



รายงานวิจัยฉบับสมบูรณ์

เงื่อนไขจำเป็นและเพียงพอสำหรับผลเฉลยที่มีประสิทธิภาพและผลลัพธ์ภาวะ
คู่กันของปัญหาค่าเหมาะหลายค่าและการประยุกต์ในปัญหาค่า

The necessary and sufficient conditions for efficient solutions and duality
to Multi-objective and Robust Multi-objective Optimization prob-
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พฤษภาคม 2562

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และมหาวิทยาลัยนเรศวร

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. และมหาวิทยาลัยนเรศวร ไม่จำเป็นต้องเห็นด้วยเสมอไป)

บทคัดย่อ

ในงานวิจัยนี้ เราได้ศึกษาในปัญหาค่าเหมาะหลายค่าสำหรับข้อจำกัดเชิงกรวยโดยไม่ใช้ความเป็นเซตคอนเวกซ์ของเซตที่เป็นไปได้ และ ความเป็นคอนเวกซ์ของฟังก์ชันจุดประสงค์รวมทั้งฟังก์ชันข้อจำกัดด้วย เราได้เริ่มต้นศึกษาเงื่อนไขข้อบังคับจำกัดสำหรับปัญหาดังกล่าว พร้อมทั้งศึกษาเงื่อนไขค่าเหมาะสมที่เป็นเงื่อนไขที่จำเป็น และ เพียงพอ (KKT) สำหรับผลเฉลยที่มีประสิทธิภาพสำหรับ weak Pareto minimum พร้อมทั้ง Pareto minimum

Abstract

This project deals with a class of differentiable multi-objective optimization problems (MOP) over cone constraints without the convexity of the feasible set, and the cone-convexity of objectives and constraint functions. We present constraint qualifications for these (MOP) problems and establish the relationships between them. We also present necessary and sufficient the Karush-Kuhn-Tucker (KKT) optimality conditions for weak Pareto minimum as well as Pareto minimum. Our main results improve some recent ones in the literature.

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Chapter 1

Main Results

1.1 Some characterizations of robust solution sets for uncertain convex optimization problems with locally Lipschitz inequality constraints

The study of characterizations of solution sets has become an important research direction for many mathematical programming problems. Based on understanding characterizations of solution sets, solution methods for solving mathematical programs that have multiple solutions can be developed. The notion of characterizations of solution sets was first introduced and studied by Mangasarian for a convex extrema problem with differentiable function [29]. Some useful examples clarifying such characterizations of solution sets can be found in [7] for characterizing the problems that have weak sharp minimum. This being a reason why several characterizations of solution sets for some classes of constrained optimization problems have appeared in the literature (see [6, 8, 13, 14, 19, 23, 32, 33, 36, 38, 39] and other references therein).

However, dealing with real-world optimization problems, the input data associated with the objective function and the constraints of programs are uncertain due to prediction error or measurement errors (see [1, 2, 3, 4]). Moreover, in many situations often we need to make decisions now before we can know the true values or have better estimations of the parameters. Robust optimization is one of the basic methodologies to protect the optimal solution that it is no longer feasible after realization of actual values of parameters. This means that any feasible points must satisfy all constraints including each set of constraints corresponding to a possible realization of the uncertain parameters from the uncertainty sets. Precisely stated, let us first consider the following optimization problem :

$$\min_{x \in \mathbb{R}^n} \{f(x) : g_i(x) \leq 0, i = 1, \dots, m\}, \quad (\text{P})$$

where $f, g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are functions. The problem (P) in the face of data uncertainty both in the objective and constraints can be written by the following optimization problems:

$$\min_{x \in \mathbb{R}^n} \{f(x, u) : g_i(x, v_i) \leq 0, i = 1, \dots, m\}, \quad (\text{UP})$$

where $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \rightarrow \mathbb{R}$, and $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are functions, u and v_i are uncertain parameters and they belong to the specified nonempty convex and compact uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^{q_0}$ and $\mathcal{V}_i \subseteq \mathbb{R}^{q_i}$, respectively. The robust (worst case) counterpart of (UP), by construction in [3], is obtained by solving the single problem:

$$\min_{x \in \mathbb{R}^n} \{\max_{u \in \mathcal{U}} f(x, u) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\}, \quad (\text{RP})$$

where the objective and constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets $\mathcal{U}$ and $\mathcal{V}_i$. The set of feasible solutions of problem (RP),

$$F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\},$$

refer to *robust feasible set* of the problem (UP). The optimal solution to the problem (RP) is known as a *robust optimal solution* of (UP). A successful treatment of the robust optimization approaches for treating convex optimization programs with data uncertainty to derive characterizations of robust optimal solution sets was given in [15, 27, 34, 35]. For issues related to optimality conditions and duality properties, see [5, 11, 16, 17, 18, 25, 26] and other references therein.

This paper is an attempt to investigate optimality conditions and to derive characterizations of robust solution sets of (UP). Unlike various related works in the literature mentioned above, in the present paper, appearing constraint functions are not convex necessarily while the robust feasible set F is convex. In this way, we refer to *convex problems without convex representation* in the sense that the constraint functions to represent the convex feasible set are non necessarily convex. Optimality conditions and characterizations of convexity of feasible set for such problems in the absent of data uncertainty can be found in [24] for differentiable case, and in [6, 14, 21] for non-differentiable case.

To the best of our knowledge, completely characterizations of robust solutions for uncertain scalar and multi-objective optimization problems over a robust convex feasible set described by non necessarily convex functions within the framework of robust optimization approach are not available in the literature. So, in this paper we examine a robust optimization framework for studying characterizations of the robust optimal solution set for uncertain convex optimization problems with a robust convex feasible set described by locally Lipschitz constraints. First, complete optimality conditions for uncertain convex optimization problems are given. In order to characterize the robust optimal solution set of a given problem, we introduce the so-called *pseudo-Lagrange function* and then, we show that pseudo-Lagrange function is constant on the robust optimal solution set. Afterwards, we then use this property to derive various characterizations of the robust optimal solution set that these are expressed in terms of convex subdifferentials, Clarke subdifferentials and Lagrange multipliers. Finally, the results are then applied to derive characterizations of weakly robust efficient solution set and properly robust efficient solution set of uncertain convex multi-objective optimization problems without convexity assumption on constraint functions.

We begin this section by fixing certain notations, definitions and preliminary results that will be used throughout the paper. We denote by $\mathbb{R}^n$ the Euclidean space with dimension n whose norm is denoted by $\|\cdot\|$ and $\langle x, y \rangle$ denotes the usual inner product between two vectors x, y in $\mathbb{R}^n$, that is, $\langle x, y \rangle = x^T y$. Let $\mathbb{R}_+^n := \{x := (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$ be non-negative orthant of $\mathbb{R}^n$. Note also that the interior non-negative orthant of $\mathbb{R}^n$ is denoted by $\text{int}\mathbb{R}_+^n$ and is defined by $\text{int}\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}$. Given a set $A \subseteq \mathbb{R}^n$, we recall that a set A is *convex* whenever $\lambda x + (1 - \lambda)y \in A$ for all $\lambda \in [0, 1]$, $x, y \in A$. A set A is said to be a *cone* if $\lambda A \subseteq A$ for all $\lambda \geq 0$. We denote the *convex hull* and the *conical hull* generated by A , by $\text{conv}A$ and $\text{cone}A$, respectively. The *normal cone* at x to a closed convex set A , denoted by $N(A, x)$, is defined by

$$N(A, x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x \rangle \leq 0, \forall y \in A\}.$$

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *convex* if for all $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}^n$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

It is a well known fact that a convex function need not be differentiable everywhere. However if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function then the one-sided or rather right-sided directional derivative always exists and is finite. The right-sided directional derivative of f at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is denoted by $f'(x; d)$, is defined as

$$f'(x; d) := \lim_{t \rightarrow 0^+} \frac{f(x + td) - f(x)}{t}.$$

It is important to note that for every fixed x the function $f'(x; \cdot)$ is a positively homogeneous convex function. The subdifferential of convex function f at x is defined as

$$\partial f(x) := \{\xi \in \mathbb{R}^n : f(y) \geq f(x) + \langle \xi, y - x \rangle, \text{ for all } y \in \mathbb{R}^n\}.$$

We now recall the following useful result, which is a subdifferential max-function rule of convex functions over a compact set, that will be used later in the paper.

Lemma 1. [15, Lemma 2.1] Let $\mathcal{U} \subseteq \mathbb{R}^p$ be a convex compact set, and let $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \rightarrow \mathbb{R}$ be a function such that for each fixed $u \in \mathcal{U}$, $f(\cdot, u)$ is a convex function on $\mathbb{R}^n$ and for each fixed $x \in \mathbb{R}^n$, $f(x, \cdot)$ is a concave function on $\mathbb{R}^{q_0}$. Then,

$$\partial \left(\max_{u \in \mathcal{U}} f(\cdot, u) \right) (\bar{x}) = \bigcup_{u \in \mathcal{U}(\bar{x})} \partial f(\cdot, u)(\bar{x}),$$

where $\mathcal{U}(\bar{x}) := \{\bar{u} \in \mathcal{U} : f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u)\}$.

Definition 2. A function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *locally Lipschitz* at $x \in \mathbb{R}^n$, if there exists a positive scalar L and a neighborhood N of x such that, for all $y, z \in N$, one has

$$|h(y) - h(z)| \leq L \|y - z\|.$$

Definition 3. [9] Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The *Clarke generalized directional derivative* of h at x in the direction $d \in \mathbb{R}^n$, denoted $h^o(x; d)$, is defined as

$$h^o(x; d) := \limsup_{\substack{y \rightarrow x \\ t \rightarrow 0^+}} \frac{h(y + td) - h(y)}{t},$$

Definition 4. [9] Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The *Clarke generalized subdifferential* of h at x , denoted by $\partial^o h(x)$, is defined as

$$\partial^o h(x) := \{\xi \in \mathbb{R}^n : h^o(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^n\}.$$

From the definition of the Clarke generalized subdifferential, it follows that

$$h^o(x; d) = \max_{\xi \in \partial^o h(x)} \langle \xi, d \rangle, \quad \forall d \in \mathbb{R}^n.$$

Definition 5. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The function h is said to be *regular* at $x \in \mathbb{R}^n$ if, for each $d \in \mathbb{R}^n$, the directional derivative $h'(x; d)$ exists and coincides with $h^o(x; d)$.

For a given compact subset $\mathcal{V}$ of $\mathbb{R}^q$ and a given function $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$, the following conditions will be considered in this paper.

- (C1) for every $x \in \mathbb{R}^n$ the function $\mathcal{V} \ni v \mapsto g(x, v)$ is upper semicontinuous;
- (C2) g is locally Lipschitz in x , uniformly for v in $\mathcal{V}$, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U of x and a constant $L > 0$ such that for all y and z in U , and $v \in \mathcal{V}$, one has

$$|g(y, v) - g(z, v)| \leq L\|y - z\|;$$

- (C3) for each $(x, v) \in \mathbb{R}^n \times \mathcal{V}$, the function $g(\cdot, v)$ is regular at x , that is,

$$g_x^o(x, v; \cdot) = g'_x(x, v; \cdot)$$

(the derivatives being with respect to x);

- (C4) set-valued map $\mathbb{R}^n \times \mathcal{V} \ni (x, v) \mapsto \partial^o g(\cdot, v)(x)$ is upper semicontinuous where $\partial^o g(\cdot, v)(x)$ denotes the Clarke subdifferential of g with respect to x .

Remark 6. In a suitable setting, if the function g is convex in x and continuous in v , the conditions (C2), (C3), and (C4) are then automatically satisfied. These conditions also hold whenever the derivative $\nabla_x g(x, v)$ with respect to x exists and is continuous in (x, v) .

Remark 7. [25] Under the conditions (C1) and (C2) the function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\psi(x) := \max\{g(x, v) : v \in \mathcal{V}\},$$

is defined and finite. Further, ψ is locally Lipschitz on $\mathbb{R}^n$, and hence for each $x \in \mathbb{R}^n$ the set $\mathcal{V}(x)$ defined as

$$\mathcal{V}(x) := \{v \in \mathcal{V} : g(x, v) = \psi(x)\},$$

is a nonempty closed subset of $\mathbb{R}^q$.

We conclude this section by the following lemmas which will be useful in our later analysis.

Lemma 8. [9] Let the function ψ be defined in Remark 7. Suppose that the conditions (C1) - (C4) are fulfilled. Then the usual one-sided directional derivative $\psi'(x; d)$ exists, and satisfies the following : for each $x, d \in \mathbb{R}^n$,

$$\begin{aligned} \psi'(x; d) &= \psi^o(x; d) = \max\{g_x^o(x, v; d) : v \in \mathcal{V}(x)\} \\ &= \max\{\langle \xi, d \rangle : \xi \in \partial^o g(\cdot, v)(x), v \in \mathcal{V}(x)\}. \end{aligned}$$

Lemma 9. [26] For a given compact convex subset $\mathcal{V}$ of $\mathbb{R}^q$ and a given function $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$, suppose that the basic conditions (C1) - (C4) are fulfilled. Further, suppose that $g(x, \cdot)$ is concave on $\mathcal{V}$, for each $x \in \mathbb{R}^n$. Then

$$\partial^o \psi(x) = \{\xi \in \mathbb{R}^n : \exists v \in \mathcal{V}(x) \text{ such that } \xi \in \partial^o g(\cdot, v)(x)\}.$$

In this section, we give a multiplier characterization for the robust optimal solution of (UP), which will play an important role in deriving characterizations of the robust optimal solution sets in the next section. Let us recall the following robust (worst case) counterpart optimization problem of (UP) :

$$\min_{x \in \mathbb{R}^n} \left\{ \max_{u \in \mathcal{U}} f(x, u) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m \right\}, \quad (\text{RP})$$

where $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \rightarrow \mathbb{R}$, and $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are given functions and for each $i = 1, 2, \dots, m$, $(u, v_i) \in \mathcal{U} \times \mathcal{V}_i \subseteq \mathbb{R}^{q_0} \times \mathbb{R}^{q_i}$, where $\mathcal{U}$ and $\mathcal{V}_i$ are the specified nonempty convex and compact uncertainty sets. The robust feasible set of (UP) is defined by

$$F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\}.$$

Assumption 10. Throughout this paper, we always assume that $F \neq \emptyset$, $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ is a convex-concave in the sense that $f(\cdot, u)$ is a convex function for any $u \in \mathcal{U}$, and $f(x, \cdot)$ is a concave function for any $x \in \mathbb{R}^n$ while $g_i(x, \cdot)$, $i = 1, \dots, m$, are concave functions for any $x \in \mathbb{R}^n$. Further, let the functions g_i , $i = 1, \dots, m$, be satisfied the conditions (C1) and (C2).

Definition 11. We say that $\bar{x} \in F$ is a robust optimal solution of (UP) if and only if $\bar{x}$ is an optimal solution of (RP).

By using Proposition 2.2 in [6], we can derive the following characterization of convexity for robust feasible set of (UP) in terms of the Clarke directional derivative. Before doing so let us denote, for each $x \in F$,

$$I(x) := \left\{ i \in \{1, \dots, m\} : \max_{v_i \in \mathcal{V}_i} g_i(x, v_i) = 0 \right\},$$

and for all $i = 1, \dots, m$,

$$\mathcal{V}_i(x) := \left\{ \bar{v}_i \in \mathcal{V}_i : g_i(x, \bar{v}_i) = \max_{v_i \in \mathcal{V}_i} g_i(x, v_i) \right\}.$$

Proposition 12. Let the system $g_i(x, v_i) \leq 0$, $\forall v_i \in \mathcal{V}_i$, $i = 1, \dots, m$, be satisfied the robust Slater constraint qualification, that is, there exists $x_0 \in \mathbb{R}^n$ such that

$$g_i(x_0, v_i) < 0, \text{ for any } v_i \in \mathcal{V}_i, i = 1, \dots, m.$$

For each $x \in F$ and $i \in I(x)$, let the function g_i be satisfied the conditions (C3), (C4), and $0 \notin \partial^o g_i(\cdot, v_i)(x)$ whenever $v_i \in \mathcal{V}_i(x)$. Then F is convex if and only if

$$F = \{y \in \mathbb{R}^n : g_{ix}^o(x, v_i; y - x) \leq 0, \forall x \in F, \forall i \in I(x), \forall v_i \in \mathcal{V}_i(x)\}.$$

Proof. For each $i = 1, \dots, m$, define a function $\psi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\psi_i(x) := \max_{v_i \in \mathcal{V}_i} g_i(x, v_i) \text{ for all } x \in \mathbb{R}^n.$$

Applying the conditions (C1) and (C2), we have, for each $i = 1, \dots, m$, ψ_i is locally Lipschitz on $\mathbb{R}^n$. To achieve the result, we will use Proposition 2.2 in [6] and then we need to justify that for any $x \in F$, ψ_i , $i \in I(x)$, are regular in the sense of Clarke and $0 \notin \partial^o \psi_i(x)$, and the system $\psi_i(x) \leq 0$, $i = 1, \dots, m$, satisfies the Slater condition. The first and the second requirements will follow from Lemma 38 and Lemma 21 that for any $x \in F$,

$$\psi_i'(x; d) = \psi_i^o(x; d) = \max\{g_{ix}^o(x, v_i; d) : v_i \in \mathcal{V}_i(x)\}, \forall i \in I(x), \quad (1.1.1)$$

and for each $i \in I(x)$

$$\begin{aligned} 0 \in \bigcap_{\substack{v_i \in \mathcal{V}_i \\ g_i(x, v_i) = 0}} \mathbb{R}^n \setminus \left(\partial^o g_i(\cdot, v_i)(x) \right) &= \mathbb{R}^n \setminus \left(\bigcup_{v_i \in \mathcal{V}_i, g_i(x, v_i) = \psi_i(x)} \partial^o g_i(\cdot, v_i)(x) \right) \\ &= \mathbb{R}^n \setminus \partial^o \psi_i(x). \end{aligned}$$

Finally, the robust Slater constraint qualification leads us to the following strict inequality

$$\psi_i(x_0) = \max\{g_i(x_0, v_i) : v_i \in \mathcal{V}_i\} < 0, \forall i = 1, \dots, m,$$

which means that the system $x \in \mathbb{R}^n$, $\psi_i(x) \leq 0$ ($i = 1, \dots, m$) satisfies the Slater's condition¹. Now applying [6, Proposition 2.2] and taking (1.1.1) into consideration, we obtain the desired results. $\square$

Remark 13. It should be noted that in Proposition 12 without robust Slater constraint qualification and $0 \notin \partial^o g_i(\cdot, v_i)(x)$ whenever $x \in F$, $i \in I(x)$, and $v_i \in \mathcal{V}_i(x)$, we easily obtain that if F is convex then

$$F \subseteq \{y \in \mathbb{R}^n : g_{ix}^o(x, v_i; y - x) \leq 0, \forall x \in F, \forall i \in I(x), \forall v_i \in \mathcal{V}_i(x)\}.$$

Furthermore, for every $x \in F$ one has

$$\partial^o g_i(\cdot, v_i)(x) \subseteq N(F, x) \text{ whenever } i \in I(x) \text{ and } v_i \in \mathcal{V}_i(x).$$

In order to establish a multiplier characterization for the robust optimal solution of (UP), we first recall a robust basic constraint qualification which was introduced in [5].

