รายงานวิจัยฉบับสมบูรณ์

โครงการ ทอพอโลยีแบบอ่อนและทฤษฎีบทครีน-มิลแมนสำหรับเซตกระชับแบบอ่อนในปริภูมิฮาดามาร์ด

โดย รองศาสตราจารย์ ดร. บัญชา ปัญญานาค

พฤษภาคม 2563

สัญญาเลขที่ RSA6080076

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ผู้วิจัย รองศาสตราจารย์ ดร. บัญชา ปัญญานาค

สังกัด ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยเชียงใหม่

กิตติกรรมประกาศ

รายงานฉบับนี้ได้จัดทำขึ้นเพื่อรายงานผลของโครงการวิจัยเรื่อง “ทอพอโลยีแบบอ่อนและทฤษฎีบทครีน-มิลแมนสำหรับเซตกระชับแบบอ่อนในปริภูมิฮาดามาร์ด” ซึ่งได้รับทุนสนับสนุนจากสำนักงานกองทุนสนับสนุนการวิจัย (สกว.) และมหาวิทยาลัยเชียงใหม่ (มช.) ผู้วิจัยต้องขอขอบคุณมา ณ โอกาสนี้

ขอขอบพระคุณ ศาสตราจารย์ ดร. สุเทพ สวนใต้ ที่ให้คำปรึกษาและให้ความช่วยเหลือด้านการวิจัยมาโดยตลอด ขอขอบคุณภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่ ที่ได้สนับสนุนการทำวิจัยครั้งนี้

รองศาสตราจารย์ ดร. บัญชา ปัญญานาค

หัวหน้าโครงการวิจัย

สารบัญ

หน้า

บทคัดย่อ

รหัสโครงการ : RSA6080076

ชื่อโครงการ : ทอพอโลยีแบบอ่อนและทฤษฎีบทครีน-มิลแมนสำหรับเซตกระชับแบบอ่อนในปริภูมิฮาดามาร์ด

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ตำบลสุเทพ อำเภอเมือง จังหวัดเชียงใหม่ 50200

ที่อยู่อีเมล : bancha.p@cmu.ac.th

ระยะเวลาโครงการ : 30 พฤษภาคม 2560 ถึง 29 พฤษภาคม 2563

ให้ $k > 0$ และ X เป็นปริภูมิ CAT(k) บริบูรณ์ซึ่งมีความยาวเส้นผ่านศูนย์กลางน้อยกว่า $\frac{\pi}{2\sqrt{k}}$ เราได้แสดงว่าถ้า C เป็นเซตย่อยคอนเวกซ์กระชับที่ไม่เป็นเซตว่างของ X แล้ว C จะเป็นเปลือกหุ้มคอนเวกซ์ปิดของเซตของจุดสุดขีดของมัน นี่คือบทขยายของทฤษฎีบทครีน-มิลแมนในปริภูมิ CAT(k) นอกจากนี้ เราได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มและแบบเดลตาของบางกระบวนการทำซ้ำสำหรับบางการส่งแบบไม่ขยายหลายค่าที่วางนัยทั่วไปในปริภูมิ CAT(0) บริบูรณ์ ผลลัพธ์ของเราขยายและปรับปรุงผลลัพธ์มากมายในวรรณกรรม

คำหลัก : ปริภูมิฮาดามาร์ด, ทอพอโลยีแบบอ่อน, การลู่เข้าแบบอ่อน, การลู่เข้าแบบเดลตา, ทฤษฎีบทครีน-มิลแมน

Abstract

Project Code : RSA6080076

Project Title : Weak topology and the Krein-Milman theorem for weakly compact sets in Hadamard spaces

Investigator : Associate Professor Dr. Bancha Panyanak
Department of Mathematics, Faculty of Science
Chiang Mai University, Chaing Mai 50200, Thailand

E-mail Address : bancha.p@cmu.ac.th

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Let $k > 0$ and X be a complete CAT(k) space whose diameter is smaller than $\frac{\pi}{2\sqrt{k}}$. We show that if C is a nonempty compact convex subset of X, then C is the closed convex hull of its set of extreme points. This is an extension of the Krein-Milman theorem in the setting of CAT(k) spaces. Furthermore, we prove strong and $\Delta-$convergence theorems of some iterative processes for some generaliezed multi-valued nonexpansive mappings in complete CAT(0) spaces. Our results extend and improve many results in the literature.

Keywords : Hadamard space, weak topology, weak convergence, $\Delta-$convergence, the Krein-Milman theorem

เนื้อหางานวิจัย

Let (X,ρ) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a function ϕ from the closed interval $[0,\rho(x,y)]$ to X such that $\phi(0)=x, \phi(\rho(x,y))=y,$ and

$$\rho(\phi(t),\phi(t'))=|t-t'| \text{ for all } t,t' \in [0,\rho(x,y)].$$

The image of ϕ is called a geodesic segment joining x and y which is denoted by $[x,y]$ when it is unique. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that $\rho(x,z)=(1-\alpha)\rho(x,y)$ and $\rho(y,z)=\alpha\rho(x,y)$. In this case, we write $z=\alpha x \oplus (1-\alpha)y$. The space (X,ρ) is said to be a geodesic space (resp. $D-$geodesic space) if every two points of X (resp. every two points of distance smaller than D) are joined by a geodesic path. A subset C of X is said to be compact if every sequence in C has a convergent subsequence whose limit is contained in C. The set C is said to be convex if C includes every geodesic segment joining any two of its points. The closed convex hull of C is defined by

$$\overline{conv}(C) := \bigcap\{A \subseteq X : C \subseteq A \text{ and } A \text{ is closed and convex}\}.$$

Let C be a convex subset of X. A subset A of C is called an extremal subset if it is nonempty, closed and satisfies the following property: if $x,y \in C$ and $\alpha x \oplus (1-\alpha)y \in A$ for some $\alpha \in (0,1)$, then $x,y \in A$. A point z in C is called an extreme point of C if $\{z\}$ is an extremal subset of C. We denote by $Ext(C)$ the set of all extreme points of C.

Given $k \geq 0$, we denote by M_k^2 the following metric spaces:

(i) if $k=0$ then M_k^2 is the Euclidean space $\mathbb{R}^2$;

(ii) if $k>0$ then M_k^2 is obtained from the spherical space by multiplying the distance function by $1/\sqrt{k}$.

A geodesic triangle $\Delta(x,y,z)$ in a geodesic space (X,ρ) consists of three points x,y,z in X (the vertices of Δ) and three geodesic segments between each pair of vertices (the edges of Δ). A comparison triangle for a geodesic triangle $\Delta(x,y,z)$ in (X,ρ) is a triangle $\overline{\Delta}(\overline{x},\overline{y},\overline{z})$ in M_k^2 such that $\rho(x,y)=d_{M_k^2}(\overline{x},\overline{y})$, $\rho(y,z)=d_{M_k^2}(\overline{y},\overline{z})$, and $\rho(x,z)=d_{M_k^2}(\overline{x},\overline{z})$. It is well known that such a comparison triangle exists if $\rho(x,y)+\rho(y,z)+\rho(x,z)<2D_k$, where $D_k=\pi/\sqrt{k}$ for $k>0$ and $D_0=\infty$. Notice also that the comparison triangle is unique up to isometry. A point $\overline{u} \in [\overline{x},\overline{y}]$ is called a comparison point for $u \in [x,y]$ if $\rho(x,u)=d_{M_k^2}(\overline{x},\overline{u})$.

A metric space (X,ρ) is said to be a CAT(k) space if it is D_k-geodesic and for each two points u,v of any geodesic triangle $\Delta(x,y,z)$ in X with $\rho(x,y)+\rho(y,z)+\rho(x,z)<2D_k$ and for their comparison points $\overline{u},\overline{v}$ in $\overline{\Delta}(\overline{x},\overline{y},\overline{z})$ the CAT(k) inequality $\rho(u,v) \leq d_{M_k^2}(\overline{u},\overline{v})$ holds.

It is known from [7] that if X is a complete CAT(k) space with $k>0$ and the diameter of X is smaller than $\dfrac{\pi}{2\sqrt{k}}$, then there exists $R>0$ such that

$$\rho^2(x,(1-\alpha)y\oplus\alpha z)\le(1-\alpha)\rho^2(x,y)+\alpha\rho^2(x,z)-\frac{R}{2}\alpha(1-\alpha)\rho^2(y,z),$$

for all x,y,z in X and $\alpha\in[0,1]$.

Let (X,ρ) be a geodesic space. The distance from a point x in X to a subset C of X is defined by $dist(x,C):=\inf\{d(x,y):y\in C\}$. The set C is called bounded if

$$diam(C):=\sup\{\rho(x,y):x,y\in C\}<\infty.$$

We denote by $CB(C)$ the family of nonempty closed bounded subsets of C, by $K(C)$ the family of nonempty compact subsets of C, and by $KC(C)$ the family of nonempty compact convex subsets of C. The Hausdorff metric on $CB(C)$ is defined by

$$H(A,B):=\max\left\{\sup_{x\in A}dist(x,B),\sup_{y\in B}dist(y,A)\right\}\text{ for }A,B\in CB(C).$$

Let $T:C\to CB(C)$ be a multi-valued mapping. A point x in C is called a fixed point of T if $x\in Tx$. Moreover, if $\{x\}=Tx$, then x is called an endpoint of T. It is denoted by $F(T)$ the set of all fixed points of T and by $E(T)$ the set of all endpoints of T. We say that T satisfies the endpoint condition if $F(T)=E(T)$. A mapping $T:C\to CB(C)$ is said to be

(i) nonexpansive [4] if $H(Tx,Ty)\le\rho(x,y)$ for all $x,y\in C$;

(ii) quasi-nonexpansive [9] if $F(T)\ne\varnothing$ and $H(Tx,Tp)\le\rho(x,p)$ for all $x\in C$ and $p\in F(T)$;

(iii) continuous if $H(Tx_n,Tx)\to 0$ whenever $x_n\to x$.

(iv) hemicompact if for any sequence $\{x_n\}$ in C such that $\lim_{n\to\infty}dist(x_n,Tx_n)=0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty}x_{n_i}=p\in C$;

(v) diametrically regular [8] if there exists a net $\{x_t\}$ in C such that $\lim_t diam(Tx_t)=0$. In this case, we call $\{x_t\}$ a diametrically regular net for T.

