รายงานวิจัยฉบับสมบูรณ์

การควบคุมแบบเปลี่ยนตามเวลาสำหรับระบบที่มีการหน่วงและไม่เป็นเชิงเส้น

Time-varying control for delayed and inherently nonlinear systems

โดยรองศาสตราจารย์ ดร. ระดม พงษ์วุฒิธรรม

เดือนพฤษภาคม 2563

สัญญาเลขที่ RSA6080027

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Time-varying control for delayed and inherently nonlinear systems

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และมหาวิทยาลัยเชียงใหม่

กิตติกรรมประกาศ

คณะผู้วิจัยใคร่ขอขอบคุณสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยเชียงใหม่ ที่ให้การสนับสนุนโครงการวิจัยนี้ ขอขอบคุณ คุณชลนภา ชื่นชมรัตน์ ที่ให้ความช่วยเหลือและทำหน้าที่ประสานงานเป็นอย่างดี ขอขอบคุณภาควิชาวิศวกรรมเครื่องกล คณะวิศวกรรมศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่ให้การสนับสนุนทางด้านสถานที่ในการดำเนินงานจนกระทั่งโครงการวิจัยนี้สำเร็จลุล่วงด้วยดี คณะวิจัยมีความหวังเป็นอย่างยิ่งว่างานวิจัยนี้จะสามารถนำไปต่อยอดเพื่อใช้ประโยชน์อย่างกว้างขวางต่อไป

บทคัดย่อ

รหัสโครงการ : RSA6080027

ชื่อโครงการ : การควบคุมแบบเปลี่ยนตามเวลาสำหรับระบบที่มีการหน่วงและไม่เป็นเชิงเส้น

ชื่อนักวิจัย : นายระดม พงษ์วุฒิธรรม คณะวิศวกรรมศาสตร์ มหาวิทยาลัย เชียงใหม่

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ระยะเวลาโครงการ : 3 ปี

ในการศึกษานี้ทำการศึกษาปัญหาการควบคุมสเตทแบบวงกว้างสำหรับระบบที่มีการหน่วง ไม่เป็นเชิงเส้น และไม่ทราบทิศทาง(เครื่องหมาย)ของตัวควบคุม เพื่อจะแก้ปัญหานี้ตัวคูณไดนามิค (เปลี่ยนแปลงตามเวลา) ถูกนำมาใช้เพื่อจัดการกับการหน่วงและตัวคูณนาสบามถูกใช้เพื่อจัดการการเครื่องหมายของตัวควบคุมที่ไม่ทราบทิศทาง ผู้วิจัยได้ทำการแก้ปัญหานี้โดยใช้ตัวควบคุมสเตทที่มีตัวคูณเปลี่ยนแปลงตามเวลาโดยการสร้างเลียปูนอฟร์-คราซอฟสกีร์แบบใหม่ซึ่งเป็นหัวใจสำคัญในการแก้ปัญหานี้ จากการศึกษาพบว่า i) ระบบมีมีการลู่เข้าแบบวงกว้างสู่จุดกำเนิด ii) สเตทถูกจำกัดแบบวงกว้าง นอกจากนี้วิธีที่ทำสามารถนำไปประยุกต์สำหรับระบบทางกลที่มีและไม่มีการหน่วง

คำหลัก : การควบคุมแบบเปลี่ยนตามเวลา, ระบบที่มีการหน่วง, ระบบไม่เป็นเชิงเส้น

รูปแบบ Abstract (บทคัดย่อ)

Project Code : RSA6080027

Project Title : Time-varying control for delayed and inherently nonlinear systems

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Project Period : 3 years

This study focuses on the problem of global state regulation with stability for time-delay nonlinear systems with unknown control directions. Using a dynamic (time-varying) gain-based method for counteracting time-delay nonlinearity and the Nussbaum gain function for dealing with unknown control directions, we develop a dynamic state feedback control strategy that solves the problem. A novel construction of Lyapunov-Krasovskii functionals is presented and plays a key role in handling nonlinearity with delayed states and unknown control directions simultaneously. The proposed dynamic state feedback compensators are shown to guarantee: i) global asymptotic convergence of the system state to the origin; and ii) global boundedness of the resulting closed-loop systems. The proposed control design can be apply to various physical systems with/without delay.

Keywords : time-varying control, delayed system, inherently nonlinear system

วัตถุประสงค์

1) Develop a delay-independent control that regulates the nonlinear system

$$
\begin{aligned}
\dot{x}_i &= \theta_i x_{i+1} + f_i(x_1, \cdots, x_i, x_1(t-d), \cdots, x_i(t-d)), \\
\dot{x}_n &= \theta_n u + f_n(x, x(t-d)), \qquad i = 1, \cdots, n-1, \\
x(s) &= \zeta(s), \quad s \in [-d, 0],
\end{aligned}
$$

without imposing any growth condition on the lower-triangular vector fields and the unknown parameters.

2) Develop a delay-independent time-varying control that regulates the p-normal form system with state delay

$$
\begin{aligned}
\dot{x}_i &= \theta_i x_{i+1}^{p_i} + f_i(x_1, \cdots, x_i, x_1(t-d), \cdots, x_i(t-d)), \\
\dot{x}_n &= \theta_n u^{p_n} + f_n(x, x(t-d)), \\
x(s) &= \zeta(s), \quad s \in [-d, 0],
\end{aligned}
$$

without imposing any growth condition on the lower-triangular vector fields and the unknown parameters.

3) Develop a delay-independent time-varying control for a physical system with state delay and test the performance the time-varying controller with actual system.

วิธีทดลอง

1) Review related publications in the area of time-varying control, nonlinear system with state delay and new discoveries in nonlinear control.

2) Formulate the control problems that can be solved and choose a mechanical system that will be implemented in a test rig.

3) Propose new time-varying controllers.

4) Test the performance of the proposed controllers by using computer simulations

5) Design a mechanical test rig to study the characteristics and performance of the propose controllers.

6) Build and assemble the test rig.

7) Gather the test results and evaluate the controller performance.

8) Prepare journal papers and reports

บทนำ

Time-delay systems extensively exist in a variety of applications including, but not limited to, network control, mechanical systems, biological systems and chemical processes. For example, models of milling processes, drilling processes and fluid flow or heating systems all exhibit time-delay phenomena. While many of these controlled plants are approximately modeled by linear systems, the work [2] presented a chemical reactor example that is described by a lower-triangular nonlinear system with time-delays in the state. To address control problems of time-delay systems, various analysis and synthesis approaches have been developed in the literature. Among them, the Lyapunov-Krasovskii and Lyapunov-Razumikhin methods are two powerful tools in the stability analysis of time-delay systems [1, 3, 16, 15]. There are primarily three types of time-delay systems that have received considerable attention. One class includes the delay in the system state [1, 12, 17, 14] and the other one contains the delay in the control input [10, 11, 5]. Of course, a more complex situation involves time-delays in both states and actuators of controlled plants. For each category of time-delay systems, many results have been obtained and reported; see, for instance, [12], [10], [11, 5, 4]. In [10], a saturation state feedback controller was proposed for global asymptotic stabilization of a chain of integrators with a delay in the input, without requiring the knowledge of the delay. In [5], control of a class of nonlinear systems with input delay was investigated with the condition that the system under consideration is forward complete. For a strict feedback system with delayed states, an attempt was first made in [12] to design a delay-independent, smooth state feedback controller. Later, it was found that the result of [12] is false, due to a circular argument in the state feedback design. Such a technical issue was addressed in [2, 4] under the assumption that the upper bound of time-delay is known, and later in [17, 19], by using dynamic instead of static state feedback. The dynamic gain-based designs or the dynamic state feedback control schemes [17, 19] have shown to be effective in counteracting the nonlinearities with delayed states, thus making it possible to remove restrictive conditions imposed on time-delay nonlinear systems, which were commonly assumed in the literature when using delay-independent static state feedback. Most of the afore-mentioned works concentrated on time-delay nonlinear systems with known control directions, e.g., the signs of all coefficients of the chain of integrator are assumed to be known. If this crucial information is not available, a new method needs to be developed for the control of time-delay systems. When no time-delay is involved, feedback design approaches have been studied for uncertain nonlinear systems with unknown control directions [18], using the so-called Nussbaum functions from universal adaptive stabilization of minimum-phase linear systems with unknown sign of high-frequency gain [13]. Since the sign of the control input often represents, for instance, motion directions of mechanical systems such as robotics modeled by the Lagrange equation and may be unknown, it is certainly important to investigate how to control time delay systems with unknown control directions.

We first focus our attention on the following class of time-delay nonlinear system with unknown control directions:

$$\begin{aligned} \dot{x}_i &= \theta_i x_{i+1} + f_i(x_1, \cdots, x_i, x_1(t-d), \cdots, x_i(t-d)), \\ \dot{x}_n &= \theta_n u + f_n(x, x(t-d)), \qquad i = 1, \cdots, n-1, \\ x(s) &= \zeta(s), \quad s \in [-d, 0], \end{aligned} \tag{1}$$

For the time-delay system with unknown control directions (1), global stabilization by delay-independent state feedback is a nontrivial problem and has not been addressed so far. There are perhaps two reasons: i) when the signs of coefficients of a chain of integrators are unknown, the design of virtual controllers is less intuitive and more involved as the uncertainties cannot be cancelled directly by a conventional backstepping design; ii) the presence of time-delay nonlinearities makes a delay-free, static state feedback law insufficient for mitigating the effects of time-delay, and hence a dynamic instead of static state feedback may be necessary. Motivated by the universal control idea [13, 6, 7, 8] and the recent development [19, 17], we propose in this work a novel construction of a set of Lyapunov-Krasovskii functionals and a delay-independent, dynamic state feedback control scheme for counteracting the effects of time-delay nonlinearities and unknown control directions in the system (1) simultaneously. With the help of the new dynamic gain-based Lyapunov-Krasovskii functionals, we are able to design a time-delay independent, dynamic state feedback compensator step-by-step, resulting in a solution to the global state regulation of the time-delay system (1) with stability. Interestingly, it is worth pointing out that the approach presented in this paper provides a new yet simpler way of designing state feedback controllers that achieve global stabilization of the nonlinear system (1) with unknown control directions, in the absence of time-delay, i.e., d = 0.

ผลการทดลอง

Dynamic State Feedback Design

In this section, we first construct a delay-independent, dynamic state feedback compensator, by means of the Nussbaum-gain function [13], a set of new Lyapunov-

Krasovskii functionals (due to the presence of unknown control directions) and the dynamic gain-based design philosophy [19].

<u>Step 1</u>: For the x_1-subsystem of (1), view the state x_2 as a virtual control and consider the Lyapunov function $V_1(x_1, l_1) = \frac{1}{2}\left(1 + \frac{1}{l_1}\right)\xi_1^2$ where l_1 is a dynamic gain to be determined in Step 2.

A direct calculation gives

$$\begin{aligned} \dot{V}_1 &= (1+\frac{1}{l_1})\xi_1[\theta_1 x_2 + f_1(x_1, x_1(t-d))] - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \\ &\leq (1+\frac{1}{l_1})\theta_1\xi_1 x_2^* + 2|\theta_1\xi_1\xi_2| + 2|\xi_1 f_1(\cdot)| - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \end{aligned} \quad (3)$$

where $\xi_2 = x_2 - x_2^*$

It can be proved that

$$2|\xi_1 f_1(\cdot)| \leq 2\xi_1^2\bar{\gamma}_1(x_1) + \xi_1^2 + \xi_1^2(t-d)\bar{\gamma}_1^{*2}(x_1(t-d)). \quad (4)$$

We now use the bound from (4) to construct the Lyapunov-Krasovskii functional

$$V_{1LK} = V_1(l_1, x_1) + \int_{t-d}^{t} \xi_1^2(s)\,\bar{\gamma}_1^{*2}(x_1(s))\,ds$$

whose time derivative satisfies (by (3)-(4))

$$\begin{aligned} \dot{V}_{1LK} \leq{}& -n\xi_1^2 + (1+\frac{1}{l_1})\theta_1\xi_1 x_2^* + c_2\xi_2^2 \\ &+\bar{c}_1\xi_1^2(1 + 2\bar{\gamma}_1(x_1) + \bar{\gamma}_1^{*2}(x_1)) - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \end{aligned} \quad (5)$$

where c_1 = 2+n: Because the sign of θ_1 is unknown, we use the idea from [13], namely, the Nussbaum function to design a controller. In fact, from (5) a virtual controller with the Nussbaum gain can be constructed as

$$\begin{aligned} x_2^* &= \xi_1 N(k_1)[1 + 2\bar{\gamma}_1(x_1) + \bar{\gamma}_1^{*2}(x_1)] \\ &:= \xi_1 N(k_1)\beta_1(x_1) \\ \dot{k}_1 &= (1+\frac{1}{l_1})\xi_1^2\beta_1(x_1), \quad k_1(0) = 1. \end{aligned} \quad (6)$$

This, together with *l*1 > 1, results in

$$\dot{V}_{1LK} \leq -n\xi_1^2 + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 + c_2\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \quad (7)$$

<u>Step 2</u>: For the (x1; x2)-subsystem of (1), treat the state x3 as a virtual control and consider the Lyapunov-Krasovskii functional

$$V_2 = V_{1LK} + \frac{1}{2l_1}k_1^2\xi_2^2 + \frac{1}{2l_1 l_2}(\xi_1^2 + k_1^2\xi_2^2) \quad (8)$$

where $l_2 > 1$ is a dynamic gain to be determined in Step 3.

In view of (7) and the properties that $l_j > 1$, j = 1; 2, we have

$$\begin{aligned}\dot{V}_2 &\le -n\xi_1^2 + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 + c_2\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2\\ &+\frac{1}{l_1}(1+\frac{1}{l_2})\theta_2 k_1^2\xi_2(x_3^* + x_3 - x_3^*) + \frac{2}{l_1}k_1^2|\xi_2 f_2(\cdot)| + \frac{2}{l_1}k_1^2|\xi_2\dot{x}_2^*|\\ &+\frac{1}{l_1 l_2}\xi_1\dot{x}_1 + \frac{2}{l_1}k_1\dot{k}_1\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}k_1^2\xi_2^2 - \frac{\dot{l}_1 l_2 + l_1\dot{l}_2}{2l_1^2 l_2^2}(\xi_1^2 + k_1^2\xi_2^2) \end{aligned} \tag{9}$$

Using $\xi_2 = x_2 - x_2^*$, (6) and the fact that $l_1 > 1$ and $k_1 > 1$, we arrive at (withthe aid of Lemmas 5.1-5.3)

$$\begin{aligned} &\frac{2}{l_1}k_1^2|\xi_2 f_2(\cdot)| \le k_1^2\xi_2^2\Upsilon_{21}(k_1,\bar{x}_2) + \frac{1}{l_1}\xi_1^2\Upsilon_{22}(k_1,x_1) + \xi_2^2(t-d)\\ &\cdot\Upsilon_{21}^*(k_1(t-d),\bar{x}_2(t-d)) + \frac{1}{l_1}\xi_1^2(t-d)\Upsilon_{22}^*(k_1(t-d),x_1(t-d))\\ &\frac{2}{l_1}k_1^2|\xi_2\dot{x}_2^*| + \frac{1}{l_1 l_2}|\xi_1\dot{\xi}_1| + \frac{2}{l_1}k_1\dot{k}_1\xi_2^2 \le k_1^2\xi_2^2\Phi_{21}(k_1,\bar{x}_2)\\ &\qquad + \frac{1}{l_1}\xi_1^2\Phi_{22}(k_1,x_1) + \frac{1}{l_1}\xi_1^2(t-d)\Phi_{22}^*(x_1(t-d))\\ &-\frac{\dot{l}_1}{2l_1^2}k_1^2\xi_2^2 - \frac{\dot{l}_1 l_2 + l_1\dot{l}_2}{2l_1^2 l_2^2}(\xi_1^2 + k_1^2\xi_2^2) \le -\frac{\dot{l}_2}{2l_1 l_2^2}(\xi_1^2 + \xi_2^2) \end{aligned} \tag{10}$$

With the help of (10), we construct the Lyapunov-Krasovskii functional

$$\begin{aligned} V_{2LK} &= V_2 + \int_{t-d}^{t} \xi_2^2(s)\Upsilon_{21}^*(k_1(s),\bar{x}_2(s))ds\\ &+\int_{t-d}^{t}\frac{1}{l_1(s)}\xi_1^2(s)[\Upsilon_{22}^*(k_1(s),x_1(s)) + \Phi_{22}^*(x_1(s))]ds \end{aligned} \tag{11}$$

With $\xi_3 = x_3 - x_3^*$, we deduce from (9)-(10) that

$$\begin{aligned} \dot{V}_{2LK} &\le -n\xi_1^2 + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2\\ &+\frac{1}{l_1}(1+\frac{1}{l_2})\theta_2 k_1^2\xi_2 x_3^* + c_3 k_1^2\xi_3^2 + k_1^2\xi_2^2[1 + c_2 + \Upsilon_{21}(k_1,\bar{x}_2)\\ &\quad+\Phi_{21}(k_1,\bar{x}_2) + \Upsilon_{21}^*(k_1,\bar{x}_2)] + \frac{1}{l_1}\xi_1^2[\Upsilon_{22}(k_1,x_1)\\ &\quad+\Phi_{22}(k_1,x_1) + \Upsilon_{22}^*(k_1,x_1) + \Phi_{22}^*(x_1)]\\ &\quad+(\frac{1}{l_1} - \frac{1}{l_1(t-d)})\xi_1^2(t-d)[\Upsilon_{22}^*(k_1(t-d),x_1(t-d))\\ &\quad+\Phi_{22}^*(x_1(t-d))] - \frac{\dot{l}_2}{2l_1 l_2^2}(\xi_1^2 + \xi_2^2) \end{aligned} \tag{12}$$

Based on (12), one can design the Riccati-like update law

$$\dot{l}_1 = \max\{-l_1^2 + l_1\rho_1(k_1,x_1),\ 0\},\ l_1(0) = 1 \tag{13}$$

$$\rho_1(k_1,x_1) = 2\big[\Upsilon_{22}(\cdot) + \Upsilon_{22}^*(\cdot) + \Phi_{22}(\cdot) + \Phi_{22}^*(\cdot)\big], \tag{14}$$

to mitigate the effects of the time-delay nonlinearity.

By construction, it is clear from (13) that

$$0 \leq \dot{l}_1 \leq l_1\rho_1(\cdot), \quad \dot{l}_1 \geq -l_1^2 + l_1\rho_1(\cdot), \quad l_1(t) \geq l_1(t-d) \geq 1. \tag{15}$$

As a consequence,

$$\begin{aligned} -\frac{\dot{l}_1}{2l_1^2}\xi_1^2 &\leq \xi_1^2 - \frac{1}{2l_1}\xi_1^2\rho_1(k_1, x_1) \\ \frac{1}{l_1} - \frac{1}{l_1(t-d)} &\leq 0. \end{aligned} \tag{16}$$

Substituting (13) and (16) into (12) leads to

$$\begin{aligned} \dot{V}_{2LK} \leq\ & -(n-1)\xi_1^2 - (n-1)k_1^2\xi_2^2 + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 \\ & + \frac{1}{l_1}(1 + \frac{1}{l_2})\theta_2 k_1^2 \xi_2 x_3^* + c_3 k_1^2 \xi_3^2 + \bar{c}_2 k_1^2 \xi_2^2 \\ & \cdot[1 + \Upsilon_{21}(k_1, \bar{x}_2) + \Phi_{21}(k_1, \bar{x}_2) + \Upsilon_{21}^*(k_1, \bar{x}_2)] \\ & - \frac{\dot{l}_1}{2l_1 l_2^2}(\xi_1^2 + \xi_2^2) \end{aligned} \tag{17}$$

Similar to Step 1, because of the unknown sign of θ_2, we need to design a virtual controller x_3^* with the Nussbaum gain as

$$\begin{aligned} x_3^* &= l_1\xi_2 N(k_2)[1 + \Upsilon_{21}(\cdot) + \Phi_{21}(\cdot) + \Upsilon_{21}^*(\cdot)] \\ &:= l_1\xi_2 N(k_2)\beta_2(k_1, \bar{x}_2) \\ \dot{k}_2 &= (1 + \frac{1}{l_2})\xi_2^2\beta_2(k_1, \bar{x}_2), \quad k_2(0) = 1, \end{aligned} \tag{18}$$

such that the inequality (17) becomes

$$\begin{aligned} \dot{V}_{2LK} \leq & -(n-1)[\xi_1^2 + k_1^2\xi_2^2] + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 \\ & + (\theta_2 N(k_2) + \bar{c}_2)k_1^2\dot{k}_2 + c_3 k_1^2 \xi_3^2 - \frac{\dot{l}_2}{2l_1 l_2^2}(\xi_1^2 + \xi_2^2). \end{aligned} \tag{19}$$

<u>Inductive Step</u>: Suppose at Step i - 1, there are a Lyapunov-Krasovskii functional **$V_{(i-1)LK}$**, a set of dynamic gains l_j j = 1,…,i-1, given by

$$\begin{aligned} \dot{l}_1 &= \max\{-l_1^2 + l_1\rho_1(k_1, x_1), 0\}, \\ \dot{l}_2 &= \max\{-l_2^2 + l_2\rho_2(l_1, \bar{k}_2, \bar{x}_2), 0\}, \\ &\ \vdots \\ \dot{l}_{i-2} &= \max\{-l_{i-2}^2 + l_{i-2}\rho_{i-2}(\bar{l}_{i-3}, \bar{k}_{i-2}, \bar{x}_{i-2}), 0\}, \end{aligned} \tag{20}$$

and a set of virtual controllers $x_1^*, \ldots, x_i^*$ with the Nussbaum gains (updated by a set of universal controllers) defined by

$$\begin{array}{ll} x_1^* = 0 & \xi_1 = x_1 - x_1^* \\ x_2^* = \xi_1 N(k_1)\beta_1(x_1) & \xi_2 = x_2 - x_2^* \\ \dot{k}_1 = (1+\frac{1}{l_1})\xi_1^2\beta_1(\cdot) & \\ \quad\vdots & \quad\vdots \\ x_i^* = (l_1\cdots l_{i-2})\xi_{i-1}N(k_{i-1}) & \xi_i = x_i - x_i^* \\ \quad \cdot\beta_{i-1}(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1}) & \\ \dot{k}_{i-1} = (1+\frac{1}{l_{i-1}})\xi_{i-1}^2\beta_{i-1}(\cdot) & \end{array} \tag{21}$$

with $\rho_j > 0$ and $\beta_j > 0$ being smooth functions, such that

$$\dot{V}_{(i-1)LK} \le -(n-(i-2))\Sigma_{j=1}^{i-1}[(\prod_{m=0}^{j-1} k_m^2)\xi_j^2] + c_i k_1^2\cdots k_{i-2}^2\xi_i^2$$
$$+\sum_{j=1}^{i-1}\Big[(\theta_j N(k_j)+\bar{c}_j)(\prod_{m=0}^{j-1}k_m^2)\dot{k}_j\Big] - \frac{\dot{l}_{i-1}}{2l_1\cdots l_{i-2}l_{i-1}^2}\sum_{j=1}^{i-1}\xi_j^2 \tag{22}$$

where $c_i > 0$ is a constant and $k_0 = 1$. Clearly, (22) reduces to (19) when $i = 3$.

Recursively, it can be shown that (22) also holds at Step i. To this end, consider the Lyapunov-Krasovskii functional

$$\begin{aligned} V_i &= V_{(i-1)LK} + \frac{1}{2l_1\cdots l_{i-1}}k_1^2\cdots k_{i-1}^2\xi_i^2 \\ &\quad+\frac{1}{2l_1\cdots l_i}\Big[\Sigma_{j=1}^{i-1}\xi_j^2 + k_1^2\cdots k_{i-1}^2\xi_i^2\Big] \end{aligned} \tag{23}$$

where $l_i > 1$ is a dynamic gain to be designed.

Using (22) and the properties that $l_j > 1$; $k_j > 1$; we have

$$\begin{aligned} \dot{V}_i \quad &\le -(n-(i-2))\Sigma_{j=1}^{i-1}\Big[(\prod_{m=0}^{j-1}k_m^2)\xi_j^2\Big] \\ &+\Sigma_{j=1}^{i-1}\Big[(\theta_j N(k_j)+\bar{c}_j)(\prod_{m=0}^{j-1}k_m^2)\dot{k}_j\Big] + c_i k_1^2\cdots k_{i-2}^2\xi_i^2 \\ &-\frac{\dot{l}_{i-1}}{2l_1\cdots l_{i-2}l_{i-1}^2}\Sigma_{j=1}^{i-1}\xi_j^2 + \frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}(1+\frac{1}{l_i})\theta_i\xi_i x_{i+1}^* \\ &+\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\theta_i\xi_i\xi_{i+1} + \xi_i f_i(\cdot) - \xi_i\dot{x}_i^*| \\ &+\frac{1}{l_1\cdots l_{i-1}}\sum_{j=1}^{i-1}\xi_j\dot{\xi}_j + \frac{2}{l_1\cdots l_{i-1}}\Big[\sum_{j=1}^{i-1}(k_j\dot{k}_j\prod_{\substack{m=1\\m\neq j}}^{i-1}k_m^2)\xi_i^2\Big] \\ &-\frac{1}{2l_1^2\cdots l_{i-1}^2}\Big[\sum_{j=1}^{i-1}(\prod_{\substack{m=1\\m\neq j}}^{i-1}l_m)\dot{l}_j\Big]k_1^2\cdots k_{i-1}^2\xi_i^2 \\ &-\frac{1}{2l_1^2\cdots l_i^2}\Big[\sum_{j=1}^{i}(\prod_{\substack{m=1\\m\neq i}}^{i}l_m)\dot{l}_j\Big][\sum_{j=1}^{i-1}\xi_j^2 + k_1^2\cdots k_{i-1}^2\xi_i^2]. \end{aligned} \tag{24}$$

The terms in (24) can be estimated and the properties of $k_i > 1$ and $l_i > 1$ as follows.

$$\begin{aligned}\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\xi_i f_i(\cdot)| \quad &\le \quad k_1^2\cdots k_{i-1}^2\xi_i^2\Upsilon_{i1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)\\ &+\frac{1}{l_1\cdots l_{i-1}}(\sum_{j=1}^{i-1}\xi_j^2)\Upsilon_{i2}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})\\ &+\xi_i^2(t-d)\Upsilon_{i1}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_i(t-d))\\ &+\frac{1}{l_1\cdots l_{i-1}}(\sum_{j=1}^{i-1}\xi_j^2(t-d))\\ &\cdot\Upsilon_{i2}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))\end{aligned}\tag{25}$$

$$\begin{aligned}\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\xi_i\dot{x}_i^*| \quad &\le \quad k_1^2\cdots k_{i-1}^2\xi_i^2\Phi_{i1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)\\ &+\frac{1}{l_1\cdots l_{i-1}}(\sum_{j=1}^{i-1}\xi_j^2)\Phi_{i2}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})\\ &+\frac{1}{l_1\cdots l_{i-1}}(\sum_{j=1}^{i-1}\xi_j^2(t-d))\\ &\cdot\Phi_{i2}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))\end{aligned}\tag{26}$$

$$\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\theta_i\xi_i\xi_{i+1}| \le k_1^2\cdots k_{i-1}^2\xi_i^2 + c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2\tag{27}$$

$$\begin{aligned}\frac{1}{l_1\cdots l_{i-1}}\sum_{j=1}^{i-1}|\xi_j\dot{\xi}_j| \quad &\le \quad k_1^2\cdots k_{i-1}^2\xi_i^2\\ &+\frac{1}{l_1\cdots l_{i-1}}(\sum_{j=1}^{i-1}\xi_j^2)\Psi_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})+\frac{1}{l_1\cdots l_{i-1}}\\ &\cdot(\sum_{j=1}^{i-1}\xi_j^2(t-d))\Psi_i^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))\end{aligned}\tag{28}$$

$$\frac{2\xi_i^2}{l_1\cdots l_{i-1}}\Big[\sum_{j=1}^{i-1}(k_j\dot{k}_j\prod_{\substack{m=1\\ m\ne j}}^{i-1}k_m^2)\Big] \le k_1^2\cdots k_{i-1}^2\xi_i^2\omega_i(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1})\tag{29}$$

$$
-\frac{1}{2l_1^2\cdots l_{i-1}^2}\Big[\sum_{j=1}^{i-1}(\prod_{\substack{m=1\\m\neq j}}^{i-1} l_m)\dot{l}_j\Big]k_1^2\cdots k_{i-1}^2\xi_i^2
$$

$$
-\frac{1}{2l_1^2\cdots l_i^2}\Big[\sum_{j=1}^{i}(\prod_{\substack{m=1\\m\neq j}}^{i} l_m)\dot{l}_j\Big][\sum_{j=1}^{i-1}\xi_j^2+k_1^2\cdots k_{i-1}^2\xi_i^2]
$$

$$
\leq \quad -\frac{\dot{l}_i}{2l_1\cdots l_{i-1}l_i^2}(\sum_{j=1}^{i}\xi_j^2) \tag{30}
$$

From the estimations above, which are related to the delay terms, one can construct the Lyapunov-Krasovskii functional

$$
V_{iLK} \quad = \quad V_i+\int_{t-d}^{t}\xi_i^2(s)\Upsilon_{i1}^*(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_i(s))ds \tag{31}
$$

$$
+\int_{t-d}^{t}\frac{1}{l_1(s)\cdots l_{i-1}(s)}\Big(\sum_{j=1}^{i-1}\xi_j^2(s)\Big)[\Upsilon_{i2}^*(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s))
$$

$$
+\Phi_{i2}^*(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s))+\Psi_i^*(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s))]ds
$$

Then, in view of (24)-(30), we have

$$
\begin{aligned}
\dot{V}_{iLK} \quad & \leq -(n-(i-2))\sum_{j=1}^{i-1}\Big[(\prod_{m=0}^{j-1}k_m^2)\xi_j^2\Big] \\
& +\sum_{j=1}^{i-1}\Big[(\theta_jN(k_j)+\bar{c}_j)(\prod_{m=0}^{j-1}k_m^2)\dot{k}_j\Big]-\frac{\dot{l}_{i-1}}{2l_1\cdots l_{i-2}l_{i-1}^2}\sum_{j=1}^{i-1}\xi_j^2 \\
& +\frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}(1+\frac{1}{l_i})\theta_i\xi_i x_{i+1}^*+c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2 \\
& +k_1^2\cdots k_{i-1}^2\xi_i^2[2+c_i+\Upsilon_{i1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)+\Phi_{i1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i) \\
& +\omega_i(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1})+\Upsilon_{i1}^*(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)]+\frac{1}{l_1\cdots l_{i-1}}(\sum_{j=1}^{i-1}\xi_j^2) \\
& \cdot[\Upsilon_{i2}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})+\Phi_{i2}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}) \\
& +\Psi_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})+\Upsilon_{i2}^*(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}) \\
& +\Phi_{i2}^*(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})+\Psi_i^*(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})] \\
& +\Big[\frac{1}{l_1\cdots l_{i-1}}-\frac{1}{l_1(t-d)\cdots l_{i-1}(t-d)}\Big](\sum_{j=1}^{i-1}\xi_j^2(t-d)) \\
& \cdot[\Upsilon_{i2}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d)) \\
& +\Phi_{i2}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1})(t-d) \\
& +\Psi_i^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))] \\
& -\frac{\dot{l}_i}{2l_1\cdots l_{i-1}l_i^2}(\sum_{j=1}^{i}\xi_j^2).
\end{aligned} \tag{32}
$$

Following the idea and design given in Step 2, we can construct (based on (32)) the delay-free gain update law

$$\dot{l}_{i-1} = \max\{-l_{i-1}^2 + l_{i-1}\rho_{i-1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}), 0\}, \tag{33}$$

with $l_{i-1}(0) = 1$, and

$$\begin{aligned}\rho_{i-1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) = 2[&\Upsilon_{i2}(\cdot) + \Upsilon_{i2}^*(\cdot) + \Phi_{i2}(\cdot) + \Phi_{i2}^*(\cdot) \\ &+\Psi_i(\cdot) + \Psi_i^*(\cdot)].\end{aligned} \tag{34}$$

By construction, it is easy to verify that

$$\begin{aligned} 0 \ &\leq \ \dot{l}_{i-1} \ \leq \ l_{i-1}\rho_{i-1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) \\ \dot{l}_{i-1} \ &\geq -l_{i-1}^2 + l_{i-1}\rho_{i-1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) \\ l_{i-1} \ &\geq \ \ l_{i-1}(t-d) \geq 1. \end{aligned} \tag{35}$$

As a consequence,

$$-\frac{\dot{l}_{i-1}}{2l_1\cdots l_{i-2}l_{i-1}^2}(\sum_{j=1}^{i-1}\xi_j^2) \ \leq \ \sum_{j=1}^{i-1}\xi_j^2 - \frac{\rho_{i-1}(\cdot)}{2l_1\cdots l_{i-1}}[\sum_{j=1}^{i-1}\xi_j^2] \tag{36}$$

$$\frac{1}{l_1\cdots l_{i-1}} - \frac{1}{l_1(t-d)\cdots l_{i-1}(t-d)} \leq 0. \tag{37}$$

Substituting (36) and (37) into (32), we obtain

$$\begin{aligned}\dot{V}_{iLK} \ \ \leq \ \ & -(n-(i-1))\sum_{j=1}^{i}(\prod_{m=0}^{j-1}k_m^2)\xi_j^2 \\ & + \sum_{j=1}^{i-1}[(\theta_j N(k_j) + \bar{c}_j)(\prod_{m=0}^{j-1}k_m^2)\dot{k}_j] + \frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}} \\ & \cdot\left(1+\frac{1}{l_i}\right)\theta_i\xi_i x_{i+1}^* + \bar{c}_i k_1^2\cdots k_{i-1}^2\xi_i^2 \\ & \cdot[1+\Upsilon_{i1}(\cdot)+\Phi_{i1}(\cdot)+\omega_i(\cdot)+\Upsilon_{i1}^*(\cdot)] \\ & -\frac{\dot{l}_i}{2l_1\cdots l_{i-1}l_i^2}\sum_{j=1}^{i}\xi_j^2 + c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2 \end{aligned} \tag{38}$$

To mitigate the effects of the unknown sign of θ_i, we design the following virtual controller with a Nussbaum gain (updated by a universal controller $\dot{k}_i$)

$$\begin{aligned} x_{i+1}^* \ &= \ (l_1\cdots l_{i-1})\xi_i N(k_i)[1+\Upsilon_{i1}(\cdot) \\ & \quad\ +\Phi_{i1}(\cdot)+\omega_i(\cdot)+\Upsilon_{i1}^*(\cdot)] \\ &:= \ (l_1\cdots l_{i-1})\xi_i N(k_i)\beta_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) \\ \dot{k}_i \ &= \ (1+\frac{1}{l_i})\xi_i^2\beta_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i). \end{aligned} \tag{39}$$

Substituting (39) into (38) leads to the claim that (22) holds at Step i. The inductive argument so far has indicated that (22) holds for *i* = *n+1* with $u = x_{n+1} - x_{n+1}^*$. As a consequence, a dynamic state feedback controller that is composed of (20) with i = n + 1 and a universal-like control law

$$\begin{aligned} u &= (l_1 \cdots l_{n-1})\xi_n N(k_n)\beta_n(\bar{l}_{n-2}, \bar{k}_{n-1}, x) \\ \dot{k}_n &= \xi_n^2 \beta_n(\bar{l}_{n-2}, \bar{k}_{n-1}, x) \end{aligned} \tag{40}$$

renders

$$\dot{V}_{nLK} \leq -\sum_{j=1}^{n}[(\prod_{m=0}^{j-1} k_m^2)\xi_j^2] + \sum_{j=1}^{n}[(\theta_j N(k_j) + \bar{c}_j)(\prod_{m=0}^{j-1} k_m^2)\dot{k}_j] \tag{41}$$

We end this subsection with an observation that the dynamic state feedback compensator designed so far, namely, (40) and (20)-(21) with i = n + 1, is exactly of the form (2).