Definition 14. Let $x \in F$ be a robust feasible solution of (UP). The *robust basic constraint qualification* is satisfied at x if

$$N(F, x) = \bigcup_{\substack{\lambda_i \geq 0, v_i \in \mathcal{V}_i \\ \lambda_i g_i(x, v_i) = 0, i=1, \dots, m}} \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, v_i)(x).$$

Now the following theorem declares a result that the robust basic constraint qualification defined in Definition 14 is a necessary and sufficient constraint qualification of a robust optimal solution for the given problem, that is, the robust basic constraint qualification holds if and only if the Lagrange multiplier conditions are satisfied for a robust optimal solution.

Theorem 15 (Characterizing the robust basic constraint qualification). Suppose that for each $x \in F$ and $i \in I(x)$, the function g_i satisfies the conditions (C3) and (C4). Then, the following statements are equivalent:

- (i) the robust basic constraint qualification holds at $\bar{x} \in F$;
- (ii) for each real-valued convex-concave function f on $\mathbb{R}^n \times \mathcal{U}$, the following statements are equivalent:
 - (a) $\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(\bar{x}, u)$ for all $x \in F$,
 - (b) there exist $\bar{u} \in \mathcal{U}$, $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ such that

$$0 \in \partial f(\cdot, \bar{u})(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}), \quad \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \quad \forall i = 1, \dots, m, \quad (1.1.2)$$

and

$$f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u). \quad (1.1.3)$$

¹the system $x \in \mathbb{R}^n$, $g_i(x) \leq 0$ ($i = 1, \dots, m$) satisfies the Slater's condition if there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$ for all $i = 1, \dots, m$.

Proof. [(i) $\Rightarrow$ (ii)] Suppose that (i) holds. Let f be a real-valued convex-concave function on $\mathbb{R}^n \times \mathcal{U}$. Firstly, we assume that (a) holds. Then, $\bar{x}$ is a solution of the following constrained convex optimization problem:

$$\text{Minimize } \max_{u \in \mathcal{U}} f(x, u) \text{ subject to } x \in F,$$

which can be equivalently expressed as,

$$0 \in \partial(\max_{u \in \mathcal{U}} f(\cdot, u))(\bar{x}) + N(F, \bar{x}).$$

By (i), there are $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ such that

$$0 \in \partial(\max_{u \in \mathcal{U}} f(\cdot, u))(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}) \text{ and } \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \forall i = 1, \dots, m.$$

Then, it follows from Lemma 1 that there exists $\bar{u} \in \mathcal{U}$ such that (1.1.2) and (1.1.3) hold.

To prove sufficiency, assume that there exist $\bar{u} \in \mathcal{U}$, $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ such that (1.1.2) and (1.1.3) hold. According to (1.1.2), we can find $\xi \in \partial f(\cdot, \bar{u})(\bar{x})$ and $\eta_i \in \partial^o g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \dots, m$, such that

$$\xi + \sum_{i=1}^m \bar{\lambda}_i \eta_i = 0. \quad (1.1.4)$$

It stems from $\xi \in \partial f(\cdot, \bar{u})(\bar{x})$ and $\eta_i \in \partial^o g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \dots, m$, we get

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) \geq \langle \xi, x - \bar{x} \rangle \quad (1.1.5)$$

and

$$g_{ix}^o(\bar{x}, \bar{v}_i; x - \bar{x}) \geq \langle \eta_i, x - \bar{x} \rangle \quad \forall i = 1, \dots, m, \quad (1.1.6)$$

for any $x \in \mathbb{R}^n$. Multiplying each of inequalities in (1.4.5) by $\bar{\lambda}_i$ and summing up the obtained inequalities with (1.2.8), we obtain that, for all $x \in \mathbb{R}^n$,

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) + \sum_{i=1}^m \bar{\lambda}_i g_{ix}^o(\bar{x}, \bar{v}_i; x - \bar{x}) \geq \langle \xi + \sum_{i=1}^m \bar{\lambda}_i \eta_i, x - \bar{x} \rangle.$$

Taking (1.4.1) into account together with the condition $\bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$, $i = 1, \dots, m$, we deduce

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g_{ix}^o(\bar{x}, \bar{v}_i; x - \bar{x}) \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Note that for each $i \in I(\bar{x})$ with $g_i(\bar{x}, \bar{v}_i) \neq 0$, $\bar{\lambda}_i = 0$. So, we consider in the case of $g_i(\bar{x}, \bar{v}_i) = 0$ for $i \in I(\bar{x})$, and hence $\bar{v}_i \in \mathcal{V}_i(\bar{x})$. By Remark 13, the last inequality becomes

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) \geq 0 \text{ for all } x \in F.$$

Thus, together with $\max_{u \in \mathcal{U}} f(x, u) \geq f(x, \bar{u})$ for all $x \in \mathbb{R}^n$ and (1.1.3), we obtain

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\bar{x}, u) \geq 0, \quad \forall x \in F.$$

It means that $\bar{x}$ is a robust optimal solution of problem (UP).

[(ii) $\Rightarrow$ (i)] The proof is similar to the one in [35, Theorem 3.1], and so is omitted. $\square$

In the uncertainty free case, we can easily obtain the following result, which was obtained by Yamamoto and Kuroiwa in [23].

Corollary 16. [23, Theorem 3.2] Let $\bar{x} \in F' := \{x \in \mathbb{R}^n : g_i(x) \leq 0, \forall i = 1, \dots, m\}$ be a feasible solution, $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, m$, be locally Lipschitz on $\mathbb{R}^n$. Assume further that for any $x \in F'$ and any $i = 1, \dots, m$ such that $g_i(x) = 0$, the function g_i is regular, and F' is convex. Then the following statement are equivalent:

$$(i) \quad N(F', \bar{x}) = \bigcup_{\substack{\lambda_i \geq 0 \\ \lambda_i g_i(\bar{x}) = 0, i=1, \dots, m}} \sum_{i=1}^m \lambda_i \partial^o g_i(\bar{x});$$

(ii) for each real-valued function f on $\mathbb{R}^n$, the following statements are equivalent:

(a) $f(x) \geq f(\bar{x})$ for all $x \in F'$;

(b) there exist $\bar{\lambda}_i \geq 0$, $i = 1, \dots, m$ such that

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\bar{x}) \text{ and } \bar{\lambda}_i g_i(\bar{x}) = 0, \forall i = 1, \dots, m.$$

Example 17. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $v_1 := (v_{1,1}, v_{1,2})$, $v_2 := (v_{2,1}, v_{2,2})$, $v_3 := (v_{3,1}, v_{3,2})$, $\mathcal{V}_1 := \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2 \leq 1\}$, $\mathcal{V}_2 := [0, 1] \times [1, 2]$, $\mathcal{V}_3 := [2, 3] \times [0, 1]$,

$$g_1(x, v_1) := v_{1,1}x_1 + v_{1,2}x_2 - x_1^3 - 2,$$

$$g_2(x, v_2) = -v_{2,1}x_1^3 + v_{2,2} \max\{-x_2, -x_2^3\},$$

$$g_3(x, v_3) = v_{3,1}x_1 + v_{3,2}x_2,$$

$$F := \{x \in \mathbb{R}^2 : g_1(x, v_1) \leq 0, g_2(x, v_2) \leq 0, g_3(x, v_3) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, 2, 3\}$$

and $\bar{x} := (0, 0)$. Then $F = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} - x_1^3 - 2 \leq 0, 2x_1 + x_2 \leq 0, -x_1 - x_2 \leq 0\}$, $I(\bar{x}) = \{2, 3\}$, $\partial^o g_2(\cdot, v_2)(\bar{x}) = \{0\} \times [-v_{2,2}, 0]$ and $\partial^o g_3(\cdot, v_3)(\bar{x}) = \{(v_{3,1}, v_{3,2})\}$. It can be observed that

$$N(F, \bar{x}) = \text{cone}\{(-1, -1), (2, 1)\}$$

and

$$\bigcup_{\substack{\lambda_i \geq 0, v_i \in \mathcal{V}_i \\ \lambda_i g_i(\bar{x}, v_i) = 0, i=1,2,3}} \sum_{i=1}^3 \lambda_i \partial^o g_i(\cdot, v_i)(\bar{x}) = \text{cone}\{(0, -1), (2, 1)\}.$$

Hence, we have the condition (i) of Theorem 45 does not hold. Thus for some convex-concave function $f : \mathbb{R}^2 \times \mathcal{U} \rightarrow \mathbb{R}$, it is impossible to characterize a sufficient condition for robust optimal solution for the following uncertain problem by using Theorem 45,

$$\text{minimize } f(x, u) \text{ subject to } x \in \mathbb{R}^2, g_i(x, v_i) \leq 0, i = 1, 2, 3.$$

Actually, let $u := (u_1, u_2)$ be an uncertain parameter belong to uncertainty set $\mathcal{U} := \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}$, and $f(x, u) := e^{x_1} - u_1x_1 - u_2x_2$. Selecting $\bar{u} := (1, 0)$, $\bar{v}_1 := (1, 0)$, $\bar{v}_2 := (1, 1)$, $\bar{v}_3 := (2, 0)$, $\bar{\lambda}_1 := 0$, $\bar{\lambda}_2 := 1$ and $\bar{\lambda}_3 := 1$ we obtain $\bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$ for all $i = 1, 2, 3$,

$$f(\bar{x}, \bar{u}) = 1 = \max_{u \in \mathcal{U}} f(\bar{x}, u)$$

and

$$(0, 0) \in \{(-2, 0)\} + \{0\} \times [-1, 0] + \{(2, 0)\} = \partial f(\cdot, \bar{u})(\bar{x}) + \sum_{i=1}^3 \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}).$$

However, by taking $x := (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \in F$, $\max_{u \in \mathcal{U}} f(\bar{x}, u) = e^{1/2} - 1 < 1 = f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u)$ which shows $\bar{x}$ is not a minimizer of $\max_{u \in \mathcal{U}} f(\cdot, u)$ on F . $\square$

In this section, we will establish some characterizations of robust optimal solution set in terms of a given robust solution point of the given problem.

We begin by recalling the following constrained convex optimization problem in the face of data uncertainty (UP):

$$\min_{x \in \mathbb{R}^n} \{f(x, u) : g_i(x, v_i) \leq 0, i = 1, \dots, m\}, \quad (\text{UP})$$

where $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ is a convex-concave function, the functions $g_i, i \in I$, satisfy the condition (C1) and (C2), $g_i(x, \cdot) : \mathcal{V}_i \rightarrow \mathbb{R}, i \in I$, are concave functions for any $x \in \mathbb{R}^n$, and the robust feasible set F is convex. Assume that the robust solution set of the problem (UP), denoted by

$$S := \{a \in F : \max_{u \in \mathcal{U}} f(a, u) \leq \max_{u \in \mathcal{U}} f(x, u), \forall x \in F\},$$

is nonempty. In what follows, for any given $y \in \mathbb{R}^n$, $\lambda := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m$, $u \in \mathcal{U}$, $v_i \in \mathcal{V}_i$, $i \in I$ and $v := (v_1, \dots, v_m)$, we introduce the so-called *pseudo Lagrangian-type function* $L^P(\cdot, y, \lambda, u, v)$ by, for all $x \in \mathbb{R}^n$,

$$L^P(x, y, \lambda, u, v) := f(x, u) + \sum_{i \in I(y)} \lambda_i g_{ix}^o(y, v_i; x - y).$$

Now, show that the pseudo Lagrangian-type function associated with a Lagrange multiplier vector and uncertainty parameters according to a solution is constant on S .

Proposition 18. *Assume all conditions of Theorem 45 hold. Let $a \in S$ be a robust optimal solution fulfilling the robust basic constraint qualification. Then there exist a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$, and uncertainty parameters $u^a \in \mathcal{U}$, $v_i^a \in \mathcal{V}_i, i \in I$, such that for any $x \in S$, $\lambda_i^a g_{ix}^o(a, v_i^a; x - a) = 0, \forall i \in I(a)$, $f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)$, and $L^P(\cdot, a, \lambda^a, u^a, v^a)$ is constant on S .*

Proof. It follows from $a \in S$ and Theorem 45 that there exist a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$, and uncertainty parameters $u^a \in \mathcal{U}$, $v_i^a \in \mathcal{V}_i, i \in I$, satisfying the conditions (1.1.2) and (1.1.3). Then, it stems from the fact that $\partial^o g_i(\cdot, v_i^a)(a) = \partial g_{ix}^o(a, v_i^a; \cdot - a)(a)$ for all $i \in I(a)$ and (1.1.2), we get

$$0 \in \partial f(\cdot, u^a)(a) + \sum_{i \in I(a)} \lambda_i^a \partial g_{ix}^o(a, v_i^a; \cdot - a)(a) \subseteq \partial L^P(\cdot, a, \lambda^a, u^a, v^a)(a),$$

which is noting else than

$$f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \geq f(a, u^a) = \max_{u \in \mathcal{U}} f(a, u) \text{ for all } x \in \mathbb{R}^n. \quad (1.1.7)$$

Notice that

$$\max_{u \in \mathcal{U}} f(x, u) = \max_{u \in \mathcal{U}} f(a, u), \text{ for any } a \in S \text{ and } x \in S, \quad (1.1.8)$$

and taking this into account, (1.1.7) deduces $\sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \geq 0$, for any $x \in S$. Let us notice that for indices $i \in I(a)$ such that $\lambda_i^a > 0$, we have $g_i(a, v_i^a) = 0$, and consequently, $v_i^a \in \mathcal{V}_i(a)$. This in turn, by Remark 13, implies that

$$\lambda_i^a g_{ix}^o(a, v_i^a; x - a) = 0, \forall i \in I(a). \quad (1.1.9)$$

Now, we prove that

$$f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u), \text{ for any } x \in S. \quad (1.1.10)$$

In fact, by (1.1.7) and (1.4.9), we get the assertion

$$\max_{u \in \mathcal{U}} f(x, u) \geq f(x, u^a) \geq \max_{u \in \mathcal{U}} f(a, u).$$

This together with (1.4.8), (1.1.10) holds. Therefore, for any $x \in S$, (1.1.3), (1.4.8), (1.4.9) and (1.1.10) entail

$$\begin{aligned} L^P(x, a, \lambda^a, u^a, v^a) &= f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \\ &= f(x, a) = \max_{u \in \mathcal{U}} f(x, u) = \max_{u \in \mathcal{U}} f(a, u) = f(a, u^a), \end{aligned}$$

showing that $L^P(\cdot, a, \lambda^a, u^a, v^a)$ is constant on S , and this completes the proof. $\square$

Remark 19. It is worth noting that if $g_i(\cdot, v_i)$, $i \in I$, are convex functions for any $v_i \in \mathcal{V}_i$ then, for each $i \in I$, Proposition 18 gives

$$\lambda_i^a g_i(x, v_i^a) - \lambda_i^a g_i(a, v_i^a) \geq \lambda_i^a g'_i(a, v_i^a; x - a) = \lambda_i^a g_i^o(a, v_i^a; x - a) = 0 \text{ for any } x \in S.$$

This together with $x \in F$ and $\lambda_i^a g_i(a, v_i^a) = 0$, $i \in I$, arrives $\lambda_i^a g_i(x, v_i^a) = 0$, $i \in I$. Furthermore, it yields

$$\begin{aligned} L^P(x, a, \lambda^a, u^a, v^a) &= f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \\ &= f(x, u^a) \\ &= f(x, u^a) + \sum_{i=1}^m \lambda_i^a g_i(x, v_i^a), \quad \forall x \in S. \end{aligned}$$

This shows that pseudo Lagrangian-type function collapses to the well-known Lagrangian-type function on the robust solution set S .