A mapping $T:C\to CB(C)$ is said to satisfy condition (E) [3] if there exists $\mu\ge1$ such that

$$dist(x,Ty)\le\mu\, dist(x,Tx)\rho(x,y)\text{ for all }x,y\in C.$$

Let $\{x_n\}$ be a bounded sequence in a geodesic space (X,ρ). The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}):=\inf\left\{\limsup_{n\to\infty}\rho(x_n,x):x\in X\right\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}):=\left\{x\in X:\limsup_{n\to\infty}\rho(x_n,x)=r(\{x_n\})\right\}.$$

It is known from [2] that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A bounded sequence $\{x_n\}$ in X is said to $\Delta-$converge to x in X if $A(\{x_{n_i}\})=\{x\}$ for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$. In this case, we write $\Delta-\lim_{n\to\infty} x_n = x$ and call x the $\Delta-$limit of $\{x_n\}$.

One of the fundamental and celebrated results in functional analysis related to extreme points is the Krein-Milman theorem. In [5], the authors proved that every compact convex subset of a locally convex Hausdorff space is the closed convex hull of its set of extreme points. This result was extended to a special class of metric spaces, namely, CAT(0) spaces, by Niculescu [6] in 2007. Notice that Niculescu's result can be applied to CAT(k) spaces with $k \le 0$ since any CAT(k) space is a CAT(k') space for $k' \ge k$ (see e.g., [1]). However, the result for $k>0$ is still unknown. In this project, we extend Niculescu's result to the setting of CAT(k) spaces with $k>0$. We also prove strong and $\Delta-$convergence theorems of some iterative process for some generaliezed multi-valued nonexpansive mappings in complete CAT(0) spaces. Our main discoveries are the following theorems.

Theorem 1. Let $k>0$ and (X,ρ) be a complete CAT(k) space whose diameter is smaller than $\frac{\pi}{2\sqrt{k}}$. If C is a nonempty compact convex subset of X, then every extremal subset of C has an extreme point.

Theorem 2. Let $k>0$ and (X,ρ) be a complete CAT(k) space whose diameter is smaller than $\frac{\pi}{2\sqrt{k}}$. If C is a nonempty compact convex subset of X, then C is the closed convex hull of its set of extreme points.

Theorem 3. Let C be a nonempty bounded closed convex subset of a complete CAT(0) space (X,ρ) and $T:C\to K(C)$ be a nonexpansive mapping which is diametrically regular with a diametrically regular net $\{x_t\}$. Suppose that $\lim_t dist(x_t, Tx_t)=0.$ Then T has an endpoint in C.

Theorem 4. Let C be a nonempty closed convex subset of a complete CAT(0) space (X,ρ) and $T:C\to K(C)$ be a nonexpansive mapping. Fix $u\in C$. For each $t\in(0,1)$, let x_t be a fixed point of $G_t: C\to K(C)$ defined by $G_t(x):=(1-t)u\oplus tTx$. Suppose that $\{x_t\}$ is diametrically regular for T. Then T has an endpoint if and only if $\{x_t\}$ is bounded as $t\to 1$. In this case, $\{x_t\}$ converges strongly to the unique point $\bar{x}$ in $E(T)$ such that $\rho(u,\bar{x})=\min\{\rho(u,e): e\in E(T)\}$.

Theorem 5. Let C be a nonempty closed convex subset of a complete CAT(0) space (X,ρ) and let $\{T_i\}$ be a countable family of continuous and quasi-nonexpansive multi-valued mappings of C into $CB(C)$ with $\bigcap_{i=1}^{\infty} F(T_i) \neq \varnothing$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Given $x_1 \in C$, and let $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \lambda_n^{(0)} y_n^{(0)} \oplus \lambda_n^{(1)} y_n^{(1)} \oplus \cdots \oplus \lambda_n^{(n)} y_n^{(n)} \quad \text{for all } n \in \mathbb{N},$$

where $y_n^{(0)} = x_n$, $y_n^{(i)} \in T_i x_n$ and the sequences $\left\{\lambda_n^{(i)}\right\} \subset (0,1)$ satisfying $\sum_{i=0}^{n} \lambda_n^{(i)} = 1$ and $\lim_{n\to\infty} \lambda_n^{(i)}$ exist for all $i \in \mathbb{N} \cup \{0\}$. Assume that one member of the family $\{T_i\}$ is hemicompact. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.

Theorem 6. Let C be a nonempty closed convex subset of a complete CAT(0) space (X,ρ) and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of C into $KC(C)$ satisfying condition (E). Assume that $\bigcap_{i=1}^{\infty} F(T_i) \neq \varnothing$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Given $x_1 \in C$, and let $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \lambda_n^{(0)} y_n^{(0)} \oplus \lambda_n^{(1)} y_n^{(1)} \oplus \cdots \oplus \lambda_n^{(n)} y_n^{(n)} \quad \text{for all } n \in \mathbb{N},$$

where $y_n^{(0)} = x_n$, $y_n^{(i)} \in T_i x_n$ and the sequences $\left\{\lambda_n^{(i)}\right\} \subset (0,1)$ satisfying $\sum_{i=0}^{n} \lambda_n^{(i)} = 1$ and $\lim_{n\to\infty} \lambda_n^{(i)}$ exist for all $i \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ $\Delta-$converges to a common fixed point of $\{T_i\}$.

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Output ที่ได้รับจากโครงการ

ผลงานวิจัยที่ได้รับการตีพิมพ์ในวารสารวิชาการระดับนานาชาติทั้งหมด 3 เรื่อง

1) Suthep Suantai, Bancha Panyanak, and Withun Phuengrattana, A new one-step iterative process for approximating common fixed points of a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces, Bulletin of the Iranian Mathematical Society, Volume 43, no. 5, 2017, 1127 – 1141 (Impact Factor = 0.313)

2) Bancha Panyanak, On the Krein-Milman theorem in CAT(k) spaces, Carpathian Journal of Mathematics, Volume 34, no. 3, 2018, 401 – 404 (Impact Factor = 0.58)

3) Bancha Panyanak and Suthep Suantai, Diametrically Regular Mappings and Browder’s Theorem Without the Endpoint Condition, Numerical Functional Analysis and Optimization, Volume 41, no. 4, 2020, 495 – 505 (Impact Factor = 0.59)

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ผลงานวิจัยเรื่องที่ 1

Suthep Suantai, Bancha Panyanak, and Withun Phuengrattana,
A new one-step iterative process for approximating common fixed points of a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces, Bulletin of the Iranian Mathematical Society, Volume 43, no. 5, 2017, 1127 – 1141 (Impact Factor = 0.313)

ผลงานวิจัยเรื่องที่ 2

Bancha Panyanak, On the Krein-Milman theorem in CAT(k) spaces, Carpathian Journal of Mathematics, Volume 34, no. 3, 2018, 401 – 404 (Impact Factor = 0.58)

ผลงานวิจัยเรื่องที่ 3

Bancha Panyanak and Suthep Suantai, Diametrically Regular Mappings and Browder's Theorem Without the Endpoint Condition, Numerical Functional Analysis and Optimization, Volume 41, no. 4, 2020, 495 – 505 (Impact Factor = 0.59)

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Title:

A new one-step iterative process for approximating common fixed points of a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces

Author(s):

S. Suantai, B. Panyanak and W. Phuengrattana

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A NEW ONE-STEP ITERATIVE PROCESS FOR APPROXIMATING COMMON FIXED POINTS OF A COUNTABLE FAMILY OF QUASI-NONEXPANSIVE MULTI-VALUED MAPPINGS IN CAT(0) SPACES

S. SUANTAI, B. PANYANAK AND W. PHUENGRATTANA*

(Communicated by Ali Ghaffari)

Abstract. In this paper, we propose a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in a CAT(0) space. We also prove strong and *Delta*-convergence theorems of the proposed iterative process under some control conditions. Our main results extend and generalize many results in the literature.

Keywords: Fixed point, quasi-nonexpansive multi-valued mappings, CAT(0) spaces.
MSC(2010): Primary: 47H09; Secondary: 47H10; 47J25.

1. Introduction

Let (X, d) be a metric space. A *geodesic* joining x to y (where $x, y \in X$) is a map γ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(l) = y$ and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. Thus γ is an isometry and $d(x, y) = l$. The image of γ is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. We write $\alpha x \oplus (1-\alpha)y$ for the unique point z in the geodesic segment joining from x to y such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(y, z) = \alpha d(x, y)$ for $\alpha \in [0, 1]$. The space X is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset D of X is said to be *convex* if D includes every geodesic segment joining any two of its points.

Following [3], a metric space X is said to be a CAT(0) *space* if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane $\mathbb{E}^2$. It is well known that any complete,

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*Corresponding author.

simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces [3], R-trees [18], the complex Hilbert ball with a hyperbolic metric [15], and many others. It follows from [3] that CAT(0) spaces are uniquely geodesic metric spaces. The fixed point theory in CAT(0) spaces was first studied by Kirk [16, 17]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, there have been many researches concerning the existence and the convergence of fixed points for single-valued and multi-valued mappings in such spaces (e.g., see [2, 5, 7, 8, 18–20]).

The study of fixed points for nonexpansive multi-valued mappings using the Pompeiu-Hausdorff metric was initiated by Markin [21]. Different iterative processes have been used to approximate fixed points of nonexpansive and quasi-nonexpansive multi-valued mappings; in particular, Sastry and Babu [24] considered Mann and Ishikawa iterative processes for a multi-valued mapping T with a fixed point p and proved that these iterative processes converge to a fixed point q of T under certain conditions in Hilbert spaces. Moreover, they illustrated that fixed point q may be different from p. Later, in 2007, Panyanak [22] generalized the results of Sastry and Babu [24] to uniformly convex Banach spaces and proved a convergence theorem of Mann iterative processes for a mapping defined on a noncompact domain. Since then, the strong convergence of the Mann and Ishikawa iterative processes for multi-valued mappings has been rapidly developed, and many papers have appeared (e.g., see [6,12,25,28]). Among other things, Shahzad and Zegeye [26] defined two types of Ishikawa iterative processes and proved strong convergence theorems for such iterative processes involving quasi-nonexpansive multi-valued mappings in uniformly convex Banach spaces. Recently, Abkar and Eslamian [1] established strong and *Delta*-convergence theorems for the multi-step iterative process for a finite family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

In this paper, motivated by the above results, we propose a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces and prove strong and *Delta*-convergence theorems for the proposed iterative process in CAT(0) spaces. We finally provide an example to support our main result.

2. **Preliminaries**

For a nonempty set X, we let $\mathcal{P}(X)$ be the power set of X and $2^X = \mathcal{P}(X) - \{\varnothing\}$. For a metric space (X, d), $x \in X$, and $A, B \in 2^X$, let $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$, $\operatorname{dist}(x, B) = \inf\{d(x, y) : y \in B\}$, and $h(A, B) = \sup\{\operatorname{dist}(x, B) : x \in A\}$.

