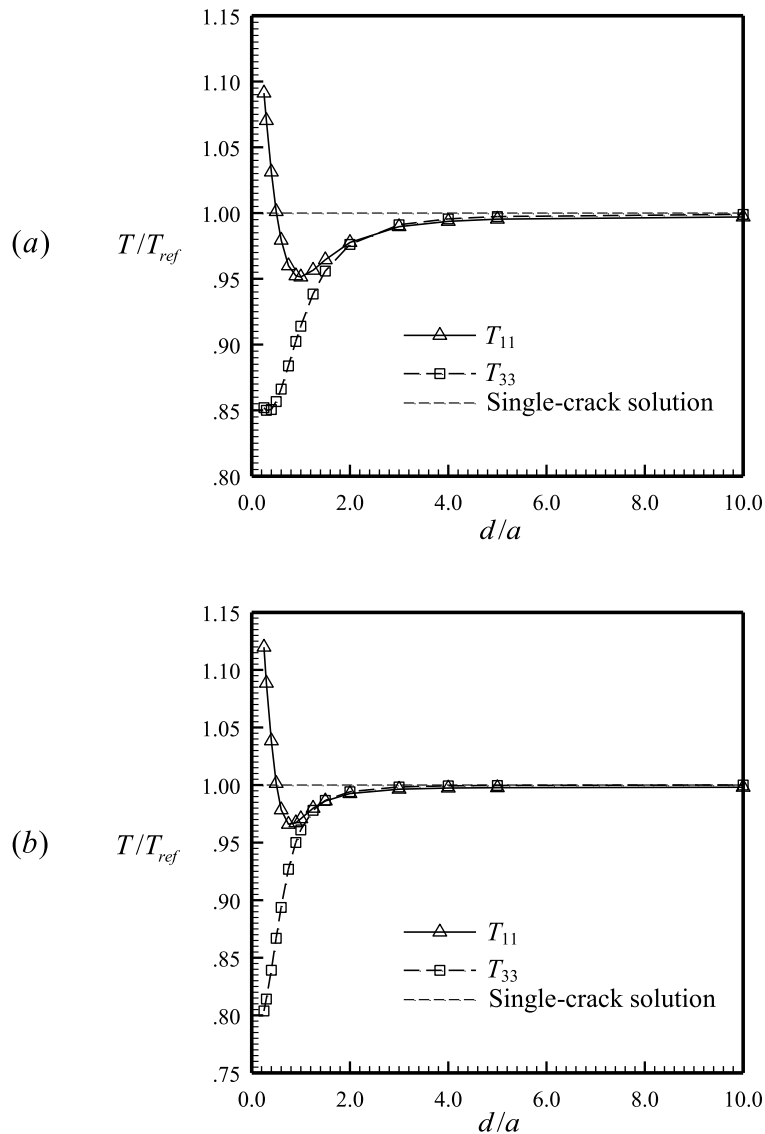


Fig. 17. Normalized T-stress components T_{11} and T_{33} for a pair of identical penny-shaped cracks under remote uniaxial stress plotted as a function of d/a . Results are reported for (a) the isotropic case and (b) the transversely isotropic case, and they are normalized by the solution for a single penny-shaped crack problem [46].

6. Conclusions and remarks

A numerical procedure, based principally upon a boundary integral equation method, has been implemented to calculate T-stress components of cracks in anisotropic, linearly elastic, infinite media. The key governing integral equations have been formulated in terms of primary unknown data on the crack surface involving the sum-of and the jump-in the crack-face displacements. In addition to being applicable to general crack geometries, loading conditions, and material anisotropy, the integral equations only contain weakly singular kernels which are convenient for both their interpretation and for their numerical implementation. A symmetric Galerkin boundary element method (SGBEM) and standard Galerkin technique have been successfully adopted to determine unknown crack-face data and, when supplemented by certain direct post-processing formulae, the T-stress components along the crack front can be accurately extracted from the sum of the crack-face displacement. We note that the accuracy of the approximation has been enhanced by using special interpolation functions to approximate the near-front jump in the crack-face displacement.

Results from an extensive numerical study have indicated that the proposed technique yields highly accurate T-stress components; in particular, numerical solutions exhibit excellent agreement with existing reference solutions for various benchmark cases. The convergence of numerical results has been confirmed by performing the analysis via a series of meshes with significantly different levels of refinement, and it has been observed that very accurate T-stress components can be obtained from relatively coarse meshes. Indeed, only a weak dependence on the level of refinement has been observed and, notably, only a relatively small number of degrees of freedom are required in the discretization to obtain solutions with quality comparable to those generated by significantly finer meshes. In addition to the capability to treat general material anisotropy and loading conditions, the technique has been successfully implemented in a general framework allowing the treatment of complex crack problems involving nonplanar and multiple cracks. Finally, it should be remarked that various components established in the present work such as the integral formulation, the solution methodology, and numerical implementations provide an essential basis for further enhancing the technique to be capable of modeling cracks in finite bodies and coupled-field media such as piezoelectric materials.

Acknowledgments

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Appendix. Asymptotic crack-tip field for two-dimensional case

Consider a two-dimensional, semi-infinite, traction-free crack that lies along the negative $\bar{x}_1$ -axis as shown schematically in Fig. A.1, and let $\{r, \theta\}$ be a polar coordinate system with origin at the crack tip. For an isotropic material, the displacements corresponding to the in-plane loading (i.e., mode I and mode II loading conditions) are given by (e.g., [1,66])

$$\begin{aligned} \begin{Bmatrix} u_1(r, \theta) \\ u_2(r, \theta) \end{Bmatrix} &= \begin{Bmatrix} u_1^0 \\ u_2^0 \end{Bmatrix} + \omega^0 r \begin{Bmatrix} -\sin \theta \\ \cos \theta \end{Bmatrix} + \frac{1}{2\mu} \sum_{n=1}^{\infty} \begin{Bmatrix} [\kappa + \frac{n}{2} + (-1)^n] \cos \frac{n\theta}{2} - \frac{n}{2} \cos \frac{(n-4)\theta}{2} \\ [\kappa - \frac{n}{2} - (-1)^n] \sin \frac{n\theta}{2} + \frac{n}{2} \sin \frac{(n-4)\theta}{2} \end{Bmatrix} a_n r^{n/2} \\ &+ \frac{1}{2\mu} \sum_{n=1}^{\infty} \begin{Bmatrix} [-\kappa - \frac{n}{2} + (-1)^n] \sin \frac{n\theta}{2} + \frac{n}{2} \sin \frac{(n-4)\theta}{2} \\ [\kappa - \frac{n}{2} + (-1)^n] \cos \frac{n\theta}{2} + \frac{n}{2} \cos \frac{(n-4)\theta}{2} \end{Bmatrix} b_n r^{n/2} \end{aligned} \quad (\text{A.1})$$

where $\{u_1^0, u_2^0\}$ correspond to a (possible) rigid translation of the crack; ω^0 represents a (possible) rigid rotation of the crack about the crack tip; μ is the elastic shear modulus; $\kappa = 3 - 4\nu$, $(3 - \nu)/(1 + \nu)$ for plane strain and plane stress,

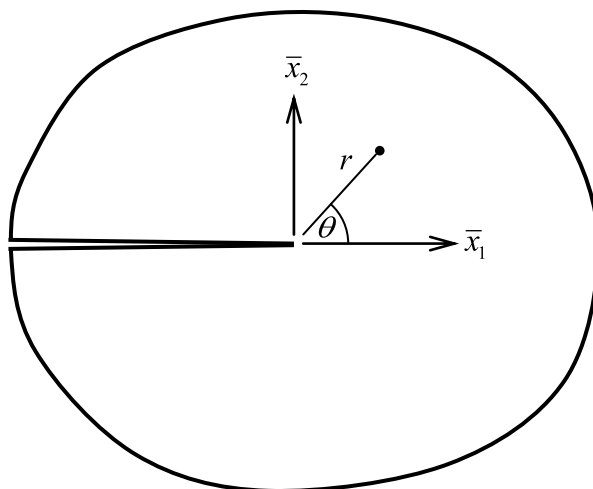


Fig. A.1. Schematic of region surrounding crack-tip and corresponding local and polar coordinate systems.

respectively; ν is Poisson's ratio; and a_n and b_n are unknown constants. The near-tip strain and stress fields can be readily obtained from the strain–displacement relationship and the constitutive law as

$$\begin{Bmatrix} \varepsilon_{11}(r, \theta) \\ \varepsilon_{22}(r, \theta) \\ \varepsilon_{12}(r, \theta) \end{Bmatrix} = \frac{a_1}{8\mu\sqrt{r}} \begin{Bmatrix} (2\kappa - 3) \cos \frac{\theta}{2} + \cos \frac{5\theta}{2} \\ (2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{5\theta}{2} \\ -\sin \frac{\theta}{2} + \sin \frac{5\theta}{2} \end{Bmatrix} + \frac{b_1}{8\mu\sqrt{r}} \begin{Bmatrix} (2\kappa + 1) \sin \frac{\theta}{2} + \sin \frac{5\theta}{2} \\ (2\kappa - 5) \sin \frac{\theta}{2} - \sin \frac{5\theta}{2} \\ -3 \cos \frac{\theta}{2} - \cos \frac{5\theta}{2} \end{Bmatrix} + \frac{a_2}{2\mu} \begin{Bmatrix} \kappa + 1 \\ \kappa - 3 \\ 0 \end{Bmatrix} + \mathcal{O}(\sqrt{r}) \quad (\text{A.2})$$

$$\begin{Bmatrix} \sigma_{11}(r, \theta) \\ \sigma_{22}(r, \theta) \\ \sigma_{12}(r, \theta) \end{Bmatrix} = \frac{a_1}{4\sqrt{r}} \begin{Bmatrix} 3 \cos \frac{\theta}{2} + \cos \frac{5\theta}{2} \\ 5 \cos \frac{\theta}{2} - \cos \frac{5\theta}{2} \\ -\sin \frac{\theta}{2} + \sin \frac{5\theta}{2} \end{Bmatrix} + \frac{b_1}{4\sqrt{r}} \begin{Bmatrix} 7 \sin \frac{\theta}{2} + \sin \frac{5\theta}{2} \\ \sin \frac{\theta}{2} - \sin \frac{5\theta}{2} \\ -3 \cos \frac{\theta}{2} - \cos \frac{5\theta}{2} \end{Bmatrix} + 4a_2 \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + \mathcal{O}(\sqrt{r}) \quad (\text{A.3})$$

It is evident from (A.2) and (A.3) that the finite part of the strain and the stress at the crack tip, denoted respectively by $\boldsymbol{\varepsilon}^{finite}$, $\boldsymbol{\sigma}^{finite}$, are given by

$$\boldsymbol{\varepsilon}^{finite} = \begin{Bmatrix} \varepsilon_{11}^{finite} \\ \varepsilon_{22}^{finite} \\ \varepsilon_{12}^{finite} \end{Bmatrix} = \frac{a_2}{2\mu} \begin{Bmatrix} \kappa + 1 \\ \kappa - 3 \\ 0 \end{Bmatrix} \quad (\text{A.4})$$

$$\boldsymbol{\sigma}^{finite} = \begin{Bmatrix} \sigma_{11}^{finite} \\ \sigma_{22}^{finite} \\ \sigma_{12}^{finite} \end{Bmatrix} = \begin{Bmatrix} 4a_2 \\ 0 \\ 0 \end{Bmatrix} \quad (\text{A.5})$$

It is worth noting that both σ_{12}^{finite} and σ_{22}^{finite} are known a priori and vanish, in this particular case, due to the traction-free condition on both crack faces. In addition, the finite part $\boldsymbol{\sigma}^{finite}$ can be related to the finite part $\boldsymbol{\varepsilon}^{finite}$ by

the Hooke's law for isotropic materials, i.e.,

$$\boldsymbol{\sigma}^{finite} = \bar{\mathbf{E}} \boldsymbol{\varepsilon}^{finite} \quad (\text{A.6})$$

where the modulus matrix $\bar{\mathbf{E}}$ is given by

$$\bar{\mathbf{E}} = \mu \begin{bmatrix} (\kappa + 1)/(\kappa - 1) & (3 - \kappa)/(\kappa - 1) & 0 \\ (3 - \kappa)/(\kappa - 1) & (\kappa + 1)/(\kappa - 1) & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (\text{A.7})$$

With $\mathbf{u}^+ = \mathbf{u}(r, \pi)$ and $\mathbf{u}^- = \mathbf{u}(r, -\pi)$ being the displacement of the upper and lower crack faces, respectively, we define the sum and difference of the crack-face displacements by $\Sigma \mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$ and $\Delta \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$. From (A.1), it can be shown that

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} = \frac{\kappa + 1}{\mu} \sqrt{r} \sum_{k=1}^{\infty} (-1)^k \begin{Bmatrix} b_{2k-1} \\ -a_{2k-1} \end{Bmatrix} r^{k-1} \quad (\text{A.8})$$

$$\begin{Bmatrix} \Sigma u_1 \\ \Sigma u_2 \end{Bmatrix} = \begin{Bmatrix} 2u_1^0 \\ 2u_2^0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -2\omega^0 r \end{Bmatrix} + \frac{\kappa + 1}{\mu} \sum_{k=1}^{\infty} (-1)^k \begin{Bmatrix} a_{2k} \\ b_{2k} \end{Bmatrix} r^k \quad (\text{A.9})$$

It is worth noting that the sum of the crack-face displacements $\Sigma \mathbf{u}$ is only in terms of integral powers of r while the difference of the crack-face displacements $\Delta \mathbf{u}$ involves only fractional powers of r . Applying (A.4) and (A.9), $\varepsilon_{11}^{finite}$ of the finite part strain at the crack-tip can be computed from Σu_1 by

$$\varepsilon_{11}^{finite} = \frac{1}{2} \lim_{r \rightarrow 0} \frac{\Sigma u_1}{\partial x_1} \quad (\text{A.10})$$

By enforcing the crack-face traction boundary conditions, the traction generated by the stress field (A.3) on the upper and lower crack faces, denoted respectively by $\mathbf{t}^+(r)$ and $\mathbf{t}^-(r)$, must be equal to the prescribed zero traction on the upper and lower crack faces, denoted respectively by $\mathbf{t}^{0+}(r) = \mathbf{0}$ and $\mathbf{t}^{0-}(r) = \mathbf{0}$, i.e.,

$$\mathbf{t}^+(r) = \begin{Bmatrix} -\sigma_{12}(r, \pi) \\ -\sigma_{22}(r, \pi) \end{Bmatrix} = \begin{Bmatrix} t_1^{0+}(r) \\ t_2^{0+}(r) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{A.11})$$

$$\mathbf{t}^-(r) = \begin{Bmatrix} \sigma_{12}(r, -\pi) \\ \sigma_{22}(r, -\pi) \end{Bmatrix} = \begin{Bmatrix} t_1^{0-}(r) \\ t_2^{0-}(r) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{A.12})$$

Besides satisfying the traction-free condition for this particular case, Eqs. (A.11)–(A.12) also suggest that for cracks under general crack-face tractions, the components σ_{12}^{finite} and σ_{22}^{finite} of the finite part stress at the crack-tip are known a priori and can be obtained from

$$\sigma_{12}^{finite} = - \lim_{r \rightarrow 0} t_1^{0+}(r) \quad (\text{A.13})$$

$$\sigma_{22}^{finite} = - \lim_{r \rightarrow 0} t_2^{0+}(r) \quad (\text{A.14})$$

The relations (A.10) and (A.13)–(A.14) allow the unknown σ_{11}^{finite} , commonly termed the T-stress T , to be solved from (A.6) provided that the sum of the crack-face displacements Σu_1 is known. The generalization of above results to the case of anisotropic elastic media can be obtained using results reported by Hoenig [65].

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Investigation of Generalized SIFs of cracks in 3D piezoelectric media under various crack-face conditions

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ABSTRACT This paper investigates the influence of crack geometry, crack-face and loading conditions, and permittivity of a medium inside the crack gap on intensity factors of planar and non-planar cracks in linear piezoelectric media. A weakly singular boundary integral equation method together with the near-front approximation is adopted to accurately determine the intensity factors. Obtained results indicate that the non-flat crack surface, electric field, and permittivity of the medium inside a crack gap play a crucial role on the behaviour of intensity factors. The mode-I stress intensity factors (K_I) for two representative non-planar cracks under different crack-face conditions are found significantly different and they possess both upper and lower bounds. In addition, K_I for impermeable and semi-permeable non-planar cracks treated depends strongly on the electric field whereas those of impermeable, permeable, and semi-permeable penny-shaped cracks are identical and independent of the electric field. The stress/electric intensity factors predicted by permeable and energetically consistent models are, respectively, independent of and dependent on the electric field for the penny-shaped crack and the two representative non-planar cracks. Also, the permittivity of the medium inside the crack gap strongly affects the intensity factors for all crack configurations considered except for K_I of the semi-permeable penny-shaped crack.

KEYWORDS Crack-face conditions, Intensity factors, Non-flat cracks, Permittivity, Piezoelectric media, SGBEM

1 Introduction

Piezoelectric materials are widely employed in various engineering and industrial applications (e.g., actuators, sensors, transducers, etc.). Extensive studies have been carried out in the past decades to gain an in-depth understanding of responses and behaviour of such materials in both macroscopic and microscopic scales (e.g., [1-10]). Besides the desirable mechanical-electrical coupling feature, the piezoelectric materials possess a major drawback in that they are quite brittle in nature and susceptible to induced fractures, either during the manufacturing process or during usage. Therefore, extensive investigations are required to gain insights into the fracture behaviour of piezoelectric materials. One of the most challenging tasks in the modelling of fractures in piezoelectric media is developing a physically suitable mathematical model capable of simulating actual conditions on the crack surface to accurately mimic the real physical phenomena. Several models of crack-face conditions have been proposed, and their pros and cons were extensively discussed. For instance, an electrically permeable condition was introduced by Parton [11]; an electrically impermeable condition was proposed by Deeg [12]; a semi-permeable condition was employed by Hao and Shen [13], and an energetically consistent boundary condition was proposed by Landis [14]. Among these electric boundary conditions, an energetically consistent boundary

condition seems to represent more realistic cracks than the permeable, impermeable, and semi-permeable models (an extensive review can be found in the work of Rungamornrat et al. [15]). Nevertheless, most previous publications focused mainly on cracks with simple geometries (viz. Griffith cracks, straight cracks, kinked cracks/branched cracks, inclined cracks, edge cracks, and inclined edge cracks) in two-dimensional bodies and planar cracks of various shapes embedded in an infinite and/or finite three-dimensional domains under simple loading conditions (e.g., [16-44]), while relatively few studies relevant to the investigation of non-planar and multiple cracks have been reported, whereby most of them only concerned the development of computational tools for fracture analysis (e.g., [44-49]).

Selected useful results from previous investigations can be summarized as follows. For two-dimensional boundary value problems, Xu and Rajapakse [25] proposed an analytical solution for an arbitrarily oriented elliptical void embedded in infinite piezoelectric media. By reducing void solutions to crack solutions, they found that under varying remote electric fields and with relatively small applied remote mechanical stress, the electric intensity factors of semi-permeable and permeable cracks are nearly identical, and both results are significantly different from the solution of impermeable cracks when a crack is not parallel to the poling direction of the piezoelectric material. In contrast, when the crack is parallel to the poling direction, the electric

intensity factors of impermeable and semi-permeable cracks are identical, and the solution of permeable cracks vanishes. Finally, it is interesting to note that the electric intensity factors for three types of crack-face conditions (i.e. impermeable, permeable, and semi-permeable) are identical when the remote electric field is nearly equal to zero. This condition exists whether the crack is parallel to the poling direction of the piezoelectric material or not. Wang and Jiang [26] studied a slit crack embedded in an infinite piezoelectric medium that is subjected to far-field mechanical and electric loadings. They found that for positive applied electric fields, the electric intensity factor of the semi-permeable crack varies from the permeable solution (the lower bound) to the impermeable solution (the upper bound) as the applied tensile stress increases. Later, Wang and Jiang [31] studied an arbitrarily-oriented crack in infinite media subjected to mechanical and electric loads. They found that there exists a discrepancy between the electric intensity factors computed from semi-permeable and permeable boundary conditions when the tensile mechanical loads is high and the poling direction is perpendicular to the crack surfaces. This finding contradicts the result obtained by Xu and Rajapakse [25]. However, Wang and Jiang [31] pointed out that the discrepancy exists due to the different levels of applied remote mechanical load. This implies that the levels of applied mechanical and electrical loads play an important role in the fracture behaviour of piezoelectric cracked media. In addition, Wang and Jiang [31] also noted that for positive applied electric fields, high mechanical loads, and when the poling direction is perpendicular to the crack surfaces, the electric intensity factor for the impermeable and permeable conditions represent the upper and lower bounds of the result, respectively, for the semi-permeable condition. Conversely, if the applied electric field is negative, such solution for impermeable and permeable conditions are the lower and upper bounds of the result for the semi-permeable condition, respectively, and it can be seen that the electric intensity factor for the permeable model is independent of the electric field. Subsequently, Denda and Mansukh [46] investigated the interaction of multiple cracks in two-dimensions using the boundary element method. However, their development is still limited to multiple straight cracks under impermeable and permeable boundary conditions.

For three-dimensional boundary value problems, Li and Lee [32] studied a penny-shaped crack embedded in a transversely isotropic infinite piezoelectric medium subjected to remote uniform tensile stress and constant electric field using three crack-face conditions (i.e., permeable, impermeable, and semi-permeable conditions) adopted at the crack surfaces. They found that the mode-I stress intensity factors for a penny-shaped crack under such three-crack models (i.e. permeable, impermeable, and semi-permeable conditions) are identical and depend only on mechanical loading. However, the electric intensity factors for those three crack models are completely different. The electric intensity factor for semi-permeable cracks depends on various parameters (i.e. mechanical loading, electrical loading, material properties, and permittivity of the medium inside the crack gap), whereas the electric intensity factor for the impermeable condition depends only on the electrical

loading. In contrast, the electric intensity factor for the permeable solution does not depend on the electrical loading but depends on the material properties and mechanical loading. These useful conclusions are in agreement with the results obtained by several investigators (e.g., [18, 20, 33, 35]).

Chiang and Weng [35] investigated a similar problem. They proposed a critical state that consists of the critical stress and applied electric induction vector. They pointed out that when the applied electric induction vector is positive and the remote tensile stress is less than the critical stress, the electric intensity factor for the semi-permeable model falls between the result of the impermeable model (that is the upper bound) and the result of the permeable model (that is the lower bound.) However, under a positive electric induction vector and when the remote tensile stress is greater than the critical stress, the electric intensity factor for the impermeable model becomes the lower bound whereas the result for the permeable becomes the upper bound of the results for the semi-permeable condition.

Recently, Li et al. [39] found that the mode-I stress intensity factor (as well as the electric intensity factor) for a penny-shaped crack under an energetically-consistent boundary condition depends on various parameters besides the mechanical loading conditions (i.e. mechanical loading, electric loading, material properties, and the permittivity of the medium inside the crack gap). This finding differs from the results obtained from the impermeable, permeable, and semi-permeable boundary conditions where the mode-I stress intensity factor depends only on mechanical loading. Moreover, as pointed out by several investigators (e.g., Li and Chen [50], Motola and Banks-Sills [51], and Li et al. [39]), the non-zero mechanical traction proposed by Landis [14] that is induced by the crack opening displacement, the jump in the electric potential, and the permittivity of the medium inside the crack gap, try to pull the upper and lower crack surfaces to close.

As mentioned above, most previous publications focused on modelling and analysing planar cracks. A question arises as to the difference between the fracture behaviour of planar and non-planar cracks (or curvilinear cracks and/or multiple cracks) in such materials under the four types of crack-face conditions. Recently, Rungamornrat and Mear [48] used a numerical technique called a weakly singular symmetric Galerkin boundary element method (SGBEM) to determine the stress intensity factors and electric intensity factor of an arbitrarily shaped crack embedded in three-dimensional, generally anisotropic, linear infinite piezoelectric media. They found that the fracture behaviour of planar and non-planar cracks under the impermeable condition are different. More specifically, while mechanical loading can cause stress intensity factors and electric intensity factors and vice versa, as observed for non-planar cracks (i.e. spherical cap cracks), there is no couple loadings between the mechanical and electrical loadings found in planar cracks (i.e. penny-shaped cracks) (e.g., [20, 32, 35]). Nevertheless, their investigation was restricted to impermeable cracks in infinite media.

Sanz et al. [47] and Solis et al. [49], the pioneers of investigations into intensity factors of transversely isotropic finite piezoelectric cracked bodies, revealed that the mode-I

stress intensity factor and electric intensity factor can be produced by uniform mechanical traction or uniform electrical loadings. Again, these findings are unlikely in planar cracks embedded in an infinite piezoelectric domain under the impermeable assumption. However, the works of Sanz et al. [47] and Solis et al. [49] were limited to the impermeable crack-face condition only. Phongtinnaboot et al. [52] generalized the work of Rungamornrat and Mear [48] to investigate the stress intensity factors and electric intensity factor of cracked finite piezoelectric domains. The electrically impermeable assumption was still adopted for the crack surface.

Most recently, Rungamornrat et al. [15] successfully generalized the work of Rungamornrat and Mear [48] to analyse cracked infinite piezoelectric domains under the four types of electrical boundary conditions (i.e. permeable, impermeable, semi-permeable, and an energetically consistent boundary condition.) While they proposed a technique that can solve cracks of arbitrary shapes and under various electrical boundary conditions, they focused only on computational aspects and a full discussion of the role of electrical boundary condition models was not presented. There are several crucial issues that arise and pose several nontrivial research problems in this specific area, and therefore, still require rigorous investigations, such as the difference between the fracture behaviour of planar and non-planar cracks under the four types of crack-face conditions and a wide range of electrical/mechanical loads; the suitability of semi-permeable and energetically consistent conditions compared to extreme crack-face conditions such as permeable and impermeable models; the influence of the dielectric permittivity of a medium inside the crack gap, etc. The current study is motivated by these outstanding challenges to fully explore the influence of several parameters such as crack geometry, crack-face conditions, loading conditions, and the permittivity of the medium inside the crack gap on the stress and electric intensity factors along the cracked front of cracked piezoelectric bodies. Three representative configurations of cracks, including a penny-shaped crack, a spherical cap crack, and a cylindrical crack, and four models of electrical boundary conditions (viz. permeable, impermeable, semi-permeable, and an energetically consistent boundary condition) are considered in the present study.

2 Problem description

Let us consider an isolated crack in a three-dimensional, homogeneous, linear piezoelectric, infinite medium Ω as shown in Fig. 1. The crack is represented by two identical surfaces S^+ and S^- with their corresponding outward unit normal vectors $\mathbf{n}^+$ and $\mathbf{n}^-$, respectively. The body is assumed to be free of body force and body charge, and can be subjected to mechanical and electrical loadings on either the crack surface or the remote boundary. The elastic constants, the piezoelectric constants, and the dielectric permittivities of the constituent material are fully prescribed and denoted in a concise form by E_{ijkl} . The boundary

conditions on the crack surface can be simulated by one of the following four models: impermeable, permeable, semi-permeable, and energetically consistent models. The first two models ignore the contribution of the permittivity of the medium inside the crack gap, whereas such contributions are considered by the semi-permeable and energetically consistent models, and are assumed to be prescribed and defined by κ_c . It should be noted here that the notations and nomenclatures utilized by Rungamornrat et al. [15] are adopted in this study and detailed explanations will not be provided for brevity.

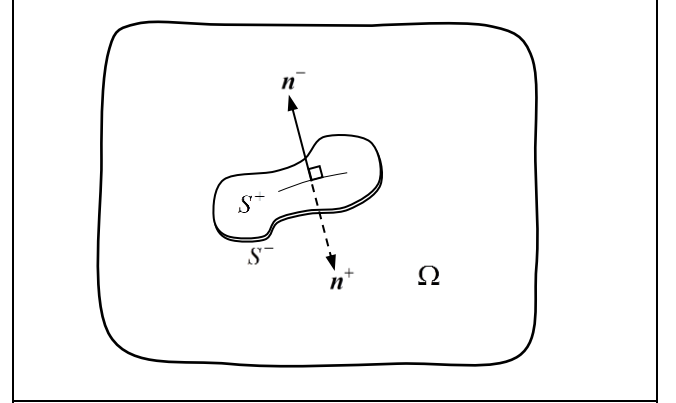


Fig. 1 Schematic of a three-dimensional, linear piezoelectric, infinite medium containing a crack

For impermeable cracks, the generalized tractions on the upper and lower crack surfaces, denoted by t_i^+ and t_i^- , are fully prescribed whereas the jump in the crack-face generalized displacement, defined by $\Delta u_i \equiv u_i^+ - u_i^-$, are unknown a priori. **This implies that the permittivity of the medium inside the crack gap is negligible or in other words the medium between the upper and lower crack surfaces is treated as an insulator** (e.g., Deeg [12]). For permeable cracks, the mechanical tractions on both crack surfaces t_i^+, t_i^- , the zero jump in the crack-face electrical potential $\Delta u_4 \equiv u_4^+ - u_4^- = 0$, and the zero sum of the crack-face electric charges $\Sigma t_4 \equiv t_4^+ + t_4^- = 0$ are fully prescribed, whereas the jump in the crack-face displacement, $\Delta u_i \equiv u_i^+ - u_i^-$ and the crack-face electric charge $t_4^+ = -t_4^-$ are unknown a priori. **For this particular case, the permittivity of a medium inside the crack gap is assumed indefinitely large** (e.g., Parton [11]). For semi-permeable cracks, the mechanical tractions on both crack surfaces, t_i^+, t_i^- are fully prescribed **and the sum of the surface electric charges vanishes (i.e., $\Sigma t_4 = 0$)** whereas the jump in the crack-face generalized displacement Δu_i and the crack-face electric charge $t_4^+ = -t_4^-$ are unknown a priori and satisfy the **following relation**

$$t_4^+ \Delta u_i n_i^+ = -\kappa_c \Delta u_4 \quad (1)$$

where κ_c is the dielectric permittivity of the medium inside the crack (e.g., Hao and Shen [13]). **It should be noted**

that if the permittivity inside the crack is significantly large (i.e., $\kappa_c \rightarrow \infty$), the electrically semi-permeable condition will converge to the conducting boundary condition (e.g., [32, 35]). Here and in what follows, the lower case indices range from 1 to 3; the upper case indices range from 1 to 4, and repeated indices imply a summation over their range. Note that the superscript “+” and “-” are employed to emphasize the quantities associated with the upper crack surface S^+ and the lower crack surface S^- , respectively.

For energetically consistent cracks (e.g., Landis [14]), the mechanical tractions on both crack surfaces t_i^+, t_i^- can be partitioned into two parts: $\sigma_i^+ + \tau_i^+$ and $\sigma_i^- + \tau_i^-$ where $\{\sigma_i^+, \sigma_i^-\}$ and $\{\tau_i^+, \tau_i^-\}$ are the normal and shear tractions, respectively. For this crack model, the mechanical shear tractions $\{\tau_i^+, \tau_i^-\}$ are fully prescribed whereas the jump in the crack-face generalized displacement Δu_I , the mechanical normal tractions $\{\sigma_i^+, \sigma_i^-\}$, and the surface electric charge $t_4^+ = -t_4^-$ are unknown a priori but satisfy the condition (1) and the following relation

$$\sigma_j^+ = (1/2)\kappa_c n_j^+ (\Delta u_4)^2 / (\Delta u_i n_i^+)^2 \quad (2)$$

Extensive details of the electrical boundary conditions used in the present investigation can also be found in the work of Rungamornrat et al. [15].

3 Formulation

A set of basic equations for linear piezoelectric materials and the boundary integral equations governing an isolated crack of arbitrary shape embedded in a three-dimensional, generally anisotropic, linear infinite piezoelectric medium under various types of crack-face conditions are summarized below.

3.1 Basic field equations

The electro-mechanical behaviour of a piezoelectric medium is governed by the classical theory of linear piezoelectricity. The basic field equations for a vanishing distributed source can be expressed in a concise form as

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0; \quad \sigma_{ij} = E_{ijkl} \frac{\partial u_k}{\partial x_l} \quad (3)$$

where σ_{ij} represents the generalized stress (i.e., the stress tensor σ_{ij} and the electric induction vector σ_{i4}); u_k represents the generalized displacement (i.e., the displacement u_k and the electric potential u_4), and E_{ijkl} denotes the generalized moduli (i.e., elastic constants E_{ijkl} , piezoelectric constants E_{i4kl} , and dielectric permittivities $-E_{i44l}$). The generalized traction at any point on a smooth surface with a unit normal $\mathbf{n}$ is given by $t_j = \sigma_{ij} n_i$ (i.e.,

the mechanical traction $t_j = \sigma_{ij} n_i$ and the surface electric charge $t_4 = \sigma_{i4} n_i$).

3.2 Boundary integral equations

Boundary integral equation methods have been found efficient for modelling cracks in homogenous, linear, infinite media because in the absence of a distributed source, the key governing equation involves only unknowns on the crack surface. This positive feature renders the solution methodology computationally efficient than domain-based techniques such as the conventional finite element and finite difference methods. In the present study, the well-known, symmetric Galerkin boundary element method (SGBEM) developed by Rungamornrat and Mear [48] and Rungamornrat et al. [15] is selected as key computational procedure in the analysis.

For isolated cracks in a linear piezoelectric, infinite medium that is free of remote loading, the generalized stress σ_{IK} at any interior point $\mathbf{x}$ can be expressed in terms of the crack-faced data via the following standard integral relation

$$\sigma_{IK}(\mathbf{x}) = - \int_{S^+} S_{IK}^I(\xi - \mathbf{x}) \Sigma t_J(\xi) dA(\xi) + \int_{S^+} \Sigma_{IJ}^{IK}(\xi - \mathbf{x}) n_i(\xi) \Delta u_J(\xi) dA(\xi) \quad (4)$$

where $\Sigma t_J = t_J^+ + t_J^-$ and $\Delta u_J = u_J^+ - u_J^-$ denote the sum of the crack-face generalized traction and the jump in the crack-face generalized displacement, respectively, and $S_{IJ}^P(\xi - \mathbf{x})$ and $\Sigma_{IJ}^{IK}(\xi - \mathbf{x})$ are known fundamental solutions for an un-cracked whole space. While the integral relation (4) can be directly used to form a boundary integral equation sufficient for determining the unknown relative crack-face generalized displacement, the resulting integral equations contain both the strongly singular kernel $S_{IJ}^P(\xi - \mathbf{x})$ and the hyper-singular kernel $\Sigma_{IJ}^{IK}(\xi - \mathbf{x})$ and this, as a direct consequence, requires nontrivial numerical treatments in the solution procedure (e.g., [53-57]). To avoid such difficulties, an alternative singularity-reduced boundary integral equation for the generalized crack-face traction proposed by Rungamornrat et al. [15] and Rungamornrat and Mear [48] can be utilized. To construct such weakly singular integral equation, they employed the following special decompositions of the kernels $S_{IJ}^P(\xi - \mathbf{x})$ and $\Sigma_{IJ}^{IK}(\xi - \mathbf{x})$:

$$S_{IJ}^P(\xi - \mathbf{x}) = H_{IJ}^P(\xi - \mathbf{x}) + \varepsilon_{ism} \frac{\partial G_{mJ}^P(\xi - \mathbf{x})}{\partial \xi_s} \quad (5)$$

$$\Sigma_{IJ}^{IK}(\xi - \mathbf{x}) = -E_{IJKl} \delta(\xi - \mathbf{x}) + \varepsilon_{ism} \frac{\partial}{\partial \xi_s} \varepsilon_{lrs} \frac{\partial}{\partial \xi_r} C_{mJ}^{IK}(\xi - \mathbf{x}) \quad (6)$$

where ε_{ism} is the standard alternating symbol; $\delta(\xi - \mathbf{x})$ is a Dirac-delta function centered at $\mathbf{x}$; $H_{IJ}^P(\xi - \mathbf{x}) = -\delta_{JP}(\xi_i - x_i) / 4\pi r^3$ with δ_{JP} denoting the generalized Kronecker delta and $r = \|\xi - \mathbf{x}\|$; and $G_{mJ}^P(\xi - \mathbf{x})$ and $C_{mJ}^{tK}(\xi - \mathbf{x})$ are given, for generally anisotropic piezoelectric materials, by

$$G_{mJ}^P(\xi - \mathbf{x}) = \frac{\varepsilon_{mqa} E_{qJKl}}{8\pi^2 r} \oint_{z,r=0} (\mathbf{z}, \mathbf{z})_{KP}^{-1} z_a z_l ds(\mathbf{z}) \quad (7)$$

$$C_{mJ}^{tK}(\xi - \mathbf{x}) = \frac{A_{mtsl}^{KJPQ}}{8\pi^2 r} \oint_{z,r=0} (\mathbf{z}, \mathbf{z})_{PQ}^{-1} z_s z_l ds(\mathbf{z}), \quad (8)$$

$$A_{mtsl}^{KJPQ} = \varepsilon_{aum} \varepsilon_{adt} \left\{ E_{uKPs} E_{dJQl} - \frac{1}{4} E_{dJKu} E_{lPQs} \right\}$$

in which $\mathbf{z}$ is a unit vector, $(\mathbf{z}, \mathbf{z})_{PQ} = z_m E_{mPQn} z_n$, $(\mathbf{z}, \mathbf{z})^{-1}$ is the inverse of $(\mathbf{z}, \mathbf{z})$; and the line integrals are carried out along a unit circle on the plane normal to the vector $\mathbf{r} = \xi - \mathbf{x}$. It is remarked that the two-point functions $G_{mJ}^P(\xi - \mathbf{x})$ and $C_{mJ}^{tK}(\xi - \mathbf{x})$ are only singular at $\xi = \mathbf{x}$ of $\mathcal{O}(1/r)$.

By applying the decompositions (5) and (6) to the integral relation (4) together with carrying out the integration by parts via Stokes' theorem, it results in the singularity-reduced integral relation for the generalized stress

$$\begin{aligned} \sigma_{IK}(\mathbf{x}) = & \varepsilon_{lrt} \frac{\partial}{\partial x_r} \int_{S^+} G_{tK}^J(\xi - \mathbf{x}) \Sigma t_J(\xi) dA(\xi) \\ & + \varepsilon_{lrt} \frac{\partial}{\partial x_r} \int_{S^+} C_{mJ}^{tK}(\xi - \mathbf{x}) D_m \Delta u_J(\xi) dA(\xi) \\ & - \int_{S^+} H_{IK}^J(\xi - \mathbf{x}) \Sigma t_J(\xi) dA(\xi) \end{aligned} \quad (9)$$

Finally, a weakly singular, weak-form boundary integral equation for the generalized traction reported in Rungamornrat et al. [15] and Rungamornrat and Mear [48] can be obtained from (9) by a limiting process, a standard weighted residual technique, and the integration by parts via Stokes' theorem. Its final explicit form, in terms of the crack-face data, is given by

$$\begin{aligned} - \int_{S^+} D_t \tilde{v}_K(\mathbf{y}) \int_{S^+} C_{mJ}^{tK}(\xi - \mathbf{y}) D_m \Delta u_J(\xi) dA(\xi) dA(\mathbf{y}) \\ = \frac{1}{2} \int_{S^+} \tilde{v}_K(\mathbf{y}) \Delta t_K(\mathbf{y}) dA(\mathbf{y}) \\ + \int_{S^+} D_t \tilde{v}_K(\mathbf{y}) \int_{S^+} G_{tK}^J(\xi - \mathbf{y}) \Sigma t_J(\xi) dA(\xi) dA(\mathbf{y}) \\ + \int_{S^+} \tilde{v}_K(\mathbf{y}) \int_{S^+} H_{IK}^J(\xi - \mathbf{y}) n_i(\mathbf{y}) \Sigma t_J(\xi) dA(\xi) dA(\mathbf{y}) \end{aligned} \quad (10)$$

where $\tilde{v}_K$ is a sufficiently smooth test function; $D_m(\cdot) = n_i \varepsilon_{ism} \partial(\cdot) / \partial \xi_s$ is a surface differential operator with

ε_{ism} denoting the standard alternating symbol; $\Sigma t_K = t_K^+ + t_K^-$ and $\Delta t_K = t_K^+ - t_K^-$ are the sum of and jump in the generalized traction across the crack surface; and $\{C_{mJ}^{tK}, G_{tK}^J, n_i H_{IK}^J\}$ are known kernels. It should be noted that the development of the boundary integral equation in (10) is independent of the crack configurations and the mechanical and electrical boundary conditions on the crack surface and, as a result, it is valid for the four types of cracks described above. In addition, since it contains complete information on the crack surface (i.e. Δt_K , Σt_K , and Δu_K), the boundary integral equation in (10) is sufficient for solving the primary unknowns on the crack surface when all prescribed data and known relations associated with each type of crack are properly incorporated. From a theoretical and computational point of view, all the double-surface integrals in (10) exist in an ordinary sense since all involved kernels $\{C_{mJ}^{tK}, G_{tK}^J, n_i H_{IK}^J\}$ are only weakly singular and of $\mathcal{O}(1/r)$. This feature allows unknown data on the crack surface to be approximated by continuous interpolation functions. Although the integral equation in (10) is applicable only to an infinite domain that is free from the electro-mechanical remote loading, such situations can readily be obtained using the principle of superposition (see clear explanations in Rungamornrat et al. [15]).