In the sequel, we are now in a position to establish the characterizations of the robust solution set for problem (UP) in terms of convex subdifferentials, Clarke subdifferentials and Lagrange multipliers. But before doing so it will thus be convenient to denote the following :

$$\begin{aligned} \tilde{I}(a) &:= \{i \in I(a) : \lambda_i^a > 0\}, \\ C(x) &:= \{\xi \in \partial f(\cdot, u^a)(a) : \langle \xi, x - a \rangle \geq 0\} \text{ for any given } x \in F. \end{aligned}$$

Theorem 1 (Characterizing the robust solution set). *Assume all conditions of Theorem 45 hold. Let $a \in S$ be a robust optimal solution fulfilling the robust basic constraint qualification. Then there exist a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$, and uncertainty parameters $u^a \in \mathcal{U}$, $v_i^a \in \mathcal{V}_i$, $i \in I$, such that the robust solution set for the problem (UP) is characterized by*

$$S = S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = S_7,$$

where

$$\begin{aligned}
S_1 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\
&\quad \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \cap \partial f(\cdot, u^a)(a); \\
&\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\
S_2 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\
&\quad \langle \zeta, a - x \rangle \geq 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \cap \partial f(\cdot, u^a)(a); \\
&\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\
S_3 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\
&\quad \langle \xi, x - a \rangle = \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); \\
&\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\
S_4 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\
&\quad \langle \xi, x - a \rangle = \langle \zeta, a - x \rangle \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); \\
&\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\
S_5 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\
&\quad \langle \xi, x - a \rangle \leq \langle \zeta, a - x \rangle \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); \\
&\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\
S_6 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\
&\quad \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x); \\
&\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\
S_7 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\
&\quad \langle \zeta, a - x \rangle \geq 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x); \\
&\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}.
\end{aligned}$$

Proof. Evidently, the following containments hold:

$$\begin{aligned}
S_1 &\subseteq S_2 \subseteq S_7, \\
S_1 &\subseteq S_6 \subseteq S_7, \\
S_1 &\subseteq S_3 \subseteq S_4 \subseteq S_5 \subseteq S_7.
\end{aligned}$$

Hence, we only have to show that $S \subseteq S_1$ and $S_7 \subseteq S$. In order to establish $S \subseteq S_1$, let $x \in S$ be arbitrarily given. It follows from (1.1.2), we therefore obtain vectors $\zeta \in \partial f(\cdot, u^a)(a)$ and $\xi_i \in \partial^o g_i(\cdot, v_i^a)(a)$, $i \in I(a)$, such that

$$\zeta + \sum_{i \in I(a)} \lambda_i^a \xi_i = 0 \quad (1.1.11)$$

(since $\lambda_i^a = 0$ for $i \notin I(a)$). According to $\zeta \in \partial f(\cdot, u^a)(a)$, $\xi_i \in \partial^o g_i(\cdot, v_i^a)(a)$, $i \in I(a)$, and $x, a \in S$, one has

$$f(x, u^a) - f(a, u^a) \geq \langle \zeta, x - a \rangle \quad (1.1.12)$$

and

$$g_{ix}^o(a, v_i^a; x - a) \geq \langle \xi_i, x - a \rangle, \forall i \in I(a). \quad (1.1.13)$$

Once we have shown, in Proposition 18, that $\lambda_i^a g_{ix}^o(a, v_i^a; x - a) = 0, \forall i \in I(a)$, after multiplying both sides of (1.4.12) by $\lambda_i^a, i \in I(a)$ we get

$$0 \geq \langle \lambda_i^a \xi_i, x - a \rangle, \forall i \in \tilde{I}(a).$$

Summing up these inequalities and using (1.4.10) we obtain that

$$0 \geq \left\langle \sum_{i \in I(a)} \lambda_i^a \xi_i, x - a \right\rangle = \langle -\zeta, x - a \rangle. \quad (1.1.14)$$

Again, it follows from Proposition 18 that

$$f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u), \quad (1.1.15)$$

and for each $i \in \tilde{I}(a)$, $\max_{\eta_i \in \partial^o g_i(\cdot, v_i^a)(a)} \langle \eta_i, x - a \rangle = g_{ix}^o(a, v_i^a; x - a) = 0$, the latter which in turn leads to there exists $\eta_i \in \partial^o g_i(\cdot, v_i^a)(a)$ such that

$$\langle \eta_i, x - a \rangle = 0.$$

On the one hand, taking (1.1.3) and (1.4.16) into account (1.4.11) we obtain

$$\langle \zeta, x - a \rangle \leq f(x, u^a) - f(a, u^a) = \max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(a, u) = 0.$$

This together with (1.4.14) arrives at

$$\langle \zeta, x - a \rangle = 0.$$

Now, we only need to prove that $\zeta \in \partial f(\cdot, u^a)(x)$. In fact, for any $y \in \mathbb{R}^n$,

$$\begin{aligned} f(y, u^a) - f(x, u^a) &= f(y, u^a) - f(a, u^a) \\ &\geq \langle \zeta, y - a \rangle \\ &= \langle \zeta, y - x \rangle + \langle \zeta, x - a \rangle = \langle \zeta, y - x \rangle, \end{aligned}$$

which means $\zeta \in \partial f(\cdot, u^a)(x)$ and so, $x \in S_1$. This proves $S \subseteq S_1$.

To obtain $S_7 \subseteq S$, we now let x be arbitrary point of S_7 . It follows that $x \in F$, and it is easy to see that

$$\max_{u \in \mathcal{U}} f(a, u) - \max_{u \in \mathcal{U}} f(x, u) = f(a, u^a) - f(x, u^a) \geq \langle \zeta, a - x \rangle \geq 0.$$

The last inequality together with the fact that $a \in S$ gives $x \in S$, and the proof is complete. $\square$

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1.2 Necessary and sufficient KKT optimality conditions in non-convex multi-objective optimization problems with cone constraints

Multi-objective (vector-valued) optimization is a subject of mathematical programming that extensively studied and applied in various decision-making contexts like economics, human decision making, control engineering, transportation and many others. We refer the reader to [24, 18, 19]. For comprehensive treatment of theoretical issues concerning multi-objective optimization can be found in [17, 13, 9, 2]. In the multi-objective setting, the scalar concept of optimality does not apply directly due to the fact that all the objectives can not be simultaneously optimized with a single solution. To this effect, we must decide which objective to improve, and so compromise solutions must be considered. In this way, we refer to a weak Pareto optimum (resp. Pareto optimum [15]) which usually uses coordinate-wise ordering (induced by the positive orthant as ordering cone) to examine the objective vectors. However, in real-world multi-objective problems concerning especially fractional programming even computational aspects of Pareto optimum, not only the coordinate-wise ordering appears but also the cone defining the lexicographic partial order is of practical interest [7]. This being a reason, study of multi-objective optimization problems involving general ordering cones has gained attention. Precisely stated, in this paper we will mainly focus on the problem of multi-objective optimization problem with cone constraint:

$$\begin{aligned} K - \text{Minimize } \mathbf{f}(\mathbf{x}) \\ \text{subject to } \mathbf{x} \in \mathbb{R}^n, -\mathbf{g}(\mathbf{x}) \in Q, \end{aligned} \tag{MOP}$$

where $\mathbf{f} := (f_1, \dots, f_p)^T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $\mathbf{g} := (g_1, \dots, g_m)^T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, are differentiable functions, K and Q are closed convex cones with non-empty interiors in $\mathbb{R}^p$ and $\mathbb{R}^m$, respectively. Let

$$\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{g}(\mathbf{x}) \in Q\} \tag{1.2.1}$$

be the set of all feasible solutions of (MOP). The notation “ K – Minimize ” refers to the *weak Pareto* (resp. *Pareto optimum*) with respect to the ordering cone K for the problem (MOP), namely a point $\mathbf{x}^* \in \mathcal{X}$ such that for every $\mathbf{x} \in \mathcal{X}$, $\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \notin \text{int}K$ (resp. $\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \notin K \setminus \{\mathbf{0}\}$).

Recall that a feasible point $\mathbf{x}^* \in \mathcal{X}$ is said to be a *KKT point* if there exist multipliers $\boldsymbol{\lambda} \in K^* \setminus \{\mathbf{0}\}$ and $\boldsymbol{\mu} \in Q^*$ such that the following Karush-Kuhn-Tucker (KKT) optimality conditions hold:

$$(i) \quad \boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0};$$

$$(ii) \quad \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0,$$

when $K^*, K^* := \{\mathbf{z} \in \mathbb{R}^p : \mathbf{x}^T \mathbf{z} \geq 0 \text{ for all } \mathbf{x} \in K\}$, denotes the *dual (positive polar) cone* of K . In this paper, the above feasible point $\mathbf{x}^*$ is also called a *non-trivial KKT point* if the corresponding $\boldsymbol{\mu}$ is a non-zero vector.

The search for weak Pareto (resp. Pareto optimum) to (MOP) has been carried out through the study of the KKT optimality conditions provided that some constraint qualifications hold, and of the convexity of the functions $\mathbf{f}$ and $\mathbf{g}$. In the current work, with the introduction of scalar convex optimization without convexity of constraint functions by Lasserre [12], Suneja et al. [21] successfully obtained a new direction to the search for weak Pareto (resp. Pareto optimum) of multi-objective convex optimization problem. The authors showed that even if the convex feasible set is not necessarily described by cone-convex constraint, the *Slater-type cone constraint qualification*² and additionally a mild non-degeneracy conditions (see Assumption 1 in the next section) render the KKT optimality conditions both necessary and sufficient. For non-smooth versions of Lasserre’s result, see [6, 14]. A more recent exhaustive treatment of constraint qualifications can be found in [23, 5].

Recently, Ho [8] went further in the case of scalar differentiable problems but moreover without the convexity of the feasible set and of the functions that are involved, and necessary and sufficient KKT optimality conditions are then considered in relation to the presence of convexity of the level sets of objective function. The aim of this brief paper is to extend the Ho’s results to the problem (MOP). And in order to do that, we are going to consider the feasible point $\mathbf{x}^*$ under the question in which satisfies the following property:

$$\forall \mathbf{x} \in \mathcal{X}, \exists t_n \rightarrow 0^+ \text{ such that } \mathbf{x}^* + t_n(\mathbf{x} - \mathbf{x}^*) \in \mathcal{X}. \quad (1.2.2)$$

Admittedly, some non-convex sets that satisfy the condition (1.4.1) will illustrate in Example 31 in Section 3. Further, we will see that the condition to impose on the cone constraints in this paper suffers from of Ho’s results in the point that the *Slater’s condition*³ is non-necessarily to be assumed. Actually, in order to prove our results we need to assume only non-degeneracy at a point $\mathbf{x}^*$. The connections among non-degeneracy condition, Slater-type cone constraint qualification, and Slater’s condition are investigated ones.

The rest of the paper is organized as follows. In Sect. ?? we recall some basic definitions and points out important results that will be used later in the paper. Section ?? presents relationships among constraint qualifications of multi-objective optimization problem (MOP) over cone constraint (1.2.1) and establishes necessary and sufficient KKT optimality conditions for a feasible point under the question to be a weak Pareto minimum of (MOP). We finally give sufficient

²The feasible set $\mathcal{X}$ as in (1.2.1) is said to satisfy *Slater-type cone constraint qualification* [9] at $\mathbf{x} \in \mathcal{X}$ if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{g}(\mathbf{x}) + \nabla \mathbf{g}(\mathbf{x})(\hat{\mathbf{x}} - \mathbf{x}) \in -\text{int}Q$.

³The feasible set $\mathcal{X}$ as in (1.2.1) is said to satisfy *Slater’s condition* if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $-\mathbf{g}(\hat{\mathbf{x}}) \in \text{int}Q$.

conditions for guaranteeing a weak Pareto minimum to be a Pareto minimum of the problem (MOP).

In this section, we briefly overview some notations, basic definitions, and preliminary results which will be used throughout the paper. All spaces under consideration are n -dimensional Euclidean space $\mathbb{R}^n$. All vectors are considered to be column vectors which can be transposed to a row vector by the superscript T . A nonempty subset K of $\mathbb{R}^p$ is said to be a *cone* if $tK \subseteq K$ for all $t \geq 0$. For a set A in $\mathbb{R}^n$, by $\text{int}A$ (resp. $\text{cone}A$) we will denote the *interior* (resp. *conical hull*) of the set A . We say A is *convex* whenever $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in A$ for all $t \in [0, 1]$, $\mathbf{x}_1, \mathbf{x}_2 \in A$. The *normal cone* to a closed convex set A at $\mathbf{x} \in A$, denoted by

$$N(A, \mathbf{x}) := \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}^T(\mathbf{y} - \mathbf{x}) \leq 0, \forall \mathbf{y} \in A\}.$$

A set $A \subseteq \mathbb{R}^n$ is called *strictly convex* at $\mathbf{x} \in A$ if $\mathbf{u}^T(\mathbf{y} - \mathbf{x}) < 0$ for every $\mathbf{y} \in A \setminus \{\mathbf{x}\}$ and $\mathbf{u} \in N(A, \mathbf{x}) \setminus \{\mathbf{0}\}$. It is worth noting that the strict convexity of A at some point $\mathbf{x}$ does not guarantee the convexity of A . For instance, the set $A := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 > 0\} \cup \{(0, 0)^T\}$ is strictly convex at $(0, 0)^T$ while A is not convex.

For a closed convex cone $K \subseteq \mathbb{R}^p$, a vector valued function $\mathbf{f} := (f_1, \dots, f_p)^T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is said to be *K-convex* (*K-pseudoconvex* [1, 22]) at a point $\mathbf{x}^* \in \mathbb{R}^n$ if for every $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^*) - \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \in K$$

(resp. $-\nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \notin \text{int}K \Rightarrow \mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \in \text{int}K$), where $\nabla \mathbf{f}(\mathbf{x}^*) := (\nabla f_1(\mathbf{x}^*), \dots, \nabla f_p(\mathbf{x}^*))^T$ is the $p \times n$ Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}^*$ and for each $k = 1, 2, \dots, p$, $\nabla f_k(\mathbf{x}^*) = \left(\frac{\partial f_k(\mathbf{x}^*)}{\partial x_1}, \frac{\partial f_k(\mathbf{x}^*)}{\partial x_2}, \dots, \frac{\partial f_k(\mathbf{x}^*)}{\partial x_n} \right)^T$ is the $n \times 1$ gradient vector of f_k at $\mathbf{x}^*$. If $\mathbf{f}$ is *K-convex* (*K-pseudoconvex*) at every point $\mathbf{x}^* \in \mathbb{R}^n$ then $\mathbf{f}$ is said to be *K-convex* (resp. *K-pseudoconvex*) on $\mathbb{R}^n$.

Now, let us recall the following results which will be useful in the sequel.

Lemma 20. [9, Lemma 3.21, p. 77] Let K be a convex cone in $\mathbb{R}^p$.

(i) If K is closed, then

$$K = \{\mathbf{x} \in \mathbb{R}^p : \mathbf{x}^T \mathbf{z} \geq 0 \text{ for all } \mathbf{z} \in K^*\}.$$

(ii) If $\text{int}K \neq \emptyset$, then

$$\text{int}K = \{\mathbf{x} \in \mathbb{R}^p : \mathbf{x}^T \mathbf{z} > 0 \text{ for all } \mathbf{z} \in K^* \setminus \{\mathbf{0}\}\}.$$

Lemma 21. [20, Lemma 1] Consider the problem (MOP). If $\mathbf{x}^* \in \mathcal{X}$ is a weak Pareto minimum of (MOP), then there exist $\boldsymbol{\lambda} \in K^*$ and $\boldsymbol{\mu} \in Q^*$ not both zero such that

$$\left(\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \right) (\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in \mathbb{R}^n$$

and

$$\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0.$$

Now, we recall the following important result which can be found in [11] and will play a key role in deriving a feasible point to be a weak Pareto minimum as well as a Pareto minimum of (MOP).

Proposition 22. [11, Proposition 2.2.] Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable at $\mathbf{x}^*$ with $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$. Then:

(i) $N(L_f^<(\mathbf{x}^*), \mathbf{x}^*) = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} = r \nabla f(\mathbf{x}^*), \text{ for some } r \geq 0\}$ provided that

$$L_f^<(\mathbf{x}^*) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < f(\mathbf{x}^*)\}$$

is convex.

(ii) $N(L_f(\mathbf{x}^*), \mathbf{x}^*) = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} = r \nabla f(\mathbf{x}^*), \text{ for some } r \geq 0\}$ provided that

$$L_f(\mathbf{x}^*) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq f(\mathbf{x}^*)\}$$

is convex.

We conclude this section by the following useful lemma, which will be crucial in the sequel.