We now recall some definitions of continuity for multi-valued mappings (see [4,14] for more details). Let (X,d) and (Y,d) be metric spaces. A multi-valued mapping $T : X \to 2^Y$ is said to be

- *Hausdorff upper semi-continuous* at x if for each $\varepsilon > 0$, there is $\delta > 0$ such that $h(Ty, Tx) < \varepsilon$ for each $y \in B(x, \delta)$;
- *Hausdorff lower semi-continuous* at x if for each $\varepsilon > 0$, there is $\delta > 0$ such that $h(Tx, Ty) < \varepsilon$ for each $y \in B(x, \delta)$;
- *continuous* at x if T is Hausdorff upper and lower semi-continuous at x.

We say that the multi-valued mapping T is continuous if it is continuous at each point in X.

Let D be a nonempty subset of a metric space X. Let $CB(D)$ and $KC(D)$ denote the families of nonempty closed bounded subsets and nonempty compact convex subsets of D, respectively. The *Pompeiu-Hausdorff distance* [23] on $CB(D)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A)\right\} \text{ for } A, B \in CB(D),$$

where $\operatorname{dist}(x, D) = \inf\{d(x, y) : y \in D\}$ is the distance from a point x to a subset D.

Note that a continuous multi-valued mapping behaves like a continuous single-valued mapping [14], that is, if a multi-valued mapping $T : D \to CB(D)$ is continuous then for every sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} H(Tx_n, Tx) = 0$.

The set of fixed points of a multi-valued mapping $T : D \to CB(D)$ will be denoted by $F(T) = \{x \in D : x \in Tx\}$.

Definition 2.1. A multi-valued mapping $T : D \to CB(D)$ is said to be

(i) *nonexpansive* [21] if $H(Tx, Ty) \le d(x, y)$, for all $x, y \in D$,

(ii) *quasi-nonexpansive* [24] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \le d(x, p)$, for all $x \in D$ and $p \in F(T)$,

(iii) *hemicompact* if for any sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = p \in D$.

Definition 2.2. A multi-valued mapping $T : D \to CB(D)$ is said to satisfy *condition* (E_μ) provided that

$$\operatorname{dist}(x, Ty) \le \mu \operatorname{dist}(x, Tx) + d(x, y)$$

for each $x, y \in D$. We say that T satisfies *condition* (E) whenever T satisfies (E_μ) for some $\mu \ge 1$.

Remark 2.3. From the above definitions, it is clear that:

(i) if T is nonexpansive, then T satisfies the condition (E_1);

(ii) if D is compact, then T is hemicompact.

Although the condition (E) implies the quasi-nonexpansiveness for single-valued mappings [13], but it is not true for multi-valued mappings as the following example.

Example 2.4 ([27, Example 1]). Let $D = [0, \infty)$ and $T : D \to CB(D)$ be defined by

$$Tx = [x, 2x] \text{ for all } x \in D.$$

Then T satisfies condition (E) and is not quasi-nonexpansive.

Notice also that the classes of (multi-valued) quasi-nonexpansive mappings, continuous mappings and mappings satisfying condition (E) are different (see Examples 2.5-2.7).

Example 2.5 ([13, Example 2]). Let $D = [-1, 1]$ and $T : D \to CB(D)$ be defined by

$$Tx = \begin{cases} \left\{ \frac{x}{1+|x|} \sin(\frac{1}{x}) \right\} & \text{if } \ x \neq 0; \\ \{0\} & \text{if } \ x = 0. \end{cases}$$

Then T is quasi-nonexpansive and does not satisfy condition (E).

Example 2.6 ([5, p. 984]). Let $D = [0, 1]$ and $T : D \to CB(D)$ be defined by

$$Tx = \begin{cases} \{x^2\} & \text{if } \ 0 \leq x < 1; \\ \{0\} & \text{if } \ x = 1. \end{cases}$$

Then T is quasi-nonexpansive and is not continuous. Notice also that the mapping $Tx = \{x^2\}$ on $[0, 1]$ is continuous but is not quasi-nonexpansive nor satisfies condition (E).

Example 2.7 ([13, Example 3]). Let $D = [-2, 1]$ and $T : D \to CB(D)$ be defined by

$$Tx = \begin{cases} \left\{ \frac{|x|}{2} \right\} & \text{if } \ -2 \leq x < 1; \\ \left\{ -\frac{1}{2} \right\} & \text{if } \ x = 1. \end{cases}$$

Then T satisfies condition (E) and is not continuous.

The notion of the asymptotic center can be introduced in the general setting of a CAT(0) space X as follows: Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A\left(\{x_n\}\right) = \left\{x \in X : r\left(x, \{x_n\}\right) = r\left(\{x_n\}\right)\right\}.$$

It is known by [10] that in a CAT(0) space, the asymptotic center $A\left(\{x_n\}\right)$ consists of exactly one point.

We now give the definition and collect some basic properties of the Δ-convergence which will be used in the sequel.

Definition 2.8 ([19]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.9 ([19]). *Every bounded sequence in a CAT(0) space has a Δ-convergent subsequence.*

Lemma 2.10 ([9]). *If D is a nonempty closed convex subset of a CAT(0) space X and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.11 ([11]). *Let $\{x_n\}$ be a sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.12 ([3]). *Let X be a geodesic metric space. The following are equivalent:*

(i) *X is a CAT(0) space.*

(ii) *X satisfies the (CN) inequality: If $x, y \in X$ and $\frac{x\oplus y}{2}$ is the midpoint of x and y, then*

$$d\left(z, \frac{x\oplus y}{2}\right)^2 \leq \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2, \ \textit{for all } z \in X.$$

The following lemma is a generalization of the (CN) inequality which can be found in [11].

Lemma 2.13. *Let X be a CAT(0) space. Then*

$$d(z, \lambda x \oplus (1-\lambda)y)^2 \leq \lambda d(z,x)^2 + (1-\lambda)d(z,y)^2 - \lambda(1-\lambda)d(x,y)^2,$$

for any $\lambda \in [0,1]$ and $x, y, z \in X$.

In 2012, Dhompongsa et al. [8] introduced the following notation in CAT(0) spaces: Let $x_1, \ldots, x_n$ be points in a CAT(0) space X and $\lambda_1, \ldots, \lambda_n \in (0,1)$ with $\sum_{i=1}^{n} \lambda_i = 1$, we write

(2.1)

$$\bigoplus_{i=1}^{n} \lambda_i x_i := (1-\lambda_n)\left(\frac{\lambda_1}{1-\lambda_n}x_1 \oplus \frac{\lambda_2}{1-\lambda_n}x_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n}x_{n-1}\right) \oplus \lambda_n x_n.$$

The definition of $\bigoplus$ is an ordered one in the sense that it depends on the order of points $x_1, \dots, x_n$. Under (2.1) we obtain that

$$d\left(\bigoplus_{i=1}^{n} \lambda_i x_i, y\right) \leq \sum_{i=1}^{n} \lambda_i d(x_i, y) \text{ for each } y \in X.$$

3. Main results

In this section, we first introduce a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces. Let D be a nonempty closed convex subset of a CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \bigoplus_{i=0}^{n} \lambda_n^{(i)} y_n^{(i)}, \text{ for all } n \in \mathbb{N}, \tag{3.1}$$

where $y_n^{(0)} = x_n$, $y_n^{(i)} \in T_i x_n$ and the sequences $\{\lambda_n^{(i)}\} \subset (0,1)$ satisfying $\sum_{i=0}^{n} \lambda_n^{(i)} = 1$.

Note that, if we put

$$W_n^{(m)} = \bigoplus_{i=0}^{m} \delta_n^{(i,m)} y_n^{(i)},$$

where $\delta_n^{(i,m)} = \frac{\lambda_n^{(i)}}{\sum_{j=0}^{m} \lambda_n^{(j)}}$ for $i = 0, 1, \dots, m$, then we get

$$\begin{aligned}
&W_n^{(m)} \\
&= \left(1 - \delta_n^{(m,m)}\right) \left(\frac{\delta_n^{(0,m)}}{1 - \delta_n^{(m,m)}} x_n \oplus \frac{\delta_n^{(1,m)}}{1 - \delta_n^{(m,m)}} y_n^{(1)} \oplus \cdots \oplus \frac{\delta_n^{(m-1,m)}}{1 - \delta_n^{(m,m)}} y_n^{(m-1)} \right) \\
&\quad \oplus \delta_n^{(m,m)} y_n^{(m)} \\
&= \left(1 - \delta_n^{(m,m)}\right) \left(\delta_n^{(0,m-1)} x_n \oplus \delta_n^{(1,m-1)} y_n^{(1)} \oplus \cdots \oplus \delta_n^{(m-1,m-1)} y_n^{(m-1)} \right) \oplus \delta_n^{(m,m)} y_n^{(m)} \\
&= \left(1 - \delta_n^{(m,m)}\right) \left(\frac{\lambda_n^{(0)}}{\sum\limits_{j=0}^{m-1} \lambda_n^{(j)}} x_n \oplus \frac{\lambda_n^{(1)}}{\sum\limits_{j=0}^{m-1} \lambda_n^{(j)}} y_n^{(1)} \oplus \cdots \oplus \frac{\lambda_n^{(m-1)}}{\sum\limits_{j=0}^{m-1} \lambda_n^{(j)}} y_n^{(m-1)} \right) \oplus \delta_n^{(m,m)} y_n^{(m)} \\
&= \left(1 - \delta_n^{(m,m)}\right) W_n^{(m-1)} \oplus \delta_n^{(m,m)} y_n^{(m)} \\
&= \frac{\sum\limits_{j=0}^{m-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{m} \lambda_n^{(j)}} W_n^{(m-1)} \oplus \frac{\lambda_n^{(m)}}{\sum\limits_{j=0}^{m} \lambda_n^{(j)}} y_n^{(m)}.
\end{aligned}$$

Therefore, the following result holds:

$$W_n^{(m)} = \frac{\sum_{j=0}^{m-1} \lambda_n^{(j)}}{\sum_{j=0}^{m} \lambda_n^{(j)}} W_n^{(m-1)} \oplus \frac{\lambda_n^{(m)}}{\sum_{j=0}^{m} \lambda_n^{(j)}} y_n^{(m)}. \tag{3.2}$$

The following two lemmas are useful and crucial for our main theorems.