Remark In the case when θi's are known constants, all the k_i = 1. Then, the Lyapunov inequality (41) reduces to

$$\dot{V}_{nLK} \leq -\sum_{i=1}^{n} \xi_j^2 \leq 0, \tag{42}$$

from which it is concluded that global asymptotic state regulation of the time-delay nonlinear system (1) and boundedness of the closed-loop system are achieved by the delay-independent, dynamic state feedback compensator (40) and (20)-(21), with k_i = 1, $N(k_i)$ = constant for i = 1;…, n,

Asymptotic State Regulation with Boundedness

In this subsection, we complete the proof of Theorem 2.1 by showing that the universallike, dynamic state feedback controller (40) and (20) designed in subsection A ensures not only the convergence of the system state x but also boundedness of the resulting closed-loop system.

First of all, from the Lyapunov inequalities (22) and (41) it is concluded that $k_i(t)$, i = 1,…, n; are bounded.

By the boundedness of $k_i(t)$, 1< I < n; it follows from (41) that

$$\frac{V_{nLK}(t)}{\prod_{m=1}^{n-1} k_m^2(0)} + \int_0^t \xi_n^2 ds \leq \sum_{j=1}^{n} \int_{k_j(0)}^{k_j(t)} \frac{(\theta_j N(k_j) + \bar{c}_j)}{(\prod_{m=j}^{n-1} k_m^2)} dk_j + c, \tag{43}$$

thus implying the boundedness of $\int_0^t \xi_n^2\, ds$. Repeating the same argument for the Lyapunov-Krasovskii functionals $V_{(n-1)LK}$, …, V_{1LK}; we can conclude that $\int_0^t \xi_i^2\, ds$, i=2,…,n are bounded. On the other hand, from the inequality (43) and the boundedness of $k_i(t)$, i = 1,…,n, it is straightforward to prove that the Lyapunov-Krasovskii functional V_{nLK}

evaluated on the solution trajectory of the closed-loop system is bounded. In view of the construction of V_{nLK}, in particular, (31) and (23), we deduce that the boundedness of V_{nLK} implies the boundedness of x_1, $\frac{k_1^2 \dots k_{i-1}^2}{l_1 \dots l_{i-1}} \xi_i^2$, I = 1,…,n.

Keeping the boundedness of x_1 and k_1 in mind, it is trivial to verify that the gain l_1 designed by (13)-(14) is monotone non-deceasing. Moreover, l_1 is also bounded. In fact, if it is unbounded, then $\lim_{t\to\infty} l_1(t) = +\infty$. By continuity of ρ_1, ρ_1 is bounded due to the boundedness of k_1 and x_1. As a consequence, there is a time instant T > 0 such that $-l_1^2 + l_1\rho_1(k_1, x_1) \leq$. This, together with (15), results in $\dot{l_1} = 0$, which contradicts to the unboundedness of l_1. Therefore, l_1 must be bounded. This, combined with the boundedness of k_1, implies the boundedness of x_2^* and $\xi_2 = x_2 - x_2^*$ and so does x_2. With the help of the boundedness of k_i(1 < i < n); the boundedness of l_i and x_i can be proved in the iterative manner of $x_2 \to l_2 \to x_3 \to \cdots \to l_{n-1} \to x_n$ by using (20) and (21). In conclusion, all the signals of the closed-loop system (1)-(40)-(20) are bounded.

Finally, note that $\dot{\xi_i}$, i = 1 ,…, n are also bounded and $\int_0^{+\infty} \xi_i^2 \, dt < +\infty$. It is thus deduced from the Barbalat's lemma that ξ_i , i = 1 ,…, n converge to zero as t ! +1. This, in view of the coordinate transformation (21), implies that the state x tends to the origin. In this way, the proof is completed.

Example 1 Consider the time-delay planar system with unknown control directions:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \theta_2 u + x_2^2(t-d). \end{aligned}$$

Using the proposed design procedure, the following controller

$$\begin{aligned} u &= k^2 \cos(k)\xi_2[2l_1 + \frac{61}{4} + (\frac{17}{4} + \xi_2^2 + x_1^2)^2] \\ \dot{k} &= \xi_2^2[2l_1 + \frac{61}{4} + (\frac{17}{4} + \xi_2^2 + x_1^2)^2], \end{aligned}$$

can globally regulate the closed-loop system.

The simulations of the trajectories (x_1, x_2) and (l_1; k) of the closed-loop system are shown in the figure below, with the parameters θ_2 = -1, d = 1:25 and the initial condition ($x_1(0)$, $x_2(0)$) = (0:75, -1:25). Notably, the proposed controller is independent of the time-delay, and hence it also works for a large delay d as long as d is finite.

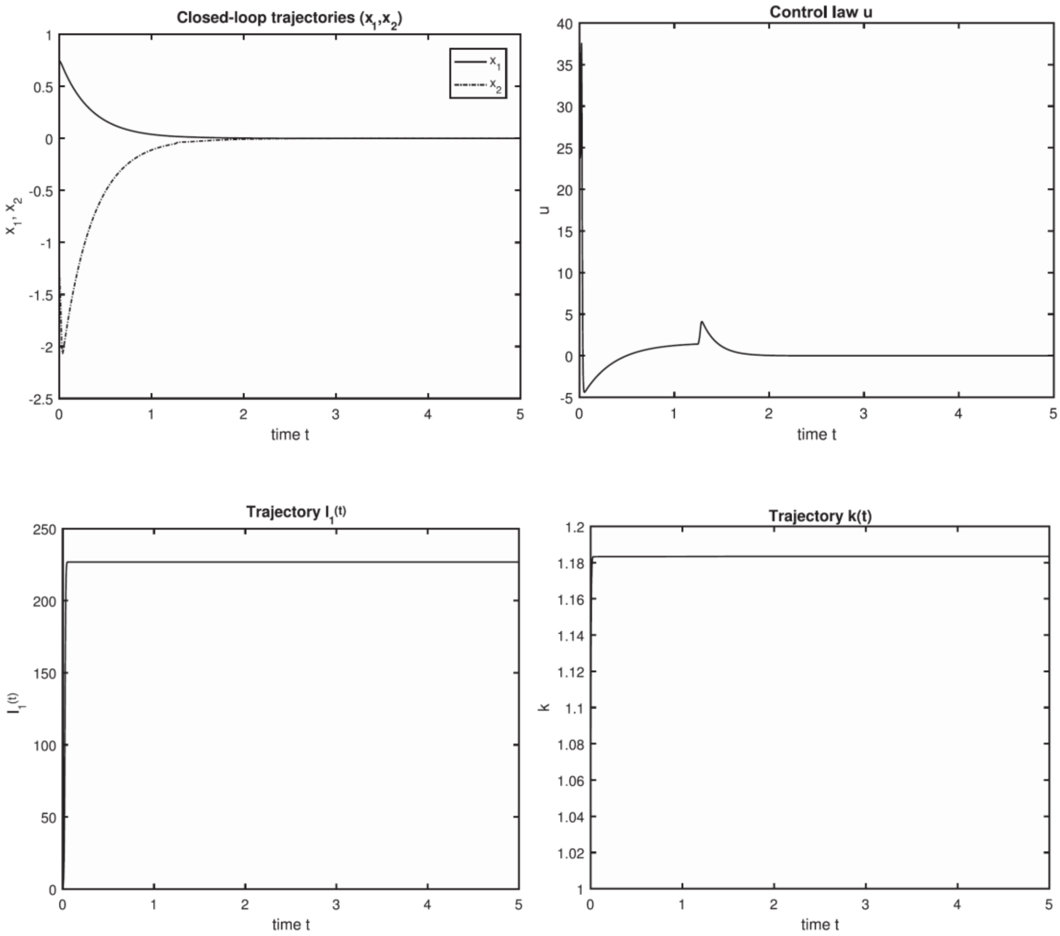

Fig 1: example 1 state tragetories of closed-loop system

Extension to P-normal form

Control of time-delay systems is a frequently encountered problem in various real world applications. In fact, network systems, chemical processes, biological systems, milling processes, drilling processes and fluid flow, to name just a few, all involve the time-delay issue. This paper first considers a family of time-delay nonlinear systems with unknown control directions of the form

$$\begin{aligned}
\dot{x}_i &= \theta_i x_{i+1}^{p_i} + f_i(x_1, \cdots, x_i, x_1(t-d), \cdots, x_i(t-d)),\\
\dot{x}_n &= \theta_n u^{p_n} + f_n(x, x(t-d)),\\
x(s) &= \zeta(s), \quad s \in [-d, 0],
\end{aligned} \tag{44}$$

When the time-delay system (44) has a known control direction, the global stabilization problem has been addressed recently by non-smooth state feedback, although the nonlinear system (44) is in general not stabilizable, even locally, by smooth state feedback (this is true even if the time-delay d = 0, due to the presence of the uncontrollable/unstable linearization at the origin).

Nonsmooth Dynamic State Feedback With The Nussbaum Functions

In this section, we adapt the idea from universal control coupled with the feedback control strategy, to design a delay-free, dynamic state compensator that achieves global asymptotic state regulation with boundedness for the time-delay nonlinear system (44) with unknown control direction. As we shall see, the proposed dynamic compensator contains two sets of dynamic state feedback control laws. One of them is capable of mitigating the effects of the unknown control direction, while the other one is able to counteract the time-delay nonlinearities of the system (44). Notably, the idea of utilizing two sets of gain update laws has been explored in the area of adaptive control of nonlinear systems with unknown parameters by output feedback. In this work, we demonstrate how a similar philosophy can be applied to effectively control the time-delay system (44) with unknown control direction.

Step 1: For the x_1-subsystem of the time-delay system (44) with the unknown sign of $\theta 1$, one can regard x_2 as a virtual control. Similarly to the previous design, define $\xi_1 = x_1$ and construct the Lyapunov function $V_1(x_1, l_1) = \frac{1}{2}\left(1 + \frac{1}{l_1}\right)\xi_1^2$ is a dynamic gain to be designed in Step 2. Then, a direct computation gives

$$\begin{aligned} \dot{V}_1 \quad \leq \quad & (1 + \frac{1}{l_1})\theta_1 \xi_1 x_2^{*p_1} - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \\ & +2\bar{c}|\xi_1\xi_2| + 2\left|\xi_1 f_1(x_1, x_1(t-d))\right|, \end{aligned} \tag{45}$$

$$2|\xi_1 f_1(\cdot)| \leq 2\xi_1^2\bar{\gamma}_1(x_1) + \xi_1^2 + \xi_1^2(t-d)\,\bar{\gamma}_1^{*2}(x_1(t-d)) \tag{46}$$

where $\xi_2 = x_2 - x_2^*$

Construct the Lyapunov-Krasovskii functiona

$$V_{1LK} = V_1(x_1, l_1) + \int_{t-d}^{t} \xi_1^2(s)\,\bar{\gamma}_1^{*2}(x_1(s))\,ds.$$

From (45)-(46), it follows that

$$\begin{aligned} \dot{V}_{1LK} \quad \leq \quad & -n\xi_1^2 + (1 + \frac{1}{l_1})\theta_1\xi_1 x_2^{*p_1} - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \\ & +\xi_1^2(2 + n + 2\bar{\gamma}_1(\cdot) + \bar{\gamma}_1^{*2}(\cdot)) + c_2\xi_2^2. \end{aligned} \tag{47}$$

To cope with the unknown sign of θ_1, we use the Nussbaum function for the design of a virtual controller. Specically, a virtual controller with the Nussbaum gain can be constructed as

$$\begin{aligned} x_2^{*p_1} &= \xi_1 N(k_1)(2+n+2\bar{\gamma}_1(\cdot)+\gamma_1^{*2}(\cdot)) := \xi_1 N(k_1)\beta_1(x_1) \\ \dot{k}_1 &= (1+\frac{1}{l_1})\xi_1^2\beta_1(x_1), \quad k_1(0)=1. \end{aligned} \tag{48}$$

This, together with l_1>1, results in

$$\dot{V}_{1LK} \leq -n\xi_1^2+(\theta_1 N(k_1)+1)\dot{k}_1+c_2\xi_2^2-\frac{\dot{l}_1}{2l_1^2}\xi_2^2. \tag{49}$$

Step 2: For the (x1; x2) subsystem of the time-delay system (44) with the unknown sign of θ_2, we construct the Lyapunov-Krasovskii functional

$$\begin{aligned} V_2 &= V_{1LK}+\frac{1}{l_1}k_1^2W_2(\cdot)+\frac{1}{l_1l_2}\Big[\frac{\xi_1^2}{2}+k_1^2W_2(\cdot)\Big] \\ W_2(k_1,x_1,x_2) &= \int_{x_2^*}^{x_2}(s^{p_1}-x_2^{*p_1})^{2-1/p_1}ds, \end{aligned} \tag{50}$$

where l_2>1 is a dynamic gain to be designed in the next step. Following the same argument in previous section, one can prove that $W_2(k_1; x_1; x_2)$ is C^1 and its partial derivatives are

$$\begin{aligned} \frac{\partial W_2}{\partial x_2} &= \xi_2^{2-1/p_1}, \\ \frac{\partial W_2}{\partial x_1} &= -(2-\frac{1}{p_1})\frac{\partial x_2^{*p_1}}{\partial x_1}\int_{x_2^*}^{x_2}(s^{p_1}-x_2^{*p_1})^{1-1/p_1}ds \\ \frac{\partial W_2}{\partial k_1} &= -(2-\frac{1}{p_1})\frac{\partial x_2^{*p_1}}{\partial k_1}\int_{x_2^*}^{x_2}(s^{p_1}-x_2^{*p_1})^{1-1/p_1}ds. \end{aligned} \tag{51}$$

Since l_j >1, it is deduced from (49) and (51) that

$$\begin{aligned} \dot{V}_2 \leq\ & -n\xi_1^2+(\theta_1 N(k_1)+1)\dot{k}_1+c_2\xi_2^2-\frac{\dot{l}_1}{2l_1^2}\xi_2^2 \\ & +\frac{k_1^2}{l_1}(1+\frac{1}{l_2})\theta_2\xi_2^{2-1/p_1}(x_3^{*p_2}+x_3^{p_2}-x_3^{*p_2}) \\ & +\frac{2}{l_1}\left|k_1^2\left[\xi_2^{2-1/p_1}f_2(\cdot)+\frac{\partial W_2}{\partial x_1}\dot{x}_1+\frac{\partial W_2}{\partial k_1}\dot{k}_1\right]+k_1\dot{k}_1W_2(\cdot)\right| \\ & +\frac{1}{l_1l_2}\xi_1\dot{x}_1-\frac{\dot{l}_1}{l_1^2}k_1^2W_2(\cdot)-\frac{\dot{l}_1l_2+l_1\dot{l}_2}{l_1^2l_2^2}(\frac{\xi_1^2}{2}+k_1^2W_2(\cdot)). \end{aligned} \tag{52}$$

From $\xi_2 = x_2^{p_1} - x_2^{*p_1}$, (48) and (51), it is not difficult to obtain,

$$
\begin{aligned}
&\tfrac{2k_1^2}{l_1}|\xi_2^{2-1/p_1}f_2(\cdot)| \leq k_1^2\xi_2^2\Upsilon_{21}(k_1,x_1,x_2)+\frac{1}{l_1}\xi_1^2\Upsilon_{22}(k_1,x_1)\\
&\quad+\frac{1}{l_1}\xi_1^2(t-d)\Upsilon_{22}^*(k_1(t-d),x_1(t-d))\\
&\quad+\xi_2^2(t-d)\Upsilon_{21}^*(k_1(t-d),x_1(t-d),x_2(t-d)),\\
&\tfrac{2k_1^2}{l_1}\left|\tfrac{\partial W_2}{\partial x_1}\dot{x}_1+\tfrac{\partial W_2}{\partial k_1}\dot{k}_1\right|+\tfrac{2}{l_1}k_1\dot{k}_1W_2(\cdot)+\tfrac{1}{l_1l_2}\xi_1\dot{x}_1\\
&\quad\leq k_1^2\xi_2^2\Phi_{21}(k_1,x_1,x_2)+\frac{1}{l_1}\xi_1^2\Phi_{22}(k_1,x_1)\\
&\quad+\frac{1}{l_1}\xi_1^2(t-d)\Phi_2^*(x_1(t-d)),
\end{aligned}
\tag{53}
$$

One can construct the Lyapunov-Krasovskii functional

$$
\begin{aligned}
V_{2LK} &= V_2+\int_{t-d}^{t}\xi_2^2(s)\Upsilon_{21}^*(k_1(s),x_1(s),x_2(s))ds\\
&\quad+\int_{t-d}^{t}\frac{1}{l_1(s)}\xi_1^2(s)[\Upsilon_{22}^*(k_1(s),x_1(s))+\Phi_2^*(x_1(s))]ds
\end{aligned}
\tag{54}
$$

Then, it is deduced from (52) and (53) that

$$
\begin{aligned}
&\dot{V}_{2LK}\leq -n\xi_1^2-(n-1)k_1^2\xi_2^2+(\theta_1N(k_1)+1)\dot{k}_1-\frac{\dot{l}_1}{2l_1^2}\xi_2^2\\
&+\frac{1}{l_1}\xi_1^2\left[\Upsilon_{22}(k_1,x_1)+\Upsilon_{22}^*(k_1,x_1)+\Phi_{22}(k_1,x_1)+\Phi_2^*(x_1)\right]\\
&+\frac{k_1^2}{l_1}(1+\frac{1}{l_2})\theta_2\xi_2^{2-1/p_1}x_3^{*p_2}+\frac{2\bar{c}}{l_1}k_1^2|\xi_2^{2-1/p_1}(x_3^{p_2}-x_3^{*p_2})|\\
&+k_1^2\xi_2^2\left[c_2+(n-1)+\Upsilon_{21}(k_1,x_1,x_2)+\Upsilon_{21}^*(k_1,x_1,x_2)\right.\\
&\left.+\Phi_{21}(k_1,x_1,x_2)\right]-\frac{\dot{l}_2}{l_1l_2^2}(\frac{\xi_1^2}{2}+W_2(\cdot)).
\end{aligned}
\tag{55}
$$

The inequality above is derived by neglecting the negative terms that are related to $\dot{l}_1$. From (55), it is not difficult to show that the dynamic state compensator

$$
\dot{l}_1=\max\{-l_1^2+l_1\rho_1(k_1,x_1),\ 0\},\ l_1(0)=1, \tag{56}
$$

$$
\rho_1(k_1,x_1)=2\left[\Upsilon_{22}(\cdot)+\Upsilon_{22}^*(\cdot)+\Phi_{22}(\cdot)+\Phi_2^*(\cdot)\right] \tag{57}
$$

can counteract the effect of the time-delay nonlinearity. In fact, by construction the gain l_1 satisfies

$$
0\leq\dot{l}_1\leq l_1\rho_1(\cdot),\ \dot{l}_1\geq -l_1^2+l_1\rho_1(\cdot), l_1\geq l_1(t-d)\geq 1 \tag{58}
$$

As a consequence,

$$
-\frac{\dot{l}_1}{2l_1^2}\xi_1^2\leq\xi_1^2-\frac{1}{2l_1}\xi_1^2\rho_1(k_1,x_1) \tag{59}
$$

Moreover,

$$\frac{2\bar{c}_2}{l_1}k_1^2\left|\xi_2^{2-1/p_1}(x_3^{p_2}-x_3^{*p_2})\right| \le \bar{c}_2 k_1^2\xi_2^2 + c_3 k_1^2 \xi_3^2, \tag{60}$$

where $\xi_3 = x_3^{p_1 p_2} - x_3^{*p_1 p_2}$.

Substituting (59) and (60) into (55), we arrive at

$$\begin{aligned}\dot{V}_{2LK} \le & -n\xi_1^2 - (n-1)k_1^2\xi_2^2 + (\theta_1 N(k_1)+1)\dot{k}_1 + c_3 k_1^2\xi_3^2 \\ & + \frac{k_1^2}{l_1}(1+\frac{1}{l_2})\theta_2\xi_2^{2-1/p_1}x_3^{*}p_2 + k_1^2\xi_2^2[c_2+\bar{c}_2+n-1 \\ & + \Upsilon_{21}(\cdot)+\Upsilon_{21}^*(\cdot)+\Phi_{21}(\cdot)] - \frac{\dot{l}_2}{l_1 l_2^2}(\frac{\xi_1^2}{2}+W_2(\cdot))\end{aligned} \tag{61}$$

Similar to Step 1, because of the unknown sign of θ_2, we design the virtual controller

$$\begin{aligned} x_3^{*p_2} &= l_1 N(k_2)\xi_2^{1/p_1}[\bar{c}_2 + n + \Upsilon_{21}(\cdot)+\Upsilon_{21}^*(\cdot)+\Phi_{21}(\cdot)] \\ &:= l_1 N(k_2)(\xi_2\beta_2(k_1,x_1,x_2))^{1/p_1} \\ \dot{k}_2 &= (1+\frac{1}{l_2})\xi_2^2\beta_2^{1/p_1}(k_1,x_1,x_2), \quad k_2(0)=1. \end{aligned} \tag{62}$$

with the Nussbaum gain k_2 that is updated dynamically. Clearly, the dynamic compensator (62) leads to

$$\begin{aligned}\dot{V}_{2LK} \le & -(n-1)(\xi_1^2+k_1^2\xi_2^2) + (\theta_1 N(k_1)+1)\dot{k}_1 + c_3 k_1^2\xi_3^2 \\ & + (\theta_2 N(k_2)+1)k_1^2\dot{k}_2 - \frac{\dot{l}_2}{l_1 l_2^2}(\frac{\xi_1^2}{2}+W_2(\cdot)).\end{aligned} \tag{63}$$

<u>Inductive Step</u>: At step i-1, assume that there are a Lyapunov-Krasovskii functional $V_{(i-1)LK}$, a set of dynamic gains $l_j > 1$, j = 1,…,i-1, updated by

$$\begin{aligned} \dot{l}_1 &= \max\{-l_1^2 + l_1\rho_1(k_1,x_1),0\}, \\ \dot{l}_2 &= \max\{-\alpha_2 l_2^2 + l_2\rho_2(l_1,k_1,k_2,x_1,x_2),0\}, \\ &\vdots \\ \dot{l}_{i-2} &= \max\{-\alpha_{i-2}l_{i-2}^2 + l_{i-2}\rho_{i-2}(\bar{l}_{i-3},\bar{k}_{i-2}\bar{x}_{i-2}),0\}, \end{aligned} \tag{64}$$

with $\alpha_j = \frac{1}{2^{p_1 \cdots p_{j-1}-1}}$, and a set of non-smooth but C^0 virtual controllers with the Nussbaum gains, given by

$$\begin{array}{lll} x_1^* & = 0 & \xi_1 = x_1 - x_1^* \\ x_2^{*p_1} & = \xi_1 N(k_1)\beta_1(x_1) & \xi_2 = x_2^{p_1} - x_2^{*p_1} \\ \dot{k}_1 & = (1+\frac{1}{l_1})\xi_1^2\beta_1(\cdot) & \\ & \vdots & \vdots \\ x_i^{*p_1\cdots p_{i-1}} & = (l_1\cdots l_{i-2}N(k_{i-1}))^{p_1\cdots p_{i-2}} & \xi_i = x_i^{p_1\cdots p_{i-1}} \\ & \cdot\xi_{i-1}\beta_{i-1}(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1}) & \quad -x_i^{*p_1\cdots p_{i-1}} \\ \dot{k}_{i-1} & = (1+\frac{1}{l_{i-1}})\xi_{i-1}^2\beta_{i-1}^{1/p_1\cdots p_{i-2}}(\cdot) & \end{array} \tag{65}$$

such that

$$
\begin{aligned}
\dot{V}_{(i-1)LK} \leq & -(n-(i-2))\Sigma_{j=1}^{i-1}[(\prod_{m=0}^{j-1} k_m^2)\xi_j^2] \\
& +c_i k_1^2 \cdots k_{i-2}^2 \xi_i^2 + \Sigma_{j=1}^{i-1}\Big[(\theta_j N(k_j)+1)(\prod_{m=0}^{j-1} k_m^2)\dot{k}_j\Big] \\
& -\frac{\dot{l}_{i-1}}{l_1 \cdots l_{i-2} l_{i-1}^2}(\frac{\xi_1^2}{2}+\sum_{j=2}^{i-1} W_j(\bar{l}_{j-2}, \bar{k}_{j-1}, \bar{x}_j)) \qquad (66)
\end{aligned}
$$

where $c_i > 0$ is a constant and $k_0 = 1$. Clearly, (66) reduces to (63) when i = 3. We claim that (66) also holds at Step i. To prove this claim, consider the Lyapunov-Krasovskii functional

$$
\begin{aligned}
V_{iLK} = & V_i + \int_{t-d}^{t} \xi_i^2(s)\Upsilon_{i1}^*(\bar{l}_{i-2}(s), \bar{k}_{i-1}(s), \bar{x}_i(s))ds \\
& + \int_{t-d}^{t} \frac{1}{l_1(s)\cdots l_{i-1}(s)}[\xi_1^2(s)+\sum_{j=2}^{i-1}(x_j(s)-x_j^*(s))^{2p_1\cdots p_{j-1}}] \\
& \cdot\big[\Upsilon_{i2}^*(\bar{l}_{i-2}(s), \bar{k}_{i-1}(s), \bar{x}_{i-1}(s)) \\
& +\Phi_{i2}^*(\bar{l}_{i-2}(s), \bar{k}_{i-1}(s), \bar{x}_{i-1}(s)) \\
& +\Psi_i^*(\bar{l}_{i-2}(s), \bar{k}_{i-1}(s), \bar{x}_{i-1}(s))\big]ds \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
V_i = & V_{(i-1)LK} + \frac{k_1^2 \cdots k_{i-1}^2}{l_1 \cdots l_{i-1}} W_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i) \\
& + \frac{1}{l_1 \cdots l_i}[\frac{\xi_1^2}{2}+\sum_{j=2}^{i-1} W_j(\bar{l}_{j-2}, \bar{k}_{j-1}, \bar{x}_j) \qquad (68) \\
& + k_1^2 \cdots k_{i-1}^2 W_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i)] \\
W_i = & \int_{x_i^*}^{x_i} \left(s^{p_1\cdots p_{i-1}} - x_i^{*p_1\cdots p_{i-1}}\right)^{2-1/(p_1\cdots p_{i-1})} ds,
\end{aligned}
$$

where l_i>1 is a dynamic gain to be designed, and

$$
m_i(x_i - x_i^*)^{2p_1\cdots p_{i-1}} \leq W_i(\cdot) \leq (2^{p_1\cdots p_{i-1}}-1)\xi_i^2 \qquad (69)
$$

for a positive constant m_i.

Repeating the same argument in Step 2, there are the delay-free gain update law

$$
\dot{l}_{i-1}=\max\{-\alpha_{i-1} l_{i-1}^2 + l_{i-1}\rho_{i-1}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}), 0\} \qquad (70)
$$

and

$$
\begin{aligned}
\rho_{i-1}(\cdot) = & \frac{1}{M_{i-1}}\big[\Upsilon_{i2}(\cdot)+\Upsilon_{i2}^*(\cdot)+\Phi_{i2}(\cdot)+\Phi_{i2}^*(\cdot) \\
& +\Psi_i(\cdot)+\Psi_i^*(\cdot)\big], \\
M_{i-1} = & \min\{1/2, m_2, \cdots, m_{i-1}\}. \qquad (71)
\end{aligned}
$$

and a non-smooth virtual controller with the Nussbaum gain

$$
\begin{aligned}
x_{i+1}^{*p_i} &= l_1 \cdots l_{i-1} N(k_i) \xi_i^{1/(p_1 \cdots p_{i-1})} \Big[2 + c_i + \bar{c}_i + n - i \\
&\quad + \Upsilon_{i1}(\cdot) + \Upsilon_{i1}^{*}(\cdot) + \Phi_{i1}(\cdot) + \omega_i(\cdot) \Big] \\
&:= l_1 \cdots l_{i-1} N(k_i) (\xi_i \beta_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i))^{1/(p_1 \cdots p_{i-1})} \\
\dot{k}_i &= (1 + \frac{1}{l_i}) \xi_i^2 \beta_i(\cdot)^{1/(p_1 \cdots p_{i-1})} \qquad (72)
\end{aligned}
$$

In addition, (66), also holds at Step i. Using the claim for i = n+1 with $u = x_{n+1} = x_{n+1}^*$, we conclude that the dynamic state feedback controller that is composed of (64) with i = n + 1 and

$$
\begin{aligned}
u &= (l_1 \cdots l_{n-1} N(k_n))^{\frac{1}{p_n}} \left(\xi_n \beta_n(\bar{l}_{n-2}, \bar{k}_{n-1}, x) \right)^{\frac{1}{(p_1 \cdots p_n)}} \\
\dot{k}_n &= \xi_n^2 \beta_n(\bar{l}_{n-2}, \bar{k}_{n-1}, x)^{\frac{1}{(p_1 \cdots p_{n-1})}} \qquad (73)
\end{aligned}
$$

is such that

$$
\dot{V}_{nLK} \leq -\sum_{j=1}^{n} [(\prod_{m=0}^{j-1} k_m^2) \xi_j^2] + \sum_{j=1}^{n} [(\theta_j N(k_j) + 1)(\prod_{m=0}^{j-1} k_m^2) \dot{k}_j]. \qquad (74)
$$

State regulation and boundedness of closed-loop trajectories can be proved in a similar fashion to the proof in the previous section. Therefore, we can conclude that the problem of state regulation of (44) can be solved by (73).

สรุปและวิจารณ์ผลการทดลอง

In this study, we have investigated the problem of global state regulation with stability for nonlinear systems with both time-delay uncertainties and unknown control directions. A delay-free, dynamic state feedback control strategy has been developed based on the dynamic gain-based design technique [19] and the idea of universal control with the Nussbaum function [13]. The proposed dynamic state feedback compensators consists of two sets of gain update laws, which are a reminiscent of the work [6, 7, 8] on universal control of nonlinear systems with unknown parameters by output feedback. One set of gain update laws is a Riccati-type, effective in counteracting the time-delay nonlinearities, while the other set of dynamic update laws is an universal control-like using the Nussbaum function, capable of mitigating the effects of unknown control directions. In contrast to the work [19], a set of new Lyapunov-Krasovskii functionals have been constructed in this paper, in order to cope with both time-delay uncertainties and unknown control directions simultaneously. It has been shown that the proposed dynamic state feedback control scheme can be extended to a class of p-normal form in which a new continuous controller has to be used since this type of systems is not stabilizable, even locally, by smooth state feedback.

ข้อเสนอแนะสำหรับงานวิจัยในอนาคต

For future study, this type of controller can possibly be applied to control traffic of an interconnected network system where delay is naturally occurred. The main advantage of this control design is the delay free design. Since the controller does not depend on the delay of the system, it is possible to use a single controller to control many system with different delay.

เอกสารอ้างอิง

[1] K. Gu, V. Kharitonov and J. Chen, Stability of Time-Delay Systems. Boston: Birkhauser, 2003.

[2] C. Hua, X. Liu and X. Guan, “Backstepping control for nonlinear systems withtime delays and applications to chemical reactor systems,” IEEE Trans. Ind. Electron., vol. 56, pp. 3723-3732 (2009).

[3] M. Jankovic, “Control Lyapunov-Razumikhin functions and robust stabilization of time delay systems,” IEEE TAC, vol. 46, 1048-1060, 2001.

[4] I. Karafyllis and Z. Jiang, “Necessary and sufficient Lyapunov-like conditions for robust nonlinear stabilization,” ESAIM Control Optim. Calc. Var., vol. 16, pp. 887-928 (2010).

[5] M. Krstic, “Input delay compensation for forward complete and strict feedforward nonlinear systems,” IEEE TAC vol. 55, pp. 287-303 (2010).

[6] H. Lei and W. Lin, “Universal adaptive control of nonlinear systems with unknown growth rate by output feedback,” Automatica vol. 42, pp. 1783-1789 (2006).

[7] H. Lei and W. Lin, “Adaptive regulation of uncertain nonlinear systems by output feedback: a universal control approach,” Systems and Control Letters, Vol. 56, pp. 529-537 (2007)

[8] H. Lei and W. Lin, “Adaptive control of nonlinear systems with unknown parameters by output feedback: a non-identifier based method,” Lecture Notes in Control and Information Sciences, A. Astolfi and L. Marconi eds., Springer-Verlag, Heidelberg, pp. 445-463 (2007).

[9] W. Lin and C. Qian, “Adaptive control of nonlinearly parameterized systems: The smooth feedback case,” IEEE Trans. Automat. Contr., Vol. 47, pp. 1249-1266 (2002).

[10] F. Mazenc, S. Mondie and S. I. Niculescu, “Global asymptotic stabilization for chains of integrators with a delay in the input,” IEEE Trans. Automat. Contr., vol. 48, pp. 57-63, 2003.

[11] F. Mazenc, S. Mondie and R. Francisco,“Global asymptotic stabilization of feedforward systems with delay in the input, ” Proc. of the 42nd IEEE CDC, Maui, Hawaii, pp. 4020-4025 (2003).

[12] S. Nguang, “Robust stabilization of a class of time-delay nonlinear systems,”IEEE Trans. Automat. Contr., vol. 45, 756-762 (2000).

[13] R. D. Nussbaum, “Some remarks on a conjecture in parameter adaptive control,” Systems & Control Letters, Vol. 3, pp. 243-246 (1983).