Lemma 23. Let $\mathcal{X}$ be as in (1.2.1). Assume that the condition (1.4.1) is satisfied at a feasible point $\mathbf{x}^* \in \mathcal{X}$. Then for every $\boldsymbol{\mu} \in Q^* \setminus \{\mathbf{0}\}$ for which $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$, one has

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) \leq 0 \text{ for all } \mathbf{v} \in \mathcal{X}.$$

Proof. Suppose on contrary that there exists $\mathbf{v} \in \mathcal{X}$ such that $(\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*))(\mathbf{v} - \mathbf{x}^*) > 0$. Then, by the first order approximation together with the condition (1.4.1), we can find some t_n small enough such that

$$\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) = \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + t_n \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) + o(t_n) > 0, \quad (1.2.3)$$

where $\frac{o(t)}{t} \rightarrow 0$ as $t \rightarrow 0^+$, and $\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*) \in \mathcal{X}$. The latter means that $-\mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) \in Q$ and consequently, $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) \leq 0$, which contradicts (1.2.3). $\square$

In this section, we present the constraint qualifications that are used to derive the KKT conditions for (MOP) and their connections. Afterward, we will establish necessary and sufficient KKT optimality conditions for a weak Pareto minimum of (MOP). In addition, we also establish sufficient conditions for guaranteeing a weak Pareto minimum to be a Pareto minimum of the problem (MOP).

At first, we recall one of constraint qualifications the so-called *non-degeneracy condition* at some feasible point $\mathbf{x}^* \in \mathcal{X}$ in the vector setting, which introduced in [21].

Assumption 1 : (Non-degeneracy condition [21]) Consider (MOP), for every $\boldsymbol{\mu} \in Q^* \setminus \{\mathbf{0}\}$,

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \neq \mathbf{0} \text{ whenever } \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0.$$

Remark 24 (Sufficient condition for non-degeneracy condition to be valid). Note that if the Slater-type cone constraint qualification at $\mathbf{x}^*$ holds, then the non-degeneracy condition is satisfied at $\mathbf{x}^*$. Indeed, if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q$, then for every $\boldsymbol{\mu} \in Q^* \setminus \{\mathbf{0}\}$ for which $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$, one has $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) = \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) < 0$ which implies that $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \neq \mathbf{0}$.

The following example shows that the Slater-type cone constraint qualification is not implied by the non-degeneracy condition alone.

Example 25. Let $\mathbf{x} := (x_1, x_2)^T \in \mathbb{R}^2$, $Q := \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $\mathbf{g}(\mathbf{x}) := (x_2^3 + x_2 - x_1, x_1 - x_2)^T$. We see that $\mathbf{g}(-3, -2) = (-7, -1)^T \in -\text{int}Q$, that is, Slater's condition holds. Also, one has $\nabla \mathbf{g}(\mathbf{x}) = \begin{pmatrix} -1 & 3x_2^2 + 1 \\ 1 & -1 \end{pmatrix}$ and a short calculation shows that the

non-degeneracy holds at $\mathbf{x}^* := (0, 0)^T \in \mathcal{X}$, while the Slater-type cone constraint qualification is invalid at $\mathbf{x}^*$. In fact, we can not find out $\hat{\mathbf{x}} := (\hat{x}_1, \hat{x}_2)^T \in \mathbb{R}^2$ such that

$$\begin{pmatrix} -\hat{x}_1 + \hat{x}_2 \\ \hat{x}_1 - \hat{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q.$$

□

Remark 26. The Slater's condition can also be guaranteed by the Slater-type cone constraint qualification at some point $\mathbf{x}^*$ as well. To see this, it follows from the Slater-type cone constraint qualification that $\nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q - \mathbf{g}(\mathbf{x}^*)$ for some $\hat{\mathbf{x}} \in \mathbb{R}^n$. This together with the fact that

$$\frac{\mathbf{g}(\mathbf{x}^* + t(\hat{\mathbf{x}} - \mathbf{x}^*)) - \mathbf{g}(\mathbf{x}^*)}{t} = \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) + o(t),$$

where $\frac{o(t)}{t} \rightarrow \mathbf{0}$ as $t \rightarrow 0^+$, for some $t_0 > 0$ sufficiently small, it holds

$$\mathbf{g}(\mathbf{x}^* + t_0(\hat{\mathbf{x}} - \mathbf{x}^*)) \in (1 - t_0)\mathbf{g}(\mathbf{x}^*) - t_0\text{int}Q \subseteq -\text{int}Q.$$

Hence, the Slater's condition has been justified.

Now, we present some sufficient conditions for the Slater-type cone constraint qualification to be valid.

Theorem 2. Let $\mathcal{X}$ be as in (1.2.1). Assume that the Slater's condition holds and the condition (1.4.1) is satisfied at a feasible point $\mathbf{x}^* \in \mathcal{X}$. If the non-degeneracy condition holds at $\mathbf{x}^*$, then the Slater-type cone constraint qualification also holds at $\mathbf{x}^*$.

Proof. Suppose that the non-degeneracy condition holds at $\mathbf{x}^*$. Assume on contrary that for every $\mathbf{x} \in \mathbb{R}^n$, one has $\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \notin -\text{int}Q$, equivalently,

$$-[\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbb{R}^n - \mathbf{x}^*)] \cap \text{int}Q = \emptyset.$$

So, by the Eidelheit separation theorem, there exists $\boldsymbol{\mu} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ such that

$$\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + \boldsymbol{\mu}^T \mathbf{y} \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall \mathbf{y} \in Q. \quad (1.2.4)$$

By taking $\mathbf{x} = \mathbf{x}^*$ and $\mathbf{y} = \mathbf{0}$ in (1.2.4), we would have $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$. Hence, with regard to (1.2.4) with $\mathbf{x} = \mathbf{x}^*$, we get $\boldsymbol{\mu} \in Q$. Therefore, in view of (1.2.4), we find a vector $\boldsymbol{\mu} \in Q^* \setminus \{\mathbf{0}\}$ with $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$ such that

$$\hat{\boldsymbol{\mu}}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \quad (1.2.5)$$

On the other hand, by assumption, there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $-\mathbf{g}(\hat{\mathbf{x}}) \in \text{int}Q$. Then, since $\mathbf{g}$ is continuous at $\hat{\mathbf{x}}$, there exists $r > 0$ such that $\mathbf{g}(\hat{\mathbf{x}} + r\mathbf{u}) \subseteq -Q$ for all $\mathbf{u} \in \square := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$. Consequently, $\hat{\mathbf{x}} + r\mathbf{u} \in \mathcal{X}$ for all $\mathbf{u} \in \square$. So, as $\mathbf{x}^* \in \mathcal{X}$ and $\mathbf{x}^*$ satisfies the condition (1.4.1), we conclude from Lemma 23 that

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} + r\mathbf{u} - \mathbf{x}^*) \leq 0, \quad \forall \mathbf{u} \in \square. \quad (1.2.6)$$

In particular, put $\mathbf{u} = \mathbf{0} \in \square$, one has $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \leq 0$. Thus, with regard to (1.2.5), $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) = 0$, and hence we deduce from (1.2.6) that

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)\mathbf{u} \leq 0, \quad \forall \mathbf{u} \in \square.$$

So, $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)$ must ultimately be zero vector, which contradicts the validity of non-degeneracy condition at $\mathbf{x}^*$. □

Remark 27. In the absence of the condition (1.4.1) at $\mathbf{x}^*$, the validity of both Slater and the non-degeneracy conditions at $\mathbf{x}^*$ does not guarantee the validity of Slater-type cone constraint qualification at $\mathbf{x}^*$. We note that, in Example 25, the condition (1.4.1) is invalid at $\mathbf{x}^* = (0, 0)^T$.

Remark 28. In the case of $Q = \mathbb{R}_+^m := \{(x_1, x_2, \dots, x_m)^T \in \mathbb{R}^m : x_i \geq 0, \forall i = 1, \dots, m\}$, non-degeneracy conditions at $\mathbf{x}^*$ can be view as the *Mangasarian-Fromovitz constraint qualification*⁴ at $\mathbf{x}^*$ and non-degeneracy conditions at $\mathbf{x}^*$ in [12, 8] as well. Indeed,

$$\begin{aligned} & \exists \mathbf{v} \in \mathbb{R}^n \text{ such that } \nabla g_i(\mathbf{x}^*)^T \mathbf{v} < 0, \forall i \in I(\mathbf{x}^*) \\ \Leftrightarrow & \mathbf{0} \notin \text{cone}\{\nabla g_i(\mathbf{x}^*) : i \in I(\mathbf{x}^*)\} \\ \Leftrightarrow & \forall \boldsymbol{\mu} := (\mu_1, \mu_2, \dots, \mu_m)^T \in \mathbb{R}_+^m \setminus \{\mathbf{0}\} \text{ with } \mu_i g_i(\mathbf{x}^*) = 0, i = 1, 2, \dots, m, \text{ one has } \sum_{i=1}^m \mu_i \nabla g_i(\mathbf{x}^*) \neq \mathbf{0}, \end{aligned}$$

and for each $i \in \{1, 2, \dots, m\}$, by taking $\boldsymbol{\mu} := \mathbf{e}_i$, where $\mathbf{e}_i$ is the unit vector in $\mathbb{R}^m$ with the i th component is 1 and the others 0, one has $\nabla g_i(\mathbf{x}^*) \neq \mathbf{0}$ whenever $i \in I(\mathbf{x}^*)$. Note that Slater-type cone constraint qualification at $\mathbf{x}^*$ also is equivalent to the *Robinson constraint qualification*⁵ at $\mathbf{x}^*$ [4, Lemma 2.99, p. 69]. Then, as the considered set $\{\mathbf{x} \in \mathbb{R}^n, g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$ is not necessarily convex, one can notice that Theorem 2 extends [5, Theorem 2.1] to non-convex setting on the set $\{\mathbf{x} \in \mathbb{R}^n, g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$.

Now, we are in the position to give necessary and sufficient KKT optimality conditions for a weak Pareto minimum of (MOP).

Theorem 3. Consider the problem (MOP) and let both Assumption 1 and the condition (1.4.1) be satisfied at a feasible point $\mathbf{x}^*$.

- (i) If $\mathbf{x}^*$ is a weak Pareto minimum of (MOP) then $\mathbf{x}^*$ is a KKT point.
- (ii) Conversely, if $\mathbf{x}^*$ is a non-trivial KKT point with multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, and $L_{\boldsymbol{\lambda}^T \mathbf{f}}^<(\mathbf{x}^*)$ is convex then $\mathbf{x}^*$ is a weak Pareto minimum of (MOP).

Proof. (i) Let $\mathbf{x}^* \in \mathcal{X}$ be a weak Pareto minimum of (MOP). By Lemma 21, there exist $\boldsymbol{\lambda} \in K^*$ and $\boldsymbol{\mu} \in Q^*$ not both zero such that $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$ and

$$\left(\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \right) (\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in \mathbb{R}^n. \quad (1.2.7)$$

As the inequality (1.2.7) holds for every $\mathbf{x} \in \mathbb{R}^n$, we conclude that

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0} \text{ and } \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0.$$

Moreover, we assert that $\boldsymbol{\lambda} = \mathbf{0}$. Otherwise, it follows in turn that $\boldsymbol{\mu} \neq \mathbf{0}$, which stands in contradiction to Assumption 1, and therefore, $\boldsymbol{\lambda} \neq \mathbf{0}$.

- (ii) Let $\mathbf{x}^* \in \mathcal{X}$ be an arbitrary non-trivial KKT point, i.e.,

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0}; \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0,$$

⁴The set $\{\mathbf{x} \in \mathbb{R}^n, g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$ is said to satisfy the *Mangasarian-Fromovitz constraint qualification* [4] at $\mathbf{x}^*$ if there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\nabla g_i(\mathbf{x}^*)^T \mathbf{v} < 0$ for each $i \in I(\mathbf{x}^*) := \{i \in \{1, 2, \dots, m\} : g_i(\mathbf{x}^*) = 0\}$.

⁵One says that the set $\{\mathbf{x} \in \mathbb{R}^n, g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$ satisfies the *Robinson constraint qualification* at $\mathbf{x}^*$ if $\mathbf{0} \in \text{int}\{\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbb{R}^n - \mathbf{x}^*) + \mathbb{R}_+^m\}$ when $\mathbf{g}(\mathbf{x}) := (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_m(\mathbf{x}))^T$.

for some non-zero vectors $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$. This together with Assumption 1 implies that $\lambda^T \nabla f(x^*)$ must ultimately be non-zero vector. It can be seen that if the set $L_{\lambda^T f}^<(x^*)$ is empty, then x^* actually is a weak Pareto minimum of (MOP). In fact, if x^* is not a weak Pareto minimum of (MOP), there exists $x \in \mathcal{X}$ such that $f(x^*) - f(x) \in \text{int}K$. So, by the virtue of Lemma 38, $\lambda^T f(x^*) > \lambda^T f(x)$, which contradicts to the fact that $L_{\lambda^T f}^<(x^*) = \emptyset$. Let us consider in the case $L_{\lambda^T f}^<(x^*) \neq \emptyset$. Applying Proposition 22(i) with $f(x) := \lambda^T f(x)$, we obtain that

$$\lambda^T \nabla f(x^*)(u - x^*) \leq 0, \quad \forall u \in L_{\lambda^T f}^<(x^*). \quad (1.2.8)$$

Therefore, by Lemma 23,

$$\lambda^T \nabla f(x^*)(v - x^*) = -\mu^T \nabla g(x^*)(v - x^*) \geq 0, \quad \forall v \in \mathcal{X}. \quad (1.2.9)$$

Note that,

$$\{y \in \mathbb{R}^n : f(x^*) - f(y) \in \text{int}K\} \subseteq L_{\lambda^T f}^<(x^*).$$

Thus, in order to obtain that x^* is a weak Pareto minimum of (MOP), it suffices to show that $\mathcal{X} \subseteq \mathbb{R}^n \setminus L_{\lambda^T f}^<(x^*)$ or consequently, $L_{\lambda^T f}^<(x^*) \cap \mathcal{X} = \emptyset$. Suppose, ad absurdum, $L_{\lambda^T f}^<(x^*) \cap \mathcal{X} \neq \emptyset$. Thus, from (1.2.8) and (1.4.5) we get the assertion $\lambda^T \nabla f(x^*)(w - x^*) = 0$ for any $w \in L_{\lambda^T f}^<(x^*) \cap \mathcal{X}$. Furthermore, as the set $L_{\lambda^T f}^<(x^*)$ being open, for each $d \in \mathbb{R}^n$ we can find $t > 0$ small enough such that $w + td \in L_{\lambda^T f}^<(x^*)$. Hence,

$$t\lambda^T \nabla f(x^*)d = \lambda^T \nabla f(x^*)(w + td - x^*) - \lambda^T \nabla f(x^*)(w - x^*) \leq 0.$$

This means $\lambda^T \nabla f(x^*) = 0$, a contradiction. Thus, $L_{\lambda^T f}^<(x^*) \cap \mathcal{X} = \emptyset$, and x^* is a weak Pareto minimum of (MOP) as desired. $\square$

We now demonstrate with the following an example to guarantee that Theorem 45 is indicated to be conveniently applied in some cases where Theorem 3.1 and Theorem 3.2 of [21] cannot be used even when the feasible set $\mathcal{X}$ is convex.

Example 29. Consider the following multi-objective optimization problem (MOP) over cones:

$$\begin{aligned} K - \quad & \text{Minimize } f(x) := (x + 1, x^3 - 5x^2 + 8x - 3)^T \\ & \text{subject to } x \in \mathcal{X} := \{x \in \mathbb{R} : -g(x) \in Q\}, \end{aligned}$$

where $g(x) := (x - 1, x^2 - x - 1)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq x_1\}$. A straightforward calculation shows that:

- $\mathcal{X} = [2, +\infty)$,
- $K^* = K$,
- $Q^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_2 \leq -x_1\}$,
- $x^* := 2$ satisfies the non-trivial KKT conditions by taking $\lambda := (2, 0)^T$ and $\mu := (1, -1)^T$,
- $L_{\lambda^T f}^<(x^*) = (-\infty, 2)$ is convex,
- It is easily to seen that Assumption 1 and the Condition 1.4.1 are satisfied.

Applying Theorem 45 (ii), we can conclude that x^* is a weak Pareto minimum of (MOP). However, it can be checked that $\mathbf{g}$ is not Q -convex, i.e.

$$\mathbf{g}(1) - \mathbf{g}(2) - \nabla \mathbf{g}(2)(1 - 2) = (0, 1)^T \notin Q,$$

but the feasible set $\mathcal{X}$ is convex. Furthermore, the function $\mathbf{f}$ is not K -pseudoconvex at $x^* := 2$, because if we take $x = 0$ then

$$-\nabla \mathbf{f}(x^*)(x - x^*) = (2, 0)^T \notin \text{int}K, \text{ but } \mathbf{f}(x^*) - \mathbf{f}(x) = (2, 4)^T \in \text{int}K.$$

Hence, the corresponding results [21] is not applicable. $\square$

Note that the multiplier vector $\boldsymbol{\mu}$ is assumed to be non-zero vector (the non-triviality of the KKT conditions) in order to ensure that $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$ in Theorem 45(ii). The following example demonstrates that this assumption cannot be dropped.