Lemma 3.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, consider the sequence $\{x_n\}$ generated by (3.1). Then, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$.*

Proof. For $p \in \cap_{i=1}^{\infty} F(T_i)$, we have by (3.1) that

$$\begin{aligned}
d(x_{n+1}, p) &= d\left(\bigoplus_{i=0}^{n} \lambda_n^{(i)} y_n^{(i)}, p\right) \\
&\leq \sum_{i=0}^{n} \lambda_n^{(i)} d(y_n^{(i)}, p) \\
&= \sum_{i=0}^{n} \lambda_n^{(i)} \operatorname{dist}(y_n^{(i)}, T_i p) \\
&\leq \sum_{i=0}^{n} \lambda_n^{(i)} H(T_i x_n, T_i p) \\
&\leq \sum_{i=0}^{n} \lambda_n^{(i)} d(x_n, p) \\
&= d(x_n, p).
\end{aligned}$$

This implies that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$. □

Lemma 3.2. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, consider the sequence $\{x_n\}$ generated by (3.1). If $\lim_{n\to\infty} \lambda_n^{(i)}$ exists for all $i \in \mathbb{N} \cup \{0\}$ and lies in $(0, 1)$, then $\lim_{n\to\infty} dist(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$.*

Proof. For each $p \in \cap_{i=1}^{\infty} F(T_i)$, we obtain by (3.1) that

$$d(x_{n+1}, p) = d\left(\bigoplus_{i=0}^{n} \lambda_n^{(i)} y_n^{(i)}, p\right) = d\left(\bigoplus_{i=0}^{n} \frac{\lambda_n^{(i)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} y_n^{(i)}, p\right) = d(W_n^{(n)}, p).$$

It follows by Lemma 2.13 and (3.2) that

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d\left(\frac{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} W_n^{(n-1)} \oplus \frac{\lambda_n^{(n)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} y_n^{(n)}, p\right)^2 \\
&\le \frac{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} d(W_n^{(n-1)}, p)^2 + \frac{\lambda_n^{(n)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} d(y_n^{(n)}, p)^2 \\
&\quad - \frac{\lambda_n^{(n)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} \frac{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&= \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, p)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&= \sum_{j=0}^{n-1} \lambda_n^{(j)} d\left(\frac{\sum\limits_{j=0}^{n-2} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}} W_n^{(n-2)} \oplus \frac{\lambda_n^{(n-1)}}{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}} y_n^{(n-1)}, p\right)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
&\quad - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&\le \sum_{j=0}^{n-1} \lambda_n^{(j)} \left(\frac{\sum\limits_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, p)^2 + \frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(y_n^{(n-1)}, p)^2\right) \\
&= \sum_{j=0}^{n-1} \lambda_n^{(j)} d\left(\frac{\sum\limits_{j=0}^{n-2} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}} W_n^{(n-2)} \oplus \frac{\lambda_n^{(n-1)}}{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}} y_n^{(n-1)}, p\right)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
&\quad - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \sum_{j=0}^{n-1} \lambda_n^{(j)} \left(\frac{\sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, p)^2 + \frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(y_n^{(n-1)}, p)^2 \right. \\
& \left. - \frac{\sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} \frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, y_n^{(n-1)})^2 \right) + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
& - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
=\ & \sum_{j=0}^{n-2} \lambda_n^{(j)} d(W_n^{(n-2)}, p)^2 + \lambda_n^{(n-1)} d(y_n^{(n-1)}, p)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
& - \frac{\lambda_n^{(n-1)} \sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, y_n^{(n-1)})^2 - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
\leq\ & \sum_{j=0}^{n-3} \lambda_n^{(j)} d(W_n^{(n-3)}, p)^2 + \lambda_n^{(n-2)} d(y_n^{(n-2)}, p)^2 + \lambda_n^{(n-1)} d(y_n^{(n-1)}, p)^2 \\
& + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 - \frac{\lambda_n^{(n-2)} \sum_{j=0}^{n-3} \lambda_n^{(j)}}{\sum_{j=0}^{n-2} \lambda_n^{(j)}} d(W_n^{(n-3)}, y_n^{(n-2)})^2 \\
& - \frac{\lambda_n^{(n-1)} \sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, y_n^{(n-1)})^2 - \lambda_n^{(N)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
\vdots\ & \\
\leq\ & \lambda_n^{(0)} d(W_n^{(0)}, p)^2 + \sum_{k=1}^{n} \lambda_n^{(k)} d(y_n^{(k)}, p)^2 - \sum_{k=1}^{n} \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2 \\
\leq\ & \sum_{k=0}^{n} \lambda_n^{(k)} d(x_n, p)^2 - \sum_{k=1}^{n} \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2 \\
=\ & d(x_n, p)^2 - \sum_{k=1}^{n} \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2.
\end{aligned}
$$

This implies that

$$\sum_{k=1}^{n} \frac{\lambda_n^{(k)} \sum\limits_{j=0}^{k-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{k} \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{3.3}$$

Since $0 < \lambda_n^{(0)} \leq \sum_{j=0}^{k} \lambda_n^{(j)} \leq 1$ for all $k = 1, 2, \ldots, n$, we have $0 < \lambda_n^{(0)} \lambda_n^{(k)} \leq \lambda_n^{(k)} \sum_{j=0}^{k} \lambda_n^{(j)}$. So, $0 < \lambda_n^{(0)} \lambda_n^{(k)} \leq \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}}$ for all $k = 1, 2, \ldots, n$. Then (3.3) becomes

$$\sum_{k=1}^{n} \lambda_n^{(0)} \lambda_n^{(k)} d(W_n^{(k-1)}, y_n^{(k)})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{3.4}$$

By Lemma 3.1 and the condition $\lim_{n\to\infty} \lambda_n^{(i)}$ exists for all $i \in \mathbb{N} \cup \{0\}$ and lies in $(0, 1)$, we get that

$$\lim_{n\to\infty} d(x_n, y_n^{(1)}) = 0 \ \text{ and } \ \lim_{n\to\infty} d(W_n^{(k-1)}, y_n^{(k)}) = 0 \ \text{ for all } k \geq 2. \tag{3.5}$$

Then, for $k \geq 2$, we have

$$\begin{aligned}
d(x_n, y_n^{(k)}) &\leq d(x_n, W_n^{(k-1)}) + d(W_n^{(k-1)}, y_n^{(k)}) \\
&= d\left(x_n, \bigoplus_{i=0}^{k-1} \frac{\lambda_n^{(i)}}{\sum\limits_{j=0}^{k-1} \lambda_n^{(j)}} y_n^{(i)} \right) + d(W_n^{(k-1)}, y_n^{(k)}) \\
&\leq \sum_{i=0}^{k-1} \frac{\lambda_n^{(i)}}{\sum\limits_{j=0}^{k-1} \lambda_n^{(j)}} d(x_n, y_n^{(i)}) + d(W_n^{(k-1)}, y_n^{(k)}) \\
&= \sum_{i=1}^{k-1} \frac{\lambda_n^{(i)}}{\sum\limits_{j=0}^{k-1} \lambda_n^{(j)}} d(x_n, y_n^{(i)}) + d(W_n^{(k-1)}, y_n^{(k)}).
\end{aligned}$$

This implies by (3.5) that $\lim_{n\to\infty} d(x_n, y_n^{(k)}) = 0$ for all $k \geq 1$. Since $\text{dist}(x_n, T_i x_n) \leq d(x_n, y_n^{(i)})$ for all $i \in \mathbb{N}$, it follows that $\lim_{n\to\infty} \text{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. □

In what follows we get a Δ-convergence theorem for a countable family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued* *mappings of D into $KC(D)$ satisfying the condition (E). Assume that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. Suppose that $\lim_{n\to\infty} \lambda_n^{(i)}$ exists for all $i \in \mathbb{N} \cup \{0\}$ and lies in $(0,1)$. Then, the sequence $\{x_n\}$ generated by (3.1) Δ-converges to a common fixed point of $\{T_i\}$.*

Proof. By Lemmas 3.1 and 3.2, we have $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$ and $\lim_{n\to\infty} \operatorname{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. Thus the sequence $\{x_n\}$ is bounded. We put $\omega_\Delta(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Let $u \in \omega_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.9, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{j\to\infty} u_{n_j} = z \in D$. We will show that $z \in T_1 z$. Since $T_1 z$ is compact, for all $j \in \mathbb{N}$, we can choose $y_{n_j} \in T_1 z$ such that $d(u_{n_j}, y_{n_j}) = \operatorname{dist}(u_{n_j}, T_1 z)$ and $\{y_{n_j}\}$ has a convergent subsequence $\{y_{n_k}\}$ with $\lim_{k\to\infty} y_{n_k} = q \in T_1 z$. By condition (E), we have

$$\operatorname{dist}(u_{n_k}, T_1 z) \le \mu \operatorname{dist}(u_{n_k}, T_1 u_{n_k}) + d(u_{n_k}, z).$$

Then we have

$$\begin{aligned} d(u_{n_k}, q) &\le d(u_{n_k}, y_{n_k}) + d(y_{n_k}, q) \\ &= \operatorname{dist}(u_{n_k}, T_1 z) + d(y_{n_k}, q) \\ &\le \mu \operatorname{dist}(u_{n_k}, T_1 u_{n_k}) + d(u_{n_k}, z) + d(y_{n_k}, q). \end{aligned}$$

This implies that

$$\limsup_{k\to\infty} d(u_{n_k}, q) \le \limsup_{k\to\infty} d(u_{n_k}, z).$$

By the uniqueness of asymptotic centers, we have $z = q \in T_1 z$. Similarly, it can be shown that $z \in T_i z$ for all $i = 2, \ldots, N$. Then, $z \in \cap_{i=1}^{\infty} F(T_i)$ and so $\lim_{n\to\infty} d(x_n, z)$ exists. Suppose that $u \neq z$. By the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{j\to\infty} d(u_{n_j}, z) &< \limsup_{j\to\infty} d(u_{n_j}, u) \\ &\le \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, z) \\ &= \limsup_{n\to\infty} d(x_n, z) \\ &= \limsup_{j\to\infty} d(u_{n_j}, z). \end{aligned}$$

This is a contradiction, hence $u = z \in \cap_{i=1}^{\infty} F(T_i)$. This shows that $\omega_\Delta(x_n) \subset \cap_{i=1}^{\infty} F(T_i)$.

Next, we show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{p\}$ and let $A(\{x_n\}) = \{q\}$. Since $p \in \omega_\Delta(x_n) \subset \cap_{i=1}^{\infty} F(T_i)$, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists. By Lemma

2.11, we obtain that $p = q$. Hence, the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\{T_i\}$. $\square$

The following result is a strong convergence theorem for a countable family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

Theorem 3.4. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of continuous and quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. Let the sequence $\{x_n\}$ generated by (3.1) with $\lim_{n\to\infty} \lambda_n^{(i)}$ exist for all $i \in \mathbb{N} \cup \{0\}$ and lie in $(0,1)$. Assume that one member of the family $\{T_i\}$ is hemicompact. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.*

Proof. By Lemma 3.2, $\lim_{n\to\infty} \operatorname{dist}(x_n, T_i x_n)$ for all $i \in \mathbb{N}$. Without loss of generality, we assume that T_1 is hemicompact. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = p \in D$. By continuity of T_i, we have $\lim_{j\to\infty} \operatorname{dist}(x_{n_j}, T_i x_{n_j}) = \operatorname{dist}(p, T_i p)$ for all $i \in \mathbb{N}$. This implies that $\operatorname{dist}(p, T_i p) = 0$ for all $i \in \mathbb{N}$ and hence $p \in \cap_{i=1}^{\infty} F(T_i)$. It follows by Lemma 3.1 that $\{x_n\}$ converges strongly to p. $\square$

Remark 3.5. Since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$ (see [3]), all our results immediately apply to any CAT(κ) space with $\kappa \leq 0$.