4 Solution technique

By adopting a standard discretization procedure commonly employed in the weakly singular SGBEM, the boundary integral equation for the generalized traction (10) can be transformed into a discrete system of linear algebraic equations

$$\mathbf{C} \Delta \mathbf{U} + (\mathbf{G} + \mathbf{H}) \Sigma \mathbf{T} + \mathbf{L} \Delta \mathbf{T} = \mathbf{0} \quad (11)$$

where matrices $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{H}$ result directly from the discretization of the double surface integrals containing the kernels C_{mJ}^{tK} , G_{tK}^J and H_{IK}^J , respectively; the matrix $\mathbf{L}$ corresponds to a single surface integral containing the jump in the crack-face generalized traction; $\Delta \mathbf{U}$ is a vector of nodal quantities associated with the jump in the crack-face generalized displacement, and $\Sigma \mathbf{T}$ and $\Delta \mathbf{T}$ are vectors of nodal quantities of the sum of and the jump in the crack-face generalized traction, respectively. The construction of all matrices $\{\mathbf{C}, \mathbf{G}, \mathbf{H}, \mathbf{L}\}$ follows directly the work of [15, 48]. The approximation of the near-front, jump in the crack-face generalized displacement is enhanced by using special crack-tip elements along the crack front (e.g., [15, 48, 58]) and the interpolation technique is employed to efficiently evaluate involved kernels for general anisotropic materials (e.g., [15, 48]). In addition to allowing the use of relatively coarse meshes in the discretization while yielding sufficiently accurate results, the former also provides the direct means to extract the intensity factors in terms of extra nodal data solved along the crack front. Details of such relevant solution procedure can be found in the work of Rungamornrat and Mear [48] and Rungamornrat et al. [15].

To obtain the unknown crack-face data, the system of linear equations (11) is solved together with the prescribed crack-face boundary conditions. For both electrically permeable and impermeable cracks, the final system of governing equations after incorporating the crack-face conditions is linear and its solution can be readily obtained via a selected efficient linear solver. On the contrary, the final governing equations for the case of electrically semi-permeable and energetically consistent cracks are nonlinear due to the nonlinear crack-face conditions and they are efficiently solved via standard Newton iterative scheme. Details of the solution procedure can be found in the work of Rungamornrat et al. [15]. Once a set of nodal data $\Delta \mathbf{U}$ is determined, it is directly used to post-process for both the stress and electric intensity factors along the crack-front.

5 Numerical results and discussion

First, computed results for certain simple cases such as a circular crack under simple electrical/mechanical loads are benchmarked with available reference solutions to validate the formulation of the governing integral equations and numerical implementations of the weakly singular SGBEM and the post-process for the stress intensity factors and the electric intensity factor. After verifying the proposed technique, we then thoroughly investigated the influence of non-flat geometry of the crack surface, remote electrical load, and permittivity of the medium inside the crack gap on the intensity factors along the crack front of representative non-flat cracks.

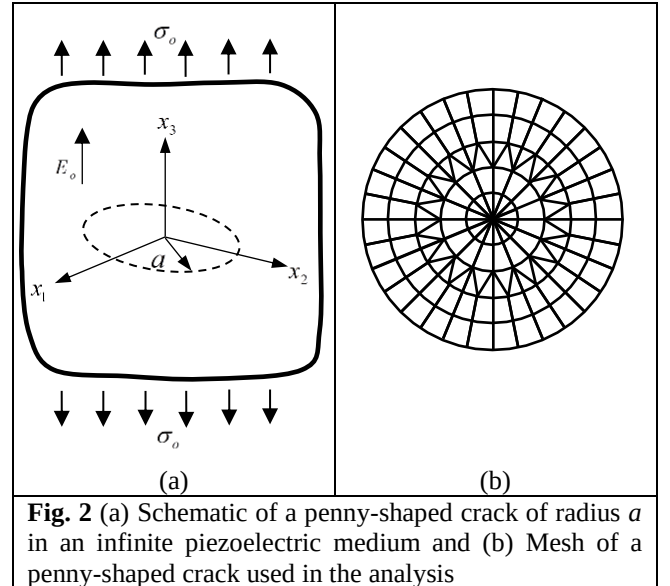
In the numerical study, a series of meshes is constructed first to investigate the convergence of the computed intensity factors and a sufficiently fine mesh is then chosen to ensure the convergence of the solution. In the discretization, the majority of the crack surface is discretized by standard 6-node and 8-node isoparametric elements, whereas the region adjacent to the crack boundary is discretized by 9-node special crack-tip elements (see [15, 48, 58]). The representative linear piezoelectric materials selected in the analysis are transversely isotropic and all material constants are taken to be those of PZT-4, as shown in Table 1.

Table 1 Generalized moduli of PZT-4 (e.g., [39]). The plan of isotropy is taken normal to the x_3 -axis.

Elastic constants ($\times 10^9$ Pa)	E_{1111}	139.00
	E_{1122}	77.80
	E_{1133}	74.30
	E_{3333}	113.00
	E_{1313}	25.60
Piezoelectric constants (C/m ²)	E_{1143}	-6.98
	E_{3343}	13.80
	E_{1341}	13.40
Dielectric permittivities ($\times 10^{-9}$ C/(Vm))	$-E_{1441}$	6.00
	$-E_{3443}$	5.47

5.1 Verification

To verify the implemented weakly singular SGBEM, consider first a penny-shaped crack of radius a embedded in a piezoelectric infinite medium made of PZT-4, as shown in Fig. 2(a). The medium is loaded by a uniform remote tension σ_o that is varied from 10 to 100 MPa, whereas the electric field E_o is fixed at 2.5 MV/m, and the dielectric permittivity of the medium inside the crack gap is chosen as $\kappa_c = 5\kappa_o$ where $\kappa_o = 8.85 \times 10^{-12}$ C/(Vm) is the permittivity of the air. The poling direction and the axis of material symmetry are taken along the x_3 -axis. The mesh shown in Fig. 2(b), after the convergence is confirmed, is used in the verification. Numerical results for the normalized mode-I stress intensity factor and electric intensity factor are reported in Figs. 3(a) and 3(b), respectively, along with the analytical solutions ([32, 39]).



It is shown that the numerical results obtained are in good agreement with the analytical solution for all four crack-face conditions considered. In addition, it can be inferred from Fig. 3(a) that the mode-I stress intensity factors K_I for the permeable, impermeable, and semi-permeable crack models are identical for the entire range of the applied uniform remote tension σ_o treated. These findings are in agreement with the work of previous investigators (e.g. Li and Lee [32] and Chiang and Weng [35]) who pointed out that the mode-I stress intensity factors for the three crack-face conditions depend only on the mechanical loading but are independent of electrical loading and the permittivity of the medium inside the crack gap.

In addition, it can be concluded from Fig. 3(a) that the results predicted by the energetically consistent model is less than those predicted by the other three crack models for the entire range of σ_o . This is because, as pointed out by several investigators (e.g., [50, 51]), the unknown mechanical traction acting on both the crack surfaces that is induced by the opening displacement of the crack, the jump in the

electric potential, and the permittivity of the medium inside the crack gap, try to pull the upper and lower crack surfaces closed. However, when the uniform remote tension increases, the mode-I stress intensity factor predicted by the energetically consistent model gets close to the results of the other three crack models.

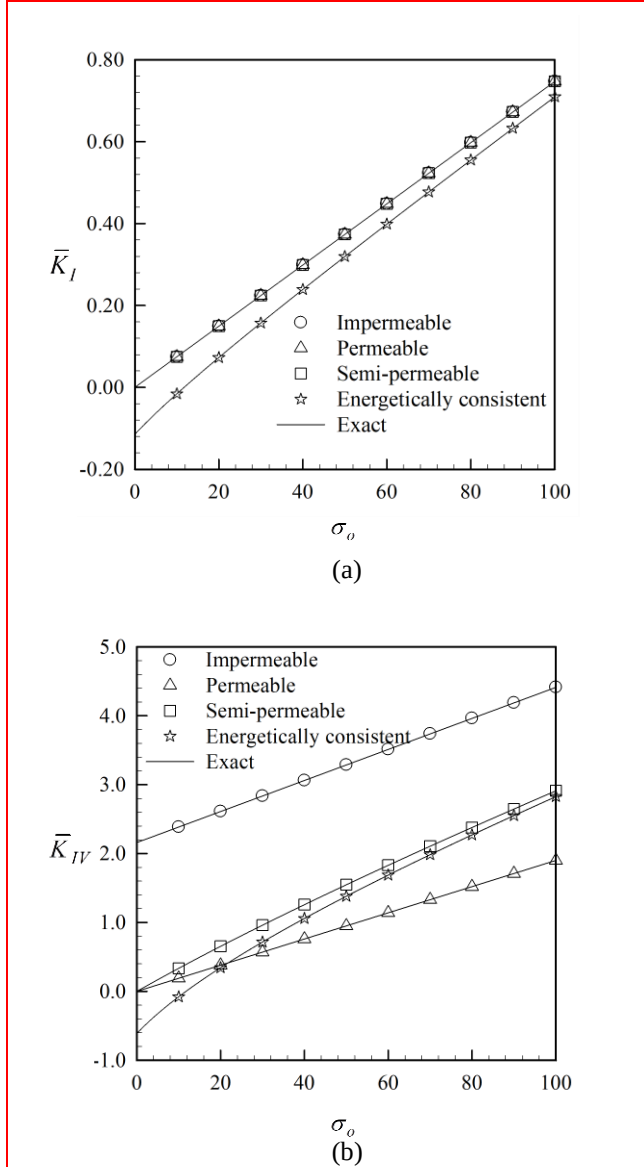


Fig. 3 Normalized intensity factors for a penny-shaped crack in an infinite piezoelectric medium subjected to remote uniform tensile stress σ_o and constant electric field E_o under four types of crack assumptions: (a) $\bar{K}_I = (K_I E_{3343}) / (E_{1111} E_{1441} E_o \sqrt{a})$ and (b) $\bar{K}_{IV} = K_{IV} / (E_{1441} E_o \sqrt{a})$

In contrast to K_I , the electric intensity factor K_{IV} strongly depends on the electrical boundary conditions adopted on the crack surface. Results in Fig. 3(b) show that the electric intensity factors of the impermeable and permeable cracks serve, respectively, as the upper and lower bounds of the results generated by the semi-permeable and energetically consistent models. However, when the uniform

remote tension σ_o is relatively small, the electric intensity factor K_{IV} for the energetically consistent model is lower than that for the permeable model. This implies that the level of applied mechanical load has a significant influence on the lower bound of K_{IV} for the penny-shaped crack subjected to uniform loading. As a result, one can see that the lower bound of K_{IV} switches from the energetically consistent solution to the permeable solution as the applied mechanical loading increases. In addition, it is obvious that the electric intensity factor K_{IV} for the semi-permeable and energetically consistent cracks are nearly identical and both results approach the result for the impermeable model when the applied remote stress increases.

5.2 Influence of crack geometry

To explore the behaviour of the intensity factors along the boundary of non-flat cracks, two representative cracks, one for a spherical cap crack (shown in Fig 4) and the other for a cylindrical crack (shown in Fig 6), are considered. The latter and the former are chosen to represent cracks possessing the curvature in a single direction and two directions, respectively. In the numerical study, following parameters are employed: (i) various half-subtended angles of the spherical cap crack and the cylindrical crack (defined further below in Sections 5.2.1 and 5.2.2) ranging from 5° to 90° are considered; (ii) the uniform remote tension and the uniform electric field are taken as 50 MPa and 2.5 MV/m, respectively; and (iii) the permittivity of the medium inside the crack gap for both semi-permeable and energetically consistent crack-face conditions are taken as $\kappa_c = 5\kappa_o$ where $\kappa_o = 8.85 \times 10^{-12}$ C/(Vm) denotes the permittivity of the air.

5.2.1 Spherical cap crack

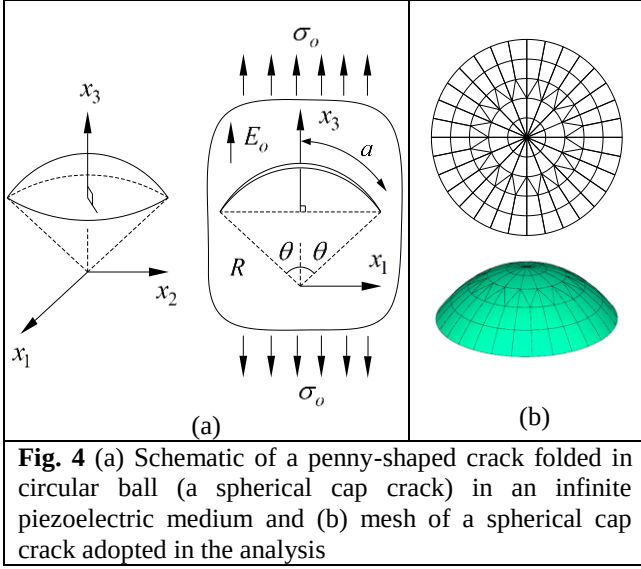
Consider a spherical cap crack embedded in a transversely isotropic infinite piezoelectric medium of PZT-4, as shown schematically in Fig. 4(a). The medium is subjected to a uniform remote tension σ_o and a uniform electric field E_o along the x_3 -axis. The axis of material symmetry and the poling direction are chosen to direct along the x_3 -axis. The geometry of the crack surface is defined by

$$x_1 = R \sin \gamma \cos \beta, \quad x_2 = R \sin \gamma \sin \beta, \quad x_3 = R \cos \gamma \quad (12)$$

where R denotes the radius of the spherical cap crack, $\beta \in [0, 2\pi]$, and $\gamma \in [0, \theta]$ with θ denoting the half subtended angle of the spherical surface. It is remarked that the spherical cap crack considered here can be viewed as a penny-shaped crack of radius $a = R\theta$ bended over the surface a sphere of radius R . In the following investigation, a mesh of the crack surface containing 144 elements is adopted, as shown in Fig. 4(b).

The normalized non-zero stress intensity factors K_I and K_{II} , and the electric intensity factor K_{IV} under the four crack-face conditions are shown in Figs 5(a)-5(c),

respectively. It is found that when the half subtended angle of the crack surface is relatively small, the behaviour of the intensity factors of the spherical cap crack is similar to that of the penny-shaped crack. However, when the half subtended angle is sufficiently large, such behaviour of the special cap crack is significantly different from that of the penny-shaped crack.



The results reported in Fig 5(a) indicate that when the half subtended angle of the crack surface is relatively small (e.g. $\theta = 5^\circ, 10^\circ, 15^\circ$), the distribution of K_I under the four crack models is similar to that of K_I for the penny-shaped crack. In particular, K_I of the impermeable, permeable, and semi-permeable cracks are nearly identical and serve as the upper bound, whereas K_I of the energetically consistent model is less than those of the other three crack models and, therefore, serves as the lower bound solution. However, when the half subtended angle is sufficiently large, the distributions of K_I for the impermeable, permeable, and semi-permeable crack models are obviously different and the solution predicted by the permeable model forms the upper bound, whereas K_I for the impermeable model becomes the lower bound instead of the result generated by the energetically consistent model. It can be concluded that the lower bound switches from the energetically consistent solution to the impermeable solution as the half subtended angle increases.

It is important to remark that while the mode-I stress intensity factors of the penny-shaped crack under the impermeable, permeable, and semi-permeable crack-face conditions are identical as indicated in [32, 39], such results for the spherical cap crack are not identical when the half subtended angle is sufficiently large. The discrepancy depends primarily on the half subtended angle. This implies that the non-planar feature of crack surface has a significant influence on the mode-I stress intensity factor of the three crack models. According to the numerical results presented later in Section 5.3, it was also found that non-flat crack

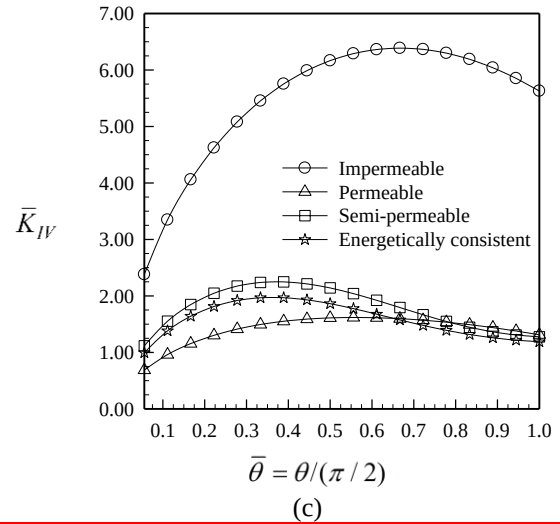
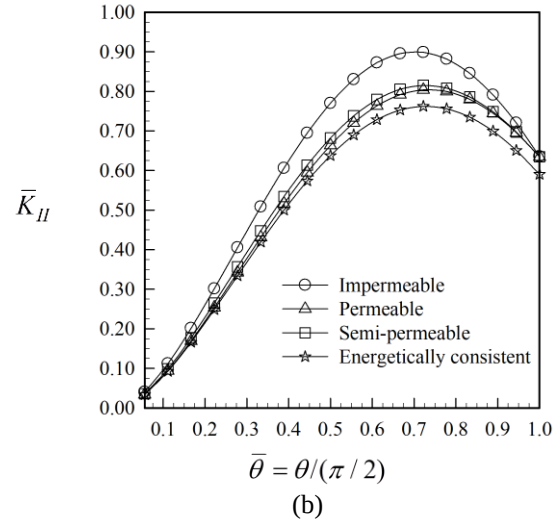
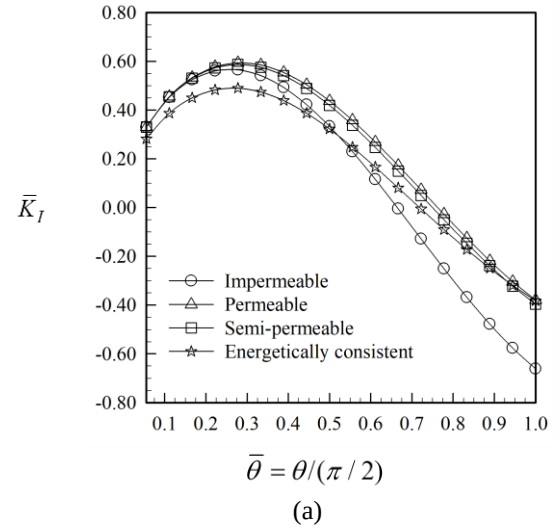


Fig. 5 Dependence of the normalized intensity factors on half-subtended angles for the spherical cap crack in an infinite piezoelectric medium: (a) $\bar{K}_I = K_I/(\sigma_o \sqrt{R})$ (b) $\bar{K}_{II} = K_{II}/(\sigma_o \sqrt{R})$ and (c) $\bar{K}_{IV} = (K_{IV} E_{3333})/(\sigma_o E_{3343} \sqrt{R})$

surfaces leads to the dependence of the mode-I stress intensity factor K_I on the electric field for both the impermeable and semi-permeable models, while the electric field has no effect on K_I of the penny-shaped crack under the impermeable, permeable, and semi-permeable crack-face conditions. This conclusion is supported by the significant difference among results generated by the three crack models (i.e., impermeable, permeable, and semi-permeable models) when the half subtended angle of the spherical cap crack is sufficiently large.

While the lower bound of the mode-I stress intensity factor for the spherical cap crack switches when the half subtended angle is sufficiently large, the upper and lower bounds of the mode-II stress intensity factor for the spherical cap crack do not. Results in Fig. 5(b) indicates that the mode-II stress intensity factor K_{II} for the impermeable and energetically consistent cracks serve, respectively, as the upper and lower bounds for (nearly) the entire range of the normalized half subtended angle $\bar{\theta}$, except when θ is either very small or large (i.e. $\theta = 5^\circ$ and $\theta = 90^\circ$). In particular, the numerical result indicates that when the half-subtended angle is relatively small (e.g., $\theta = 5^\circ$), results generated by the four crack-face models are nearly identical and almost zero. This finding is similar to that for the penny-shaped crack subjected to uniform remote loading where the mode-II stress intensity factors for the four crack-face models vanish. However, such result becomes clearly distinct when the half subtended angles become larger (e.g., $\theta = 40^\circ, 45^\circ, 50^\circ, 55^\circ$) for the four crack-face models, and it can be seen that the impermeable solution is significantly different from those of the other three crack-face models. Finally, it should be noted that when the half subtended angle is relative large (i.e., $\theta = 90^\circ$), the mode-II stress intensity factor K_{II} for the impermeable, permeable, and semi-permeable cracks are nearly identical.

Similar to the mode-I stress intensity factor but unlike the mode-II stress intensity factor, the lower bound of the electric intensity factor K_{IV} for the spherical cap crack switches as the half subtended angle increases. Results in Fig. 5(c) indicate that the electric intensity factors K_{IV} for the four crack-face models are different, and that the permeable solution represents the lower bound when the half subtended angle is relatively small. However, when the half subtended angle becomes larger (e.g., $\theta = 65^\circ, 70^\circ$), the energetically consistent solution becomes the lower bound instead for the permeable solution. Moreover, it is obvious that when the half subtended angle of the spherical cap crack is relatively small, K_{IV} of the spherical cap crack for the four crack-face models possesses a similar behaviour to that of the penny-shaped crack, where the permeable solution of K_{IV} serves as the lower bound. However, K_{IV} of the spherical cap crack and the penny-shaped crack obviously exhibit different behaviour when the half subtended angle is sufficiently large, and it can be seen that the lower bound switches from the permeable solution to the energetically consistent solution. Finally, it can be inferred from Figs. 3(b) and 5(c) that K_{IV} for the impermeable model is still the

upper bound for both the penny-shaped crack and the spherical cap crack for the entire range of the half subtended angle θ considered.

5.2.2 Cylindrical crack

Consider a cylindrical crack with the radius of curvature R embedded in a transversely isotropic infinite piezoelectric medium made of PZT-4, as illustrated in Fig. 6(a). The piezoelectric medium is subjected to a uniform remote tension σ_o and a uniform electric field E_o along the x_3 -axis. The axis of the material symmetry and the poling direction coincide with the x_3 -axis.

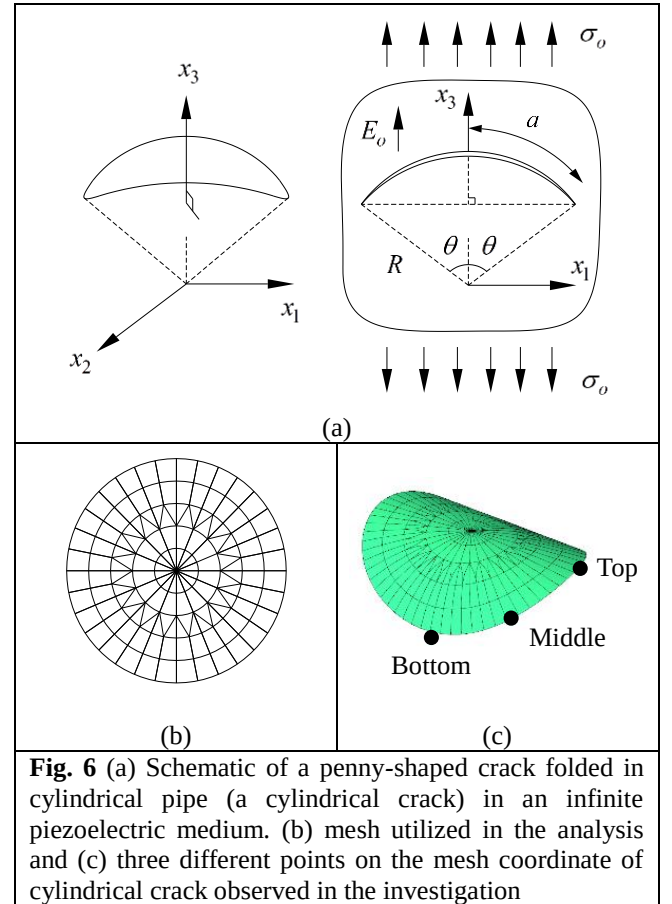
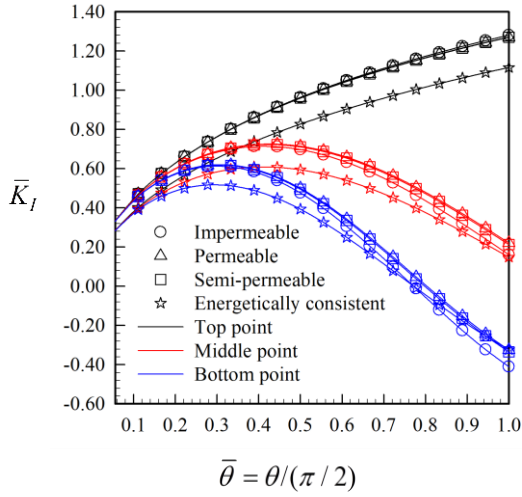


Fig. 6 (a) Schematic of a penny-shaped crack folded in cylindrical pipe (a cylindrical crack) in an infinite piezoelectric medium. (b) mesh utilized in the analysis and (c) three different points on the mesh coordinate of cylindrical crack observed in the investigation

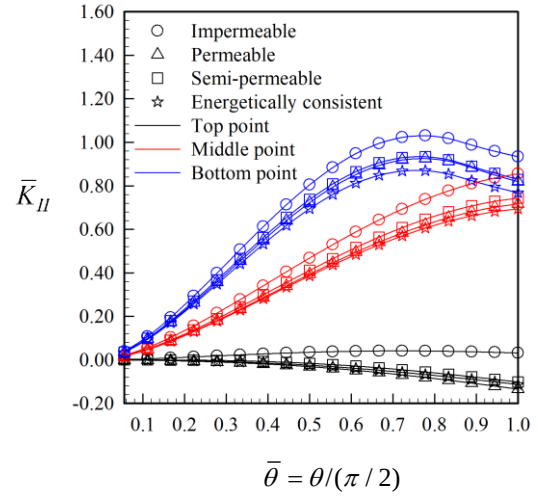
The geometry of the crack surface is defined by

$$x_1 = R \sin \gamma, \quad x_2 \in [-d, d], \quad x_3 = R \cos \gamma \quad (13)$$

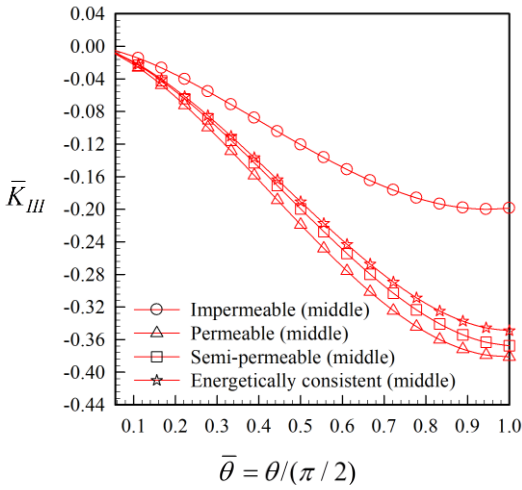
where $\gamma \in [0, \theta]$, θ denotes the half subtended angle of the crack surface, and $d = R\sqrt{\theta^2 - \gamma^2}$. It is worth noting that the cylindrical crack considered here can be obtained directly by bending a penny-shaped crack of radius $a = R\theta$ over the surface of a cylinder of radius R . In the present study, a mesh of the cylindrical crack containing 144 elements is used, as shown in Fig 6(b). The normalized stress intensity factors $\{K_I, K_{II}, K_{III}\}$ and the electric intensity



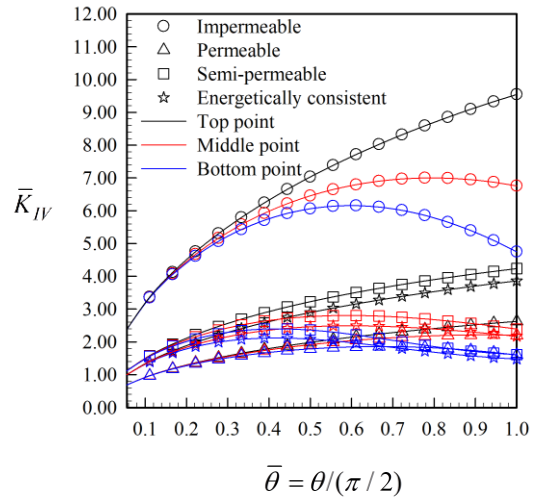
(a)



(b)



(c)



(d)

Fig. 7 Dependence of the normalized intensity factors on the half-subtended angles for a cylindrical crack in an infinite piezoelectric medium: (a) $\bar{K}_I = K_I/(\sigma_o\sqrt{R})$ (b) $\bar{K}_{II} = K_{II}/(\sigma_o\sqrt{R})$ (c) $\bar{K}_{III} = K_{III}/(\sigma_o\sqrt{R})$ and (d) $\bar{K}_{IV} = (K_{IV}E_{3333})/(\sigma_oE_{3343}\sqrt{R})$

factor K_{IV} at three different points (i.e., top, middle, and bottom points) located on the cylindrical crack as shown in Fig. 6(c) are reported in Figs 7(a)-7(d) for four crack-face models.

Results in Fig 7(a) indicate that the mode-I stress intensity factor K_I for the top point is greater than those for the middle and bottom points of the cylindrical crack. Moreover, it can be inferred from Fig 7(a) that K_I under the four types of crack-face conditions obtained from the top and bottom points of the cylindrical crack are similar to the results for the penny-shaped crack and spherical cap crack, whereas such intensity factors obtained for the middle point of the cylindrical crack are different from those for the penny-shaped crack and the spherical cap crack. In particular, it was found that for the top point in the cylindrical crack, K_I for the impermeable, permeable, and semi-permeable cracks are nearly identical, whereas that for the energetically

consistent model is less than those for the other three crack-face models and, therefore, serves as the lower bound for the entire range of the normalized half-subtended angle $\bar{\theta}$. While K_I at the top point of the cylindrical crack obtained from the four crack models are similar to that of the penny-shaped crack, the results at the bottom points has the similar behaviour to that of the spherical cap crack. It was also found that K_I at the bottom point of the cylindrical crack obtained from the impermeable, permeable and semi-permeable models are not identical, and that the permeable solution serves as the upper bound, whereas the lower bound of K_I switches from the energetically consistent model to the impermeable model when the subtended angle of the cylindrical crack increases. In contrast to the top and bottom points of the cylindrical crack, K_I at the middle point under the four crack-face conditions are different from the results obtained for the penny-shaped crack and the spherical cap

crack. In particular, it can be seen that results for the impermeable, permeable, and semi-permeable crack-face models are not nearly identical as those observed at the top point and that the permeable solution serves as the upper bound, whereas the energetically consistent solution represents the lower bound for the entire range of the normalized half-subtended angle $\bar{\theta}$.

While the mode-II stress intensity factor K_{II} for the penny-shaped crack subjected to uniform loading identically vanishes along the crack front, it is non-zero and completely different for the top, middle, and bottom points of the cylindrical crack. Results in Fig. 7(b) indicate that K_{II} obtained at the bottom point is greater than those at the middle and top points of the cylindrical crack. In addition, it is obvious that when the half-subtended angle is relatively small (e.g., $\theta = 5^\circ$), the mode-II stress intensity factors at all three points of the cylindrical crack for all four crack-face models are nearly identical and almost vanish. However, when the half-subtended angle increases, the stress intensity factors K_{II} for the four crack-face models are different and their lower and upper bounds can be obviously seen. More specifically, it can be inferred from Fig. 7(b) that when the half-subtended angle is sufficiently large, the impermeable crack model yields the upper bound of K_{II} at all three points of the cylindrical crack, whereas the permeable and energetically consistent crack models provide the lower bound of K_{II} at the top point and the remaining two points (i.e., middle and bottom points), respectively. This indicates that the non-planar feature of the crack surface has the significant influence on the lower bound and the value of the mode-II stress intensity factor for the cylindrical crack. As a result, it can be seen that K_{II} obtained from the four crack models are different and that the upper and lower bounds exist when the half-subtended angle becomes larger. In addition, the lower bounds of K_{II} are not the same for the top and the remaining two points of the cylindrical crack. Finally, it can be observed that when the half-subtended angle of the cylindrical crack is sufficiently large, K_{II} at the top point of the cylindrical crack obtained from the impermeable model is positive and quite different from those obtained from the other three models which clearly attain the negative values.

While the mode-III stress intensity factors K_{III} at the top and bottom points of the cylindrical crack for the four crack-face conditions vanish due to the symmetry, it does not vanish at the middle point of the cylindrical crack and possesses the same negative sign. Results in Fig. 7(c) also indicate that K_{III} obtained from the semi-permeable and energetically consistent crack-face models fall between the impermeable and permeable solutions which constitute the upper and lower bounds, respectively. Moreover, it was also observed that when the half-subtended angle is relatively small (i.e., $\theta = 5^\circ$), K_{III} for the four crack-face conditions tends to be zero. This finding is expected since the mode-III stress intensity factor vanishes for any planar cracks (e.g. penny-shaped cracks) subjected to uniform loading. Finally, it can be pointed out that, similar to the mode-II

stress intensity factor K_{II} , K_{III} for the impermeable model is significantly different from those generated by the other three crack-face models when the half-subtended angle becomes larger.

The electric intensity factor K_{IV} at the top, middle, and bottom points of the cylindrical crack are shown in Fig. 7(d). It was found that K_{IV} at the top point is greater than those at the middle and bottom points. Moreover, obtained numerical results also indicate that the behaviour of K_{IV} at the top and bottom points of the cylindrical crack are similar to those for the penny-shaped crack and the spherical cap crack, respectively, whereas K_{IV} at the middle point of the cylindrical crack is significantly different from those of the penny-shaped and spherical cap cracks. More specifically, it was found that at the top point of the cylindrical crack, the electric intensity factor K_{IV} for the four crack models are completely different according to the electrical boundary conditions adopted on the crack surface. It can be seen that the impermeable and permeable crack models yield, respectively, the upper and lower bounds of K_{IV} to the solutions from the semi-permeable and energetically consistent models. This finding is similar to the results obtained for the penny-shaped crack. Obtained results also indicate that the electric intensity factor K_{IV} at the bottom point of the cylindrical crack obtained from the four crack models are different from those of the penny-shaped crack, but are similar to those of the spherical cap crack. In addition, the impermeable crack model yields the upper bound of K_{IV} for the entire range of the normalized half-subtended angle $\bar{\theta}$, whereas the lower bound fluctuates between solutions generated by the permeable and energetically consistent crack models as $\bar{\theta}$ increases. Finally, it can be seen that when the half-subtended angle is large (e.g., $\theta = 75^\circ, 80^\circ, 85^\circ, 90^\circ$), the electric intensity factor K_{IV} of the permeable crack is nearly identical to that of the semi-permeable crack. In contrast to the top and bottom points, K_{IV} obtained at the middle point of the cylindrical crack is different from those of the penny-shaped and spherical cap cracks. It was found that the permeable crack provides the lower bound solution of K_{IV} for almost the entire range of the normalized half-subtended angle $\bar{\theta}$ except when the subtended angle is significantly large (e.g., $\theta = 85^\circ, 90^\circ$). Solutions generated by the permeable and energetically consistent crack models are nearly identical and serve as the lower bound. It is interesting to note that though the lower bound of K_{IV} for all three points of the cylindrical crack are completely different; the impermeable crack model always yields the upper bound of K_{IV} for all those three points.

5.3 Influence of remote electrical loading

To explore the influence of remote electrical loading on the intensity factors along the crack front of non-flat cracks, the same spherical cap crack and cylindrical crack embedded in

a transversely isotropic infinite piezoelectric medium made of PZT-4 as shown in Figs. 4 and 6, respectively, are considered. In the numerical study, following parameters are employed: (i) the half-subtended angle for both cracks is taken to be $\theta = 45^\circ$; (ii) the uniform remote tension σ_o is taken to be 50 MPa whereas the uniform electric field E_o varies from -4.5 to 4.5 MV/m; (iii) the permeable, impermeable, semi-permeable, and energetically consistent crack models are considered; and (iv) the permittivity of the medium inside the crack gap for both semi-permeable and energetically consistent crack models are taken to be $\kappa_c = 5\kappa_o$ where $\kappa_o = 8.85 \times 10^{-12}$ C/(Vm) is the permittivity of air. In the analysis of the spherical cap crack and cylindrical crack, the meshes shown in Figs. 4(b) and 6(b) are employed, respectively. In addition, results for a penny-shaped crack of radius $a = R\theta$ are also obtained and used in the comparison and discussion.

5.3.1 Spherical cap crack

The normalized non-zero stress intensity factors K_I , K_{II} and the electric intensity factor K_{IV} under four types of crack-face conditions are shown in Figs. 8(a)-8(c), respectively. It can be seen from the results reported in Fig. 8(a) that the mode-I stress intensity factor K_I for the impermeable and semi-permeable spherical cap cracks depends strongly on the electric field E_o . These findings are in contrast with those observed for planar cracks (i.e., a penny-shaped crack) where K_I for impermeable, permeable, and semi-permeable cracks are identical and independent of the electric field. Moreover, it was also observed that K_I for permeable planar and non-planar cracks are independent of the electric field. This implies that the curvature of non-planar cracks leads to the dependence of the electric field on K_I for both the impermeable and semi-permeable crack-face conditions. However, such effect plays no role on the value of K_I for the permeable condition. Moreover, it can be inferred from Fig. 8(a) that the mode-I stress intensity factor generated from the energetically consistent model strongly depends on the electric field for both the penny-shaped crack and the spherical cap crack. In addition, obtained numerical results also indicate that the energetically consistent crack-face model yields the lower bound of K_I for the penny-shaped crack subjected to the uniform loading for the entire range of the electric field considered, while the lower bound of the spherical cap crack for the positive electric field fluctuates between the impermeable and energetically consistent solutions by showing a turning point at about $E_o = 2.5$ MV/m. Moreover, it was also observed that K_I for all four crack-face conditions are nearly identical when the electric field vanishes, i.e. $E_o = 0$ MV/m. This finding is equally true for both the penny-shaped and spherical cap cracks, as shown in Fig. 8(a). Similar to the mode-I stress intensity factor, Fig. 8(b) indicates that the mode-II stress intensity factor K_{II} for the permeable spherical cap crack is

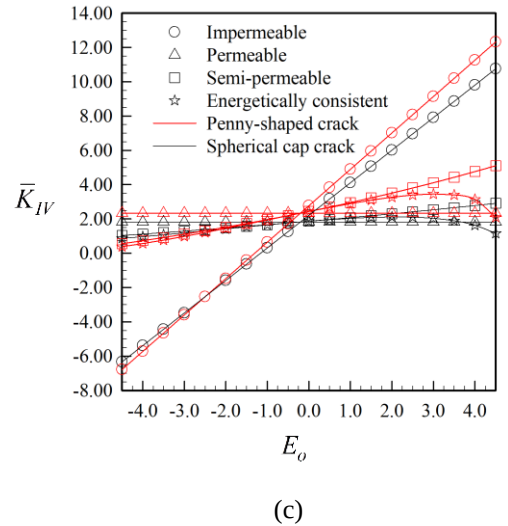
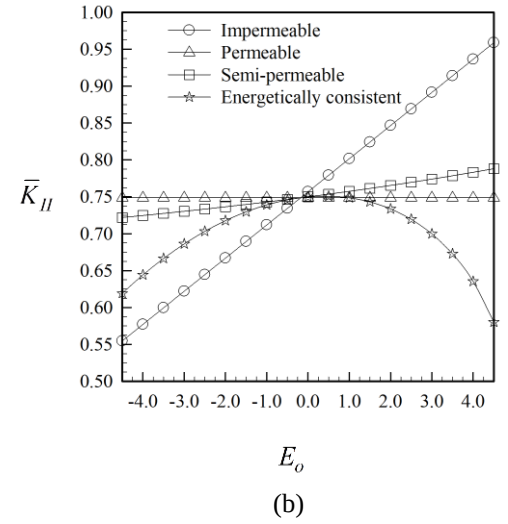
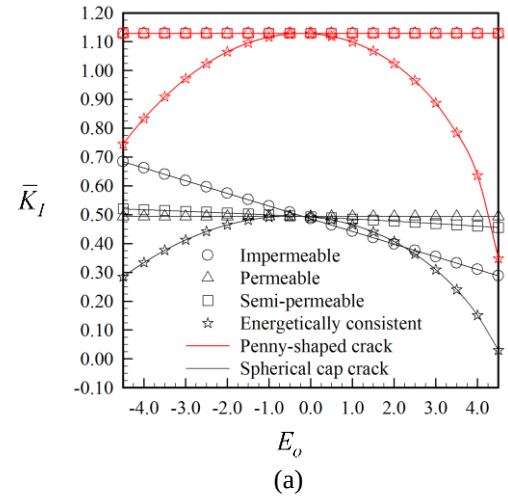


Fig. 8 Dependence of the normalized intensity factors on the electric field for a spherical cap crack in an infinite piezoelectric medium: (a) $\bar{K}_I = K_I/(\sigma_o \sqrt{a})$ (b) $\bar{K}_{II} = K_{II}/(\sigma_o \sqrt{a})$ and (c) $\bar{K}_{IV} = (K_{IV} E_{3333})/(\sigma_o E_{3343} \sqrt{a})$

independent of the electric field, whereas K_{II} of the impermeable, semi-permeable, and energetically consistent spherical cap cracks depend strongly on the electric field.