Example 30. Let $\mathbf{f}(x) := (x + 1, -(x - 2)^3)^T$, $\mathbf{g}(x) := (x^2 - 1, 2x - 1)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geq -x_1, x_1 \geq 0\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq x_2, x_1 \geq 0\}$. It is not hard to check that $\mathcal{X} = [1, 2]$, $x^* := 2$ is a KKT point with $\boldsymbol{\lambda} := (0, -1)^T$ and $\boldsymbol{\mu} := (0, 0)^T$, and all the conditions in Theorem 45 (ii) are fulfilled. However x^* is not even a weak Pareto minimum, i.e., if we take $x := \frac{3}{2}$ then $\mathbf{f}(x^*) - \mathbf{f}(x) = (3, 0)^T - (\frac{5}{2}, \frac{1}{8})^T = (\frac{1}{2}, -\frac{1}{8})^T \in \text{int}K$. The main reason is that x^* is not a non-trivial KKT point. $\square$

To appreciate Theorem 45 we present an example that is applicable while the aforementioned result in [21] is not.

Example 31. Consider the following multi-objective optimization problem (MOP) over cones :

$$\begin{aligned} K - \quad & \text{Minimize } \mathbf{f}(x) := (x^2 - 1, -x^3 + 5x^2 - 8x + 5)^T \\ & \text{subject to } x \in \mathcal{X} := \{x \in \mathbb{R} : -\mathbf{g}(x) \in Q\}, \end{aligned}$$

where $\mathbf{g}(x) := (x^3 + x^2 + x, x^3 + 2x^2 - 5x + 8)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0, x_2 \leq x_1\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq x_1\}$. Evidently, $\mathbf{f}$, and $\mathbf{g}$ are not K , and Q -convex, respectively. Indeed, $\mathbf{f}(1) - \mathbf{f}(0) - \nabla \mathbf{f}(0)(1 - 0) = (1, 4)^T \notin K$, and $\mathbf{g}(1) - \mathbf{g}(0) - \nabla \mathbf{g}(0)(1 - 0) = (2, 3)^T \notin Q$. It is easy to verify that $\mathcal{X} = [0, 2] \cup [4, +\infty)$. Then we have already seen that the feasible set $\mathcal{X}$ is not convex. Therefore, the results in [21] cannot be applicable. However, it is not hard to verify that

- $K^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_2 \geq -x_1\}$,
- $Q^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_2 \leq -x_1\}$,
- $x^* := 0$ satisfies the non-trivial KKT conditions by taking $\boldsymbol{\lambda} := (1, -1)^T$ and $\boldsymbol{\mu} := (-8, 0)^T$,
- Assumption 1 and the condition 1.4.1 are satisfied,
- $L_{\boldsymbol{\lambda}^T \mathbf{f}}^<(x^*) = (-\infty, 0)$, which is convex.

Hence, Theorem 45 (ii) indicates that x^* is a weak Pareto minimum of (MOP). $\square$

Next, we will see now how the convexity of $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ together with the strict convexity of $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ at a non-trivial KKT point $\mathbf{x}^*$ possess $\mathbf{x}^*$ to be a Pareto minimum of (MOP).

Theorem 32. Consider the problem (MOP) and let both Assumption 1 and the condition (1.4.1) be satisfied at a feasible point $\mathbf{x}^*$. If $\mathbf{x}^*$ is a non-trivial KKT point with multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ is convex, and additionally $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ is strictly convex at $\mathbf{x}^*$, then $\mathbf{x}^*$ is a Pareto minimum of (MOP).

Proof. In a similar manner of the second argument as the proof of Theorem 45, by the KKT conditions and Proposition 22(ii), we arrive at the following assertion

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) \geq 0 \geq \boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{u} - \mathbf{x}^*), \quad \forall \mathbf{v} \in \mathcal{X}, \forall \mathbf{u} \in L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*), \quad (1.2.10)$$

and $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$. To establish the desired results, we argue first by using Lemma 38 that

$$\{\mathbf{y} \in \mathbb{R}^n : \mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{y}) \in K \setminus \{\mathbf{0}\}\} \subseteq L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*) \setminus \{\mathbf{x}^*\}.$$

Thus, we only need to justify this containment

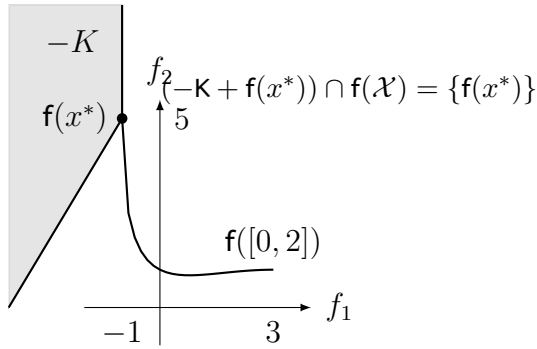
$$\mathcal{X} \subseteq \mathbb{R}^n \setminus (L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*) \setminus \{\mathbf{x}^*\})$$

We argue by contradiction that there exists some $\mathbf{w} \in \mathcal{X}$ such that $\mathbf{w} \neq \mathbf{x}^*$ and $\mathbf{w} \in L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$. Taking (1.4.6) into account we actually have

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{w} - \mathbf{x}^*) = 0.$$

Furthermore, as $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) \in N(L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*), \mathbf{x}^*) \setminus \{\mathbf{0}\}$ (by the second inequality in (1.4.6)) and $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ is strictly convex set, then $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{w} - \mathbf{x}^*) < 0$. This is a contradiction, and thereby implying that $\mathbf{x}^*$ is a Pareto minimum of (MOP). $\square$

Remark 33. In Example 31 with $\boldsymbol{\lambda} := (1, -1)^T$, it is evident that $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ is strictly convex at $\mathbf{x}^* := \mathbf{0}$, by Theorem 50, and hence $\mathbf{x}^*$ is a Pareto minimum of (MOP) (see the below figure).



In Example 31, $\mathbf{x}^* := \mathbf{0}$ is a Pareto minimum of (MOP).

Remark 34. It should be noted that to obtain a Pareto minimum from a drawback (see [9, 21] and other references therein), the multiplier vector $\boldsymbol{\lambda}$ in KKT conditions need to be taken from the strict positive dual cone of K , K^{s*} , which defined as

$$K^{s*} := \{\mathbf{z} \in \mathbb{R}^n : \mathbf{x}^T \mathbf{z} > 0 \text{ for all } \mathbf{x} \in K \setminus \{\mathbf{0}\}\}.$$

However, in this case study the multiplier vector $\boldsymbol{\lambda}$ is not necessarily to take from the strict positive dual cone. In fact, as K defined in Example 31 and $\boldsymbol{\lambda} := (1, -1)^T$, Then elementary calculations give us

$$K^{s*} = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 > 0, x_2 > -x_1\}$$

and so, $\boldsymbol{\lambda} \notin K^{s*}$.

Now we give an example showing that the strict convexity of $L_{\lambda^{Tf}}(x^*)$ with corresponding multiplier λ is essential for $\mathbf{x}^*$ under the question to be a Pareto minimum of (MOP) in Theorem 50.

Example 35. Let $\mathbf{x} := (x_1, x_2)^T \in \mathbb{R}^2$, $\mathbf{f}(\mathbf{x}) := (x_1^2, x_2 - x_1)^T$, $\mathbf{g}(\mathbf{x}) := (-x_1^3 + 3x_1 + x_2, x_1 - x_2)^T$ and $K = Q := \{\mathbf{x} \in \mathbb{R}^2 : x_1 \leq 0, x_2 \geq 0\}$. It is easy to check that the feasible set $\mathcal{X}$ is not convex and the condition (1.4.1) is valid at $\mathbf{x}^* := (1, 1)^T \in \mathcal{X}$. Then elementary calculations give us

- $K^* = Q^* = K$,
- $\mathbf{g}(\mathbf{x}^*) = (3, 0)^T$, $\nabla \mathbf{g}(\mathbf{x}^*) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $\mathbf{f}(\mathbf{x}^*) = (1, 0)^T$, $\nabla \mathbf{f}(\mathbf{x}^*) = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$,
- $\mathbf{x}^*$ satisfies Assumption 1 and the non-trivial KKT conditions by taking $\lambda = \mu := (0, 1)^T$,
- $L_{\lambda^{Tf}}^<(\mathbf{x}^*) = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 < x_1\}$ and $L_{\lambda^{Tf}}(\mathbf{x}^*) = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq x_1\}$ are convex sets.

By Theorem 45 (ii), we can conclude that $\mathbf{x}^*$ is a weak Pareto minimum of (MOP). However, the set $L_{\lambda^{Tf}}(\mathbf{x}^*)$ is not a strictly convex set at $\mathbf{x}^*$, i.e., it is clear that $N(L_{\lambda^{Tf}}(\mathbf{x}^*), \mathbf{x}^*) = \{(-r, r)^T \in \mathbb{R}^2 : r \geq 0\}$. So, by taking $\mathbf{u} := (-1, 1)^T \in N(L_{\lambda^{Tf}}(\mathbf{x}^*), \mathbf{x}^*) \setminus \{(0, 0)^T\}$ and $\mathbf{y} := (2, 2)^T \in L_{\lambda^{Tf}}(\mathbf{x}^*) \setminus \{(0, 0)^T\}$, $\mathbf{u}^T(\mathbf{y} - \mathbf{x}^*) = 0$. Actually, a point $\mathbf{x}^*$ is not even a Pareto minimum, i.e., if we take $\bar{\mathbf{x}} := (-2, -2)^T \in \mathcal{X}$, one has

$$\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\bar{\mathbf{x}}) = (-3, 0)^T \in K \setminus \{(0, 0)^T\}.$$

Remark 36. It is worth noting that the convexity of $L_{\lambda^{Tf}}^<(\mathbf{x}^*)$ (resp. $L_{\lambda^{Tf}}(\mathbf{x}^*)$) in Theorem 45 (resp. in Theorem 50) can be viewed as a generalized quasiconvexity of $\mathbf{f}$ at $\mathbf{x}^*$ due to the notion of **-quasiconvexity* [10] in the sense that for each $\lambda \in K^*$ the function $\lambda^T \mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ is quasiconvex⁶. It is quite clear from the definition that **-quasi-convexity* of $\mathbf{f}$ guarantees the convexity of the level set $L_{\lambda^{Tf}}^<(\mathbf{x}^*)$ or of $L_{\lambda^{Tf}}(\mathbf{x}^*)$. In fact, the function $\mathbf{f}$ in Example 35 is not a **-quasiconvexity*, i.e., by taking $\lambda := (-1, 1)^T \in K^*$ and $\mathbf{x} := (1, 1)^T$, the sublevel set $L_f(\mathbf{x})$ is non-convex. For related conditions for cone quasiconvex mappings we refer the reader to [13, 16, 3].

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⁶A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *quasiconvex* if its sublevel set $L_f(\mathbf{x})$ at $\mathbf{x}$ is convex for all $\mathbf{x} \in \mathbb{R}^n$ or, equivalently, the strict sublevel set $L_f^<(\mathbf{x})$ at $\mathbf{x}$ is convex for all $\mathbf{x} \in \mathbb{R}^n$.

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1.3 Characterizing robust weak sharp solution sets of convex optimization problems with uncertainty

The notion of a weak sharp minimizer in general mathematical programming problems was first introduced in [1]. It is an extension of a sharp minimizer (or equivalently, strongly unique minimizer) in [2] to include the possibility of non-unique solution set. It has been acknowledged that the weak sharp minimizer plays important roles in stability/sensitivity analysis and convergence analysis of a wide range of numerical algorithms in mathematical programming (see [3, 4, 5, 6, 7, 8] and references therein).

In the context of optimization, much attention has been paid to concerning sufficient and/or necessary conditions for weak sharp minimizers/solutions and characterizing weak sharp solution sets (of such weak sharp minimizers) in various types of problems. Particularly, the study of characterizations of the weak sharp solution sets covers both single-objective and multi-objective optimization problems (see, [9, 10, 11, 12] and references therein) and, recently, is extended to mathematical programs with inequality constraints and semi-infinite programs (see, e.g., [13, 14]). As it might be seen, the study of characterizations of the weak sharp solution sets has been popular in many optimization problems. *How about the issue of this study, particularly, in a robust optimization?*

Robust (convex) optimization has been known as an important class of convex optimization deals with uncertainty in the data of the problems [15, 16]. The goal of robust optimization is to immunize an optimization problem against uncertain parameters in the problem. In the last two decades, it has been through a rapid development owing to the practical requirement and its effective implementation in real-world applications of optimization (see, e.g., [17, 18, 19, 20, 21] and the references therein). A successful treatment of the robust optimization approaches to convex optimization problems under data uncertainty was given in ([15, 16, 22, 23, 24]).

While the characterizations of optimal solution sets have been in the limelight presently, there has been no research concerning the characterizations robust weak sharp solution sets for such problems. Indeed, a robust weak sharp solution of an uncertain optimization problem is the weak sharp minimizer of the robust counterpart of such problem. Our main goal in this paper is to establish characterizations of the robust weak sharp solution set of the convex optimization problem under the data uncertainty.

This paper is organized as follows. In section 2, we recall the basic definitions. In Section 3, we establish necessary conditions for a robust weak sharp solution, constancy of Lagrangian-type function on the robust weak sharp solution set, and some characterizations of robust weak sharp solution sets are established respectively. Some properties of subdifferentials of convex functions and the (RSCQ), which was introduced in [24], are employed in the section. Finally, in section 4, we consider the characterizations of the robust weak sharp weakly efficient solutions for the multi-objective optimization problem under data uncertainty.

Throughout the paper, let $\mathbb{R}^n$, $n \in \mathbb{N}$, be the n -dimensional Euclidean space, and the inner product and the norm of $\mathbb{R}^n$ are denoted respectively by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. The symbol $B(x, r)$

stands for the open ball centered at $x \in \mathbb{R}^n$ with the radius $r > 0$ while the $\mathcal{B}_{\mathbb{R}^n}$ stands for the closed unit ball in $\mathbb{R}^n$. For a nonempty subset $A \subseteq \mathbb{R}^n$, we denote the notations of the closure, boundary and convex hull of A by $\text{cl}A$, $\text{bd}A$, and $\text{co}A$, respectively. In particular, when $\lambda x \in E \subseteq \mathbb{R}^n$ for every $\lambda \geq 0$ and every $x \in E$, the set E in $\mathbb{R}^n$ is said to be a cone. A dual cone E^* of the cone E is given as $E^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in E\}$. Observe that the dual cone E^* is always closed and convex (regardless of E).

In general, for a given nonempty set $A \subseteq \mathbb{R}^n$, the indicator function $\delta_A : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ of A and the support function $\sigma_A : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ of A are, respectively, defined by

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A; \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\sigma_A(x^*) := \sup_{x \in A} \langle x^*, x \rangle.$$

The distance function $d_A : \mathbb{R}^n \rightarrow \mathbb{R}_+ : [0, +\infty)$ is defined by

$$d_A(x) := \inf_{y \in A} \|x - y\|.$$

A normal cone of the set A at the point x is the following set:

$$N_A(x) = \begin{cases} \{y \in \mathbb{R}^n : \langle y, a - x \rangle \leq 0 \text{ for all } a \in A\}, & \text{if } x \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$

The normal cone $N_A(x)$ is always closed and convex for any set A .

For any extended real-valued function $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}} := [-\infty, +\infty]$ the following notations stand, respectively, for its effective domain and epigraph:

$$\text{dom}h := \{x \in \mathbb{R}^n : h(x) < +\infty\},$$

and

$$\text{epi}h := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq r\}.$$

The function h is said to be a proper function if and only if $h(x) > -\infty$ for every $x \in \mathbb{R}^n$ and $\text{dom}h$ is nonempty. Further, it is said to be a convex function if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y),$$

or equivalently, $\text{epi}h$ is convex. On the other hand, the function h is said to be a concave function if and only if $-h$ is a convex function. In the case of vector valued function, let $\tilde{h} : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a given function and $D \subseteq \mathbb{R}^p$ is a convex set. The function $\tilde{h}$ is said to be D -convex if and only if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$\tilde{h}(\lambda x + (1 - \lambda)y) - \lambda \tilde{h}(x) - (1 - \lambda)\tilde{h}(y) \in -D.$$

Simultaneously, the function h is called a lower semicontinuous at $x \in \mathbb{R}^n$ if for every sequence $\{x_k\} \subseteq \mathbb{R}^n$ converging to x ,

$$h(x) \leq \liminf_{k \rightarrow \infty} h(x_k).$$

Equivalently,

$$h(x) \leq \liminf_{y \rightarrow x} h(y),$$

where the term on the right-hand side of the inequality denotes the lower limit of the function h defined as

$$\liminf_{y \rightarrow x} h(y) = \lim_{r \downarrow 0} \inf_{y \in B(x,r)} h(y).$$

For any proper and convex function $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, the subdifferential of h at $\hat{x} \in \text{dom} h$, is defined by

$$\partial h(\hat{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, x - \hat{x} \rangle \leq h(x) - h(\hat{x}), \forall x \in \mathbb{R}^n\}.$$

More specifically, for each $\varepsilon \geq 0$, the ε -subdifferential of h at $\hat{x} \in \text{dom} h$, is defined by

$$\partial_\varepsilon h(\hat{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, x - \hat{x} \rangle \leq h(x) - h(\hat{x}) + \varepsilon, \forall x \in \mathbb{R}^n\}.$$

It is obvious that for $\varepsilon \geq \varepsilon'$, we have $\partial_{\varepsilon'} h(\hat{x}) \subseteq \partial_\varepsilon h(\hat{x})$. In particular, if h is a proper lower semicontinuous convex function, then for every $\hat{x} \in \text{dom} h$, the ε -subdifferential $\partial_\varepsilon h(\hat{x})$ is a nonempty closed convex set and

$$\partial h(\hat{x}) = \bigcap_{\varepsilon > 0} \partial_\varepsilon h(\hat{x}).$$

If $x \notin \text{dom} h$, then we set $\partial h(x) = \emptyset$. Simultaneously, for the nonempty subset A of $\mathbb{R}^n$ we get $\partial \delta_A(x) = N_A(x)$ and $\partial d_A(x) = \mathcal{B}_{\mathbb{R}^n} \cap N_A(x)$.