Finally, we give a numerical example supporting Theorems 3.3 and 3.4.

Example 3.6. Let X be a real line with the Euclidean norm and $D = [0,1]$. For $x \in D$, $i = 1, 2, \ldots$, we define mappings T_i on D as follows:

$$T_i x = \left[0, \frac{x}{i}\right] \text{ for all } i \in \mathbb{N}.$$

Let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \bigoplus_{i=0}^{n} \lambda_n^{(i)} y_n^{(i)}, \text{ for all } n \in \mathbb{N}, \tag{3.6}$$

where $y_n^{(0)} = x_n$, $y_n^{(i)} \in T_i x_n$ and the sequences $\{\lambda_n^{(i)}\}$ defined by

$$\lambda_n^{(i)} = \begin{cases} \dfrac{1}{2^{i+1}}\left(\dfrac{n}{n+1}\right), & n \geq i+1 \\ 1 - \dfrac{n}{n+1}\left(\sum_{k=1}^{n} \dfrac{1}{2^k}\right), & n = i \\ 0, & n < i. \end{cases}$$

Obviously, T_i is quasi-nonexpansive and satisfies condition (E) for all $i \in \mathbb{N}$ and $T_i(0) = \{0\}$ such that $\cap_{i=1}^{\infty} F(T_i) = \{0\}$. It can be observed that all the assumptions of Theorems 3.3 and 3.4 are satisfied.

For any arbitrary $x_1 \in D = [0,1]$, we put $y_n^{(i)} = \frac{x_n}{5i}$ for all $i \in \mathbb{N}$. Then, we rewrite the algorithm (3.6) as follows:

$$x_{n+1} = \lambda_n^{(0)} x_n + \frac{\lambda_n^{(1)} x_n}{5} + \frac{\lambda_n^{(2)} x_n}{10} + \cdots + \frac{\lambda_n^{(n)} x_n}{5n}, \quad \text{for all } n \in \mathbb{N},$$

where

$$\left(\lambda_n^{(i)}\right) = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{3}{8} & \frac{3}{16} & \frac{3}{32} & \frac{11}{32} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ \frac{n}{2(n+1)} & \frac{n}{4(n+1)} & \frac{n}{8(n+1)} & \frac{n}{16(n+1)} & \frac{n}{32(n+1)} & \frac{n}{64(n+1)} & \cdots & \frac{n}{2^i(n+1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

The values of the sequence $\{x_n\}$ with different n are reported in Table 1.

TABLE 1. The values of the sequence $\{x_n\}$ in Example 3.6.

	$x_1 = 0.11$	$x_1 = 0.95$
n	x_n	x_n
1	0.1100000	0.9500000
2	0.0440000	0.3800000
3	0.0183333	0.1583333
4	0.0081545	0.0704253
5	0.0037986	0.0328065
6	0.0018280	0.0157875
7	0.0009008	0.0077801
8	0.0004520	0.0039036
9	0.0002300	0.0019863
10	0.0001184	0.0010222
$\vdots$	$\vdots$	$\vdots$
17	0.0000014	0.0000118
18	0.0000007	0.0000064
19	0.0000004	0.0000034
20	0.0000002	0.0000019

From Table 1, it is clear that $\{x_n\}$ converges to 0, where $\{0\} = \cap_{i=1}^{\infty} F(T_i)$.

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(Suthep Suantai) Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.
E-mail address: `suthep.s@cmu.ac.th`

(Bancha Panyanak) Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.
E-mail address: `bancha.p@cmu.ac.th`

(Withun Phuengrattana) Department of Mathematics, Faculty of Science and Technology, Nakhon Pathom Rajabhat University, Nakhon Pathom 73000, Thailand, and Research Center for Pure and Applied Mathematics, Research and Development Institute, Nakhon Pathom Rajabhat University, Nakhon Pathom 73000, Thailand.
E-mail address: `withun_ph@yahoo.com`

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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

On the Krein-Milman theorem in CAT(κ) spaces

BANCHA PANYANAK

ABSTRACT. Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space whose diameter smaller than $\frac{\pi}{2\sqrt{\kappa}}$. It is shown that if K is a nonempty compact convex subset of X, then K is the closed convex hull of its set of extreme points. This is an extension of the Krein-Milman theorem to the general setting of CAT(κ) spaces.

1. INTRODUCTION AND PRELIMINARIES

One of the fundamental and celebrated results in functional analysis related to extreme points is the Krein-Milman theorem. In [5], the authors proved that every compact convex subset of a locally convex Hausdorff space is the closed convex hull of its set of extreme points. This result was extended to a special class of metric spaces, namely, CAT(0) spaces, by Niculescu [6] in 2007. Notice that Niculescu's result can be applied to CAT(κ) spaces with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$ (see e.g., [1]). However, the result for $\kappa > 0$ is still unknown. In this paper, we extend Niculescu's result to the setting of CAT(κ) spaces with $\kappa > 0$.

Let $(\mathcal{P}, \preceq)$ be a partially ordered set. An element $p_0 \in \mathcal{P}$ is *maximal* in $\mathcal{P}$ if for each $p \in \mathcal{P}$, the following implication holds:

$$p_0 \preceq p \implies p_0 = p.$$

Similarly, an element $q_0 \in \mathcal{P}$ is *minimal* in $\mathcal{P}$ if for each $p \in \mathcal{P}$, the following implication holds:

$$p \preceq q_0 \implies p = q_0.$$

An upper bound (resp. *A lower bound*) of a nonempty subset $\mathcal{Q}$ of $\mathcal{P}$ is an element $p \in \mathcal{P}$ such that $q \preceq p$ (resp. $p \preceq q$) for all $q \in \mathcal{Q}$. A nonempty subset $\mathcal{C}$ of $\mathcal{P}$ is called a *chain* in $\mathcal{P}$ if any two elements p and q in $\mathcal{C}$ are comparable, that is, $p \preceq q$ or $q \preceq p$.

Lemma 1.1. (Zorn) *If $(\mathcal{P}, \preceq)$ is a partially ordered set such that every chain in $\mathcal{P}$ has an upper (resp. lower) bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal (resp. minimal) element.*

Let (X, ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a function ξ from the closed interval $[0, \rho(x, y)]$ to X such that $\xi(0) = x, \xi(l) = y$, and $\rho(\xi(t), \xi(t')) = |t - t'|$ for all $t, t' \in [0, \rho(x, y)]$. The image of ξ is called a *geodesic segment* joining x and y which is unique, denoted by $[x, y]$. This means that $z \in [x, y]$ if and only if there exists $\alpha \in [0, 1]$ such that

$$\rho(x, z) = (1 - \alpha)\rho(x, y) \text{ and } \rho(y, z) = \alpha\rho(x, y).$$

In this case, we write $z = \alpha x \oplus (1 - \alpha)y$. The space (X, ρ) is said to be a *geodesic space* (resp. *$D-$geodesic space*) if every two points of X (resp. every two points of distance smaller than

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D) are joined by a geodesic path. A subset C of X is said to be *convex* if C includes every geodesic segment joining any two of its points.

Now we introduce the model spaces M^2_κ, for more details on these spaces the reader is referred to [1, 3, 4, 8, 9]. We denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^3$. By $\mathbb{S}^2$ we denote the unit sphere in $\mathbb{R}^3$, that is the set $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$. The *spherical distance* on $\mathbb{S}^2$ is defined by

$$d_{\mathbb{S}^2}(x, y) := \arccos\langle x, y\rangle \text{ for all } x, y \in \mathbb{S}^2.$$

Definition 1.2. Given $\kappa \geq 0$, we denote by M^2_κ the following metric spaces:

(i) if $\kappa = 0$ then M^2_κ is the Euclidean space $\mathbb{E}^2$;

(ii) if $\kappa > 0$ then M^2_κ is obtained from the spherical space $\mathbb{S}^2$ by multiplying the distance function by $1/\sqrt{\kappa}$.

A *geodesic triangle* $\triangle(x, y, z)$ in a geodesic space (X, ρ) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x, y, z)$ in (X, ρ) is a triangle $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ in M^2_κ such that

$$\rho(x, y) = d_{M^2_\kappa}(\bar{x}, \bar{y}),\ \rho(y, z) = d_{M^2_\kappa}(\bar{y}, \bar{z}),\ \text{and } \rho(z, x) = d_{M^2_\kappa}(\bar{z}, \bar{x}).$$

It is well known that such a comparison triangle exists if $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$ for $\kappa > 0$ and $D_0 = \infty$. Notice also that the comparison triangle is unique up to isometry. A point $\bar{u} \in [\bar{x}, \bar{y}]$ is called a *comparison point* for $u \in [x, y]$ if $\rho(x, u) = d_{M^2_\kappa}(\bar{x}, \bar{u})$.

A metric space (X, ρ) is said to be a *CAT(κ) space* if it is $D_\kappa-$geodesic and for each two points u, v of any geodesic triangle $\triangle(x, y, z)$ in X with $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_\kappa$ and for their comparison points $\bar{u}, \bar{v}$ in $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ the *CAT(κ) inequality*

$$\rho(u, v) \leq d_{M^2_\kappa}(\bar{u}, \bar{v}),$$

holds. Notice also that Pre-Hilbert spaces, $\mathbb{R}-$trees, Euclidean buildings are examples of CAT(κ) spaces (see [1, 2]).

Recall that a geodesic space (X, ρ) is said to be *R−convex* for $R \in (0, 2]$ ([7]) if for any three points $x, y, z \in X$, we have

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - \frac{R}{2}\alpha(1-\alpha)\rho^2(y, z). \tag{1.1}$$

The following lemmas will be needed.

Lemma 1.3. ([7]) *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with* $\operatorname{diam}(X) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0, \pi/2)$. Then (X, ρ) is $R-$convex for $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.*

Lemma 1.4. ([1]) *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with* $\operatorname{diam}(X) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0, \pi/2)$. Then*

$$\rho((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)\rho(x, z) + \alpha\rho(y, z),$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

Let (X, ρ) be a geodesic space. The *distance* from a point x in X to a subset C of X is defined by

$$\operatorname{dist}(x, C) := \inf\{\rho(x, y) : y \in C\}.$$

The set C is *bounded* if $\operatorname{diam}(C) := \sup\{\rho(x, y) : x, y \in C\} < \infty$.