While the fracture behaviour of planar cracks (i.e. penny-shaped cracks) and non-planar cracks (i.e. spherical cap cracks) are different for the mode-I stress intensity factor when using the impermeable and semi-permeable conditions, Fig. 8(c) indicates that the electric intensity factor K_{IV} for the spherical cap crack is similar to K_{IV} of the penny-shaped crack under all four crack-face conditions. More specifically, K_{IV} is independent of the electric field for both the permeable penny-shaped and spherical cap cracks, whereas K_{IV} for the impermeable, semi-permeable, and energetically consistent crack-face models depend strongly on the electric field for both crack configurations.

According to the numerical results for the penny-shaped and spherical cap cracks subjected to remote uniform loading, the non-zero intensity factors $\{K_I, K_{II}, K_{IV}\}$ for the spherical cap crack and the non-zero intensity factors $\{K_I, K_{IV}\}$ for the penny-shaped crack are all independent of the electric field under the permeable condition.

5.3.2 Cylindrical cracks

The normalized stress intensity factors K_I , K_{II} and K_{III} and the electric intensity factor K_{IV} under four types of crack-face conditions are shown in Figs. 9(a)-9(d), respectively. It can be seen from the results reported in Fig. 9(a) that the mode-I stress intensity factor K_I at the top point is greater than those at the middle and bottom points of the cylindrical crack. Moreover, it can be inferred from Fig. 9(a) that K_I at the top point of the impermeable, permeable, and semi-permeable cracks are nearly identical, whereas that of the energetically consistent crack strongly depends on the electric field. These findings are similar to the case of the penny-shaped crack subjected to uniform loading as shown in Fig 8(a). In contrast, the behaviour of K_I obtained from the three crack models for the remaining two points (i.e., middle and the bottom points) of the cylindrical crack is different from that of the penny-shaped crack but similar to that of the spherical cap crack. Fig. 9(a) indicates that the impermeable, permeable, and semi-permeable models yield slightly different K_I at the middle point but clearly different at the bottom point of the cylindrical crack. In addition, it can be seen that K_I for the impermeable and semi-permeable cases depend on the electric field whereas K_I for the permeable case is independent of the electric field. Finally, it should be noted that K_I generated from the energetically consistent model strongly depends on the electric field for all three points of the cylindrical crack, similar to that of the penny-shaped crack.

While the mode-II stress intensity factor K_{II} vanishes identically for the penny-shaped crack subjected to uniform loading, it is non-zero for the top, middle, and bottom points. Fig. 9(b) indicates that K_{II} at the bottom point is greater than those at the middle and top points of the cylindrical

crack. Moreover, it can be inferred from Fig 9(b) that K_{II} for the impermeable, semi-permeable, and energetically consistent models depend on the electric field and vary according to the electrical boundary condition adopted on the crack surface (except when the electric field is equal to zero, at the point where K_{II} for the four crack models are nearly identical), whereas K_{II} for the permeable model is independent of the electric field. These findings are similar for all three points of the cylindrical crack, as shown in Fig. 9(b). While the mode-III stress intensity factor K_{III} at the top and bottom points of the cylindrical crack vanish, Fig. 9(c) shows that it does not vanish at the middle point. It was observed that, similar to the mode-II stress intensity factor, K_{III} at the middle point of the cylindrical crack for the impermeable, semi-permeable, and energetically consistent models depend on the electric field, whereas that for the permeable model is clearly independent of the electric field.

The electric intensity factor K_{IV} obtained at the top, middle, and bottom points of the cylindrical crack are shown in Fig. 9(d). It was observed that, for the positive electric field, K_{IV} at the top point is greater than those at the middle and bottom points. Moreover, it was found that the behaviour of K_{IV} at all three points of the cylindrical crack is similar to that of the spherical cap crack and the penny-shaped crack. More specifically, the numerical results indicate that K_{IV} for the impermeable, semi-permeable, and energetically consistent models depend strongly on the electric field, whereas K_{IV} at all three points of the cylindrical crack for the permeable model is independent of the electric field.

Finally, it is important to note that the intensity factors $\{K_I, K_{II}, K_{III}, K_{IV}\}$ at the top, middle, and bottom points of the cylindrical crack under the permeable crack-face condition are independent of the electric field. This similar conclusion was also observed for the case of a penny-shaped crack and a spherical cap crack.

5.4 Influence of permittivity of the medium inside the crack gap

To explore the influence of the permittivity of the medium inside the crack gap on the intensity factors along the crack front for both the spherical cap crack and the cylindrical crack (see Figs. 4 and 6) under semi-permeable and energetically consistent crack-face conditions, following parameters were used for investigation: (i) variation in the permittivity of the medium inside the crack gap, defined by $\kappa_c = \alpha \kappa_o$ where α is the relative permittivity and $\kappa_o = 8.85 \times 10^{-12}$ C/(Vm) is the permittivity of air, is considered by varying α ; (ii) the half-subtended angle is taken as $\theta = 45^\circ$; (iii) the uniform remote tension and the uniform electric field are taken as, 50 MPa and 2.5 MV/m, respectively; and (iv) the impermeable and permeable crack-face conditions are also considered in addition to the semi-permeable, and energetically consistent case to represent the case $\alpha = 0$ and $\alpha = \infty$, respectively. Again, the meshes shown in Figs. 4(b) and 6(b) are employed in the analysis.

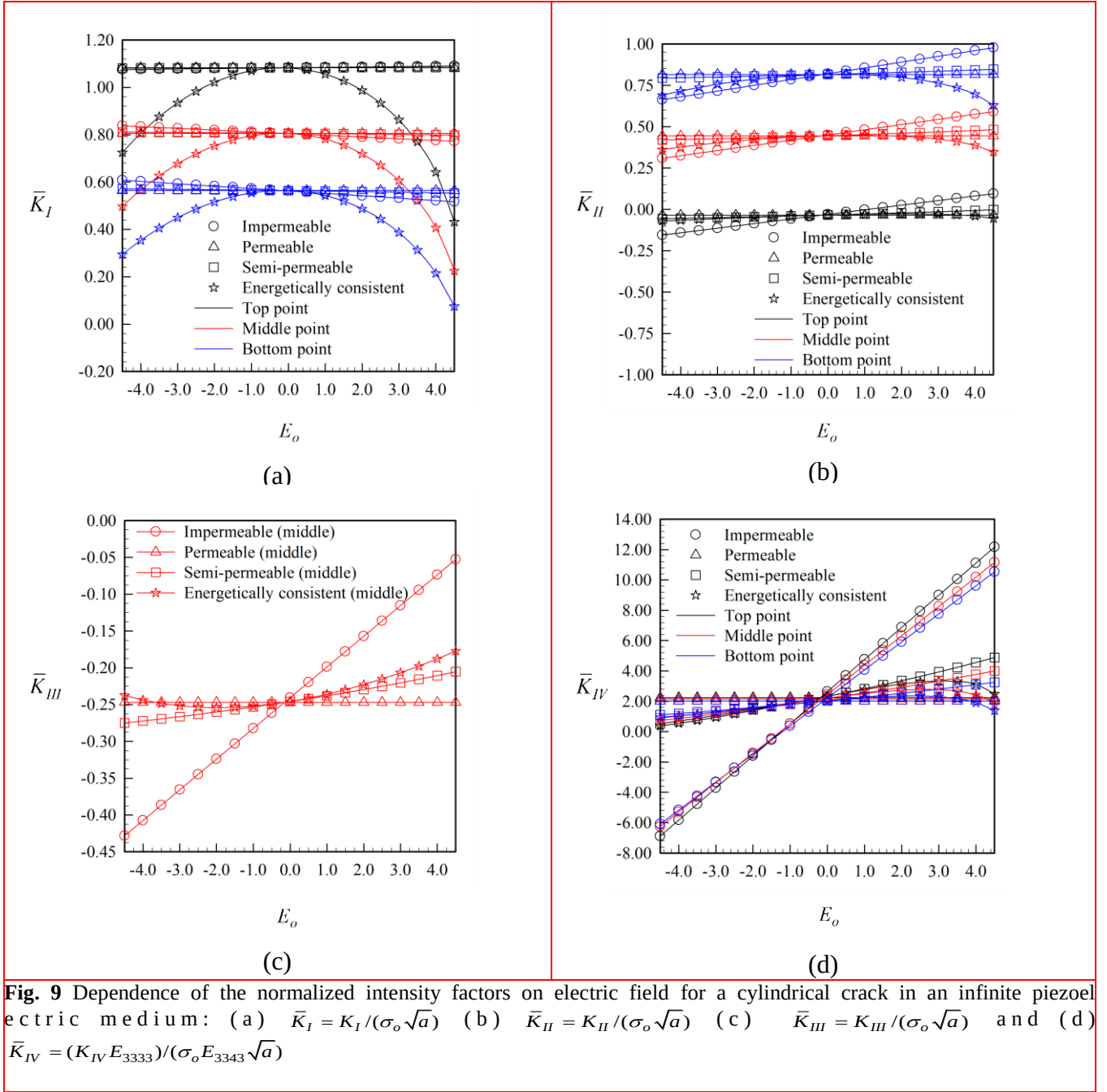


Fig. 9 Dependence of the normalized intensity factors on electric field for a cylindrical crack in an infinite piezoelectric medium: (a) $\bar{K}_I = K_I/(\sigma_o\sqrt{a})$ (b) $\bar{K}_{II} = K_{II}/(\sigma_o\sqrt{a})$ (c) $\bar{K}_{III} = K_{III}/(\sigma_o\sqrt{a})$ and (d) $\bar{K}_{IV} = (K_{IV}E_{3333})/(\sigma_oE_{3343}\sqrt{a})$

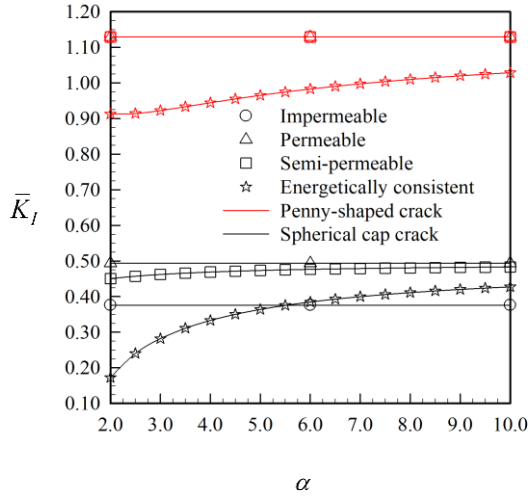
Results for a penny-shaped crack of radius $a = R\theta$ are also obtained for the comparison purpose.

5.4.1 Spherical cap crack

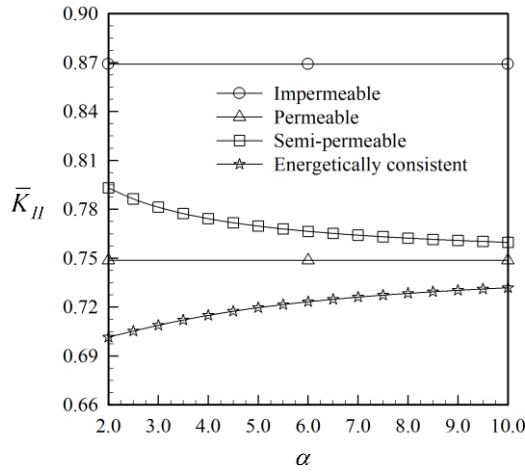
The influence of the permittivity of the medium inside the spherical cap crack on the non-zero intensity factors, $\{K_I, K_{II}, K_{IV}\}$ are reported in Figs. 10(a)-10(c), respectively. It was observed that the mode-I stress intensity factor K_I of the penny-shaped crack under the impermeable, permeable, and semi-permeable crack-face conditions are identical and clearly independent of the medium inside the crack gap, while K_I of the same crack predicted by the energetically consistent model strongly depends on the medium inside the crack gap and it serves as

the lower bound solution for the entire range of the relative permittivity α .

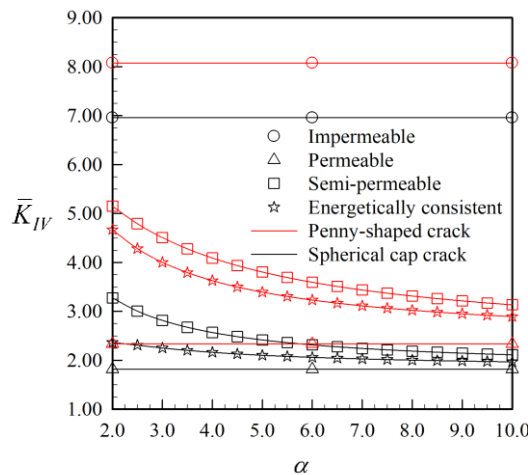
Unlike the penny-shaped crack, Fig. 10(a) shows that the results for the spherical cap crack obtained from the impermeable, permeable, and semi-permeable models are not identical and that the results for the semi-permeable crack depend on the medium inside the crack gap, whereas the results for the impermeable and permeable cases are completely different. Also, one can see that K_I for the semi-permeable crack varies from the impermeable solution to the permeable solution (which clearly represents the upper bound solution) as the value of the permittivity of the medium increases. This implies that the non-planar feature of the crack surface has the significant influence on the mode-I stress intensity factor of the three crack models. In addition, the numerical results also indicate that the lower



(a)



(b)



(c)

Fig 10 Dependence of the normalized intensity factors on relative permittivity α of the medium inside the crack gap for a spherical cap crack in an infinite piezoelectric medium: (a) $\bar{K}_I = K_I / (\sigma_o \sqrt{a})$ (b) $\bar{K}_{II} = K_{II} / (\sigma_o \sqrt{a})$ and (c) $\bar{K}_{IV} = (K_{IV} E_{3333}) / (\sigma_o E_{3343} \sqrt{a})$

bound of K_I for the special cap crack fluctuate between the energetically consistent and the impermeable models at a turning point of about $\alpha = 5.5$.

Figs. 10(b) and 10(c) indicate that the mode-II stress intensity factor K_{II} and the electric intensity factor K_{IV} of the spherical cap crack under four crack-face conditions are significantly different according to the electrical boundary condition adopted on the crack surface. Moreover, it can be observed from obtained results that K_{II} and K_{IV} are strongly dependent on the permittivity of medium inside the crack gap for both the semi-permeable and energetically consistent models. Such the results for both crack-face conditions approach the permeable solution as the permittivity of the medium increases. In addition, it can be seen that the impermeable solutions represent the upper bound for both K_{II} and K_{IV} while the energetically consistent solution is the lower bound for the mode-II stress intensity factor K_{II} and the permeable solution serves as the lower bound for the electric intensity factor K_{IV} . Finally, it is important to note that the distribution of K_{IV} of the spherical cap crack under four crack-face conditions is similar to that of the penny-shaped crack.

5.4.2 Cylindrical cracks

The normalized stress intensity factors K_I , K_{II} , K_{III} and the electric intensity factor K_{IV} under four types of crack-face conditions are reported in Figs. 11(a)-11(d), respectively. Results in Fig. 11(a) indicate that the mode-I stress intensity factor K_I at the top point is greater than the middle and bottom points of the cylindrical crack. It can be also observed from Fig. 11(a) that K_I for the semi-permeable and energetically consistent models depends strongly on the medium inside the crack gap. In addition, the energetically consistent solution is significantly different from those of the other three crack-face models and it serves as the lower bound for the entire range of the relative permittivity α . This similar behaviour is observed for the top, middle, and bottom points of the cylindrical crack. It is interesting to remark that the upper bounds of K_I at the three points of the cylindrical crack are not the same. More specifically, results at the top point for the impermeable, permeable, and semi-permeable crack-face conditions are nearly identical and serve as the upper bound, whereas the permeable solutions represent the upper bounds for both the middle and bottom points of the cylindrical crack.

Finally, it can be concluded that the lower and upper bounds of K_I for the penny-shaped crack, spherical cap crack, and cylindrical crack under four crack-face conditions are different. For the penny-shaped crack, the three crack-face models (i.e., impermeable, permeable, and semi-permeable models), and the energetically consistent crack-face model yield the upper and lower bound solutions, respectively, whereas, for the spherical cap crack, the permeable solution serves as the upper bound while the lower bound fluctuates between the energetically consistent and impermeable solutions as the permittivity of the medium

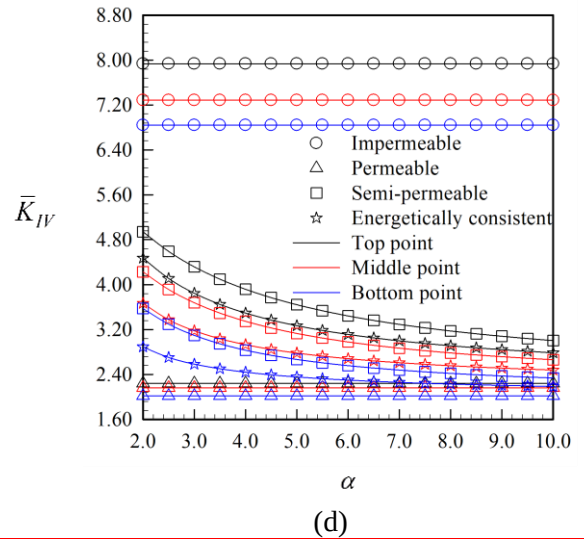
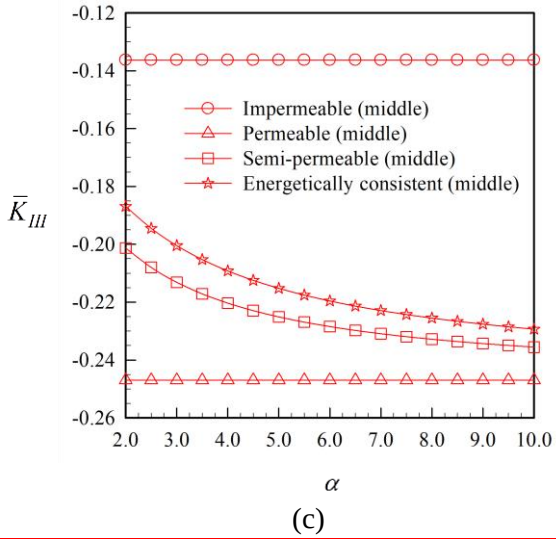
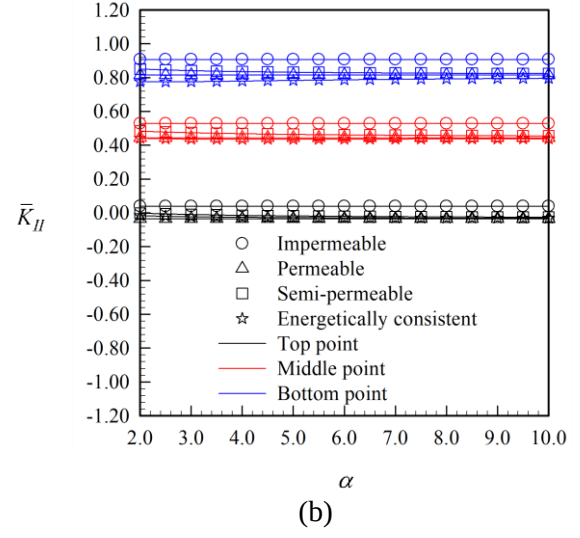
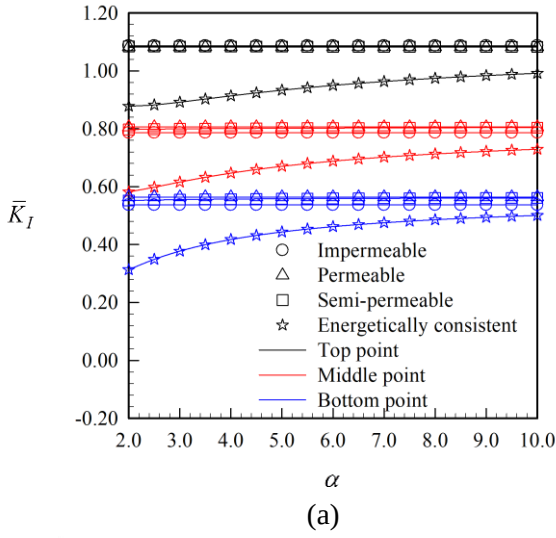


Fig. 11 Dependence of the normalized intensity factors on the relative permittivity, α of the crack medium for a cylindrical crack in an infinite piezoelectric medium : (a) $\bar{K}_I = K_I / (\sigma_o \sqrt{a})$ (b) $\bar{K}_{II} = K_{II} / (\sigma_o \sqrt{a})$ (c) $\bar{K}_{III} = K_{III} / (\sigma_o \sqrt{a})$ and (d) $\bar{K}_{IV} = (K_{IV} E_{3333}) / (\sigma_o E_{3343} \sqrt{a})$

inside the crack gap increases. However, the lower bound does not fluctuate for the cylindrical crack, and it can be observed that the energetically consistent solution serves as the lower bound for all three points of the cylindrical crack. One can also see that the semi-permeable solution is independent of the permittivity of the medium inside the crack gap for the penny-shaped crack but strongly depends on such quantity for both the spherical cap crack and the cylindrical crack.

Results in Fig. 11(b) indicate that the mode-II stress intensity factor K_{II} at the bottom point is greater than those at the middle and top points of the cylindrical crack. Similar to the mode-I stress intensity factor, K_{II} for the semi-permeable and energetically consistent conditions is also strongly dependent on the permittivity of medium inside the crack gap. Results for both models vary from the impermeable solution to the permeable solution as the

permittivity of the medium inside the crack gap increases. This finding is true for all three points of the cylindrical crack. In addition, one can observe that the impermeable solutions represent the upper bound for all three points of the cylindrical crack whereas the permeable solution serves as the lower bound for the top point and the energetically consistent solution represents the lower bound for the middle and bottom points for the cylindrical crack. This implies that the distribution of K_{II} for the middle and bottom points of the cylindrical crack is similar to that of the special cap crack. As a result, one can see that the upper and lower bounds of the middle and bottom points of the cylindrical crack are the same as those for the spherical cap crack.

From results of the non-zero mode-III stress intensity factor K_{III} at the middle point of the cylindrical crack shown in Fig. 11(c), the semi-permeable and energetically consistent solutions, again, depend strongly on the

permittivity of the medium inside the crack gap and both results varies from the impermeable solution to the permeable solution as the permittivity of the medium increases. Clearly, the impermeable solution serves as the upper bound whereas the permeable solution represents the lower bound for the entire range of the relative permittivity α .

Results reported in Fig. 11(d) indicate that the electric intensity factor K_{IV} at the top point is greater than those at the middle and bottom points of the cylindrical crack. Moreover, it was found that the behaviour of K_{IV} at the three points of the cylindrical crack is similar to that of the penny-shaped crack and the spherical cap crack. More specifically, K_{IV} generated from the semi-permeable and energetically consistent models strongly depends on the relative permittivity α of the medium inside the crack gap and both results vary from the impermeable solution that is the upper bound to the permeable solution that is the lower bound as α increases.

6 Conclusions

In the present study, we have fully investigated the influence of half subtended angle on the intensity factors of three crack configurations including the penny-shaped crack, the spherical cap crack, and the cylindrical crack under impermeable, permeable, semi-permeable, and energetically consistent crack-face conditions. Obtained results have indicated that the mode-I stress intensity factor K_I and the electric intensity factor K_{IV} at the top and bottom points of the cylindrical crack have the similar behaviour to that of the penny-shaped crack and the spherical cap crack, respectively, whereas $\{K_I, K_{IV}\}$ at the middle point of the cylindrical crack have different trend when compared with those for the penny-shaped crack and the spherical cap crack. However, for the relatively small half-subtended angle (e.g., $\theta = 5^\circ, 10^\circ$), the characteristics of $\{K_I, K_{IV}\}$ for the spherical cap crack and the cylindrical crack are similar to those for the penny-shaped crack.

For the penny-shaped crack subjected to a uniform loading, the mode-II and mode-III stress intensity factors vanish identically for all four types of crack-face conditions considered due to the symmetry. In contrast, due to the non-flat feature of the crack surface, the mode-II stress intensity factor of the spherical cap crack and the cylindrical crack and the mode-III stress intensity factor of the cylindrical crack do not vanish along the crack front. It was also found that when the half-subtended angle of the spherical cap crack and the cylindrical crack are sufficiently large, the non-zero mode-II and mode-III stress intensity factors under the four crack-face conditions are different according to the electric boundary condition adopted on the crack surface. As a result, the upper and lower bounds of the mode-II and mode-III stress intensity factors could be seen. In particular, the impermeable solutions represents the upper bound of the mode-II stress intensity factor of the spherical cap crack and the three representative points for the cylindrical crack, and

as well as the upper bound of the mode-III stress intensity factor at the middle point of the cylindrical crack. However, the energetically consistent solution represents the lower bound of the mode-II stress intensity factor of the spherical cap crack and the middle and bottom points of the cylindrical crack, whereas the permeable solution serves as the lower bound of the mode-II and mode-III stress intensity factors at the top and middle points of the cylindrical crack, respectively.

Results from the investigating of the influence of the electric field on the intensity factors for three representative crack configurations under four different crack-face conditions have indicated that the non-zero intensity factors $\{K_I, K_{II}, K_{III}, K_{IV}\}$ under the permeable and energetically consistent crack-face conditions for both planar (i.e. penny-shaped crack) and non-planar cracks (i.e. spherical cap and cylindrical cracks) were independent and dependent of the electric field, respectively. In addition, it was found that the influence of the electric field on the electric intensity factor K_{IV} for the penny-shaped crack was similar to that for the spherical cap crack and cylindrical crack under four crack-face conditions. In particular, K_{IV} obtained from the impermeable, semi-permeable, and energetically consistent models depends strongly on the electric field, whereas K_{IV} of the permeable crack was independent of the electric field. On the contrary, the behaviour of the mode-I stress intensity factor K_I under the impermeable and semi-permeable conditions of the planar crack (i.e. penny-shaped) and the non-planar cracks (i.e. spherical cap crack, and the middle and bottom points of the cylindrical crack) were different. In particular, K_I of the impermeable and semi-permeable spherical cap crack depends strongly on the electric field whereas such results for the penny-shaped crack were independent of the electric field. This implies that the non-flat crack surface renders the dependence of the electric field on K_I for both the impermeable and semi-permeable crack-face conditions.

Finally, it can be concluded that all the intensity factors $\{K_I, K_{II}, K_{III}, K_{IV}\}$ for the three representative crack configurations under the energetically consistent crack-face condition strongly depend on the permittivity of the medium inside the crack gap. The same behaviour can also be observed for the semi-permeable crack-face condition, except for the mode-I stress intensity factor K_I of the penny-shaped crack that was found independent of the medium inside the crack gap.

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ANALYSIS FOR GENERALIZED INTENSITY FACTORS AND T-STRESSES OF PERMEABLE CRACKS IN 3D LINEAR PIEZOELECTRIC MEDIA

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ABSTRACT

This paper presents the development of an efficient and accurate computational procedure for analysis of cracks in three-dimensional, linear, piezoelectric, infinite medium. The formulation is established in a general framework allowing permeable cracks of arbitrary shape, material anisotropy, and general loading conditions to be treated. The final governing equations are given in terms of a pair of weakly singular, weak-form boundary integral equations involving only unknowns on the crack surface. In the solution procedure, a weakly singular boundary integral equation method together with a standard Galerkin finite element technique is adopted to solve a system of integral equations. Several components including the near-front approximation, special numerical integration, evaluation of two point kernels for general anisotropy, and linear solvers for both dense and sparse systems are properly integrated into the proposed method to enhance its capability. The generalized intensity factors and T-stresses along the crack front are then post-processed directly from the jump in and the sum of the crack-face generalized displacement. The implemented numerical scheme is then fully tested for various scenarios and a selected set of results are reported to demonstrate both the accuracy and its computational performance.

Keywords: BIEM, generalized T-stresses, permeable cracks, generalized intensity factors, piezoelectric media.

1. INTRODUCTION

Piezoelectric materials are widely used in various engineering and industrial applications (e.g., sensors and actuators). Unfortunately, these smart materials are brittle and susceptible to fracture, either during the manufacturing process or during the usage. Therefore, it is necessary to investigate the fracture-based failure mechanism of such materials. Based on the theory of linear piezoelectric fracture mechanics (LPFM), besides the generalized intensity factors, the generalized T-stresses

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also play an important roles in characterizing the near-tip behavior of cracked piezoelectric bodies and ignorance of such parameters in the fracture modeling can lead to inaccurate results or mislead the response prediction; extensive review of applications of generalized T-stresses can be found in the work of Subsathaphol (2013) and Pich (2015). On the basis of an extensive literature survey, available solution techniques have been well established for and focused mainly on the calculation of the generalized intensity factors (e.g., Li and Lee, 2004a; Rungamornrat and Mear, 2008; Rungamornrat et al., 2015), but those capable of determining the generalized T-stresses along the crack front are still relatively few and required further enhancement. In particular, most of existing studies were restricted mainly to two-dimensional boundary value problems or three-dimensional problems with simple settings (e.g., Li and Lee, 2004b; Pich et al., 2016) and simple crack-face condition such as an electrically impermeable crack (e.g., Subsathaphol et al., 2015)

In this paper, an efficient and accurate numerical method based on the weakly singular, boundary integral equation method, developed by Rungamornrat and Mear (2008) and Subsathaphol et al. (2015), is generalized to be capable of determining both the generalized intensity factors and the generalized T-stress components of arbitrary shaped, permeable cracks embedded in three-dimensional, generally anisotropic, linear piezoelectric media.

2. PROBLEM DESCRIPTION AND BASIC FIELD EQUATIONS

Consider an arbitrary shaped crack embedded in a three-dimensional, homogeneous, generally anisotropic, linear piezoelectric, infinite medium Ω as shown schematically in Figure 1. The crack surface consists of two geometrically coincident surfaces (e.g., upper and lower crack surfaces), denoted by S_c^+ and S_c^- with the outward unit normal vectors $\mathbf{n}^+$ and $\mathbf{n}^-$, respectively. The medium, while free of the body force and body electric charge, is subjected to both mechanical and electrical loadings acting either to the crack surfaces or the remote boundary.

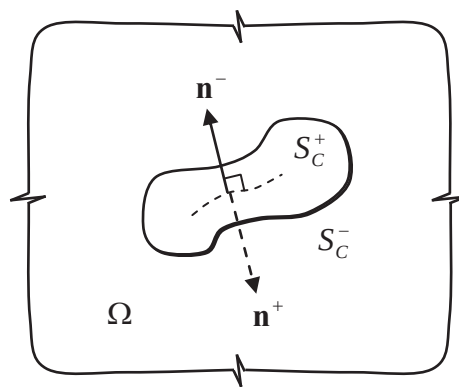


Figure 1: Piezoelectric infinite medium containing crack.

The electro-mechanical behavior of a piezoelectric medium that is free of distributed sources (i.e., zero body force and zero electric charge) is governed by following field equations

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0 \quad ; \quad \sigma_{ij} = E_{ijkl} \frac{\partial u_k}{\partial x_l} \quad (1)$$

where σ_{iJ} is the generalized stress (consisting of both the mechanical stress σ_{ij} and the electrical induction σ_{i4}), u_K is the generalized displacement (consisting of both the displacement u_k and the electric potential u_4), and E_{iJKL} is the generalized moduli (a collection of elastic moduli $E_{ijkl} = E_{jikl} = E_{ijlk} = E_{lkji}$, piezoelectric constants $E_{i4kl} = E_{lk4i}$, and dielectric permittivities $-E_{i44i} = -E_{l44i}$). The generalized traction at any point on the smooth surface with a unit normal vector $\mathbf{n}$ is given by $t_J = \sigma_{iJ}n_i$ (i.e., the mechanical traction $t_j = \sigma_{ij}n_i$ and the surface electric charge $t_4 = \sigma_{i4}n_i$). Here and in what follows, the lower case indices range from 1 to 3; the upper case indices range from 1 to 4; and repeated indices imply the summation over their range. Readers are referred to the work of Rungamornrat and Mear (2008) for more details about the convention and nomenclature. For a permeable crack, the jump in the crack-face displacement Δu_i and the surface electric charge t_4^+ are unknown a priori whereas the mechanical tractions t_i^+, t_i^- acting to both crack surfaces are fully prescribed and the jump in the crack-face electric potential Δu_4 and the sum of the crack-face electric charges, denoted by $\Sigma t_4 \equiv t_4^+ + t_4^-$, vanish.

3. GOVERNING BOUNDARY INTEGRAL EQUATIONS

A boundary value problem for an infinite cracked medium is formulated in terms of a pair of weakly singular, weak-form boundary integral equations, one for the sum of the crack-face generalized displacement and the other for the jump in the crack-face generalized traction. These weak-form integral equations were established by Rungamornrat and Mear (2008) and are given explicitly by

$$\begin{aligned} \frac{1}{2} \int_{s_c^+} \tilde{t}_K(\mathbf{y}) \Sigma u_K(\mathbf{y}) dS(\mathbf{y}) &= \int_{s_c^+} \tilde{t}_K(\mathbf{y}) \int_{s_c^+} U_J^K(\xi - \mathbf{y}) \Sigma t_J(\xi) dS(\xi) dS(\mathbf{y}) \\ &+ \int_{s_c^+} \tilde{t}_K(\mathbf{y}) \int_{s_c^+} G_{mJ}^K(\xi - \mathbf{y}) D_m \Delta u_J(\xi) dS(\xi) dS(\mathbf{y}) \\ &- \int_{s_c^+} \tilde{t}_K(\mathbf{y}) \int_{s_c^+} H_{iJ}^K(\xi - \mathbf{y}) n_i(\xi) \Delta u_J(\xi) dS(\xi) dS(\mathbf{y}) \end{aligned} \quad (2)$$

$$\begin{aligned} -\frac{1}{2} \int_{s_c^+} \tilde{v}_K(\mathbf{y}) \Delta t_K(\mathbf{y}) dS(\mathbf{y}) &= \int_{s_c^+} D_t \tilde{v}_K(\mathbf{y}) \int_{s_c^+} C_{mJ}^{tK}(\xi - \mathbf{y}) D_m \Delta u_J(\xi) dS(\xi) dS(\mathbf{y}) \\ &+ \int_{s_c^+} D_t \tilde{v}_K(\mathbf{y}) \int_{s_c^+} G_{tK}^J(\xi - \mathbf{y}) \Sigma t_J(\xi) dS(\xi) dS(\mathbf{y}) \\ &+ \int_{s_c^+} \tilde{v}_K(\mathbf{y}) \int_{s_c^+} H_{iK}^J(\xi - \mathbf{y}) n_i(\mathbf{y}) \Sigma t_J(\xi) dS(\xi) dS(\mathbf{y}) \end{aligned} \quad (3)$$

where $D_m = n_i \varepsilon_{ism} \partial / \partial \xi_s$ denotes the surface differential operator; $\tilde{t}_K$ and $\tilde{v}_K$ are sufficiently smooth test functions; ε_{ism} is a standard alternating symbol, Σu_J , Δu_J are the sum of and the jump in the generalized displacement across the crack surface; Σt_J , Δt_J are the sum of and the jump in the generalized traction across the crack surface. Furthermore, they possess several important properties; for instant, all involved kernels $\{U_J^P, G_{mJ}^P, H_{iJ}^P n_i, C_{mJ}^{tK}\}$ are only weakly singular of order $\mathcal{O}(1/r)$ where $r = \|\xi - \mathbf{y}\|$ and they are applicable to bodies made of generally anisotropic, piezoelectric

materials and containing cracks of arbitrary configuration and under general crack-face boundary conditions. All these positive features allow a weakly singular boundary integral equation method to be established to construct an approximate solution of the associated boundary value problem. Details for the development of above boundary integral equations can be found in Rungamornrat and Mear (2008).

4. SOLUTION METHODOLOGY

The weak-form boundary integral equations (2) and (3) are solved by the numerical technique based on the well-known symmetric Galerkin boundary element method and standard Galerkin finite element procedure. Details of such techniques (e.g., discretization, numerical integration of double surface integrals, evaluation of weakly singular kernels, and determination of generalized intensity factors and T-stresses) are briefly summarized as follows.

4.1. Discretization

Discretization of the weak-form boundary integral equations (2) and (3) via Galerkin approximation leads to the following two systems of linear algebraic equations:

$$[\mathbf{D}][\Sigma\mathbf{U}] = [\mathbf{U} \quad \mathbf{G} + \mathbf{H}] \begin{bmatrix} \Sigma\mathbf{T} \\ \Delta\mathbf{U} \end{bmatrix} \quad (4)$$

$$[\mathbf{C}][\Delta\mathbf{U}] = [\mathbf{G} + \mathbf{H} \quad \hat{\mathbf{D}}] \begin{bmatrix} \Sigma\mathbf{T} \\ \Delta\mathbf{T} \end{bmatrix} \quad (5)$$

where matrices $\mathbf{C}$, $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{U}$ correspond directly to the double surface integrals involving the kernels C_{mJ}^{tK} , G_{mJ}^P , $H_{ij}^P n_i$ and U_J^P , respectively; the matrices $\mathbf{D}$ and $\hat{\mathbf{D}}$ correspond to the single surface integrals; $\Sigma\mathbf{U}$ and $\Delta\mathbf{U}$ are vectors of nodal quantities associated with the sum of and the jump in the generalized displacement, respectively; and $\Sigma\mathbf{T}$ and $\Delta\mathbf{T}$ are vectors of nodal quantities of the sum of and the jump in the generalized traction across the crack surface, respectively. In the discretization, both the test and trial functions are approximated by standard, element-based, C^0 interpolation functions except in a local region near the crack boundary where the special C^0 interpolation functions, generated by special crack-tip elements (see Rungamornrat and Mear, 2008), are utilized to capture the right asymptotic behavior of the jump in the generalized displacement in the neighborhood of the crack front. These special crack-tip elements have two desirable features including that (i) their shape functions are properly enriched by a square-root function allowing the near-front field to be captured accurately and (ii) extra degrees of freedoms placed on nodes along the crack front are directly related to the generalized intensity factors to be determined (also see Rungamornrat and Mear, 2008; Phongtinnaboot et al., 2011).

4.2. Numerical Integration

The matrices $\mathbf{D}$ and $\hat{\mathbf{D}}$ involving only single surface integrals can be computed efficiently by standard Gaussian quadrature. On the contrary, numerical evaluation of entries of the matrices $\{\mathbf{C}$,

$\mathbf{G}, \mathbf{H}, \mathbf{U}$ requires a special care and, generally, consumes significantly more computation cost due to the double surface integrals involved. Such double surface integrals over a pair of elements resulting from the discretization can be regular, nearly singular, or weakly singular depending on the distance between the two elements in that pair. The regular double surface integrals over a pair of remote elements contain a well behaved integrand and can be integrated efficiently by standard Gaussian quadrature. For double surface integrals over a pair of identical elements, the integrand is clearly weakly singular and Gaussian quadrature alone is inefficient for evaluating this type of integrals. In the present study, the integrand regularization via a family of variable transformations is employed first to remove the singularity (e.g., Xiao, 1997) and the resulting integrals are then integrated by standard Gaussian quadrature. For the last type of double surface integrals over a pair of relatively close elements, the integrand experiences very rapid variation in the region where the distance between the source and field points is nearly zero. To efficiently evaluate these integrals by Gaussian quadrature, the same type of variable transformations as those employed for the weakly singular case is applied to regularize the integrand (e.g., Xiao, 1997).

4.3. Efficient evaluation of kernels

While the numerical evaluation of a weakly singular kernel $H_{ij}^P n_i$ can be obtained using a standard procedure since H_{ij}^P is given in terms of element functions and the unit normal vector n_i can be readily obtained from the element data, the evaluation of the weakly singular kernels $\{U_J^P, G_{mJ}^P, C_{mJ}^{tK}\}$ must be properly treated since they are defined in terms of a closed contour integral for any pair of source and field points arising from the numerical quadrature. To overcome this situation, the interpolation technique, proposed by Rungamornrat and Mear (2008), is adopted to approximate values of those weakly singular kernels. Accuracy of this approximation can readily be enhanced by either increasing the number of nodal points or the degree of the polynomial basis functions.

4.4. Solver for permeable cracks

To obtain the unknown crack-face data for permeable cracks, the two systems of linear algebraic equations (4) and (5) are solved by using the following procedure. The nodal vectors $\Delta \mathbf{T}$ and $\Delta \mathbf{U}$ are first partitioned into $[\Delta \mathbf{T}_m \ \Delta \mathbf{T}_e]$ and $[\Delta \mathbf{U}_m \ \Delta \mathbf{U}_e]$ where $\Delta \mathbf{T}_m, \Delta \mathbf{U}_m$ are vectors containing mechanical nodal quantities and $\Delta \mathbf{T}_e, \Delta \mathbf{U}_e$ are vectors containing electrical nodal quantities. For the permeable crack, the sum of the generalized traction ($\Sigma \mathbf{T}$), the jump in the mechanical traction ($\Delta \mathbf{T}_m$) and the jump in the electric potential across the crack ($\Delta \mathbf{U}_e$) are known a priori whereas the sum of the generalized displacement ($\Sigma \mathbf{U}$), the jump in the displacement ($\Delta \mathbf{U}_m$) and the jump in the surface electric charge ($\Delta \mathbf{T}_e$) are unknown. As a result, the system (5) is solved first to obtain the unknown vectors $\Delta \mathbf{U}_m$ and $\Delta \mathbf{T}_e$ by using a partitioning scheme together with a preconditioning conjugate gradient method. Once $\Delta \mathbf{T}$ and $\Delta \mathbf{U}$ are fully determined, the system (4) can now be solved to obtain the unknown vector $\Sigma \mathbf{U}$ by using a preconditioning conjugate gradient method.