[14] P. Ordaz, O. J. Santos-Sanchez, Omar-Jacobo; L. Rodriguez-Guerrero, and A. Gonzlez-Facundo, “Nonlinear stabilization for a class of time delay systems via inverse optimality approach,” ISA Transactions, Vol. 67, pp. 1-8 (2017).

[15] P. Pepe, “On Sontag’s formula for the input-to-state practical stabilization of retarded control-affine systems, ” Systems & Control Letters, Vol. 62, pp. 1018-1025 (2013).

[16] J. P. Richard, “Time-delay systems: An overview of some recent advances and open problems,” Automatica, vol. 39, 1667-1694 (2003). 16

[17] X. Zhang, W. Lin and Y. Lin, “Dynamic partial state feedback control of cascade systems with time-delay, Automatica, Vol. 77, pp. 370-379 (2017).

[18] X. Ye, “Asymptotic regulation of uncertain nonlinear systems with unknown control directions,” Automatica, vol. 35, 929-935, 1999.

[19] X. Zhang, W. Lin and Y. Lin, “Nonsmooth Control of Time-Delay Nonlinear Systems by Dynamic State Feedback,” Proc. of the 54th IEEE CDC, Osaka, Japan, 7715-7722 (2015). Also, IEEE Trans. Automat. Contr., vol. 62, 438-444 (2017).

Output จากโครงการวิจัยที่ได้รับทุนจาก สกว.

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   2) Pongvuthithum, R., Rattanamongkhonkun, K., Lin, W., Asymptotic Regulation of Time-Delay Nonlinear Systems with Unknown Control Directions, IEEE Transactions on Automatic Control, 63 (5), pp. 1495-1502 (2018). published
   3) Lin, W., Rattanamongkhonkun, K., Pongvuthithum, R., LgV-Type adaptive controllers for uncertain non-affine systems and application to a DC-Microgrid with PV and battery, IEEE Transactions on Automatic Control, 64 (5), pp. 2182-2189 (2019). published

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Nonsmooth feedback stabilization of a class of nonlinear systems with unknown control direction and time delay

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Summary

The problem of global stabilization by nonsmooth state feedback is investigated for a family of time-delay nonlinear systems with unknown control direction. Using the idea of the Nussbaum function from universal control, we present a delay-free, nonsmooth dynamic state feedback strategy to achieve asymptotic state regulation with boundedness of the closed-loop system. The proposed control scheme allows one to design a set of Lyapunov-Krasovskii functionals and nonsmooth dynamic compensators simultaneously, by the technique of adding a power integrator. It is shown that while the effects of unknown control direction can be mitigated by the Nussbaum-type gains, the strong nonlinearity with time delay can be effectively dealt with by nonsmooth state feedback controllers whose gains are updated dynamically.

KEYWORDS

dynamic state compensation, nonlinear systems, nonsmooth feedback, time-delay, unknown control directions

1 | INTRODUCTION

Control of time-delay systems is a frequently encountered problem in various real-world applications. In fact, network systems, chemical processes, biological systems, milling processes, drilling processes, and fluid flow, to name just a few, all involve the time-delay issue.

This paper first considers a family of time-delay nonlinear systems with unknown control directions of the form

$$
\begin{aligned}
\dot{x}_i &= \theta_i x_{i+1}^{p_i} + f_i\left(x_1, \ldots, x_i, x_1(t-d), \ldots, x_i(t-d)\right), \\
\dot{x}_n &= \theta_n u^{p_n} + f_n\left(x, x(t-d)\right), \\
x(s) &= \zeta(s), \quad s \in [-d, 0],
\end{aligned} \tag{1}
$$

where $i = 1, \ldots, n-1, x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state an input, respectively. The constant $d \geq 0$ is an unknown time-delay of the system, $p_i > 0$ are odd integers, $f_i : \mathbb{R}^{2i} \rightarrow \mathbb{R}$ are C^1 mappings with $f_i(0,0) = 0$, and $\zeta(s) \in \mathbb{R}^n$ is a continuous function defined on $[-d, 0]$. The coefficients $\theta_i \neq 0, 1 \leq i \leq n$, are unknown constants whose bound is known. They represent unknown control direction and can be either positive or negative. The motivation for studying the unknown control direction problem was already explained in the context of universal control (see, for instance, the works of Nussbaum[1] and Willems and Byrnes[2]) where even for the one-dimensional linear system

$$\dot{x} = ax + bu, \qquad x \in R,$$

with a and $b \neq 0$ being unknown constants, it was illustrated how adaptive control of the linear system (a, b) is naturally connected to the problem of unknown control direction.

When the time-delay system (1) has a known control direction (eg, $\theta_i = 1$ for $i = 1, \dots, n$), the global stabilization problem has been addressed recently by nonsmooth state feedback,[3] although the nonlinear system (1) is in general not stabilizable, even locally, by smooth state feedback (this is true even if the time delay $d = 0$, due to the presence of the uncontrollable/unstable linearization at the origin).

For the analysis and synthesis of time-delay systems,[4-6] the Lyapunov-Krasovskii and Lyapunov-Razumikhin methods are two popular tools that have been found wide applications. In the literature, research of time-delay systems can be classified primarily into three different categories. The first category of study focuses on the time delay in the system state,[4] whereas the second one is aimed at the time delay in the control input.[7,8] The last category addresses a general case where the time delay is present in both the control input and the system state. For each category of time-delay nonlinear control problems, substantial progress has been made and various results have been obtained (see, for instance, other related works[3,7,9] and the references therein).

Following the line of the research in the work of Zhang et al,[3] we study in this work the global stabilization of the time-delay nonlinear system (1) by nonsmooth state feedback in the presence of unknown control directions. Most of the aforementioned works concentrated on time-delay nonlinear systems with known control directions, ie, the signs of all coefficients of the chain of "nonlinear integrators" are assumed to be known. If this crucial information is not available, a new feedback design method needs to be developed for the control of time-delay systems. When no time delay is involved and the linearized system is controllable (eg, $d = 0$ and $p_i = 1, i = 1, \dots, n$ in (1)), a feedback control scheme based on the Nussbaum functions[1] was proposed in the work of Ye[10] for a class of lower-triangular systems. It was shown that the idea of the Nussbaum functions is effective[1,2] in dealing with the unknown control direction issue.

Note that the sign of the control input often represents motion direction of mechanical systems (for example, robotics modeled by the Lagrange equation) and may be unknown. Therefore, it is certainly important necessary to investigate the question of how to control time-delay nonlinear systems when control directions are not known. Motivated by the universal control idea[1,11-14] and the recent development,[3] we propose in this work a delay-free, nonsmooth dynamic state feedback compensation scheme, together with the idea of Nussbaum functions, to globally stabilize the time-delay nonlinear system (1) with unknown control directions. In particular, an iterative algorithm is developed for the construction of a set of Lyapunov-Krasovskii functionals as well as a delay-free, dynamic state compensator that mitigates the effects of time-delay nonlinearities and unknown control direction in the nonlinear system (1) simultaneously. More specifically, global state regulation of the time-delay system (1) with boundedness of all the signals is guaranteed by the proposed nonsmooth dynamic compensator. Based on this main result, we further show how it can be generalized to a much boarder class of time-delay nonlinear systems with uncertainty, under a homogeneous-like growth condition that can be viewed as a natural extension of the well-known lower-triangular condition. Finally, a simple but nontrivial example is presented to illustrate the significance of the finding obtained in this paper.

Notation. Denote $\bar{v}_i = [v_1, \dots, v_i]^T \in \mathbb{R}^i$, for $i = 1, \dots, n$. For instance, $\bar{x}_i = [x_1, \dots, x_i]^T$, $\bar{x}_i(t-d) = [x_1(t-d), \dots, x_i(t-d)]^T$ and $\bar{l}_i = [l_1, \dots, l_i]^T$. A Nussbaum function $N(k) = k^2\cos(k)$, which is obviously an even function, will be used throughout this work. It is not difficult to verify that it satisfies the following properties: (i) $\lim_{k\to+\infty}\sup \frac{1}{k}\int_0^k N(s)ds = +\infty$; (ii) $\lim_{k\to+\infty}\inf \frac{1}{k}\int_0^k N(s)ds = -\infty$.

2 | PRELIMINARY

This section collects a number of useful lemmas to be frequently used in this paper.

Lemma 1. (See the works of Qian and Lin[15,16])
For positive real numbers m, n and a real-valued function $\pi(x, y) > 0$, the following inequality holds $\forall x, y \in \mathbb{R}$:

$$|x|^m|y|^n \leq \frac{m}{m+n}\pi(x, y)|x|^{m+n} + \frac{n}{m+n}\pi^{-m/n}(x, y)|y|^{m+n}. \tag{2}$$

Lemma 2. (See the works of Lin and Qian[17])
For a C^0 function $f(x,y)$, there are smooth functions $a(x) \geq 0$, $b(y) \geq 0$, $c(x) \geq 1$ and $d(y) \geq 1$, such that

$$|f(x,y)| \leq a(x) + b(y), \quad |f(x,y)| \leq c(x)d(y). \tag{3}$$

Lemma 3. (See the works of Qian and Lin[15,16])
Let $x, y \in \mathbb{R}$ and $p \geq 1$ be an integer. Then,

$$\begin{aligned} &|x+y|^p \leq 2^{p-1}\left|x^p + y^p\right|, \\ &(|x|+|y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}(|x|+|y|)^{\frac{1}{p}}. \end{aligned} \tag{4}$$

If p is an odd positive integer, then

$$|x-y|^p \leq 2^{p-1}|x^p - y^p|. \tag{5}$$

Lemma 4. (See the work of Zhang et al[3])
For a C^0 function $f(x,y)$ and a positive integer k, there exist smooth functions $g(x) \geq 0$ and $h(y) \geq 0$, such that

$$f(x,y)\left(|x|^k + |y|^k\right) \leq g(x)|x|^k + h(y)|y|^k. \tag{6}$$

Lemma 5. (See the work of Zhang et al[3])
For the C^1 function $f_i(\bar{x}_i, \bar{x}_i(t-d))$ with $f_i(0,0) = 0$, there exist smooth functions $\bar{\gamma}_{ij}(x_j) \geq 0$ and $\bar{\gamma}^*_{ij}(x_j(t-d)) \geq 0$, $j = 1, \dots, i$, such that

$$|f_i(\cdot)| \leq \sum_{j=1}^{i}\left(\bar{\gamma}_{ij}(x_j)|x_j| + \bar{\gamma}^*_{ij}(x_j(t-d))|x_j(t-d)|\right). \tag{7}$$

3 | NONSMOOTH DYNAMIC STATE FEEDBACK WITH THE NUSSBAUM FUNCTIONS

In this section, we adapt the idea from universal control,[1,11-14] coupled with the feedback control strategy in the work of Zhang et al,[3] to design a delay-free, dynamic state compensator that achieves global asymptotic state regulation with boundedness for the time-delay nonlinear system (1) with unknown control direction. As we shall see, the proposed dynamic compensator contains two sets of dynamic state feedback control laws. One of them is capable of mitigating the effects of the unknown control direction, whereas the other one is able to counteract the time-delay nonlinearities of the system (1). Notably, the idea of utilizing two sets of gain update laws has been explored in the area of adaptive control of nonlinear systems with unknown parameters by output feedback.[11-14] In this work, we demonstrate how a similar philosophy can be applied to effectively control the time-delay system (1) with unknown control direction.

Theorem 1. *For the time-delay nonlinear system (1) whose control directions are not known, there exists a delay-free, dynamic state feedback controller of the form*

$$\dot{L} = \eta(L,k,x), \quad \dot{k} = h(L,k,x), \quad u = \alpha(L,k,x), \tag{8}$$

with $\alpha(L,k,0) = 0$, such that the system state x converges to the origin while maintaining boundedness of the closed-loop system, where $\eta : \mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n-1}$, $h : \mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ and $\alpha : \mathbb{R}^{n-1}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ are C^0 mappings.

Proof. We apply the adding of a power integrator technique,[15,16] together with the idea of utilizing the Nussbaum functions[1] and dynamic gains,[3,11-14] to design a delay-free, nonsmooth dynamic state compensator (8) that does the job. □

Step 1: For the x_1-subsystem of the time-delay system (1) with the unknown sign of θ_1, one can regard x_2 as a virtual control. Define $\xi_1 = x_1$ and construct the Lyapunov function $V_1(x_1, l_1) = \frac{1}{2}(1+\frac{1}{l_1})\xi_1^2$, where $l_1(\cdot) \geq 1$ is a dynamic gain to be designed in Step 2. Then, a direct computation gives

$$\dot{V}_1 \leq \left(1+\frac{1}{l_1}\right)\theta_1\xi_1 x_2^{*p_1} - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 + 2\bar{c}|\xi_1\xi_2| + 2\left|\xi_1 f_1(x_1, x_1(t-d))\right|, \tag{9}$$

where $\xi_2 = x_2^{p_1} - x_2^{*p_1}$.

In view of Lemma 5, we have

$$|f_1(\cdot)| \le \bar{\gamma}_1(x_1)|x_1| + \bar{\gamma}_1^*(x_1(t-d))|x_1(t-d)|,$$

for some smooth functions $\bar{\gamma}_1(\cdot) \ge 0$ and $\bar{\gamma}_1^*(\cdot) \ge 0$. Hence,

$$2|\xi_1 f_1(\cdot)| \le 2\xi_1^2\bar{\gamma}_1(x_1) + \xi_1^2 + \xi_1^2(t-d)\bar{\gamma}_1^{*2}(x_1(t-d)). \tag{10}$$

Use the bound $\bar{\gamma}_1^*(\cdot)$ to construct the Lyapunov-Krasovskii functional

$$V_{1LK} = V_1(x_1, l_1) + \int_{t-d}^{t} \xi_1^2(s)\bar{\gamma}_1^{*2}(x_1(s))\,ds.$$

From (9)-(10), it follows that

$$\dot{V}_{1LK} \le -n\xi_1^2 + \left(1 + \frac{1}{l_1}\right)\theta_1\xi_1 x_2^{*p_1} - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 + \xi_1^2\left(2 + n + 2\bar{\gamma}_1(\cdot) + \bar{\gamma}_1^{*2}(\cdot)\right) + c_2\xi_2^2. \tag{11}$$

To cope with the unknown sign of θ_1, we use the Nussbaum function[1] for the design of a virtual controller. Specifically, a virtual controller with the Nussbaum gain can be constructed as

$$\begin{aligned} x_2^{*p_1} &= \xi_1 N(k_1)\left(2 + n + 2\bar{\gamma}_1(\cdot) + \gamma_1^{*2}(\cdot)\right) := \xi_1 N(k_1)\beta_1(x_1) \\ \dot{k}_1 &= \left(1 + \frac{1}{l_1}\right)\xi_1^2\beta_1(x_1), \quad k_1(0) = 1. \end{aligned} \tag{12}$$

This, together with $l_1(\cdot) \ge 1$, results in

$$\dot{V}_{1LK} \le -n\xi_1^2 + (\theta_1 N(k_1) + 1)\dot{k}_1 + c_2\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2. \tag{13}$$

Step 2: For the (x_1, x_2)-subsystem of the time-delay system (1) with the unknown sign of θ_2, we construct the Lyapunov-Krasovskii functional

$$\begin{aligned} V_2 &= V_{1LK} + \frac{1}{l_1}k_1^2 W_2(\cdot) + \frac{1}{l_1 l_2}\left[\frac{\xi_1^2}{2} + k_1^2 W_2(\cdot)\right] \\ W_2(k_1, x_1, x_2) &= \int_{x_2^*}^{x_2} \left(s^{p_1} - x_2^{*p_1}\right)^{2-1/p_1} ds, \end{aligned} \tag{14}$$

where $l_2(\cdot) \ge 1$ is a dynamic gain to be designed in the next step.

Following the argument in the works of Qian and Lin,[15,16] one can prove that $W_2(k_1, x_1, x_2)$ is C^1 and its partial derivatives are

$$\begin{aligned} \frac{\partial W_2}{\partial x_2} &= \xi_2^{2-1/p_1}, \\ \frac{\partial W_2}{\partial x_1} &= -\left(2 - \frac{1}{p_1}\right)\frac{\partial x_2^{*p_1}}{\partial x_1}\int_{x_2^*}^{x_2}\left(s^{p_1} - x_2^{*p_1}\right)^{1-1/p_1} ds \\ \frac{\partial W_2}{\partial k_1} &= -\left(2 - \frac{1}{p_1}\right)\frac{\partial x_2^{*p_1}}{\partial k_1}\int_{x_2^*}^{x_2}\left(s^{p_1} - x_2^{*p_1}\right)^{1-1/p_1} ds. \end{aligned} \tag{15}$$

Moreover, $m_2(x_2 - x_2^*)^{2p_1} \le W_2(k_1, x_1, x_2) \le (2^{p_1} - 1)\xi_2^2$, for a positive constant m_2.

Since $l_j \geq 1$, it is deduced from (13) and (15) that

$$\dot{V}_2 \leq -n\xi_1^2 + (\theta_1 N(k_1)+1)\dot{k}_1 + c_2\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 + \frac{k_1^2}{l_1}\left(1+\frac{1}{l_2}\right)\theta_2\xi_2^{2-1/p_1}\left(x_3^{*p_2}+x_3^{p_2}-x_3^{*p_2}\right)$$
$$+\frac{2}{l_1}\left|k_1^2\left[\xi_2^{2-1/p_1}f_2(\cdot)+\frac{\partial W_2}{\partial x_1}\dot{x}_1+\frac{\partial W_2}{\partial k_1}\dot{k}_1\right]+k_1\dot{k}_1W_2(\cdot)\right|+\frac{1}{l_1l_2}\xi_1\dot{x}_1-\frac{\dot{l}_1}{l_1^2}k_1^2W_2(\cdot)-\frac{\dot{l}_1l_2+l_1\dot{l}_2}{l_1^2l_2^2}\left(\frac{\xi_1^2}{2}+k_1^2W_2(\cdot)\right). \tag{16}$$

From $\xi_2 = x_2^{p_1} - x_2^{*p_1}$, (12) and (15), it is not difficult to obtain (by Lemma 1 and Lemmas 3-5)

$$\frac{2k_1^2}{l_1}\left|\xi_2^{2-1/p_1}f_2(\cdot)\right| \leq k_1^2\xi_2^2\Upsilon_{21}(k_1,x_1,x_2)+\frac{1}{l_1}\xi_1^2\Upsilon_{22}(k_1,x_1)+\frac{1}{l_1}\xi_1^2(t-d)\Upsilon_{22}^*\left(k_1(t-d),x_1(t-d)\right)$$
$$+\xi_2^2(t-d)\Upsilon_{21}^*\left(k_1(t-d),x_1(t-d),x_2(t-d)\right),$$
$$\frac{2k_1^2}{l_1}\left|\frac{\partial W_2}{\partial x_1}\dot{x}_1+\frac{\partial W_2}{\partial k_1}\dot{k}_1\right|+\frac{2}{l_1}k_1\dot{k}_1W_2(\cdot)+\frac{1}{l_1l_2}\xi_1\dot{x}_1 \leq k_1^2\xi_2^2\Phi_{21}(k_1,x_1,x_2)+\frac{1}{l_1}\xi_1^2\Phi_{22}(k_1,x_1)+\frac{1}{l_1}\xi_1^2(t-d)\Phi_2^*(x_1(t-d)), \tag{17}$$

where $\Upsilon_{2j}(\cdot) \geq 0$, $\Upsilon_{2j}^*(\cdot) \geq 0$, $\Phi_{2j}(\cdot) \geq 0$, and $\Phi_2^*(\cdot) \geq 0, j = 1, 2$, are smooth functions. Using the bounds $\Upsilon_{2j}^*(\cdot)$ and $\Phi_2^*(\cdot)$ thus obtained, one can construct the Lyapunov-Krasovskii functional

$$V_{2LK} = V_2 + \int_{t-d}^{t}\xi_2^2(s)\Upsilon_{21}^*\left(k_1(s),x_1(s),x_2(s)\right)ds + \int_{t-d}^{t}\frac{1}{l_1(s)}\xi_1^2(s)\left[\Upsilon_{22}^*\left(k_1(s),x_1(s)\right)+\Phi_2^*(x_1(s))\right]ds. \tag{18}$$

Then, it is deduced from (16) and (17) that

$$\dot{V}_{2LK} \leq -n\xi_1^2-(n-1)k_1^2\xi_2^2+(\theta_1N(k_1)+1)\dot{k}_1-\frac{\dot{l}_1}{2l_1^2}\xi_1^2+\frac{1}{l_1}\xi_1^2\left[\Upsilon_{22}(k_1,x_1)+\Upsilon_{22}^*(k_1,x_1)+\Phi_{22}(k_1,x_1)+\Phi_2^*(x_1)\right]$$
$$+\frac{k_1^2}{l_1}\left(1+\frac{1}{l_2}\right)\theta_2\xi_2^{2-1/p_1}x_3^{*p_2}+\frac{2\bar{c}}{l_1}k_1^2\left|\xi_2^{2-1/p_1}\left(x_3^{p_2}-x_3^{*p_2}\right)\right|+k_1^2\xi_2^2[c_2+(n-1)+\Upsilon_{21}(k_1,x_1,x_2)$$
$$+\Upsilon_{21}^*(k_1,x_1,x_2)+\Phi_{21}(k_1,x_1,x_2)]-\frac{\dot{l}_2}{l_1l_2^2}\left(\frac{\xi_1^2}{2}+W_2(\cdot)\right). \tag{19}$$

The inequality above is derived by neglecting the negative terms that are related to $\dot{l}_1$ and using the facts that $-k_1^2W_2(\cdot) \leq -W_2(\cdot)$ and $\frac{1}{l_1}-\frac{1}{l_1(t-d)} \leq 0$ (see (22)).

From (19), it is not difficult to show that the dynamic state compensator

$$\dot{l}_1 = \max\left\{-l_1^2+l_1\rho_1(k_1,x_1),\ 0\right\},\ l_1(0)=1, \tag{20}$$

$$\rho_1(k_1,x_1) = 2\left[\Upsilon_{22}(\cdot)+\Upsilon_{22}^*(\cdot)+\Phi_{22}(\cdot)+\Phi_2^*(\cdot)\right] \tag{21}$$

can counteract the effect of the time-delay nonlinearity. In fact, by construction the gain l_1 satisfies

$$0 \leq \dot{l}_1 \leq l_1\rho_1(\cdot),\ \ \dot{l}_1 \geq -l_1^2+l_1\rho_1(\cdot),\ \ l_1 \geq l_1(t-d) \geq 1. \tag{22}$$

As a consequence,

$$-\frac{\dot{l}_1}{2l_1^2}\xi_1^2 \leq \xi_1^2-\frac{1}{2l_1}\xi_1^2\rho_1(k_1,x_1). \tag{23}$$

Moreover,

$$\frac{2\bar{c}_2}{l_1}k_1^2\left|\xi_2^{2-1/p_1}(x_3^{p_2}-x_3^{*p_2})\right| \leq \bar{c}_2k_1^2\xi_2^2+c_3k_1^2\xi_3^2, \tag{24}$$

where $\xi_3 = x_3^{p_1p_2}-x_3^{*p_1p_2}$, $\bar{c}_2$ and c_3 are positive constants.

Substituting (23) and (24) into (19), we arrive at

$$\dot{V}_{2LK} \leq -(n-1)\xi_1^2-(n-1)k_1^2\xi_2^2+(\theta_1N(k_1)+1)\dot{k}_1+c_3k_1^2\xi_3^2+\frac{k_1^2}{l_1}\left(1+\frac{1}{l_2}\right)\theta_2\xi_2^{2-1/p_1}x_3^{*p_2}$$
$$+k_1^2\xi_2^2\left[c_2+\bar{c}_2+n-1+\Upsilon_{21}(\cdot)+\Upsilon_{21}^*(\cdot)+\Phi_{21}(\cdot)\right]-\frac{\dot{l}_2}{l_1l_2^2}\left(\frac{\xi_1^2}{2}+W_2(\cdot)\right). \tag{25}$$

Similar to Step 1, because of the unknown sign of θ_2, we design the virtual controller

$$
\begin{aligned}
x_3^{*p_2} &= l_1 N(k_2)\xi_2^{1/p_1}\left[c_2+\bar{c}_2+n-1+\Upsilon_{21}(\cdot)+\Upsilon_{21}^{*}(\cdot)+\Phi_{21}(\cdot)\right]\\
&:= l_1 N(k_2)(\xi_2\beta_2(k_1,x_1,x_2))^{1/p_1}\\
\dot{k}_2 &= \left(1+\frac{1}{l_2}\right)\xi_2^2\beta_2^{1/p_1}(k_1,x_1,x_2), \quad k_2(0)=1
\end{aligned} \tag{26}
$$

with the Nussbaum gain k_2 that is updated dynamically. Clearly, the dynamic compensator (26) leads to

$$
\dot{V}_{2LK} \leq -(n-1)\left(\xi_1^2+k_1^2\xi_2^2\right)+(\theta_1 N(k_1)+1)\dot{k}_1+c_3k_1^2\xi_3^2+(\theta_2 N(k_2)+1)k_1^2\dot{k}_2-\frac{\dot{l}_2}{l_1 l_2^2}\left(\frac{\xi_1^2}{2}+W_2(\cdot)\right). \tag{27}
$$

Inductive Step: At step $i-1$, assume that there are a Lyapunov-Krasovskii functional $V_{(i-1)LK}$, a set of dynamic gains $l_j(\cdot) \geq 1,\ j=1,\ \dots,i-1$, updated by

$$
\begin{aligned}
\dot{l}_1 &= \max\left\{-l_1^2+l_1\rho_1(k_1,x_1),0\right\},\\
\dot{l}_2 &= \max\left\{-\alpha_2 l_2^2+l_2\rho_2(l_1,k_1,k_2,x_1,x_2),0\right\},\\
&\vdots\\
\dot{l}_{i-2} &= \max\left\{-\alpha_{i-2}l_{i-2}^2+l_{i-2}\rho_{i-2}\left(\bar{l}_{i-3},\bar{k}_{i-2}\bar{x}_{i-2}\right),0\right\},
\end{aligned} \tag{28}
$$

with $\alpha_j = 1/(2^{p_1\cdots p_{j-1}}-1)$, and a set of nonsmooth but C^0 virtual controllers $x_1^*,\ \dots,x_i^*$, with the Nussbaum gains, given by

$$
\begin{aligned}
x_1^* &= 0 & \xi_1 &= x_1-x_1^*\\
x_2^{*p_1} &= \xi_1 N(k_1)\beta_1(x_1) & \xi_2 &= x_2^{p_1}-x_2^{*p_1}\\
\dot{k}_1 &= \left(1+\frac{1}{l_1}\right)\xi_1^2\beta_1(\cdot) & &\\
&\vdots & &\vdots\\
x_i^{*p_1\cdots p_{i-1}} &= (l_1\cdots l_{i-2}N(k_{i-1}))^{p_1\cdots p_{i-2}}\xi_{i-1}\beta_{i-1}\left(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1}\right) & \xi_i &= x_i^{p_1\cdots p_{i-1}}-x_i^{*p_1\cdots p_{i-1}}\\
\dot{k}_{i-1} &= \left(1+\frac{1}{l_{i-1}}\right)\xi_{i-1}^2\beta_{i-1}^{1/p_1\cdots p_{i-2}}(\cdot), & &
\end{aligned} \tag{29}
$$

with $\rho_j(\cdot) > 0$ and $\beta_j(\cdot) > 0$ being smooth functions, such that

$$
\begin{aligned}
\dot{V}_{(i-1)LK} \leq{}& -(n-(i-2))\sum_{j=1}^{i-1}\left[\left(\prod_{m=0}^{j-1}k_m^2\right)\xi_j^2\right]+c_ik_1^2\cdots k_{i-2}^2\xi_i^2+\sum_{j=1}^{i-1}\left[(\theta_j N(k_j)+1)\left(\prod_{m=0}^{j-1}k_m^2\right)\dot{k}_j\right]\\
&-\frac{\dot{l}_{i-1}}{l_1\cdots l_{i-2}l_{i-1}^2}\left(\frac{\xi_1^2}{2}+\sum_{j=2}^{i-1}W_j\left(\bar{l}_{j-2},\bar{k}_{j-1},\bar{x}_j\right)\right),
\end{aligned} \tag{30}
$$

where $c_i > 0$ is a constant and $k_0 = 1$. Clearly, (30) reduces to (27) when $i = 3$.

We claim that (30) also holds at Step i. To prove this claim, consider the Lyapunov-Krasovskii functional

$$
\begin{aligned}
V_i &= V_{(i-1)LK}+\frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}W_i\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i\right)+\frac{1}{l_1\cdots l_i}\left[\frac{\xi_1^2}{2}+\sum_{j=2}^{i-1}W_j\left(\bar{l}_{j-2},\bar{k}_{j-1},\bar{x}_j\right)+k_1^2\cdots k_{i-1}^2W_i\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i\right)\right]\\
W_i &= \int_{x_i^*}^{x_i}\left(s^{p_1\cdots p_{i-1}}-x_i^{*p_1\cdots p_{i-1}}\right)^{2-1/(p_1\cdots p_{i-1})}ds,
\end{aligned} \tag{31}
$$

where $l_i(\cdot) \geq 1$ is a dynamic gain to be designed. Similar to the argument in Step 2, one can show that $W_i(\cdot) = W_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i)$ is C^1. Moreover,

$$
\begin{aligned}
&\frac{\partial W_i}{\partial x_i} = \xi_i^{2-1/(p_1\cdots p_{i-1})} \\
&\frac{\partial W_i}{\partial x_j} = -\left(2 - \frac{1}{p_1\cdots p_{i-1}}\right) \frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial x_j} \int_{x_i^*}^{x_i} \left(s^{p_1\cdots p_{i-1}} - x_i^{*p_1\cdots p_{i-1}}\right)^{1-\frac{1}{(p_1\cdots p_{i-1})}} ds \\
&\frac{\partial W_i}{\partial k_j} = -\left(2 - \frac{1}{p_1\cdots p_{i-1}}\right) \frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial k_j} \int_{x_i^*}^{x_i} \left(s^{p_1\cdots p_{i-1}} - x_i^{*p_1\cdots p_{i-1}}\right)^{1-\frac{1}{(p_1\cdots p_{i-1})}} ds \\
&\frac{\partial W_i}{\partial l_j} = -\left(2 - \frac{1}{p_1\cdots p_{i-1}}\right) \frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial l_j} \int_{x_i^*}^{x_i} \left(s^{p_1\cdots p_{i-1}} - x_i^{*p_1\cdots p_{i-1}}\right)^{1-\frac{1}{(p_1\cdots p_{i-1})}} ds \\
&m_i\left(x_i - x_i^*\right)^{2p_1\cdots p_{i-1}} \leq W_i(\cdot) \leq (2^{p_1\cdots p_{i-1}} - 1)\xi_i^2, \quad 1 \leq j \leq i-1,
\end{aligned}
\tag{32}
$$

for a positive constant m_i.

Analogous to the derivation of (19), using the facts that $l_j \geq 1$ and $-k_1^2\cdots k_{i-1}^2 W_i(\cdot) \leq -W_i(\cdot)$, we deduce from (30)-(32) that (by neglecting the negative terms which are related to $\dot{l}_j, j = 1, \ldots, i-1$)

$$
\begin{aligned}
\dot{V}_i \leq &-(n-(i-2)) \sum_{j=1}^{i-1} \left[\left(\prod_{m=0}^{j-1} k_m^2\right) \xi_j^2\right] + c_i k_1^2\cdots k_{i-2}^2 \xi_i^2 + \sum_{j=1}^{i-1} \left[(\theta_j N(k_j) + 1)\left(\prod_{m=0}^{j-1} k_m^2\right) \dot{k}_j\right] \\
&- \frac{\dot{l}_{i-1}}{l_1\cdots l_{i-2} l_{i-1}^2} \left(\frac{\xi_1^2}{2} + \sum_{j=2}^{i-1} W_j\left(\bar{l}_{j-2}, \bar{k}_{j-1}, \bar{x}_j\right)\right) + \frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}} \left(1 + \frac{1}{l_i}\right) \left[\left|\xi_i^{2-\frac{1}{(p_1\cdots p_{i-1})}} f_i(\cdot)\right|\right. \\
&+ \theta_i \xi_i^{2-\frac{1}{(p_1\cdots p_{i-1})}} \left(x_{i+1}^{*p_i} - x_{i+1}^{p_i} + x_{i+1}^{*p_i}\right) + \left.\left|\sum_{j=1}^{i-1} \frac{\partial W_i}{\partial x_j}\dot{x}_j + \sum_{j=1}^{i-1} \frac{\partial W_i}{\partial k_j}\dot{k}_j + \sum_{j=1}^{i-2} \frac{\partial W_i}{\partial l_j}\dot{l}_j\right|\right] \\
&+ \frac{2}{l_1\cdots l_{i-1}} \left(\sum_{j=1}^{i-1} \left(k_j \dot{k}_j \prod_{\substack{m=1 \\ m\neq j}}^{i-1} k_m^2\right)\right) W_i(\cdot) + \frac{1}{l_1\cdots l_i} \left[\sum_{j=2}^{i-1} \left(\sum_{m=1}^{j} \frac{\partial W_j}{\partial x_m}\dot{x}_m + \sum_{m=1}^{j-1} \frac{\partial W_j}{\partial k_m}\dot{k}_m + \sum_{m=1}^{j-2} \frac{\partial W_j}{\partial l_m}\dot{l}_m\right) + \xi_1 \dot{x}_1\right] \\
&- \frac{\dot{l}_i}{l_1\cdots l_{i-1} l_i^2} \left(\frac{\xi_1^2}{2} + \sum_{j=2}^{i} W_j(\cdot)\right).
\end{aligned}
\tag{33}
$$

Using an argument similar to the work of Zhang et al,[3] we obtain the estimations (34)-(38) (see Appendix A for details)

$$
\begin{aligned}
\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}} \left|\xi_i^{2-1/(p_1\cdots p_{i-1})} f_i\right| \leq\ & k_1^2\cdots k_{i-1}^2 \xi_i^2 \Upsilon_{i1}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i\right) + \xi_i^2(t-d)\Upsilon_{i1}^*\left(\bar{l}_{i-2}(t-d), \bar{k}_{i-1}(t-d), \bar{x}_i(t-d)\right) \\
&+ \frac{1}{l_1\cdots l_{i-1}} \left[\xi_1^2 + \sum_{j=2}^{i-1} \left(x_j - x_j^*\right)^{2p_1\cdots p_{j-1}}\right] \Upsilon_{i2}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right) \\
&+ \frac{1}{l_1\cdots l_{i-1}} \left[\sum_{j=2}^{i-1} \left(x_j(t-d) - x_j^*(t-d)\right)^{2p_1\cdots p_{j-1}} + \xi_1^2(t-d)\right] \\
&\Upsilon_{i2}^*\left(\bar{l}_{i-2}(t-d), \bar{k}_{i-1}(t-d), \bar{x}_{i-1}(t-d)\right),
\end{aligned}
\tag{34}
$$

$$\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left|\sum_{j=1}^{i-1}\frac{\partial W_i}{\partial x_j}\dot{x}_j+\sum_{j=1}^{i-1}\frac{\partial W_i}{\partial k_j}\dot{k}_j+\sum_{j=1}^{i-2}\frac{\partial W_i}{\partial l_j}\dot{l}_j\right|$$
$$\le k_1^2\cdots k_{i-1}^2\xi_i^2\Phi_{i1}\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i\right)+\frac{1}{l_1\cdots l_{i-1}}\left[\xi_1^2+\sum_{j=2}^{i-1}\left(x_j-x_j^*\right)^{2p_1\cdots p_{j-1}}\right]\Phi_{i2}\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right) \tag{35}$$
$$+\frac{1}{l_1\cdots l_{i-1}}\left[\sum_{j=2}^{i-1}\left(x_j(t-d)-x_j^*(t-d)\right)^{2p_1\cdots p_{j-1}}+\xi_1^2(t-d)\right]\Phi_{i2}^*\left(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d)\right),$$

$$\frac{2}{l_1\cdots l_{i-1}}\left[\sum_{j=1}^{i-1}\left(2k_j\dot{k}_j\prod_{\substack{m=1\\m\neq j}}^{i-1}k_m^2\right)\right]W_i(\cdot)\le k_1^2\cdots k_{i-1}^2\xi_i^2\omega_i\left(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1}\right), \tag{36}$$

$$\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left|\theta_i\xi_i^{2-\frac{1}{p_1\cdots p_{i-1}}}\left(x_{i+1}^{p_i}-x_{i+1}^{*p_i}\right)\right|\le k_1^2\cdots k_{i-1}^2\left(\bar{c}_i\xi_i^2+c_{i+1}\xi_{i+1}^2\right), \tag{37}$$

$$\frac{1}{l_1\cdots l_i}\left|\xi_1\dot{x}_1+\sum_{j=2}^{i-1}\left[\sum_{m=1}^{j}\frac{\partial W_j}{\partial x_m}\dot{x}_m+\sum_{m=1}^{j-1}\frac{\partial W_j}{\partial k_m}\dot{k}_m+\sum_{m=1}^{j-2}\frac{\partial W_j}{\partial l_m}\dot{l}_m\right]\right|$$
$$\le\frac{1}{l_1\cdots l_{i-1}}\Psi_i\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right)\left[\xi_1^2+\sum_{j=2}^{i-1}\left(x_j-x_j^*\right)^{2p_1\cdots p_{j-1}}\right]+k_1^2\cdots k_{i-1}^2\xi_i^2 \tag{38}$$
$$+\frac{1}{l_1\cdots l_{i-1}}\Psi_i^*\left(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d)\right)\left[\xi_1^2(t-d)+\sum_{j=2}^{i-1}\left(x_j(t-d)-x_j^*(t-d)\right)^{2p_1\cdots p_{j-1}}\right],$$

where $\Upsilon_{ij}(\cdot)\ge 0$, $\Upsilon_{ij}^*(\cdot)\ge 0$, $\Phi_{ij}(\cdot)\ge 0$, $\Phi_{ij}^*(\cdot)\ge 0$, $\Psi_i(\cdot)\ge 0$ and $\Psi_i^*(\cdot)\ge 0,\omega_i(\cdot)\ge 0\, j=1,2$, are smooth functions.