Definition 1.5. Let $f : C \to \mathbb{R}$ be a function. Then

(i) f is said to be *convex* if $f(\alpha x \oplus (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $\alpha \in [0,1]$ and $x, y \in C$;

(ii) f is said to be *strictly convex* if $f(\alpha x \oplus (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$ for all $\alpha \in (0,1)$ and $x, y \in C$ with $x \neq y$.

Let A be a nonempty subset of X. The *closed convex hull* of A is defined by

$$\overline{\mathrm{conv}}(A) := \bigcap\{B \subseteq X : A \subseteq B \text{ and } B \text{ is closed and convex}\}.$$

Let C be a convex subset of X. A subset A of C is called an *extremal subset* if it is nonempty, closed and satisfies the following property: If $x, y \in C$ and $\alpha x \oplus (1-\alpha)y \in A$ for some $\alpha \in (0,1)$, then $x, y \in A$. Notice that if A is an extremal subset of B and B is an extremal subset of C, then A is an extremal subset of C. A point z in C is called an *extreme point* of C if $\{z\}$ is an extremal subset of C. We denote by $\mathrm{Ext}(C)$ the set of all extreme points of C.

Example 1.6. In the Euclidean space $\mathbb{R}^2$, the square $A := \{(x,y) : |x| \leq 1, |y| \leq 1\}$ has four extreme points while the strip $B := \{(x,y) : 0 \leq x \leq 1, y \in \mathbb{R}\}$ does not have an extreme point.

2. Main results

We begin this section by proving a crucial lemma.

Lemma 2.1. *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with $\mathrm{diam}(X) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. If K is a nonempty compact convex subset of X, then every extremal subset of K has an extreme point.*

Proof. Let $\mathcal{C}$ be the family of all nonempty extremal subset of K. Since $K \in \mathcal{C}$, it follows that $\mathcal{C}$ is nonempty and it can be partially ordered by set inclusion. By Zorn's Lemma, $\mathcal{C}$ has a minimal element, say M. It is enough to show that M consists of exactly one point. Suppose that it contains at least two points, say x_0 and y_0. Let $f : M \to \mathbb{R}$ be defined by

$$f(x) := \rho^2(x_0, x) \text{ for all } x \in M.$$

Since $x_0 \neq y_0$, f is not a constant function. By (1.1), f is strictly convex. Let $M_0 := \{x \in M : f(x) = \sup_{y \in M} f(y)\}$. Since f is continuous and K is compact, M_0 is nonempty. Notice also that it is a closed proper subset of M. Next, we show that M_0 is an extremal subset of M. Let $x', x'' \in M$ and M_0 contains a point $(1-\alpha)x' \oplus \alpha x''$ for some $\alpha \in (0,1)$. By (1.1), we have

$$\begin{aligned} \sup_{y \in M} f(y) &= f((1-\alpha)x' \oplus \alpha x'') \\ &\leq (1-\alpha)f(x') + \alpha f(x'') - \alpha(1-\alpha)\rho^2(x', x'') \\ &\leq \sup_{y \in M} f(y) - \alpha(1-\alpha)\rho^2(x', x''), \end{aligned}$$

which implies that $x' = x'' \in M_0$. Thus $M_0 \in \mathcal{C}$ which contradicts to the minimality of M and hence the proof is complete. $\square$

Theorem 2.2. *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with $\mathrm{diam}(X) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. If K is a nonempty compact convex subset of X, then $\overline{conv}(Ext(K)) = K$.*

Proof. (This proof is patterned after the proof of Theorem 3.1 in [6]). By Lemma 2.1, $\mathrm{Ext}(K) \neq \emptyset$. Clearly, $\overline{\mathrm{conv}}(\mathrm{Ext}(K)) \subseteq K$. Suppose that $\overline{\mathrm{conv}}(\mathrm{Ext}(K)) \neq K$. Let $g : K \to \mathbb{R}$ be defined by $g(x) := \mathrm{dist}(x, \overline{\mathrm{conv}}(\mathrm{Ext}(K)))$ and let $K_0 := \{x \in K : g(x) = \sup_{y \in K} g(y)\}$. Since g is continuous and K is compact, K_0 is nonempty. Notice also that it is a closed subset of K. Since $\overline{\mathrm{conv}}(\mathrm{Ext}(K)) \neq K$, we get that $\sup\{g(y) : y \in K\} > 0$. By Lemma 1.4 for $x, y \in K$, $\alpha \in [0, 1]$ and $z \in \overline{\mathrm{conv}}(\mathrm{Ext}(K))$ we have

$$\rho((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)\rho(x,z) + \alpha\rho(y,z),$$

which implies that g is convex. Notice also that K_0 is an extremal subset of K. Again, by Lemma 2.1 there is a point z in $K_0 \cap \mathrm{Ext}(K)$. Thus $0 = g(z) = \sup\{g(y) : y \in K\}$ which is a contradiction. Hence $\overline{\mathrm{conv}}(\mathrm{Ext}(K)) = K$. □

As a consequence of Theorem 2.2, we obtain the following corollary.

Theorem 2.3. ([6, Theorem 1]) *Let (X, ρ) be a complete CAT(0) space and K be a nonempty compact convex subset of X. Then $\overline{conv}(Ext(K)) = K$.*

Proof. It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space (see e.g., [1]). Thus (K, ρ) is a CAT(0) space and hence it is a CAT(κ) space for all $\kappa > 0$. Notice also that it is $R-$convex for $R = 2$. Since K is bounded, we can choose $\varepsilon \in (0, \pi/2)$ and $\kappa > 0$ such that $\mathrm{diam}(K) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$. The conclusion follows from Theorem 2.2. □

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Center of Excellence in Mathematics and Applied Mathematics
Department of Mathematics
Faculty of Science, Chiang Mai University
Chiang Mai 50200 Thailand
E-mail address: bancha.p@cmu.ac.th

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Diametrically Regular Mappings and Browder's Theorem Without the Endpoint Condition

B. Panyanak and S. Suantai

Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand

ABSTRACT

In this paper, we introduce the class of diametrically regular mappings and prove that it includes the class of multi-valued mappings having endpoints. We also prove the existence of endpoints and the convergence of an iterative process for mappings of this class. Our approach provides a new idea of proving Browder's theorem without the endpoint condition.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and ρ be the metric induced by $\langle\cdot,\cdot\rangle$. Let $g : C \to C$ be a single-valued nonexpansive mapping and $F(g) := \{x \in C : g(x) = x\}$. Fix $u \in C$. Consider the following convex optimization problem:

$$\min \rho(u, v) \qquad \text{subject to} \qquad v \in F(g). \tag{1}$$

One of the fundamental and celebrated results for finding the solution of (1) was given by Browder [1] in 1967.

Theorem 1.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space and $g : C \to C$ be a nonexpansive mapping. Fix $u \in C$. For each $t \in (0, 1)$, let x_t be defined by*

$$x_t := (1-t)u \oplus tg(x_t).$$

Then $\{x_t\}$ converges strongly as $t \to 1$ to the unique solution of (1).

If $T : C \to \mathcal{K}(C)$ is a multi-valued nonexpansive mapping, then the mapping $G : C \to \mathcal{K}(C)$ defined by $G(x) := (1-t)u \oplus tT(x)$ is a multi-valued

CONTACT B. Panyanak bancha.p@cmu.ac.th Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand.

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contractive mapping. By Nadler's theorem [2], for each $t \in (0,1)$ there exists x_t in C such that

$$x_t \in G(x_t) = (1-t)u \oplus tT(x_t). \tag{2}$$

Browder's theorem has many useful applications in mathematical science, for instance, in fixed point theory and optimization theory. A natural question arises whether Browder's theorem can be extended to multi-valued nonexpansive mappings. The first result regarding this question was proved by Pietramala [3] in 1991. He gave the strong convergence of $\{x_t\}$ defined by (2) under the endpoint condition. Since then the strong convergence of $\{x_t\}$ in various classes of spaces has been developed and many papers have appeared, see e.g., [4–10]. However, all those results require the endpoint condition which is a strong one. In particular, Dhompongsa et al. [4] proved the following theorem.

Theorem 1.2. *Let C be a nonempty closed convex subset of a Hadamard space X and $T : C \to \mathcal{K}(C)$ be a nonexpansive mapping satisfying the endpoint condition. Fix $u \in C$ and let $\{x_t\}$ be defined by (2). Then T has an endpoint if and only if $\{x_t\}$ is bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly to the unique endpoint of T which is nearest u.*

It was quickly noted by Dhompongsa et al. [11] that the endpoint condition in Theorem 1.2 can be omitted if the space X is restricted to a complete $\mathbb{R}-$tree. Unfortunately, the class of $\mathbb{R}-$trees does not include real Hilbert spaces. Therefore, there is no any result in real Hilbert spaces which extends Browder's theorem to multi-valued nonexpansive mappings without the endpoint condition.

In this paper, we introduce the concept of diametrically regular mappings and prove that it is more general than the concept of endpoint condition for multi-valued nonexpansive mappings in Hadamard spaces. We also prove the existence of endpoints for diametrically regular mappings and show that if the net $\{x_t\}$ defined by (2) is diametrically regular for T, then $\{x_t\}$ converges strongly to the unique endpoint of T which is nearest u. This method provides a new idea of proving Browder's theorem without the endpoint condition.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers. Let (X,ρ) be a metric space and $x, y \in X$. A *geodesic* joining x to y is a mapping ϕ from a closed interval $[0, L]$ to X such that $\phi(0) = x, \phi(L) = y,$ and $\rho(\phi(s), \phi(t)) = |s-t|$ for all $s, t \in [0, L]$. The image of ϕ is called a *geodesic segment* joining x and y which when unique is denoted by $[x, y]$. The space (X,ρ) is said to be a *geodesic space* if every two points in X are joined by a geodesic, and X is said to be *uniquely* *geodesic* if for each x and y in X there is exactly one geodesic joining them. A subset C of X is said to be *convex* if every pair of points x, y in C can be joined by a geodesic in X and the image of every such geodesic is contained in C. A geodesic space (X, ρ) is said to be a *Hadamard space* if X is a complete metric space and for each $x, y, z \in X,$ one has

$$\rho^2(x, m) \leq \frac{1}{2}\rho^2(x, y) + \frac{1}{2}\rho^2(x, z) - \frac{1}{4}\rho^2(y, z),$$

where m is the midpoint of $[y, z]$. For other equivalent definitions and basic properties of Hadamard spaces, we refer the reader to standard texts such as [12, 13]. It is well-known that every Hadamard space is uniquely geodesic. Notice also that Pre-Hilbert spaces, $\mathbb{R}-$trees, Euclidean buildings are examples of Hadamard spaces, see [12, 14].