4.5. Calculation of generalized intensity factors and T-stresses

Once the unknown crack-face data are obtained, the generalized intensity factors and generalized T-stress components can be post-processed using the information of the jump in and the sum of the generalized displacement, respectively. The explicit formula to compute those fracture data can be found in the work of Rungamornrat and Mear (2008) and Subsathaphol (2013), respectively.

5. NUMERICAL RESULTS

Results of two selected examples including a penny-shaped crack and a spherical cap crack are presented in this paper. The first example is chosen to verify the formulation and numerical implementation of the weakly singular boundary integral equation method by comparing computed results with available benchmark solutions. The second problem, which is more complex, is treated to demonstrate the capability of the current technique to model non-planar crack.

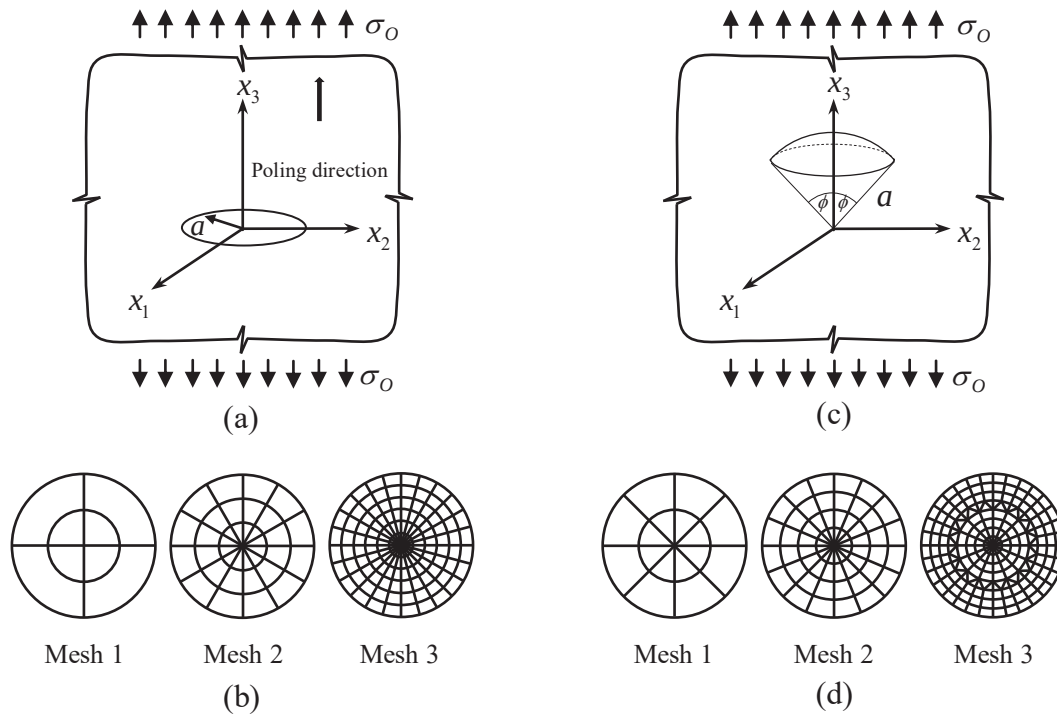


Figure 2: (a) Penny-shaped crack in piezoelectric infinite medium, (b) three meshes of penny-shaped crack used in analysis, (c) spherical cap crack in piezoelectric infinite medium, and (d) three meshes of spherical cap crack used in analysis.

Consider a penny-shaped crack of a radius a contained in a transversely isotropic piezoelectric infinite medium with the axis of material symmetry and poling direction coincident with the x_3 -axis as shown in Figure 2(a). The medium is loaded by a remote uniform tensile stress $\sigma_{33}^\infty = \sigma_o = 1 \times 10^6 \text{ N/m}^2$. In a numerical study, material properties of PZT-5H are employed (Rungamornrat and Mear, 2008) and three meshes containing 8, 36 and 144 elements are adopted as shown in Figure 2(b). Numerical results normalized by the analytical solutions (Li and Lee, 2004a; Pich et al., 2016) are reported in Table 1. It is evident from this set of results that numerical

solutions are in excellent agreement with the analytical solutions and exhibit only weak dependence on the level of discretization. In particular, use of special crack-tip elements along the crack front allows sufficiently coarse meshes to be used while still yielding results of good accuracy.

Table 1: Normalized generalized intensity factors $\{K_I, K_{IV}\}$ and T-stress components $\{T_{11}, T_{33}\}$ of penny-shaped crack in piezoelectric infinite medium for permeable crack

Mesh	K_I/K_I^{exact}	K_{IV}/K_{IV}^{exact}	T_{11}/T_{11}^{exact}	T_{33}/T_{33}^{exact}
1	0.9945	0.9945	0.9985	0.9989
2	1.0005	1.0005	0.9982	0.9992
3	1.0006	1.0006	0.9968	0.9986

Next, consider a spherical cap crack embedded in a transversely isotropic piezoelectric infinite medium (PZT-5H) as shown in Figure 2(c). The geometry of the crack surface is parameterized by

$$x_1 = a \sin \gamma \cos \beta, \quad x_2 = a \sin \gamma \sin \beta, \quad x_3 = a \cos \gamma \quad (6)$$

where a denotes a radius of the spherical cap crack, $\beta \in [0, 2\pi]$, and $\gamma \in [0, \phi]$ with ϕ denoting the half subtended angle of the surface. The axis of material symmetry and the poling direction are chosen to direct along the x_3 -axis and the cracked medium is subjected to a uniform uniaxial remote stress $\sigma_{33}^\infty = \sigma_o = 1 \times 10^6 \text{ N/m}^2$. In the analysis, three meshes of the crack surface shown in Figure 2(d) are utilized and the half subtended angle $\phi = 30^\circ$ is chosen. It should be noted that the coarsest mesh only contains 16 elements whereas the intermediate and finest meshes contain 64 and 208 elements, respectively. Numerical results for nonzero generalized intensity factors and T-stress components, normalized by results generated from the Mesh-3, are reported in Table 2. It is found that the generalized intensity factors and T-stress components obtained from the Mesh-1 and Mesh-2 are in very good agreement with those obtained from the finest mesh. This weak mesh-dependent behavior should result directly from the use of special crack-tip element to model the jump in the generalized displacement along the crack-front.

Table 2: Normalized generalized intensity factors $\{K_I, K_{II}, K_{IV}\}$ and T-stress components $\{T_{11}, T_{33}, T_{14}\}$ of spherical cap crack in piezoelectric infinite medium for permeable crack

Mesh	K_I/K_I^{ref}	K_{II}/K_{II}^{ref}	K_{IV}/K_{IV}^{ref}	T_{11}/T_{11}^{ref}	T_{33}/T_{33}^{ref}	T_{14}/T_{14}^{ref}
1	0.9993	0.9981	0.9990	0.9966	0.9989	0.9963
2	0.9998	0.9999	0.9999	0.9994	0.9998	0.9996
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

6. CONCLUSIONS

A weakly singular, boundary integral equation method has been developed for calculating both the generalized intensity factors and T-stress components of arbitrary-shaped permeable cracks in three-dimensional, homogeneous, generally anisotropic, linear piezoelectric, infinite media. The final governing equation has been formulated in terms of a pair of boundary integral equations, one

for the sum of the crack-face generalized displacement and the other for jump in the crack-face generalized traction. Due to the weakly singular nature of boundary integral equations, continuous interpolation functions have been employed in the discretization. Special crack-tip elements have been adopted to enhance the approximation of a near-front relative crack-face displacement. Once the unknown crack-face data are solved, the generalized intensity factors and the generalized T-stress components have been post-processed using the information of the jump in and the sum of the generalized displacement in a local region adjacent to the crack front. Results from extensive numerical experiments indicate that the proposed technique is computationally robust and promising. In particular, it yields highly accurate numerical solutions for the generalized intensity factors and the generalized T-stresses along the crack front with only weak dependence on the level of mesh refinement.

7. ACKNOWLEDGMENTS

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International Conference Article No. 2

Title: Efficient BIEM for analysis of SIFs and T-stresses for cracks in 3D linear elastic whole space

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Efficient BIEM for Analysis of SIFs and T-stresses for Cracks in 3D Linear Elastic Whole Space

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ABSTRACT

This paper presents a regularized boundary integral equation method for the analysis of both stress intensity factors and T-stresses for cracks in a three-dimensional, linear elastic, isotropic whole space. The governing equations are formulated using a pair of weak-form integral equations for the displacement and the traction. The key feature of such integral equations, in addition to the weak singularity of all involved singular integrals, is that they contain a complete set of unknown crack-face data allowing a simple and direct calculation of the essential fracture information along the edge of the crack in the post-process. Numerical procedures, based upon a standard Galerkin finite element technique and a weakly singular symmetric Galerkin boundary element method, are implemented to solve a system of integral equations for the unknown crack-face displacements. Special interpolation functions are also employed to further enhance the accuracy of the approximate near-front relative crack-face displacements. Two explicit formulae are proposed to extract the stress intensity factors and the T-stresses along the crack front directly from the solved unknown crack-face data. A selected set of results is then reported to demonstrate the accuracy, the convergence and the computational robustness of the proposed technique.

INTRODUCTION

In the past three decades, two-parameter-based fracture models involving the information from both singular and nonsingular terms (known as the stress intensity factors (SIFs) and T-stress, respectively) in the asymptotic near-front expansion of the stress field have been increasingly utilized by various researchers in the fracture modeling and simulations (e.g., Tvergaard, 2008; Cheng et al., 2012). This is due to the fact that the integration of the T-stress information can further enhance the modeling capability in comparison with those using only the SIFs (e.g., Ayatollahi et al., 2002; Sedinghiani et al., 2011; Cheng et al., 2012). To facilitate the use of such two-parameter-based fracture models in the modeling, efficient and powerful solution procedures have been developed for determining the essential data associated with the singular and nonsingular terms. One of numerical techniques commonly employed in the analysis of fracture problems is the method based upon the boundary integral equations (BIEMs). The positive feature of this particular class of techniques results directly from the reduction of a spatial dimension of the key governing equations and that the primary unknowns involve only the boundary and crack-face data. Applications of BIEMs to perform the stress analysis of elastic cracked bodies with the primary objective to extract the stress intensity factors along the crack front have been well recognized in the literature (e.g., Xu and Ortiz, 1993; Li et al., 1998; Frangi et al., 2002; Rungamornrat and Mear, 2008b) whereas those capable of computing the T-stress are relatively few and most of them are restricted to two-dimensional problems or simple crack configurations (e.g., Sladek et al., 1997; Sutradhar and Paulino, 2004; Shah et al., 2005; Yu et al., 2006; Tran, 2010). Note, in addition, that the efficiency of the post-process algorithms for extracting the T-stress data from the primary unknowns solved from the key governing boundary integral equations still requires further enhancement. Several techniques employed in most of previous studies involve the evaluation of both path independent integrals and fields within the domain (e.g., e.g., Sladek et al., 1997; Sutradhar and Paulino, 2004; Shah et al., 2005; Yu et al., 2006). Recently, a technique based upon a pair of weakly singular, weak-form displacement and traction integral equations for cracks to solve for the crack-face displacement data and using such information to directly post-process for both SIFs and T-stress was successfully implemented by Tran (2010). However, such work is still limited only to cracks in two-dimensional, linear elastic media. In the present study, the work of Tran (2010) is further extended, within the context of linear elasticity, to have the capability to treat cracks in three-dimensional media.

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PROBLEM FORMULATION

Consider a three-dimensional body Ω occupying a whole Euclidean space $\mathbb{R}^3$ (i.e., $\Omega = \mathbb{R}^3$) and containing isolated cracks as shown schematically in Fig. 1. Each crack, in the reference or undeformed state, are represented by two geometrically identical surfaces S^+ and S^- with opposite outward unit normal vectors, denoted by $\mathbf{n}^+$ and $\mathbf{n}^-$, respectively. The body Ω is made of a homogeneous isotropic linearly elastic material, free of body force, and subjected to a uniform remote stress field $\hat{\sigma}^\infty$ and crack-face tractions. In the present study, only cracks that can be represented by piecewise smooth surface are considered.

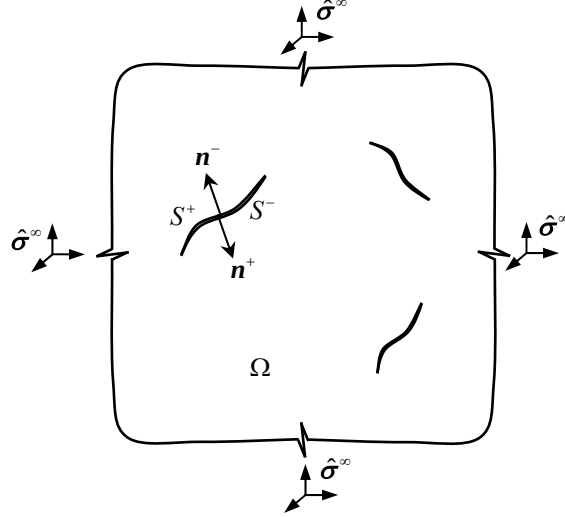


Fig.1. Schematic of isolated cracks in linear whole space under crack-face and remote loading

The displacement and stress at any point $\mathbf{x}$ within the body can be related to the crack-face data and the uniform remote stress field by

$$u_p(\mathbf{x}) = u_p^\infty(\mathbf{x}) + \int_{S^+} U_j^p(\mathbf{y} - \mathbf{x}) \Sigma t_j(\mathbf{y}) dS(\mathbf{y}) - \int_{S^+} S_{mk}^p(\mathbf{y} - \mathbf{x}) n_m^+(\mathbf{y}) \Delta u_k(\mathbf{y}) dS(\mathbf{y}) \quad (1)$$

$$\sigma_{ij}(\mathbf{x}) = \sigma_{ij}^\infty(\mathbf{x}) - \int_{S^+} S_{ij}^p(\mathbf{y} - \mathbf{x}) \Sigma t_p(\mathbf{y}) dS(\mathbf{y}) + \int_{S^+} \Sigma_{ij}^{lk}(\mathbf{y} - \mathbf{x}) n_l^+(\mathbf{y}) \Delta u_k(\mathbf{y}) dS(\mathbf{y}) \quad (2)$$

where u_p^∞ is the displacement due to the remote stress field; U_j^p and S_{ij}^p are displacement and stress fundamental solutions; $\Sigma_{ij}^{lk} = E_{ijpm} \partial S_{lk}^p(\mathbf{y} - \mathbf{x}) / \partial y_m$ with E_{ijpm} denoting the elasticity tensor; Σt_j is the sum of crack-face traction; and Δu_k is the jump in the crack-face displacement. The pair of integral relations can be used to compute the displacement and stress within the body once the crack-face data are fully known. To form a set of equations sufficient for solving the unknown crack-face data, the two relations (1) and (2) are used as the basis along with the regularization procedure proposed by Rungamornrat and Mear (2008a) and the standard weighted residual technique. Following such procedure, it finally yields a pair of weakly singular weak-form integral equations for both the crack-face displacement and traction:

$$\int_{S^+} \tilde{t}_p(\mathbf{z}) \left\{ \frac{1}{2} \Sigma u_p(\mathbf{z}) - u_p^\infty(\mathbf{z}) \right\} dS(\mathbf{z}) = \int_{S^+} \tilde{t}_p(\mathbf{z}) \int_{S^+} U_j^p(\mathbf{y} - \mathbf{z}) \Sigma t_j^0(\mathbf{y}) dS(\mathbf{y}) dS(\mathbf{z}) - \int_{S^+} \tilde{t}_p(\mathbf{z}) \int_{S^+} n_i^+(\mathbf{y}) H_{ij}^p(\mathbf{y} - \mathbf{z}) \Delta u_j(\mathbf{y}) dS(\mathbf{y}) dS(\mathbf{z}) \\ + \int_{S^+} \tilde{t}_p(\mathbf{z}) \int_{S^+} G_{mj}^p(\mathbf{y} - \mathbf{z}) D_m \Delta u_j(\mathbf{y}) dS(\mathbf{y}) dS(\mathbf{z}) \quad (3)$$

$$\int_{S^+} \tilde{v}_k(\mathbf{z}) \left\{ \sigma_{lk}^\infty n_l^+(\mathbf{z}) - \frac{1}{2} \Delta t_k(\mathbf{z}) \right\} dS(\mathbf{z}) = \int_{S^+} D_i \tilde{v}_k(\mathbf{z}) \int_{S^+} G_{ij}^k(\mathbf{y} - \mathbf{z}) \Sigma t_j(\mathbf{y}) dS(\mathbf{y}) dS(\mathbf{z}) + \int_{S^+} D_i \tilde{v}_k(\mathbf{z}) \int_{S^+} C_{mj}^{lk}(\mathbf{y} - \mathbf{z}) D_m \Delta u_j(\mathbf{y}) dS(\mathbf{y}) dS(\mathbf{z}) \\ + \int_{S^+} \tilde{v}_k(\mathbf{z}) \int_{S^+} H_{lk}^j(\mathbf{y} - \mathbf{z}) n_l^+(\mathbf{z}) \Sigma t_j(\mathbf{y}) dS(\mathbf{y}) dS(\mathbf{z}) \quad (4)$$

where $D_m = n_i \varepsilon_{ism} \partial / \partial \xi_s$ is the surface differential operator; Σu_p and Δt_k are sum of the crack-face displacement and jump in the crack-face traction, respectively; $\tilde{t}_p$ and $\tilde{v}_k$ are sufficiently smooth test functions; and all involved kernels are defined by

$$H_{lk}^j(\mathbf{y}-\mathbf{z}) = -\frac{y_l - z_l}{4\pi r^3} \delta_{kj} \quad (5)$$

$$U_j^p(\mathbf{y}-\mathbf{z}) = \frac{1}{16\pi(1-\nu)\mu r} \left[(3-4\nu)\delta_{pj} + \frac{(y_p - z_p)(y_j - z_j)}{r^2} \right] \quad (6)$$

$$G_{ij}^k(\boldsymbol{\xi}-\mathbf{y}) = \frac{1}{8\pi(1-\nu)r} \left[(1-2\nu)\varepsilon_{tkj} + \varepsilon_{ajt} \frac{(y_a - z_a)(y_p - z_p)}{r^2} \right] \quad (7)$$

$$C_{mj}^{tk}(\boldsymbol{\xi}-\mathbf{y}) = \frac{1}{8\pi(1-\nu)r} \left[(1-\nu)\delta_{ik}\delta_{mj} + 2\nu\delta_{km}\delta_{ij} - \delta_{kj}\delta_{im} - \frac{(y_j - z_j)(y_k - z_k)}{r^2} \delta_{im} \right] \quad (8)$$

where μ is the elastic shear modulus; ν is Poisson's ratio; $r = \|\mathbf{y}-\mathbf{z}\|$; and δ_{pj} and ε_{ijk} are standard Kronecker-delta and alternating symbols, respectively. It is evident that all involved kernels in (3) and (4) are weakly singular of order $1/r$.

SOLUTION PROCEDURE AND POST-PROCESS

A weakly singular SGBEM (e.g., Rungamornrat and Mear, 2008b) is adopted to solve the weakly singular, weak-form traction integral equation (4) for the jump in the crack-face displacement. In the discretization, special interpolation functions constructed by special crack-tip elements (e.g., Li *et al.*, 1998) are utilized to enhance the approximation of the near-front relative crack-face displacement. The weak-form displacement integral equation (3) along with the obtained crack-face data Δu_k is then solved by standard Galerkin technique for the sum of the crack-face displacement. In the implementations, special quadrature rules are employed to handle both weakly singular and nearly singular double surface integrals resulting from the discretization and a selected efficient linear solver is employed to solve a dense system of linear algebraic equations. After all unknown crack-face data are solved, two explicit formulae are employed to calculate the stress intensity factors and T-stress. Specifically, the formula proposed by (Rungamornrat and Mear, 2008b) is adopted to post-process the jump in the crack-face displacement for the stress intensity factor whereas the formula analogous to that presented by Tran (2010) for two-dimensional crack problems is extended to post-process the sum of the crack-face displacement for the T-stress along the crack-front.

PRELIMINARY RESULTS

A set of preliminary results associated with a crack problem possessing an analytical solution is presented, here, only to demonstrate the accuracy and convergence of the proposed numerical technique. The full aspect of the computational performance of the developed technique is still under investigation. Consider a whole space containing a penny-shaped crack of radius a , made of an isotropic material with Poisson's ratio $\nu = 0.3$, and subjected to a uniform axial tension σ_0 in the direction normal to the crack surface as shown in Fig. 2(a). In the analysis, three meshes of the crack surface shown in Fig. 2(b) are employed. Results for the mode-I stress intensity factor (K_I) and the non-zero T-stresses (T_{11}, T_{33}) are normalized by the exact solutions (e.g., Li *et al.*, 1998; Rungamornrat and Pinitpanich, 2016) and then reported in Table 1 for $\theta = 0^\circ, 45^\circ, 90^\circ$ where θ is the angle measured from the x_1 -axis. It is seen from this set of results that the numerical solutions for both the stress intensity factor and the T-stresses show the good convergence and are weakly dependent on the level of refinement. In addition, the computed solutions are in excellent agreement with the benchmark results even the coarse mesh with only eight elements are used in the discretization. This should result directly from the use of special interpolation functions in the approximation of the near-front relative crack-face displacement.

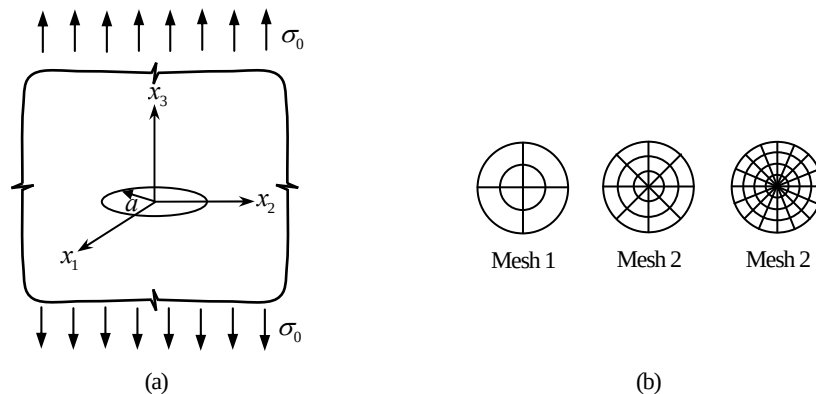


Fig.2. Schematic of (a) penny-shaped crack in elastic whole space under remote stress σ_0 and (b) meshes used in the analysis.

Table .1 Normalized stress intensity factor and normalized T-stresses of penny shaped crack in isotropic elastic whole space under uniform remote tension.

Mesh	K_I / K_I^{exc}			T_{11} / T_{11}^{exc}			T_{33} / T_{33}^{exc}		
	$\theta = 0^\circ$	$\theta = 45^\circ$	$\theta = 90^\circ$	$\theta = 0^\circ$	$\theta = 45^\circ$	$\theta = 90^\circ$	$\theta = 0^\circ$	$\theta = 45^\circ$	$\theta = 90^\circ$
1	0.9922	0.9968	0.9922	1.0003	0.9958	1.0003	0.9966	1.0010	0.9966
2	1.0008	1.0008	1.0008	0.9961	0.9961	0.9961	0.9986	0.9986	0.9986
3	1.0008	1.0008	1.0008	0.9966	0.9966	0.9966	0.9991	0.9991	0.9991

CONCLUSIONS

An efficient and accurate weakly singular boundary integral equation method is successfully established for analysis of isolated cracks in an isotropic, linear elastic whole space. An explicit post-process algorithm is also implemented to directly extract the stress intensity factors and T-stress along the crack front in terms of the crack-face displacement data. Results from the preliminary numerical study have indicated that the proposed technique is computationally promising and robust. The extensive exploration of its computational performance is still required and currently under investigation.

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International Conference Article No. 3

Title: Influence of surface and couple stresses on response of surface-loaded elastic half-plane

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Influence of Surface and Couple Stresses on Response of Surface-loaded Elastic Half-plane

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Abstract

A mathematical model integrating both Gurtin-Murdoch surface elasticity and consistent couple stress theories is proposed to simulate the simultaneous effects of the surface energy and the material microscopic structure on the mechanical response of an elastic half-plane under arbitrarily surface normal loadings. The displacement-based governing equations for the bulk material and the top material layer are established and then solved, via the method of Fourier integral transform together with the prescribed boundary conditions, to obtain the closed form solution of the elastic field in the transform space. To obtain solutions in the physical space, an efficient quadrature is adopted to evaluate all involved integrals associated with the Fourier transform inversion. A selected set of results is reported and they have indicated that both the surface and couple stresses significantly influences the elastic field within the bulk when the size of the loading region is comparable to the internal length scales of the surface material and the bulk.

Keywords: Elastic half-plane, Surface stresses, Couple stresses, Surface elasticity, Couple stress elasticity.

Introduction

In past several decades, micro- and nano-technologies have received increasingly growing attention due to their vast applications in various disciplines. In the field of material sciences and engineering, understanding the fundamental characteristics and mechanical behavior of materials at those tiny scales is considered essential in the design procedure and fabrication of micro- and nano-scale devices and systems such as MEMS and NEMS. Unfortunately, many difficulties and challenges arise in the study of small-size objects since the response at those scales is significantly complex, generally size-dependent, and mostly influenced by various actions such as the surface free energy [1-3], existing defects and flaws, and material microscopic structures [4]. Experimental-based approaches are ones of the most popular candidates extensively and successfully employed to investigate the physical phenomena in a tiny scale (e.g., [5-8]). While results gained from those approaches have been found reliable and closely resembling the actual response, tests can only be performed in fully equipped laboratories and significant amount of resources associated with sophisticated testing setups and procedures and preparations of specimens must also be paid. This therefore renders purely experimental-based approaches less cost efficient in comparison with those combined with theoretical-based simulations. The latter, once properly equipped with physically admissible and sufficiently validated mathematical models, can be used not only to obtain the first-order prediction of the actual phenomena but also to assist the reduction of the number of cases to be considered in the experiments.

It has been well recognized that classical size-independent theories in continuum mechanics adopted specifically for simulating mechanical response of macro-scale problems have failed to simulate situations when the external length scale (e.g., size of loading region, crack length, contact length, etc.) is comparable to the internal length scale of materials (e.g., granular distance, lattice parameters). Attempts have been devoted to modify/enhance existing continuum-based mathematical models by integrating the influences observed in a small-scale before used in the simulations. Several continuum-based models have been proposed and successfully employed to capture the size-dependent behavior due to the presence of both surface-free energy and microstructures of constituting materials. For instance, the model established by Gurtin and his colleagues (e.g., [9-11]), called the theory of surface elasticity, has been successfully utilized to capture the surface stress effects. Due to its mathematical simplicity and capability in handling small-scale influence, the Gurtin-Murdoch model has become popular and extensively applied to investigate various problems in mechanics, e.g., thin films [12-13], thin plates [14], dislocations [15], nano-scaled elastic layer [16-18], half-space [19-20], and layered elastic half-space [21]. To handle the influence of material microstructures, various theories have been considered including the Cosserat theory [22], the couple stress theory [23-25], the strain gradient elasticity theory [26-28], the modified couple stress theory [29], and the consistent couple stress theory [30]. During the past decades, these theories have been extensively employed in the simulations and modeling of nano/micro-structured systems, especially for small-scaled beams and plates [31-33] and the size-dependent contact problems of elastic solids [34-36].

Towards the modeling of micro-/nano-scale layered media, results from an extensive literature survey have indicated that most of existing studies considered separately either the effect of the surface-free energy or the influence of the microscopic structures of constituting materials (e.g., [13, 14-21, 34-39]). Applications of both Gurtin-Murdoch surface elasticity and the couple stress theory to simultaneously handling those small-scale influences, especially within the framework of surface and contact mechanics, have not been well recognized and this, as a consequence, leaves a significant gap of knowledge for further investigations. The idea of integrating both the surface-free energy and the micro/nano-structure of the bulk material in the modeling has been found in the study of size-dependent responses of nano-scale structures such as nano-beams (e.g., [40]) and nano-plates (e.g., [41-43]). This set of investigations not only confirms the applicability of the two theories but also provides the useful basis and ingredients essential for the extension to treat nano-scale problems of interest.

The present study aims to investigate the mechanical response of an elastic half-plane loaded on its surface by taking the influence of both surface and couple stresses into account. Gurtin-Murdoch surface elasticity theory is employed together with the consistent couple stress theory to form the underlying mathematical model and the analytical solution of elastic fields is obtained via the method of Fourier integral transform and a selected efficient numerical quadrature. The complete elastic fields under the simultaneous effects of surface stresses and couple stresses within the half-plane are thoroughly studied.

Problem Formulation

Consider a linearly elastic half-plane loaded by an arbitrarily distributed normal traction p over the length $2a$ on the top surface as shown schematically in Fig. 1. For convenience, a reference Cartesian coordinate system $\{x, y, z; O\}$ is chosen such that the origin is at the center of the loading region and the x , y , and z -axes direct rightward, downward, and normal to the half-plane, respectively. The bulk material is assumed homogeneous and its response is described by the consistent couple stress theory (e.g., [30]) whereas the material

layer at the top surface of the half-plane is governed by the theory of surface elasticity (e.g., [9-11]). In the present study, it is assumed that the body force and the body couple are negligible and the plane-strain deformation prevails.

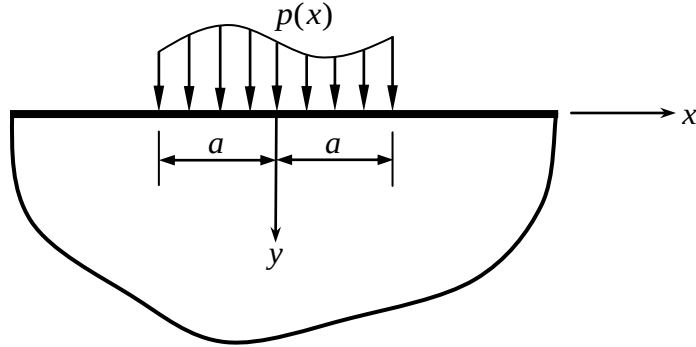


Figure 1. Schematic of an elastic half-plane loaded on its surface by normal traction

For the bulk medium, basic field equations (i.e., equilibrium equations, constitutive laws, and kinematics) from the consistent couple stress theory (e.g., [30]) when specialized to the two-dimensional body under the plane strain condition are given by

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \quad \frac{\partial \mu_{xz}}{\partial x} + \frac{\partial \mu_{yz}}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0 \quad (1)$$

$$\sigma_{xx} = \frac{2\mu}{1-2\nu} [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}], \quad \sigma_{yy} = \frac{2\mu}{1-2\nu} [\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy}]$$

$$\sigma_{xy} = 2\mu\varepsilon_{xy} - 2\mu\ell^2\Delta\omega, \quad \sigma_{yx} = 2\mu\varepsilon_{xy} + 2\mu\ell^2\Delta\omega \quad (2)$$

$$\mu_{xz} = -8\mu\ell^2\kappa_{xz}, \quad \mu_{yz} = -8\mu\ell^2\kappa_{yz}$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \quad (3)$$

$$\omega = \frac{1}{2} \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right), \quad \kappa_{xz} = -\frac{1}{2} \frac{\partial \omega}{\partial x}, \quad \kappa_{yz} = -\frac{1}{2} \frac{\partial \omega}{\partial y}$$

where $\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{yx}\}$ are the force-stress components; $\{\mu_{xz}, \mu_{yz}\}$ are the couple-stress components; $\{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}\}$ and ω are the infinitesimal strain components and the rotation about the z -axis, respectively; $\{\kappa_{xz}, \kappa_{yz}\}$ represent the curvature components; $\{u_x, u_y\}$ are the in-plane displacement components; μ and ν are the elastic shear modulus and Poisson's ratio of the bulk material, respectively; ℓ is a length-scale parameter in the couple stress elasticity; and Δ denotes two-dimensional Laplacian operator. It is worth noting that the presence of the couple stresses renders the force-stress tensor non-symmetric and this is in contrast to the classical linear elasticity. Note also that by setting the parameter $\ell = 0$, one can readily recover the classical case.

For the material layer at the top surface of the half-plane, basic equations governing its response can be established from Gurtin-Murdoch surface elasticity theory [9-11] and, when specialized to this particular case, they are given by

$$\frac{\partial \sigma_{xx}^s}{\partial x} + t_x^s = 0, \quad \frac{\partial \sigma_{xy}^s}{\partial x} + t_y^s + p = 0 \quad (4)$$

$$\sigma_{xx}^s = \tau^s + \kappa^s \varepsilon_{xx}^s, \quad \sigma_{xy}^s = \tau^s \frac{\partial u_y^s}{\partial x} \quad (5)$$

$$\varepsilon_{xx}^s = \frac{\partial u_x^s}{\partial x} \quad (6)$$

where the superscript ‘s’ is utilized to designate the surface quantities; $\{\mu^s, \lambda^s\}$ and τ^s denote surface Lamé constants and the residual surface tension, respectively; $\kappa^s = 2\mu^s + \lambda^s$ denote the surface elastic modulus; and $\{t_x^s, t_y^s\}$ are tractions acting to the bottom side of the material layer induced by the underlying bulk material.

Solution Procedure

A set of displacement-based governing equations for the bulk material can be readily obtained by properly combining all basic field equations (1)-(3) and its general solution can be established in a closed-form via the method of Fourier integral transform (e.g., [44]). The final expression for the in-plane displacements $\{u_x, u_y\}$ is given by

$$u_x(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[Ai|\xi| e^{-|\xi|y} - Bi(\kappa - |\xi|y) e^{-|\xi|y} + Ci\zeta e^{\frac{-\zeta}{\ell}y} \right] e^{-i\xi x} d\xi \quad (7)$$

$$u_y(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[A\xi e^{-|\xi|y} + B\xi y e^{-|\xi|y} + C\ell \xi e^{\frac{-\zeta}{\ell}y} \right] e^{-i\xi x} d\xi \quad (8)$$

where $i = \sqrt{-1}$ denotes an imaginary number; ξ is a transform parameter; $\zeta = \sqrt{1 + \xi^2 \ell^2}$; $\kappa = 3 - 4\nu$; and A, B, C are arbitrary unknown functions of ξ to be determined from boundary conditions. The general solution for the rotation ω , the force stresses $\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{yx}\}$, and the couple stresses $\{\mu_{xz}, \mu_{yz}\}$ can be obtained from (7)-(8) together with (2)-(3).

To form sufficient conditions for determining the unknown functions A, B, C , the continuity of the displacements and tractions at the interface of the bulk and the material layer is enforced along with the surface equations (4)-(6). For the case of a constant residual surface tension, the following set of three boundary conditions is obtained:

$$\tau^s \frac{\partial^2 u_y}{\partial x^2} \Big|_{y=0} + \sigma_{yy} \Big|_{y=0} + p(x) = 0 \quad (9)$$

$$\kappa^s \frac{\partial^2 u_x}{\partial x^2} \Big|_{y=0} + \sigma_{yx} \Big|_{y=0} = 0 \quad (10)$$

$$\mu_{yz} \Big|_{y=0} = 0 \quad (11)$$

It is worth noting that the couple traction boundary condition (11) results directly from that the material layer cannot resist the bending moment. By substituting (7)-(8) and the general

solution for the force and couple stresses into (9)-(11), it gives rise to a system of three linear algebraic equations:

$$\begin{bmatrix} \frac{\tau^s \xi^3}{\mu} + 2\xi|\xi| & 2\xi(2\nu-1) & \frac{\tau^s \ell \xi^3}{\mu} + 2\xi\xi\zeta \\ i\xi^2(\Lambda|\xi|+2) & -i[\Lambda\xi^2(3-4\nu)+4|\xi|(1-\nu)] & i\xi^2(\Lambda\xi+2\ell) \\ 0 & 8i\ell^2\xi^2(\nu-1) & 2i\xi\zeta \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \frac{1}{\mu} \begin{Bmatrix} \hat{p}(\xi) \\ 0 \\ 0 \end{Bmatrix} \quad (12)$$

where $\Lambda = \kappa^s / \mu$ is defined as the material length-scale parameter corresponding to the presence of the surface stresses and $\hat{p}(\xi)$ is the Fourier transform of the prescribed normal traction $p(x)$ given by

$$\hat{p}(\xi) = \int_{-\infty}^{+\infty} p(x) e^{i\xi x} dx \quad (13)$$

By solving the system of linear equations (12), the unknown functions A, B, C are obtained in a closed-form as functions of ξ , two length-scale parameters Λ and ℓ , and transformed traction $\hat{p}(\xi)$. Substituting these functions into the general solutions gives the integral expressions for the displacements, rotation, force-stress and couple-stress fields at any location within the half-plane. Interestingly, it can be observed that all elastic field quantities contain both Λ and ℓ representing the length-scale parameters corresponding to the presence of the surface stresses and couple stresses, respectively. Therefore, it is anticipated that the influence of simultaneous effects of the surface and couple stresses can be captured in predicted solutions as well as the size-dependent behavior.

Numerical Results and Discussion

To obtain numerical results for elastic field within the bulk material, standard Gaussian quadrature is adopted to efficiently evaluate all involved integrals resulting from Fourier integral inversion. In the numerical study, following material properties associated with Silicon [45] (i.e., $\nu = 0.33$, $\mu = 40.22$ GPa, $\lambda = 78.08$ GPa, $\mu^s = 2.78$ N/m, $\lambda^s = 4.49$ N/m, $\tau^s = 0.61$ N/m) and the length-scale parameter associated with the presence of couple stresses $\ell = 50$ nm [46] are chosen. A representative surface load, corresponding to a uniformly distributed normal pressure $p(x) = p_0$ over the length $2a$, is chosen and its Fourier transform is given explicitly by

$$\hat{p}(\xi) = \frac{2 \sin(a\xi)}{\xi} p_0 \quad (14)$$

Numerical results for the force stresses and couple stresses of an elastic half-plane under the uniformly distributed normal pressure $p(x) = p_0$ and the simultaneous influence of both surface and couple stresses are reported in Fig. 2 and Fig. 3, respectively. The stress distributions along the positive x -direction of the surface-loaded half-plane are considered at different normalized depths, $y/a \in \{0.2; 0.4; 0.8\}$ with $a = 0.5\Lambda$ and $\Lambda = 0.25$ nm. Results

represented by the dash lines denote the classical solutions corresponding to $\ell = \Lambda = 0$. It is worth noting that while the classical solutions are independent of the length scale Λ , the use of Λ in the normalization is only for the purpose of comparison.