With the help of the bounds $\Upsilon_{ij}^*(\cdot)$, $\Phi_{ij}^*(\cdot)$, and $\Psi_i^*(\cdot)$ thus obtained, which are related to the delay terms, we construct the Lyapunov-Krasovskii functional

$$V_{iLK}=V_i+\int_{t-d}^{t}\xi_i^2(s)\Upsilon_{i1}^*\left(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_i(s)\right)ds+\int_{t-d}^{t}\frac{1}{l_1(s)\cdots l_{i-1}(s)}\left[\xi_1^2(s)+\sum_{j=2}^{i-1}\left(x_j(s)-x_j^*(s)\right)^{2p_1\cdots p_{j-1}}\right] \tag{39}$$
$$\cdot\left[\Upsilon_{i2}^*\left(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s)\right)+\Phi_{i2}^*\left(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s)\right)+\Psi_i^*\left(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s)\right)\right]ds.$$

From (33)-(38) and the fact that $\frac{1}{l_1\cdots l_{i-1}(t)}\le\frac{1}{l_1(t-d)\cdots l_{i-1}(t-d)}$ and $k_i\ge 1, i=1,\ldots,i-1$, a straightforward but tedious calculation gives

$$\dot{V}_{iLK}\le-(n-(i-2))\sum_{j=1}^{i-1}\left[\left(\prod_{m=0}^{j-1}k_m^2\right)\xi_j^2\right]+\sum_{j=1}^{i-1}\left[(\theta_jN(k_j)+1)\left(\prod_{m=0}^{j-1}k_m^2\right)\dot{k}_j\right]-\frac{\dot{l}_{i-1}}{l_1\cdots l_{i-2}l_{i-1}^2}\left(\frac{\xi_1^2}{2}+\sum_{j=2}^{i-1}W_j(\cdot)\right)$$
$$+\frac{1}{l_1\cdots l_{i-1}}\left[\xi_1^2+\sum_{j=2}^{i-1}\left(x_j-x_j^*\right)^{2p_1\cdots p_{j-1}}\right]\left[\Upsilon_{i2}\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right)+\Upsilon_{i2}^*\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right)\right.$$
$$\left.+\Phi_{i2}\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right)+\Phi_{i2}^*\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right)+\Psi_i\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right)+\Psi_i^*\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right)\right]$$
$$+\frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left(1+\frac{1}{l_i}\right)\theta_i\xi_i^{2-1/(p_1\cdots p_{i-1})}x_{i+1}^{*p_i}+c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2+k_1^2\cdots k_{i-1}^2\xi_i^2\left[1+c_i+\bar{c}_i+\Upsilon_{i1}\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i\right)\right.$$
$$\left.+\Upsilon_{i1}^*\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i\right)+\Phi_{i1}\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i\right)+\omega_i\left(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1}\right)\right]-\frac{\dot{l}_i}{l_1\cdots l_{i-1}l_i^2}\left(\frac{\xi_1^2}{2}+\sum_{j=2}^{i}W_j(\cdot)\right). \tag{40}$$

Based on the inequality above, one can design the delay-free gain update law

$$\dot{l}_{i-1}=\max\left\{-\alpha_{i-1}l_{i-1}^2+l_{i-1}\rho_{i-1}\left(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}\right),\ 0\right\}, \tag{41}$$

with $l_{i-1}(0)=1$, $\alpha_{i-1}=1/(2^{p_1\cdots p_{i-2}}-1)$ and

$$\rho_{i-1}(\cdot)=\frac{1}{M_{i-1}}\left[\Upsilon_{i2}(\cdot)+\Upsilon_{i2}^*(\cdot)+\Phi_{i2}(\cdot)+\Phi_{i2}^*(\cdot)+\Psi_i(\cdot)+\Psi_i^*(\cdot)\right],\quad M_{i-1}=\min\left\{\frac{1}{2},m_2,\ldots,m_{i-1}\right\}. \tag{42}$$

By construction, the gain thus constructed satisfies

$$0 \le \dot{l}_{i-1} \le l_{i-1}\rho_{i-1}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right) \tag{43}$$

$$\dot{l}_{i-1} \ge -\alpha_{i-1} l_{i-1}^2 + l_{i-1}\rho_{i-1}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right). \tag{44}$$

Using (32) and (43)-(44), it is not difficult to prove that

$$\frac{-\dot{l}_{i-1}}{l_1 \cdots l_{i-2} l_{i-1}^2}\left(\frac{\xi_1^2}{2} + \sum_{j=2}^{i-1} W_j(\cdot)\right) \le \sum_{j=1}^{i-1}\left[\left(\prod_{m=0}^{j-1} k_m^2\right)\xi_j^2\right] - \frac{M_{i-1}\rho_{i-1}(\cdot)}{l_1 \cdots l_{i-1}}\left[\xi_1^2 + \sum_{j=2}^{i-1}\left(x_j - x_j^*\right)^{2p_1 \cdots p_{j-1}}\right]. \tag{45}$$

Substituting (42) into (40) yields

$$\begin{aligned}\dot{V}_{iLK} \le{}& -(n-(i-1))\sum_{j=1}^{i}\left(\prod_{m=0}^{j-1} k_m^2\right)\xi_j^2 + \sum_{j=1}^{i-1}\left[(\theta_j N(k_j)+1)\left(\left(\prod_{m=0}^{j-1} k_m^2\right)\right)\dot{k}_j\right] + \frac{k_1^2 \cdots k_{i-1}^2}{l_1 \cdots l_{i-1}}\left(1+\frac{1}{l_i}\right)\theta_i \xi_i^{2-1/(p_1 \cdots p_{i-1})} x_{i+1}^{*p_i}\\ &+ k_1^2 \cdots k_{i-1}^2 \xi_i^2\left[2 + c_i + \bar{c}_i + n - i + \Upsilon_{i1}(\cdot) + \Upsilon_{i1}^*(\cdot) + \Phi_{i1}(\cdot) + \omega_i(\cdot)\right]\\ &- \frac{\dot{l}_i}{l_1 \cdots l_{i-1} l_i^2}\left(\frac{\xi_1^2}{2} + \sum_{j=2}^{i} W_j(\cdot)\right) + c_{i+1} k_1^2 \cdots k_{i-1}^2 \xi_{i+1}^2.\end{aligned} \tag{46}$$

In view of (46), one can design the nonsmooth virtual controller with the Nussbaum gain

$$\begin{aligned}x_{i+1}^{*p_i} &= l_1 \cdots l_{i-1} N(k_i)\xi_i^{1/(p_1 \cdots p_{i-1})}\left[2 + c_i + \bar{c}_i + n - i + \Upsilon_{i1}(\cdot) + \Upsilon_{i1}^*(\cdot) + \Phi_{i1}(\cdot) + \omega_i(\cdot)\right]\\ &:= l_1 \cdots l_{i-1} N(k_i)\left(\xi_i \beta_i\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i\right)\right)^{1/(p_1 \cdots p_{i-1})}\\ \dot{k}_i &= \left(1+\frac{1}{l_i}\right)\xi_i^2 \beta_i^{1/(p_1 \cdots p_{i-1})}(\cdot).\end{aligned} \tag{47}$$

Using the claim for $i = n+1$ with $u = x_{n+1} = x_{n+1}^*$, we conclude that the dynamic state feedback controller that is composed of (28) with $i = n+1$ and

$$\begin{aligned}u &= (l_1 \cdots l_{n-1} N(k_n))^{\frac{1}{p_n}}\left(\xi_n \beta_n\left(\bar{l}_{n-2}, \bar{k}_{n-1}, x\right)\right)^{\frac{1}{(p_1 \cdots p_n)}}\\ \dot{k}_n &= \xi_n^2 \beta_n^{\frac{1}{(p_1 \cdots p_{n-1})}}\left(\bar{l}_{n-2}, \bar{k}_{n-1}, x\right)\end{aligned} \tag{48}$$

is such that

$$\dot{V}_{nLK} \le -\sum_{j=1}^{n}\left[\left(\prod_{m=0}^{j-1} k_m^2\right)\xi_j^2\right] + \sum_{j=1}^{n}\left[(\theta_j N(k_j)+1)\left(\prod_{m=0}^{j-1} k_m^2\right)\dot{k}_j\right]. \tag{49}$$

4 | ASYMPTOTIC STATE REGULATION WITH STABILITY

In this section, we use the Lyapunov-Krasovskii inequality (49) to complete the proof of Theorem 1. In particular, it is shown that the proposed dynamic state feedback controller (48) and (28) can regulate the system state to the origin while maintaining the boundedness of the closed-loop system.

We begin with the introduction of a key lemma whose proof can be found in Appendix B, based on the Lyapunov-Krasovskii inequalities (30) and (49).

Lemma 6. *The Nussbaum gains $k_i(t), i = 1, \dots, n$, given by (47) are bounded $\forall t \in [0, +\infty)$.*

With the aid of Lemma 6, we deduce from (47) that $\xi_i^2(t) \le \dot{k}_i(t)$ because, by construction, $\beta_i(\cdot) \ge 1$ and $l_i(t) \ge 1$. Hence, $\int_0^{+\infty} \xi_i^2 ds \le k_i(+\infty) - k_i(0) = c$.

On the other hand, (49) and the boundedness of $k_i(t), 1 \leq i \leq n$, imply that

$$V_{nLK}(t) \leq \sum_{j=1}^{n} \int_0^t |\theta_j N(k_j(s)) + 1| \left(\prod_{m=0}^{j-1} k_m^2(s) \right) \dot{k}_j(s) ds + V_{nLK}(0)$$
$$\leq c_1 \sum_{j=1}^{n} \int_0^t \dot{k}_j(s) ds + c_2 \leq C. \tag{50}$$

In view of (39) and (31), it is clear that the boundedness of $V_{nLK}(\cdot)$ on $[0, +\infty)$ implies the boundedness of x_1, $\frac{k_1^2 \cdots k_{i-1}^2}{l_1 \cdots l_{i-1}} W_i(\cdot)$, $i = 2, \ldots, n$. Using the estimation of $W_i(\cdot)$ in (32), one concludes that x_1 and $\frac{k_1^2 \cdots k_{i-1}^2}{l_1 \cdots l_{i-1}} (x_i - x_i^*)^{2p_1 \cdots p_{i-1}}$, $i = 2, \ldots, n$, are also bounded.

Because x_1 and k_1 are bounded and the gain $l_1(\cdot)$ given by (20), (21) is monotone nondecreasing, then $l_1(\cdot)$ must be bounded. If not, $\lim_{t \to +\infty} l_1(t) = +\infty$. By continuity of $\rho_1(\cdot)$, $\rho_1(k_1, x_1)$ is bounded. Consequently, there is a time instant $T > 0$ such that $-l_1^2 + l_1 \rho_1(k_1, x_1) \leq 0$ on $[T, +\infty)$. This, together with (22), yields $\dot{l}_1 = 0$ on $[T, +\infty)$, which contradicts to the unboundedness of $l_1(\cdot)$. In conclusion, $l_1(\cdot)$ is bounded. The boundedness of $l_1(\cdot)$ and k_1 implies the boundedness of x_2^* as well as $x_2 - x_2^*$. As such, x_2 is also bounded. Similarly, one can prove the boundedness of $l_i(\cdot)$ and x_i in the following recursive manner: $x_2 \to l_2 \to x_3 \to \cdots \to l_{n-1} \to x_n$, by the boundedness of $k_i(\cdot), i = 1, \ldots, n$, (28) and the estimation (32). Therefore, all the signals of the closed-loop system (1)-(48)-(28) are bounded $\forall t \in [0, +\infty)$.

To prove the convergence of the system state, we observe that $\dot{\xi}_i, i = 1, \ldots, n$ are also bounded and $\int_0^{+\infty} \xi_i^2(t) dt < +\infty$. By the Barbalat's lemma, it is concluded that $\xi_i, i = 1, \ldots, n$ converge to zero. This, in view of the coordinate transformation (29), implies that all the states $x_1(t), \ldots, x_n(t)$ converge to zero as well, thus completing the proof of Theorem 1.

Because the proposed nonsmooth control scheme is based on the Lyapunov-Krasovskii functional method, it is not surprising that Theorem 1 is robust with respect to the uncertainty. With this observation in mind, Theorem 1 can be extended to a larger family of uncertain time-delay systems dominated by a homogeneous system with time delay. In fact, the following more general result also holds.

Theorem 2. *Consider a family of uncertain time-delay systems with unknown control directions*

$$\dot{x}_i = \theta_i x_{i+1}^{p_i} + \phi_i (x, x(t-d), t), \quad i = 1, \ldots, n, \tag{51}$$

where $x_{n+1} = u$ *and* $\phi_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *is a continuous mapping. Assume that the uncertain function* ϕ_i *satisfies the homogeneous growth condition*

$$|\phi_i (x, x(t-d), t)| \leq \gamma_i (\bar{x}_i, \bar{x}_i(t-d)) \Big(|x_1|^{\frac{1}{p_1 \cdots p_{i-1}}} + |x_2|^{\frac{1}{p_2 \cdots p_{i-1}}} + \cdots + |x_{i-1}|^{\frac{1}{p_{i-1}}} + |x_i| + |x_1(t-d)|^{\frac{1}{p_1 \cdots p_{i-1}}}$$
$$+ \cdots + |x_{i-1}(t-d)|^{\frac{1}{p_{i-1}}} + |x_i(t-d)| \Big), \; i = 1, \ldots, n, \tag{52}$$

with $\gamma_i(\bar{x}_i, \bar{x}_i(t-d)) \geq 0$ *being a known smooth function. Then, there is a delay-free, nonsmooth but* C^0 *dynamic state feedback (8) that steers the state x to zero and keeps the boundedness of the closed-loop system (8)-(51).*

Under the homogeneous growth condition (52), the proof of Theorem 2 can be carried out, with some subtle modifications, by means of an argument analogue to that of Theorem 1. For this reason, the details are left to the reader as an exercise.

From Theorem 2, it is straightforward to deduce the following robust stabilization result obtained in the work of Pongvuthithum et al[18] recently.

Corollary 1. *Consider a family of uncertain time-delay systems with controllable linearization and unknown control directions*

$$\dot{x}_i = \theta_i x_{i+1} + \phi_i (x, x(t-d), t), \quad i = 1, \ldots, n, \tag{53}$$

where $x_{n+1} = u$ *and* $\phi_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *is a* C^0 *uncertain function satisfying the condition*

$$|\phi_i (x, x(t-d), t)| \leq \gamma_i (\bar{x}_i, \bar{x}_i(t-d)) \left[\sum_{j=1}^{i} (|x_j| + |x_j(t-d)|) \right], \tag{54}$$

where $\gamma_i(\bar{x}_i, \bar{x}_i(t-d)) \geq 0$ *is a known smooth function. Then, there is a delay-free, nonsmooth but* C^0 *dynamic state feedback (8) driving the state x to zero and ensuring the boundedness of the closed-loop system (8)-(53).*

Remark 1. When the nonlinear system (1) or (51) contains multiple delays, the design of a delay-independent controller remains almost same, except that multiple Lyapunov-Krasovskii functionals with different time delays need to be used. Specifically, the Lyapunov-Krasovskii functional $\int_{t-d}^{t} K(s)ds$ should be replaced by $\int_{t-d_i}^{t} K(s)ds$ in each step of the recursive design. Hence, for the nonlinear system (1) or (51) with multiple delays $d_1, \dots, d_m$, (39) can be replaced by

$$V_{iLK} = V_i + \sum_{l=1}^{m}\left(\int_{t-d_l}^{t} \xi_i^2(s)\Upsilon_{i1}^*(\cdot)ds + \int_{t-d_l}^{t} \frac{1}{l_1(s)\cdots l_{i-1}(s)} \cdot \left[\xi_1^2(s) + \sum_{j=2}^{i-1}\left(x_j(s) - x_j^*(s)\right)^{2p_1\cdots p_{j-1}}\right]\left[\Upsilon_{i2}^*(\cdot) + \Phi_{i2}^*(\cdot) + \Psi_i^*(\cdot)\right] ds\right).$$

Of course, a similar philosophy can be used to handle the general case when every subsystem of (1) involves different time delays.

Remark 2. The assumption that the bound C of unknown coefficients $\theta_i, i = 1, \dots, n$ is known is used only for a technical convenience and can indeed be removed. When the bound C is unknown, a similar design procedure can be carried out with slightly different estimations of the right-hand side of $\dot{V}_{(i-1)LK}$ in (30) so that the term $(\theta_j N(k_j) + 1)$ is replaced by $(\theta_j N(k_j) + C_j)$, where C_j is an unknown constant. Due to the characteristics of the Nussbaum function and the monotone property of the adaptive gains $k_j, 1 \le j \le n$, the same argument in Appendix B can also be used for the stability proof.

We end this section with a simple but nontrivial example that demonstrates how a Nussbaum gain needs to be introduced in order to deal with the problem of unknown control direction.

Example 1. Consider a time-delay system in the plane, with strong nonlinearity and unknown control directions, of the form

$$\begin{aligned}\dot{x}_1 &= \theta_1 x_2^3 + x_1\\ \dot{x}_2 &= \theta_2 u + \frac{1}{2}x_2^3(t-d),\end{aligned} \tag{55}$$

where $\theta_1, \theta_2 \neq 0$ are unknown constants whose signs are also unknown (either positive or negative) and represents unknown directions of the actuator. Note that the time-delay system under consideration involves not only an unknown control direction but also strong nonlinearities. The latter requires the use of a nonsmooth rather than smooth feedback control strategy. As a matter of fact, even in the case when control directions are known (eg, $\theta_1 = \theta_2 = 1$) and no time delay is involved (ie, $d = 0$), it is known that the planar system cannot be controlled by any smooth state feedback, even locally, and a nonsmooth feedback must be used.

Following the control scheme proposed in Section 3, we first consider the Lyapunov function $V_1(x_1, l_1) = \frac{1}{2}(1 + \frac{1}{l_1})\xi_1^2$, where $\xi_1 = x_1$ and the gain l_1 is updated by

$$\dot{l}_1 = \max\left\{-l_1^2 + l_1\rho(k_1, x_1),\ 0\right\}, \quad l_1(0) = 1, \tag{56}$$

with $\rho_1(k_1, x_1) \ge 0$ being a smooth function to be determined later on.

For the x_1-subsystem, it is clear that the nonsmooth virtual control law $x_2^{*3} = 2x_1N(k_1)$, with $\dot{k}_1 = 2(1 + \frac{1}{l_1})x_1^2$, globally asymptotically regulates it.

Define $\xi_2 = x_2^3 - x_2^{*3} = x_2^3 - 2x_1N(k_1)$. From (56), it is easy to see that $l_1(\cdot) \ge 1$ and $\dot{l}_1 \ge -l_1^2 + l_1\rho_1(k_1, x_1)$. Moreover,

$$\dot{V}_1 \le -2x_1^2 + (\theta_1 N(k_1) + 1)\,\dot{k}_1 - \frac{1}{2l_1}\xi_1^2\rho_1(k_1, x_1). \tag{57}$$

Then, consider the Lyapunov-Krasovskii functional

$$V_{2LK} = V_1(x_1, l_1) + \frac{k_1^2}{l_1}\int_{x_2^*}^{x_2} \left(s^3 - x_2^{*3}\right)^{2-1/3} ds + \int_{t-d}^{t} \frac{1}{l_1(s)}\left(\xi_2^6(s) + 2\left(k_1^6 x_1^3(s)\right)^2\right) ds. \tag{58}$$

Following the design procedure in Step 2, one can find a dynamic state compensator that consists of (56) and

$$
\begin{aligned}
u &= N(k_2)\xi_2^{1/3}\left(2\left(k_1^2+2k_1\right)^2+\frac{10}{3}\left(1+\frac{1}{l_1}\right)x_1^2+l_1+\xi_2^4\right) \\
\dot{k}_2 &= \frac{1}{l_1}\xi_2^2\left(2\left(k_1^2+2k_1\right)^2+\frac{10}{3}\left(1+\frac{1}{l_1}\right)x_1^2+l_1+\xi_2^4\right),
\end{aligned}
\tag{59}
$$

with $\rho_1(k_1,x_1) = 2(2x_1^4k_1^{12}+4k_1^6+\frac{5}{3}(1+\frac{1}{l_1})x_1^2k_1^2)$ in (56) and $N(k_2) = k_2^2\cos(k_2)$, such that

$$
\dot{V}_{2LK} \le -x_1^2 - k_1^2\xi_2^2 + (\theta_1 N(k_1)+1)\,\dot{k}_1 + \frac{(\theta_2 N(k_2)+1)\,k_1^2\dot{k}_2}{l_1},
$$

from which it is deduced, as shown in Section 4, that the delay-free controller (59) and (56) achieves asymptotic state regulation and maintains the boundedness of the closed-loop system (55), (56), and (59), without the information of the sign of the parameter $\theta_i, i = 1, 2$. The simulation results of the closed-loop system (55)-(56)-(59) are shown in Figures 1 to 3 with $\theta_1 = -1, \theta_2 = 1, d = 0.1$, and $(x_1(0), x_2(0), l(0), k_1(0), k_2(0)) = (-0.15, 0.25, 1, 1, 1)$.

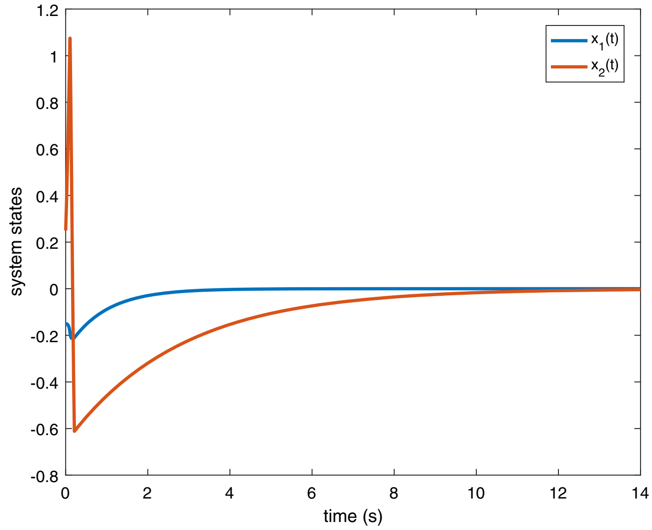

FIGURE 1 State trajectories (x_1, x_2) of the closed-loop system (55)-(56)-(59) [Colour figure can be viewed at wileyonlinelibrary.com]

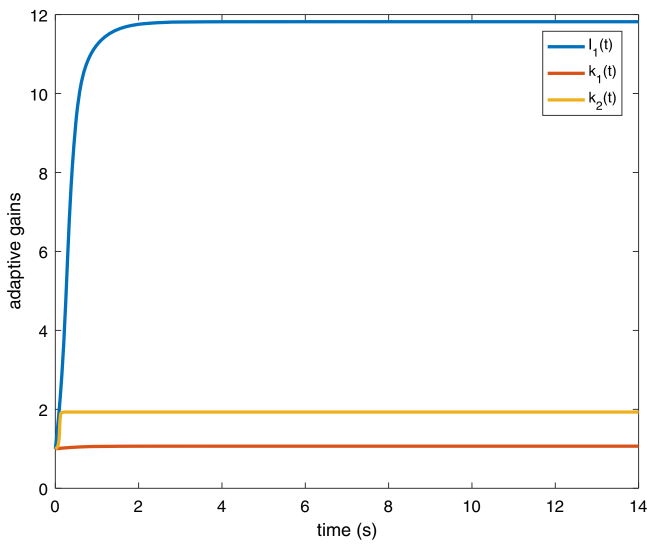

FIGURE 2 Dynamic gains (k, l_1) of the system (55)-(56)-(59) [Colour figure can be viewed at wileyonlinelibrary.com]

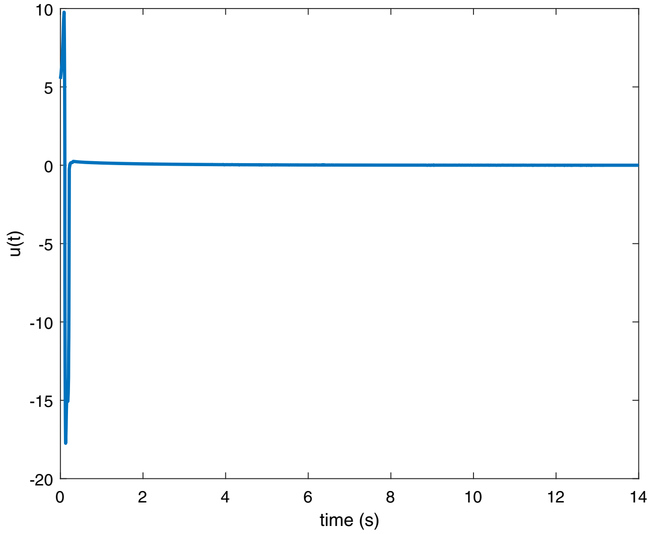

FIGURE 3 Control input *u* of the system (55)-(56)-(59) [Colour figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSIONS

A delay-free, nonsmooth dynamic state feedback control scheme has been proposed in this paper to deal with a family of uncertain time-delay systems with strong nonlinearities and unknown control direction. To cope with the effects of time-delay nonlinearities and unknown control direction, we have introduced, respectively, two sets of gains that need to be updated online, in a dynamic manner. One of them is the Nussbaum-type gains from universal control,[1] making it possible to mitigate the effect of unknown control direction, whereas the other one is borrowed the idea from the dynamic state feedback control method,[3] which can counteract the time-delay effects via a delay-free nonsmooth controller. Global asymptotic state regulation with boundedness of the closed-loop system has been proved to be possible, thanks to the construction of a set of new Lyapunov-Krasovskii functionals that are different from the previous ones in the literature, due to the involvement of the Nussbaum functions.

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REFERENCES

1. Nussbaum RD. Some remarks on a conjecture in parameter adaptive control. *Syst Control Lett*. 1983;3(5):243-246.
2. Willems JC, Byrnes CI. *Global Adaptive Stabilization in the Absence of Information on the Sign of the High Frequency Gain*. Vol. 62. Berlin, Germany: Springer-Verlag; 1984:49-57. *Lecture Notes in Control and Information Sciences*.
3. Zhang X, Lin W, Lin Y. Nonsmooth feedback control of time-delay systems: a dynamic gain based approach. *IEEE Trans Autom Control*. 2017;62(1):438-444.
4. Gu K, Kharitonov V, Chen J. *Stability of Time-Delay Systems*. Boston, MA: Birkhäuser; 2003.
5. Jankovic M. Control Lyapunov-Razumikhin functions and robust stabilization of time delay systems. *IEEE Trans Autom Control*. 2001;46(7):1048-1060.
6. Richard JP. Time-delay systems: an overview of some recent advances and open problems. *Automatica*. 2003;39(10):1667-1694.
7. Mazenc F, Mondie S, Francisco R. Global asymptotic stabilization of feedforward systems with delay in the input. In: Proceedings of the 42nd IEEE CDC; 2003; Maui, HI.

8. Bekiaris-Liberis N, Krstic M. Compensation of state-dependent input delay for nonlinear systems. *IEEE Trans Autom Control*. 2013;58(2):275-289.
9. Zhang X, Boukas EK, Lui Y, Baron L. Asymptotic stabilization of high-order feedforward systems with delays in the input. *Int J Robust Nonlinear Control*. 2010;20(12):1395-1406.
10. Ye X. Asymptotic regulation of time-varying uncertain nonlinear systems with unknown control directions. *Automatica*. 1999;35(5):929-935.
11. Lei H, Lin W. Universal adaptive control of nonlinear systems with unknown growth rate by output feedback. *Automatica*. 2006;42(10):1783-1789.
12. Lei H, Lin W. Adaptive regulation of uncertain nonlinear systems by output feedback: a universal control approach. *Syst Control Lett*. 2007;56(7-8):529-537.
13. Lei H, Lin W. Adaptive robust stabilization of a family of uncertain nonlinear systems by output feedback: the non-polynomial case. In: Proceedings of the 2007 American Control Conference; 2007; New York, NY.
14. Lei H, Lin W. Adaptive control of nonlinear systems with unknown parameters by output feedback: a non-identifier based method. In: Astolfi A, Marconi L, eds. *Lecture Notes in Control and Information Sciences*. Heidelberg, Germany: Springer-Verlag; 2007:445-463.
15. Qian C, Lin W. Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization. *Syst Control Lett*. 2001;42(3):185-200.
16. Qian C, Lin W. A continuous feedback approach to global strong stabilization of nonlinear systems. *IEEE Trans Autom Control*. 2001;46(7):1061-1079.
17. Lin W, Qian C. Adaptive control of nonlinearly parameterized systems: a nonsmooth feedback framework. *IEEE Trans Autom Control*. 2002;47(5):757-774.
18. Pongvuthithum R, Rattanamongkhonkun K, Lin W. Asymptotic regulation of time-delay nonlinear systems with unknown control directions. *IEEE Trans Autom Control*. 2018;63(5):1495-1502.

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APPENDIX A

This section gives the proof of inequalities (34), (35), and (38). We begin by introducing two propositions.

Proposition 1. *There is a smooth function $A_i(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}) \geq 0$ such that for $i = 2, \dots, n$,*

$$|\xi_i| \leq 2^{p_1\cdots p_{i-1}-1}|x_i - x_i^*|^{p_1\cdots p_{i-1}} + \left(|\xi_1| + \sum_{j=2}^{i-1}\left|x_j - x_j^*\right|^{p_1\cdots p_{j-1}}\right)A_i(\cdot). \tag{A1}$$

Proof. In view of (29) and (4), we have

$$\begin{aligned} |\xi_2| &= \left|\left(x_2 - x_2^* + x_2^*\right)^{p_1} - x_2^{*p_1}\right| \\ &\leq 2^{p_1-1}\left|x_2 - x_2^*\right|^{p_1} + \left(2^{p_1-1}+1\right)\left|x_2^{*p_1}\right| \\ &\leq 2^{p_1-1}\left|x_2 - x_2^*\right|^{p_1} + \left(2^{p_1-1}+1\right)|\xi_1 N(k_1)|\,\beta_1(x_1), \end{aligned} \tag{A2}$$

which indicates that (A1) holds for $k = 2$. Now, assume that (A1) holds when $k = i - 1$. From (29) and (4), it can be deduced that

$$\begin{aligned} |\xi_i| &= \left|\left(x_i - x_i^* + x_i^*\right)^{p_1\cdots p_{i-1}} - x_i^{*p_1\cdots p_{i-1}}\right| \\ &\leq 2^{p_1\cdots p_{i-1}-1}\left|x_i - x_i^*\right|^{p_1\cdots p_{i-1}} + \left(2^{p_1\cdots p_{i-1}-1}+1\right)\left(l_1\cdots l_{i-2}\,|N(k_{i-1})|\right)^{p_1\cdots p_{i-2}}|\xi_{i-1}|\beta_{i-1}\left(\bar{l}_{i-3}, \bar{k}_{k-2}, \bar{x}_{i-1}\right). \end{aligned} \tag{A3}$$

Substituting the estimation $|\xi_{i-1}|$ into (A3), it can be verified that (A1) also holds for $k = i$. □

Proposition 2. *For a C^∞ function $\gamma(x_i) \geq 0$, there are C^∞ functions $B_{ij}(\cdot) \geq 0, j = 1, 2$, such that*

$$\gamma(x_i)|x_i|^{p_1\cdots p_{i-1}} \leq \left|x_i - x_i^*\right|^{p_1\cdots p_{i-1}}B_{i1}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_i\right) + \left(|\xi_1| + \sum_{j=2}^{i-1}\left|x_j - x_j^*\right|^{p_1\cdots p_{j-1}}\right)B_{i2}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right). \tag{A4}$$

Proof. By Lemmas 3 and 4, there are smooth functions $\gamma_1(x_i - x_i^*) \geq 0$ and $\gamma_2(x_i^*) \geq 0$, such that

$$\begin{aligned}\gamma(x_i)|x_i|^{p_1\cdots p_{i-1}} &= \gamma\left(x_i - x_i^* + x_i^*\right)\left|x_i - x_i^* + x_i^*\right|^{p_1\cdots p_{i-1}} \\ &\leq \gamma_1\left(x_i - x_i^*\right)\left|x_i - x_i^*\right|^{p_1\cdots p_{i-1}} + \gamma_2\left(x_i^*\right)\left|x_i^{*p_1\cdots p_{i-1}}\right| \\ &\leq \gamma_1\left(x_i - x_i^*\right)\left|x_i - x_i^*\right|^{p_1\cdots p_{i-1}} + \gamma_2\left(x_i^*\right)\left(l_1\cdots l_{i-2}\left|N(k_{i-1})\right|\right)^{p_1\cdots p_{i-2}}|\xi_{i-1}|\beta_{i-1}\left(\bar{l}_{i-3}, \bar{k}_{i-2}, \bar{x}_{i-1}\right).\end{aligned} \tag{A5}$$

Substituting (29) and (A1) into (A5) yields (A4). □

With the aid of Propositions 1 and 2, we are able to prove inequalities (34), (35), and (38).