The conjugate function $h^* : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ of any $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is defined by

$$h^*(x^*) := \sup_{x \in \mathbb{R}^n} \{\langle x^*, x \rangle - h(x)\}$$

for all $x \in \mathbb{R}^n$. The function h^* is lower semicontinuous convex irrespective of the nature of h but for h^* to be proper, we need h to be a proper convex function.

Next, let us recall some basic concepts dealing a DC problem/programming. A DC function is the difference of two convex functions. The minimization (or maximization) problem of a DC function is called a DC problem, i.e., the DC problem concerned about finding

$$\inf_{x \in \mathbb{R}^n} h(x) := f(x) - \phi(x)$$

where $f, \phi : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex. Note that the function h is DC and it is not expected to be convex.

It shall be found later that some DC problems are considered and their properties, in particular the following lemma, are employed.

Lemma 37. [25] Let $h_1, h_2 : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be two proper lower semicontinuous convex functions. Then

1. A point $\hat{x} \in \text{dom} h_1 \cap \text{dom} h_2$ is a (global) minimizer of the DC problem :

$$\inf_{x \in \mathbb{R}^n} \{h_1(x) - h_2(x)\}$$

if and only if for any $\varepsilon \geq 0$, $\partial_\varepsilon h_2(\hat{x}) \subseteq \partial_\varepsilon h_1(\hat{x})$.

2. If $\hat{x} \in \text{dom } h_1 \cap \text{dom } h_2$ is a local minimizer of the DC problem :

$$\inf_{x \in \mathbb{R}^n} \{h_1(x) - h_2(x)\}$$

then $\partial h_2(\hat{x}) \subseteq \partial h_1(\hat{x})$.

Lemma 38. [19] Let $\mathcal{U} \subseteq \mathbb{R}^p$ be a convex compact set, and $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ be a function such that, $f(\cdot, u)$ is a convex function for any $u \in \mathcal{U}$, and $f(x, \cdot)$ is a concave function for any $x \in \mathbb{R}^n$. Then,

$$\partial \left(\max_{u \in \mathcal{U}} f(\cdot, u) \right) (\hat{x}) = \bigcup_{u \in \mathcal{U}(\hat{x})} \partial f(\cdot, u)(\hat{x}),$$

where

$$\mathcal{U}(\hat{x}) := \left\{ \hat{u} \in \mathcal{U} : f(\hat{x}, \hat{u}) = \max_{u \in \mathcal{U}} f(\hat{x}, u) \right\}.$$

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Let $D \subseteq \mathbb{R}^p$ be a nonempty closed convex cone. Consider the following convex optimization problem:

$$\min f(x) \text{ s.t. } x \in C, g(x) \in -D \quad (\text{P})$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a D -convex function. The feasible set of (P) is defined by

$$K_0 := \{x \in C : g(x) \in -D\}.$$

The problem (P) in the face of data uncertainty both in the objective and constraints can be captured by the following *uncertain optimization problem* :

$$\min \{f(x, u) : x \in C, g(x, v) \in -D\}. \quad (\text{UP})$$

where $\mathcal{U} \subseteq \mathbb{R}^p$ and $\mathcal{V} \subseteq \mathbb{R}^q$ are convex and compact uncertainty sets, $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ is a given real-valued function such that, for any uncertain parameter $u \in \mathcal{U}$, $f(\cdot, u)$ is convex as well as $f(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathcal{V} \rightarrow \mathbb{R}^m$ is a vector-valued function such that, for any uncertain parameter $v \in \mathcal{V}$, $g(\cdot, v)$ is D -convex as well as $g(x, \cdot)$ is D -concave for any $x \in \mathbb{R}^n$. The uncertain sets can be apprehended in the sense that the parameter vectors u and v are not known exactly at the time of the decision.

For examining the uncertain optimization problem (UP), one usually associates with its *robust (worst-case) counterpart*, which is the following problem:

$$\min \left\{ \max_{u \in \mathcal{U}} f(x, u) : x \in C, g(x, v) \in -D, \forall v \in \mathcal{V} \right\}. \quad (\text{RUP})$$

It is worth observing here that the robust counterpart, which is termed as the robust optimization problem, finds a worst-case possible solution that can be immunized opposed the data uncertainty.

The problem (RUP) is said to be feasible if the *robust feasible set* K is nonempty where it is denoted by

$$K := \{x \in C : g(x, v) \in -D, \forall v \in \mathcal{V}\}. \quad (1.3.1)$$

Now, we recall the following concept of solutions, which was introduced in [26].

Definition 39. [26] A point $\hat{x} \in K$ is said to be a *robust optimal solution* for (UP) if it is an optimal solution for (RUP), i.e., for all $x \in K$,

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq 0.$$

The *robust optimal solution set* of (UP) is the set which consists of all robust optimal solutions of (UP) and is given by

$$S := \left\{ x \in K : \max_{u \in \mathcal{U}} f(x, u) \leq \max_{u \in \mathcal{U}} f(y, u), \forall y \in K \right\}.$$

In this paper, using the idea of weak sharp minimizer, and the robust optimal solution, we introduce a new concept of solutions for (UP), which related to the sharpness, namely the robust weak sharp solution.

Definition 40. A point $\hat{x} \in K$ is said to be a (or an optimal) *weak sharp solution* for (RUP) if there exist a real number $\eta > 0$ such that for all $x \in K$,

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(x)$$

where $\tilde{K} := \left\{ x \in K : \max_{u \in \mathcal{U}} f(x, u) = \max_{u \in \mathcal{U}} f(\hat{x}, u) \right\}$.

Definition 41. A point $\hat{x} \in K$ is said to be a (or an optimal) *robust weak sharp solution* for (UP) if it is a weak sharp solution for (RUP). The *robust weak sharp solution set* of (UP) is given by

$$\tilde{S} := \left\{ \hat{x} \in K : \exists \eta > 0 \text{ s.t. } \max_{u \in \mathcal{U}} f(y, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(y), \forall y \in K \right\}.$$

Throughout the paper, we assume that $\tilde{S}$ is nonempty.

Remark 42. It is worthwhile to be noted that every robust weak sharp solution for (UP) is a robust optimal solution. In general, the reverse implication need not to be valid.

1.4 Characterizations of robust weak sharp solutions

In this section, we establish some optimality conditions for the robust weak sharp solution in convex uncertain optimization problems as well as obtain characterizations of the robust weak sharp solution sets for the considered problems. For any $\hat{x} \in \mathbb{R}^n$, we use the following notations:

$$\mathcal{U}(\hat{x}) := \left\{ \hat{u} \in \mathcal{U} : f(\hat{x}, \hat{u}) = \max_{u \in \mathcal{U}} f(\hat{x}, u) \right\},$$

and

$$\mathcal{V}(\hat{x}) := \left\{ \hat{v} \in \mathcal{V} : g(\hat{x}, \hat{v}) = \max_{v \in \mathcal{V}} g(\hat{x}, v) \right\}.$$

The following definition, which was introduced in [24], plays a vital role in determining characterizations of robust optimal weak sharp solution sets.

Definition 43. [24] The robust type subdifferential constraint qualification (RSCQ) is said to be satisfied at $\hat{x} \in K$ if

$$\partial \delta_K(\hat{x}) \subseteq \partial \delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial(\mu g)(\cdot, v)(\hat{x}).$$

Remark 44. In an excellent work, [24], Sun et. al. introduced the (RSCQ) and then obtained some characterizations of the the robust optimal solution set, for an uncertain convex optimization problem.

Although it has been used as a guideline for dealing with the (UP), our attention is paid to characterizing the sets containing the robust weak sharp solutions of such problem. Furthermore, the presence of the term $d_{\tilde{K}}(x)$ in this paper has led us to deal with some different tools and methods from those in work of Sun et.al.

The following theorem presents that the robust type subdifferential constraint qualification (RSCQ) defined in Definition 43 is fulfilled if and only if optimality conditions for a robust weak sharp solution of (UP) are satisfied.

Theorem 45. Let $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ satisfy the following properties :

1. for any $u \in \mathcal{U}$ and $v \in \mathcal{V}$, $f(\cdot, u)$ is convex and continuous as well as $g(\cdot, v)$ is D -convex on $\mathbb{R}^n$;
2. for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ is concave on $\mathcal{U}$ and $g(x, \cdot)$ is D -concave on $\mathcal{V}$.

Then, the following statements are equivalent:

- (a) The (RSCQ) is fulfilled at $\hat{x} \in K$;
- (b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp solution of (UP) if and only if there exists a positive constant η such that

$$\begin{aligned} & N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n} \\ & \subseteq \bigcup_{u \in \mathcal{U}(\hat{x})} \partial f(\cdot, u)(\hat{x}) + \partial \delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial ((\mu g)(\cdot, v))(\hat{x}). \end{aligned} \quad (1.4.1)$$

Proof. (a) $\Rightarrow$ (b) Assume that the (RSCQ) is satisfied at $\hat{x} \in K$. Let $\hat{x}$ be a robust weak sharp solution of (UP). Consequently, there exists $\eta > 0$ such that

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(x). \quad (1.4.2)$$

By (1.4.2), we obtain that for all $x \in K$,

$$\begin{aligned} \max_{u \in \mathcal{U}} f(x, u) + \delta_K(x) - \eta d_{\tilde{K}}(x) & \geq \max_{u \in \mathcal{U}} f(\hat{x}, u) \\ & = \max_{u \in \mathcal{U}} f(\hat{x}, u) + \delta_K(\hat{x}) - \eta d_{\tilde{K}}(\hat{x}), \end{aligned}$$

thereby implying that, for all $\xi_d \in \partial \eta d_{\tilde{K}}(x)$,

$$\begin{aligned} & \left(\max_{u \in \mathcal{U}} f(\cdot, u) + \delta_K \right)(x) - \left(\max_{u \in \mathcal{U}} f(\cdot, u) + \delta_K \right)(\hat{x}) \\ & \geq \eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x}) \\ & \geq \langle \xi_d, x - \hat{x} \rangle. \end{aligned}$$

Thus, $\xi_d \in \partial (\max_{u \in \mathcal{U}} f(\cdot, u) + \delta_K)(\hat{x})$. Hence,

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial \left(\max_{u \in \mathcal{U}} f(\cdot, u) + \delta_K \right)(\hat{x}).$$

As $\max_{u \in \mathcal{U}} f(\cdot, u)$ is continuous on $\mathbb{R}^n$ and δ_K is proper lower semicontinuous convex on $\mathbb{R}^n$, we have

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial(\max_{u \in \mathcal{U}} f(\cdot, u))(\hat{x}) + \partial\delta_K(\hat{x}).$$

It can be noted that $\partial d_{\tilde{K}}(x) = N_{\tilde{K}}(x) \cap \mathcal{B}_{\mathbb{R}^n}$. Since (RSCQ) is satisfied at $\hat{x}$, we have the following:

$$\begin{aligned} N_{\tilde{K}}(x) \cap \mathcal{B}_{\mathbb{R}^n} &= \partial(\eta d_{\tilde{K}})(\hat{x}) \\ &\subseteq \bigcup_{u \in \mathcal{U}(\hat{x})} \partial f(\cdot, u)(\hat{x}) + \partial\delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial(\mu g)(\cdot, v)(\hat{x}), \end{aligned}$$

which implies that (1.4.1) holds.

Conversely, assume that there is a positive number η such that (1.4.1) holds. Since $N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$ always contains 0, it is a nonempty set and so is $\bigcap_{\varepsilon > 0} \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$. Thus, for any $\varepsilon \geq 0$, $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \neq \emptyset$. Let $\varepsilon > 0$ be arbitrary and let $\xi \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$. Then for any $x \in K$,

$$\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x}) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \quad (1.4.3)$$

Note that $0 \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$. It follows that

$$\eta d_{\tilde{K}}(\hat{x}) \leq \inf_{x \in \mathbb{R}^n} \eta d_{\tilde{K}}(x) + \varepsilon \leq \inf_{x \in K} \eta d_{\tilde{K}}(x) + \varepsilon.$$

Above inequality and (1.4.3) imply that

$$0 \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \quad (1.4.4)$$

Simultaneously, there exist $\hat{u} \in \mathcal{U}(\hat{x})$, $\hat{\mu} \in D^*$, $\hat{v} \in \mathcal{V}(\hat{x})$

$\xi_f \in \partial f(\cdot, \hat{u})(\hat{x})$, $\xi_\delta \in \partial\delta_C(\hat{x})$, and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$ such that

$$\xi_f + \xi_\delta + \xi_{\hat{\mu}g} = 0, \quad (1.4.5)$$

and for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} f(x, \hat{u}) - f(\hat{x}, \hat{u}) &\geq \langle \xi_f, x - \hat{x} \rangle, \\ \delta_C(x) - \delta_C(\hat{x}) &\geq \langle \xi_\delta, x - \hat{x} \rangle, \text{ and} \\ (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) &\geq \langle \xi_{\hat{\mu}g}, x - \hat{x} \rangle. \end{aligned}$$

Adding these above inequalities implies that for each $x \in K$

$$f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle 0, x - \hat{x} \rangle = 0.$$

Since $\hat{u}$ belongs to $\mathcal{U}(\hat{x})$, for each $x \in K$, above inequality becomes

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq 0.$$

This along with $(\hat{\mu}g)(x, \hat{v}) \leq 0$, $(\hat{\mu}g)(\hat{x}, \hat{v}) = 0$, and (1.4.5) imply

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq 0, \quad (1.4.6)$$

for all $x \in K$. Observe that, combining inequalities (1.4.4) and (1.4.6) leads to

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon, \quad \forall x \in K.$$

This means $\xi \in \partial_\varepsilon(\max_{u \in \mathcal{U}} f(\cdot, u))(\hat{x})$, and so $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial_\varepsilon(\max_{u \in \mathcal{U}} f(\cdot, u))(\hat{x})$. Since the inclusion holds for arbitrary $\varepsilon \geq 0$, it follows from the Lemma 37 that $\hat{x}$ is a minimizer of the DC problem: $\inf_{x \in \mathbb{R}^n} \{\max_{u \in \mathcal{U}} f(x, u) - \eta d_{\tilde{K}}(x)\}$ and hence for any $x \in K$

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) - (\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x})) \geq 0.$$

Therefore, for any $x \in K$,

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(x).$$

This means $\hat{x}$ is a robust weak sharp solution of (UP).

(b) $\Rightarrow$ (a) Let $\xi_\delta \in \partial \delta_K(\hat{x})$ be given. Then, we have

$$0 = \delta_K(x) - \delta_K(\hat{x}) \geq \langle \xi_\delta, x - \hat{x} \rangle$$

holds for all $x \in K$. Let $\bar{\eta} > 0$ be given, and then, set $f(x, u) := -\langle \xi_\delta, x \rangle + \bar{\eta} d_{\tilde{K}}(x)$. Thus, for any $x \in K$,

$$\begin{aligned} \max_{u \in \mathcal{U}} f(x, u) - \bar{\eta} d_{\tilde{K}}(x) &= -\langle \xi_\delta, x \rangle \\ &\geq -\langle \xi_\delta, \hat{x} \rangle + \bar{\eta} d_{\tilde{K}}(\hat{x}) \\ &= \max_{u \in \mathcal{U}} f(\hat{x}, u). \end{aligned}$$

Thus, $\hat{x}$ is a robust weak sharp solution of (UP). By hypothesis, there is $\eta := \bar{\eta}$ such that (1.4.1) is fulfilled. Since for any $u \in \mathcal{U}$, $\partial f(\cdot, u)(\hat{x}) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x})$, we obtain that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u} \in \mathcal{U}(\hat{x})$, $\hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$x^* \in \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x}) + \partial \delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

As $0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, we obtain

$$\xi_\delta \in \partial \delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

It follows that

$$\xi_\delta \in \partial \delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}),$$

and so we get the desired inclusion. Therefore, the proof is complete. $\square$

Remark 46. In [27], the necessary conditions for weak sharp minima in cone constrained optimization problems, which can be captured by weak sharp minima in cone constrained robust optimization problems, were established by means of upper Studniarski or Dini directional derivatives. With the result in Theorem 45, the mentioned necessary conditions are established by an alternative method different from the referred work.

The following result is established easily by means of the basic concepts of variational analysis.

Corollary 47. Let $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^p$ satisfying the following properties:

1. for any $u \in \mathcal{U}$, and $v \in \mathcal{V}$, $f(\cdot, u)$ is convex and continuous as well as $g(\cdot, v)$ is D -convex on $\mathbb{R}^n$;
2. for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ is concave on $\mathcal{U}$ and $g(x, \cdot)$ is D -concave on $\mathcal{V}$, respectively.