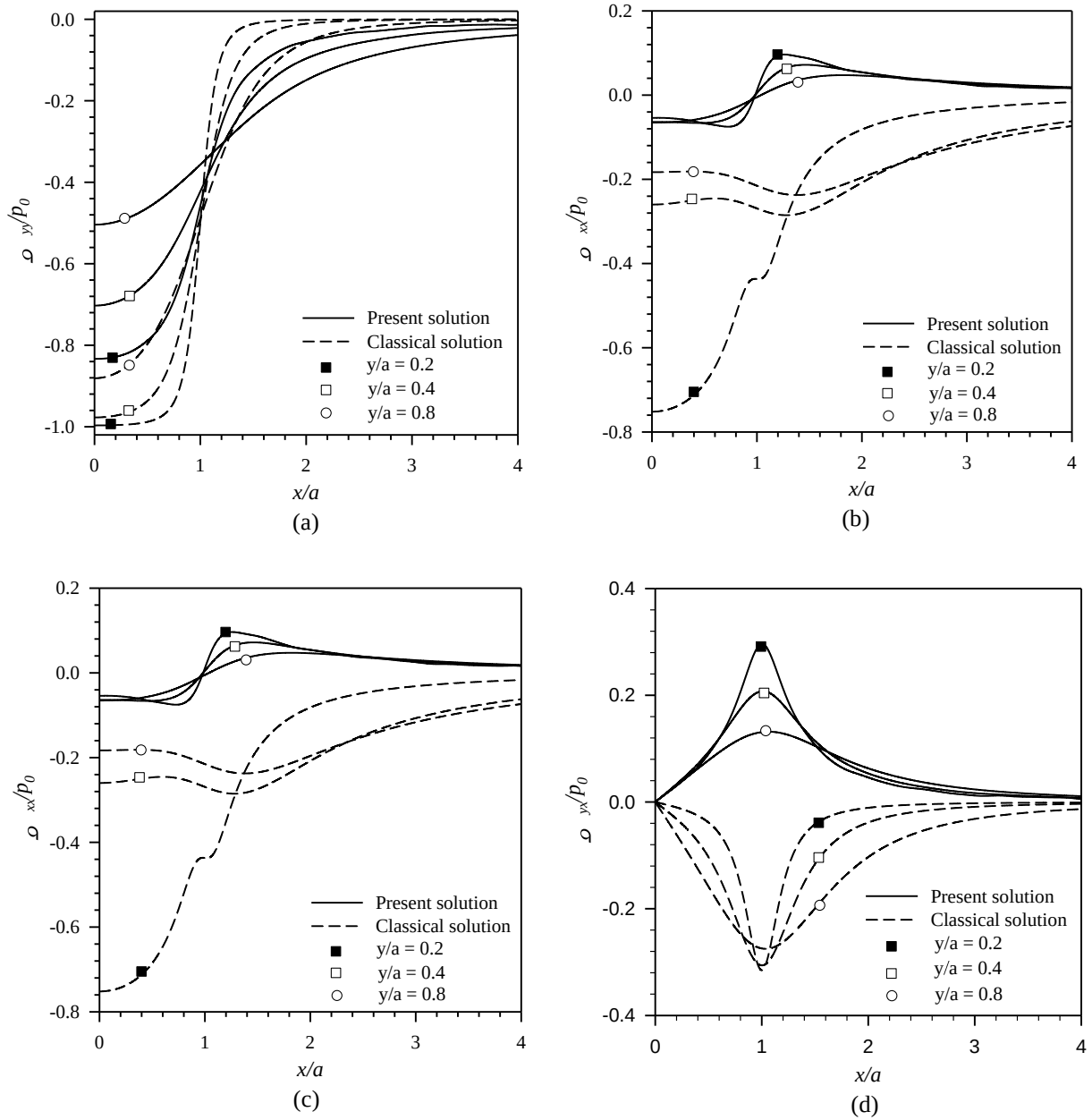


Figure 2. Normalized force stresses of elastic half-plane under uniformly distributed normal traction at various depths

Although the normalized vertical stress σ_{yy}/p_0 and the shear stress σ_{xy}/p_0 from the proposed model and the classical solution display similar trends for all values of the normalized coordinate x/a as displayed in Fig. 2(a) and Fig. 2(c), the magnitude of the normalized stresses with the influence of the surface and couple stresses are lower than that of the classical case. It seems that the presence of the surface and couple stresses renders the elastic medium stiffer. In contrast, the normalized horizontal stress σ_{xx}/p_0 and the shear stress

σ_{yx}/p_0 possess different characteristic in comparison with the classical solutions. In addition, the non-symmetric character of the force-stress tensor is confirmed by results shown in Figs. 2(c) and 2(d). Variations of the couple stresses μ_{xz}, μ_{yz} within the elastic half-plane at various depths are also displayed in Fig. 3. For the classical case, the couple stresses within the bulk vanish identically. It is also evident from this set of results that solutions predicted by the proposed model exhibit the significant departure from the classical solutions.

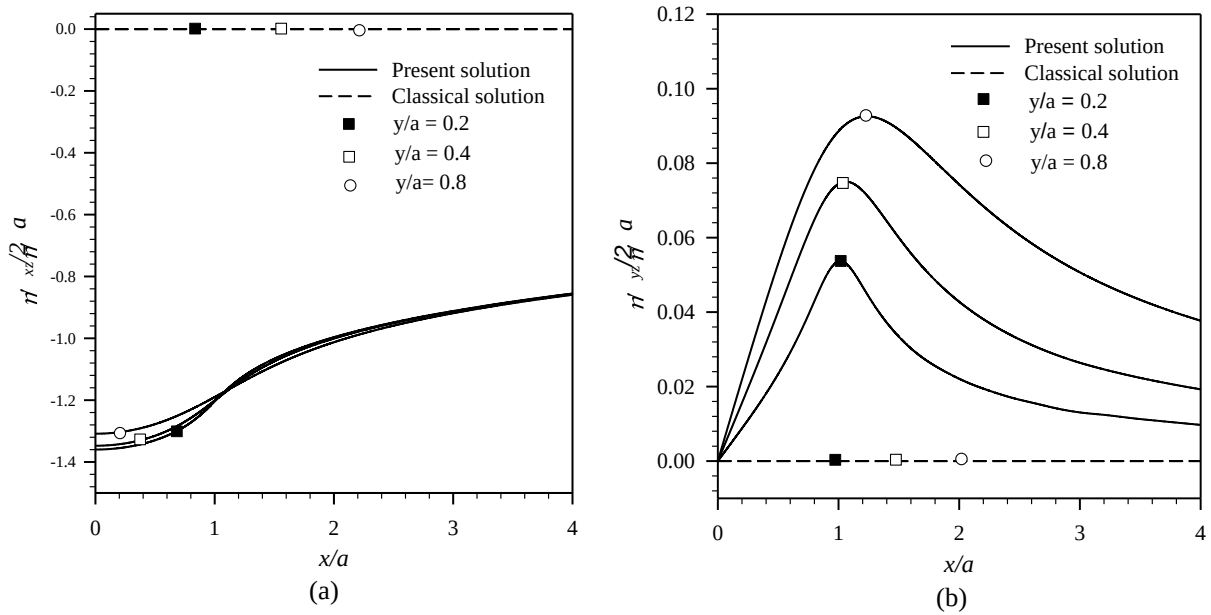


Figure 3. Normalized couple stresses of elastic half-plane under uniformly distributed normal traction at various depths

Conclusion and Remarks

An elastic half-plane under the plane-strain deformation and loaded on its surface by the normal traction has been studied by integrating the influence of both surface stresses and couple stresses. Gurtin-Murdoch surface elasticity theory and the consistent couple stress theory have been used in the problem formulation and the closed-form integral expressions of the elastic field have been derived via the method of Fourier integral transform. An efficient quadrature has been adopted to evaluate all involved integrals resulting from the Fourier transform inversion. A set of preliminary results has indicated the significant influence of the surface and couple stresses on the behavior of predicted solutions; in particular, the obvious deviation from the classical solutions has been observed. The size-dependence behavior of predicted responses and the effects of the two material length scales are also of key interest and still under investigation. It should be remarked that the analytical solutions established in the present study can be used either as benchmark results in the verification process or as the fundamental solutions in the formulation of related problems such as contact and indentation problems.

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Analysis of Cracked Body Strengthened by Adhesively Bonded Patches by BEM-FEM Coupling

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Abstract

This paper presents an efficient numerical technique capable of handling the stress analysis of three-dimensional cracked bodies strengthened by adhesively bonded patches. The proposed technique is implemented within the framework of the coupling of the weakly singular boundary integral equation method and the standard finite element procedure. The former is applied to efficiently treat the elastic body containing cracks whereas the latter is adopted to handle both the adhesive layers and patches. The approximation of the near-front relative crack-face displacement is enhanced by using local interpolation functions that can capture the right asymptotic behaviour. This also offers the direct calculation of the stress intensity factors along the crack front. A selected set of results is reported to demonstrate the capability of the proposed technique and the influence of various parameters on the performance of the strengthening.

Keywords: Boundary integral equation method, Bonded patches, Cracked body, Finite element method, Stress intensity factors

Introduction

Sustainability and integrity of engineering structures are ones among various crucial issues that must be properly integrated in the design procedure and the subsequent maintenance stage. It has been known that the presence of cracks/flaws/damages is one of the major causes of subsequent failures of components, parts of or the entire structures leading finally to the loss of their functions. For such reason, the strengthening of damaged/cracked structures at the earliest time as soon as they are detected during inspections is considered essential. Furthermore, the control of cracked structures after being strengthened is also one of vital tasks to evaluate the selected strengthening method so that those strengthened structures can maintain their bearing capacity.

Replacement of cracked or damaged parts can be time-consuming and expensive, and, in addition, requires a high level of expertise. It is usually more cost effective to strengthen those damaged components by using patching techniques. One of those methods, with adhesively bonded repairs, has been widely employed in practices. Many theoretical and empirical investigations have demonstrated the advantages of this particular method relative to other existing techniques for strengthening cracked structures due mainly to its cost effectiveness (e.g., [1-6]). In addition to their high stiffness and strength, the patches are structurally efficient and induce much less damages to the strengthened structures.

In the modeling point of view via a theory of linear elasticity and linear elastic fracture mechanics, the stress intensity factors along the crack front can be significantly reduced after

the cracked body is strengthened by attaching a patch over the cracked region (e.g., [6-19]). This is due to the fact that some of externally applied loads exerted to the cracked bodies are either shared by or transferred directly to the patches via the adhesive shear layers rendering the enhancement of stiffness near the patching region and the reduction of relative crack-face displacements. As a direct consequence, the growth of the cracks can be delayed or even ceased if the strengthening is properly designed. Evidences from past studies have indicated that various parameters including the dimensions and material properties of the strengthened structure, the patches, and the adhesive layers significantly affects the efficiency and effectiveness of the strengthening (e.g., [6-19]). The full investigation to understand the role of those parameters is considered essential and can assist designers/engineers in the optimization of their designs. While experimental studies offer an excellent means to establish a set of results reflecting real responses, the methods themselves consume a significant amount of resources and are quite limited to test settings. In particular, to assess efficiency and influence of the patches in the experiments, a large set of testing specimens including unpatched and patched cracked structures for various specimen configurations must be considered. An alternative approach is to adopt computer-based simulations via reliable mathematical models to perform such extensive parametric study. It is remarked, however, that to accomplish such an important task within a broad and general framework (e.g., three-dimensional settings, large-scale and complex cases), powerful and computationally efficient tools are prerequisite.

On the basis of an extensive literature survey, most of existing studies toward the development of computational techniques to assist the analysis and simulations of cracked components repaired by adhesively bonded patches were focused extensively on the two-dimensional framework and quite specific to certain settings such as the repair configurations and types of repaired structures. The enhancement of existing modeling tools to be capable of handling more complex and general scenarios such as fully three-dimensional and large scale problems is challenging and requires further extensive investigations.

Problem Formulation

Consider a three-dimensional, finite body containing both embedded and surface-breaking cracks and strengthened or reinforced by adhesively bonded patches as shown schematically in Figure 1. The cracked body is made of a homogeneous, generally anisotropic, linearly elastic material. The ordinary boundary of the cracked body, denoted by S_0 , consists of a surface S_u on which the displacement $\mathbf{u}^{bu}$ is prescribed, a surface S_t on which the traction $\mathbf{t}^{bt}$ is known a priori, and a flat or planar surface S_a on which the patch is attached. The surface of displacement discontinuity used to describe the crack in the undeformed state (i.e., stress-free state) is represented by a pair of geometrically identical surfaces, denoted by S_c^+ and S_c^- , and, in the present study, the attention is restricted only to the case that the crack surface is subjected to the point-wise self-equilibrated tractions; i.e., the prescribed tractions $\mathbf{t}^{b+}, \mathbf{t}^{b-}$ acting respectively to the surfaces S_c^+, S_c^- satisfy the condition $\mathbf{t}^{b+} + \mathbf{t}^{b-} = \mathbf{0}$. Each patch is made of a homogeneous, linear elastic material and fully adhered to the cracked body on the surface S_a by means of an adhesive bonding material. The prescribed traction exerted to the surface of the patch, opposite to the attached surface, is denoted by $\mathbf{t}^{p0}$. In the present study, the thickness of the patches and adhesive layers is assumed relatively small in comparison with the characteristic dimension of the repaired body; as a result, the bending stiffness of the patches can be considered negligible whereas only the shear resistance is treated for the adhesive layers.

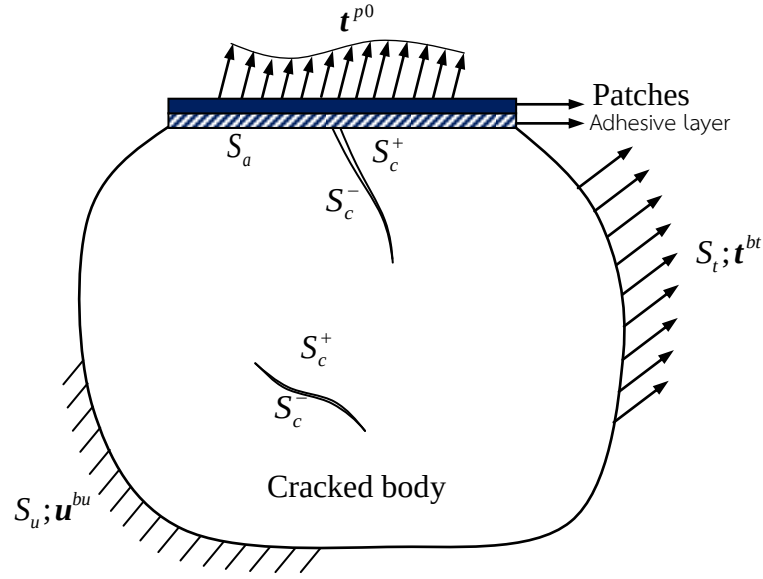


Figure 1. Schematic of a body containing embedded and surface-breaking cracks and strengthened by adhesively bonded patches

Since the thickness of each patch is sufficiently small and the bending effect can be ignored, its responses can then be properly modeled by a two-dimensional, plane-stress, linear elasticity theory. The final governing equation formulated in a local, two-dimensional, Cartesian coordinate system containing the patch by taking the in-plane displacement vector $\bar{\mathbf{u}}^{*p}$ as the primary unknown is given by

$$h_p \mathbf{L}^T \mathbf{C}^{*p} \mathbf{L} \bar{\mathbf{u}}^{*p} + \bar{\mathbf{s}}^{*pa} + \bar{\mathbf{s}}^{*p0} = \mathbf{0} \quad (1)$$

where $\mathbf{L}$ is a conventional, two-dimensional differential operator transforming $\bar{\mathbf{u}}^{*p}$ into a vector containing independent in-plane strain components; the superscript “ T ” denotes the matrix transpose; h_p is the thickness of the patch; $\bar{\mathbf{s}}^{*p0}$ is a vector containing local components of the prescribed shear traction $\mathbf{s}^{p0}$ (i.e., the tangential component vector of the prescribed traction $\mathbf{t}^{p0}$ on the plane of the patch), $\bar{\mathbf{s}}^{*pa}$ is a vector containing local components of the unknown shear traction exerted by the adhesive layer, and $\mathbf{C}^{*p}$ is the elastic in-plane modulus matrix for the plane-stress case. An alternative weak-form of (1) can be readily established via a standard weighted residual technique and used as the basis in the discretization procedure.

From the assumption that the adhesive layer can transfer only shear across its thickness, the state of strain for the entire layer can be completely described by the out-of-plane shear strain. Since the thickness of the adhesive layer is infinitesimal in comparison with its planar dimensions, it is legitimate to assume that the out-of-plane shear strain components are uniform across the thickness or, equivalently, the in-plane displacement varies linearly across the thickness. The out-of-plane shear stress induced within the layer $\bar{\boldsymbol{\sigma}}^{*a}$ is then governed by

$$\bar{\boldsymbol{\sigma}}^{*a} = \frac{G_a}{h_a} (\bar{\mathbf{u}}^{*ap} - \bar{\mathbf{u}}^{*ab}) \quad (2)$$

where h_a is the thickness of the adhesive layer, G_a denotes the elastic shear modulus of the adhesive material, and $\bar{\mathbf{u}}^{*ap}$ and $\bar{\mathbf{u}}^{*ab}$ are values of the in-plane displacement at its interfaces connecting to the patch and the cracked body, respectively. Similarly, the weak-form statement of (2) can also be formulated by the weighted residual technique.

For the cracked body, the theory of linear elasticity with the absence of the body force is adopted and the key governing equations are formulated within the framework of boundary integral equations. In particular, the following pair of weakly singular, weak-form boundary integral equations for the displacements and tractions, proposed by Rungamornrat and Mear (2008a), is utilized to form a system of integral equations governing the unknown data on the boundary and crack surface:

$$\begin{aligned} \frac{1}{2} \int_{S_0} \tilde{\tau}_p(\mathbf{y}) u_p^b(\mathbf{y}) dS(\mathbf{y}) &= \int_{S_0} \tilde{\tau}_p(\mathbf{y}) \int_{S_0} U_j^p(\boldsymbol{\xi} - \mathbf{y}) t_j^b(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \\ &+ \int_{S_0} \tilde{\tau}_p(\mathbf{y}) \int_S G_{mj}^p(\boldsymbol{\xi} - \mathbf{y}) D_m v_j^b(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \\ &- \int_{S_0} \tilde{\tau}_p(\mathbf{y}) \int_S n_i(\boldsymbol{\xi}) H_{ij}^p(\boldsymbol{\xi} - \mathbf{y}) v_j^b(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \end{aligned} \quad (3)$$

$$\begin{aligned} -\frac{1}{2} \int_S \tilde{u}_k(\mathbf{y}) \tau_k^b(\mathbf{y}) dS(\mathbf{y}) &= \int_S D_t \tilde{u}_k(\mathbf{y}) \int_S C_{mj}^{tk}(\boldsymbol{\xi} - \mathbf{y}) D_m v_j^b(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \\ &+ \int_S D_t \tilde{u}_k(\mathbf{y}) \int_{S_0} G_{tk}^j(\boldsymbol{\xi} - \mathbf{y}) t_j^b(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \\ &+ \int_S \tilde{u}_k(\mathbf{y}) \int_{S_0} n_l(\mathbf{y}) H_{lk}^j(\boldsymbol{\xi} - \mathbf{y}) t_j^b(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) dS(\mathbf{y}) \end{aligned} \quad (4)$$

where $S \equiv S_0 \cup S_c^+$ denotes the total boundary of the cracked body; $\tilde{\tau}_p$ is any sufficiently smooth test function defined on the ordinary boundary S_0 ; $\tilde{u}_k$ is any sufficiently smooth test function defined on the total boundary S ; u_p^b and t_j^b are components of the displacement and traction on the ordinary boundary S_0 of the cracked body; n_i are components of the outward unit normal vector to the total boundary S ; $D_m = n_i \varepsilon_{ism} \partial / \partial \xi_s$ or $D_m = n_i \varepsilon_{ism} \partial / \partial y_s$ denotes the surface differential operator; U_j^p , C_{mj}^{tk} , G_{mj}^p , and H_{ij}^p are known fundamental solutions (see details of development and explicit expressions in Rungamornrat and Mear (2008a)); and v_j^b and τ_k^b are data defined by

$$v_j^b(\boldsymbol{\xi}) = \begin{cases} u_j^b(\boldsymbol{\xi}), & \boldsymbol{\xi} \in S_0 \\ \Delta u_j^b(\boldsymbol{\xi}), & \boldsymbol{\xi} \in S_c^+ \end{cases} \quad (5)$$

$$\tau_k^b(\boldsymbol{\xi}) = \begin{cases} t_k^b(\boldsymbol{\xi}), & \boldsymbol{\xi} \in S_0 \\ \Delta t_k^b(\boldsymbol{\xi}), & \boldsymbol{\xi} \in S_c^+ \end{cases} \quad (6)$$

in which $\Delta u_j^b = u_j^+ - u_j^-$ denotes the relative crack-face displacement and $\Delta t_k^b(\boldsymbol{\xi}) = t_k^{0+} - t_k^{0-}$ denotes the jump in the crack-face traction. In particular, for the self-equilibrated crack-face tractions, it yields $\Delta t_k^b(\boldsymbol{\xi}) = 2t_k^{0+}$. To form a system of integral equations governing all

unknown data on the boundary and the crack surface, the displacement boundary integral equation (3) is applied to the surface S_u with $\tilde{\tau}_p \equiv 0$ on $S_t \cup S_a$ whereas the traction boundary integral equation (4) is adopted for the remaining surface $S_t \cup S_a \cup S_c^+$ with $\tilde{u}_k \equiv 0$ on S_u .

A system of governing equations for the whole repaired cracked body shown in Figure 1 can now be obtained by combining the weak-form equations governing all patches, the weak-form equations governing all adhesive layers, and those governing the cracked body together with the continuity of the displacement and the traction along all material interfaces. The final system contains the following unknown functions: the shear stress within the adhesive layers $\bar{\sigma}^{*a}$, the in-plane displacement of the patch $\bar{u}^{*p}$, the displacement $\mathbf{u}^{ba}$ on the surface S_a , the displacement $\mathbf{u}^{bt}$ on the surface S_t , the traction $\mathbf{t}^{bu}$ on the surface S_u , and the relative crack-face displacement $\Delta \mathbf{u}^b$.

Numerical Implementations

To discretize the governing weak-form equations for the patches and the adhesive layers, a standard finite element procedure for two-dimensional problems (e.g., [21-23]) is adopted. The unknown shear stress within the adhesive layer $\bar{\sigma}^{*a}$, the unknown in-plane displacement of the patch $\bar{u}^{*p}$, the unknown displacement on the surface of the cracked body $\mathbf{u}^{ba}$ and all involved test functions are approximated using standard basis functions constructed locally on a finite element mesh consisting of standard, isoparametric, C^0 -elements.

To discretize the weakly-singular, weak-form integral equations governing the cracked body, Galerkin-based procedure similar to that proposed by Rungamornrat and Mear (2008b) is implemented. Due to the weakly singular feature of all involved integrals, both the trial and test functions can be approximated by a set of continuous basis functions constructed locally on a finite element mesh. In particular, standard isoparametric C^0 elements are employed everywhere in the solution discretization except in a local region of the crack surface adjacent to the crack front where special crack-tip elements, originally proposed by Li *et al.* (1998) and used later by Rungamornrat and Mear (2008b) to treat cracks in anisotropic media, are adopted. Element shape functions of such special crack-tip elements were properly enriched to contain the square-root-type behavior and accurately capture the near-front relative crack-face displacement (also see details in Yates *et al.* (2010) and Rungamornrat *et al.* (2019) for the structure of the near-front elastic field). Special quadrature rules proposed by Xiao (1998) are implemented to handle both weakly singular and nearly singular integrals and the efficient interpolation-based algorithm similar to that employed by Rungamornrat and Mear (2008b) is adopted to calculate all involved fundamental solutions for generally anisotropic materials.

The final system of linear algebraic equations resulting from the discretization of the governing equations of the patches, the adhesive layers, and the cracked body is solved by a selected efficient linear solver. The stress intensity factors along the crack front are then extracted directly from the solved relative crack-face displacement data together with the properties of the special crack-tip elements via the post-process formula proposed by Rungamornrat and Mear (2008b).

Numerical Results

To verify the implemented technique and also provide a set of results from a preliminary parametric study on the strengthening of cracked bodies, the following representative problem is chosen in numerical simulations. Consider a cube of an isotropic linearly elastic material that occupies the region $[-w, w] \times [-w, w] \times [-w, w]$ in space and contains a penny-shaped

crack of radius a as shown schematically in Figure 2. The crack lies on a plane $x_3 = 0$ with its center located at point $(0.4w, 0, 0)$. The crack front can be parametrized in terms of the angular position $\theta \in [0, 2\pi]$ by

$$x_1 = 0.4w - a \cos \theta, \quad x_2 = a \sin \theta, \quad x_3 = 0 \quad (7)$$

The cube is loaded by a uniform normal traction $t_3 = \sigma_0$ on the face $x_3 = w$ and the uniform normal traction $t_3 = -\sigma_0$ on the face $x_3 = -w$. To strengthen the cracked body, a patch of uniform thickness h_p is bonded to its entire face $x_1 = w$ by the adhesive layer of uniform thickness h_a . In the numerical study, the aspect ratio $a/w = 0.5$ and Young's modulus and Poisson's ratio given in Table 1 are considered and three meshes shown in Figure 3 are adopted.

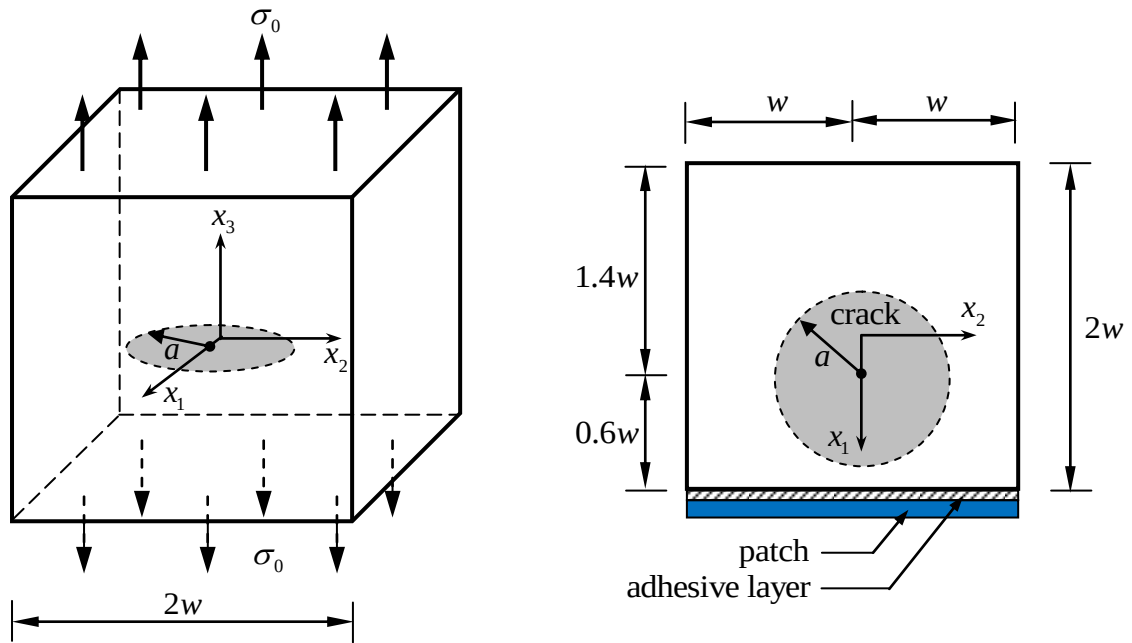


Figure 2. Schematic of cube material containing near-surface penny-shaped crack and strengthening by adhesively bonded patch

Table 1. Young's modulus and Poisson's ratio for cracked body, patch, and adhesive layer used in parametric study

Materials	Young's modulus ($\times 10^6$ psi)	Poisson's ratio
Cracked body	2.0	0.25
Patch	17.4	0.25
Adhesive layer	0.1	0.33

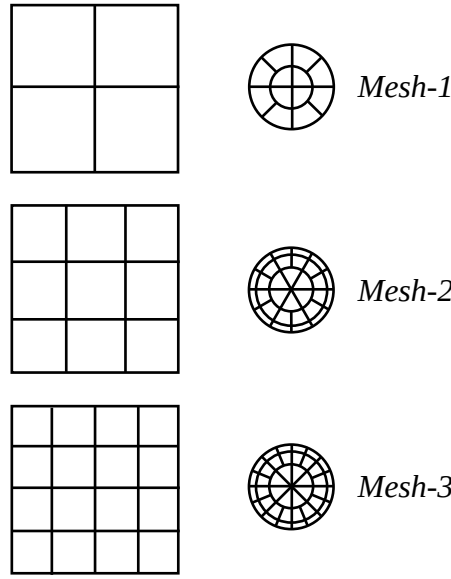


Figure 3. Three meshes adopted in analysis (only mesh of each face of cube is shown and it is identical to those for patch and adhesive layer)

Due to the symmetry and the loading condition considered, only the mode-I stress intensity factor (K_I) is non-zero. The normalized K_I obtained from the three meshes are reported in Figure 4 along with those generated by ABAQUS for the cases with and without the strengthening. It can be concluded from this set of results that numerical solutions converge as the mesh is refined and the good agreement between the converged and reference solution (with the difference within a fraction of one percent) is observed. Note in particular that relatively coarse meshes such as the Mesh-1 and Mesh-2 can also yield quite accurate results; this is due mainly to the use of special crack-tip elements in the approximation of the near-front relative crack-face displacement. After the cracked body is strengthened by the adhesively bonded patch, the stress intensity factor is significantly reduced especially in the region near the bonded patch.

After fully tested, the proposed technique can be further applied to study the influence of various strengthening parameters (e.g., thickness of the adhesive layer and thickness of the patch) on the strengthening performance. For instance, to explore the influence of the patch thickness on the reduction of the stress intensity factor of the crack in the representative problem, simulations can be carried out for different values of h_p while all other parameters remain fixed. A plot of the normalized mode-I stress intensity factor resulting from such simulations are reported in Figure 5, as examples, for $h_p/w = 0.00, 0.01, 0.02, 0.03, 0.04$. Besides the expected reduction of the stress intensity factor as the patch thickness increases (due to the increase in the stiffness after the strengthening), this piece of information is potentially useful in the selection of the patch thickness to confine the stress intensity factor below the tolerance or to prevent the subsequent crack growth. Similarly, the influence of the thickness of the adhesive layer on the response after the strengthening can also be investigated by carrying out simulations for various values of h_a while maintaining all other parameters. Results shown in Figure 6 are for the representative cracked body with three different values of the thickness of the adhesive layer (i.e., $h_a/w = 0.001, 0.005, 0.01$). It is evident that as the thickness of the adhesive layer increases, the apparent stiffness of the cracked body after strengthening tends to decrease.

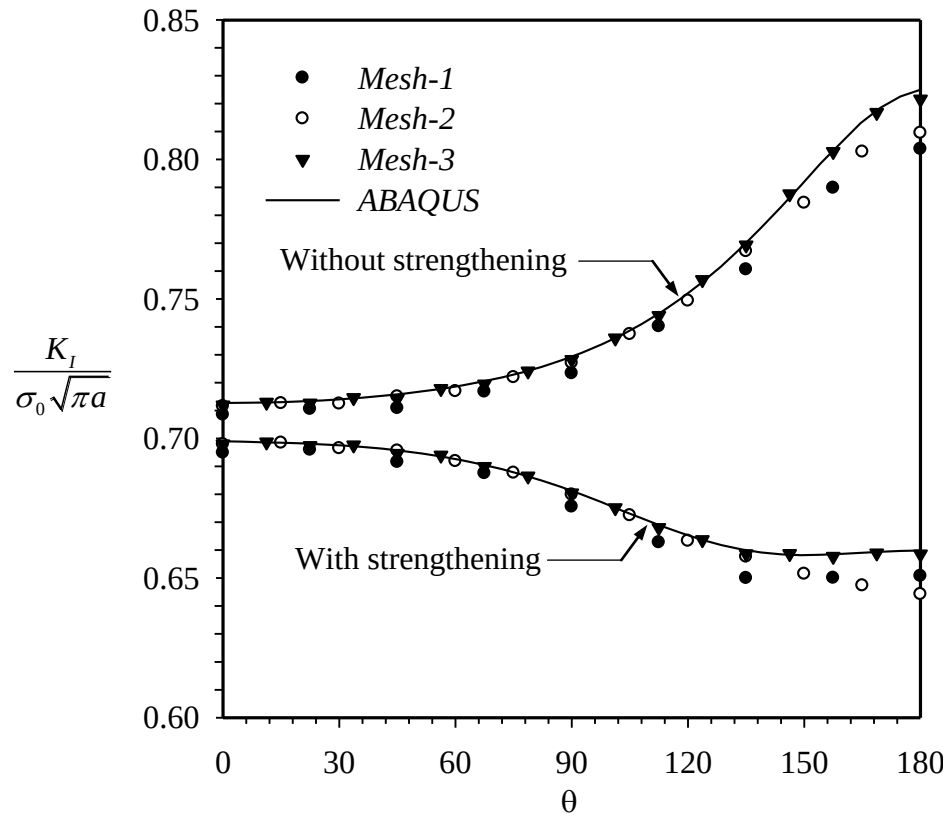


Figure 4. Normalized mode-I stress intensity factors of near-surface penny-shaped crack in cube of material under uniform normal traction σ_0 on its upper and lower faces. Results for the case of strengthening are reported for $h_a / w = 0.001$ and $h_p / w = 0.01$.

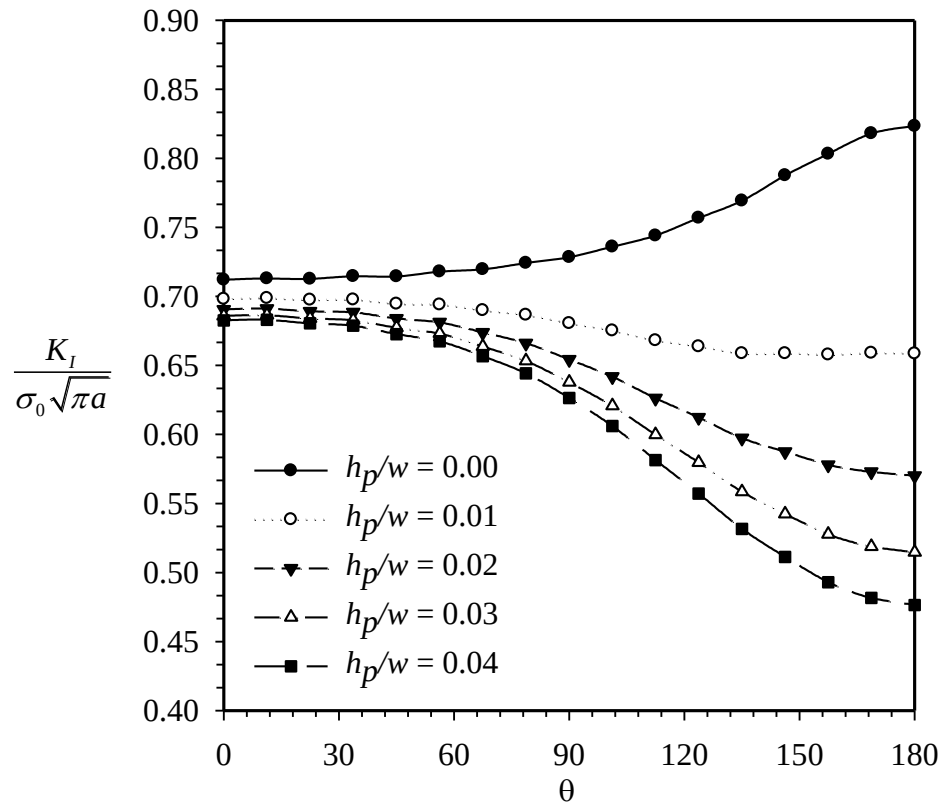


Figure 5. Influence of thickness of patch on normalized mode-I stress intensity factors for near-surface penny-shaped crack in cube of material

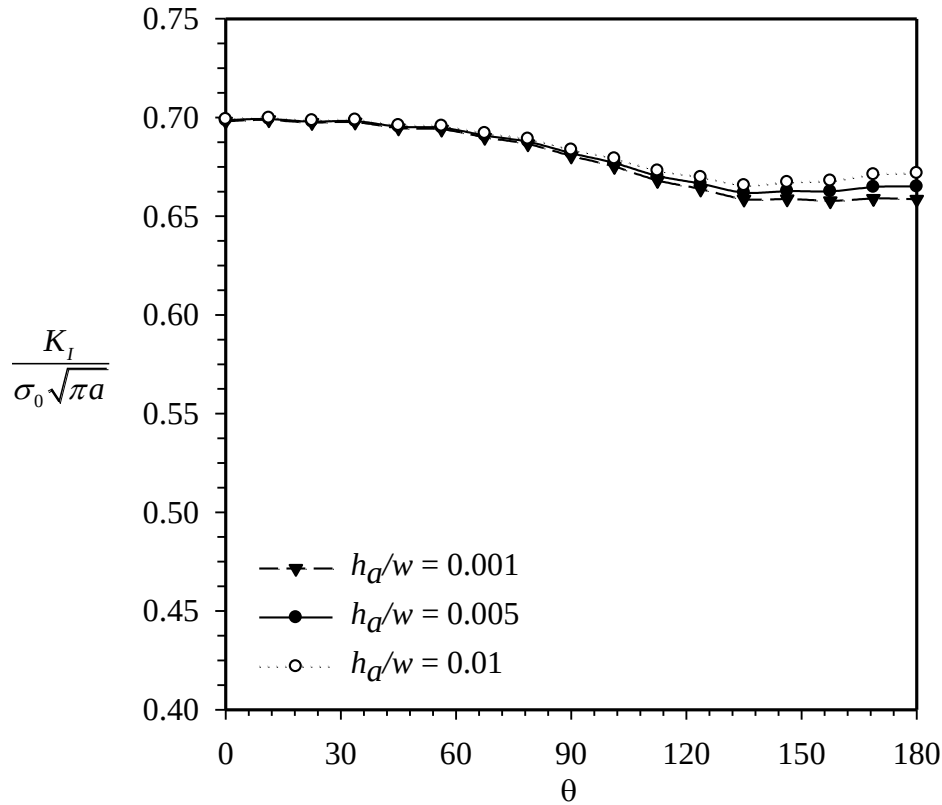


Figure 6. Influence of thickness of adhesive layer on normalized mode-I stress intensity factors for near-surface penny-shaped crack in cube of material

Conclusion and Remarks

The efficient and accurate BIE-FE coupling technique has been successfully implemented for the analysis of three-dimensional cracked bodies strengthened by adhesively bonded patch. The boundary integral equation method has been adopted to efficiently treat the cracked body whereas the standard finite element method has been utilized to handle both the adhesive layer and the patch. The near-front approximation of the relative crack-face displacement has been enhanced by means of using special crack-tip elements and this allows relatively coarse meshes to be employed in the discretization while still yielding sufficiently accurate fracture data along the crack front. Results from a numerical study have indicated that numerical solutions obtained from the proposed technique possess the good convergence behaviour and are of excellent agreement with reliable benchmark solutions. In addition, the preliminary parametric study has shown that the stress intensity factor along the crack front is significantly reduced as the thickness of the patch increases while the reverse trend has been observed for the adhesive layer.

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Analysis of Near Interface Cracks by Weakly Singular Boundary Integral Equation Method

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Abstract

This paper presents a weakly singular boundary integral equation method for analysis of sub-interface cracks in a three-dimensional, linearly elastic, bi-material domain. The formulation is established in a general framework allowing finite bodies, general material anisotropy and loading conditions, arbitrarily shaped cracks, and curved material interface to be treated. A system of integral equations governing the unknown data on the boundary, the crack surface and the material interface are established using a pair of weakly singular, weak-form displacement and traction integral equations and the continuity along the material interface. A symmetric Galerkin boundary element method together with the finite element technique is implemented to discretize the governing integral equations. Special near-front approximation is employed to enhance the discretization of the relative crack-face displacement in the neighborhood of the crack front. The solved crack-face data is then used to post-process for the essential fracture information such as the stress intensity factors. A selected set of results is then presented to demonstrate the accuracy, convergence and capability of the proposed technique

Keywords: anisotropy, BIEM, bi-materials, stress intensity factors, sub-interface cracks

1. Introduction

Bi-materials and multilayer-materials are a well-known class of composite materials fabricated by combining two or more constituents joining at the material interfaces. These materials have been widely used in various engineering and industrial applications (e.g., automobile, aircraft and aerospace structures, dissimilar metal welded joint, pavement structures, etc.) due to their attractive mechanical properties, strength-to-weight ratio, and excellent resistance to severe environmental conditions such as

corrosion and chemical attacks. Besides their attractive features, these materials are often prone to damage/failure resulting primarily from the initiation or presence of fractures/flaws. For instance, fiber reinforced ceramics which are employed under high temperature conditions are very brittle and, generally, possess very low fracture toughness. During applications, cracks may initiate from a brittle zone and then propagate toward the material interface. This finally can lead to ultimate failure of components or loss of their functions. To be capable of avoiding such failure-prone mechanism in the design procedure, the fundamental understanding of fracture behavior of this class of materials is of importance and still requires further investigations.

A classical theory of linear elasticity along with the concept of linear elastic fracture mechanics is commonly utilized to construct a proper mathematical model for simulating fracture-based failure mechanism for a class of brittle materials (e.g., [1, 9]). In such a theory, the fracture process is postulated to be governed by the dominant stress field in an immediate region surrounding the crack front (e.g., [1, 9]). Stress intensity factors, known as parameters used to completely describe the dominant stress field along the crack boundary, are commonly used in various single-parameter models to investigate/simulate fracture responses such as crack advances and fatigue growth (e.g., [11, 23]). For those models to be applicable for various problems under general scenarios, powerful and robust procedures for accurate determination of the stress intensity factors are therefore required. In past several decades, the computational technology for the analysis of cracks in linear elastic bodies with the primary aim to calculate the stress intensity factors has been extensively well-established (e.g., [2, 7-8, 14, 16-17, 20-22, 28-29]). Unlike cracks in a single-material domain, work related to the modeling of cracks in bi-materials/multi-materials is found relatively few and most of existing work is limited mainly to simple crack and domain geometries, loading conditions, and material models such as

isotropic and transversely isotropic elastic materials (e.g., [4-6, 10, 12, 15, 24, 26]). More complex scenarios should be also treated to further enhance the modeling capability to simulate more practical, large scale problems.

In the present study, an efficient boundary integral equation method is developed for the analysis of cracks located near the material interface. Both the formulation and implementation are established in a general framework allowing a multi-material finite body made of generally anisotropic solids, containing arbitrarily shaped cracks, and subjected to general loading conditions to be readily treated. A domain decomposition technique is employed to partition a multi-material domain along the material interface into sub-domains. A pair of weakly singular weak-form boundary integral equations for both the displacement and traction (e.g., [19, 21]) is then applied to construct a set of integral equations governing unknown boundary and crack-face data. A weakly singular symmetric Galerkin boundary element method (e.g., [20]) and standard Galerkin finite element technique along with the special near-front approximation and the conforming discretization along the material interface are implemented to solve the corresponding set of governing equations. The solved crack-face data in a local region along the crack front is directly employed in the post-process to obtain the essential fracture data such as the stress intensity factors and T-stresses (e.g., [17, 18, 20]).