Proof of (34). Using the inequality (2) and Lemma 5, one can find smooth functions $\hat{\gamma}_{ij}(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1})$ and $\hat{\gamma}_{ij}^*(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1})$ so that

$$\left|\xi_i^{2-1/(p_1\cdots p_{i-1})} f_i(\cdot)\right| \leq 2i\frac{2p_1\cdots p_{i-1} - 1}{p_1\cdots p_{i-1}}\xi_i^2 + \frac{1}{2p_1\cdots p_{i-1}}\sum_{j=1}^{i}\left[\hat{\gamma}_{ij}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right)\xi_j^2 + \hat{\gamma}_{ij}^*\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right)\xi_j^2(t-d)\right].$$

This, combined with Proposition 1, implies that there are smooth functions $\Upsilon_{ij}(\cdot) \geq 0$ and $\Upsilon_{ij}^*(\cdot) \geq 0, j = 1, 2$, such that inequality (34) holds. □

Proof of (35). Similar to the proof in the works of Qian and Lin,[15,16] one has

$$\begin{aligned}\left|\frac{\partial W_i}{\partial x_j}\right| &\leq a_i|\xi_i|\left|\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial x_j}\right|, \ j = 1, \ldots, i-1, \\ \left|\frac{\partial W_i}{\partial k_j}\right| &\leq a_i|\xi_i|\left|\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial k_j}\right|, \ j = 1, \ldots, i-1, \\ \left|\frac{\partial W_i}{\partial l_j}\right| &\leq a_i|\xi_i|\left|\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial l_j}\right|, \ j = 1, \ldots, i-2,\end{aligned} \tag{A6}$$

where a_i is a positive constant. Moreover, it follows from (29) that

$$\begin{aligned}\left|\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial x_j}\right||\dot{x}_j| &\leq \sum_{m=1}^{i}|\xi_m|\phi_{i1}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right) + \sum_{m=1}^{i-1}|\xi_m(t-d)|\,\phi_{i1}^*\left(\bar{l}_{i-2}(t-d), \bar{k}_{i-1}(t-d), \bar{x}_{i-1}(t-d)\right) \\ \left|\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial k_j}\right| &\leq \sum_{m=1}^{i-1}|\xi_m|\phi_{i2}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right) \\ \left|\frac{\partial x_i^{*p_1\cdots p_{i-1}}}{\partial l_j}\right| &\leq \sum_{m=1}^{i-1}|\xi_m|\phi_{i3}\left(\bar{l}_{i-2}, \bar{k}_{i-1}, \bar{x}_{i-1}\right)\end{aligned} \tag{A7}$$

where $\phi_{ir}(\cdot) \geq 0, r = 1, 2, 3$, are smooth functions.

Using Lemma 1, Propositions 1 and 2, and Lemma 5, it is not difficult to conclude that (35) holds. □

The proof of inequality (38) can be carried out in a manner similar to that of (34) and (35) and, hence, is omitted.

APPENDIX B

To prove Lemma 6, we need the following proposition.

Proposition 3. *Let θ be an unknown constant with unknown sign and $k(t)$ and $\sigma(t)$ be any positive monotone nondecreasing functions well defined on $[0, t_f)$. Then, the following inequality holds for some constant ϵ_θ:*

$$\int_0^t \theta\frac{\cos(k(s))}{\sigma(s)}\dot{k}(s)ds \leq \epsilon_\theta, \quad \forall t \in [0, t_f). \tag{B1}$$

Proof. Obviously, (B1) holds when $\sigma(t)$ or $k(t)$ is bounded on $[0, t_f]$. When both $\sigma(t)$ and $k(t)$ are unbounded on $[0, t_f]$, $\sigma(t)$ and $k(t)$ must have a finite escape time at t_f. Then, two cases need to be considered.

Case i: If $\theta \geq 0$, in this case, there exists a time sequence $\{t_r\}$ such that $k(t_r) = (2r-1)\frac{\pi}{2}$, $r = 1, 2, \ldots$. Clearly, $\cos(k(t_r)) = 0$ and $\cos(k(t)) \leq 0$, $\forall t \in [t_{2m-1}, t_{2m}]$, while $\cos(k(t)) \geq 0$, when $t \in [t_{2m}, t_{2m+1}]$, for $m = 1, 2, \ldots$.

With this in mind, we assume that without loss of generality, the time $t \in [0, t_{2m+3})$. Then, (B1) can be written as

$$
\begin{aligned}
\int_0^t \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds &= \int_0^{t_1} \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds + \int_{t_{2m+1}}^{t} \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds \\
&\quad + \sum_{l=1}^{m}\left(\int_{t_{2l-1}}^{t_{2l}} \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds + \int_{t_{2l}}^{t_{2l+1}} \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds\right) \\
&\leq \int_0^{t_1} \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds + \int_{t_{2m+1}}^{t} \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds \\
&\quad + \frac{\theta}{\sigma(t_2)}\left[\int_{t_1}^{t_2} \cos(k(s))\,\dot{k}(s)ds + \int_{t_2}^{t_3} \cos(k(s))\,\dot{k}(s)ds\right] + \cdots \\
&\quad + \frac{\theta}{\sigma(t_{2m})}\left[\int_{t_{2m-1}}^{t_{2m}} \cos(k(s))\,\dot{k}(s)ds + \int_{t_{2m}}^{t_{2m+1}} \cos(k(s))\,\dot{k}(s)ds\right] \\
&\equiv \int_0^{t_1} \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds + \int_{t_{2m+1}}^{t} \frac{\theta\cos(k(s))}{\sigma(s)}\dot{k}(s)ds.
\end{aligned}
$$

As a consequence,

$$
\int_0^t \theta\frac{\cos(k(s))}{\sigma(s)}\dot{k}(s)ds \leq \left[\frac{\pi\theta}{2\sigma(0)} + \frac{2\pi\theta}{\sigma(t_{2m+1})}\right] \leq \frac{5\pi\theta}{2\sigma(0)},
$$

as $t < t_{2m+3}$. Hence, Proposition 3 is true when $\theta \geq 0$.

Case ii: If $\theta < 0$, an analogous argument with the obvious modification that $k(t_r) = (2r+1)\frac{\pi}{2}$, for $r = 1, 2, \ldots$, leads to the same conclusion. Thus, Proposition 3 also holds when $\theta < 0$. □

Proof of Lemma 6. Lemma 6 can be proved by a contradiction argument. If $k_i(t), i = 1, \ldots, n$ are unbounded, let $[0, t_{f_i})$ be the maximal interval of $k_i, i = 1, \ldots, n$. Then, the following statement holds.

- *Claim (1):* $t_{f_1} \geq t_{f_2} \geq t_{f_3} \geq \cdots \geq t_{f_n}$.

This conclusion is proved by an inductive argument.

Step 1: For $n = 2$, suppose that Claim (1) is not true. Then, $t_{f_1} < t_{f_2}$. This implies that $k_2(t)$ is bounded on $[0, t_{f_1})$. Using (13) and the property that $\xi_2^2 \leq \dot{k}_2$ from (12), we have

$$
\dot{V}_{1LK} \leq (\theta_1 N(k_1) + 1)\,\dot{k}_1 + c_2\dot{k}_2, \quad \forall t \in [0, t_{f_1}). \tag{B2}
$$

This, in view of $N(k) = k^2\cos k$ and

$$
\int N(k)dk = k^2\sin k + 2k\cos k - 2\sin k + c, \tag{B3}
$$

leads to

$$
0 \leq \frac{V_{1LK}(t)}{k_1(t)} \leq \theta_1 k_1(t)\sin(k_1(t)) + C_1, \quad \forall t \in [0, t_{f_1}), \tag{B4}
$$

where C_1 is a constant.

Because $\lim_{t\to t_{f_1}} k_1(t) = +\infty$, the right-hand side of (B4) can become negative when $k_1(t)$ is sufficiently large, regardless of θ_1. This is clearly a contradiction. Therefore, $t_{f1} \geq t_{f2}$.

Step i: Suppose that for $n = i$, $i \geq 3$, $t_{f_1} \geq \cdots \geq t_{f_i}$ is true, but $t_{f_i} \geq t_{f_{i+1}}$ does not hold. That is, $t_{f_i} < t_{f_{i+1}}$. As such, $k_{i+1}(t)$ is bounded on $[0, t_{f_i})$. Using (30) with index i being replaced by $i + 1$, dividing both sides by $k_1^2 \cdots k_{i-1}^2(t)$ and then integrating from 0 to t, $\forall t \in [0, t_{f_i})$, we obtain

$$\begin{aligned}\int_0^t \frac{\dot{V}_{iLK} ds}{\prod_{m=0}^{i-1} k_m^2(s)} &\leq \sum_{j=1}^{i-1} \int_0^t \frac{\theta_j N(k_j(s)) + 1}{\prod_{m=j}^{i-1} k_m^2(s)} \dot{k}_j(s) ds + \int_0^t (\theta_i N(k_i(s)) + 1) \dot{k}_i(s) ds + c_{i+1} \int_0^t \dot{k}_{i+1}(s) ds \\ &\leq \sum_{j=1}^{i-1} \int_0^t \frac{\theta_j \cos(k_j(s))}{\prod_{m=j+1}^{i-1} k_m^2(s)} \dot{k}_j(s) ds + \int_0^t \theta_i N(k_i) dk_i + k_i(t) + c. \end{aligned} \tag{B5}$$

Due to the monotone increasing property of $\prod_{m=j}^{i-1} k_m^2(t)$ on $[0, t_{f_i})$, the following inequality holds:

$$0 \leq \frac{1}{\prod_{m=0}^{i-1} k_m^2(t)} \int_0^t \dot{V}_{iLK} ds \leq \int_0^t \frac{\dot{V}_{iLK} ds}{\prod_{m=0}^{i-1} k_m^2(s)}. \tag{B6}$$

By Proposition 3, the first term in (B5) is bounded by a constant. This, together with (B3) and (B6), results in

$$0 < \frac{V_{iLK}(t)}{\prod_{m=0}^{i-1} k_m^2(t) k_i(t)} n \leq \theta_i k_i(t) \sin(k_i(t)) + C, \ \forall t \in [0, t_{fi}). \tag{B7}$$

Since $\lim_{t \to t_{f_i}} k_i(t) = +\infty$, the right-hand side of (B7) can become negative when $k_i(t)$ is large enough. This is a contradiction. Hence, $t_{fi} \geq t_{f_{i+1}}$. In this way, we have inductively proved Claim (1). In addition, the argument above, in particular, (B5)-(B7) also leads to

- *Claim (2):* $k_i(t)$ is bounded if $k_{i+1}(t)$ is bounded.

In view of Claim (2), it is clear that to prove Lemma 6, we only need to prove that k_n is bounded $\forall t \in [0, +\infty)$. If $k_n(t)$ is unbounded and only defined on $[0, t_{f_n})$, consider inequality (49). The same argument as done in Step i or (B5) leads to (B7) with $i = n$. That is, $\forall t \in [0, t_{f_n})$,

$$0 < \frac{V_{nLK}(t)}{\prod_{m=0}^{n-1} k_m^2(t) k_n(t)} \leq \theta_n k_n(t) \sin(k_n(t)) + C_n, \tag{B8}$$

where C_n is a constant. Similar to the proof in (B4), a contradiction can be found from (B8). Hence, $k_n(t)$ must be bounded on $[0, t_{f_n}]$. Consequently, the maximal interval $[0, t_{fn})$ can be extended to $[0, +\infty)$. In other words, $k_n(t)$ is bounded $\forall t \geq 0$. By Claim (2), $k_{n-1}(t)$ is also bounded on $[0, +\infty)$. Inductively, it is concluded that $k_i(t), i = 1, \ldots, n$ are bounded on $[0, +\infty)$. $\square$

Asymptotic Regulation of Time-Delay Nonlinear Systems With Unknown Control Directions

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***Abstract*—This paper studies the problem of global state regulation with stability for time-delay nonlinear systems with unknown control directions. Using a dynamic gain-based method for counteracting time-delay nonlinearity and the Nussbaum-gain function for dealing with unknown control directions, we develop a dynamic state feedback control strategy that solves the problem. A novel construction of Lyapunov–Krasovskii functionals is presented and plays a key role in handling nonlinearity with delayed states and unknown control directions simultaneously. The proposed dynamic state feedback compensators are shown to guarantee 1) global asymptotic convergence of the system state to the origin and 2) global boundedness of the resulting closed-loop systems.**

***Index Terms*—Asymptotic state regulation, boundedness, dynamic state compensator, nonlinear systems, time delay, unknown control directions.**

I. Introduction

Time-delay systems extensively exist in a variety of applications including, but not limited to, network control, mechanical systems, biological systems, and chemical processes. For example, models of milling processes, drilling processes, and fluid flow or heating systems all exhibit time-delay phenomena. While many of these controlled plants are approximately modeled by linear systems, the work [2] presented a chemical reactor example that is described by a lower triangular nonlinear system with time delays in the state.

To address control problems of time-delay systems, various analysis and synthesis approaches have been developed in the literature. Among them, the Lyapunov–Krasovskii and Lyapunov–Razumikhin methods are two powerful tools in the stability analysis of time-delay systems [1], [3], [15], [16]. There are primarily three types of time-delay systems that have received considerable attention. One class includes the delay in the system state [1], [12], [14], [17] and the other one contains the delay in the control input [5], [10], [11]. Of course, a more complex situation involves time delays in both states and actuators of controlled plants. For each category of time-delay systems, many results have been obtained and reported; see, for instance, [4], [5], [10]–[12]. In [10], a saturation state feedback controller was proposed for global asymptotic stabilization of a chain of integrators with a delay in the input, without requiring the knowledge of the delay. In [5], control of a class of nonlinear systems with input delay was investigated with the condition that the system under consideration is forward complete. For a strict feedback system with delayed states, an attempt was first made in [12] to design a delay-independent, smooth state feedback controller. Later, it was found that the result of [12] is false, due to a circular argument in the state feedback design. Such a technical issue was addressed in [2] and [4] under the assumption that the upper bound of time delay is known, and later in [17] and [19], by using *dynamic* instead of static state feedback. The dynamic gain-based designs or the dynamic state feedback control schemes [17], [19] have shown to be effective in counteracting the nonlinearities with delayed states, thus making it possible to remove restrictive conditions imposed on time-delay nonlinear systems, which were commonly assumed in the literature when using delay-independent static state feedback.

Most of the aforementioned works concentrated on time-delay nonlinear systems with known control directions, e.g., the signs of all coefficients of the chain of integrator are assumed to be known. If this crucial information is not available, a new method needs to be developed for the control of time-delay systems. When no time delay is involved, feedback design approaches have been studied for uncertain nonlinear systems with unknown control directions [18], using the so-called Nussbaum functions from universal adaptive stabilization of minimum-phase linear systems with unknown sign of high-frequency gain [13]. Since the sign of the control input often represents, for instance, motion directions of mechanical systems such as robotics modeled by the Lagrange equation and may be unknown, it is certainly important to investigate how to control time-delay systems with unknown control directions.

In this paper, we first focus our attention on the following class of time-delay nonlinear system with unknown control directions:

$$\begin{aligned}\dot{x}_i &= \theta_i x_{i+1} + f_i(x_1,\ldots,x_i,x_1(t-d),\ldots,x_i(t-d)),\\ \dot{x}_n &= \theta_n u + f_n(x,x(t-d)), \qquad i=1,\ldots,n-1,\\ x(s) &= \zeta(s), \quad s\in[-d,0]\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state and input, respectively. The constant $d \geq 0$ is an unknown time-delay of the system, $f_i : \mathbb{R}^{2i} \to \mathbb{R}$ are C^1 mappings with $f_i(0,\ldots,0)=0$, and $\zeta(s) \in \mathbb{R}^n$ is a continuous function defined on $[-d,0]$. The coefficients $\theta_i \neq 0$, $1 \leq i \leq n$, are unknown constants whose signs are also unknown, but bounded by a known constant $\bar{c}$. For example, in the planar case, θ_1 may be 1 or -10 while θ_2 can be -2 or 3.

For the time-delay system with unknown control directions (1), global stabilization by delay-independent state feedback is a nontrivial problem and has not been addressed so far. There are perhaps two reasons, which are as follows.

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1) When the signs of coefficients of a chain of integrators are unknown, the design of virtual controllers is less intuitive and more involved as the uncertainties cannot be cancelled directly by a conventional backstepping design.
2) The presence of time-delay nonlinearities makes a delay-free, static state feedback law insufficient for mitigating the effects of time delay, and hence a *dynamic* instead of *static* state feedback may be necessary.

Motivated by the universal control idea [6]–[8], [13] and the recent development [17], [19], we propose in this paper a novel construction of a set of Lyapunov–Krasovskii functionals and a delay-independent, *dynamic state* feedback control scheme for counteracting the effects of time-delay nonlinearities and unknown control directions in the system (1) simultaneously. With the help of the new dynamic gain-based Lyapunov–Krasovskii functionals, we are able to design a time-delay independent, dynamic state feedback compensator step-by-step, resulting in a solution to the global state regulation of the time-delay system (1) with stability. Interestingly, it is worth pointing out that the approach presented in this paper provides a new yet simpler way of designing state feedback controllers that achieve global stabilization of the nonlinear system (1) with unknown control directions, in the absence of time delay, i.e., $d = 0$.

Notations: In this paper, an even Nussbaum function $N(k) = k^2\cos(k)$ is chosen, which clearly satisfies the properties: 1) $\lim_{k\to\infty}\sup\ \frac{1}{k}\int_0^k N(s)ds = +\infty$; 2) $\lim_{k\to\infty}\inf\ \frac{1}{k}\int_0^k N(s)ds = -\infty$. For the sake of convenience, denote $\bar{v}_i = [v_1,\ldots,v_i]^T \in \mathbb{R}^i$ for $i = 1,\ldots,n$. For example, $\bar{x}_i = [x_1,\ldots,x_i]^T$, $\bar{x}_i(t-d) = [x_1(t-d),\ldots,x_i(t-d)]^T$, $\bar{l}_i = [l_1,\ldots,l_i]^T$, and $\bar{k}_i = [k_1,\ldots,k_i]^T$, where l_i and k_i are controller gains to be designed step-by-step in the next section.

II. Universal Control-Based Dynamic State Feedback

In this section, we employ the idea of universal control [6]–[8], [13] to design a *dynamic state* feedback compensator that is composed of two sets of gain update laws, in the spirit of [6]–[8]. A set of them is expected to mitigate the effects of the unknown control directions present in the system (1), while the other set is capable of counteracting the time-delay nonlinearities of the system (1). The idea of utilizing two sets of gain update laws have been explored in a different context, particularly, in the case of adaptive output feedback stabilization of nonlinear systems with unknown parameters [6]–[8]. In this paper, we further explore the potential/power of the idea by showing its new application in the control of the time-delay system (1) with unknown control directions. The main result of this paper is the following theorem.

Theorem 2.1: For the time-delay system with unknown control directions (1), there exists a delay-independent, dynamic state feedback controller

$$\dot{L} = \eta(L,K,x),\quad \dot{k} = h(L,K,x),\quad u = \alpha(L,K,x) \tag{2}$$

with $\alpha(L,K,0) = 0$, such that the state x is regulated to the origin while keeping all the signals of the closed-loop system bounded.

Remark 2.2: As shown in Section II-A, a delay-independent, dynamic state feedback compensator (2) can be explicitly designed and given by, for instance, (40) and (20)–(21), with $L = (l_1,\ldots,l_{n-1})^T$ and $K = (k_1,\ldots,k_n)$.

The proof of Theorem 2.1 is divided into two parts. The first part contains a recursive procedure for the design of a universal-like dynamic state compensator (2), while the second part provides stability analysis of the closed-loop system.

A. Dynamic State Feedback Design

In this section, we first construct a delay-independent, dynamic state feedback compensator, by means of the Nussbaum-gain function [13], a set of new Lyapunov–Krasovskii functionals (due to the presence of unknown control directions) and the dynamic gain-based design philosophy [19].

Step 1: For the x_1-subsystem of (1), view the state x_2 as a virtual control and consider the Lyapunov function $V_1(x_1,l_1) = \frac{1}{2}(1+\frac{1}{l_1})\xi_1^2$ with $\xi_1 = x_1$, where $l_1(\cdot) \geq 1$ is a dynamic gain to be determined in Step 2.

A direct calculation gives

$$\begin{aligned}\dot{V}_1 &= \left(1+\frac{1}{l_1}\right)\xi_1[\theta_1 x_2 + f_1(x_1,x_1(t-d))] - \frac{\dot{l}_1}{2l_1^2}\xi_1^2\\ &\leq \left(1+\frac{1}{l_1}\right)\theta_1\xi_1 x_2^* + 2|\theta_1\xi_1\xi_2| + 2|\xi_1 f_1(\cdot)| - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \end{aligned} \tag{3}$$

where $\xi_2 = x_2 - x_2^*$.

From Lemma 4.3 in the Appendix, it is clear that there exist smooth functions $\bar{\gamma}_1(x_1) \geq 0$ and $\bar{\gamma}_1^*(x_1(t-d)) \geq 0$ such that $|f_1(x_1,x_1(t-d))| \leq \bar{\gamma}_1(x_1)|x_1| + \bar{\gamma}_1^*(x_1(t-d))|x_1(t-d)|$. Consequently,

$$2|\xi_1 f_1(\cdot)| \leq 2\xi_1^2\bar{\gamma}_1(x_1) + \xi_1^2 + \xi_1^2(t-d)\bar{\gamma}_1^{*2}(x_1(t-d)). \tag{4}$$

In addition, $2|\theta_1\xi_1\xi_2| \leq \xi_1^2 + c_2\xi_2^2$, for a constant $c_2 > 0$.

We now use the bound $\bar{\gamma}_1^{*2}(\cdot)$ from (4) to construct the Lyapunov–Krasovskii functional

$$V_{1LK} = V_1(l_1,x_1) + \int_{t-d}^{t}\xi_1^2(s)\,\bar{\gamma}_1^{*2}(x_1(s))\,ds$$

whose time derivative satisfies [by (3) and (4)]

$$\begin{aligned}\dot{V}_{1LK} \leq\ & -n\xi_1^2 + \left(1+\frac{1}{l_1}\right)\theta_1\xi_1 x_2^* + c_2\xi_2^2\\ & + \bar{c}_1\xi_1^2(1+2\bar{\gamma}_1(x_1)+\bar{\gamma}_1^{*2}(x_1)) - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \end{aligned} \tag{5}$$

where $\bar{c}_1 = 2+n$. Because the sign of θ_1 is unknown, we use the idea from [13], namely, the Nussbaum function to design a controller. In fact, from (5) a virtual controller with the Nussbaum gain can be constructed as

$$\begin{aligned} x_2^* &= \xi_1 N(k_1)[1+2\bar{\gamma}_1(x_1)+\bar{\gamma}_1^{*2}(x_1)]\\ &:= \xi_1 N(k_1)\beta_1(x_1)\\ \dot{k}_1 &= \left(1+\frac{1}{l_1}\right)\xi_1^2\beta_1(x_1),\quad k_1(0) = 1. \end{aligned} \tag{6}$$

This, together with $l_1(\cdot) \geq 1$, results in

$$\dot{V}_{1LK} \leq -n\xi_1^2 + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 + c_2\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \tag{7}$$

Step 2: For the (x_1,x_2)-subsystem of (1), treat the state x_3 as a virtual control and consider the Lyapunov–Krasovskii functional

$$V_2 = V_{1LK} + \frac{1}{2l_1}k_1^2\xi_2^2 + \frac{1}{2l_1 l_2}(\xi_1^2 + k_1^2\xi_2^2) \tag{8}$$

where $l_2 \geq 1$ is a dynamic gain to be determined in Step 3.

In view of (7) and the properties that $l_j \geq 1, j = 1, 2$, we have

$$\dot{V}_2 \leq -n\xi_1^2 + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 + c_2\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 + \frac{1}{l_1}\left(1+\frac{1}{l_2}\right)\theta_2 k_1^2 \xi_2(x_3^* + x_3 - x_3^*) + \frac{2}{l_1}k_1^2|\xi_2 f_2(\cdot)| + \frac{2}{l_1}k_1^2|\xi_2\dot{x}_2^*| + \frac{1}{l_1 l_2}\xi_1\dot{x}_1 + \frac{2}{l_1}k_1\dot{k}_1\xi_2^2 - \frac{\dot{l}_1}{2l_1^2}k_1^2\xi_2^2 - \frac{\dot{l}_1 l_2 + l_1\dot{l}_2}{2l_1^2 l_2^2}(\xi_1^2 + k_1^2\xi_2^2). \tag{9}$$

Using $\xi_2 = x_2 - x_2^*$, (6) and the fact that $l_1(\cdot) \geq 1$ and $k_1(\cdot) \geq 1$, we arrive at (with the aid of Lemmas 4.1–4.3)

$$\frac{2}{l_1}k_1^2|\xi_2 f_2(\cdot)| \leq k_1^2\xi_2^2\Upsilon_{21}(k_1,\bar{x}_2) + \frac{1}{l_1}\xi_1^2\Upsilon_{22}(k_1,x_1) + \xi_2^2(t-d) \cdot \Upsilon_{21}^*(k_1(t-d),\bar{x}_2(t-d)) + \frac{1}{l_1}\xi_1^2(t-d)\Upsilon_{22}^*(k_1(t-d),x_1(t-d))$$

$$\frac{2}{l_1}k_1^2|\xi_2\dot{x}_2^*| + \frac{1}{l_1 l_2}|\xi_1\dot{\xi}_1| + \frac{2}{l_1}k_1\dot{k}_1\xi_2^2 \leq k_1^2\xi_2^2\Phi_{21}(k_1,\bar{x}_2) + \frac{1}{l_1}\xi_1^2\Phi_{22}(k_1,x_1) + \frac{1}{l_1}\xi_1^2(t-d)\Phi_{22}^*(x_1(t-d)) - \frac{\dot{l}_1}{2l_1^2}k_1^2\xi_2^2 - \frac{\dot{l}_1 l_2 + l_1\dot{l}_2}{2l_1^2 l_2^2}(\xi_1^2 + k_1^2\xi_2^2) \leq -\frac{\dot{l}_2}{2l_1 l_2^2}(\xi_1^2+\xi_2^2) \tag{10}$$

where $\Upsilon_{2j}(\cdot) \geq 0$, $\Upsilon_{2j}^*(\cdot) \geq 0$, $\Phi_{2j}(\cdot) \geq 0$, and $\Phi_{22}^*(\cdot) \geq 0$, $j = 1, 2$, are smooth functions.

With the help of the smooth functions $\Upsilon_{2j}^*(\cdot)$ and $\Phi_{22}^*(\cdot)$ obtained from (10), we construct the Lyapunov–Krasovskii functional

$$V_{2LK} = V_2 + \int_{t-d}^{t}\xi_2^2(s)\Upsilon_{21}^*(k_1(s),\bar{x}_2(s))ds + \int_{t-d}^{t}\frac{1}{l_1(s)}\xi_1^2(s)[\Upsilon_{22}^*(k_1(s),x_1(s)) + \Phi_{22}^*(x_1(s))]ds \tag{11}$$

Observing that $\frac{1}{l_1}(1+\frac{1}{l_2})k_1^2|\theta_2\xi_2(x_3 - x_3^*)| \leq k_1^2\xi_2^2 + c_3 k_1^2\xi_3^2$, with $\xi_3 = x_3 - x_3^*$ and $c_3 > 0$, we deduce from (9) and (10) that

$$\dot{V}_{2LK} \leq -n\xi_1^2 + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 + \frac{1}{l_1}\left(1+\frac{1}{l_2}\right)\theta_2 k_1^2\xi_2 x_3^* + c_3 k_1^2\xi_3^2 + k_1^2\xi_2^2[1 + c_2 + \Upsilon_{21}(k_1,\bar{x}_2) + \Phi_{21}(k_1,\bar{x}_2) + \Upsilon_{21}^*(k_1,\bar{x}_2)] + \frac{1}{l_1}\xi_1^2[\Upsilon_{22}(k_1,x_1) + \Phi_{22}(k_1,x_1) + \Upsilon_{22}^*(k_1,x_1) + \Phi_{22}^*(x_1)] + \left(\frac{1}{l_1} - \frac{1}{l_1(t-d)}\right)\xi_1^2(t-d)[\Upsilon_{22}^*(k_1(t-d),x_1(t-d)) + \Phi_{22}^*(x_1(t-d))] - \frac{\dot{l}_2}{2l_1 l_2^2}(\xi_1^2+\xi_2^2). \tag{12}$$

Based on (12), one can design the Riccati-like update law

$$\dot{l}_1 = \max\{-l_1^2 + l_1\rho_1(k_1,x_1),\ 0\},\ l_1(0) = 1 \tag{13}$$

$$\rho_1(k_1,x_1) = 2\left[\Upsilon_{22}(\cdot) + \Upsilon_{22}^*(\cdot) + \Phi_{22}(\cdot) + \Phi_{22}^*(\cdot)\right] \tag{14}$$

to mitigate the effects of the time-delay nonlinearity.

By construction, it is clear from (13) that

$$0 \leq \dot{l}_1 \leq l_1\rho_1(\cdot),\ \ \dot{l}_1 \geq -l_1^2 + l_1\rho_1(\cdot),\ \ l_1(t) \geq l_1(t-d) \geq 1. \tag{15}$$

As a consequence

$$-\frac{\dot{l}_1}{2l_1^2}\xi_1^2 \leq \xi_1^2 - \frac{1}{2l_1}\xi_1^2\rho_1(k_1,x_1)$$
$$\frac{1}{l_1} - \frac{1}{l_1(t-d)} \leq 0. \tag{16}$$

Substituting (13) and (16) into (12) leads to

$$\dot{V}_{2LK} \leq -(n-1)\xi_1^2 - (n-1)k_1^2\xi_2^2 + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 + \frac{1}{l_1}\left(1+\frac{1}{l_2}\right)\theta_2 k_1^2\xi_2 x_3^* + c_3 k_1^2\xi_3^2 + \bar{c}_2 k_1^2\xi_2^2 \cdot[1 + \Upsilon_{21}(k_1,\bar{x}_2) + \Phi_{21}(k_1,\bar{x}_2) + \Upsilon_{21}^*(k_1,\bar{x}_2)] - \frac{\dot{l}_1}{2l_1 l_2^2}(\xi_1^2+\xi_2^2) \tag{17}$$

where $\bar{c}_2 = c_2 + n$. Similar to Step 1, because of the unknown sign of θ_2, we need to design a virtual controller x_3^* with the Nussbaum gain as

$$x_3^* = l_1\xi_2 N(k_2)[1 + \Upsilon_{21}(\cdot) + \Phi_{21}(\cdot) + \Upsilon_{21}^*(\cdot)] := l_1\xi_2 N(k_2)\beta_2(k_1,\bar{x}_2)$$
$$\dot{k}_2 = \left(1+\frac{1}{l_2}\right)\xi_2^2\beta_2(k_1,\bar{x}_2),\quad k_2(0) = 1 \tag{18}$$

such that the inequality (17) becomes

$$\dot{V}_{2LK} \leq -(n-1)[\xi_1^2 + k_1^2\xi_2^2] + (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 + (\theta_2 N(k_2) + \bar{c}_2)k_1^2\dot{k}_2 + c_3 k_1^2\xi_3^2 - \frac{\dot{l}_2}{2l_1 l_2^2}(\xi_1^2+\xi_2^2). \tag{19}$$

Inductive Step: Suppose at Step $i-1$, there are a Lyapunov–Krasovskii functional $V_{(i-1)LK}$, a set of dynamic gains $l_j(\cdot) \geq 1 = l_j(0),\ j = 1,\ldots,i-1$, given by

$$\dot{l}_1 = \max\{-l_1^2 + l_1\rho_1(k_1,x_1),0\},$$
$$\dot{l}_2 = \max\{-l_2^2 + l_2\rho_2(l_1,\bar{k}_2,\bar{x}_2),0\},$$
$$\vdots \tag{20}$$
$$\dot{l}_{i-2} = \max\{-l_{i-2}^2 + l_{i-2}\rho_{i-2}(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-2}),0\}$$

and a set of virtual controllers $x_1^*,\ldots,x_i^*$ with the Nussbaum gains (updated by a set of universal controllers) defined by

$$\begin{array}{ll} x_1^* = 0 & \xi_1 = x_1 - x_1^* \\ x_2^* = \xi_1 N(k_1)\beta_1(x_1) & \xi_2 = x_2 - x_2^* \\ \dot{k}_1 = \left(1+\frac{1}{l_1}\right)\xi_1^2\beta_1(\cdot) & \\ \vdots & \vdots \\ x_i^* = (l_1\cdots l_{i-2})\xi_{i-1}N(k_{i-1}) & \xi_i = x_i - x_i^* \\ \cdot\beta_{i-1}(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1}) & \\ \dot{k}_{i-1} = \left(1+\frac{1}{l_{i-1}}\right)\xi_{i-1}^2\beta_{i-1}(\cdot) & \end{array} \tag{21}$$

with $\rho_j(\cdot) > 0$ and $\beta_j(\cdot) > 0$ being smooth functions, such that

$$\dot{V}_{(i-1)LK} \le -(n-(i-2))\Sigma_{j=1}^{i-1}\left[\left(\prod_{m=0}^{j-1} k_m^2\right)\xi_j^2\right] + c_i k_1^2\cdots k_{i-2}^2\xi_i^2 + \sum_{j=1}^{i-1}\left[(\theta_j N(k_j)+\bar{c}_j)\left(\prod_{m=0}^{j-1} k_m^2\right)\dot{k}_j\right] - \frac{\dot{l}_{i-1}}{2l_1\cdots l_{i-2}l_{i-1}^2}\sum_{j=1}^{i-1}\xi_j^2 \tag{22}$$

where $c_i > 0$ is a constant and $k_0 = 1$. Clearly, (22) reduces to (19) when $i = 3$.