The following two below statements are equivalent:

- (a) The (RSCQ) is fulfilled at $\hat{x} \in K$;
- (b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp solution of (UP) if and only if there exists a real number $\eta > 0$ such that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u} \in \mathcal{U}(\hat{x})$, $\hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ yield

$$x^* \in \partial f(\cdot, \hat{u}) + \partial \delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}), \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0. \quad (1.4.7)$$

The result, which deals with a special case that $\mathcal{U}$ and $\mathcal{V}$ are singleton sets, can be obtained easily and be presented as follows:

Corollary 48. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and continuous and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is D -convex. The following statements are equivalent:

1. The (SCQ) is fulfilled at $\hat{x} \in K$
2. $\hat{x} \in \mathbb{R}^n$ is a weak sharp solution of (P) if and only there exists a real number $\eta > 0$ such that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{\mu} \in D^*$ such that

$$x^* \in \partial f(\hat{x}) + \partial \delta_C(\hat{x}) + \partial(\hat{\mu}g)(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}) = 0. \quad (1.4.8)$$

Next, a characterization of robust weak sharp solution sets in terms of a given robust weak sharp solution point of our considered problem is also illustrated in this section. In order to present the mentioned characterization, we first prove that the Lagrangian-type function associated with fixed Lagrange multiplier and uncertainty parameters corresponding to a robust weak sharp solution is constant on the robust weak sharp solution set under suitable conditions. In what follows, let $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $\mu \in D^*$. The Lagrangian-type function $\mathcal{L}(\cdot, \mu, u, v)$ is given by

$$\mathcal{L}(x, \mu, u, v) = f(x, u) + (\mu g)(x, v), \quad \forall x \in \mathbb{R}^n.$$

Now, we denote by

$$\tilde{S} := \left\{ x \in K : \exists \eta > 0 \text{ s.t. } \max_{u \in \mathcal{U}} f(y, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta d_{\tilde{K}}(y), \forall y \in K \right\}.$$

the robust weak sharp solution set of (UP), and then we prove that the Lagrangian-type function associated with a Lagrange multiplier corresponding to a robust weak sharp solution is constant on the robust weak sharp solution set.

Theorem 49. Let $\hat{x} \in \tilde{S}$ be given. Suppose that the (RSCQ) is satisfied at $\hat{x}$. Then, there exist uncertainty parameters $\hat{u} \in \mathcal{U}$, $\hat{v} \in \mathcal{V}$, and Lagrange multiplier $\hat{\mu} \in D^*$, such that for any $x \in \tilde{S}$,

$$(\hat{\mu}g)(x, \hat{v}) = 0, \quad \hat{u} \in \mathcal{U}(x), \quad \text{and } \mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) \text{ is a constant on } \tilde{S}.$$

Proof. Since $\hat{x} \in \tilde{S}$ with the real number $\eta_1 > 0$ and the (RSCQ) is satisfied at this point $\hat{x}$, by Theorem 45 we have that (1.4.1) holds for $\eta := \eta_1$. Clearly $N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$ contains 0, then it is nonempty and so is any $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ where $\varepsilon > 0$. Let $\varepsilon > 0$ and $x^* \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ be arbitrary. Again, we obtain that there exist $\hat{u} \in \mathcal{U}$, $\hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that (1.4.1) is fulfilled. Let $x \in \tilde{S}$ be arbitrary, then we have

$$f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle x^*, x - \hat{x} \rangle,$$

and so

$$f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle x^*, x - \hat{x} \rangle - \varepsilon. \quad (1.4.9)$$

Since $f(\cdot, u)$ and $g(\cdot, v)$ are convex, for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$ respectively,

$$x^* \in \partial_\varepsilon(f(\cdot, u) + \lambda g(\cdot, v))(\hat{x}).$$

Therefore, we obtain $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial_\varepsilon(f(\cdot, u) + \lambda g(\cdot, v))(\hat{x})$, and so

$$f(x, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - \eta d_{\tilde{K}}(x) \geq f(\hat{x}, \hat{u}) = \max_{u \in \mathcal{U}} f(\hat{x}, u). \quad (1.4.10)$$

Note that, as $x \in \tilde{S}$, there exists $\eta_2 > 0$ such that

$$\max_{u \in \mathcal{U}} f(y, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta_2 d_{\tilde{K}}(y), \quad \forall y \in \tilde{S},$$

and so

$$\max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta_2 d_{\tilde{K}}(\hat{x}) = \max_{u \in \mathcal{U}} f(x, u). \quad (1.4.11)$$

From $\hat{\mu} \in D^*$, $g(x, \hat{v}) \in -D$, and (1.4.10), it is not hard to see that

$$(\hat{\mu}g)(x, \hat{v}) = 0. \quad (1.4.12)$$

Then, by (1.4.10) and the positivity of $\eta d_{\tilde{K}}(x)$, we see that

$$\max_{u \in \mathcal{U}} f(x, u) \geq f(x, \hat{u}) \geq \max_{u \in \mathcal{U}} f(\hat{x}, u) + \eta d_{\tilde{K}}(x) \geq \max_{u \in \mathcal{U}} f(\hat{x}, u), \quad (1.4.13)$$

which together with (1.4.11) leads to

$$\max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}). \quad (1.4.14)$$

It follows that $\mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) = f(\hat{x}, \hat{u})$, which is constant. Since $x \in \tilde{S}$ was arbitrary, we finish the proof. $\square$

Theorem 50. For the problem (UP), let $\tilde{S}$ be the robust weak sharp solutions set of (UP) and $\hat{x}$ belongs to it. Suppose that the (RSCQ) is satisfied at $\hat{x} \in \tilde{S}$. Then, there exist uncertain parameters $\hat{u} \in \mathcal{U}$, $\hat{v} \in \mathcal{V}$ and Lagrange multiplier $\hat{\mu} \in D^*$ such that

$$\begin{aligned} \tilde{S} = & \left\{ x \in K : \exists \eta > 0, \exists \xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(\hat{x}) \cap \partial_\varepsilon f(\cdot, \hat{u})(x), \exists \varepsilon > \eta d_{\tilde{K}}(x), \right. \\ & \left. \langle \xi_f, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x), (\hat{\mu}g)(x, \hat{v}) = 0, \max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}) \right\}. \end{aligned} \quad (1.4.15)$$

Proof. ($\subseteq$) Let $x \in \tilde{S}$ be given. Then there exists $\eta > 0$ such that (1.4.1) holds. Hence, there exist $\xi_f \in \partial f(\cdot, \hat{u})(x)$, $\xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$ such that

$$0 = \xi_f + \xi_\delta + \xi_{\hat{\mu}g} \text{ since } 0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}, \quad (1.4.16)$$

and

$$(\hat{\mu}g)(\hat{x}, \hat{v}) = 0. \quad (1.4.17)$$

Since $\xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$,

$$\delta_C(x) - \delta_C(\hat{x}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle \xi_\delta + \xi_{\hat{\mu}g}, x - \hat{x} \rangle. \quad (1.4.18)$$

By the same fashion in the proof of Theorem 45, we have

$$(\hat{\mu}g)(x, \hat{v}) = (\hat{\mu}g)(\hat{x}, \hat{v}) = 0,$$

and

$$\max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}).$$

Therefore, it follows from (1.4.18) that

$$0 \geq \langle \xi_\delta + \xi_{\hat{\mu}g}, x - \hat{x} \rangle,$$

and so by (1.4.16), we obtain

$$\eta d_{\tilde{K}}(x) \geq \langle \xi_f, \hat{x} - x \rangle.$$

Simultaneously, since $\xi_f \in \partial f(\cdot, \hat{u})(\hat{x})$, we have

$$\langle \xi_f, \hat{x} - x \rangle \geq f(\hat{x}, \hat{u}) - f(x, \hat{u}).$$

By (1.4.14) in the proof of Theorem 45, we obtain

$$\langle \xi_f, \hat{x} - x \rangle \geq \max_{u \in \mathcal{U}} f(\hat{x}, \hat{u}) - \max_{u \in \mathcal{U}} f(x, u) \geq 0 = \eta d_{\tilde{K}}(x). \quad (1.4.19)$$

Hence, we have that $\langle \xi_f, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x)$. Now, we prove that for $\xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(x)$, there is an $\varepsilon > \eta d_{\tilde{K}}(x) \geq 0$. In fact, we can show that for any $y \in \mathbb{R}^n$,

$$\langle \xi_f, y - x \rangle = \langle \xi_f, y - \hat{x} \rangle + \langle \xi_f, \hat{x} - x \rangle \leq \langle \xi_f, y - \hat{x} \rangle$$

as $\langle \xi_f, \hat{x} - x \rangle \leq 0$. Since $\xi_f \in \partial f(\cdot, \hat{u})(\hat{x})$ and $f(x, \hat{u}) = f(\hat{x}, \hat{u})$ by (1.4.13) and (1.4.11),

$$\langle \xi_f, y - x \rangle \leq f(y, \hat{u}) - f(\hat{x}, \hat{u}) = f(y, \hat{u}) - f(x, \hat{u}),$$

which means $\xi_f \in \partial f(\cdot, \hat{u})(x)$.

($\supseteq$) Let

$$x \in \left\{ x \in K : \exists \eta > 0, \exists \xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(\hat{x}) \cap \partial_\varepsilon f(\cdot, \hat{u})(x), \exists \varepsilon > \eta d_{\tilde{K}}(x), \right. \\ \left. \langle \xi_f, x - \hat{x} \rangle = \eta d_{\tilde{K}}(x), (\mu g)(x, \hat{v}) = 0, \max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}) \right\}.$$

Since $\hat{x} \in \tilde{S}$, it is clear that $\eta d_{\tilde{K}}(\hat{x}) = 0$. By assumption and $\xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(x)$ for some $\varepsilon > 0$, we get

$$\begin{aligned}
 -\eta d_{\tilde{K}}(\hat{x}) &= 0 \\
 &= \langle \xi_f, \hat{x} - x \rangle - \eta d_{\tilde{K}}(x) \\
 &\leq f(\hat{x}, \hat{u}) - f(x, \hat{u}) + \varepsilon - \eta d_{\tilde{K}}(x) \\
 &= f(\hat{x}, \hat{u}) - f(x, \hat{u}) - \eta d_{\tilde{K}}(x) + \eta d_{\tilde{K}}(x) \\
 &= f(\hat{x}, \hat{u}) - f(x, \hat{u}).
 \end{aligned} \tag{1.4.20}$$

Therefore, we obtain

$$\max_{u \in \mathcal{U}} f(x, u) \leq \max_{u \in \mathcal{U}} f(x, u) + \eta d_{\tilde{K}}(\hat{x}).$$

Since $\hat{x} \in \tilde{S}$ and $x \in K$, the conclusion that $x \in \tilde{S}$ is satisfied. $\square$

In the case that $D := \mathbb{R}_+$, which is a closed convex (and pointed) cone in $\mathbb{R}$, the problem is reduced to be an inequality constrain problem. Suppose that $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ is a function such that $f(\cdot, u)$ is convex for any $u \in \mathcal{U}$ and $f(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$ as well as $g : \mathbb{R}^n \times \mathcal{V} \rightarrow \mathbb{R}$ is a function such that $g(\cdot, v)$ is convex for any $v \in \mathcal{V}$ and $g(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$. Here, the problem (UP) is represented as

$$\min \{f(x, u) : g(x, v) \leq 0, \forall v \in \mathcal{V}\},$$

and its robust counter part is

$$\min \left\{ \max_{u \in \mathcal{U}} f(x, u) : g(x, v) \leq 0, \forall v \in \mathcal{V} \right\}.$$

In this case, we can see that *robust feasible set* K is denoted by

$$K := \{x \in \mathbb{R}^n : g(x, v) \leq 0, \forall v \in \mathcal{V}\}.$$

1.5 Applications to multi-objective optimization

In this section, in order to apply our general results of the previous section, we investigate the class multi-objective optimization problem

$$\min_{x \in \mathbb{R}^n} \{(f_1(x), f_2(x), \dots, f_l(x)) : x \in C, g(x) \in -D\}, \tag{MP}$$

where where $C \subseteq \mathbb{R}^n$ is a nonempty convex set, $D \subseteq \mathbb{R}^m$, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function for any $i \in I$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a D -convex function. The feasible set of (MP) is defined by

$$K_0 := \{x \in C : g(x) \in -D\}.$$

The problem (MP) in the face of data uncertainty both in the objective and constraint can be captured by the following multi-objective optimization problem

$$\min_{x \in \mathbb{R}^n} \{(f_1(x, u_1), f_2(x, u_2), \dots, f_l(x, u_l)) : x \in C, g(x, v) \in -D\}, \tag{UMP}$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}, i = 1, \dots, l$, and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$. $u_i, i = 1, \dots, l$, and v are uncertain parameters, and they belong to the corresponding convex and compact uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^p$, and $\mathcal{V} \subseteq \mathbb{R}^q$. Suppose that for any $u_i \in \mathcal{U}_i, i \in I$, the function $f_i(\cdot, u_i)$ is convex on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, $f_i(x, \cdot)$ is concave on $\mathcal{U}_i, i \in I$. Besides, suppose that for any $v \in \mathcal{V}$, the function $g(\cdot, v)$ is D -convex on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, $g(x, \cdot)$ is D -concave on $\mathcal{V}$.

Similarly, we obtain some characterizations of the robust weak sharp weakly efficient solutions of (UMP) by using investigation of its robust (worst case) counterpart:

$$\min_{x \in \mathbb{R}^n} \left\{ \left(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \dots, \max_{u_l \in \mathcal{U}_l} f_l(x, u_l) \right) : x \in C, g(x, v) \in -D \right\} \quad (\text{RUMP})$$

where the *robust feasible set* of (UMP) is also defined by

$$K := \{x \in C : g(x, v) \in -D, \}.$$

Now, we recall the following concepts of robust weak sharp weakly efficient solutions in multi-objective optimization, which can be found in the literature; see e.g., [21] and [12].

Definition 51. [21] A point $\hat{x} \in K$ is said to be a *weakly robust efficient solution* of for (UMP) if it is a weakly efficient solution solution for (RUMP) i.e., there does not exist $x \in K$ such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i), \text{ for all } i \in I.$$

Definition 52. [12] A point feasible element $\hat{x}$ is said to be a *weak sharp efficient solution* for (MP) if there exists a real number $\eta > 0$ such that for any $x \in K$

$$\max_{1 \leq k \leq l} \{f_k(x) - f_k(\hat{x})\} \geq \eta d_{\hat{K}}(x)$$

where $\hat{K} := \{x \in K : f(x) = f(\hat{x})\}$.

Now, we introduce a new concept of solution, which related to the sharpness, namely the robust weak sharp weakly efficient solutions.

Definition 53. A point $\hat{x} \in K$ is said to be a *weak sharp weakly efficient solution* for (RUMP) if and only if there exist a real number $\eta > 0$ such that there does not exist $y \in K \setminus \{\hat{x}\}$ satisfying

$$\max_{u_i \in \mathcal{U}_i} f_i(y, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < \eta d_{\tilde{K}}(y), \text{ for all } i \in I,$$

or equivalently, for all $x \in K$

$$\max_{i \in I} \left\{ \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_{\tilde{K}}(x)$$

where $\tilde{K} := \left\{ x \in K : \max_{u \in \mathcal{U}} f_i(x, u) = \max_{u \in \mathcal{U}} f_i(\hat{x}, u), i \in I \right\}$.

Definition 54. A point $\hat{x} \in K$ is said to be a *robust weak sharp weakly efficient solution* for (UMP) if it is a weakly weak sharp weakly efficient solution for (RUMP).

The following lemma is useful for establishing our results in this section.

Lemma 55. [30] Let $\mathcal{U}_1, \dots, \mathcal{U}_l$ be nonempty convex and compact sets of $\mathbb{R}^p$ and for any $u_i \in \mathcal{U}_i, i \in I$, the function $f_i(\cdot, u_i) : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex as well as for any $x \in \mathbb{R}^n, f_i(x, \cdot) : \mathcal{U}_i \rightarrow \mathbb{R}$ be concave where $i \in I$. Then, for any $\lambda_i \geq 0, i \in I$,

$$\partial \left(\max_{u \in \prod_{i \in I} \mathcal{U}_i(\hat{x})} \sum_{i \in I} \lambda_i f_i(\cdot, u_i) \right) (\hat{x}) = \bigcup_{u \in \prod_{i \in I} \mathcal{U}_i(\hat{x})} \sum_{i \in I} \lambda_i (f_i(\cdot, u_i)) (\hat{x}),$$

where

$$\prod_{i \in I} \mathcal{U}_i(\hat{x}) := \left\{ (\hat{u}_1, \dots, \hat{u}_l) \in \prod_{i \in I} \mathcal{U}_i : \sum_{i \in I} \lambda_i f_i(\hat{x}, \hat{u}_i) = \max_{u \in \prod_{i \in I} \mathcal{U}_i} \sum_{i \in I} \lambda_i f_i(\hat{x}, u_i) \right\}$$

Now, by using the similar methods of Section 3, we can characterize the corresponding robust weak sharp weakly efficient solutions of (UMP).