2. Problem Formulation

Consider a three-dimensional, linear elastic, finite body Ω as shown in Figure 1. The body consists of two regions, denoted by Ω^1 and Ω^2 , in which the region Ω^γ ($\gamma = 1, 2$) is made from a homogeneous, generally anisotropic, linear elastic material with the prescribed elastic constants E_{ijkl}^γ and they are perfectly bonded along the interface denoted by S_I . Here and in what follows, the superscript γ takes the value 1 and 2 and the summation is not implied for this particular repeated superscript. In the present study, the body is assumed to be free of the body force and only sub-interface cracks are considered (i.e., cracks intersecting or aligning with the interface S_I are not of interest). The geometry of cracks embedded in the region Ω^γ are represented by two geometrically identical, smooth surfaces $S_C^{+\gamma}$ and $S_C^{-\gamma}$. The outward unit normal vector at any point $\xi^{\gamma+} \in S_C^{+\gamma}$ and its coincident point $\xi^{\gamma-} \in S_C^{-\gamma}$ are denoted, respectively, by $\mathbf{n}^{\gamma+}$ and $\mathbf{n}^{\gamma-}$ and they clearly satisfy $\mathbf{n}^{\gamma+} = -\mathbf{n}^{\gamma-}$. On both crack surfaces $S_C^{+\gamma}$ and $S_C^{-\gamma}$, the traction is fully prescribed. The ordinary boundary of the body, denoted by S_0 , can be decomposed into $S_0 = S_0^1 \cup S_0^2$

S_0^γ is the boundary of the region Ω^γ . The boundary S_0^γ can be further divided into a surface S_T^γ on which the traction is fully prescribed and a surface S_U^γ on which the displacement is fully prescribed. It is important to remark, here, that the formulation to follow is demonstrated only for a domain consisting of two materials for brevity; the generalization to treat multiple material domains follows immediately the same procedure.

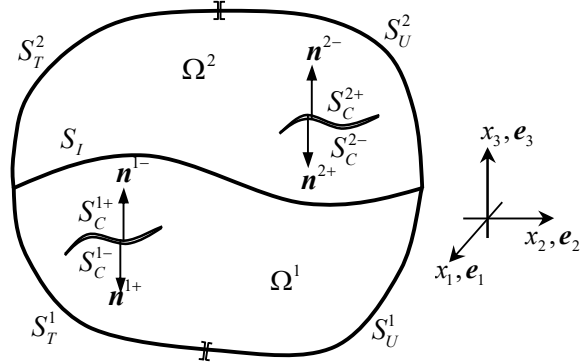


Fig. 1. Schematic of cracked body comprising two regions made from different anisotropic linear elastic materials

2.1 Regularized integral equations for homogeneous domain

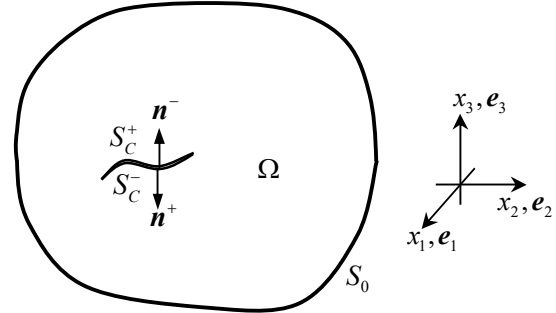


Fig. 2. Schematic of homogeneous, anisotropic, linearly elastic finite body containing crack

For a finite cracked body Ω that is made of a homogeneous, linear elastic material and free of the body force as shown in Figure 2. The total boundary of the body consists of the ordinary boundary S_0 and two geometrically identical surfaces S_C^+ and S_C^- . By following the work of Rungamornrat and Mear [19], the boundary integral representations for the displacement and stress at any interior point $\mathbf{x} \in \Omega$ take the form

$$u_p(\mathbf{x}) = \int_S U_j^p(\xi - \mathbf{x}) \tau_j(\xi) dA(\xi) - \int_S H_{ij}^p(\xi - \mathbf{x}) n_i(\xi) v_j(\xi) dA(\xi) + \int_S G_{mj}^p(\xi - \mathbf{x}) D_m v_j(\xi) dA(\xi) \quad (1)$$

$$\sigma_{ik}(\mathbf{x}) = \epsilon_{irt} \frac{\partial}{\partial x_r} \left\{ \int_S G_{ik}^p(\xi - \mathbf{x}) \tau_j(\xi) dA(\xi) + \int_S C_{mj}^{ik}(\xi - \mathbf{x}) D_m v_j(\xi) dA(\xi) \right\} - \int_S H_{ik}^l(\xi - \mathbf{x}) \tau_j(\xi) dA(\xi) \quad (2)$$

where $S = S_0 \cup S_C^+$ denotes the total surface, $\mathcal{E}_{ism}$ denotes the standard alternating tensor, $D_m(\cdot) = n_i \mathcal{E}_{ism} \partial(\cdot) / \partial \xi_s$ stands for a surface differential operator, and the boundary and crack-face data τ_j and v_j are defined by

$$\tau_j(\xi) = \begin{cases} t_j(\xi), & \xi \in S_0 \\ \Sigma t_j(\xi) \equiv t_j^+(\xi) + t_j^-(\xi), & \xi \in S_C^+ \end{cases} \quad (3)$$

$$v_j(\xi) = \begin{cases} u_j(\xi), & \xi \in S_0 \\ \Delta u_j(\xi) \equiv u_j^+(\xi) - u_j^-(\xi), & \xi \in S_C^+ \end{cases} \quad (4)$$

in which $t_j^+(\xi)$ and $u_j^+(\xi)$ are traction and displacement components at any point $\xi \in S_C^+$ whereas $t_j^-(\xi)$ and $u_j^-(\xi)$ are traction and displacement components at its coincident point on the other crack surface S_C^- . Explicit expressions for the fundamental solutions H_{ij}^p , U_j^p , G_{mj}^p and C_{mj}^{tk} appearing in (3) and (4) for general anisotropic, linear elastic materials are given explicitly in [19, 20]. It is worth noting that the function H_{ij}^p is singular only at $\xi = x$ of $\mathcal{O}(1/r^2)$ whereas the functions U_j^p , G_{mj}^p and C_{mj}^{tk} are singular only at $\xi = x$ of $\mathcal{O}(1/r)$ where $r = \|\xi - x\|$.

To form a set of boundary integral equations governing all unknown data on the total boundary of the body including the crack surface, a standard procedure similar to that proposed by Rungamornrat and Mear [19] is employed. For instance, the displacement boundary integral equation is established by first taking the limit $x \rightarrow y \in S$ of (1) and then establishing the weak form via the standard weighted residual technique. Similarly, the traction boundary integral equation is formed by first using (2) to form a product $\sigma_{ik}(x)n_i(y)$ and taking the limit $x \rightarrow y \in S$ to obtain the traction boundary integral equation, then casting the weak form via the standard weighted residual technique, and finally performing the integration by parts to regularize all integrals containing strongly singular kernels of $\mathcal{O}(1/r^2)$. A final pair of weak-form displacement and traction boundary integral equations is given by

$$\frac{1}{2} \int_S \tilde{t}_p(y) \bar{v}_p(y) dA(y) = \int_S \tilde{t}_p(y) \int_S U_j^p(\xi - y) \tau_j(\xi) dA(\xi) dA(y) \quad (5)$$

$$- \int_S \tilde{t}_p(y) \int_S H_{ij}^p(\xi - y) n_i(\xi) v_j(\xi) dA(\xi) dA(y) + \int_S \tilde{t}_p(y) \int_S G_{mj}^p(\xi - y) D_m v_j(\xi) dA(\xi) dA(y)$$

$$\frac{1}{2} \int_S \tilde{v}_k(y) \bar{\tau}_k(y) dA(y) = - \int_S D_i \tilde{v}_k(y) \int_S C_{mj}^{tk}(\xi - y) D_m v_j(\xi) dA(\xi) dA(y) \quad (6)$$

$$- \int_S D_i \tilde{v}_k(y) \int_S G_{ik}^j(\xi - y) \tau_j(\xi) dA(\xi) dA(y) - \int_S \tilde{v}_k(y) \int_S H_{ik}^j(\xi - y) n_i(\xi) \tau_j(\xi) dA(\xi) dA(y)$$

where $\tilde{t}_p$ and $\tilde{v}_k$ are sufficiently smooth test functions rendering the integrability of all involved integrals and $\bar{v}_p$ and $\bar{\tau}_k$ are data associated with the displacement and traction on the boundary and crack surface defined by

$$\bar{v}_j(y) = \begin{cases} u_j(y), & y \in S_0 \\ \Sigma u_j(y) \equiv u_j^+(y) + u_j^-(y), & y \in S_C^+ \end{cases} \quad (7)$$

$$\bar{\tau}_j(y) = \begin{cases} t_j(y), & y \in S_0 \\ \Delta t_j(y) \equiv t_j^+(y) - t_j^-(y), & y \in S_C^+ \end{cases} \quad (8)$$

It is important to remark that both the displacement and traction boundary integral equations (5) and (6) contain only weakly singular kernels of $\mathcal{O}(1/r)$ and all involved integrals are integrable in the sense of Riemann.

2.2 Symmetric formulation for cracks in bi-material domain

First, the body is partitioned along the material interface S_I into two sub-regions Ω^1 and Ω^2 as shown in Figure 3. The total boundary of each sub-region Ω^γ ($\gamma = 1, 2$) consists of the crack surfaces $S_C^{+\gamma}, S_C^{-\gamma}$ and the ordinary boundary including the surface S_T^γ on which the traction is prescribed, the surface $S_U^\gamma = S_0^\gamma - S_T^\gamma$ on which the displacement is prescribed, and the interface S_I^γ on which both the displacement and traction are unknown a priori. Remark in particular that the interfaces S_I^1 and S_I^2 resulting from the partition have the same geometry as that of the material interface S_I but possess the opposite outward unit normal vector (i.e., $n_I^1 = -n_I^2$).

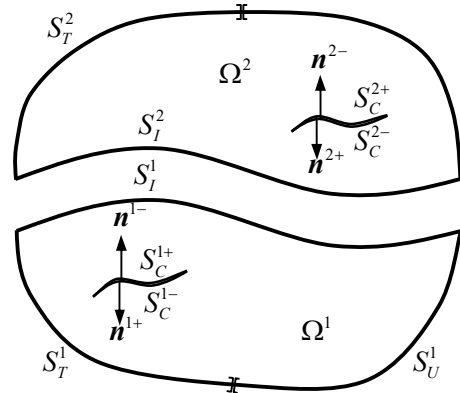


Fig. 3. Schematic of two sub-regions resulting from partition of bi-material domain containing cracks shown in Fig.1

To establish the governing integral equations for each sub-regions Ω^γ ($\gamma = 1, 2$), the weak-form displacement boundary integral equation (5) is utilized along the interface S_I^γ and the boundary on which the displacement is fully prescribed whereas the weak-form traction boundary integral equation (6) is employed along the

interface S_I^γ and the boundary and crack surfaces on which the traction is known a priori. A final set of weak-form boundary integral equations is given by

$$\begin{aligned}
& \mathcal{A}_{UU}^\gamma(\tilde{t}^\gamma, t^\gamma) + \mathcal{B}_{UT}^\gamma(\tilde{t}^\gamma, u^\gamma) + \mathcal{B}_{UC}^\gamma(\tilde{t}^\gamma, \Delta u^\gamma) + \mathcal{A}_{UI}^\gamma(\tilde{t}^\gamma, t_I^\gamma) \\
& + \mathcal{B}_{UI}^\gamma(\tilde{t}^\gamma, u_I^\gamma) = \mathcal{R}_U^\gamma(\tilde{t}^\gamma) \\
& \mathcal{B}_{UT}^\gamma(t^\gamma, \tilde{v}^\gamma) + \mathcal{C}_{TT}^\gamma(\tilde{v}^\gamma, u^\gamma) + \mathcal{C}_{TC}^\gamma(\tilde{v}^\gamma, \Delta u^\gamma) + \mathcal{B}_{TI}^\gamma(t_I^\gamma, \tilde{v}^\gamma) \\
& + \mathcal{C}_{TI}^\gamma(\tilde{v}^\gamma, u_I^\gamma) = \mathcal{R}_T^\gamma(\tilde{v}^\gamma) \\
& \mathcal{B}_{UC}^\gamma(t^\gamma, \tilde{v}^\gamma) + \mathcal{C}_{TC}^\gamma(\tilde{v}^\gamma, u^\gamma) + \mathcal{C}_{CC}^\gamma(\tilde{v}^\gamma, \Delta u^\gamma) + \mathcal{B}_{IC}^\gamma(t_I^\gamma, \tilde{v}^\gamma) \\
& + \mathcal{C}_{CI}^\gamma(\tilde{v}^\gamma, u_I^\gamma) = \mathcal{R}_C^\gamma(\tilde{v}^\gamma) \\
& \mathcal{A}_{UU}^\gamma(\tilde{t}_I^\gamma, t_I^\gamma) + \mathcal{B}_{UT}^\gamma(\tilde{t}_I^\gamma, u_I^\gamma) + \mathcal{B}_{UC}^\gamma(\tilde{t}_I^\gamma, \Delta u_I^\gamma) + \mathcal{A}_{UI}^\gamma(\tilde{t}_I^\gamma, t_I^\gamma) \\
& + \mathcal{B}_{UI}^\gamma(\tilde{t}_I^\gamma, u_I^\gamma) = \mathcal{R}_U^{I\gamma}(\tilde{t}_I^\gamma) + \mathcal{D}_I^\gamma(\tilde{t}_I^\gamma, u_I^\gamma) \\
& \mathcal{B}_{UT}^\gamma(t_I^\gamma, \tilde{u}_I^\gamma) + \mathcal{C}_{TI}^\gamma(\tilde{u}_I^\gamma, u_I^\gamma) + \mathcal{C}_{CI}^\gamma(\tilde{u}_I^\gamma, \Delta u_I^\gamma) + \mathcal{B}_{UI}^\gamma(t_I^\gamma, \tilde{u}_I^\gamma) \\
& + \mathcal{C}_{UI}^\gamma(\tilde{u}_I^\gamma, u_I^\gamma) = \mathcal{R}_T^{I\gamma}(\tilde{u}_I^\gamma) - \mathcal{D}_I^\gamma(\tilde{u}_I^\gamma, t_I^\gamma)
\end{aligned} \quad (9)$$

where all involved bilinear and linear integral operators, with $p, q \in \{T, U, C, I\}$ and $\gamma \in \{1, 2\}$, are defined by

$$\mathcal{A}_{pq}^\gamma(X, Y) = \int_{S_p^\gamma} X_j(y) \int_{S_q^\gamma} U_i^j(\xi - y) Y_i(\xi) dA(\xi) dA(y) \quad (10)$$

$$\begin{aligned}
\mathcal{B}_{pq}^\gamma(X, Y) &= \int_{S_p^\gamma} X_k(y) \int_{S_q^\gamma} G_{mj}^k(\xi - y) D_m Y_j(\xi) dA(\xi) dA(y) \\
&- \int_{S_p^\gamma} X_k(y) \int_{S_q^\gamma} n_m(\xi) H_{mj}^k(\xi - y) Y_j(\xi) dA(\xi) dA(y)
\end{aligned} \quad (11)$$

$$\mathcal{C}_{pq}^\gamma(X, Y) = \int_{S_p^\gamma} D_i X_k(y) \int_{S_q^\gamma} C_{mj}^{yk}(\xi - y) D_m Y_j(\xi) dA(\xi) dA(y) \quad (12)$$

$$\mathcal{D}_p^\gamma(X, Y) = \frac{1}{2} \int_{S_p^\gamma} X_i(y) Y_i(y) dA(y) \quad (13)$$

$$\mathcal{R}_U^\gamma(\tilde{t}^\gamma) = \mathcal{D}_U^\gamma(\tilde{t}^\gamma, u^\gamma) - \mathcal{A}_{UT}^\gamma(\tilde{t}^\gamma, t^\gamma) - \mathcal{A}_{UC}^\gamma(\tilde{t}^\gamma, \Sigma t^\gamma) - \mathcal{B}_{UI}^\gamma(\tilde{t}^\gamma, u^\gamma) \quad (14)$$

$$\mathcal{R}_T^\gamma(\tilde{v}^\gamma) = -\mathcal{D}_T^\gamma(\tilde{v}^\gamma, t^\gamma) - \mathcal{B}_{TI}^\gamma(t^\gamma, \tilde{v}^\gamma) - \mathcal{B}_{CT}^\gamma(\Sigma t^\gamma, \tilde{v}^\gamma) - \mathcal{C}_{TU}^\gamma(\tilde{v}^\gamma, u^\gamma) \quad (15)$$

$$\mathcal{R}_C^\gamma(\tilde{v}^\gamma) = -\mathcal{D}_C^\gamma(\tilde{v}^\gamma, \Delta t^\gamma) - \mathcal{B}_{TC}^\gamma(t^\gamma, \tilde{v}^\gamma) - \mathcal{B}_{CC}^\gamma(\Sigma t^\gamma, \tilde{v}^\gamma) - \mathcal{C}_{CU}^\gamma(\tilde{v}^\gamma, u^\gamma) \quad (16)$$

$$\mathcal{R}_U^{I\gamma}(\tilde{t}_I^\gamma) = -\mathcal{A}_{IT}^\gamma(\tilde{t}_I^\gamma, t_I^\gamma) - \mathcal{A}_{IC}^\gamma(\tilde{t}_I^\gamma, \Sigma t_I^\gamma) - \mathcal{B}_{UI}^\gamma(\tilde{t}_I^\gamma, u_I^\gamma) \quad (17)$$

$$\mathcal{R}_T^{I\gamma}(\tilde{u}_I^\gamma) = -\mathcal{B}_{TI}^\gamma(t_I^\gamma, \tilde{u}_I^\gamma) - \mathcal{B}_{CI}^\gamma(\Sigma t_I^\gamma, \tilde{u}_I^\gamma) - \mathcal{C}_{IU}^\gamma(\tilde{u}_I^\gamma, u_I^\gamma) \quad (18)$$

By combining the boundary integral equations (9) for each sub-region Ω^γ ($\gamma = 1, 2$) and then enforcing the continuity conditions of the displacement, traction, and test functions along the interface, i.e.,

$$u_I^1 = u_I^2 \equiv u_I, \quad \tilde{u}_I^1 = \tilde{u}_I^2 \equiv \tilde{u}_I \quad (19)$$

$$t_I^1 = -t_I^2 \equiv t_I, \quad \tilde{t}_I^1 = -\tilde{t}_I^2 \equiv \tilde{t}_I \quad (20)$$

It yields a final system of boundary integral equations governing the unknown data on the boundary, the material interface, and the crack surface of cracked bi-material domains

$$\begin{aligned}
& \mathcal{B}_{UT}^1(t^1, \tilde{v}^1) + \mathcal{C}_{TT}^1(\tilde{v}^1, u^1) + \mathcal{C}_{TC}^1(\tilde{v}^1, \Delta u^1) + \mathcal{B}_{TI}^1(t_I^1, \tilde{v}^1) \\
& + \mathcal{C}_{TI}^1(\tilde{v}^1, u_I^1) = \mathcal{R}_T^1(\tilde{v}^1)
\end{aligned}$$

$$\begin{aligned}
& \mathcal{B}_{UC}^1(t^1, \tilde{v}^1) + \mathcal{C}_{TC}^1(\tilde{v}^1, u^1) + \mathcal{C}_{CC}^1(\tilde{v}^1, \Delta u^1) + \mathcal{B}_{IC}^1(t_I^1, \tilde{v}^1) \\
& + \mathcal{C}_{CI}^1(\tilde{v}^1, u_I^1) = \mathcal{R}_C^1(\tilde{v}^1) \\
& \mathcal{A}_{UU}^2(\tilde{t}^2, t^2) + \mathcal{B}_{UT}^2(\tilde{t}^2, u^2) + \mathcal{B}_{UC}^2(\tilde{t}^2, \Delta u^2) - \mathcal{A}_{UI}^2(\tilde{t}^2, t_I^1) \\
& + \mathcal{B}_{UI}^2(\tilde{t}^2, u_I^1) = \mathcal{R}_U^2(\tilde{t}^2) \\
& \mathcal{B}_{UT}^2(t^2, \tilde{v}^2) + \mathcal{C}_{TT}^2(\tilde{v}^2, u^2) + \mathcal{C}_{TC}^2(\tilde{v}^2, \Delta u^2) - \mathcal{B}_{TI}^2(t_I^1, \tilde{v}^2) \\
& + \mathcal{C}_{TI}^2(\tilde{v}^2, u_I^1) = \mathcal{R}_T^2(\tilde{v}^2) \\
& \mathcal{B}_{UC}^2(t^2, \tilde{v}^2) + \mathcal{C}_{TC}^2(\tilde{v}^2, u^2) + \mathcal{C}_{CC}^2(\tilde{v}^2, \Delta u^2) - \mathcal{B}_{IC}^2(t_I^1, \tilde{v}^2) \\
& + \mathcal{C}_{CI}^2(\tilde{v}^2, u_I^1) = \mathcal{R}_C^2(\tilde{v}^2) \\
& \mathcal{A}_{UU}^1(\tilde{t}_I^1, t_I^1) + \mathcal{B}_{UT}^1(\tilde{t}_I^1, u_I^1) + \mathcal{B}_{UC}^1(\tilde{t}_I^1, \Delta u_I^1) - \mathcal{A}_{UI}^1(\tilde{t}_I^1, t_I^2) - \mathcal{B}_{TI}^1(\tilde{t}_I^1, u_I^2) \\
& - \mathcal{B}_{IC}^1(\tilde{t}_I^1, \Delta u_I^2) + \mathcal{A}_{II}^*(\tilde{t}_I^1, t_I^1) + \mathcal{B}_{II}^*(\tilde{t}_I^1, u_I^1) = \mathcal{R}_U^1(\tilde{t}_I^1) \\
& \mathcal{B}_{UT}^1(t_I^1, u_I^1) + \mathcal{C}_{TI}^1(\tilde{u}_I^1, u_I^1) + \mathcal{C}_{CI}^1(\tilde{u}_I^1, \Delta u_I^1) + \mathcal{B}_{UI}^1(t_I^1, u_I^1) + \mathcal{C}_{UI}^1(\tilde{u}_I^1, u_I^2) \\
& + \mathcal{C}_{CI}^2(\tilde{u}_I^1, \Delta u_I^2) + \mathcal{B}_{II}^*(t_I^1, \tilde{u}_I^1) + \mathcal{C}_{II}^*(\tilde{u}_I^1, u_I^1) = \mathcal{R}_T^1(\tilde{u}_I^1)
\end{aligned} \quad (21)$$

where additional bilinear and linear operators appearing in (21) are defined by

$$\mathcal{A}_{II}^*(\tilde{t}_I^1, t_I^1) = \mathcal{A}_{II}^1(\tilde{t}_I^1, t_I^1) + \mathcal{A}_{II}^2(\tilde{t}_I^1, t_I^1) \quad (22)$$

$$\mathcal{B}_{II}^*(\tilde{t}_I^1, u_I^1) = \mathcal{B}_{II}^1(\tilde{t}_I^1, u_I^1) - \mathcal{B}_{II}^2(\tilde{t}_I^1, u_I^1) \quad (23)$$

$$\mathcal{C}_{II}^*(\tilde{u}_I^1, u_I^1) = \mathcal{C}_{II}^1(\tilde{u}_I^1, u_I^1) + \mathcal{C}_{II}^2(\tilde{u}_I^1, u_I^1) \quad (24)$$

$$\mathcal{R}_U^1(\tilde{t}_I^1) = \mathcal{R}_U^{I1}(\tilde{t}_I^1) - \mathcal{R}_U^{I2}(\tilde{t}_I^1) \quad (25)$$

$$\mathcal{R}_T^1(\tilde{u}_I^1) = \mathcal{R}_T^{I1}(\tilde{u}_I^1) + \mathcal{R}_T^{I2}(\tilde{u}_I^1) \quad (26)$$

It should be evident that the set of boundary integral equations (21) is in a symmetric form.

2.3 Near-front stress field

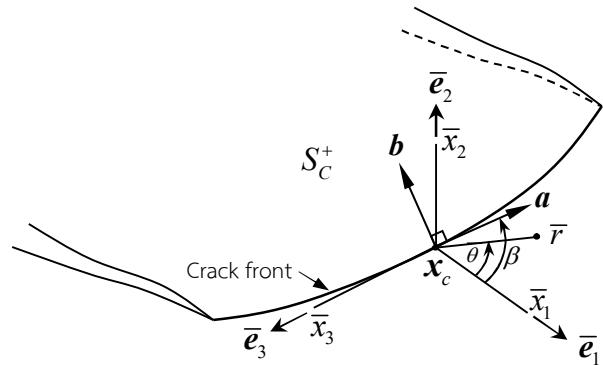


Fig. 4. Local coordinate system utilized in determination of stress intensity factors

In linear elastic fracture analysis, essential fracture parameters, commonly, termed the stress intensity factors, are of primary interest in addition to the elastic field induced within the domain. The former is known to sufficiently describing the singular part of the near-front stress field whereas the latter represents the non-zero nonsingular part of stress along the crack front. By following the primitive work of [25], the near-front stress field in a region surrounding a point x_c admits the representation

$$\bar{\sigma}_{ij}(x_c; \bar{r}, \bar{\theta}) = \frac{1}{\sqrt{2\pi\bar{r}}} \sum_{n=1}^3 K_n(x_c) \bar{\sigma}_{ij}^{K(n)}(\bar{\theta}) + \sum_{m=0}^{\infty} \bar{r}^{m/2} \bar{\sigma}_{ij}^{(m)}(x_c; \bar{\theta}) \quad (27)$$

where all quantities appearing in (27) are referred to the local Cartesian coordinate system $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ with origin at point $\mathbf{x}_c$ and a set of orthonormal basis vectors $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$ shown in Figure 4 (in particular, the unit vector $\bar{\mathbf{e}}_1$ is normal to the crack front and contained in the tangent plane of the crack surface at $\mathbf{x}_c$, $\bar{\mathbf{e}}_2 = -\mathbf{n}^+(\mathbf{x}_c)$, and $\bar{\mathbf{e}}_3 = \bar{\mathbf{e}}_1 \times \bar{\mathbf{e}}_2$; $(\bar{r}, \bar{\theta})$ are polar coordinates associated with the local two-dimensional Cartesian coordinate system $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2\}$; $K_1(\mathbf{x}_c)$, $K_2(\mathbf{x}_c)$ and $K_3(\mathbf{x}_c)$ are mode-I, mode-II, and mode-III stress intensity factors at the point $\mathbf{x}_c$; $\bar{\sigma}_{ij}^{K(n)}$ and $\bar{\sigma}_{ij}^{(m)}$ are angular-dependent functions that can be completely obtained from the near-front eigen-analysis. It is apparent from the expansion (27) that the first term is singular at $\mathbf{x}_c$ of $\mathcal{O}(1/\sqrt{\bar{r}})$, the second term is independent of the coordinates $\bar{r}$ and $\bar{\theta}$, and all remaining terms identically vanish along the crack front.

To calculate the stress intensity factors, the formula proposed by Barnett and Asaro [3] and Xu [28] is adopted. Such expression allows $K_1(\mathbf{x}_c)$, $K_2(\mathbf{x}_c)$ and $K_3(\mathbf{x}_c)$ to be calculated directly in terms of the relative crack-face displacements in the vicinity of the crack front as follows

$$k_i(\mathbf{x}_c) = \frac{\sqrt{2\pi}}{4} B_{ij} \lim_{\bar{x}_1 \rightarrow 0^+, \bar{x}_2 \rightarrow \mathbf{x}_c} \left(\frac{\Delta \bar{u}_j(\bar{\mathbf{x}})}{\sqrt{-\bar{x}_1}} \right) \quad (28)$$

where $k_1 = K_2$, $k_2 = K_1$, $k_3 = K_3$, $\Delta \bar{u}_j$ are components of the relative crack-face displacements referring to the local coordinate system $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ and B_{ij} are material-dependent constants defined by

$$B_{ij} = \frac{1}{2\pi} \int_0^{2\pi} [(a, a)_{ij} - (a, b)_{im} (b, b)_{mn}^{-1} (b, a)_{nj}] d\beta \quad (29)$$

in which $\mathbf{a}$ and $\mathbf{b}$ are orthonormal vectors contained in the plane $\bar{x}_3 = 0$; β is the angle between $\mathbf{a}$ and $\bar{\mathbf{e}}_1$ as indicated in Figure 4; and $(a, b)_{ij} = a_m E_{mijn} b_n$.

3. Solution Procedure

A well-known, weakly singular, symmetric Galerkin boundary element method (SGBEM) is utilized to construct approximate solutions of (21). Standard Galerkin approximation together with the finite element procedure is utilized in the discretization of the unknown data, test functions, and geometries of the ordinary boundary S_0 , the material interface S_I and the crack surfaces S_C^{1+} and S_C^{2+} . In particular, all unknown data and test functions for each sub-region Ω^γ and on the material interface are approximated by

$$\mathbf{u}^\gamma = (\Phi_i^\gamma)^T \mathbf{U}^\gamma, \tilde{\mathbf{v}}^\gamma = (\Phi_i^\gamma)^T \mathbf{V}^{\gamma\gamma} \text{ on } S_I^\gamma \quad (30)$$

$$\mathbf{t}^\gamma = (\Phi_u^\gamma)^T \mathbf{T}^\gamma, \tilde{\mathbf{t}}^\gamma = (\Phi_u^\gamma)^T \tilde{\mathbf{T}}^\gamma \text{ on } S_u^\gamma \quad (31)$$

$$\Delta \mathbf{u}^\gamma = (\Phi_c^\gamma)^T \Delta \mathbf{U}^\gamma, \tilde{\mathbf{v}}^\gamma = (\Phi_c^\gamma)^T \mathbf{V}^{c\gamma} \text{ on } S_C^{\gamma+} \quad (32)$$

$$\mathbf{u}_I = (\Phi_I)^T \mathbf{U}_I, \tilde{\mathbf{u}}_I = (\Phi_I)^T \tilde{\mathbf{U}}_I \text{ on } S_I \quad (33)$$

$$\mathbf{t}_I = (\Phi_I)^T \mathbf{T}_I, \tilde{\mathbf{t}}_I = (\Phi_I)^T \tilde{\mathbf{T}}_I \text{ on } S_I \quad (34)$$

where Φ_i^γ , Φ_u^γ , Φ_c^γ , and Φ_I are vectors of nodal basis functions defined on the surfaces S_T^γ , S_U^γ , S_C^γ , and S_I , respectively; $\mathbf{U}^\gamma$, $\mathbf{T}^\gamma$, $\Delta \mathbf{U}^\gamma$, $\mathbf{U}_I$ and $\mathbf{T}_I$ are vectors of unknown nodal quantities associated with the displacement on S_T^γ , the traction on S_U^γ , the relative crack-face displacement on $S_C^{\gamma+}$, the displacement on S_I and the traction on S_I , respectively; $\mathbf{V}^{\gamma\gamma}$, $\tilde{\mathbf{T}}^\gamma$, $\mathbf{V}^{c\gamma}$, $\tilde{\mathbf{U}}_I$ and $\tilde{\mathbf{T}}_I$ are vectors of arbitrary nodal quantities; and the superscript "T" denotes the transpose operator. Due to the weakly singular nature of all involved integrals, all nodal basis functions are constructed locally on each C^0 -element in a finite element mesh using standard finite element procedure. In addition, to enhance the near-front approximation, special local C^0 -interpolation functions defined over special crack-tip elements (see details in the work [14] and [20]) are employed to discretize the relative crack-face displacement in all elements adjacent to the crack front.

Substituting the approximations (30)-(34) into the system of governing equations (21) and then exploiting the arbitrariness of $\mathbf{V}^{\gamma\gamma}$, $\tilde{\mathbf{T}}^\gamma$, $\mathbf{V}^{c\gamma}$, $\tilde{\mathbf{U}}_I$ and $\tilde{\mathbf{T}}_I$ lead to the following system of linear algebraic equations

$$\begin{bmatrix} \mathbf{A}_{UU}^1 & \mathbf{B}_{UT}^1 & \mathbf{B}_{UC}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{UI}^1 & \mathbf{B}_{II}^1 \\ (\mathbf{B}_{UT}^1)^T & \mathbf{C}_{TT}^1 & \mathbf{C}_{TC}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{B}_{II}^1)^T & \mathbf{C}_{TI}^1 \\ (\mathbf{B}_{UC}^1)^T & (\mathbf{C}_{TC}^1)^T & \mathbf{C}_{CC}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{B}_{II}^1)^T & \mathbf{C}_{CI}^1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{UU}^2 & \mathbf{B}_{UT}^2 & \mathbf{B}_{UC}^2 & \mathbf{A}_{UI}^2 & \mathbf{B}_{II}^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{B}_{UT}^2)^T & \mathbf{C}_{TT}^2 & \mathbf{C}_{TC}^2 & (\mathbf{B}_{II}^2)^T & \mathbf{C}_{TI}^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{B}_{UC}^2)^T & (\mathbf{C}_{TC}^2)^T & \mathbf{C}_{CC}^2 & (\mathbf{B}_{II}^2)^T & \mathbf{C}_{CI}^2 \\ (\mathbf{A}_{UI}^1)^T & \mathbf{B}_{II}^1 & \mathbf{B}_{IC}^1 & (\mathbf{A}_{UI}^2)^T & \mathbf{B}_{II}^2 & \mathbf{B}_{IC}^2 & \mathbf{A}_{II}^1 & \mathbf{B}_{II}^1 \\ (\mathbf{B}_{II}^1)^T & (\mathbf{C}_{TI}^1)^T & (\mathbf{C}_{CI}^1)^T & (\mathbf{B}_{II}^2)^T & (\mathbf{C}_{TI}^2)^T & (\mathbf{C}_{CI}^2)^T & (\mathbf{B}_{II}^1)^T & \mathbf{C}_{II}^1 \end{bmatrix} \begin{bmatrix} \mathbf{T}^1 \\ \mathbf{U}^1 \\ \Delta \mathbf{U}^1 \\ \mathbf{T}^2 \\ \mathbf{U}^2 \\ \Delta \mathbf{U}^2 \\ \mathbf{T}_I \\ \mathbf{U}_I \end{bmatrix} = \begin{bmatrix} \mathbf{R}_U^1 \\ \mathbf{R}_T^1 \\ \mathbf{R}_C^1 \\ \mathbf{R}_U^2 \\ \mathbf{R}_T^2 \\ \mathbf{R}_C^2 \\ \mathbf{R}_I^1 \\ \mathbf{R}_I^2 \end{bmatrix} \quad (35)$$

where the sub-matrices $\mathbf{A}_{pq}^\gamma$, $\mathbf{B}_{pq}^\gamma$, $\mathbf{C}_{pq}^\gamma$, $\mathbf{A}_{II}^*$, $\mathbf{B}_{II}^*$ and $\mathbf{C}_{II}^*$ for $p, q \in \{T, U, C, I\}$ are obtained from the discretization of the bilinear integral operators $\mathcal{A}_{pq}^\gamma$, $\mathcal{B}_{pq}^\gamma$, $\mathcal{C}_{pq}^\gamma$, $\mathcal{A}_{II}^*$, $\mathcal{B}_{II}^*$ and $\mathcal{C}_{II}^*$, respectively, and the sub-vectors $\mathbf{R}_p^\gamma$ and $\mathbf{R}_p^I$ for $p \in \{T, U, C, I\}$ are obtained from the discretization of the linear integral operators $\mathcal{R}_p^\gamma$ and $\mathcal{R}_p^I$, respectively. To construct the coefficient matrix and known vector, two computational issues, one associated with the numerical integration of weakly singular and nearly singular double surface integrals and the other corresponding to the numerical evaluation

of kernels for general anisotropy, must be properly treated. In the present study, the former is achieved by adopting the special quadrature combining a series of variable transformations and Gaussian quadrature proposed by Xiao [27] whereas the latter is handled by utilizing the interpolation technique based on a finite element interpolant proposed by Rungamornrat and Mear [20] to mainly avoid the direct integration of the closed contour integrals. The system of linear equations (35) is essentially symmetric resulting from the symmetry of the formulation and is solved numerically using a selected indirect linear solver such as a conjugate gradient method. The relative crack-face displacement for cracks in each sub-region is then obtained once ΔU^r is determined.

By employing the formula (28) along with the near-front approximation of the relative crack-face displacement on special 9-node crack-tip elements (see [14] and [20]) and Taylor series expansion, the explicit expression for the stress intensity factors take the form

$$k_i(\mathbf{x}_c) = \sqrt{\frac{\pi}{2\gamma}} B_{ij} [\hat{\mathbf{u}}(\mathbf{x}_c) \cdot \bar{\mathbf{e}}_j] \quad (36)$$

where all involved parameters are defined by

$$\gamma = -\bar{\mathbf{e}}_1 \cdot \frac{\partial \mathbf{r}_c}{\partial \eta}(\xi_c, -1) \quad (37)$$

$$\mathbf{r}_c = \sum_{i=1}^9 \mathbf{x}^{(i)} \psi^{(i)}(\xi, \eta) - \mathbf{x}_c \quad (38)$$

$$\hat{\mathbf{u}}(\mathbf{x}_c) = \sum \Delta \mathbf{u}^{(i)} \psi^{(i)}(\xi_c, -1) \quad (39)$$

in which $(\xi, \eta) \in [-1, 1] \times [-1, 1]$ are natural coordinates used to define the master element; $\mathbf{x}^{(i)}$ denotes the i^{th} nodal point of the crack-tip element; $\psi^{(i)}(\xi, \eta)$ are shape functions associated with the i^{th} node of a standard 9-node quadratic element; $\Delta \mathbf{u}^{(i)}$ denotes the extra degree of freedom associated with the i^{th} node of the crack-tip element located along the crack front; and the summation appearing in (39) is taken only for nodes along the crack front. It is apparent from (36) that once ΔU^r for each sub-region is obtained, the information $\Delta \mathbf{u}^{(i)}$ can be readily obtained and used to directly post-process the stress intensity factor without carrying out the limiting process.

4. Preliminary Results

In this section, preliminary results for a representative problem are reported and compared with existing benchmark solutions to demonstrate the accuracy and convergence of numerical solutions generated by the proposed technique.

Consider a linear elastic body occupying the region $[-d, d] \times [-d, d] \times [-d, d]$ as shown schematically in Figure

5. The upper part of this body (which occupies the region $[-d, d] \times [-d, d] \times [0, d]$) is made of an isotropic material with Young's modulus E_1 and Poisson's ratio ν_1 (called "Material-1") whereas the remaining part (which occupies the region $[-d, d] \times [-d, d] \times [-d, 0]$) is made of a different isotropic material with Young's modulus E_2 and Poisson's ratio ν_2 (called "Material-2"). An elliptical crack is embedded within the Material-2 and oriented such that its boundary (i.e., the crack front) can be parameterized by

$$x_1 = 0, x_2 = a \cos \theta, x_3 = -h - b \sin \theta \quad (40)$$

where a and b are major and minor semi-axes, respectively; h is the distance from the crack center to the material interface; and $\theta \in [0, 2\pi]$ denotes the angular position of point along the crack front. The body is free of the body force and loaded by a uniformly distributed, self-equilibrated normal traction t_0 on both crack surfaces. In the numerical study, $h/b = 2$, $a/b = 2$ and $\nu_1 = \nu_2 = 0.3$ are considered and three meshes, indicated in Figure 6, are utilized to investigate the convergence of numerical solutions.

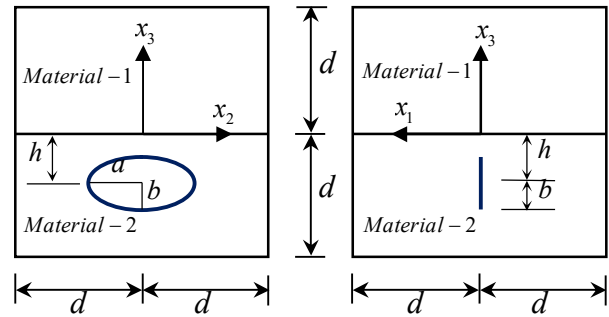


Fig. 5. Bi-material finite cube containing embedded elliptical crack under uniformly distributed normal traction.

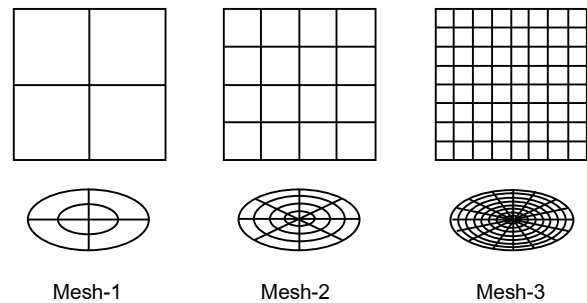


Fig. 6. Three meshes adopted in analysis. Meshes for each face of cube and material interface are identical whereas mesh for crack is shown below that of the boundary.