Recursively, it can be shown that (22) also holds at Step i. To this end, consider the Lyapunov–Krasovskii functional

$$V_i = V_{(i-1)LK} + \frac{1}{2l_1\cdots l_{i-1}}k_1^2\cdots k_{i-1}^2\xi_i^2 + \frac{1}{2l_1\cdots l_i}\left[\Sigma_{j=1}^{i-1}\xi_j^2 + k_1^2\cdots k_{i-1}^2\xi_i^2\right] \tag{23}$$

where $l_i(\cdot) \ge 1$ is a dynamic gain to be designed.

Using (22) and the properties that $l_j \ge 1,\ k_j \ge 1$, we have

$$\begin{aligned}\dot{V}_i \le\ & -(n-(i-2))\Sigma_{j=1}^{i-1}\left[\left(\prod_{m=0}^{j-1} k_m^2\right)\xi_j^2\right] \\ & +\Sigma_{j=1}^{i-1}\left[(\theta_j N(k_j)+\bar{c}_j)\left(\prod_{m=0}^{j-1} k_m^2\right)\dot{k}_j\right] + c_i k_1^2\cdots k_{i-2}^2\xi_i^2 \\ & -\frac{\dot{l}_{i-1}}{2l_1\cdots l_{i-2}l_{i-1}^2}\Sigma_{j=1}^{i-1}\xi_j^2 + \frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left(1+\frac{1}{l_i}\right)\theta_i\xi_i x_{i+1}^* \\ & +\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\theta_i\xi_i\xi_{i+1} + \xi_i f_i(\cdot) - \xi_i\dot{x}_i^*| \\ & +\frac{1}{l_1\cdots l_{i-1}}\sum_{j=1}^{i-1}\xi_j\dot{\xi}_j + \frac{2}{l_1\cdots l_{i-1}}\left[\sum_{j=1}^{i-1}\left(k_j\dot{k}_j\prod_{\substack{m=1\\m\ne j}}^{i-1}k_m^2\right)\xi_i^2\right] \\ & -\frac{1}{2l_1^2\cdots l_{i-1}^2}\left[\sum_{j=1}^{i-1}\left(\prod_{\substack{m=1\\m\ne j}}^{i-1}l_m\right)\dot{l}_j\right]k_1^2\cdots k_{i-1}^2\xi_i^2 \\ & -\frac{1}{2l_1^2\cdots l_i^2}\left[\sum_{j=1}^{i}(\prod_{\substack{m=1\\m\ne j}}^{i}l_m)\dot{l}_j\right]\left[\sum_{j=1}^{i-1}\xi_j^2 + k_1^2\cdots k_{i-1}^2\xi_i^2\right].\end{aligned} \tag{24}$$

The terms in (24) can be estimated by using Lemma 4.2 and the properties of $k_i \ge 1$ and $l_i \ge 1$ as follows:

$$\begin{aligned}\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\xi_i f_i(\cdot)| \le\ & k_1^2\cdots k_{i-1}^2\xi_i^2\Upsilon_{i1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i) \\ & +\frac{1}{l_1\cdots l_{i-1}}\left(\sum_{j=1}^{i-1}\xi_j^2\right)\Upsilon_{i2}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}) \\ & +\xi_i^2(t-d)\Upsilon_{i1}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_i(t-d)) \\ & +\frac{1}{l_1\cdots l_{i-1}}\left(\sum_{j=1}^{i-1}\xi_j^2(t-d)\right) \\ & \cdot\Upsilon_{i2}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\xi_i\dot{x}_i^*| \le\ & k_1^2\cdots k_{i-1}^2\xi_i^2\Phi_{i1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i) \\ & +\frac{1}{l_1\cdots l_{i-1}}\left(\sum_{j=1}^{i-1}\xi_j^2\right)\Phi_{i2}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}) \\ & +\frac{1}{l_1\cdots l_{i-1}}\left(\sum_{j=1}^{i-1}\xi_j^2(t-d)\right) \\ & \cdot\Phi_{i2}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))\end{aligned} \tag{26}$$

$$\frac{2k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}|\theta_i\xi_i\xi_{i+1}| \le k_1^2\cdots k_{i-1}^2\xi_i^2 + c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2 \tag{27}$$

$$\begin{aligned}\frac{1}{l_1\cdots l_{i-1}}\sum_{j=1}^{i-1}|\xi_j\dot{\xi}_j| \le\ & k_1^2\cdots k_{i-1}^2\xi_i^2 \\ & +\frac{1}{l_1\cdots l_{i-1}}\left(\sum_{j=1}^{i-1}\xi_j^2\right)\Psi_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}) + \frac{1}{l_1\cdots l_{i-1}} \\ & \cdot\left(\sum_{j=1}^{i-1}\xi_j^2(t-d)\right)\Psi_i^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))\end{aligned} \tag{28}$$

$$\frac{2\xi_i^2}{l_1\cdots l_{i-1}}\left[\sum_{j=1}^{i-1}\left(k_j\dot{k}_j\prod_{\substack{m=1\\m\ne j}}^{i-1}k_m^2\right)\right] \le k_1^2\cdots k_{i-1}^2\xi_i^2\omega_i \times(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1}) \tag{29}$$

$$\begin{aligned}& -\frac{1}{2l_1^2\cdots l_{i-1}^2}\left[\sum_{j=1}^{i-1}\left(\prod_{\substack{m=1\\m\ne j}}^{i-1}l_m\right)\dot{l}_j\right]k_1^2\cdots k_{i-1}^2\xi_i^2 \\ & -\frac{1}{2l_1^2\cdots l_i^2}\left[\sum_{j=1}^{i}\left(\prod_{\substack{m=1\\m\ne j}}^{i}l_m\right)\dot{l}_j\right]\left[\sum_{j=1}^{i-1}\xi_j^2 + k_1^2\cdots k_{i-1}^2\xi_i^2\right] \\ & \le -\frac{\dot{l}_i}{2l_1\cdots l_{i-1}l_i^2}\left(\sum_{j=1}^{i}\xi_j^2\right)\end{aligned} \tag{30}$$

where $\Upsilon_{ij}(\cdot) \ge 0,\ \Upsilon_{ij}^*(\cdot) \ge 0,\ \Phi_{ij}(\cdot) \ge 0,\ \Phi_{ij}^*(\cdot) \ge 0,\ \Psi_i(\cdot) \ge 0,\ \Psi_i^*(\cdot) \ge 0,\ j = 1, 2$, and $\omega_i(\cdot) \ge 0$ are smooth functions.

With the aid of the bounding functions $\Upsilon_{ij}^*(\cdot)$, $\Phi_{ij}^*(\cdot)$, and $\Psi_i^*(\cdot)$ from the estimations above, which are related to the delay terms, one can construct the Lyapunov–Krasovskii functional

$$\begin{aligned}V_{iLK} =\ & V_i + \int_{t-d}^{t}\xi_i^2(s)\Upsilon_{i1}^*(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_i(s))ds \\ & +\int_{t-d}^{t}\frac{1}{l_1(s)\cdots l_{i-1}(s)}\left(\sum_{j=1}^{i-1}\xi_j^2(s)\right)[\Upsilon_{i2}^*(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s)) \\ & +\Phi_{i2}^*(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s))+\Psi_i^*(\bar{l}_{i-2}(s),\bar{k}_{i-1}(s),\bar{x}_{i-1}(s))]ds.\end{aligned} \tag{31}$$

Then, in view of (24)–(30), we have

$$\begin{aligned}
\dot{V}_{iLK} \leq& -(n-(i-2))\sum_{j=1}^{i-1}\left[\left(\prod_{m=0}^{j-1}k_m^2\right)\xi_j^2\right]\\
&+\sum_{j=1}^{i-1}\left[(\theta_j N(k_j)+\bar{c}_j)\left(\prod_{m=0}^{j-1}k_m^2\right)\dot{k}_j\right]-\frac{\dot{l}_{i-1}}{2l_1\cdots l_{i-2}l_{i-1}^2}\sum_{j=1}^{i-1}\xi_j^2\\
&+\frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\left(1+\frac{1}{l_i}\right)\theta_i\xi_i x_{i+1}^*+c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2\\
&+k_1^2\cdots k_{i-1}^2\xi_i^2[2+c_i+\Upsilon_{i1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)+\Phi_{i1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)\\
&+\omega_i(\bar{l}_{i-3},\bar{k}_{i-2},\bar{x}_{i-1})+\Upsilon_{i1}^*(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)]+\frac{1}{l_1\cdots l_{i-1}}\left(\sum_{j=1}^{i-1}\xi_j^2\right)\\
&\cdot[\Upsilon_{i2}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})+\Phi_{i2}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})\\
&+\Psi_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})+\Upsilon_{i2}^*(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})\\
&+\Phi_{i2}^*(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})+\Psi_i^*(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})]\\
&+\left[\frac{1}{l_1\cdots l_{i-1}}-\frac{1}{l_1(t-d)\cdots l_{i-1}(t-d)}\right]\left(\sum_{j=1}^{i-1}\xi_j^2(t-d)\right)\\
&\cdot[\Upsilon_{i2}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))\\
&+\Phi_{i2}^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))\\
&+\Psi_i^*(\bar{l}_{i-2}(t-d),\bar{k}_{i-1}(t-d),\bar{x}_{i-1}(t-d))]\\
&-\frac{\dot{l}_i}{2l_1\cdots l_{i-1}l_i^2}\left(\sum_{j=1}^{i}\xi_j^2\right). \qquad (32)
\end{aligned}$$

Following the idea and design given in Step 2, we can construct [based on (32)] the delay-free gain update law

$$\dot{l}_{i-1}=\max\{-l_{i-1}^2+l_{i-1}\rho_{i-1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}),0\} \qquad (33)$$

with $l_{i-1}(0)=1$, and

$$\begin{aligned}
\rho_{i-1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1}) =\ & 2[\Upsilon_{i2}(\cdot)+\Upsilon_{i2}^*(\cdot)+\Phi_{i2}(\cdot)+\Phi_{i2}^*(\cdot)\\
&+\Psi_i(\cdot)+\Psi_i^*(\cdot)]. \qquad (34)
\end{aligned}$$

By construction, it is easy to verify that

$$\begin{aligned}
0 \leq\ \dot{l}_{i-1}\ &\leq\ l_{i-1}\rho_{i-1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})\\
\dot{l}_{i-1}\ &\geq\ -l_{i-1}^2+l_{i-1}\rho_{i-1}(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_{i-1})\\
l_{i-1}\ &\geq\ l_{i-1}(t-d)\geq 1. \qquad (35)
\end{aligned}$$

As a consequence,

$$-\frac{\dot{l}_{i-1}}{2l_1\cdots l_{i-2}l_{i-1}^2}\left(\sum_{j=1}^{i-1}\xi_j^2\right)\leq\sum_{j=1}^{i-1}\xi_j^2-\frac{\rho_{i-1}(\cdot)}{2l_1\cdots l_{i-1}}\left[\sum_{j=1}^{i-1}\xi_j^2\right] \qquad (36)$$

$$\frac{1}{l_1\cdots l_{i-1}}-\frac{1}{l_1(t-d)\cdots l_{i-1}(t-d)}\leq 0. \qquad (37)$$

Substituting (36) and (37) into (32), we obtain

$$\begin{aligned}
\dot{V}_{iLK} \leq& -(n-(i-1))\sum_{j=1}^{i}\left(\prod_{m=0}^{j-1}k_m^2\right)\xi_j^2\\
&+\sum_{j=1}^{i-1}\left[(\theta_j N(k_j)+\bar{c}_j)\left(\prod_{m=0}^{j-1}k_m^2\right)\dot{k}_j\right]+\frac{k_1^2\cdots k_{i-1}^2}{l_1\cdots l_{i-1}}\\
&\cdot\left(1+\frac{1}{l_i}\right)\theta_i\xi_i x_{i+1}^*+\bar{c}_i k_1^2\cdots k_{i-1}^2\xi_i^2\\
&\cdot[1+\Upsilon_{i1}(\cdot)+\Phi_{i1}(\cdot)+\omega_i(\cdot)+\Upsilon_{i1}^*(\cdot)]\\
&-\frac{\dot{l}_i}{2l_1\cdots l_{i-1}l_i^2}\sum_{j=1}^{i}\xi_j^2+c_{i+1}k_1^2\cdots k_{i-1}^2\xi_{i+1}^2 \qquad (38)
\end{aligned}$$

where $\bar{c}_i=2+c_i+(n-i)$.

To mitigate the effects of the unknown sign of θ_i, we design the following virtual controller with a Nussbaum gain (updated by a universal controller $\dot{k}_i$)

$$\begin{aligned}
x_{i+1}^* &= (l_1\cdots l_{i-1})\xi_i N(k_i)[1+\Upsilon_{i1}(\cdot)\\
&\quad+\Phi_{i1}(\cdot)+\omega_i(\cdot)+\Upsilon_{i1}^*(\cdot)]\\
&:= (l_1\cdots l_{i-1})\xi_i N(k_i)\beta_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i)\\
\dot{k}_i &= \left(1+\frac{1}{l_i}\right)\xi_i^2\beta_i(\bar{l}_{i-2},\bar{k}_{i-1},\bar{x}_i). \qquad (39)
\end{aligned}$$

Substituting (39) into (38) leads to the claim that (22) holds at Step i. The inductive argument so far has indicated that (22) holds for $i=n+1$ with $u=x_{n+1}=x_{n+1}^*$. As a consequence, a dynamic state feedback controller that is composed of (20) with $i=n+1$ and a universal-like control law

$$\begin{aligned}
u &= (l_1\cdots l_{n-1})\xi_n N(k_n)\beta_n(\bar{l}_{n-2},\bar{k}_{n-1},x)\\
\dot{k}_n &= \xi_n^2\beta_n(\bar{l}_{n-2},\bar{k}_{n-1},x) \qquad (40)
\end{aligned}$$

renders

$$\begin{aligned}
\dot{V}_{nLK} \leq& -\sum_{j=1}^{n}\left[\left(\prod_{m=0}^{j-1}k_m^2\right)\xi_j^2\right]\\
&+\sum_{j=1}^{n}\left[(\theta_j N(k_j)+\bar{c}_j)\left(\prod_{m=0}^{j-1}k_m^2\right)\dot{k}_j\right]. \qquad (41)
\end{aligned}$$

We end this section with an observation that the dynamic state feedback compensator designed so far, namely, (40) and (20)–(21) with $i=n+1$, is exactly of the form (2) with $L=(l_1,\ldots,l_{n-1})^T$ and $K=(k_1,\ldots,k_n)$.

Remark 2.3: In the case when θ_i's are known constants (e.g., $\theta_i=1$), all the $k_i=1$ and $\dot{k}_i=0$. Then, the Lyapunov inequality (41) reduces to

$$\dot{V}_{nLK}\leq-\sum_{j=1}^{n}\xi_j^2\leq 0 \qquad (42)$$

from which it is concluded that global asymptotic state regulation of the time-delay nonlinear system (1) and boundedness of the closed-loop system are achieved by the delay-independent, dynamic state feedback compensator (40) and (20)–(21), with $k_i = 1$, $\dot{k}_i = 0$ and $N(k_i) =$ constant for $i = 1, \ldots, n$.

B. Asymptotic State Regulation With Boundedness

In this section, we complete the proof of Theorem 2.1 by showing that the universal-like, dynamic state feedback controller (40) and (20) designed in Section II-A ensures not only the convergence of the system state x but also boundedness of the resulting closed-loop system.

First of all, from the Lyapunov inequalities (22) and (41) it is concluded that $k_i(t), i = 1 \cdots n$, are bounded. The proof of this claim is given in Appendix B.

By the boundedness of $k_i(t), 1 \le i \le n$, it follows immediately from (39) that $\xi_i^2(t) \le \dot{k}_i(t)$ because $\beta_i(\cdot) \ge 1$ and $l_i(t) \ge 1$. Hence, $\int_0^{+\infty} \xi_i^2 ds \le k_i(+\infty) - k_i(0) = C$.

On the other hand, it follows from (41) that

$$\begin{aligned} V_{nLK}(t) &\le \sum_{j=1}^{n} \int_0^t |\theta_j N(k_j(s)) + \bar{c}_j| \left(\prod_{m=0}^{j-1} k_m^2(s) \right) \dot{k}_j(s) ds + c, \\ &\le \bar{c} \sum_{j=1}^{n} \int_0^t \dot{k}_j(s) ds + c. \end{aligned} \tag{43}$$

From the inequality (43) and the boundedness of $k_i(t), i = 1 \cdots n$, it is straightforward to prove that the Lyapunov–Krasovskii functional $V_{nLK}(\cdot)$ evaluated on the solution trajectory of the closed-loop system is bounded $\forall t \in [0, +\infty)$. In view of the construction of $V_{nLK}(\cdot)$, in particular, (31) and (23), we deduce that the boundedness of $V_{nLK}(\cdot)$ implies the boundedness of x_1, $\frac{k_1^2 \cdots k_{i-1}^2}{l_1 \cdots l_{i-1}} \xi_i^2$, $i = 1, \ldots, n$.

Keeping the boundedness of x_1 and k_1 in mind, it is trivial to verify that the gain $l_1(\cdot)$ designed by (13) and (14) is monotone nondeceasing. Moreover, $l_1(\cdot)$ is also bounded. In fact, if it is unbounded, then $\lim_{t \to +\infty} l_1(t) = +\infty$. By continuity of $\rho_1(\cdot)$, $\rho_1(k_1, x_1)$ is bounded due to the boundedness of k_1 and x_1. As a consequence, there is a time instant $T > 0$ such that $-l_1^2 + l_1 \rho_1(k_1, x_1) \le 0$ on $[T, +\infty)$. This, together with (15), results in $\dot{l}_1 = 0$ on $[T, +\infty)$, which contradicts to the unboundedness of $l_1(\cdot)$. Therefore, $l_1(\cdot)$ must be bounded. This, combined with the boundedness of k_1, implies the boundedness of x_2^* and $\xi_2 = x_2 - x_2^*$, and so does x_2. With the help of the boundedness of $k_i (1 \le i \le n)$, the boundedness of $l_i(\cdot)$ and x_i can be proved in the iterative manner of $x_2 \to l_2 \to x_3 \to \cdots \to l_{n-1} \to x_n$, by using (20) and (21). In conclusion, all the signals of the closed-loop system (1)–(40)–(20) are bounded $\forall t \in [0, +\infty)$.

Finally, note that $\dot{\xi}_i$, $i = 1, \ldots, n$ are also bounded and $\int_0^{+\infty} \xi_i^2(t) dt < +\infty$. It is thus deduced from the Barbalat's lemma that ξ_i, $i = 1, \ldots, n$ converge to zero as $t \to +\infty$. This, in view of the coordinate transformation (21), implies that the state x tends to the origin as $t \to +\infty$. In this way, the proof of Theorem 2.1 is completed.

III. Extension and Discussion

So far Theorem 2.1 has been established for the time-delay system (1) with unknown control directions. Due to the robust nature of the Lyapunov– Krasovskii functional based design in Section III, it is easy to show that Theorem 2.1 can be extended to a family of uncertain nonlinear systems with time delay, as long as their bounding system has a lower triangular structure.

Corollary 3.1: For the following family of uncertain time-delay systems with unknown control directions

$$\dot{x}_i = \theta_i x_{i+1} + \phi_i(x, x(t-d), u, t), \quad i = 1, \ldots, n \tag{44}$$

where $x_{n+1} = u$ and $\phi_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping with uncertainty satisfying, for $i = 1, \ldots, n$

$$|\phi(x, x(t-d), u, t)| \le \gamma_i(\bar{x}_i, \bar{x}_i(t-d)) \left(\sum_{j=1}^{i} (|x_j| + |x_j(t-d)|) \right) \tag{45}$$

with $\gamma_i(\bar{x}_i, \bar{x}_i(t-d)) \ge 0$ being a known smooth function. Then, there is a delay-independent, dynamic state feedback compensator of the form (2), driving the system state x to the origin while keeping boundedness of the closed-loop system (44) and (2).

A difference between Theorem 2.1 and Corollary 3.1 lies in that the former requires the controlled plant (1) to be precisely known, while the latter needs no accurate information of the time-delay system (44), i.e., $\phi_i(x, x(t-d), u, t)$ may involve uncertainty but does need the knowledge of the bounding system, or, $\gamma_i(\bar{x}_i, \bar{x}_i(t-d))$ in (45).

The next result illustrates how a simplified delay-free, dynamic state feedback compensator, with a set of reduced control gains, can be designed to achieve global asymptotic state regulation with stability for the time-delay nonlinear system (45) with uncertainty.

Corollary 3.2: Under the growth condition (45), a family of time-delay uncertain systems (44) with unknown control directions is globally asymptotically regulated by the delay-free, dynamic state feedback compensator

$$\begin{aligned} u &= \xi_n N(k_n) \beta_n(\bar{l}_{n-1}, \bar{k}_{n-1}, x), \quad \beta_n(\cdot) > 0, \\ \dot{k}_n &= \frac{1}{l_1 \cdots l_{n-1}} \xi_n^2 \beta_n(\bar{l}_{n-1}, \bar{k}_{n-1}, x) \end{aligned} \tag{46}$$

where the gains l_i and $k_i (1 \le i \le n-1)$ are updated by (20) with the coordinate transformation

$$\begin{aligned} & x_1^* = 0 && \xi_1 = x_1 - x_1^* \\ & x_2^* = \xi_1 N(k_1) \beta_1(x_1) && \xi_2 = x_2 - x_2^* \\ & \dot{k}_1 = \left(1 + \frac{1}{l_1}\right) \xi_1^2 \beta_1(\cdot) && \\ & \quad \vdots && \quad \vdots \\ & x_n^* = \xi_{n-1} N(k_{n-1}) \beta_{n-1}(\bar{l}_{n-2}, \bar{k}_{n-2}, \bar{x}_{n-1}) && \xi_n = x_n - x_n^* \\ & \dot{k}_{n-1} = \frac{1}{l_1 \cdots l_{n-2}} \left(1 + \frac{1}{l_{n-1}}\right) \xi_{n-1}^2 \beta_{n-1}(\cdot). && \end{aligned} \tag{47}$$

Remark 3.3: Compared with the dynamics state feedback law (40) and (20) with (21), the dynamic state controller given by Corollary 3.2 is simpler and has much smaller gains, by removing $l_1 \cdots l_{n-1}$ from the controller (40) and $l_1 \cdots l_{i-1}$ from the virtual controllers (21), and reducing the gains of the universal control laws $\dot{k}_i$ simultaneously.

Corollary 3.2 can be proved in a manner similar to that of Theorem 2.1. However, the proof is less intuitive and involves more subtle/tedious estimations. Details are omitted for the reason of space.

The following example demonstrates the application of Corollary 3.2, showing how a delay-free, dynamic state feedback compensator can be designed.

Example 3.4: Consider the time-delay planar system with unknown control directions

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \theta_2 u + x_2^2(t-d). \end{aligned} \tag{48}$$

To handle the nonlinearity with the delayed state, we use the Lyapunov function $V_1 = \frac{1}{2}(1 + \frac{1}{l_1})\xi_1^2$ with $\xi_1 = x_1$, and introduce the gain update law

$$\dot{l}_1 = \max\{-l_1^2 + l_1 \rho_1(k_1, x_1), 0\}, \quad l_1(0) = 1 \tag{49}$$

where $\rho_1(k_1, x_1) \ge 0$ is a smooth function to be given later.

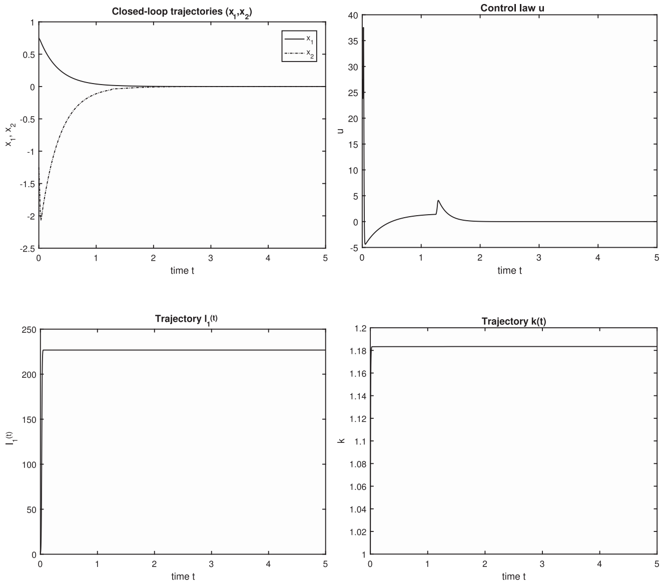

Fig. 1. State trajectories of the closed-loop system (50)-(54).

For the planar system (48), design the virtual controller $x_2^* = -3\xi_1$ and define $\xi_2 = x_2 - x_2^*$. Using the fact that $l_1 \geq 1$ and $\dot{l}_1 \geq -l_1^2 + l_1\rho_1(k_1, x_1)$ leads to

$$\dot{V}_1 = \left(1 + \frac{1}{l_1}\right)\xi_1 x_2 - \frac{\dot{l}_1}{2l_1^2}\xi_1^2 \leq -2\xi_1^2 + \xi_2^2 - \frac{1}{2l_1}\rho_1(k_1, x_1)\xi_1^2.$$

Since the sign of x_2 is known, the virtual controller x_2^* is designed directly without using the Nussbaum gain, i.e., simply set $k_1 = 1$.

Following a design procedure similar to the one presented in Section II, we construct the Lyapunov–Krasovskii functional

$$\begin{aligned} V_{2LK} &= V_1 + \frac{1}{2l_1}\xi_2^2 + \int_{t-d}^{t} \frac{1}{l_1(s)}\xi_2^2(s)\left[\frac{17}{4} + x_1^2 + \xi_2^2\right]^2 ds \\ &\quad + \int_{t-d}^{t} \frac{1}{l_1(s)}\xi_1^2(s)[10 + x_1^2(s)]^2 ds. \end{aligned} \tag{50}$$

Then, a direct calculation gives

$$\begin{aligned} \dot{V}_{2LK} &\leq -2\xi_1^2 - 2\xi_2^2 - \frac{1}{2l_1}\rho_1(k_1, x_1)\xi_1^2 + \frac{1}{l_1}\theta_2\xi_2 u \\ &\quad + \frac{1}{l_1}\xi_2^2\left[2l_1 + \frac{49}{4} + \left(\frac{17}{4} + \xi_2^2 + x_1^2\right)^2\right] \\ &\quad + \frac{1}{l_1}x_1^2\left[\frac{9}{2} + (10 + x_1^2)^2\right] \end{aligned} \tag{51}$$

from which it is deduced that the delay-free dynamic state controller

$$\begin{aligned} \rho_1(k_1, x_1) &= 2\left[\frac{9}{2} + (10 + x_1^2)^2\right] \\ u &= k^2\cos(k)\xi_2\left[2l_1 + \frac{61}{4} + \left(\frac{17}{4} + \xi_2^2 + x_1^2\right)^2\right] \\ \dot{k} &= \xi_2^2\left[2l_1 + \frac{61}{4} + \left(\frac{17}{4} + \xi_2^2 + x_1^2\right)^2\right] \end{aligned} \tag{52}$$

together with the gain update law (49), asymptotically regulates the state (x_1, x_2) of the system (48) to $(0, 0)$, while keeping the signals (l_1, k, x_1, x_2) of the closed-loop system bounded.

The simulations of the trajectories (x_1, x_2) and (l_1, k) of the closed-loop system (48)–(49) and (52) are shown in the figure above, with the parameters $\theta_2 = -1$, $d = 1.25$ and the initial condition $(x_1(0), x_2(0)) = (0.75, -1.25)$. Notably, the proposed dynamic compensator (49) and (52) is independent of the time delay, and hence it also works for a large delay d in the nonlinear system (48), as long as d is finite.

IV. Conclusion

In this paper, we have investigated the problem of global state regulation with stability for nonlinear systems with both time-delay uncertainties and unknown control directions. A delay-free, dynamic state feedback control strategy has been developed based on the dynamic gain-based design technique [19] and the idea of universal control with the Nussbaum function [13]. The dynamic state feedback compensators proposed in Theorem 2.1 or Corollary 3.1 consist of two sets of gain update laws, which are a reminiscent of the work [6]–[8] on universal control of nonlinear systems with unknown parameters by output feedback. One set of gain update laws is a Riccati-type, effective in counteracting the time-delay nonlinearities, while the other set of dynamic update laws is an universal control-like using the Nussbaum function, capable of mitigating the effects of unknown control directions. In contrast to the work in [19], a set of new Lyapunov–Krasovskii functionals have been constructed in this paper, in order to cope with both time-delay uncertainties and unknown control directions simultaneously. It has been shown that the proposed dynamic state feedback control scheme guarantees not only the convergence of the system state to the origin but also global boundedness of the resultant closed-loop system.

Appendix A

This Appendix collects three lemmas that are used in this paper.

Lemma 4.1: [9], [20] Let $x \in R^n, y \in R^m$ and $f : R^n \times R^m \to R$ be a continuous function. Then, there are smooth functions $a(x) \geq 0$, $b(y) \geq 0$, $c(x) \geq 1$ and $d(y) \geq 1$, such that

$$|f(x, y)| \leq a(x) + b(y), \quad |f(x, y)| \leq c(x)\, d(y). \tag{53}$$

Lemma 4.2: [19] Let $x \in R^n, y \in R^m$ and $f : R^n \times R^m \to R$ be a real-valued continuous function. Then, there exist smooth functions $g(x) \geq 0$ and $h(y) \geq 0$, such that

$$f(x, y)(\|x\| + \|y\|) \leq g(x)\|x\| + h(y)\|y\|. \tag{54}$$

Lemma 4.3: For $\bar{x}_i \in R^i$ and $\bar{x}_i(t-d) \in R^i$ denoting $\bar{x}_i$ at time $t - d$, let $f_i : R^i \times R^i \to R$ be a real-valued continuous function with $f_i(\bar{0}, \bar{0}) = 0$. Then, there exist smooth functions $\bar{\gamma}_{ij}(\bar{x}_j) \geq 0$ and $\bar{\gamma}_{ij}^*(\bar{x}_j(t-d)) \geq 0, j = 1, \ldots, i$, such that

$$\begin{aligned} |f_i(\bar{x}_i, \bar{x}_i(t-d))| &\leq \Sigma_{j=1}^{i}(\bar{\gamma}_{ij}(x_j)\|\bar{x}_j\| \\ &\quad + \bar{\gamma}_{ij}^*(\bar{x}_j(t-d))\|\bar{x}_j(t-d)\|) \end{aligned} \tag{55}$$

The last lemma is a direct consequence of the mean value theorem with an integration remainder, as shown in [19], [20].

Appendix B

The boundedness of $k_i(t), i = 1, \ldots, n$ on $[0, +\infty)$ can be proved by a contradiction argument. For simplicity, we first prove the claim for the case of $n = 2$. In this case, if $k_i(t), i = 1, 2$ are unbounded, let $[0, t_{f1})$ and $[0, t_{f2})$ be the maximum intervals of $k_1(t)$ and $k_2(t)$, respectively. Then,

- *Fact (a): $t_{f1} \geq t_{f2}$ must hold.*

If not, $t_{f1} < t_{f2}$. This implies that $k_2(t)$ is bounded on $[0, t_{f_1})$. From (7) and the property that $\xi_2^2 \le \dot{k}_2$ (by (18)), we have

$$\dot{V}_{1LK} \le (\theta_1 N(k_1) + \bar{c}_1)\dot{k}_1 + c_2\dot{k}_2, \quad \forall t \in [0, t_{f_1}). \tag{56}$$

This, together with the relation that

$$\int N(k)dk = k^2 \sin k + 2k\cos k - 2\sin k + c, \tag{57}$$

results in

$$0 \le \frac{V_{1LK}(t)}{k_1(t)} \le \theta_1 k_1(t)\sin(k_1(t)) + C_1, \ \ \forall t \in [0, t_{f_1}) \tag{58}$$

where C_1 is a constant.

Since $\lim_{t\to t_{f_1}} k_1(t) = +\infty$, the right hand side of (58) can become negative when $k_1(t)$ is large enough, regardless of θ_1. This clearly leads to a contradiction. Hence, $t_{f1} \ge t_{f2}$.

As a consequence of Fact (a), $k_1(t)$ is well-defined on $[0, t_{f2})$. In addition, the argument above, in particular, (56)–(58), also implies that

- *Fact (b):* $k_1(t)$ is bounded if $k_2(t)$ is bounded.

With this in mind, we only need to prove that k_2 is bounded. If $k_2(t)$ is unbounded and only define on $[0, t_{f_2})$, consider the inequality (41) with $n = 2$, or, equivalently, (19) with $\dot{l}_2 = 0$ and $\xi_3 = 0$. Dividing $k_1^2(t)$ on the both sides of (41) and integrating from 0 to $t, \forall t \in [0, t_{f2})$, we arrive at

$$\int_0^t \frac{\dot{V}_{2LK}(s)}{k_1^2(s)}ds \le \int_0^t \left(\theta_1 \cos(k_1) + \frac{\bar{c}_1}{k_1^2}\right) dk_1 + \int_{k_2(0)}^{k_2(t)} (\theta_2 N(k_2) + \bar{c}_2)\, dk_2,$$

which, combined with (57) and the monotone increasing property of $k_1(t)$ on $[0, t_{f2})$, yields

$$0 \le \frac{V_{2LK}(t)}{k_1^2(t)k_2(t)} \le \theta_2 k_2(t)\sin(k_2(t)) + C_2, \ \forall t \in [0, t_{f2}) \tag{59}$$

where C_2 is a constant.

Similar to the proof in (58), a contradiction can be drawn from (59). Thus, $k_2(t)$ must be bounded for all $0 \le t \le t_{f2}$. As a such, the maximal interval $[0, t_{f2})$ of $k_2(t)$ can be extended to $[0, +\infty)$. In other words, $k_2(t)$ is bounded $\forall t \ge 0$. By Fact (b), $k_1(t)$ is also bounded on $[0, +\infty)$.