Let C be a nonempty closed convex subset of a Hadamard space (X, ρ). It follows from Proposition 2.4 of [12] that for each x in X, there exists a unique point x_0 in C such that

$$\rho(x, x_0) = \mathrm{dist}(x, C) := \inf\{\rho(x, c) : c \in C\}.$$

In this case, x_0 is called the *unique nearest point* of x in C. By Lemma 2.1 of [15], for each $x, y \in X$ and $t \in [0, 1],$ there exists a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(y, z) = (1-t)\rho(x, y). \tag{3}$$

We denote by $(1-t)x \oplus ty$ the unique point z satisfying (3). The following lemma can be found in [15].

Lemma 2.1. *Let (X, ρ) be a Hadamard space, $x, y, z \in X$ and $t \in [0, 1]$. Then the following inequalities hold:*

(i) $\rho((1-t)x \oplus ty, z) \leq (1-t)\rho(x, z) + t\rho(y, z);$
(ii) $\rho^2((1-t)x \oplus ty, z) \leq (1-t)\rho^2(x, z) + t\rho^2(y, z) - t(1-t)\rho^2(x, y).$

Let C be a nonempty subset of a Hadamard space (X, ρ) and $x \in X$. The *radius* of C relative to x is defined by

$$R(x, C) := \sup\{\rho(x, y) : y \in C\}.$$

The *diameter* of C is defined by

$$\mathrm{diam}(C) := \sup\{\rho(x, y) : x, y \in C\}.$$

The set C is called *bounded* if $\mathrm{diam}(C) < \infty$. It is denoted by $\mathcal{K}(C)$: the family of nonempty compact subsets of C. The *Pompeiu-Hausdorff distance* on $\mathcal{K}(C)$ is given by

$$\mathcal{H}(A, B) := \max\left\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\right\} \quad \text{for all} \quad A, B \in \mathcal{K}(C).$$

A multi-valued mapping $T: C \to \mathcal{K}(X)$ is said to be *contractive* if there exists a constant $\lambda \in [0, 1)$ such that

$$\mathcal{H}(T(x), T(y)) \leq \lambda \ \rho(x, y) \quad \text{for all} \quad x, y \in C. \tag{4}$$

If (4) is valid when $\lambda = 1$, then T is said to be *nonexpansive*. The mapping T is called a *single-valued mapping* if $T(x)$ is a singleton for every x in C. An element x in C is called a *fixed point* of T if $x \in T(x)$. Moreover, if $\{x\} = T(x)$, then x is called an *endpoint* of T. It is denoted by $F(T)$: the set of all fixed points of T and by $E(T)$: the set of all endpoints of T. The existence of endpoints for multi-valued nonexpansive mappings was studied by many researchers, see e.g., [16–19]. Notice also that if C is a nonempty closed convex subset of a Hadamard space and $T: C \to \mathcal{K}(X)$ is a nonexpansive mapping, then $E(T)$ is a closed convex subset of C, see [18].

A multi-valued mapping $T: C \to \mathcal{K}(X)$ is said to be *diametrically regular* if there exists a net $\{x_\alpha\}$ in C such that $\lim_\alpha \operatorname{diam}(T(x_\alpha)) = 0$. In this case, we call $\{x_\alpha\}$ a *diametrically regular net* for T. The mapping T is said to satisfy the *endpoint condition* if $F(T) = E(T)$. Obviously, if T has an endpoint then T is diametrically regular.

The following result shows that the diametric regularity of a nonexpansive mapping T is weaker than the endpoint condition.

Proposition 2.2. *Let C be a nonempty closed convex subset of a Hadamard space X, $u \in C$, and $T: C \to \mathcal{K}(C)$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then the following statement holds:*
($*$) if T satisfies the endpoint condition, then T has a diametrically regular net $\{x_t\}$ in C such that $x_t \in (1-t)u \oplus tT(x_t)$ for all $t \in (0, 1)$.

Proof. Let $\{x_t\}$ be defined by (2) and let $\tilde{x}$ be the unique nearest point of u in $E(T)$. By Theorem 1.2, $\lim_{t\to 1} x_t = \tilde{x}$. For $v, w \in T(x_t)$ we have

$$\begin{aligned} \rho(v, w) &\leq \rho(v, \tilde{x}) + \rho(\tilde{x}, w) \\ &\leq \operatorname{dist}(v, T(\tilde{x})) + \operatorname{dist}(w, T(\tilde{x})) \\ &\leq 2\mathcal{H}(T(x_t), T(\tilde{x})) \\ &\leq 2\rho(x_t, \tilde{x}). \end{aligned}$$

This implies that $\lim_{t\to 1} \operatorname{diam}(T(x_t)) = 0$ and hence $\{x_t\}$ is diametrically regular for T. ■

The endpoint condition in Proposition 2.2 is necessary as shown in the following example. Notice also that the converse of ($*$) does not hold, see Example 3.7.

Example 2.3. Let $C = [0, 1]$ and $T : C \to \mathcal{K}(C)$ be defined by

$$T(x) := [0, 1] \quad \text{for all} \quad x \in C.$$

Then T is a nonexpansive mapping with $F(T) = [0, 1]$ and $E(T) = \emptyset$. Since $\operatorname{diam}(T(x)) = 1$ for all $x \in C$, there is no a net $\{x_t\}$ in C which is diametrically regular for T.

Let $\{x_n\}$ be a bounded sequence in a Hadamard space (X, ρ). The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) := \inf\{\limsup_{n\to\infty} \rho(x_n, x) : x \in X\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) := \{x \in X : \limsup_{n\to\infty} \rho(x_n, x) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [20] that in a Hadamard space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to $\Delta-$converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta-\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{\Delta} x$ and call x the $\Delta-$limit of $\{x_n\}$. Now, we collect some basic properties of $\Delta-$convergence.

Lemma 2.4. *Let C be a nonempty closed convex subset of a Hadamard space X and $T : C \to \mathcal{K}(C)$ be a nonexpansive mapping. Then the following statements hold:*

- (i) [21, Page 3690] *Every bounded sequence in X always has a $\Delta-$convergent subsequence.*
- (ii) [18, Lemma 4.6] *If $\{x_n\}$ is a sequence in C and $x \in X$, then the conditions $x_n \xrightarrow{\Delta} x$, $\operatorname{dist}(x_n, T(x_n)) \to 0$, and $\operatorname{diam}(T(x_n)) \to 0$ imply $x \in E(T)$.*

Let (X, ρ) be a metric space. We denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a *vector*. The *quasi-linearization* is a mapping $\langle \cdot, \cdot \rangle$ from $(X \times X) \times (X \times X)$ to $\mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle := \frac{1}{2}[\rho^2(a, d) + \rho^2(b, c) - \rho^2(a, c) - \rho^2(b, d)] \quad \text{for all} \quad a, b, c, d \in X.$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle, \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that (X, ρ) satisfies the *Cauchy-Schwarz inequality* if

$$|\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle| \leq \rho(a, b)\rho(c, d) \quad \text{for all} \quad a, b, c, d \in X.$$

It is known from Corollary 3 of [22] that every Hadamard space satisfies the Cauchy-Schwarz inequality. Some other properties of quasi-linearization are included as the following lemma.

Lemma 2.5. *Let (X,ρ) be a Hadamard space. Then the following statements hold:*

(i) [23, Theorem 3.1] *Let C be a nonempty closed convex subset of X, $u \in X$ and $x \in C$. Then x is the unique nearest point of u in C if and only if $\langle \overrightarrow{xu}, \overrightarrow{vx} \rangle \geq 0$ for all $v \in C$.*

(ii) [24, Lemma 2.10] *Let $p, q \in X$. For each $t \in [0,1]$, we set $u_t = (1-t)p \oplus tq$. Then, for $x \in X$, we have*

$$\langle \overrightarrow{u_t x}, \overrightarrow{u_t x} \rangle \leq (1-t)\langle \overrightarrow{px}, \overrightarrow{u_t x} \rangle + t\langle \overrightarrow{qx}, \overrightarrow{u_t x} \rangle.$$

(iii) [25, Theorem 2.6] *A sequence $\{x_n\}$ in X $\Delta-$converges to x in X if and only if $\limsup_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{ux} \rangle \leq 0$ for all $u \in X$.*

3. Main results

This section is begun by proving the existence of endpoints for diametrically regular mappings. For this we will make use of the following facts.

Lemma 3.1. [18, Theorem 4.7] *Let C be a nonempty bounded closed convex subset of a Hadamard space (X,ρ) and $T : C \to \mathcal{K}(X)$ be a nonexpansive mapping. Then T has an endpoint if and only if $\inf\{R(x, T(x)) : x \in C\} = 0$.*

Lemma 3.2. [18, Proposition 2.4] *Let C be a nonempty subset of a Hadamard space (X,ρ), $\{x_\alpha\}$ be a net in C and $T : C \to \mathcal{K}(X)$ be a mapping. Then $\lim_\alpha R(x_\alpha, T(x_\alpha)) = 0$ if and only if $\lim_\alpha \operatorname{dist}(x_\alpha, T(x_\alpha)) = 0$ and $\lim_\alpha \operatorname{diam}(T(x_\alpha)) = 0$.*

As an immediate consequence of Lemmas 3.1 and 3.2, we can obtain the following theorem.

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of a Hadamard space (X,ρ) and $T : C \to \mathcal{K}(X)$ be a nonexpansive mapping which is diametrically regular with a diametrically regular net $\{x_\alpha\}$. Suppose that $\lim_\alpha \operatorname{dist}(x_\alpha, T(x_\alpha)) = 0$. Then T has an endpoint in C.*

The following example shows that the condition $\lim_\alpha \operatorname{dist}(x_\alpha, T(x_\alpha)) = 0$ in Theorem 3.3 cannot be omitted.

Example 3.4. [18] Let $X = \mathbb{R}$, $C = [0,1]$ and $T : C \to \mathcal{K}(X)$ be defined by

$$T(x) = [0, 1-x] \quad \text{for all} \quad x \in C.$$

Then T is a nonexpansive mapping with $\lim_{x\to1} \operatorname{diam}(T(x)) = 0$. However,

$$\lim_{x\to1} \operatorname{dist}(x, T(x)) = 1.$$

Notice also that T does not have an endpoint in C.

Now we prove the main theorem.