First, the analysis is carried out for the case that the dimension of the body is much larger than the crack size (i.e., $d \gg a, b$) and computed results are then compared with those generated by Noda [15] for an infinite case. The normalized

mode-I stress intensity factor obtained from all three meshes is reported along the crack front as a function of the angular position θ in Figure 7 for $E_1/E_2 = 0$ and Figure 8 for $E_1/E_2 = \infty$. It is evident from these results that the computed stress intensity factors along the crack front converge to the reference solution as the level of discretization is refined and, in particular, relatively coarse meshes with only few degrees of freedom to discretize the crack surface can also provide reasonably accurate results. The good quality of numerical solutions results directly from the use of special crack-tip elements to enhance the approximation of the near-front field.

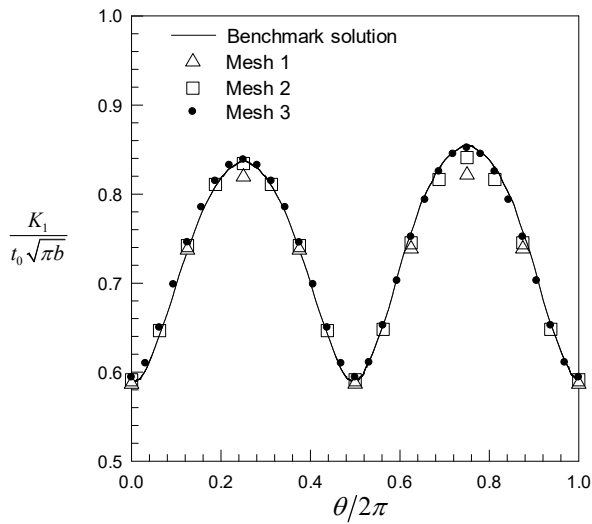


Fig. 7. Normalized mode-I stress intensity factor of elliptical crack embedded in infinite bi-material domain. Results are reported for $E_1/E_2 = 0$.

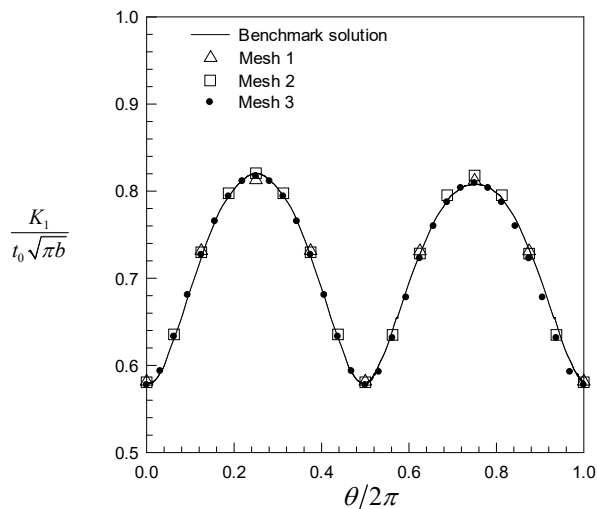


Fig. 8. Normalized mode-I stress intensity factor of elliptical crack embedded in infinite bi-material domain. Results are reported for $E_1/E_2 = \infty$.

mode-I stress intensity factors for $E_1/E_2 = 0$ and $E_1/E_2 = \infty$ are shown in Figure 9 and Figure 10, respectively. It is seen from this set of results that the numerical solutions exhibit the good convergence behavior and the relatively coarse mesh can yield sufficiently accurate results. To further explore the influence of the modulus ratio E_1/E_2 on both the value and distribution of the stress intensity factor along the crack front, the analysis is performed for various values of E_1/E_2 and obtained results are shown in Figure 10. It can be concluded, for this particular case, that the stress intensity factor attains the absolute maximum at $\theta = \pi/2$ and local maximum at $\theta = 3\pi/2$ except for the case $E_1/E_2 = 0$ where the stress intensity factors at $\theta = \pi/2$ and $\theta = 3\pi/2$ are identical. The latter results directly from that the crack is located at the center of the Material-2. It is evident that the increase in stiffness of the Material-1 from zero generally reduces the crack opening displacement of the entire crack and also the mode-I stress intensity factor along the crack front. The reduction is more significant at points closer to the material interface.

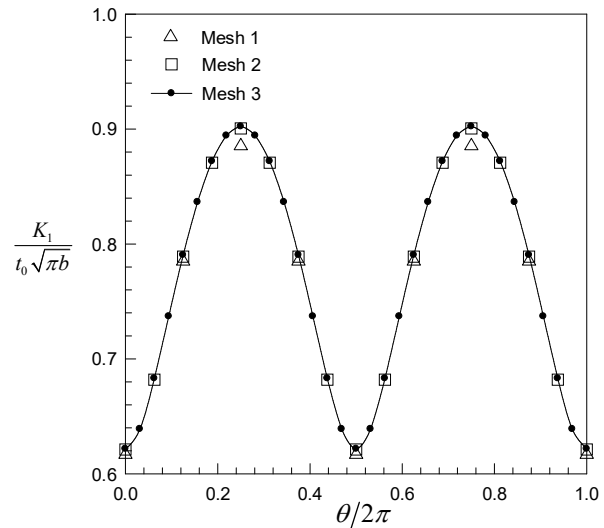


Fig. 9. Normalized mode-I stress intensity factor of elliptical crack embedded in finite cube with $d/b = 4$. Results are reported for $E_1/E_2 = 0$.

5. Conclusion

An efficient and robust numerical technique, based upon the weakly singular boundary integral equation method, has been successfully implemented for the analysis of near-interface or sub-interface cracks in multi-material finite bodies. The formulation has been established within a general framework allowing general data such as the body and crack geometries, general material anisotropy, general loading and boundary conditions to be treated. Besides the efficiency gained directly from the reduction of the spatial discretization, the accuracy of the proposed technique has

Once the implemented technique is verified, it is employed to solve the finite cube with $d/b = 4$. Results of the normalized

been further enhanced by exploiting special interpolations along the crack front to capture the dominant near-front field. Preliminary results from the present study have indicated that the integration of this type of elements in the solution discretization significantly reduces the requirement of exploiting very fine mesh to accurately capture the near-front field. In addition, it also provides the direct means to post-process for the stress intensity factors from the relative crack-face displacement in the vicinity of the crack front.

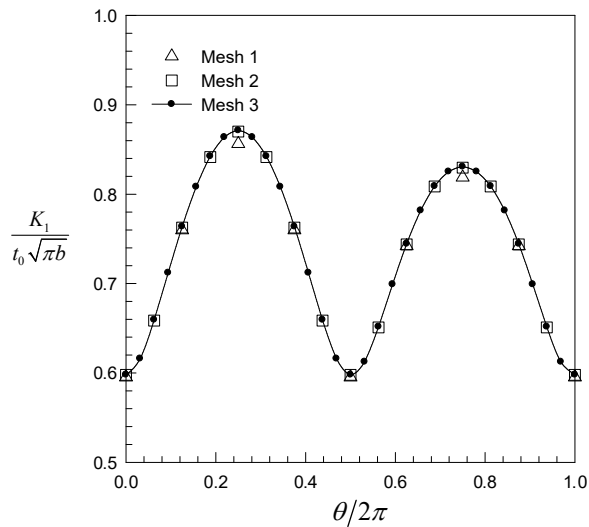


Fig. 10. Normalized mode-I stress intensity factor of elliptical crack embedded in finite cube with $d/b = 4$. Results are reported for $E_1/E_2 = \infty$.

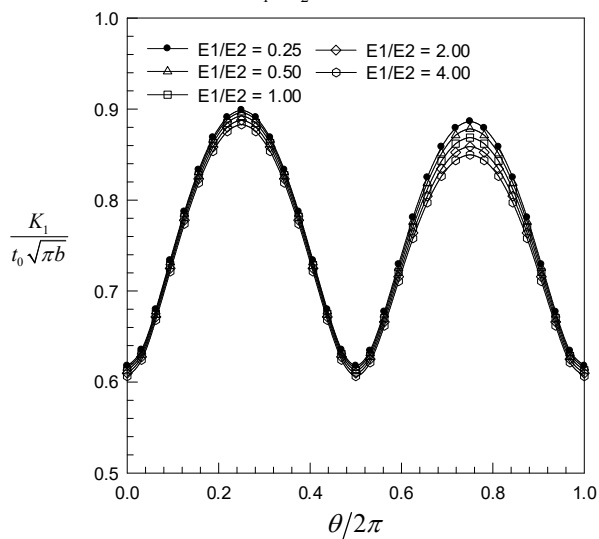


Fig. 11. Normalized mode-I stress intensity factor of elliptical crack embedded in finite cube with $d/b = 4$. Results are obtained from Mesh-3 and reported for various values of E_1/E_2 .

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Calculations of SIFs for Sub-interface Cracks in Bi-Material

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Abstract

This paper presents an efficient numerical procedure for calculating the stress intensity factors of sub-interface cracks in a linearly elastic, bi-material, whole space. A system of governing equations is formulated within the framework of boundary integral equations and its final form possesses two attractive features including the weakly singular property of all involved integrals and no direct contribution of integrals over the material interface. The latter feature stems from that all kernels are derived by directly taking the presence of material interface and the displacement discontinuity into account. In this sense, the jump in the crack-face displacements constitutes only the primary unknowns. A weakly singular symmetric Galerkin boundary element method together with the special near-front approximation is implemented to solve the unknown crack-face data and such information is then used to directly extract the stress intensity along the crack front. A selected set of results is then reported to demonstrate the accuracy, convergence and capability of the proposed numerical technique.

Keywords: BIEM, Bi-material, SGBEM, stress intensity factors, sub-interface cracks.

1. Introduction

Layered materials belong to a well-known class of composites fabricated by combining two or more constituents joining at the material interfaces. These materials have been found in various engineering and industrial applications (e.g., automobiles, aircraft and aerospace structures, dissimilar metal welded joints, pavement structures, etc.) due to their attractive mechanical properties, strength-to-weight ratio, and excellent resistance to severe environmental conditions such as corrosion and chemical attacks. Besides those positive features, such materials are often

prone to damage/failure resulting primarily from the presence of fractures/flaws. For instance, fiber reinforced ceramics which are employed under high temperature conditions are very brittle and, generally, possess relatively low fracture toughness. During applications, cracks may initiate from a brittle zone and then propagate toward the material interface. This finally can lead to ultimate failure of components or loss of their functions. To be capable of avoiding such failure-prone mechanism in the design procedure, the fundamental understanding of fracture behavior of this class of materials is of importance and still requires further investigations.

A classical theory of linear elasticity along with the concept of linear elastic fracture mechanics is commonly used to construct a proper mathematical model for simulating fracture-based failure mechanism for brittle materials [1, 9]. In such theory, the fracture process is postulated to be governed by the dominant stress field in an immediate region surrounding the crack front. Stress intensity factors, known as parameters used to completely describe the dominant stress field along the crack boundary, are commonly used in various single-parameter fracture models to investigate/simulate fracture responses such as crack advances and fatigue growth [11, 23]. For those models to be applicable for various problems under general scenarios, powerful and robust procedures for accurate determination of the stress intensity factors are therefore required. In past several decades, the computational technology for the analysis of cracks in linear elastic bodies with the primary aim to calculate the stress intensity factors has been extensively well-established [2, 7-8, 14, 16-17, 20-22, 28-29]. Unlike cracks in a single-material domain, work related to the modeling of cracks in multi-material domain has been found relatively few and most of existing work was limited mainly to simple crack and domain geometries, loading conditions, and material models such as isotropic and

transversely isotropic elastic materials [4-6, 10, 12, 15, 24, 26]. Recently, Watanavit et al. [xx] implemented a boundary-element-based method to analyze near-interface (or sub-interface) cracks in three-dimensional multi-material bodies. The formulation was established based upon the framework of a finite body and the general material anisotropy. While the technique was found capable of treating near-interface cracks in a quite general context, the computational cost can become significant, when the outer boundary and material interfaces are relatively large, due to the need to treat the unknowns on those surfaces. The influence of the outer boundary of the cracked medium plays an insignificant role on the values of the stress intensity factors when the crack size is relatively small in comparison with the distance to the boundary. Such situation arises naturally for cracks induced very near the material interface. Results from the past study have indicated that the discrepancy of solutions obtained by treating the cracked medium in that circumstance as a finite body (including the outer boundary) and as an infinite domain (ignoring the outer boundary) is insignificant. However, within the context of boundary integral equation methods, the computational cost associated with the treatment of the outer boundary is nontrivial.

The present study aims to offer a boundary-element-based numerical technique that is computationally efficient for handling the stress analysis of near-interface cracks buried deeply from the boundary. The integral formulation is established specifically to handle the remote boundary and, also, integrate the influence of the material interface into the involved fundamental solutions. Such feature renders the discretization of the solution to be carried out only on the crack surface. The procedure is implemented as an in-house computer code and in a sufficiently general framework capable of modeling cracks of arbitrary shapes and subjected to general crack-face loadings.

2. Problem Formulation

Consider a whole space Ω containing an isolated crack as shown schematically in Figure 1. The whole space Ω consists of two half-spaces, denoted by Ω^1 and Ω^2 , adhered perfectly at their flat boundary. Each half-space is made of a homogeneous, isotropic, linear elastic material with the elastic shear modulus and Poisson's ratio denoted by E_1, E_2 and ν_1, ν_2 , respectively. The crack geometry, in the undeformed state, is represented by two geometrically coincident surfaces, denoted by S^+ and S^- , and they are differentiated by their associated outward unit normal vectors $\mathbf{n}^+$ and $\mathbf{n}^-$ satisfying the relation $\mathbf{n}^+ = -\mathbf{n}^-$. In the present study, only near-interface cracks are considered and they are assumed to be

contained in the half-space Ω^1 and possess the piecewise smooth crack surface S^+ (i.e., $\mathbf{n}^+$ is well-defined almost everywhere on S^+ except a region of measure zero). The cracked medium is free of the body force and remote loading and loaded by arbitrarily distributed, self-equilibrated tractions on the crack surface. Cartesian coordinate system $\{x_1, x_2, x_3; O\}$ with the $x_1 - x_2$ plane contained in the material interface and x_3 directing upward to the half-space Ω^1 is selected. With this specific choice of the reference coordinate system, the plane $x_3 = 0$ represents the material interface.

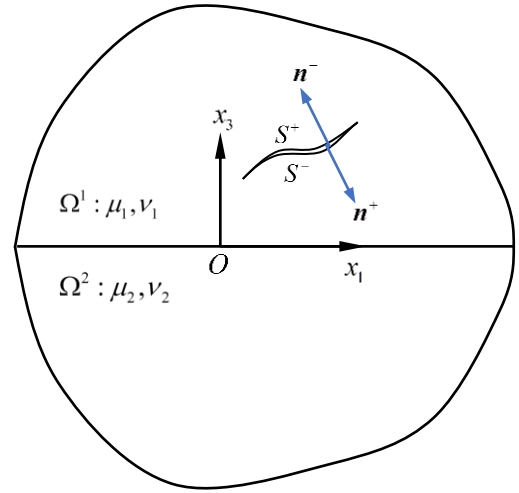


Fig.1. Schematic of bi-material whole space containing crack

2.1 Governing integral equations

Generalizing Somigliana's identity to a bi-material whole space containing an isolated crack shown in Figure 1 yields an integral relation for the displacement at any interior point $\mathbf{x}$

$$u_p(\mathbf{x}) = - \int_{S^+} \bar{S}_{ij}^p(\mathbf{y}, \mathbf{x}) n_i^+(\mathbf{y}) \Delta u_j(\mathbf{y}) dA(\mathbf{y}) \quad (1)$$

where $\bar{S}_{ij}^p(\mathbf{y}, \mathbf{x})$ denotes the stress fundamental solution of an uncracked bi-material whole space that is singular at $\mathbf{y} = \mathbf{x}$ of $\mathcal{O}(1/r^2)$ with $r = \|\mathbf{y} - \mathbf{x}\|$, and $\Delta u_j = u_j^+ - u_j^-$ is the relative crack-face displacement. By employing the integral relation (1) together with the infinitesimal strain-displacement relations and the constitutive law for linear elastic materials, it gives rise to the boundary integral relation for the stress at any interior point $\mathbf{x}$

$$\sigma_{ik}(\mathbf{x}) = \int_{S^+} \bar{\Sigma}_{ij}^{ik}(\mathbf{y}, \mathbf{x}) n_i^+(\mathbf{y}) \Delta u_j(\mathbf{y}) dA(\mathbf{y}) \quad (2)$$

where the two-point function $\bar{\Sigma}_{ij}^{ik}(\mathbf{y}, \mathbf{x})$ is defined by

$$\bar{\Sigma}_{ij}^{ik}(\mathbf{y}, \mathbf{x}) = -E_{lkpq}(\mathbf{x}) \frac{\partial \bar{\Sigma}_{ij}^p(\mathbf{y}, \mathbf{x})}{\partial x_q} \quad (3)$$

with $E_{lkpq}(\mathbf{x})$ denoting the fourth-order elasticity tensor given in terms of the elastic shear modulus and Poisson's ratio of the material at point $\mathbf{x}$. From the singularity behavior of the stress fundamental solution, the function $\bar{\Sigma}_{ij}^{ik}(\mathbf{y}, \mathbf{x})$ is clearly singular at $\mathbf{y} = \mathbf{x}$ of order $1/r^3$. The traction integral equation established directly from (2) contains the hypersingular kernel, and the implementation of the solution procedure of this equation requires nontrivial treatment regarding to the interpretation and evaluation of all involved integrals and the smoothness requirement of crack-face data. To overcome such difficulty, it is appealing to seek an alternatives, singularity-reduced, boundary integral equation. To regularize the boundary integral relation (2), a systematic technique proposed by [12] is utilized. First, the two-point function $\bar{\Sigma}_{ij}^{ik}(\mathbf{y}, \mathbf{x})$ are represented by

$$\bar{\Sigma}_{ij}^{ik}(\mathbf{y}, \mathbf{x}) = \bar{D}_{ijkl}(\mathbf{y}, \mathbf{x}) - \varepsilon_{ism} \frac{\partial}{\partial y_s} \varepsilon_{lrt} \frac{\partial}{\partial x_r} \bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x}) \quad (4)$$

where ε_{ism} denotes a standard alternating symbol; $\bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$ is a weakly singular kernel discussed further below; and $\bar{D}_{ijkl}(\mathbf{y}, \mathbf{x})$ is given by

$$\bar{D}_{ijkl}(\mathbf{y}, \mathbf{x}) = -E_{ijkl}(\mathbf{x})\delta(\mathbf{y} - \mathbf{x}) - \Lambda \delta_{ia} \delta_{ja} E_{abkl}(\mathbf{x})\delta(\mathbf{x} - \bar{\mathbf{y}}) \quad (5)$$

with $\Lambda = (\mu_1 - \mu_2) / (\mu_1 + \mu_2)$, $\delta(\mathbf{y} - \mathbf{x})$ denoting the three-dimensional Dirac-delta function centered at $\mathbf{x}$, δ_{ij} denoting the standard Kronecker-delta symbol, $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ standing for the images of points $\mathbf{x}, \mathbf{y}$ with respect the material interface $x_3 = 0$, respectively. By substituting (4) into (2) and then integrating terms containing the functions $\bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$ by parts via Stokes' theorem, it finally yields the singularity-reduced integral relation for the stress:

$$\sigma_{lk}(\mathbf{x}) = \varepsilon_{lrt} \frac{\partial}{\partial x_r} \int_{S^+} \bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x}) D_m \Delta u_j(\mathbf{y}) dA(\mathbf{y}) \quad (6)$$

where $D_m(\cdot) = \varepsilon_{mjk} n_j \partial(\cdot) / \partial y_k$ denotes a surface differential operator. It is important to remark that the fact that $\bar{D}_{ijkl}(\mathbf{y}, \mathbf{x}) = 0$ for $\mathbf{y} \neq \mathbf{x}$ and Δu_j vanishes along the crack front has been employed to establish the relation (6). By forming the linear combination $\sigma_{lk}(\mathbf{x}) n_l(\xi)$ for any smooth point $\xi \in S^+$ together with the use of (6) and then taking the limit $\mathbf{x} \rightarrow \xi \in S^+$, it leads to an integral equation

$$\frac{1}{2} \Delta t_k(\xi) = D_t \int_{S^+} \bar{C}_{mj}^{ik}(\mathbf{y}, \xi) D_m \Delta u_j(\mathbf{y}) dA(\mathbf{y}) \quad (7)$$

where $\Delta t_k \equiv t_k^+ - t_k^-$ denotes the jump in the crack-face tractions. Finally, by applying the weighted residual technique together with the integration-by-parts procedure, the weak-form version of (7) is given by

$$\begin{aligned} & \frac{1}{2} \int_{S^+} \tilde{v}_k(\xi) \Delta t_k(\xi) dA(\xi) \\ &= - \int_{S^+} D_t \tilde{v}_k(\xi) \int_{S^+} \bar{C}_{mj}^{ik}(\mathbf{y}, \xi) D_m \Delta u_j(\mathbf{y}) dA(\mathbf{y}) dA(\xi) \end{aligned} \quad (8)$$

where $\tilde{v}_k(\xi)$ is a sufficiently smooth test function defined over the crack surface S^+ . Note that the line integral along the crack front resulting from the integration by parts of the term containing the function $\bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$ vanishes due to the fact that the test function $\tilde{v}_k(\xi)$ is chosen to satisfy the closure condition $\tilde{v}_k(\xi) = 0, \xi \in \partial S^+$. It should be evident from the expression of $\bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$ shown below that the weak-form integral equation (8) contains only weakly singular kernels of order $1/r$. This integral equation is utilized as the basis for determining the unknown crack-face displacement.

2.2 Explicit expressions for $\bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$

The two-point function $\bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$ appearing in the special decomposition (4) can be obtained using key results reported in [13] via the use of symmetric and anti-symmetric operations together with the superposition method and fundamental results using Papkovitch-Neuber potentials. From the assumption that the crack surface is embedded entirely in the half-space Ω^1 and the final form of the governing integral equations (8), it only required the expression of $\bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$ only for the case $\mathbf{y}, \mathbf{x} \in \Omega^1$. The explicit expression is given by

$$\begin{aligned} \bar{C}_{mj}^{ik}(\mathbf{y}, \mathbf{x}) &= C_{mj}^{ik}(\mathbf{y} - \mathbf{x}) - \Lambda C_{ab}^{ik}(\mathbf{x} - \bar{\mathbf{y}}) \bar{\delta}_{jb} \bar{\delta}_{ma} \\ &+ \tilde{\Omega}_{mj}^{ik}(\mathbf{y}, \mathbf{x}) + \bar{\Omega}_{mj}^{ik}(\mathbf{y}, \mathbf{x}) \end{aligned} \quad (9)$$

where $C_{mj}^{ik}(\mathbf{y} - \mathbf{x})$ is a kernel for the homogeneous whole space defined by (see more details in [12, 13])

$$\begin{aligned} C_{mj}^{ik}(\mathbf{y} - \mathbf{x}) &= \frac{\mu_1}{4\pi(1-\nu_1)r} \left[(1-\nu_1) \delta_{ik} \delta_{mj} + 2\nu_1 \delta_{km} \delta_{ij} \right] \\ &- \frac{\mu_1}{4\pi(1-\nu_1)r} \left[\delta_{im} \frac{(y_j - x_j)(y_k - x_k)}{r^2} \right] \end{aligned} \quad (10)$$

and the two-point functions $\tilde{\Omega}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$ and $\bar{\Omega}_{mj}^{ik}(\mathbf{y}, \mathbf{x})$ are given explicitly by (also see details in [13])

$$\begin{aligned} \tilde{\Omega}_{mj}^{\alpha\beta}(\mathbf{y}, \mathbf{x}) &= -\varepsilon_{3\alpha\beta} (x_\gamma - y_\gamma) \frac{\partial M_{\gamma mj}}{\partial x_\beta} + \varepsilon_{\alpha 3\lambda} \frac{\partial K_{mj}}{\partial x_\lambda} \\ &+ (1-2\nu_1) \varepsilon_{3\alpha\gamma} M_{\gamma mj} \end{aligned} \quad (11)$$

$$\begin{aligned}\tilde{Q}_{mj}^{\alpha\beta}(\mathbf{y}, \mathbf{x}) = & -\varepsilon_{3\alpha\lambda}(x_\gamma - y_\gamma) \frac{\partial^2 \tilde{M}_{\gamma mj}}{\partial x_\lambda \partial x_\beta} - \varepsilon_{3\beta\alpha} \frac{\partial \tilde{M}_{\gamma mj}}{\partial x_\gamma} + \varepsilon_{\alpha\beta 3} \frac{\partial K_{mj}}{\partial x_3} \\ & + (1 - 2\nu_1) \varepsilon_{3\gamma\lambda} \left(\delta_{\alpha\beta} \frac{\partial}{\partial x_\lambda} - \delta_{\lambda\alpha} \frac{\partial}{\partial x_\gamma} - \delta_{\lambda\beta} \frac{\partial}{\partial x_\alpha} \right) \tilde{M}_{\gamma mj}\end{aligned}\quad (12)$$

$$\tilde{Q}_{mj}^{3k}(\mathbf{y}, \mathbf{x}) = \varepsilon_{3k\lambda} \frac{\partial K_{mj}}{\partial x_\lambda} \quad (13)$$

$$\begin{aligned}\bar{Q}_{mj}^{tk}(\mathbf{y}, \mathbf{x}) = & \varepsilon_{tkb} x_3 \frac{\partial P_{mj}}{\partial x_b} - \varepsilon_{tkb} \frac{\partial Q_{mj}}{\partial x_b} + (1 - 2\nu_1) \varepsilon_{tk3} P_{mj} \\ & + 2(1 - \nu_1) \delta_{3k} \varepsilon_{3\gamma t} \frac{\partial \tilde{P}_{mj}}{\partial x_\gamma}\end{aligned}\quad (14)$$

with all Greek indices ranging from 1 to 2 and repeated indices implying the summation over their range. The functions $M_{\gamma mj}, \tilde{M}_{\gamma mj}, K_{mj}$ appearing in (11)-(13) are given by

$$\begin{aligned}M_{\gamma\alpha\beta} &= M_{\gamma 3\alpha} = 0 \\ M_{\gamma\alpha 3} &= 2A\mu_2 \varpi_1 \varepsilon_{3\alpha\gamma} \bar{h}_1\end{aligned}\quad (15)$$

$$\begin{aligned}M_{\gamma 33} &= A\mu_2 \varpi_1 \varepsilon_{3\lambda\gamma} \frac{\partial \bar{h}_2}{\partial x_\lambda} \\ \tilde{M}_{\gamma\alpha\beta} &= \tilde{M}_{\gamma 3\alpha} = 0 \\ \tilde{M}_{\gamma\alpha 3} &= 2A\mu_2 \varpi_1 \varepsilon_{3\alpha\gamma} \bar{h}_2\end{aligned}\quad (16)$$

$$\begin{aligned}\tilde{M}_{\gamma 33} &= A\mu_2 \varpi_1 \varepsilon_{3\lambda\gamma} \frac{\partial \bar{h}_3}{\partial x_\lambda} \\ K_{\alpha\beta} &= -A\mu_2 \varpi_2 \varepsilon_{3\alpha\lambda} \frac{\partial^2 (\bar{r}^2 \bar{h}_2 - 3x_3 \bar{h}_3 - 3y_3 \bar{h}_3)}{\partial x_\lambda \partial x_\beta} \\ K_{\alpha 3} &= 2A\mu_2 (1 + \Gamma + \kappa_1 + \Gamma \kappa_2) \varpi_1 \varpi_2 \varepsilon_{3\alpha\lambda} \frac{\partial (\bar{r} - x_3 \bar{h}_2)}{\partial x_\lambda} \\ K_{3\alpha} &= K_{33} = 0\end{aligned}\quad (17)$$

where $\kappa_\alpha = 3 - 4\nu_\alpha$, $\Gamma = \mu_2 / \mu_1$, $\varpi_\alpha = 1 / (1 + \Gamma)(1 + \kappa_\alpha)$, $\bar{r} = \sqrt{(y_\alpha - x_\alpha)(y_\alpha + x_\alpha) + (y_3 + x_3)(y_3 + x_3)}$, and

$$\begin{aligned}\bar{h}_1 &= 1 / \bar{r} \\ \bar{h}_2 &= \int \bar{h}_1 dx_3 = \ln(\bar{r} + y_3 + x_3) \\ \bar{h}_3 &= \int \bar{h}_2 dx_3 = (x_3 + y_3) \bar{h}_2 - \bar{r}\end{aligned}\quad (18)$$

The functions $P_{mj}, \tilde{P}_{mj}, Q_{mj}$ in the expression (14) are given by

$$P_{mj} = -\frac{E_{mj} + F_{mj}}{1 + \kappa_1 \Gamma} \quad (19)$$

$$\tilde{P}_{mj} = -\frac{\tilde{E}_{mj} + \tilde{F}_{mj}}{1 + \kappa_1 \Gamma} \quad (20)$$

$$Q_{mj} = \frac{(1 - \kappa_1 \kappa_2) \tilde{E}_{mj} + [\kappa_2 (1 - \kappa_1) + \kappa_1 (\kappa_2 - 1) \Gamma] \tilde{F}_{mj}}{2(1 + \kappa_1 \Gamma)(\kappa_2 + \Gamma)} \quad (21)$$

where the functions $E_{mj}, \tilde{E}_{mj}, F_{mj}, \tilde{F}_{mj}$ and all involved constants are defined by

$$\begin{aligned}E_{\alpha\beta} &= -2H_1 \mu_2 \varepsilon_{3\alpha\lambda} \frac{\partial^2}{\partial x_\lambda \partial x_\beta} (\bar{r} - x_3 \bar{h}_2) - 2H_2 \mu_2 \varepsilon_{\alpha\beta 3} \frac{1}{\bar{r}} \\ &\quad - 2H_3 \mu_2 \varepsilon_{3\alpha\lambda} \frac{\partial^2}{\partial x_\lambda \partial x_\beta} (y_3 \bar{h}_2)\end{aligned}\quad (22)$$

$$\begin{aligned}E_{\alpha 3} &= -2H_4 \mu_2 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} \bar{h}_2 - H_5 \mu_2 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} \left(\frac{y_3}{\bar{r}} \right) \\ E_{3j} &= 0 \\ \tilde{E}_{\alpha\beta} &= -2H_1 \mu_2 \varepsilon_{3\alpha\lambda} \frac{\partial^2}{\partial x_\lambda \partial x_\beta} (y_3 \bar{h}_3 - \bar{h}_4) - 2H_2 \mu_2 \varepsilon_{\alpha\beta 3} \bar{h}_2 \\ \tilde{E}_{\alpha 3} &= -2H_4 \mu_2 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} \bar{h}_3 - H_5 \mu_2 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} (y_3 \bar{h}_2) \\ \tilde{E}_{3j} &= 0\end{aligned}\quad (23)$$

$$\begin{aligned}F_{\alpha\beta} &= -I_1 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial^2}{\partial x_\gamma \partial x_\beta} (y_3 \bar{h}_2) + I_2 \mu_1 \varepsilon_{\alpha\beta 3} \left(\frac{1}{\bar{r}} \right) \\ &\quad - I_3 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial^2}{\partial x_\gamma \partial x_\beta} \bar{h}_3\end{aligned}\quad (24)$$

$$\begin{aligned}F_{\alpha 3} &= -I_4 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} \bar{h}_2 + I_5 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} \left(\frac{y_3}{\bar{r}} \right) \\ F_{3\alpha} &= -I_6 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} \bar{h}_2; \quad F_{33} = 0 \\ \tilde{F}_{\alpha\beta} &= -I_1 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial^2}{\partial x_\gamma \partial x_\beta} (y_3 \bar{h}_3) + I_2 \mu_1 \varepsilon_{\alpha\beta 3} \bar{h}_2 \\ &\quad - I_3 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial^2}{\partial x_\gamma \partial x_\beta} \bar{h}_4\end{aligned}\quad (25)$$

$$\begin{aligned}\tilde{F}_{\alpha 3} &= -I_4 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} \bar{h}_3 + I_5 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} (y_3 \bar{h}_2) \\ \tilde{F}_{3\alpha} &= -I_6 \mu_1 \varepsilon_{3\gamma\alpha} \frac{\partial}{\partial x_\gamma} \bar{h}_3; \quad \tilde{F}_{33} = 0 \\ H_1 &= -E - 2A\varpi_2 (1 - \Gamma) \\ H_2 &= D - A \\ H_3 &= -2A\varpi_2 (3 + \Gamma) \\ H_4 &= -F + A(\Gamma - 1)(1 + \Gamma + \kappa_1 + \kappa_2 \Gamma) \varpi_1 \varpi_2 \\ H_5 &= E\end{aligned}\quad (26)$$

$$\begin{aligned}I_1 &= -2E + 8A\Gamma(3 + \Gamma) \varpi_2 \\ I_2 &= -2A\Gamma \\ I_3 &= -2A\Gamma[2\varpi_2(3 + \Gamma) + \frac{\kappa_2 - 3}{\kappa_2 + 1}] \\ I_4 &= -2E + 2A\Gamma[(1 + 2\kappa_2 + \kappa_1 \kappa_2)\Gamma + (1 + 2\kappa_1 + \kappa_1 \kappa_2)] \varpi_1 \varpi_2 \\ I_5 &= -2E \\ I_6 &= H\end{aligned}\quad (27)$$

$$\begin{aligned}A &= \frac{(\kappa_2 - \kappa_1)}{2\pi(\kappa_1 + 1)(1 + \Gamma)}; \quad D = \frac{(\kappa_2 - 3) - \Gamma(\kappa_1 - 3)}{2\pi(1 + \Gamma)(1 + \kappa_1)}; \\ E &= \frac{(1 - \Gamma)}{\pi(1 + \Gamma)(1 + \kappa_1)}; \quad F = \frac{(\kappa_2 + 1) - \Gamma(\kappa_1 + 1)}{2\pi(1 + \Gamma)(1 + \kappa_1)}; \\ H &= \frac{\Gamma(\kappa_2 - 3) - (\kappa_1 - 3)}{\pi(1 + \Gamma)(1 + \kappa_1)}\end{aligned}\quad (28)$$

2.3 Near-front displacement and stress fields

Following the work of [14-17], the near-front strain and stress fields can be expressed as

$$\begin{aligned} \bar{\varepsilon}_{ij}(\mathbf{x}_c; \bar{r}, \bar{\theta}) &= \frac{1}{\sqrt{2\pi\bar{r}}} \sum_{n=1}^3 K_n(\mathbf{x}_c) \bar{\varepsilon}_{ij}^{K(n)}(\bar{\theta}) \\ &+ \bar{\varepsilon}_{ij}^{finite}(\mathbf{x}_c) + \sum_{m=1}^{\infty} \bar{r}^{m/2} \bar{\varepsilon}_{ij}^m(\mathbf{x}_c; \bar{\theta}) \end{aligned} \quad (29)$$

$$\begin{aligned} \bar{\sigma}_{ij}(\mathbf{x}_c; \bar{r}, \bar{\theta}) &= \frac{1}{\sqrt{2\pi\bar{r}}} \sum_{n=1}^3 K_n(\mathbf{x}_c) \bar{\sigma}_{ij}^{K(n)}(\bar{\theta}) \\ &+ \bar{\sigma}_{ij}^{finite}(\mathbf{x}_c) + \sum_{m=1}^{\infty} \bar{r}^{m/2} \bar{\sigma}_{ij}^m(\mathbf{x}_c; \bar{\theta}) \end{aligned} \quad (30)$$

where $\mathbf{x}_c$ denotes a point along the crack front as indicated in Figure 2; $\{\mathbf{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ is the local reference coordinate system with the origin at $\mathbf{x}_c$, the $\bar{x}_1 - \bar{x}_2$ plane perpendicular to the crack front at $\mathbf{x}_c$, and the $\bar{x}_1 - \bar{x}_3$ plane tangent to the crack surface at $\mathbf{x}_c$; $\bar{\varepsilon}_{ij}(\mathbf{x}_c; \bar{r}, \bar{\theta})$ and $\bar{\sigma}_{ij}(\mathbf{x}_c; \bar{r}, \bar{\theta})$ are components of strain and stress in the local $\bar{x}_1 - \bar{x}_2$ plane; $(\bar{r}, \bar{\theta})$ denote the local polar coordinates of point on the local $\bar{x}_1 - \bar{x}_2$ plane; $K_1(\mathbf{x}_c), K_2(\mathbf{x}_c), K_3(\mathbf{x}_c)$ are stress intensity factors at $\mathbf{x}_c$; $\bar{\varepsilon}_{ij}^{K(n)}, \bar{\varepsilon}_{ij}^m, \bar{\sigma}_{ij}^{K(n)}$ and $\bar{\sigma}_{ij}^m$ are functions independent of $\bar{r}$; and $\bar{\varepsilon}_{ij}^{finite}$ and $\bar{\sigma}_{ij}^{finite}$ are functions independent of both $\bar{r}, \bar{\theta}$. An alternative expression for the stress intensity factors in terms of the near front relative crack-face displacement was constructed by [18] and [19] in a form

$$K_i(\mathbf{x}_c) = \frac{\sqrt{2\pi}}{4} B_{ij} \lim_{\bar{x}_1 \rightarrow 0^+} \bar{x}_1 \rightarrow \mathbf{x}_c \left(\frac{\Delta u_j(\bar{\mathbf{x}})}{\sqrt{-\bar{x}_1}} \right) \quad (31)$$

where B_{ij} is given, for isotropic materials, by

$$B_{ij} = \frac{\mu_1}{1-\nu_1} (\delta_{k1}\delta_{i1} + \delta_{k2}\delta_{i2}) + \mu_1\delta_{k3}\delta_{i3} \quad (32)$$

The formula (31) allows the stress intensity factors to be extracted directly from the relative crack-face displacement in the vicinity of the crack front and such data is the primary unknown to be determined from the traction boundary integral equation (8).

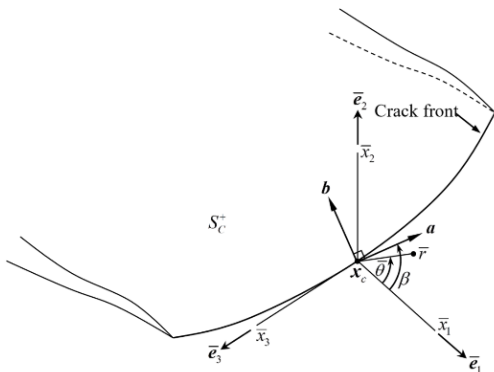


Fig.2. Geometry of crack front and local coordinate system used in calculating stress intensity factors

3. Solution Procedure

To construct an approximate solution of the weak-form traction boundary integral equation (8), a weakly singular, symmetric Galerkin boundary element method (SGBEM) similar to that employed by [13] and [20] is utilized. First, the unknown relative crack-face displacement Δu_j and the test function $\tilde{v}_k$ are approximated by

$$\Delta u_j(\mathbf{y}) = \sum_{p=1}^N u_j^{\Delta(p)} \phi^{(p)}(\mathbf{y}); \quad \tilde{v}_k(\boldsymbol{\zeta}) = \sum_{p=1}^N \tilde{v}_k^{(p)} \phi^{(p)}(\boldsymbol{\zeta}) \quad (33)$$

where N is the number of nodes in the finite element mesh representing the discretization of the crack surface S^+ ; $\phi^{(p)}$ is the nodal basis function of the p^{th} node in a finite element, $u_j^{\Delta(p)}$ denote nodal degrees of freedom of the p^{th} node; and $\tilde{v}_k^{(p)}$ are arbitrary constants of the p^{th} node. To enhance the approximation of the near-front relative crack-face displacement, special crack-tip elements developed by [13] and [20] are utilized in a region adjacent to the crack front. With the use of these crack-tip elements, the square-root behavior of Δu_j in the neighborhood of the crack front can be captured accurately using elements of relatively large size. In addition, extra degrees of freedom located along the crack front are directly related to the stress intensity factors allowing such fracture data to be extracted conveniently from those extra degrees of freedom. By substituting (33) into (8) leads to a system of linear algebraic equations:

$$\mathbf{CAU} = \mathbf{T} \quad (34)$$

where entries of the coefficient matrix $\mathbf{C}$, the unknown vector $\Delta \mathbf{U}$, and the known load vector $\mathbf{T}$ are given by

$$\begin{aligned} [\mathbf{C}]_{3(p-1)+i, 3(q-1)+j} &= \int_{S^+} D_i \phi^{(p)}(\mathbf{y}) \int_{S^+} \bar{C}_{mj}^{ii}(\boldsymbol{\zeta}, \mathbf{y}) D_m \phi^{(q)}(\boldsymbol{\zeta}) dA(\boldsymbol{\zeta}) dA(\mathbf{y}) \end{aligned} \quad (35)$$

$$[\mathbf{T}]_{3(p-1)+i} = -\frac{1}{2} \int_{S^+} \phi^{(p)}(\mathbf{y}) \Delta t_i(\mathbf{y}) dA(\mathbf{y}) \quad (36)$$

To efficiently construct all entries of $\mathbf{C}$ and $\mathbf{T}$, a standard assembly procedure together with computations performed over the finite element mesh is adopted. Also, special integral quadrature rules similar to that employed by [13] and [20] are utilized to handle both weakly singular and nearly singular double surface integrals over pairs of elements resulting from the discretization. It should be remarked that the coefficient matrix

$\mathbf{C}$ is essentially symmetric due to the symmetry of the kernel $\bar{C}_{mj}^{ti}(\xi, y)$. Once the unknown vector $\Delta \mathbf{U}$ is solved from (34) via a selected efficient linear solver, the relative crack-face displacement Δu_j over the whole crack surface can be obtained from the interpolation (33).