When $n > 2$, an analogous but more tedious proof can also be carried out. If the claim is not true, there is at least one $k_i(t)$ that is unbounded. Let $[0, t_{fi})$ be the maximum interval of $k_i(t)$. Then, we can prove that by the proposed design, $t_{f1} \ge t_{f2} \ge \cdots \ge t_{fn}$. Note that (58) also holds for the n-dimensional system. From (22) and the fact that $\xi_i^2 \le \dot{k}_i, \forall i = 1, \ldots, n$, we have

$$\dot{V}_{iLK} \le \sum_{j=1}^{i}[(\theta_j N(k_j) + \bar{c}_j)\left(\prod_{m=0}^{j-1} k_m^2\right)\dot{k}_j] + c_{i+1}k_1^2 \cdots k_{i-1}^2\dot{k}_{i+1} \tag{60}$$

Using (60) and proceeding a similar argument as done in (58) recursively from $i = 1$ to $i = n$, one can conclude that: (i) $t_{f1} \ge \cdots \ge t_{fn}$, and (ii) k_i is bounded if k_{i+1} is bounded.

Finally, similar to (60), by dividing $\prod_{m=0}^{n-1} k_m^2(t)$ on the both sides of (41) and integrating from 0 to $t, \forall t \in [0, t_{fn})$, it can be shown that $k_n(t)$ is bounded on $[0, +\infty)$. With the help of the property (ii), one can prove recursively, from $i = n-1$ to $i = 1$, that all $k_i(t)$'s are bounded $\forall t \ge 0$.

References

[1] K. Gu, V. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Boston, MA, USA: Birkhauser, 2003.

[2] C. Hua, X. Liu, and X. Guan, "Backstepping control for nonlinear systems with time delays and applications to chemical reactor systems," *IEEE Trans. Ind. Electron.*, vol. 56, no. 9, pp. 3723–3732, Sep. 2009.

[3] M. Jankovic, "Control Lyapunov-Razumikhin functions and robust stabilizationof time delay systems," *IEEE Trans. Automat. Control*, vol. 46, no. 7, pp. 1048–1060, Jul. 2001.

[4] I. Karafyllis and Z. Jiang, "Necessary and sufficient lyapunov-like conditions for robust nonlinear stabilization," *ESAIM Control Optim. Calculus Variation*, vol. 16, pp. 887–928, 2010.

[5] M. Krstic, "Input delay compensation for forward complete and strict-feedforward nonlinear systems," *IEEE Trans. Automat. Control*, vol. 55, no. 2, pp. 287–303, Feb. 2010.

[6] H. Lei and W. Lin, "Universal adaptive control of nonlinear systems with unknown growth rate by output feedback," *Automatica*, vol. 42, pp. 1783–1789, 2006.

[7] H. Lei and W. Lin, "Adaptive regulation of uncertain nonlinear systems by output feedback: A universal control approach," *Syst. Control Lett.*, vol. 56, pp. 529–537, 2007.

[8] H. Lei and W. Lin, "Adaptive control of nonlinear systems with unknown parameters by output feedback: A non-identifier based method," *Lecture Notes in Control and Information Sciences*, A. Astolfi and L. Marconi Eds. Heidelberg, Germany: Springer-Verlag, 2007, pp. 445–463.

[9] W. Lin and C. Qian, "Adaptive control of nonlinearly parameterized systems: The smooth feedback case," *IEEE Trans. Automat. Control*, vol. 47, no. 8, pp. 1249–1266, Aug. 2002.

[10] F. Mazenc, S. Mondie, and S. I. Niculescu, "Global asymptotic stabilization for chains of integrators with a delay in the input," *IEEE Trans. Automat. Control*, vol. 48, no. 1, pp. 57–63, Jan. 2003.

[11] F. Mazenc, S. Mondie, and R. Francisco, "Global asymptotic stabilization of feedforward systems with delay in the input," in *Proc. 42nd IEEE Int. Conf. Decision Control*, Maui, HI, USA, 2003, pp. 4020–4025.

[12] S. Nguang, "Robust stabilization of a class of time-delay nonlinear systems," *IEEE Trans. Automat. Control*, vol. 45, no. 4, 756–762, Apr. 2000.

[13] R. D. Nussbaum, "Some remarks on a conjecture in parameter adaptive control," *Syst. Control Lett.*, vol. 3, pp. 243–246, 1983.

[14] P. Ordaz, O. J. Santos-Sanchez, Omar-Jacobo; L. Rodriguez-Guerrero, and A. Gonzlez-Facundo, "Nonlinear stabilization for a class of time delay systems via inverse optimality approach," *ISA Trans.*, vol. 67, pp. 1–8, 2017.

[15] P. Pepe, "On Sontags formula for the input-to-state practical stabilization of retarded control-affine systems," *Syst. Control Lett.*, vol. 62, pp. 1018–1025, 2013.

[16] J. P. Richard, "Time-delay systems: An overview of some recent advances and open problems," *Automatica*, vol. 39, 1667–1694, 2003.

[17] X. Zhang, W. Lin, and Y. Lin, "Dynamic partial state feedback control of cascade systems with time-delay, *Automatica*, vol. 77, pp. 370–379, 2017.

[18] X. Ye, "Asymptotic regulation of uncertain nonlinear systems with unknown control directions," *Automatica*, vol. 35, 929–935, 1999.

[19] X. Zhang, W. Lin, and Y. Lin, "Nonsmooth control of time-delay nonlinear systems by dynamic state feedback," in *Proc. 54th IEEE Conf. Decision Control*, Osaka, Japan, 2015, 7715–7722. Also, the *IEEE Trans. Automat. Contr.*, vol. 62, no. 1, pp. 438–444, 2017.

[20] W. Lin and C. Qian, "Adaptive Control of Nonlinearly Parameterized Systems: A Nonsmooth Feedback Framework," *IEEE Trans. Automat. Contr.*, vol. 47, no. 5, pp. 757–774, 2002.

L_gV-Type Adaptive Controllers for Uncertain Non-Affine Systems and Application to a DC-Microgrid with PV and Battery

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***Abstract*—Adaptive control of general nonlinear systems with nonlinear parameterization is studied in this paper. Under the assumptions that the system has stable free dynamics and satisfies controllability-like conditions characterized by the Lie brackets of affine vector fields, it is proved that there exist L_gV-type adaptive controllers that not only asymptotically regulate the state of the nonlinearly parameterized system but also guarantee global stability of the closed-loop system. The design of L_gV-type adaptive controllers is also included. Applications of the proposed adaptive control scheme are presented, including an interesting case of a dc-microgrid with photovoltaic (PV) and battery system.**

***Index Terms*—Adaptive control, dc-microgrid with PV and Battery, nonaffine systems, nonlinear parameterization, bounded feedback, passivity.**

I. Introduction

Motivated by the recent development in the area of voltage regulation and maximal power point tracking (MPPT) control for dc-microgrid with photovoltaic (PV) and battery in island mode [20], where a dc-microgrid that consists of a PV array, a battery storage, a dc bus, dc/dc converters and loads with different voltage levels are modeled by a nonaffine nonlinear system with parameters, we investigate in this paper the problem of adaptive control of general nonlinear systems with parametric uncertainty. The objective is to develop a new adaptive control strategy based on the theory of nonaffine passive systems [10], [11], for nonlinearly parameterized systems with a general structure

$$\dot{x} = f(x, u, \theta) \tag{1}$$

$$y = h(x, u, \theta) \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$ are the system state, input and output, respectively. The parameter $\theta \in \mathbb{R}^r$ is assumed to be a constant vector, the vector fields $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^m$ are smooth with $f(0, 0, \theta) = 0$ and $h(0, 0, \theta) = 0 \ \forall \theta$.

For affine systems with stable free dynamics, global stabilization via state feedback has been studied extensively in the literature by passive systems theory. The notions of passivity and dissipativity for nonlinear systems were originally introduced in [21], which are naturally evolved from a series of studies on the positive-real transfer function, the Kalman–Yakubovitch–Popov (KYP) Lemma and their various applications in linear systems and adaptive control. Extensions of [21] to affine systems and a nonlinear analogue of the KYP lemma were obtained in [4]. For a class of affine systems whose unforced dynamics are stable, L_gV controllers were proposed in [7]. Further extensions and developments can be found in a series of papers [1], [6], [8], [9], [17], [18]. Using the concepts and synthesis techniques from passive systems, together with the geometric approach [5], a framework was developed in the paper [2] for global stabilization of minimum-phase nonlinear systems. It was proved that the aforementioned results and generalization thereof can all be unified and rederived by passivity and feedback equivalence. When affine systems involve a structural uncertainty, a robust version of nonlinear KYP lemma and its application to robust feedback stabilization were carried out in [13].

Since the seminar work of [2], attempts have been made in developing a more general passive system theory that goes beyond affine systems, for example, for nonlinear systems which are not linear in the control input, such as nonaffine systems of the form (1)-(2). In [10], a solution to the local stabilization problem was first addressed for the nonaffine system (1) without parametric uncertainty, i.e., $\theta = 0$, by means of the passivity of nonaffine systems. In the subsequent work [11], it was shown that similar to the affine case [2], [4], a passive system (1)-(2) with $\theta = 0$ is globally asymptotically stabilizable by static output feedback if it is zero-state detectable. A criterion for zero-state detectability of the nonaffine passive system (1)-(2) was characterized by the Lie brackets of the vector fields $f(x, 0)$ and $\frac{\partial f}{\partial u}(x, 0)$. Based on these results and the feedback equivalence of rendering a system (1) passive via a suitable dummy output, a controllability-like condition was derived for a nonaffine system (1) [11], under which global asymptotic stabilizability is achievable by bounded state feedback.

All the results reviewed so far have been focused on nonlinear systems without parametric uncertainty. When a nonaffine system such as the dc-microgrid with PV and battery [20] involves uncertainty or unknown parameters, how to control this type of nonaffine systems with nonlinear parameterization is certainly an interesting question that is worth of studying. In this paper, we tackle the problem and present an adaptive control strategy for global asymptotic regulation of the nonlinearly parameterized system (1) with stability. In particular, we show how the nonaffine passive systems theory [10] together with the techniques of feedback passivation and bounded control [11], can be employed to design a LgV-type adaptive controller, which solves the problem of global adaptive stabilization of general nonlinear systems with stable free dynamics. Examples and

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applications to a dc-microgrid with PV and battery [20] are presented to highlight the contribution of the paper and some key features of the $L_g V$-like adaptive control scheme.

The paper is organized as follows: In Section II, we review some basic concepts and properties of passive systems with a nonaffine structure. We then briefly discuss applications of passivity to the feedback stabilization and the characterization of a controllability-like rank condition — a dual of zero-state detectability. Using the results in Section II, we present in Section III sufficient conditions for the existence of globally stabilizing adaptive controllers. In addition, we also show how a $L_g V$-type adaptive control law can be constructed for the nonlinearly parameterized system (1). The crucial idea behind our design is the use of "small control" technique from [11]. Examples and an application to the dc-microgrid with PV and battery are presented in both Sections III and IV, respectively, for the validation of the proposed $L_g V$-like adaptive control scheme.

II. Passivity, Detectability and Bounded Feedback

We briefly review in this section some basic concepts and stability properties from passive systems theory [2], [4], [10], [11], [21], which play a vital role in the development of adaptive control for the nonlinearly parameterized system (1).

Recall that an input-output system (1)-(2) with the parameter θ is said to be *passive* if there exists a continuous nonnegative function $V : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$, with $V(0, \theta) = 0$, such that for each $\theta \in \mathbb{R}^r$

$$V_0(x(t), \theta) - V_0(x_0, \theta) \le \int_0^t y^T(s) u(s) ds, \ \forall u \in \mathbb{R}^m, \forall x_0 \in \mathbb{R}^n \tag{3}$$

where $x(t) = \phi_\theta(t, x_0, u)$ is a solution of (1) from $x(0) = x_0$.

If V_0 is C^1, the passivity inequality (3) can be simplified as

$$\dot{V}_0 \le y^T u, \quad \forall u \in \mathbb{R}^m. \tag{4}$$

Moreover, system (1)-(2) is called lossless if (4) becomes an identity.

A fundamental property of affine passive systems is characterized by the well-known KYP Lemma [4]. It has been shown that the KYP Lemma is instrument in solving the feedback equivalence problem between affine passive systems and minimum-phase nonlinear systems with relative degree $\{1, 1, \ldots, 1\}$ [2]. For the input-output passive system (1)-(2), an analogue of the KYP lemma does not exist due to the loss of an affine structure. However, a necessary condition can still be obtained for the nonaffine system (1)-(2) to be passive.

Lemma 2.1: [10] Let $\Omega_0 \triangleq \{x \in \mathbb{R}^n : L_{f_0} V_0 = 0\}$. If the parameterized input-output system (1)-(2) is passive with a C^1 storage function V_0. Then, for each $\theta \in \mathbb{R}^r$

$$L_{f_0} V_0 \le 0, \quad \forall x \in \mathbb{R}^n \tag{5}$$

$$\frac{\partial V_0}{\partial x} \frac{\partial f}{\partial u}(x, 0, \theta) = h^T(x, 0, \theta), \quad \forall x \in \ \Omega_0. \tag{6}$$

With the help of Lemma 2.1:, it is possible to characterize some intrinsic properties of the parameterized passive system (1)-(2) such as zero-state detectability, observability, and stabilizability, which are crucial in the design of globally stabilizing state feedback controllers for the nonaffine system (1).

An input-output nonlinear system of the form (1)-(2) is said to be zero-state detectable if for each $\theta \in \mathbb{R}^r$ and $x_0 = x \in \mathbb{R}^n$

$$y = h\left(\phi_\theta(t, x, u), u, \theta\right)|_{u=0} = 0 \ \forall t \ge 0 \ \Rightarrow \ \lim_{t \to \infty} \phi_\theta(t, x, 0) = 0.$$

The system (1)-(2) is zero-state observable if

$$y = h\left(\phi_\theta(t, x; u), u\right)|_{u=0} = 0 \quad \forall t \ge 0 \ \Rightarrow \ x = 0.$$

Using the notion of zero-state detectability, one can prove the following stabilization result.

Lemma 2.2 ([10]): A passive system (1)-(2) with a C^ℓ $(\ell \ge 1)$ storage function V_0, which is positive definite and proper, is globally asymptotically stabilizable by $u = -s(y)$ if it is zero-state detectable, where $s : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth functions satisfying $y^T s(y) > 0 \ \forall y \ne 0$ and $s(0) = 0$.

For a passive system with the nonaffine structure (1)-(2), zero-state detectability and observability can be characterized by the Lie derivatives and Lie brackets of the affine vector fields associated with the system (1) [10], [11]. In fact, for each $\theta \in \mathbb{R}^r$, let $f_0(x, \theta) = f(x, 0, \theta)$ and $g_i^0(x, \theta) = g_i(x, 0, \theta) = \frac{\partial f}{\partial u_i}(x, 0, \theta)$ denote the smooth vector fields in $\mathbb{R}^n$, $1 \le i \le m$. Define the Jacobian of f w.r.t. u at $u = 0$ as $g_0(x, \theta) := \frac{\partial f}{\partial u}(x, 0, \theta) = [g_1^0(x, \theta), \ldots, g_m^0(x, \theta)] \in \mathbb{R}^{n \times m}$. Using the vector fields $f_0, g_1^0, \ldots, g_m^0$, we introduce the distribution

$$D_\theta = \text{span}\left\{ad_{f_0}^k g_i^0 : \ 0 \le k \le n-1, \ 1 \le i \le m\right\}$$

and two sets Ω_θ and S_θ defined by

$$\Omega_\theta = \{x \in \mathbb{R}^n : \ L_{f_0}^k V_0(x, \theta) = 0, \ k = 1, \ldots, \ell\} \tag{7}$$

$$S_\theta = \{x \in \mathbb{R}^n : L_{f_0}^k L_\tau V_0(x, \theta) = 0, \forall \tau \in D_\theta, 0 \le k \le \ell - 1\}. \tag{8}$$

With the aid of the notations above and in view of Lemmas 2.1 and 2.2, a computable criterion can be obtained for testing zero-state detectability and observability of the parameterized passive system (1)-(2), by virtue of the affine vector fields $f_0(x, \theta)$, $g_i^0(x, \theta)$ and their Lie derivatives and Lie brackets.

Lemma 2.3 ([10]): Consider the passive system (1)-(2) with a C^1 storage function V_0, which is positive definite and proper. Then,

1) the system is zero-state detectable if $\Omega_\theta \cap S_\theta = \{0\}$. Moreover, if the system (1)-(2) is lossless, then
2) the system is zero-state observable if and only if $S_\theta = \{0\}$.

Putting Lemmas 2.3 and 2.2 together, we have the following proposition to be used in Section III.

Proposition 2.4 ([11]): Assume that the input/output nonaffine system (1)-(2) is passive with a C^1 storage function V_0, which is positive definite and proper. If $\Omega_\theta \cap S_\theta = \{0\}$, the system is globally asymptotically stabilized by the static output feedback controller $u = -s(y)$, for instance, by $u = -y$ or a small bounded feedback law $u = -\beta \frac{y}{1 + \|y\|^2}$, $\forall \beta \in (0, 1)$.

Finally, we recall the following parameter separation lemma from [15], [16] to deal with the issue of nonlinear parameterization.

Lemma 2.5: For a real-valued continuous function $f(x, \theta)$, there exist smooth scalar functions $\alpha(x) \ge 0$, $b(\theta) \ge 0$, $c(x) \ge 1$, and $d(\theta) \ge 1$, such that

$$|f(x, \theta)| \le a(x) + b(\theta) \quad \text{and} \quad |f(x, \theta)| \le c(x) d(\theta). \tag{9}$$

III. Adaptive Stabilization by State Feedback

We now study the adaptive control of the nonaffine system (1) with parametric uncertainty. In this paper, we make the following assumption that characterizes a class of nonlinear systems (1).

A1) There is a C^ℓ $(\ell \ge 1)$ function $V_0 : \mathbb{R}^n \to \mathbb{R}$, which is positive definite and proper, such that the unforced dynamics with an unknown constant vector θ, i.e., $\dot{x} = f(x, 0, \theta) \triangleq f_0(x, \theta)$ is Lyapunov stable, i.e., $L_{f_0} V_0(x) \le 0$, $\quad \forall (x, \theta) \in \mathbb{R}^n \times \mathbb{R}^r$.

As illustrated by Examples 3.8 and 3.9 or the dc-microgrid with PV and battery, many physical systems of interest satisfy the assumption (A1). In what follows, we apply the theory of passive systems reviewed in the previous section to develop an adaptive control strategy for the nonlinearly parameterized system (1) with stable free dynamics. In particular, we show how an adaptive controller that achieves global asymptotic state regulation with stability can be designed, by means of the concepts of passivity and feedback equivalence, as well as the idea of bounded feedback shown in Proposition 2.4.

We begin by observing that a smooth nonlinear system (1) can be decomposed as

$$\dot{x} = f_0(x,\theta) + g(x,u,\theta)u = f_0(x,\theta) + \Sigma_{i=1}^m g_i(x,u,\theta)u_i \quad (10)$$

or, what is the same,

$$\dot{x} = f_0(x,\theta) + g_0(x,\theta)u + \Sigma_{i=1}^m u_i(R_i(x,u,\theta)u) \quad (11)$$

where $g_0(x,\theta)$ is defined in Section II and the $n \times m$ smooth matrix $g(x,u,\theta) = \int_0^1 \frac{\partial f}{\partial \eta}(x,\eta,\theta)\Big|_{\eta=\lambda u} d\lambda$ can be obtained by the mean value theorem with an integration remainder, i.e.,

$$f(x,u,\theta) - f(x,0,\theta) = \left(\int_0^1 \frac{\partial f}{\partial \eta}(x,\eta,\theta)\Big|_{\eta=\lambda u} d\lambda\right) u$$
$$:= g(x,u,\theta)u = [g_1(\cdot),\ldots,g_n(\cdot)]u. \quad (12)$$

By the same reasoning, $R_i(x,u,\theta)$ is an $n \times m$ matrix that can be computed from the relationship

$$g_i(x,u,\theta) - g_i(x,0,\theta) = \left(\int_0^1 \frac{\partial g_i}{\partial \eta}(x,\eta,\theta)\Big|_{\eta=\lambda u} d\lambda\right) u$$
$$= R_i(x,u,\theta)u \quad (13)$$

for $i = 1,\ldots,m$. Clearly, (11) follows immediately from (10) and (13).

To address adaptive control of the nonaffine system (11) with parametric uncertainty by a L_gV-type feedback, we make the following assumption.

(A2) $g_0(x,\theta) = \frac{\partial f}{\partial u}(x,0,\theta)$ is independent of the unknown parameter θ.

Remark 3.1: The assumption (A2) basically requires that the *affine part* of the nonlinearly parameterized system (1) or (11), i.e., the term of $g_0(x,\theta)u$, be independent of the parameter θ. Clearly, a significant class of nonaffine systems with unknown parameters satisfies (A2), as shown by Examples 3.8 and 3.9 or the uncertain nonaffine systems (24) and (27).

Under the assumptions (A1) and (A2), we prove that the controllability-like condition - $\Omega_\theta \cap S_\theta = \{0\}$ is sufficient for the existence of a L_gV-type adaptive controller that adaptively stabilizes the nonaffine system (1) with parametric uncertainty. The proof is carried out by designing an adaptive law based on the idea of bounded control combined with feedback equivalence to a passive system.

Theorem 3.2: Assume that the nonaffine system (1) or (11) with parametric uncertainty satisfies the assumptions (A1) and (A2). If $\Omega_\theta \cap S_\theta = \{0\}$, the following L_gV-like adaptive control law

$$\dot{\widehat{\Theta}} = \beta \frac{\alpha(x,\widehat{\Theta})\|L_{g_0}V_0\|^2}{(1+\widehat{\Theta}^2)(1+\|L_{g_0}V_0\|^2)} \quad (14)$$

$$u(x,\widehat{\Theta}) = -\alpha(x,\widehat{\Theta})\frac{(L_{g_0}V_0(x))^T}{1+\|L_{g_0}V_0(x)\|^2} \quad (15)$$

$$\alpha(x,\widehat{\Theta}) = \frac{\beta}{m(1+\widehat{\Theta}^2)} \cdot \frac{1}{1+\rho^2(x)\|\frac{\partial V_0}{\partial x}\|^2},\ 0<\beta<1 \quad (16)$$

with $\rho(x)$ being satisfying (17), globally asymptotically steers the state x to zero while maintaining global stability of the closed-loop system (1) and (14)–(16).

Proof: By Lemma 2.5, it is easy to see that there exist a smooth function $c_i(x) \geq 1$ and a constant $d_i(\theta) \geq 1$, such that $\forall\ ||u|| \leq 1$

$$\|R_i(x,u,\theta)\| \leq \gamma_i(x,\theta) \leq c_i(x)d_i(\theta) \leq \Theta\rho(x),\ i = 1,\ldots,m \quad (17)$$

where $\rho(x) \geq \Sigma_{i=1}^m c_i(x)$ (or, $\rho(x) \geq \max_{1\leq i\leq m} c_i(x)$) is a smooth function and $\Theta = \max_{1\leq i\leq m} d_i(\theta) \geq 1$.

By the assumption (A1), there exists a C^1 Lyapunov function V_0, which is positive definite and proper, such that the unforced dynamic system $\dot{x} = f(x,0,\theta) = f_0(x,\theta)$ is globally stable. Let $\widehat{\Theta}$ be the estimate of the unknown parameter Θ. Define the estimation error $\widetilde{\Theta} = \Theta - \widehat{\Theta}$.

Now, consider the Lyapunov function

$$V(x,\widetilde{\Theta}) = V_0(x) + \frac{1}{2}\widetilde{\Theta}^2 \quad (18)$$

for the closed-loop system (11) and (14)–(16). Then,

$$\dot{V}(x,\widetilde{\Theta}) = L_{f_0}V_0(x) + L_{g_0}V_0(x)u + u^T L_{R(x,u,\theta)}V_0(x)u - \widetilde{\Theta}\dot{\widehat{\Theta}} \quad (19)$$

where the $m \times m$ matrix

$$L_{R(x,u,\theta)}V_0(x) = \begin{bmatrix} L_{R_1(x,u,\theta)}V_0(x) \\ \vdots \\ L_{R_m(x,u,\theta)}V_0(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial V_0}{\partial x}R_1(x,u,\theta) \\ \vdots \\ \frac{\partial V_0}{\partial x}R_m(x,u,\theta) \end{bmatrix}. \quad (20)$$

From (17) and (20), it follows that

$$\|L_{R(x,u,\theta)}V_0(x)\| \leq \left(m\|\frac{\partial V_0}{\partial x}\|^2\right)^{\frac{1}{2}} \left(\Sigma_{i=1}^m\|R_i(x,u,\theta)\|^2\right)^{\frac{1}{2}}$$
$$\leq \sqrt{m}\|\frac{\partial V_0}{\partial x}\|\left(m\Theta^2\rho^2(x)\right)^{\frac{1}{2}} \leq m\rho(x)\|\frac{\partial V_0}{\partial x}\|\Theta.$$

This, together with (A1), (19), and (15), yields

$$\dot{V} \leq L_{g_0}V_0(x)u(x,\widehat{\Theta}) + \|u(x,\widehat{\Theta})\|^2\|L_{R(x,u,\theta)}V_0(x)\| - \widetilde{\Theta}\dot{\widehat{\Theta}}$$
$$\leq \alpha(x,\widehat{\Theta})\frac{\|L_{g_0}V_0(x)\|^2}{1+\|L_{g_0}V_0(x)\|^2}\left[\alpha(\cdot)\frac{m\rho(x)||\frac{\partial V_0}{\partial x}||\Theta}{1+\|L_{g_0}V_0(x)\|^2} - 1\right] - \widetilde{\Theta}\dot{\widehat{\Theta}}.$$

In view of (16) and $\Theta \geq 1$, we have

$$\begin{aligned}
\dot{V} &\leq -\widetilde{\Theta}\dot{\widehat{\Theta}} + \alpha(x,\widehat{\Theta})\frac{\|L_{g_0}V_0(x)\|^2}{1+\|L_{g_0}V_0(x)\|^2} \\
&\times \left[\frac{\beta\|\frac{\partial V_0}{\partial x}\|\rho(x)}{(1+\widehat{\Theta}^2)\left[1+\|\frac{\partial V_0}{\partial x}\|^2\rho^2(x)\right]}\frac{\Theta}{(1+\|L_{g_0}V_0(x)\|^2)} - 1\right] \\
&\leq -\widetilde{\Theta}\dot{\widehat{\Theta}} - \frac{\alpha(\cdot)\|L_{g_0}V_0(x)\|^2}{1+\|L_{g_0}V_0(x)\|^2} \\
&+ \frac{\beta\alpha(\cdot)}{1+\widehat{\Theta}^2}\left[\frac{(\widehat{\Theta}+\widetilde{\Theta})\|L_{g_0}V_0(x)\|^2}{1+\|L_{g_0}V_0(x)\|^2}\right] \\
&\leq \left[\beta\frac{\widehat{\Theta}}{1+\widehat{\Theta}^2} - 1\right]\frac{\alpha(\cdot)\|L_{g_0}V_0\|^2}{1+\|L_{g_0}V_0\|^2} \\
&+ \widetilde{\Theta}\left[\frac{\beta\alpha(\cdot)\|L_{g_0}V_0\|^2}{(1+\widehat{\Theta}^2)(1+\|L_{g_0}V_0\|^2)} - \dot{\widehat{\Theta}}\right] \\
&\leq \left(\beta\frac{\widehat{\Theta}}{1+\widehat{\Theta}^2} - 1\right)\frac{\alpha(x,\widehat{\Theta})\|L_{g_0}V_0(x)\|^2}{1+\|L_{g_0}V_0(x)\|^2} \leq 0. \qquad (21)
\end{aligned}$$

The last inequality is deduced from (14) and the fact that $1 > \beta\frac{\widehat{\Theta}}{1+\widehat{\Theta}^2}$.

By the Lyapunov theorem, (21) implies that the closed-loop system (11) and (14)–(16) is globally stable. To prove asymptotic convergence of the state x, we let $\dot{V}(x,\widetilde{\Theta}) = 0$. Then, it is deduced from (19) and (21) that

$$L_{f_0}V_0(x) = 0 \quad \text{and} \quad L_{g_0}V_0(x) = 0. \qquad (22)$$

Using an inductive argument, we arrive at

$$L_{f_0}^{k+1}V_0(x) = 0,\ L_{f_0}^k L_\tau V_0(x) = 0,\ \forall\tau \in D_\theta,\ 0 \leq k \leq \ell - 1. \qquad (23)$$

By the La Salle's invariance principle, all the bounded trajectories of the closed-loop system (11)–(14)–(15) eventually approach the largest invariant set in $\{(x,\widetilde{\Theta}) : \dot{V}(x,\widetilde{\Theta}) = 0\}$, which is contained by $(S_\theta \cap \Omega_\theta) \times \mathbb{R}$.

Because $\Omega_\theta \cap S_\theta = \{0\}$, $\{(x,\widetilde{\Theta}) : \dot{V}(x,\widetilde{\Theta}) = 0\} = \{0\} \times \mathbb{R}$. This means that $\lim_{t\to+\infty} x(t) = 0$. That is, global asymptotic state regulation is achieved. ■

From Theorem 3.2, we can deduce some interesting corollaries on adaptive stabilization of nonaffine systems with parametric uncertainty. The first result is a direct consequence of Theorem 3.2.

Corollary 3.3: Consider the single-input nonlinearly parameterized system with a polynomial input

$$\dot{x} = f_0(x,\theta) + g_0(x)u + g_2(x,\theta)u^2 + \cdots + g_p(x,\theta)u^p \qquad (24)$$

where $g_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $2 \leq i \leq p$, are smooth vector fields. Suppose (A1) holds and $\Omega_\theta \cap S_\theta = \{0\}$. Then, the nonlinearly parameterized system (24) is globally adaptively stabilizable by the controller

$$\dot{\widehat{\Theta}} = \frac{\beta}{(1+\widehat{\Theta}^2)(1+\rho^2(x)\|\frac{\partial V_0}{\partial x}\|^2)}\frac{(L_{g_0}V_0(x))^2}{1+(L_{g_0}V_0(x))^2},\ 0<\beta<1$$

$$u(x,\widehat{\Theta}) = -\frac{\beta}{(1+\widehat{\Theta}^2)(1+\rho^2(x)\|\frac{\partial V_0}{\partial x}\|^2)}\frac{L_{g_0}V_0(x)}{1+(L_{g_0}V_0(x))^2} \qquad (25)$$

where $\rho(x)$ is a bounding function satisfying the inequality (26).

Proof: In the single-input case, it is clear from (11) and (24) that

$$R_1(x,u,\theta) = g_2(x,\theta) + g_3(x,\theta)u + \cdots + g_p(x,\theta)u^{p-2}.$$

By construction, $|u| \leq 1$. Thus, it follows from Lemma 2.5 that there exist smooth functions $\bar{c}_i(x) \geq 1$ and $\bar{d}_i(\theta) \geq 1$, such that

$$\begin{aligned}
\|R_1(x,u,\theta)\| &\leq \|g_2(x,\theta)\| + \|g_3(x,\theta)\|\|u\| \\
&\quad + \cdots + \|g_p(x,\theta)\|\|u\|^{p-2} \\
&\leq \Sigma_{i=2}^p \bar{c}_i(x)\bar{d}_i(\theta) \leq \Theta\rho(x), \qquad (26)
\end{aligned}$$

where $\rho(x) \geq \Sigma_{i=2}^p \bar{c}_i(x)$ is a smooth function and

$$\Theta = \max_{2\leq i\leq p} d_i(\theta) \geq 1.$$

Using the bounding function $\rho(x)$ thus obtained, Corollary 3.3 follows directly from Theorem 3.2. ■

In the multi-input case, an analogous result can also be deduced from Theorem 3.2, which will find an interesting application to the dc-microgrid with PV and battery, as illustrated in Section IV.

Corollary 3.4: Consider the multi-input nonaffine system with nonlinear parameterization

$$\begin{aligned}
\dot{x} &= f_0(x,\theta) + \Sigma_{i=1}^m g_i^0(x)u_i + \Sigma_{i_1=1}^m\Sigma_{i_2=1}^m g_{i_1i_2}(x,\theta)u_{i_1}u_{i_2} \\
&\quad + \cdots + \Sigma_{i_1=1}^m\cdots\Sigma_{i_p=1}^m g_{i_1\cdots i_p}(x,\theta)u_{i_1}\cdots u_{i_p}. \qquad (27)
\end{aligned}$$

Under (A1) and $\Omega_\theta \cap S_\theta = \{0\}$, the adaptive controller (14)–(16) not only renders the nonlinearly parameterized system (27) globally stable but also steers the state x to the origin asymptotically.

Proof: By construction, $\|u\| \leq 1$. This, together with Lemma 2.5, implies the existence of smooth functions $c_{i_1i_2}(x),\ldots,c_{i_1\ldots,i_p}(x)$, and $d_{i_1i_2}(\theta),\ldots,d_{i_1\ldots,i_p}(\theta)$, $i_j = 1,\ldots,m$, $j = 1,\ldots,p$, all of them bounded below by one, such that

$$\begin{aligned}
&\|\Sigma_{i_1=1}^m\Sigma_{i_2=1}^m g_{i_1i_2}(x,\theta)u_{i_1}u_{i_2} + \cdots + \Sigma_{i_1=1}^m\cdots\Sigma_{i_p=1}^m g_{i_1\cdots i_p}(x,\theta) \\
&\cdot u_{i_1}\cdots u_{i_p}\| \leq \Sigma_{i_1=1}^m\Sigma_{i_2=1}^m c_{i_1i_2}(x)d_{i_1i_2}(\theta) + \cdots \\
&\quad + \Sigma_{i_1=1}^m\cdots\Sigma_{i_p=1}^m c_{i_1\cdots i_p}(x)d_{i_1\cdots i_p}(\theta) \leq \Theta\rho(x) \qquad (28)
\end{aligned}$$

where $\rho(x) \geq \sum_{i_1=1}^m \sum_{i_2=1}^m c_{i_1i_2}(x) + \cdots + \sum_{i_1=1}^m \cdots ; \sum_{i_p=1}^m c_{i_1\cdots i_p}(x) \geq 1$ is a smooth function and the unknown constant

$$\Theta \geq \max\{d_{i_1\cdots i_p}(\theta)\ :\ 1 \leq i_j \leq m,\ 1 \leq j \leq p\} \geq 1.$$

Using the bounding function $\rho(x)$ in (28), it is straightforward to deduce Corollary 3.4 from Theorem 3.2. ■

Finally, based on Theorem 3.2 and the backstepping design, we can establish the following adaptive control result for a class of *weakly minimum-phase* systems with nonlinear parameterization

$$\begin{aligned}
\dot{x} &= f(x,\xi_1,\theta) \\
\dot{\xi}_1 &= \xi_2 + f_1(x,\xi_1,\theta) \\
&\vdots \\
\dot{\xi}_{q-1} &= \xi_r + f_{q-1}(x,\xi_1,\ldots,\xi_{q-1},\theta) \\
\dot{\xi}_q &= v + f_q(x,\xi_1,\ldots,\xi_q,\theta)
\end{aligned} \qquad (29)$$

where $v \in \mathbb{R}$ is the control, $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^q$ is the system state.