Theorem 3.5. *Let C be a nonempty closed convex subset of a Hadamard space (X,ρ) and $T: C \to \mathcal{K}(C)$ be a nonexpansive mapping. Fix $u \in C$ and let $\{x_t\}$ be defined by (2). Suppose that $\{x_t\}$ is diametrically regular for T. Then T has an endpoint if and only if $\{x_t\}$ is bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly to the unique point $\tilde{x}$ in $E(T)$ such that*

$$\rho(u,\tilde{x}) = \min\{\rho(u,e) : e \in E(T)\}.$$

Proof. Let $e \in E(T)$. From (2), for each $t \in (0,1)$ there exists $y_t \in T(x_t)$ such that $x_t = (1-t)u \oplus ty_t$. Since T is nonexpansive,

$$\rho(y_t,e) = \mathrm{dist}(y_t, T(e)) \leq \mathcal{H}(T(x_t), T(e)) \leq \rho(x_t,e).$$

By Lemma 2.1 (i), we have

$$\rho(x_t,e) \leq (1-t)\rho(u,e) + t\rho(y_t,e) \leq (1-t)\rho(u,e) + t\rho(x_t,e).$$

Thus $\rho(x_t,e) \leq \rho(u,e)$. Therefore, $\{x_t\}$ is bounded. Conversely, suppose that $\{x_t\}$ is bounded. Let a sequence $\{t_n\}$ in (0, 1) converging to 1 and put $x_n := x_{t_n}$. For each $n \in \mathbb{N}$, let $y_n \in T(x_n)$ be such that $x_n = (1-t_n)u \oplus t_n y_n$. Then

$$\mathrm{dist}(x_n, T(x_n)) \leq \rho(x_n,y_n) = (1-t_n)\rho(u,y_n) \to 0 \quad \text{as} \quad n \to \infty.$$

By Lemma 2.4, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $\tilde{x}$ in $E(T)$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = \tilde{x}$. This implies that $E(T) \neq \emptyset$. Let z be the unique nearest point of u in $E(T)$. We will show that $\lim_{t\to 1} x_t = z$. Let $\{x_n\}$ be as above. It suffices to show that there exists a subsequence of $\{x_n\}$ which converges strongly to z. Let $\{x_{n_k}\}$ and $\tilde{x}$ be as above. Then $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = \tilde{x}$ and $\tilde{x} \in E(T)$. By Lemma 2.5 (ii), we have

$$\begin{aligned}
\rho^2(x_{n_k},\tilde{x}) &= \langle \overrightarrow{x_{n_k}\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle \\
&\leq (1-t_{n_k})\langle \overrightarrow{u\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle + t_{n_k}\langle \overrightarrow{y_{n_k}\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle \\
&\leq (1-t_{n_k})\langle \overrightarrow{u\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle + t_{n_k}\rho(y_{n_k},\tilde{x})\rho(x_{n_k},\tilde{x}) \\
&\leq (1-t_{n_k})\langle \overrightarrow{u\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle + t_{n_k}\mathcal{H}(T(x_{n_k}), T(\tilde{x}))\rho(x_{n_k},\tilde{x}) \\
&\leq (1-t_{n_k})\langle \overrightarrow{u\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle + t_{n_k}\rho^2(x_{n_k},\tilde{x}),
\end{aligned}$$

which yields

$$\rho^2(x_{n_k},\tilde{x}) \leq \langle \overrightarrow{u\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle. \tag{5}$$

Since $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = \tilde{x}$, by Lemma 2.5 (iii), we have

$$\limsup_{k\to\infty} \langle \overrightarrow{u\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle \leq 0.$$

This, together with (5), implies that $\{x_{n_k}\}$ converges strongly to $\tilde{x}$. Next, we show that $\tilde{x} = z$. By Lemma 2.1 (ii), for each $p \in E(T)$ we have

$$\begin{aligned}\rho^2(x_{n_k}, p) &\leq (1-t_{n_k})\rho^2(u,p) + t_{n_k}\rho^2(y_{n_k}, p) - t_{n_k}(1-t_{n_k})\rho^2(u, y_{n_k}) \\ &\leq (1-t_{n_k})\rho^2(u,p) + t_{n_k}\mathcal{H}^2(T(x_{n_k}), T(p)) - t_{n_k}(1-t_{n_k})\rho^2(u, y_{n_k}) \\ &\leq (1-t_{n_k})\rho^2(u,p) + t_{n_k}\rho^2(x_{n_k}, p) - t_{n_k}(1-t_{n_k})\rho^2(u, y_{n_k}).\end{aligned}$$

This implies that

$$\rho^2(x_{n_k}, p) \leq \rho^2(u,p) - t_{n_k}\rho^2(u, y_{n_k}).$$

Taking $k \to \infty$, we get that $\rho^2(\tilde{x}, p) \leq \rho^2(u,p) - \rho^2(u, \tilde{x})$. Thus

$$0 \leq \frac{1}{2}[\rho^2(\tilde{x},\tilde{x}) + \rho^2(u,p) - \rho^2(\tilde{x},p) - \rho^2(u,\tilde{x})] = \langle \overrightarrow{\tilde{x}u}, \overrightarrow{p\tilde{x}} \rangle, \text{ for all } p \in E(T).$$

By Lemma 2.5 (i), $\tilde{x} = z$ and hence the proof is complete. ■

As an immediate consequence of Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space and $T : C \to \mathcal{K}(C)$ be a nonexpansive mapping. Fix $u \in C$ and let $\{x_t\}$ be defined by (2). Suppose that $\{x_t\}$ is diametrically regular for T. Then T has an endpoint if and only if $\{x_t\}$ is bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly to the unique point $\tilde{x}$ in E(T) such that*

$$\rho(u, \tilde{x}) = \min\{\rho(u, e) : e \in E(T)\}.$$

The following example shows that the diametric regularity of $\{x_t\}$ in Theorem 3.5 is necessary.

Example 3.7. [3, 26] Let X be the Euclidean space $\mathbb{R}^2$ and $C = [0,1] \times [0,1]$. Let $T : C \to \mathcal{K}(C)$ be defined by

$$T(a,b) := \text{ the closed convex hull of } \{(0,0), (a,0), (0,b)\}.$$

Then T is a nonexpansive mapping with $F(T) = \{(a,b) \in C : ab = 0\}$ and $E(T) = \{(0,0)\}$. Fix $u \in C$. For each $t \in (0,1)$, let $x_t = (1-t)u$. It is easy to see that $\{x_t\}$ satisfies (2) and is diametrically regular for T. By Corollary 3.6, $\lim_{t\to 1} x_t = (0,0)$ (see Table 1 for numerical experiments and Figure 1 for comparison of errors). However, in the case of $u = (1,0)$, if we let $z_t \equiv u$ then $\{z_t\}$ satisfies (2) but is not diametrically regular for T. Moreover, $\lim_{t\to 1} z_t = u \neq (0,0)$.

4. Numerical experiments and comparison of errors

Let X, C and T be as in Example 3.7. For each $n \in \mathbb{N}$, let $t_n := \frac{2^n - 1}{2^n}$ and $x_n := x_{t_n} = (1-t_n)u$, where $u \in C$ be fixed. We see that, in any case of u, the sequence $\{x_n\}$ converges to (0, 0) as n tends to ∞.

Table 1. Numerical experiments.

n	$u = (1, 0.4)$ boundary case x_n	$u = (0.3, 0.5)$ interior case x_n	$u = (0.6, 0.6)$ diagonal case x_n	$u = (0, 1)$ corner case x_n
1	(0.50000,0.20000)	(0.15000,0.25000)	(0.30000,0.30000)	(0,0.50000)
2	(0.25000,0.10000)	(0.07500,0.12500)	(0.15000,0.15000)	(0,0.25000)
3	(0.12500,0.05000)	(0.03750,0.06250)	(0.07500,0.07500)	(0,0.12500)
4	(0.06250,0.02500)	(0.01875,0.03125)	(0.03750,0.03750)	(0,0.06250)
5	(0.03125,0.01250)	(0.00938,0.01563)	(0.01875,0.01875)	(0,0.03125)
6	(0.01563,0.00625)	(0.00469,0.00781)	(0.00938,0.00938)	(0,0.01563)
7	(0.00781,0.00313)	(0.00234,0.00391)	(0.00469,0.00469)	(0,0.00781)
8	(0.00391,0.00156)	(0.00117,0.00195)	(0.00234,0.00234)	(0,0.00391)
9	(0.00195,0.00078)	(0.00059,0.00098)	(0.00117,0.00117)	(0,0.00195)
10	(0.00098,0.00039)	(0.00029,0.00049)	(0.00059,0.00059)	(0,0.00098)
11	(0.00049,0.00020)	(0.00015,0.00024)	(0.00029,0.00029)	(0,0.00049)
12	(0.00024,0.00010)	(0.00007,0.00012)	(0.00015,0.00015)	(0,0.00024)
13	(0.00012,0.00005)	(0.00004,0.00006)	(0.00007,0.00007)	(0,0.00012)
14	(0.00006,0.00002)	(0.00002,0.00003)	(0.00004,0.00004)	(0,0.00006)

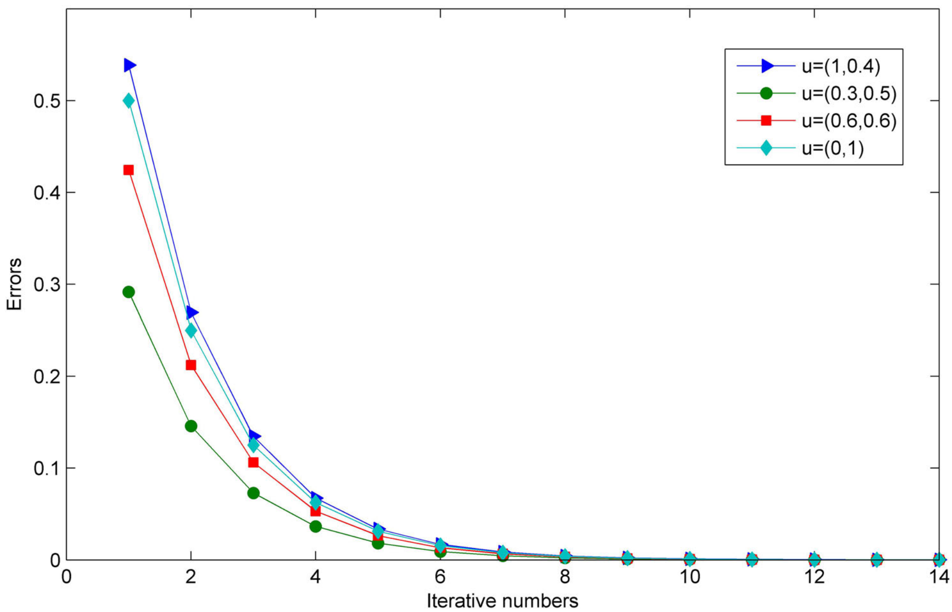

Figure 1. Comparison of errors.

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