With the use of special 9-node crack-tip elements together with the relationship (31) and Taylor series expansion, an explicit formula for computing the stress intensity factors at any point $\mathbf{x}_c$ along the crack front can be obtained as

$$K_i(\mathbf{x}_c) = \sqrt{\frac{\pi}{2\gamma}} B_{ij} \mathbf{u}^*(\mathbf{x}_c) \cdot \bar{\mathbf{e}}_j \quad (34)$$

where

$$\gamma = -\bar{\mathbf{e}}_1 \cdot \frac{\partial \mathbf{r}_c(\xi_c, -1)}{\partial \eta} \quad (35)$$

$$\mathbf{r}_c = \sum_{i=1}^9 \mathbf{x}^{(i)} \varphi^{(i)}(\xi, \eta) - \mathbf{x}_c \quad (36)$$

$$\mathbf{u}^*(\mathbf{x}_c) = \sum \mathbf{u}^{\Lambda(i)} \varphi^{(i)}(\xi_c, -1) \quad (37)$$

in which $\mathbf{x}^{(i)}$ and $\varphi^{(i)}$ denote the physical coordinates and standard shape function of the i^{th} -node of 9-node elements; $(\xi, \eta) \in [-1, 1] \times [-1, 1]$ are natural coordinates of a master element; $(\xi_c, -1)$ are natural coordinates of $\mathbf{x}_c$ in the master element; and $\mathbf{u}^{\Lambda(i)}$ are extra degrees of freedom at the i^{th} -node along the crack front. Note that the summation appearing on the right hand side of (37) is taken over only nodes of the special crack-tip element located along the crack front.

4. Numerical Results

In this section, preliminary results for a representative problem are reported and compared with existing benchmark solutions to demonstrate the accuracy and convergence of numerical solutions generated by the proposed technique.

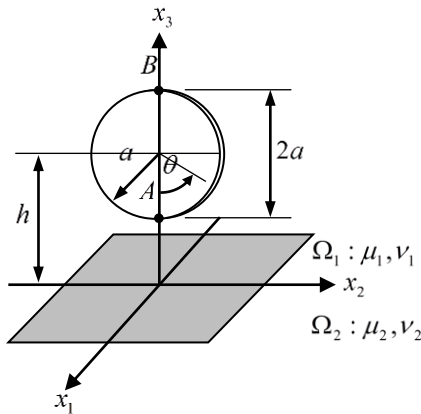


Fig.3. A vertical penny-shaped crack embedded in a bi-material whole space

Consider a penny-shaped crack of radius a embedded in a bi-material whole space as shown in Figure 3. The crack is oriented vertically such that its crack front can be parametrized by

$$x_1 = 0, x_2 = a \sin \theta, x_3 = h - a \cos \theta \quad (38)$$

where $\theta \in [0, 2\pi]$. The body is free of the body force and loaded by a uniformly distributed, self-equilibrated normal traction t_0 . To examine the convergence of numerical solutions, three meshes are adopted in the analysis as shown in Figure 4.

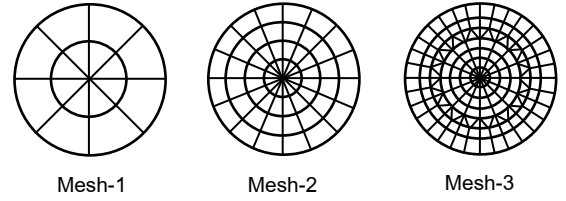


Fig.4. Three meshes of penny-shaped crack used in analysis

The analysis is carried out first for a special case when the whole space is homogeneous (by setting $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$) and obtained results are then compared with the exact solution reported by [21]. It has been found that the stress intensity factors obtained from the three meshes show excellent agreement with the reference solution; in particular, the errors are only a fraction of one percent. To allow the comparison with available benchmark solutions reported for the case of bi-materials (e.g., [7]), Poisson's ratio of the two materials are taken to be identical to 0.3 (i.e., $\nu_1 = \nu_2 = 0.3$). The computed mode-I stress intensity factors are normalized and then reported in Figure 5 for $\mu_2/\mu_1 = 0.5$ and in Figure 6 for $\mu_2/\mu_1 = 2$ as a function of normalized angular position θ/π and for $h/b \in \{1.2, 1.6, 2\}$. Note that values of the stress intensity factor taken from the work of [7] are associated with points A ($\theta = 0^\circ$) and B ($\theta = 180^\circ$).

It is evident from these results that the computed stress intensity factors along the crack front converge as the level of discretization is refined and the good agreement between the converged and benchmark solutions is observed. In particular, relatively coarse meshes with only few degrees of freedom to discretize the crack surface can also provide reasonably accurate results. The good quality of numerical solutions results directly from the use of special crack-tip elements to enhance the approximation of the near-front field. Obtained results also indicate that when the crack is contained in a stiffer material ($\mu_2/\mu_1 = 0.5$), the stress intensity increases monotonically from

the minimum value at point B (farthest from the material interface) to the maximum value at point A (closest to the material interface) and the reverse trend is observed for $\mu_2/\mu_1 = 2$. In addition, the discrepancy between the maximum and minimum stress intensity factors and its rapid variation in the neighborhood of point A becomes more significant as the crack approaches the material interface.

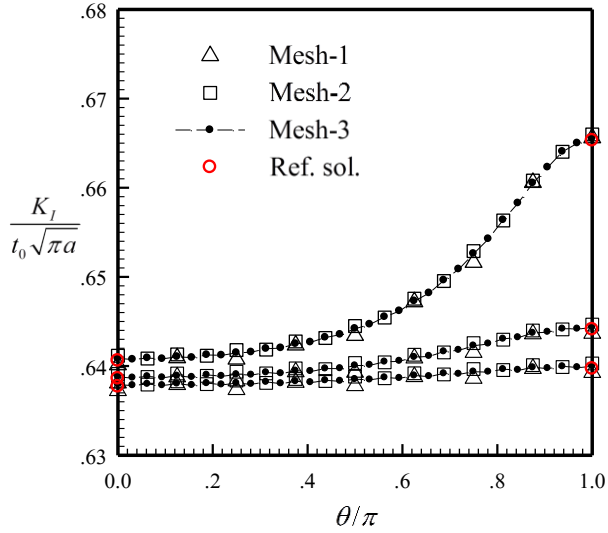


Fig.5. Normalized mode-I stress intensity factor for penny-shaped crack in bi-material whole space under uniform normal traction t_0 . Results are reported for $\mu_2/\mu_1 = 0.5$ and $h/b \in \{1.2, 1.6, 2\}$

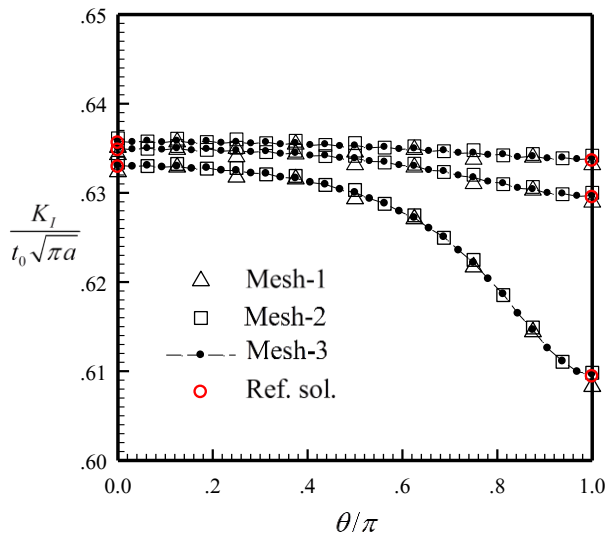


Fig.6. Normalized mode-I stress intensity factor for penny-shaped crack in bi-material whole space under uniform normal traction t_0 . Results are reported for $\mu_2/\mu_1 = 2$ and $h/b \in \{1.2, 1.6, 2\}$

5. Conclusions

An efficient and robust numerical technique, based upon the weakly singular boundary integral equation method, has been successfully implemented for the analysis of near-interface cracks in an isotropic, linear elastic bi-material whole space. The formulation has been established such that the governing integral equation involves only integrals over the crack surface with the contribution of the material contrast being integrated directly into the fundamental solutions. Besides the efficiency gained directly from the reduction of the spatial discretization, the accuracy of the proposed technique has been further enhanced by exploiting special interpolations along the crack front to capture the dominant near-front field. Preliminary results from the present study have indicated that the integration of this type of elements in the solution discretization significantly reduces the requirement of exploiting very fine mesh to accurately capture the near-front field. In addition, it also provides the direct means to post-process for the stress intensity factors from the relative crack-face displacement in the vicinity of the crack front. Obtained results also indicate the important role of the material contrast and the distance from the crack to the material interface on the value and the distribution of the stress intensity factor along the crack front.

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Size-dependence Elastic Responses of Layered Media under Surface Loading

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Abstract

This paper presents fundamental solutions of a two-dimensional, elastic layered medium under surface loadings by taking the influence of material microstructure into account. An underlying mathematical model for simulating such small-scale influence is established within the continuum-based framework via the well-known consistent couple stress theory. For each material layer, the generalized Navier's equation governing the displacement field is established and the method of Fourier integral transform is applied to derive its general solution in a transformed space. A set of boundary conditions and the continuity of fields along the material interfaces are enforced to obtain a system of linear algebraic equations governing all unknown degrees of freedom of the whole layered medium in the transformed space. An efficient quadrature is then adopted to carry out all involved integrals arising from Fourier integral transform inversion. A selected set of results is also reported not only to confirm the validity of established solutions but also to demonstrate the capability of the selected mathematical model to simulate the size-dependency when the external and internal length scales are comparable.

Keywords: Couple Stresses, Integral Transform, Layered Media, Size Dependence, Surface Loading

1. Introduction

Nowadays, micro- and nano-technologies have played a crucial role in various disciplines including natural and medical sciences, engineering, and modern industries (e.g., [1-4]). In the past several decades, small-scale materials (e.g., carbon nanotubes, lithium ion, nano-clusters, nano-crystals, etc.) and various tiny devices (e.g., MEMS, NEMS, micro- and nano- sensors, actuators, chips, etc.) have attracted significant interest due to their attractive and desirable mechanical, electrical and other

related properties (e.g., [5-8]). Particularly, at those tiny scales, their properties have been found significantly different from those at a macro scale, and clearly become size-dependent (e.g., [9]). According to the increasing applications of such technologies, extensive researches have been conducted continuously by various investigators with the primary objective to supply sufficient fundamental knowledge and the in-depth insight into those problems of interest. In particular, the investigation of micro/nano-scale objects to thoroughly interpret their fundamental properties and size-dependent behaviors has become an increasing and central interest of numerous studies (e.g., [10-14]).

While the modeling framework and corresponding solution methodology for macroscopic problems in applied mechanics have been well established and quite matured based upon the current state of the art, those established to handling problems in micro/nano scales are relatively young and this leaves a plenty of rooms for researchers to further explore/improve/enhance existing accomplishments. The issue of finding proper, physically admissible mathematical models to better capture/mimic/predict physical phenomena in such small scales with an acceptable level of accuracy is still a challenging and essential task. Also, the development of solution techniques to accurately/efficiently construct solutions of selected mathematical models is also considered fundamental as to accommodate/facilitate/enhance their applications to a broad/variety of scenarios, suitable for limited available computational environments, especially when a complex/large set of data is involved.

Nano-scale/nano-structured materials have been well recognized in surface coating applications. Both single-layer and multi-layer surface coatings have been commonly utilized to enhance the surface properties and overall performance of components. For instance, a surface coated by nano-layers of

carbon nanotubes makes it possible to absorb light waves, to reduce glare on the computer screen and even possibly make an object seem invisible. A reflective color filter device made from photonic nano-structure can be used to replace the conventional color filter on the LCD screen in order to harvest the wasted energy of the absorbed light into electricity. Likewise, a compact multilayer film structures can be applied to solar-thermal harvesting, thermoelectric detection and imaging due to its possibility in ultra-broadband and perfect omnidirectional absorption. The surface coating is also required for quantum dots (QDs) in order to improve or strengthen their performance regarding the fluorescence property and long-term stability. The nano-sized, complex metal boron-carbide (CMBC) has also been found in the application of surface coating for steel substrates to significantly enhance their surface wear resistance. Due to the vast applications of nano-scale surface coating, the fundamental understanding and the capability to assess and predict the performance of the surface after coating is essential and provide the basic information in the design procedure.

The existence of couple stresses, which was suggested to deal mathematically with the influence of micro-/nano-structure present in materials, was originally introduced in the continuum representations by Voigt [15]. This work can be considered as a natural extension of Cauchy continuum mechanics. However, a complete mathematical theory for modeling microstructure-dependent behaviors of materials was still not proposed until Cosserat and Cosserat [16] developed their theory. In such development, the interaction within the body is generally considered through the true force stress and the pseudo couple stress tensors. In this sense, each material element is considered subjected not only to the normal and tangential forces on their surface but also the couples. The corresponding consequence of integrating such the couple stresses into the system renders the loss of symmetry of the force stress tensor and, in addition, requiring the reconstruction of the force equilibrium equations as well as those for couple stresses. In their theory, both the displacement and the material micro-rotation are considered independent kinematical quantities. This causes the increase in the number of degrees of freedom from three to six and, as a result, renders the theory itself much more complicated than the one without the couple stresses. Moreover, the micro-rotations defined in Cosserat theory are inconsistent, independent, artificial, continuous micro-rotation (e.g. [17]) that cannot represent mathematically the effects which are caused by the discontinuity of microstructure in materials (e.g., [18]). The theory, therefore, did not receive much attention at that period.

Recently, Hadjefandiari and Dargush [18] revisited the original Cosserat theory and then proposed a consistent couple stress theory which can resolve all of the inconsistencies present in the original one. By purely kinematical considerations, this development involves only true kinematical quantities without adding any artificial degrees of freedom. The skew-symmetric part of the rotation gradient, termed the mean curvature tensor, was suggested to be a new deformation measure being compatible with the couple stress tensor. From this point of view, the curvature tensor becomes consistent and proper because of the vanishing of its spherical part. Another contribution of the proposed theory also indicated that the couple stress tensor must be skew-symmetric to be compatible with prescribed boundary conditions. Generally, such discoveries described above allow one to overcome the indeterminacy of the couple stress tensor and issues related to boundary conditions corresponding to the normal component of the couple traction. Furthermore, through the principle of virtual work, it was clarified that the body couple can be equivalently represented in terms of the body forces and surface tractions. This finding eliminates the inconsistency of constitutive relation for the force stress tensor in the original theory due to the presence of the body couple. It should be noted that the theory possesses only one extra material constant which can account for the effect due to the presence of couple stresses, called the couple stress effect [18].

The application of couple stress theory to investigate the size-dependent behavior of elastic medium under surface excitations was recognized in the work of Muki and Sternberg [19]. In their study, the half-plane subjected to the surface loading and indented by a flat-ended, rigid, frictionless punch were investigated. After the pioneer work of Muki and Sternberg [19], no significant progress has been recognized in the literature until the appearance of the work by Zisis et al. [20]. In their study, the plane-strain half-plane indented by a rigid, frictionless indenter with three different profiles was solved using a standard singular integral equation technique. Later, Gourgiotis and Zisis [12] developed a more general solution procedure based upon Green's function technique to reexamine a half-plane indented by punch of various profiles by taking into account the presence of couple stresses. Recently, Karuriya and Bhandakkar [21] followed the solution scheme proposed by Zisis et al. [20] to develop solutions of a plane-strain, finite thickness bonded layer indented by rigid, frictionless punches. Later, Song et al. [14] investigated the size-dependent response, via the couple stress theory, of a coated half-plane indented by a rigid, frictionless indenter of flat, parabolic, cylindrical, and wedge profiles. The treatment of more practical contact conditions over the elastic

half-plane was also considered, for instance, the sliding frictional contact by Song et al. [13] and the partial slip contact by Wang et al. [22]. It should be remarked that results from studies mentioned above clearly confirmed the strong dependence of mechanical responses on the characteristic material length scale induced by the influence of material micro-/nano-structure and the significant departure from solutions predicted by classical, size-independent continuum theories. However, most of existing studies were carried out within the limited context of two-dimensional settings associated with only the half-plane or single layer media. The extension of existing work to treat more practical scenarios such as multi-layered media is considered challenging and is the main focus of the present study.

2. Problem Formulation

Consider a two-dimensional, infinite, elastic layer as shown schematically in Figure 1. The layer is perfectly bonded to the rigid base and possesses a constant thickness. A two-dimensional, Cartesian reference coordinate system $\{x, y; O\}$ is chosen such that the origin O is located at the top surface; the x -axis directs along the infinite direction of the layer; and the y -axis directs downward. The top surface is subjected to arbitrarily distributed normal traction p_0 , arbitrarily distributed shear traction q_0 , and arbitrarily distributed couple traction m_0 over the region $x \in [-a, a]$. The layer is made of a homogeneous, isotropic, linearly elastic material and subjected to zero body force and body couple. The thickness of the medium in the direction perpendicular to the x - y plane is assumed significantly large such that the plane-strain condition prevails.

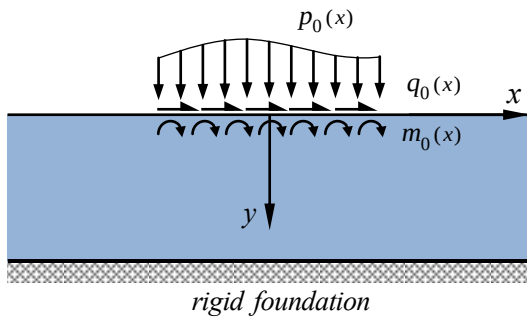


Fig.1. Schematic of two-dimensional, infinite, elastic layer rested on rigid foundation and subjected to surface loading

2.1 Basic field equations

A set of basic field equations for a body subjected to a plane-strain condition can be obtained from the consistent couple stress theory. In the absence of the body force and couple, the force and moment equilibrium equations are given explicitly by

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \quad (1)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (2)$$

$$\frac{\partial \mu_{xz}}{\partial x} + \frac{\partial \mu_{yz}}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0 \quad (3)$$

where $\{S_{xx}, S_{xy}, S_{yx}, S_{yy}\}$ are in-plane components of the force-stress tensor and $\{m_{xz}, m_{yz}\}$ are non-zero components of the couple stress tensor defined based on the coordinate system $\{x, y; O\}$. The in-plane displacements u_x and u_y , the rotation normal to the x - y plane w_z , the infinitesimal in-plane strain components $\{\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}\}$ and the curvatures κ_{xz} and κ_{yz} are related by the following linearized kinematics:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \quad (4)$$

$$\varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \quad (5)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \quad (6)$$

$$\omega_z = \frac{1}{2} \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \quad (7)$$

$$\kappa_{yz} = -\frac{1}{2} \frac{\partial \omega_z}{\partial y} \quad (8)$$

$$\kappa_{xz} = -\frac{1}{2} \frac{\partial \omega_z}{\partial x} \quad (9)$$

The constitutive relations relating the force-stresses, the couple-stresses, the infinitesimal strain, and the rotation take the form

$$\mu_{yz} = -8\eta\kappa_{yz} = 4\eta \frac{\partial \omega_z}{\partial y} \quad (10)$$

$$\mu_{xz} = -8\eta\kappa_{xz} = 4\eta \frac{\partial \omega_z}{\partial x} \quad (11)$$

$$\sigma_{xx} = \frac{2\mu}{1-2\nu} [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}] \quad (12)$$

$$\sigma_{yy} = \frac{2\mu}{1-2\nu} [\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy}] \quad (13)$$

$$\sigma_{xy} = 2\mu\varepsilon_{xy} - 2\eta\Delta\omega_z \quad (14)$$

$$\sigma_{yx} = 2\mu\varepsilon_{xy} + 2\eta\Delta\omega_z \quad (15)$$

where μ and η are Lamé constants defined in the same fashion as that in the size-independent continuum mechanics, η denotes the material constant accounting for the presence of

couple stresses, $\mathbf{D}$ denotes the two-dimensional Laplacian operator, and η is the Poisson's ratio. It is worth noting that h is an additional material parameter responsible for the length-scale effect and, if this constant is set to zero, the couple stress theory simply reduces to classical linear elasticity.

2.2 Governing equations

By substituting (4)-(9) into (10)-(15), it leads to the expressions of the force stresses and couple stresses in terms of the in-plane displacements u_x and u_y

$$\sigma_{xx} = \frac{2\mu}{1-2\nu} \left((1-\nu) \frac{\partial u_x}{\partial x} + \nu \frac{\partial u_y}{\partial y} \right) \quad (16)$$

$$\sigma_{yy} = \frac{2\mu}{1-2\nu} \left(\nu \frac{\partial u_x}{\partial x} + (1-\nu) \frac{\partial u_y}{\partial y} \right) \quad (17)$$

$$\sigma_{xy} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) - \eta \Delta \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \quad (18)$$

$$\sigma_{yx} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \eta \Delta \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \quad (19)$$

$$\mu_{xz} = 2\eta \left(\frac{\partial^2 u_y}{\partial x^2} - \frac{\partial^2 u_x}{\partial x \partial y} \right) \quad (20)$$

$$\mu_{yz} = 2\eta \left(\frac{\partial^2 u_y}{\partial x \partial y} - \frac{\partial^2 u_x}{\partial y^2} \right) \quad (21)$$

Upon substituting (16)-(21) into the force equilibrium equations (1) and (2), it gives rise to a pair of equilibrium equations in terms of displacements

$$\begin{aligned} \frac{\partial}{\partial x} \left[\frac{2\mu}{1-2\nu} \left((1-\nu) \frac{\partial u_x}{\partial x} + \nu \frac{\partial u_y}{\partial y} \right) \right] \\ + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + \eta \Delta \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \right] = 0 \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) - \eta \Delta \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \right] \\ + \frac{\partial}{\partial y} \left[\frac{2\mu}{1-2\nu} \left(\nu \frac{\partial u_x}{\partial x} + (1-\nu) \frac{\partial u_y}{\partial y} \right) \right] = 0 \end{aligned} \quad (23)$$

where terms associated with the Laplacian operator Δ can be further expanded to obtain

$$\eta \Delta \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) = \eta \left(\frac{\partial^3 u_y}{\partial x^3} - \frac{\partial^3 u_x}{\partial x^2 \partial y} + \frac{\partial^3 u_y}{\partial y^2 \partial x} - \frac{\partial^3 u_x}{\partial y^3} \right) \quad (24)$$

It is important to remark that the moment equilibrium equation (3) is not employed here as the key governing equation but it

provides the sufficient conditions for determining the skew-symmetric part of the force-stress tensor in terms of the couple-stress tensor. By substituting (24) into (22) and (23), it leads to a two-dimensional version of Navier's equations for the consistent couple stress theory

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (25)$$

where $D_{11}, D_{12}, D_{21}, D_{22}$ are linear differential operators defined by

$$D_{11} = \alpha \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - l^2 \left(\frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4} \right) \quad (26)$$

$$D_{12} = D_{21} = \beta \frac{\partial^2}{\partial x \partial y} + l^2 \left(\frac{\partial^4}{\partial x^3 \partial y} + \frac{\partial^4}{\partial y^3 \partial x} \right) \quad (27)$$

$$D_{22} = \frac{\partial^2}{\partial x^2} + \alpha \frac{\partial^2}{\partial y^2} - l^2 \left(\frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial x^4} \right) \quad (28)$$

with $\alpha = 2(1-\nu)/(1-2\nu)$, $\beta = 1/(1-2\nu)$ and $l^2 = \eta/\mu$.

3. Solution Procedures

To determine the general solution of (25) for an infinite elastic layer occupying a region $x \in (-\infty, \infty)$, $y \in [0, h]$, it is appealing to employ the method of Fourier integral transform [23]. Fourier integral transform of any function $f = f(x, y)$ defined over the region $x \in (-\infty, \infty)$, $y \in [0, h]$ and its inverse with respect to the coordinate x are defined by

$$\tilde{f}(\xi, y) = \int_{-\infty}^{\infty} f(x, y) e^{i\xi x} dx \quad (29)$$

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\xi, y) e^{-i\xi x} d\xi \quad (30)$$

By applying Fourier integral transform with respect to the coordinate x to the system (25), it leads to the homogeneous system of linear, ordinary differential equations with respect to the coordinate y :

$$\begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \begin{Bmatrix} \tilde{u}_x \\ \tilde{u}_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (31)$$

where $\tilde{D}_{11}, \tilde{D}_{12}, \tilde{D}_{21}, \tilde{D}_{22}$ are defined by

$$\tilde{D}_{11} = -\alpha \xi^2 + (1 + l^2 \xi^2) \frac{\partial^2}{\partial y^2} - l^2 \frac{\partial^4}{\partial y^4} \quad (32)$$

$$\tilde{D}_{12} = \tilde{D}_{21} = -i\xi \left[(\beta - l^2 \xi^2) \frac{\partial}{\partial y} + l^2 \frac{\partial^3}{\partial y^3} \right] \quad (33)$$

$$\tilde{D}_{22} = -\xi^2 (1 + l^2 \xi^2) + (\alpha + l^2 \xi^2) \frac{\partial^2}{\partial y^2} \quad (34)$$

The general solution of (31) can be obtained explicitly as

$$\begin{aligned} \tilde{u}_x = & -i|\xi| e^{-|\xi|(h-y)} C_1 - i(\kappa + |\xi|y) e^{-|\xi|(h-y)} C_2 \\ & + i|\xi| e^{-|\xi|y} C_3 - i(\kappa - |\xi|y) e^{-|\xi|y} C_4 \\ & - \frac{i\xi}{l} e^{-\xi(h-y)/l} C_5 + \frac{i\xi}{l} e^{-\xi y/l} C_6 \end{aligned} \quad (35)$$

$$\begin{aligned} \tilde{u}_y = & \xi e^{-|\xi|(h-y)} C_1 + \xi y e^{-|\xi|(h-y)} C_2 + \xi e^{-|\xi|y} C_3 \\ & + \xi y e^{-|\xi|y} C_4 + \xi e^{-\xi(h-y)/l} C_5 + \xi e^{-\xi y/l} C_6 \end{aligned} \quad (36)$$

where C_i ($i = 1, 2, \dots, 6$) are independent unknown function of x , $k = 3 - 4n$, and

$$\xi = \sqrt{1 + l^2 \xi^2} \quad (37)$$

By taking Fourier integral transform of (7) and (16)-(21) together with (35)-(36), it gives rise to the general solution of the rotation, the force stresses, and the couple stresses in the transform space:

$$\begin{aligned} \tilde{\omega}_z = & \frac{i}{2} (\kappa + 1) |\xi| e^{-|\xi|(h-y)} C_2 - \frac{i}{2} (\kappa + 1) |\xi| e^{-|\xi|y} C_4 \\ & + \frac{i}{2l^2} e^{-\xi(h-y)/l} C_5 + \frac{i}{2l^2} e^{-\xi y/l} C_6 \end{aligned} \quad (38)$$

$$\begin{aligned} \frac{\tilde{\sigma}_{xx}}{2\mu} = & -\xi |\xi| e^{-|\xi|(h-y)} C_1 - (3 - 2\nu + |\xi|y) \xi e^{-|\xi|(h-y)} C_2 \\ & + \xi |\xi| e^{-|\xi|y} C_3 - (3 - 2\nu - |\xi|y) \xi e^{-|\xi|y} C_4 \\ & - \frac{\xi \xi}{l} e^{-\xi(h-y)/l} C_5 + \frac{\xi \xi}{l} e^{-\xi y/l} C_6 \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{\tilde{\sigma}_{yy}}{2\mu} = & \xi |\xi| e^{-|\xi|(h-y)} C_1 + (1 - 2\nu + |\xi|y) \xi e^{-|\xi|(h-y)} C_2 \\ & - \xi |\xi| e^{-|\xi|y} C_3 + (1 - 2\nu - |\xi|y) \xi e^{-|\xi|y} C_4 \\ & + \frac{\xi \xi}{l} e^{-\xi(h-y)/l} C_5 - \frac{\xi \xi}{l} e^{-\xi y/l} C_6 \end{aligned} \quad (40)$$

$$\begin{aligned} \frac{\tilde{\sigma}_{xy}}{2\mu} = & -i\xi^2 e^{-|\xi|(h-y)} C_1 - i(2 - 2\nu + |\xi|y) |\xi| e^{-|\xi|(h-y)} C_2 \\ & - i\xi^2 e^{-|\xi|y} C_3 + i(2 - 2\nu - |\xi|y) |\xi| e^{-|\xi|y} C_4 \\ & - \frac{i\xi^2}{l^2} e^{-\xi(h-y)/l} C_5 - \frac{i\xi^2}{l^2} e^{-\xi y/l} C_6 \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{\tilde{\sigma}_{yx}}{2\mu} = & -i\xi^2 e^{-|\xi|(h-y)} C_1 - i(2 - 2\nu + |\xi|y) |\xi| e^{-|\xi|(h-y)} C_2 \\ & - i\xi^2 e^{-|\xi|y} C_3 + i(2 - 2\nu - |\xi|y) |\xi| e^{-|\xi|y} C_4 \\ & - i\xi^2 e^{-\xi(h-y)/l} C_5 - i\xi^2 e^{-\xi y/l} C_6 \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{\tilde{\mu}_{xz}}{2\mu} = & l^2 (\kappa + 1) \xi |\xi| e^{-|\xi|(h-y)} C_2 - l^2 (\kappa + 1) \xi |\xi| e^{-|\xi|y} C_4 \\ & + \xi e^{-\xi(h-y)/l} C_5 + \xi e^{-\xi y/l} C_6 \end{aligned} \quad (43)$$

$$\begin{aligned} \frac{\tilde{\mu}_{yz}}{2\mu} = & il^2 (\kappa + 1) \xi^2 e^{-|\xi|(h-y)} C_2 + il^2 (\kappa + 1) \xi^2 e^{-|\xi|y} C_4 \\ & + \frac{i\xi}{l} e^{-\xi(h-y)/l} C_5 + \frac{i\xi}{l} e^{-\xi y/l} C_6 \end{aligned} \quad (44)$$

Once the unknown functions C_i ($i = 1, 2, \dots, 6$) are determined, the general solutions (35)-(36) and (38)-(44) can be utilized together with Fourier integral transform inversion to obtain the elastic field within the layer.

To determine the unknown functions C_i ($i = 1, 2, \dots, 6$), the following boundary conditions at the top and bottom of the layer, in the transform space, must be properly enforced:

$$\tilde{\sigma}_{yy}(y=0) = \tilde{p}_0(\xi) \quad (45)$$

$$\tilde{\sigma}_{yx}(y=0) = \tilde{q}_0(\xi) \quad (46)$$

$$\tilde{\mu}_{yz}(y=0) = \tilde{m}_0(\xi) \quad (47)$$

$$\tilde{u}_x(y=h) = 0 \quad (48)$$

$$\tilde{u}_y(y=h) = 0 \quad (49)$$

$$\tilde{\omega}_z(y=h) = 0 \quad (50)$$

By using the general solutions established above, the following relations can be readily obtained

$$\begin{Bmatrix} \tilde{u}_x(y=0) \\ \tilde{u}_y(y=0) \\ \tilde{\omega}(y=0) \\ \tilde{u}_x(y=h) \\ \tilde{u}_y(y=h) \\ \tilde{\omega}(y=h) \end{Bmatrix} = \mathbf{M}(\xi) \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} \quad (51)$$

$$\begin{Bmatrix} \tilde{\sigma}_{yx}(y=0) \\ \tilde{\sigma}_{yy}(y=0) \\ \tilde{\mu}_{yz}(y=0) \\ \tilde{\sigma}_{yx}(y=h) \\ \tilde{\sigma}_{yy}(y=h) \\ \tilde{\mu}_{yz}(y=h) \end{Bmatrix} = \mathbf{N}(\xi) \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} \quad (52)$$

where $\mathbf{M}$ and $\mathbf{N}$ are 6x6-matrices that can be obtained in a closed form from the general solutions of the displacement (35)-(36), the rotation (38), the force stress components (40)-(42) and the couple stress (44). Combining (51) and (52) leads to

$$\begin{Bmatrix} \tilde{\sigma}_{yx}(y=0) \\ \tilde{\sigma}_{yy}(y=0) \\ \tilde{\mu}_{yz}(y=0) \\ \tilde{\sigma}_{yx}(y=h) \\ \tilde{\sigma}_{yy}(y=h) \\ \tilde{\mu}_{yz}(y=h) \end{Bmatrix} = \mathbf{K}(\xi) \begin{Bmatrix} \tilde{u}_x(y=0) \\ \tilde{u}_y(y=0) \\ \tilde{\omega}(y=0) \\ \tilde{u}_x(y=h) \\ \tilde{u}_y(y=h) \\ \tilde{\omega}(y=h) \end{Bmatrix} \quad (53)$$

where $\mathbf{K}(\xi)$ denotes the stiffness matrix of the layer defined by

$$\mathbf{K}(\xi) = \mathbf{N}(\xi) \mathbf{M}^{-1}(\xi) \quad (54)$$

The stiffness equation of the layer (53) and the boundary conditions (45)-(50) are sufficient for solving the unknown data $\{\tilde{u}_x, \tilde{u}_y, \tilde{\omega}\}$ at the top surface. Once such information is obtained, (51) can then be used along with the boundary conditions (48)-(50) to solve for the unknown functions C_i ($i = 1, 2, \dots, 6$).

4. Results and Discussion

In this section, a set of preliminary results is reported to validate the proposed formulation and the corresponding solution procedure. For the purpose of discussion of the material length-scale effects and the comparison of results with those from the size-independent continuum theory, a reference length l_0 is introduced such that $l = \rho l_0$ where ρ is a non-dimensional parameter. In this sense, results for the classical case must be recovered as ρ approaches zero.

First, let us consider an elastic half-plane subjected to a uniformly distributed normal traction p_0 over the region $[-a, a]$. To simulate this scenario within the context of the current formulation, the thickness of the layer h is chosen to be sufficiently large in comparison with the half-length of the loading region a (i.e., $h/a = 1000$). Results of the normalized normal force-stress components along the line of symmetry (i.e., $x = 0$) are reported as a function of the normalized coordinate y/l_0 as shown in Figure 2 for $a/l_0 = 1$ and different values of ρ together with the exact classical solution. It is seen that as ρ approaches zero (i.e., the material length scale is much smaller than the size of the loading region), obtained results

converged to the classical solution. In addition, for the case that length of the loading region is comparable to the material length scale l , the force stresses predicted by the couple stress theory significantly deviate from those of classical continuum.

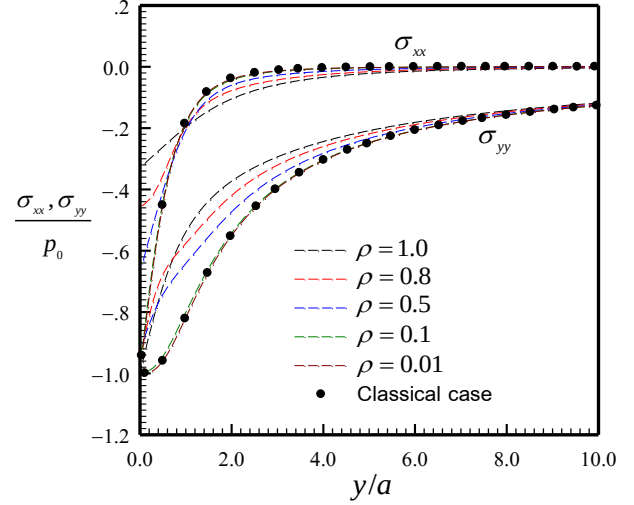


Fig.2. Normalized normal force stresses σ_{xx} and σ_{yy} along the line of symmetry of elastic half plane under uniformly distributed normal traction. Results are reported for $\nu = 0.33$ and $a/l_0 = 1$.

Next, consider an elastic layer subjected to the uniformly distributed normal traction p_0 over the region $[-a, a]$. The thickness of the layer is taken such that $a/h = 1$. To investigate the size dependency of predicted solutions, various values of the normalized half-length of the loading region a/l_0 are considered in the numerical study. An example set of results associated with the normalized vertical stress σ_{yy} along the line of symmetry is reported in Figure 3 for $\rho = 0$ and $\rho = 1$. It is evident from this set of results that solutions predicted from the couple stress theory become size dependent and significantly deviates from the classical solution (which is clearly size independent) as the size of the loading region reduces to be comparable to the material length scale l .

To further confirm the size-dependent behavior of predicted responses due to the presence of couple stresses ($\rho \neq 0$), the normalized vertical stress σ_{yy} at a selected point $x = 0, y = h/2$ is also reported as a function of the normalized half-length of the loading region a/l_0 as shown in Figure 4. It is seen that the solution predicted by the couple stress theory is size-dependent for a wide range of a/l_0 . In particular, as a/l_0 increases, the predicted solution reduces monotonically and finally converges to the classical case. Note also that in the neighborhood of $a/l_0 = 0$, the result obtained from the couple

stress theory is nearly constant and this implies the size independent behavior as observed in the classical case. While the size dependent characteristic disappears as the size of the loading region is much smaller than the material length scale, the influence of the couple stresses is still significant and cannot be ignored in the modeling.

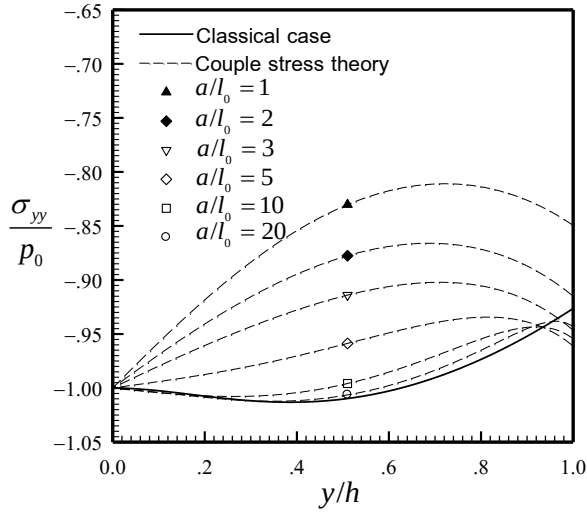


Fig.3. Normalized vertical stress σ_{yy} along the line of symmetry of elastic layer under uniformly distributed normal traction. Results are reported for $\nu = 0.33$, $h/a = 1$, $\rho \in \{0, 1\}$, and various values of a/l_0 .

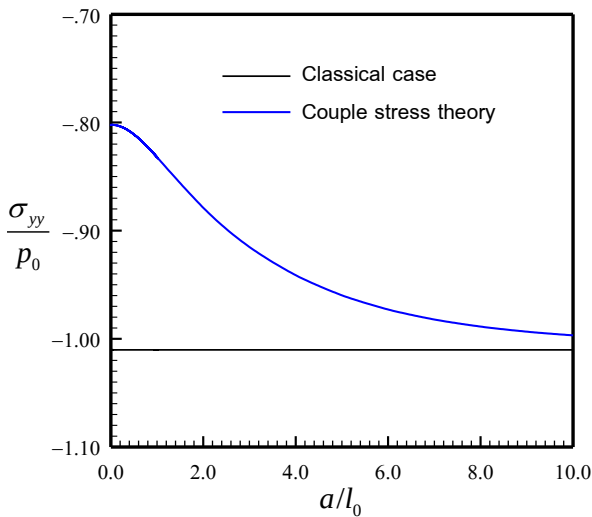


Fig.4. Normalized vertical stress σ_{yy} of elastic layer under uniformly distributed normal traction versus the normalized half-length of loading region. Results are reported for $\nu = 0.33$, $h/a = 1$, and $\rho \in \{0, 1\}$.

5. Conclusion and Remarks

A set of fundamental solutions of a two-dimensional, elastic layer under the surface loading has been established by taking the material micro-structure into account. A couple stress theory has been chosen to construct the mathematical model with the presence of the material length scale and to formulate the key equations governing the elastic field within the layer. An analytical solution procedure, based upon the method of Fourier integral transform, is adopted to obtain the closed form general solution in the transform space, and an efficient quadrature is implemented to carry out Fourier transform inversion. Results from preliminary numerical study, upon the comparison with available benchmark cases, has confirmed the correctness of the formulation and the corresponding solution procedure. It has also been found that as the size of the loading region is comparable to the material length scale (induced by the integration of the couple stresses), the predicted solutions become size dependent and deviate significantly from those of the size-independent continuum mechanics. As the size of the loading region becomes sufficiently large or small in comparison with the material length scale, the size dependent characteristics disappears. An in-depth investigation of the significant influence of the couple stresses on the behavior of the elastic field within the layer and the extension to treat a multi-layer medium is still under development.

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