Corollary 3.5: Assume that the zero-dynamic system $\dot{x} = f(x,u,\theta)$ with $u = \xi_1$ satisfies (A1), (A2) and $\Omega_\theta \cap S_\theta = \{0\}$. Then, the problem of global adaptive stabilization of the nonlinearly parameterized system (29) is solvable.

Proof: By Theorem 3.2, the adaptive controller (14)–(16) that is of the form

$$\begin{aligned}
\dot{\hat{\Theta}} &= \eta(\hat{\Theta},x), \qquad \eta(\hat{\Theta},0) = 0 \\
\xi_1^* &= \gamma(\hat{\Theta},x), \qquad \gamma(\hat{\Theta},0) = 0 \qquad (30)
\end{aligned}$$

globally adaptively stabilizes the zero-dynamic system

$$\dot{x} = f(x, \xi_1, \theta) = f_0(x, \theta) + g(x, \xi_1, \theta)\xi_1 \quad (31)$$

when the state ξ_1 is viewed as a control input. In particular, there is a Lyapunov function $V(x, \tilde{\theta})$ defined by (18), such that

$$\dot{V}(x, \widetilde{\Theta}) \leq L_{f_0} V_0(x) - \left[1 - \beta \frac{\widehat{\Theta}}{1 + \widehat{\Theta}^2}\right] \frac{\alpha(x, \widehat{\Theta}) \| L_{g_0} V_0(x) \|^2}{1 + \| L_{g_0} V_0(x) \|^2} \leq 0 \quad (32)$$

for the closed-loop system (31)–(30) with the virtual controller $\xi_1^* = \gamma(\hat{\Theta}, x)$ and $\gamma(\hat{\Theta}, 0) = 0$.

By the backstepping design, combined with the adaptive domination method [15], [16] for dealing with the nonlinear parameterization, one can prove that the augmented system

$$\begin{aligned} \dot{x} &= f\,(x, \xi_1, \theta) \\ \dot{\xi}_1 &= \xi_2 + f_1(x, \xi_1, \theta) \end{aligned} \quad (33)$$

which is obtained by adding an integrator to the zero-dynamic system (31) with the perturbation $f_1(x, \xi_1, \theta)$, is still globally stabilizable by the same form of the adaptive controller (30), in which the unknown parameter Θ and its estimation $\widehat{\Theta}$ needed to be modified accordingly, and $\eta(\hat{\Theta}, x)$ and $\gamma(\hat{\Theta}, x)$ are replaced by the new adaptive law $\eta(\hat{\Theta}, x, \xi_1)$ and the virtual controller $\xi_2^* = \gamma(\hat{\Theta}, x, \xi_1)$. In addition, the control Lyapunov function in this step is given by

$$V_1(x, \xi_1, \tilde{\theta}) = V(x, \widetilde{\Theta}) + \frac{1}{2}(\xi_1 - \xi_1^*)^2$$

when ξ_2 is treated as an input for the augmented system (33).

Inductively, one can carry out, similar to the one in [16], an adaptive domination design step-by-step, by modifying the parameter Θ and its estimation $\widehat{\Theta}$, as well as the adaptive update law $\eta(\cdot)$ and the virtual controller $\gamma(\cdot)$ at each step. At the qth step, a true adaptive controller of the form (with a bit abuse of the notations η and γ)

$$\begin{aligned} \dot{\hat{\Theta}} &= \eta(\hat{\Theta}, x, \xi_1, \ldots, \xi_q), \qquad \eta(\hat{\Theta}, 0, 0, \ldots, 0) = 0 \\ v &= \gamma(\hat{\Theta}, x, \xi_1, \ldots, \xi_q), \qquad \gamma(\hat{\Theta}, 0, 0, \ldots, 0) = 0 \end{aligned} \quad (34)$$

is found, rendering the nonlinearly parameterized system (29) globally stable and $\lim_{t \to +\infty}(x(t), \xi(t)) = 0$. ■

Remark 3.6: Analogue to the analysis in [15], [16], by Lemma 2.5 and reparameterization, we can prove that Theorem 3.2 and its corollaries remain true even if the parameter θ is a time-varying signal rather than a constant vector, as long as $\theta : \mathbb{R} \to \mathbb{R}^s$ is a *periodic* function of t, whose norm bounded by an *unknown constant* $\bar{\theta}$, In other words, Theorem 3.2 and Corollaries 3.3-3.5 are also applicable to nonlinearly parameterized systems such as (1), (27), and (29) (weakly minimum-phase), in which the parameter $\theta = \theta(t)$ represents unknown periodic signals and satisfies $||\theta(t)|| \leq \bar{\theta} \;\; \forall t \in [0, +\infty)$.

Remark 3.7: Notably, Corollary 3.5 has refined the previous results on minimum-phase systems with nonlinear parameterization. For example, the class of nonlinear systems in [16] requires that the system (29) be globally asymptotically and locally exponentially minimum phase, i.e., the zero-dynamics of (29) $\dot{x} = f(x, 0, \theta)$ is globally asymptotically and locally exponentially stable at $x = 0$, for each $\theta \in \mathbb{R}^r$. This assumption has been relaxed by a weaker condition, namely, the weakly minimum-phase property. That is, the zero-dynamics of (29) is only globally stable. The trade-off is, however, a controllability-like condition needs to be imposed on the zero-dynamic system (31).

We end this section with two examples that illustrate the applications and interesting features of the proposed adaptive controllers.

Example 3.8: Consider the single-input nonaffine system

$$\begin{aligned} \dot{x}_1 &= -\omega x_2 \\ \dot{x}_2 &= \omega x_1^3 + x_2 \sin u + u^3 \ln(1 + (\theta x_1)^2) \end{aligned} \quad (35)$$

where $\omega \neq 0$ and θ are unknown constants.

Clearly, system (35) is of the form (1) or (11) with

$$f_0(x, \omega) = \begin{bmatrix} -\omega x_2 \\ \omega x_1^3 \end{bmatrix}, \quad g_0(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

$$R(x, u, \theta) = \begin{bmatrix} 0 \\ x_2 \frac{\sin u - u}{u^2} + u \ln(1 + (\theta x_1)^2) \end{bmatrix}.$$

Note that $\frac{\sin u - u}{u^2}$ is a well-defined analytic function and bounded when u is bounded. Then, whenever $|u| \leq 1$

$$\|R(x, u, \theta)\|^2 \leq |x_2| + |\theta||x_1| \leq (1 + |\theta|)(1 + x_1^2 + x_2^2) := \Theta\rho(x).$$

The constant $\Theta = 1 + |\theta|$ is a new parameter to be estimated, and the bounding function $\rho(x) = 1 + x_1^2 + x_2^2$ will be used to design adaptive controllers for the uncertain system (35).

Now, consider the Lyapunov function $V_0(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. It is easy to see that $L_{f_0(x,\omega)} V_0(x) = 0, \; \forall \omega \in \mathbb{R}$. As a such, $\Omega_\theta = \mathbb{R}^2$ and the assumption (A1) holds. Note that (A2) is also true as $g_0(x) = [0 \; x_2]^T$ is independent of the unknown parameters (ω, θ). According to Theorem 3.2, the $L_g V$-type adaptive controller

$$\dot{\widehat{\Theta}} = \alpha(x, \widehat{\Theta}) \frac{x_2^4}{(1 + \widehat{\Theta}^2)(1 + x_2^4)}, \; u(x, \widehat{\Theta}) = -\alpha(x, \widehat{\Theta}) \frac{x_2^2}{1 + x_2^4} \quad (36)$$

with $\widehat{\Theta}$ being the estimate of $\Theta = 1 + |\theta|$ and $\alpha(x, \widehat{\Theta}) = \frac{1}{2 + 2\widehat{\Theta}^2}\left[\frac{1}{1 + (x_1^6 + x_2^2)(1 + x_1^2 + x_2^2)^2}\right]$, drives the state (x_1, x_2) of (35) to $(0, 0)$ asymptotically and maintains global stability of the closed-loop system (35)-(36), if the system (35) satisfies the condition $\Omega_\theta \cap S_\theta = \{0\}$.

It turns out that an inductive calculation based on (23) results in $x_1 = x_2 = 0$, and hence $S_\theta = \{0\}$ or $\Omega_\theta \cap S_\theta = \mathbb{R}^2 \cap \{0\} = \{0\}$. Intuitively, the adaptive controller (36) renders the system (35) globally stable as the closed-loop system (35)-(36) satisfies the inequality (21). From (21) and the La Salle's invariance principle, it is deduced that $L_{g_0} V_0(x) = x_2^2 = 0 \to \; x_2 = 0$. This, in turn, implies that $\dot{x}_2(t) = 0$ and $u = u(x, \widehat{\Theta}) = 0$. Hence, $x_1 = 0$. In other words, asymptotic state regulation with global stability is achieved.

Example 3.9: Consider the two-input nonaffine system

$$\begin{aligned} \dot{x}_1 &= \theta_1 x_2^3 + x_1 u_1 + \frac{\theta_3 x_3}{(1 + \theta_0 x_1 x_2)^2 + x_1^2 x_2^2} u_1 u_2 \\ \dot{x}_2 &= \theta_2 x_3 - \theta_1 x_1^3 + x_2 u_2 \\ \dot{x}_3 &= -\theta_2 x_2^3 \end{aligned} \quad (37)$$

with $\theta_i, \; 0 \leq i \leq 3$ being unknown constants and $\theta_2 \neq 0$.

The uncertain system (37) is of the form (27) with $m = 2$ and

$$f_0(x, \theta) = \begin{bmatrix} \theta_1 x_2^3 \\ \theta_2 x_3 - \theta_1 x_1^3 \\ -\theta_2 x_2^3 \end{bmatrix}, g_1^0(x) = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}, g_2^0(x) = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$$

$$g_{11}(\cdot) = g_{22}(\cdot) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; g_{12}(\cdot) + g_{21}(\cdot) = \begin{bmatrix} \frac{\theta_3 x_3}{(1 + \theta_0 x_1 x_2)^2 + x_1^2 x_2^2} \\ 0 \\ 0 \end{bmatrix}.$$

Choose the Lyapunov function $V_0(x_1, x_2, x_3) = \frac{1}{4}(x_1^4 + x_2^4) + \frac{1}{2}x_3^2$. It is easy to see that $L_{f_0} V_0(x) = 0$ for all the unknown parameters $(\theta_1, \theta_2) \in \mathbb{R}^2$. This indicates that (A1) holds and $\Omega_\theta = \mathbb{R}^3$.

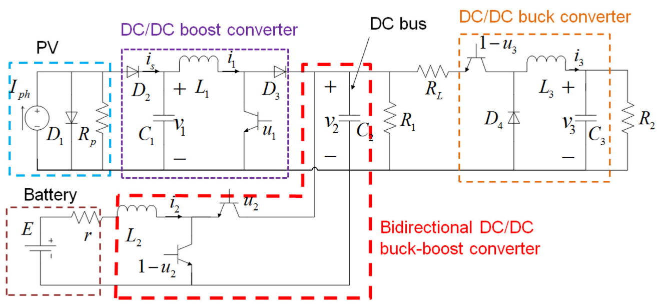

Fig. 1. DC-microgrid system with PV and battery.

In view of Corollary 3.4, the nonaffine system (37) with nonlinear parameterization is globally adaptively stabilized by an adaptive control law of the form (14)–(16), provided that $\Omega_\theta \cap S_\theta = \mathbb{R}^3 \cap S_\theta = S_\theta = \{0\}$. In what follows, we show that this is indeed the case.

By Corollary 3.4, once $S_\theta = \{0\}$, global adaptive regulation of the nonaffine system (37) with parametric uncertainty is possible. In fact, an adaptive controller that achieves asymptotic state regulation with global stability can be designed as follows.

First of all, observe that the function $(1+\theta_0 x_1 x_2)^2 + x_1^2 x_2^2$ reaches its minimal value at the hyperplane $x_1 x_2 = -\theta_0/(1+\theta_0^2)$. Hence,

$$||g_{12}(x,\theta) + g_{21}(x,\theta)|| \le |\frac{\theta_3 x_3}{(1+\theta_0 x_1 x_2)^2 + x_1^2 x_2^2}| \le \Theta |x_3|$$

where $\Theta = |\theta_3|(1+\theta_0^2)$ is an unknown constant and $\rho(x) = |x_3|$.

Let $\hat{\Theta}$ be the estimate of the parameter Θ and define the estimation error $\widetilde{\Theta} = \Theta - \hat{\Theta}$. Following the adaptive control design in the previous section, we obtain the adaptive controller

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \frac{-\alpha(x,\widehat{\Theta})}{1+x_1^8+x_2^8} \begin{bmatrix} x_1^4 \\ x_2^4 \end{bmatrix} \\ \dot{\widehat{\Theta}} &= \frac{\alpha(x,\widehat{\Theta})(x_1^8+x_2^8)}{(1+\widehat{\Theta}^2)(1+x_1^8+x_2^8)} \end{aligned} \tag{38}$$

with $\alpha(x,\widehat{\Theta}) = \frac{1}{4+4\widehat{\Theta}^2} \cdot \frac{1}{1+(x_1^6+x_2^6+x_3^2)x_3^2}$, which globally stabilizes the nonaffine system (37). In particular, it can be shown that the closed-loop system (37)–(38) satisfies

$$\dot{V}(x,\widetilde{\Theta}) \le -\left(1 - \frac{\widehat{\Theta}}{2+2\widehat{\Theta}^2}\right) \frac{\alpha(x,\widehat{\Theta})\|L_{g_0} V_0(x)\|^2}{1+\|L_{g_0} V_0(x)\|^2} \le 0 \tag{39}$$

for $V(x,\widetilde{\Theta}) = \frac{1}{4}(x_1^4+x_2^4) + \frac{1}{2}x_3^2 + \frac{1}{2}\widetilde{\Theta}^2$.

Finally, to prove asymptotic state regulation, or, equivalently, to see why $S_\theta = \{0\}$, we note that

$$L_{g_0^1} V_0(x) = x_1^4 = 0 \quad \text{and} \quad L_{g_0^2} V_0(x) = x_2^4 = 0$$

which imply that $u = [u_1(x,\widehat{\Theta})\ u_2(x,\widehat{\Theta})]^T = [0\ 0]^T$ and $\dot{x}_1 = \dot{x}_2 = 0$. This, in turn, yields $x_3 = 0$ because of $\theta_2 \neq 0$.

The discussion above shows that $S_\theta = \{0\}$, and hence $\lim_{t\to+\infty}(x_1(t), x_2(t), x_3(t)) = 0, \forall (x(0), \hat{\Theta}(0)) \in \mathbb{R}^4$.

IV. Application to a DC-Microgrid With PV and Battery

We now apply the $L_g V$-type adaptive control scheme in the last section to a dc-microgrid with PV and Battery in island mode [20], which is modeled by a nonaffine system with three inputs. The dc-microgrid consists of a PV array, a battery storage, a dc bus, three dc/dc converters and two loads that work on different voltage levels, as shown in Fig. 1.

A dc/dc boost converter is used between PV and dc bus to maximize the power output from the PV array. The battery connects to the dc bus through a bidirectional dc/dc buck-boost converter to maintain the voltage on the dc bus. A bus load directly connects to the dc bus while another load connects to the dc bus via a dc/dc buck converter, which can be viewed as a constant power load.

In what follows, we consider the case when the battery resistor r is small comparing to the loads modeled by constant resistors R_1 and R_2. Then, the battery resistor r can be neglected and a corresponding model of the dc-microgrid with PV and battery is given by the state space equation (1), i.e., $\dot{\bar{x}} = f(\bar{x}, u)$ with [20]

$$f(\bar{x},u) = \begin{bmatrix} \frac{1}{L_1}[\bar{x}_2 - \bar{x}_4 + (\bar{x}_4 + x_4^\star)u_1] \\ \frac{1}{C_1}[-\bar{x}_1 - \frac{\bar{x}_2}{R_p} - I_0 e^{a x_2^\star}(e^{a\bar{x}_2} - 1)] \\ \frac{1}{L_2}[-r\bar{x}_3 - (\bar{x}_4 + x_4^\star)u_2] \\ \frac{1}{C_2}[\bar{x}_1 - \frac{\bar{x}_4}{R_1} - \bar{x}_5 - (\bar{x}_1 + x_1^\star)u_1 \\ \quad +(\bar{x}_3 + x_3^\star)u_2 + (\bar{x}_5 + x_5^\star)u_3] \\ \frac{1}{L_3}[\bar{x}_4 - R_L\bar{x}_5 - \bar{x}_6 + \big(2(\bar{x}_5 + x_5^\star) \\ \quad -(\bar{x}_4 + x_4^\star)\big)u_3 - R_L(\bar{x}_5 + x_5^\star)u_3^2] \\ \frac{1}{C_3}[\bar{x}_5 - \frac{\bar{x}_6}{R_2}] \end{bmatrix} \tag{40}$$

where $x = [x_1, x_2, x_3, x_4, x_5, x_6]^T = [i_1, v_1, i_2, v_2, i_3, v_3]^T$ as illustrated in Fig. 1, x^* is the equilibrium and $\bar{x} = x - x^*$ is a translation, so that the dc-microgrid system in the coordinate $\bar{x}$ has an equilibrium $\bar{x} = 0 \in \mathbb{R}^6$. $I_0 > 0$ is the reverse saturation of the diode and $a = \frac{q}{nkT} > 0$ are the PV parameter described in [20], both of them are assumed to be constants.

There are three control inputs in the dc-microgrid model. The control signals are duty cycles of the converters in Fig. 1 and have a physical constraint: $0 \le u_i \le 1, i = 1, 2, 3$. As illustrated in Fig. 1, u_1 controls the power output of the PV, u_2 controls the dc bus voltage and the charge/discharge of the battery, and u_3 controls the load voltage.

The control objectives are to maximize the power output of PV panel and maintain the dc bus voltage and load voltage, which can be proved to be equivalent to a set point regulation of the voltages v_1, v_2, and v_3 in Fig. 1. Detailed discussions and derivations can be found in [20] on how the MPPT problem can be addressed by a passivity-based control strategy.

For the purpose of illustration, we consider the stabilization of the system (40) that can be expressed as (with $u = [u_1\ u_2\ u_3]^T$)

$$\dot{\bar{x}} = f_0(\bar{x}) + g_0(\bar{x})u + g_{33}(\bar{x})u_3^2 \tag{41}$$

where $g_{33}(\bar{x}) = [0\ 0\ 0\ 0\ -\frac{R_L}{L_3}(\bar{x}_5 + x_5^\star)\ 0]^T$, $g_{i_1 i_2}(\bar{x}) = 0 \in \mathbb{R}^6$, for all $i_k = 1,2,3, k = 1,2$, except for $i_1 i_2 = 33$, and, eq. (42) shown at the bottom of next page. Then, it is easy to prove that the quadratic Lyapunov function

$$V_0(\bar{x}) = \frac{1}{2}\Big(L_1\bar{x}_1^2 + C_1\bar{x}_2^2 + L_2\bar{x}_3^2 + C_2\bar{x}_4^2 + L_3\bar{x}_5^2 + C_3\bar{x}_6^2\Big)$$

which is positive definite and proper, satisfies (by neglecting the register r, i.e., $r = 0$)

$$\begin{aligned} L_{f_0} V_0(\bar{x}) &= -\frac{\bar{x}_2^2}{R_p} - I_0 e^{a x_2^\star}(e^{a\bar{x}_2} - 1)\bar{x}_2 - \frac{\bar{x}_4^2}{R_1} - R_L\bar{x}_5^2 - \frac{\bar{x}_6^2}{R_2} \\ &\le 0, \qquad \forall \bar{x} \in \mathbb{R}^6 \end{aligned} \tag{43}$$

because $a, I_0 e^{a x_2^\star}$ are positive constants and $(e^{a\bar{x}_2} - 1)\bar{x}_2 \ge 0, \forall \bar{x}_2 \in \mathbb{R}$. Consequently, the unforced dynamics of the dc-microgrid with PV and battery is globally stable.

On the other hand, note that

$$(L_{g_0} V_0)^T = \begin{bmatrix} x_4^\star \overline{x}_1 - x_1^\star \overline{x}_4 \\ -x_4^\star \overline{x}_3 + x_3^\star \overline{x}_4 \\ x_5^\star \overline{x}_4 - x_4^\star \overline{x}_5 + 2(\overline{x}_5 + x_5^\star)\overline{x}_5 \end{bmatrix}. \tag{44}$$

Then, a tedious but direct computation shows that $L_{g_0} V_0(\overline{x}) = 0$ implies that $\bar{x} = 0$, indicating that the dc-microgrid system (41) which is nonaffine is globally asymptotically stabilized by the bounded feedback law $u = -\alpha(\bar{x}) \frac{(L_{g_0} V(\overline{x}))^T}{1+\|L_{g_0} V(\overline{x})\|^2}$ if the resistors, inductors, capacitors of the dc-microgrid system, the PV parameters a and I_0, and the loads R_1 and R_2 are known constants.

When the dc-microgrid system involves parametric uncertainty, for instance, the register R_L is uncertain or even time varying but bounded by an *unknown constant* Θ, the dc-microgrid system (41) with a nonaffine structure can still be, according to Corollary 3.4, globally controlled by the $L_g V$-like adaptive regulator (14)–(16). For the purpose of illustration, hereafter we use the following set of parameters in [20] to conduct adaptive control design and simulations: $L_1 = L_2 = L_3 = 5$ mH, $C_1 = 200\,\mu$F, $C_2 = 2000$ μF, $C_3 = 300$ μF, $R_1 = 144\,\Omega$, $R_2 = 9\,\Omega$,$r = 1\,\Omega$, $E = 20\,$V, and $I_{ph} = 9\,$A, $q = 1.6 \times 10^{-19}$, $k = 1.38 \times 10^{-23}$, $a = q/(70\,\mathrm{k}(273+25)) = 0.767$, $I_0 = 10^{-9}$, $R_p = 10^6\ \Omega$. The equilibrium point is

$$x^* = [4.286\ \ 40.074\ \ 20\ \ 40.074\ \ 4.007\ \ 36.067]^T. \tag{45}$$

Using the parameters above, we design the following $L_g V$-like adaptive controller (with $\beta = 0.9$)

$$\dot{\widehat{\Theta}} = 0.9 \frac{\alpha(\bar{x}, \widehat{\Theta}) \|L_{g_0} V_0\|^2}{(1+\widehat{\Theta}^2)(1+\|L_{g_0} V_0\|^2)}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{-0.9\ \alpha(\bar{x}, \widehat{\Theta})}{1+\|L_{g_0} V_0\|^2} \begin{bmatrix} 40.074\overline{x}_1 - 4.286\overline{x}_4 \\ -40.074\overline{x}_3 + 20\overline{x}_4 \\ 4.007\overline{x}_4 - 32.06\overline{x}_5 + 2\overline{x}_5^2 \end{bmatrix} \tag{46}$$

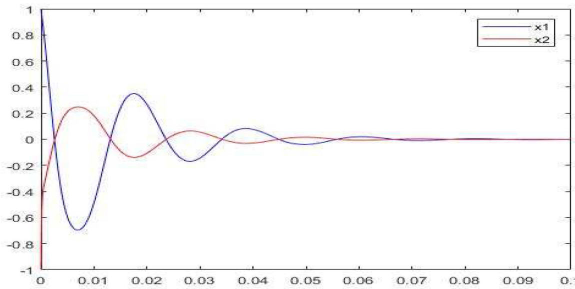

Fig. 2. Transient response of the state $(\bar{x}_1, \bar{x}_2)$.

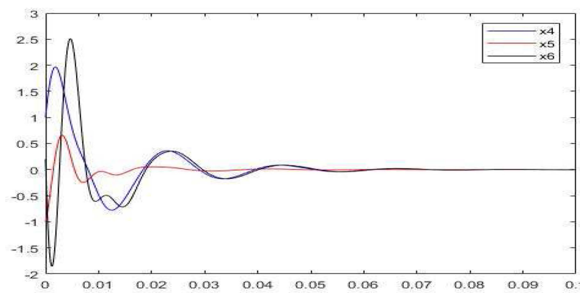

Fig. 3. Transient response of the state $(\bar{x}_4, \bar{x}_5, \bar{x}_6)$.

where

$$\|L_{g_0} V_0\|^2 = (40.074\overline{x}_1 - 4.286\overline{x}_4)^2 + (20\overline{x}_4 - 40.074\overline{x}_3)^2 + (4.007\overline{x}_4 - 32.06\overline{x}_5 + 2\overline{x}_5^2)^2$$

$$\alpha(\bar{x}, \widehat{\Theta}) = \frac{0.3}{1+\widehat{\Theta}^2} \cdot \frac{1}{1+(0.2\overline{x}_5 + 0.8014)^2 \left\| \frac{\partial V_0}{\partial \bar{x}} \right\|^2}$$

$$\frac{\partial V_0}{\partial \bar{x}} = [5\overline{x}_1\ \ 200\overline{x}_2\ \ 5\overline{x}_3\ \ 2000\overline{x}_4\ \ 5\overline{x}_5\ \ 300\overline{x}_6]. \tag{47}$$

The simulation results shown in Figs. 2–5 are conducted for the case when $R_L = 1\ \Omega$. It is demonstrated that the $L_g V$-type adaptive controller (46), (47) regulates all the states $\bar{x} = [\bar{x}_1\ \ \bar{x}_2\ \ \cdots\ \ \bar{x}_6]^T$ of the dc-microgrid (41) with PV and battery to the origin. Equivalently, the

$$f_0(\overline{x}) = f(\overline{x}, 0) = \begin{bmatrix} \frac{1}{L_1}(\overline{x}_2 - \overline{x}_4) \\ \frac{1}{C_1}(-\overline{x}_1 - \frac{1}{R_p}\overline{x}_2 - I_0 e^{a x_2^\star}(e^{a\overline{x}_2} - 1)) \\ -\frac{r}{L_2}\overline{x}_3 \\ \frac{1}{C_2}(\overline{x}_1 - \frac{1}{R_1}\overline{x}_4 - \overline{x}_5) \\ \frac{1}{L_3}(\overline{x}_4 - R_L \overline{x}_5 - \overline{x}_6) \\ \frac{1}{C_3}(\overline{x}_5 - \frac{\overline{x}_6}{R_2}) \end{bmatrix}$$

$$g_0(\overline{x}) = \begin{bmatrix} \frac{1}{L_1}(\overline{x}_4 + x_4^\star) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{L_2}(\overline{x}_4 + x_4^\star) & 0 \\ \frac{-1}{C_2}(\overline{x}_1 + x_1^\star) & \frac{1}{C_2}(\overline{x}_3 + x_3^\star) & \frac{1}{C_2}(\overline{x}_5 + x_5^\star) \\ 0 & 0 & \frac{1}{L_3}[2(\overline{x}_5 + x_5^\star) - \overline{x}_4 - x_4^\star] \\ 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

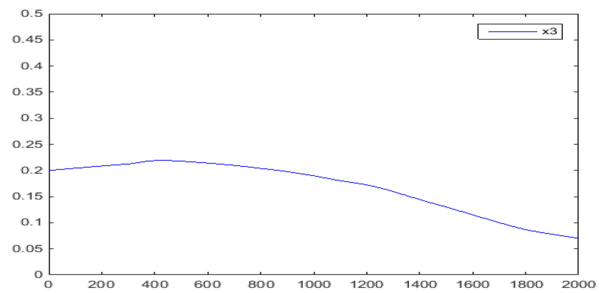

Fig. 4. Transient response of the state $\bar{x}_3$.

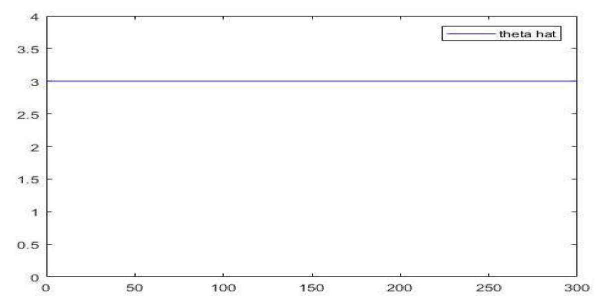

Fig. 5. Adaptive gain $\hat{\Theta}$.

system state $x = [i_1\ v_1\ i_2\ v_2\ i_3\ v_3]^T$ illustrated in Fig. 1 converges to the equilibrium point x^* defined by (45).

For the initial conditions $[\bar{x}_1\ \bar{x}_2\ \bar{x}_3\ \bar{x}_4\ \bar{x}_5\ \bar{x}_6]^T(0) = (1\ -1\ 0.2\ 1\ -1\ 0.2)^T$ and $\hat{\Theta}(0) = 3$, the simulations show that due to the nature of the low-gain feedback, the transient performance of the state $\bar{x}_3$ is poor compared with the other system states. As shown in Fig. 4, $\bar{x}_3(t)$ converges to zero very slowly. For the same reasoning, in particular, from the relationships (46) and (47), it is clear that the adaptive law (46) is updated in a "low-gain" fashion. Consequently, the adaptive gain $\hat{\Theta}(t)$ has a little transient and remains almost a constant, as indicated in Fig. 5.

V. Concluding Remarks

The problem of global adaptive stabilization was investigated in this paper for nonlinearly parameterized systems with a nonaffine structure. Under a controllability-like rank condition, together with a parametric independent requirement on the affine vector field, it was proved that a general nonaffine system with parametric uncertainty is globally adaptively stabilizable by state feedback if the unforced dynamic system is stable. Moreover, a globally stabilizing L_gV-type adaptive controller has been explicitly designed. Three examples, including the dc-microgrid with PV and battery, were presented to demonstrate the applications of the proposed adaptive controllers.

As a subsequent of this paper, future research will include: first, how to use the L_gV-like adaptive controller proposed in this paper to investigate the global adaptive regulation of a larger class of upper-triangular nonlinear systems considered in [14] with parametric uncertainty; and second, adaptive control of nonaffine systems with parametric uncertainty in discrete-time, in particular, studying if the adaptive control results obtained so far can be generalized to the discrete-time nonaffine system in [12].

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References

[1] C. I. Byrnes and A. Isidori, "New results and examples in nonlinear feedback stabilization," *Syst. Control Lett.*, vol. 12, pp. 437–442, 1989.
[2] C. I. Byrnes, A. Isidori, and J. C. Willems, "Passivity, feedback equivalence, and global stabilization of minimum phase nonlinear systems," *IEEE Trans. Automat. Control*, vol. 36, no. 11, pp. 1228–1240, Nov. 1991.
[3] C. I. Byrnes and W. Lin, "Losslessness, feedback equivalence and the global stabilization of discrete-time nonlinear systems," *IEEE Trans. Automat. Control*, vol. 39, no. 1, pp. 83–98, Jan. 1994.
[4] D. Hill and P. Moylan, "The stability of nonlinear dissipative systems," *IEEE Trans. Automat. Control*, vol. 21, no. 5, pp. 708–711, Oct. 1976.
[5] A. Isidori, *Nonlinear control systems II*. New York, NY, USA: Springer, 1999.
[6] M. Jankovic, D. Fontaine, and P. V. Kokotovic, "TORA example: Cascade- and passivity-based control designs," *IEEE Trans. Control Syst. Tech.*, vol. 4, no. 1, pp. 292–297, May 1996.
[7] V. Jurdjevic and J. P. Quinn, "Controllability and stability," *J. Diff. Equ.*, vol. 28, pp. 381–389, 1978.
[8] N. Kalouptsidis and J. Tsinias, "Stability improvement of nonlinear systems by feedback," *IEEE Trans. Automat. Contr.*, vol. 29, no. 1, pp. 364–367, Apr. 1984.
[9] K. K. Lee and A. Arapostathis, "Remarks on smooth feedback stabilization of nonlinear systems," *Syst. Control Lett.*, vol. 10, pp. 41–44, 1988.
[10] W. Lin, "Feedback stabilization of general nonlinear control systems: A passive system approach," *Syst. Control Lett.*, vol. 25, pp. 41–52, 1995.
[11] W. Lin, "Global asymptotic stabilization of general nonlinear systems with stable free dynamics via passivity and bounded feedback," *Automatica*, vol. 32, pp. 915–924, 1996.
[12] W. Lin, "Further results on global stabilization of discrete nonlinear systems," *Syst. Control Lett.*, vol. 29, pp. 51–59, 1996.
[13] W. Lin and T. Shen, "Robust passivity and feedback design for minimum-phase nonlinear systems with structural uncertainty," *Automatica*, vol. 35, pp. 35–47, 1999.
[14] W. Lin and X. J. Li, "Synthesis of upper-triangular nonlinear systems with marginally unstable free dynamics using state-dependent saturation," *Int. J. Control*, vol. 72, pp. 1078–1086, 1999.
[15] W. Lin and C. Qian, "Adaptive control of nonlinearly parameterized systems: A nonsmooth feedback framework," *IEEE Trans. Automat. Contr.*, vol. 47, no. 5, pp. 757–774, May 2002.
[16] W. Lin and C. Qian, "Adaptive control of nonlinearly parameterized systems: the smooth feedback case," *IEEE Trans. Automat. Control*, vol. 47, no. 8, pp. 1249–1266, Aug. 2002.
[17] R. Outbib and G. Sallet, "Stabilizability of the angular velocity of a rigid body revisited," *Syst. Control Lett.*, vol. 18, pp. 93–98, 1992.
[18] P. V. Kokotovic and H. J. Sussman, "A positive real condition for global stabilization on nonlinear systems," *Syst. Control Lett.*, vol. 13, pp. 125–134, 1989.
[19] A. J. Van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*. New York, NY, USA: Springer, 2000.
[20] J. W. Sun, *Voltage Regulation of DC-Microgrid with PV and Battery*, M. S. Thesis, Dept. of Elect. Eng. and Comput. Sci., Case Western Reserve Univ., Cleveland, OH, USA, May 2017.
[21] J. C. Willems, "Dissipative dynamic systems I: General theory," *Arch. Rational Mech. Anal.*, vol. 45, pp. 321–351, 1972.