Theorem 56. Let $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^l$ and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ satisfying the following properties:

1. for any $u_i \in \mathcal{U}_i, i \in I$ and $v_j \in \mathcal{V}_j, j \in J, f_i(\cdot, u_i)$ is convex and continuous as well as $g(\cdot, v)$ is D -convex on $\mathbb{R}^n$;
2. for any $x \in \mathbb{R}^n, f_i(x, \cdot)$ is concave on $\mathcal{U}_i, i \in I$ and $g(x, \cdot)$ is D -concave on $\mathcal{V}$.

Then, the following statements are equivalent:

- (a) The (RSCQ) is fulfilled at $\hat{x} \in K$;
- (b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp weakly efficient solutions of (UMP) if and only if there exists $\eta > 0$ such that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u}_i \in \mathcal{U}_i(\hat{x}), \sigma_i \geq 0, i \in I$, not all zero, $\hat{v} \in \mathcal{V}$, and $\hat{\mu} \geq 0$ such that

$$0 \in \{-x^*\} + \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) + \partial \delta_C(\hat{x}) + \partial ((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \quad (1.5.1)$$

$$(\hat{\mu}_j g_j)(\hat{x}, \hat{v}_j) = 0, \quad (1.5.2)$$

and

$$\sigma_i f_i(\hat{x}, \hat{u}_i) = \sigma_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i), i \in I. \quad (1.5.3)$$

Proof. (a) $\Rightarrow$ (b) Assume that the (RSCQ) is satisfied at $\hat{x} \in \mathbb{R}^n$. Let $\hat{x}$ be a robust weak sharp weakly efficient solutions of (UMP) i.e., there exists $\eta > 0$ such that there does not exist $y \in K \setminus \{\hat{x}\}$ satisfying

$$\max_{u_i \in \mathcal{U}_i} f_i(y, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < \eta d_{\tilde{K}}(y), \text{ for all } i \in I,$$

or equivalently, for any $x \in K$,

$$\max_{i \in I} \left\{ \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_{\tilde{K}}(x). \quad (1.5.4)$$

By (1.5.4), there is $s \in I$ such that for all $x \in K$,

$$\begin{aligned} \max_{u_s \in \mathcal{U}_s} f_s(x, u_s) + \delta_K(x) - \eta d_{\tilde{K}}(x) &\geq \max_{u_s \in \mathcal{U}_s} f(\hat{x}, u_s) \\ &= \max_{u_s \in \mathcal{U}_s} f_s(\hat{x}, u_s) + \delta_K(\hat{x}) - \eta d_{\tilde{K}}(\hat{x}). \end{aligned} \quad (1.5.5)$$

Besides, according to (1.5.5), we follow the techniques used in Theorem 45 and obtain that for any $\xi \in \partial \eta d_{\tilde{K}}(x)$,

$$\begin{aligned} \langle \xi, x - \hat{x} \rangle &\leq \max_{u_s \in \mathcal{U}_s} f_s(x, u_s) + \delta_K(x) - \max_{u_s \in \mathcal{U}_s} f_s(\hat{x}, u_s) - \delta_K(\hat{x}). \end{aligned} \quad (1.5.6)$$

Therefore,

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial \left(\max_{u_s \in \mathcal{U}_s} f_s(\cdot, u_s) + \delta_K \right) (\hat{x}), \quad (1.5.7)$$

Note that the right hand side term of above inclusion is in the subdifferential of the max function:

$$\phi(x) = \max_{i \in I} \phi_i(x) := \max_{i \in I} \left(\max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) + \delta_K \right) (x).$$

Due to the well-known fact, subdifferential of maximum of functions at x is the convex hull of the union of subdifferentials of the active functions at x , the inclusion (1.5.7) becomes

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \text{co}(\cup \{ \partial \phi_i(\hat{x}) : \phi_i(\hat{x}) = \phi(\hat{x}) \}),$$

thereby

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \sum_{i \in I(\hat{x})} \sigma_i \partial \phi_i(\hat{x}),$$

where $\sigma_i \geq 0, i \in I(\hat{x})$ with $\sum_{i \in I(\hat{x})} \sigma_i = 1$ and $I(\hat{x}) := \{k \in I : \phi_k(\hat{x}) = \phi(\hat{x})\}$. Further, setting $\hat{\sigma}_i = \sigma, i \in I(\hat{x})$, and otherwise equals to 0 leads to

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \sum_{i \in I} \hat{\sigma}_i \partial \phi_i(\hat{x}).$$

By the definition of $\phi_i, i \in I$, the continuity of $\max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i), i \in I$ and the lower semicontinuity and convexity of δ_K , we have

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \sum_{i \in I} \hat{\sigma}_i \partial \left(\max_{u_i \in \mathcal{U}_i} f(\cdot, u_i) \right) (\hat{x}) + \sum_{i \in I} \hat{\sigma}_i (\partial \delta_K(\hat{x})).$$

It follows from Lemma 55 and the hypothesis such (RSCQ) is satisfied at $\hat{x} \in K$ that

$$\begin{aligned} \partial(\eta d_{\tilde{K}})(\hat{x}) &\subseteq \bigcup_{u \in \prod_{i=1}^l \mathcal{U}_i(\hat{x})} \sum_{i \in I} \hat{\sigma}_i \partial f_i(\cdot, u_i)(\hat{x}) + \sum_{i \in I} \hat{\sigma}_i (\partial \delta_C(\hat{x})) \\ &\quad + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}). \end{aligned}$$

Because $\hat{\sigma}_i \geq 0, i = 1, 2, \dots, l$, all nonzero, thereby

$$\begin{aligned} \partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq & \bigcup_{\substack{u=(u_i)_{i=1}^l, \\ u \in \prod_{i=1}^l \mathcal{U}_i(\hat{x})}} \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, u_i)(\hat{x})) + \partial \delta_C(\hat{x}) \\ & + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}). \end{aligned}$$

As $\partial d_{\tilde{K}}(x) = N_{\tilde{K}}(x) \cap \mathcal{B}_{\mathbb{R}^n}$, we obtain (1.5.1) as desired.

Conversely, assume that there is $\eta > 0$ such that (1.5.1)-(1.5.3) hold. Then, for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u} := (\hat{u}_1, \dots, \hat{u}_l) \in \prod_{i \in I} \mathcal{U}_i(\hat{x}), \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$\begin{aligned} x^* \in & \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) + \partial \delta_C(\hat{x}) + \partial((\hat{\mu} g)(\cdot, \hat{v}))(\hat{x}), \text{ and} \\ & (\hat{\mu} g)(\hat{x}, \hat{v}) = 0. \end{aligned} \quad (1.5.8)$$

Since $0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n} = \bigcap_{\varepsilon > 0} \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$, for each positive ε , $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ is nonempty. Let $\varepsilon > 0$ and $\xi \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ be arbitrary, then for any $x \in K$

$$\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x}) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \quad (1.5.9)$$

Therefore, we obtain

$$\eta d_{\tilde{K}}(\hat{x}) \leq \inf_{x \in \mathbb{R}^n} \eta d_{\tilde{K}}(x) + \varepsilon \leq \inf_{x \in K} \eta d_{\tilde{K}}(x) + \varepsilon.$$

Above inequality and (1.5.9) imply that

$$0 \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \quad (1.5.10)$$

Further, since $0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, we have that there exist $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})), \xi_\delta \in \partial \delta_C(\hat{x})$, and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu} g)(\cdot, \hat{v}))(\hat{x})$ such that

$$\xi_f + \xi_\delta + \xi_{\hat{\mu}g} = 0. \quad (1.5.11)$$

As $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) = \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{x}), \xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu} g)(\cdot, \hat{v}))(\hat{x})$, we have

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) & \geq \langle \xi_f, x - \hat{x} \rangle, \\ \delta_C(x) - \delta_C(\hat{x}) & \geq \langle \xi_\delta, x - \hat{x} \rangle, \text{ and} \\ (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\hat{x}, \hat{v}) & \geq \langle \xi_{\hat{\mu}g}, x - \hat{x} \rangle. \end{aligned}$$

Then, adding these inequalities yields

$$\begin{aligned} \langle 0, x - \hat{x} \rangle & \leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ & \quad + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\hat{x}, \hat{v}). \end{aligned}$$

Since $\hat{u}_i$ belongs to $\mathcal{U}_i(\hat{x})$, above inequality becomes the following one:

$$0 \leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}).$$

This together with $(\hat{\mu}g)(x, \hat{v}) \leq 0$, $(\hat{\mu}g)(\hat{x}, \hat{v}) = 0$, and (1.5.11), for any $x \in K$,

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \langle 0. \quad (1.5.12)$$

By summing (1.5.12) with (1.5.9), for any $x \in K$, we obtain

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon,$$

which means $\xi \in \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) \right) (\hat{x})$, and so $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \subseteq$

$\partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) \right) (\hat{x})$. As $\varepsilon > 0$ was arbitrary, for each $x \in K$,

$$0 \leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - (\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x})),$$

which is equivalent to the following inequality: for all $x \in K$

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_{\tilde{K}}(x) \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_{\tilde{K}}(\hat{x}).$$

It follows that

$$\sum_{i \in I} \hat{\sigma}_i \left(\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_{\tilde{K}}(x) \right) \geq \sum_{i \in I} \hat{\sigma}_i \left(\max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_{\tilde{K}}(\hat{x}) \right),$$

for any $x \in K$, which yields for any $i \in I$,

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_{\tilde{K}}(x) \geq \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_{\tilde{K}}(\hat{x}), \forall x \in K.$$

Therefore, for any $x \in K$

$$\max_{i \in I} \left\{ \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_{\tilde{K}}(x).$$

This means $\hat{x}$ is a robust weak sharp weakly efficient solutions of (UMP).

(b) $\Rightarrow$ (a) Let $\bar{\eta} > 0$ be given. Consider $f_i(x, u_i) := -\langle \xi_\delta, x \rangle + \bar{\eta} d_{\tilde{K}}(x)$, $i \in I$. Thus, for any $x \in K$,

$$\begin{aligned} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \bar{\eta} d_{\tilde{K}}(x) &= -\langle \xi_\delta, x \rangle \\ &\geq -\langle \xi_\delta, \hat{x} \rangle + \bar{\eta} d_{\tilde{K}}(\hat{x}) \\ &= \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned}$$

Thus, $\hat{x}$ is a robust weak sharp weakly efficient solutions of (UMP). By hypothesis, there is $\eta := \bar{\eta}$ such that (1.5.1) is fulfilled. Since for any $u_i \in \mathcal{U}_i$, $\partial f_i(\cdot, u_i)(\hat{x}) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x})$, one has

$$\sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, u_i)(\hat{x})) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x}),$$

where $\hat{\sigma}_i \geq 0, i \in I$ and all nonzero. Thus, we obtain that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u}_i \in \mathcal{U}_i(\hat{x}), \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$x^* \in \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x}) + \partial\delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

As $0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, we obtain

$$\xi_\delta \in \partial\delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

It follows that

$$\xi_\delta \in \partial\delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}),$$

and so we get the desired inclusion. Therefore, the proof is complete. $\square$

Remark 57. (i) In [28] and [29], the authors presented the necessary condition for the local sharp efficiency for the semi-infinite vector optimization problem by using the different method with Theorem 56. In fact, they employed the exact sum rule for Fréchet subdifferentials to obtained their results.

(ii) In [31], the exact sum rule for Mordukhovich subdifferentials was used as a vital tool under some regularity and differentiability assumptions for establishing their results. This means Theorem 56 use the different method from the mentioned work.

Next, by using the similar methods of section 3, a characterization of robust weak sharp weakly efficient solution sets in terms of a given robust weak sharp weakly efficient solution point of the problem is also illustrated in this section. In order to present the mentioned characterization, we start by deriving constant Lagrangian-type property for robust weak sharp weakly efficient solution sets of (MP). In what follows, let $u = (u_1, \dots, u_l) \in \mathcal{U}_1 \times \dots \times \mathcal{U}_l, \sigma = (\sigma_1, \dots, \sigma_l) \in \mathbb{R}_+^l, v \in \mathcal{V}$ and $\mu \geq 0$. The Lagrangian-type function $\mathcal{L}(\cdot, \sigma\mu, u, v)$ is given by

$$\mathcal{L}(x, \sigma, \mu, u, v) = \sum_{i \in I} \sigma_i f_i(x, u_i) + (\mu g)(x, v), \quad \forall x \in \mathbb{R}^n.$$

Theorem 58. Let $x \in \tilde{S}$ be given. Suppose that the (RSCQ) is fulfilled at $\hat{x}$. Then, there exist a positive valued vector $\hat{\sigma} := (\hat{\sigma}_1, \dots, \hat{\sigma}_l) \in \mathbb{R}_+^l, \hat{\sigma}_i, i \in I$ all nonzero, uncertain parameters $\hat{u} := (u_1, \dots, u_l) \in \mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_l, \hat{v} \in \mathcal{V}$, and Lagrange multiplier $\hat{\mu} \geq 0$ such that for any $x \in \tilde{S}$,

$$(\hat{\mu}g)(x, \hat{v}) = 0, \hat{u} \in \mathcal{U}(x), \text{ and } \mathcal{L}(x, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) \text{ is a constant on } \tilde{S}.$$

Proof. Since $\hat{x} \in \tilde{S}$ with the real number $\eta_1 > 0$ and the (RSCQ) is satisfied at this point $\hat{x}$, by Theorem 56, (1.5.1) holds for $\eta := \eta_1$. Since $N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$ is nonempty we can let $\varepsilon > 0$ be arbitrary and $x^* \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ be given. Besides, there exist $\hat{\sigma} \in \mathbb{R}_+^l$, all nonzero, $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$

and $\hat{\mu} \in D^*$ such that (1.5.1) is fulfilled. Let $x \in \tilde{S}$ be arbitrary. By the same fashion using in the proof of Theorem 45 we have

$$\begin{aligned} \langle x^*, x - \hat{x} \rangle &\leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ &\quad + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}), \end{aligned}$$

and so

$$\begin{aligned} \langle x^*, x - \hat{x} \rangle - \varepsilon &\leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ &\quad + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}), \end{aligned} \tag{1.5.13}$$

As $f_i(\cdot, u_i), i \in I$ and $g(\cdot, v)$ are convex, for any $u_i \in \mathcal{U}_i$ and $v \in \mathcal{V}$, we have $x^* \in \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i (f_i(\cdot, u_i) + \lambda g(\cdot, v)) \right)$. Hence, one has

$$\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i (f_i(\cdot, u_i) + \lambda g(\cdot, v)) \right) (\hat{x}),$$

thereby

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) + (\hat{\mu}g)(x, \hat{v}) - \eta d_{\tilde{K}}(x) &\geq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned} \tag{1.5.14}$$

Note that, as $x \in \tilde{S}$, then there exists $\eta_2 > 0$ such that for all $y \in K$,

$$\max_{u_i \in \mathcal{U}_i} f_i(y, u_i) \geq \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\tilde{K}}(y),$$

which implies

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(y, u_i) &\geq \sum_{i \in I} \hat{\sigma}_i \left(\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\tilde{K}}(y) \right) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\tilde{K}}(y) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i), \end{aligned}$$

for all $y \in \tilde{S}$. Since $\hat{x} \in \tilde{S}$,

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i). \tag{1.5.15}$$

From $\hat{\mu} \geq 0, g(x, \hat{v}) \leq 0$, and (1.5.14), it is not hard to see that

$$(\hat{\mu}g)(x, \hat{v}) = 0. \tag{1.5.16}$$

Moreover, by (1.5.14) and the positivity of $\eta d_{\tilde{K}}(x)$, we see that

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) &\geq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) \\ &\geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + \eta d_{\tilde{K}}(x) \\ &\geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned} \quad (1.5.17)$$

This together with (1.5.16) leads to

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i). \quad (1.5.18)$$

Thus, $\mathcal{L}(\cdot, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v})$ is constant on $\tilde{S}$ as follows:

$$\begin{aligned} \mathcal{L}(x, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) &= \sum_{i \in I} \hat{\sigma}_i f_i(x, u_i) + (\hat{\mu}g)(x, \hat{v}) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + (\hat{\mu}g)(x, \hat{v}) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + (\hat{\mu}g)(x, \hat{v}) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned}$$

This completes the proof. $\square$

Theorem 59. For the problem (UMP), let $\tilde{S}$ be the robust weak sharp weakly efficient solution set of (UMP) and $\hat{x} \in \tilde{S}$. Suppose that the (RSCQ) is fulfilled at $\hat{x} \in \tilde{S}$. Then, there exist $\hat{\sigma}_i \geq 0, i \in I$, all non zero, $\hat{u} := (\hat{u}_1, \dots, \hat{u}_l) \in \mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_l, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \geq 0$ such that

$$\begin{aligned} \tilde{S} = \left\{ x \in K : \exists \eta > 0, \exists a \in \bigcap_{y \in \{x, \hat{x}\}} \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{y}), \right. \\ \left. \exists \varepsilon > \eta d_{\tilde{K}}(x), \langle a, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x), (\hat{\mu}g)(x, \hat{v}) = 0, \right. \\ \left. \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = f_i(x, \hat{u}_i), i \in I \right\}. \end{aligned}$$

Proof. ($\subseteq$) Let $x \in \tilde{S}$ be given. Then there exists $\eta > 0$ such that (1.5.1) holds. Thus, there exist $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \geq 0$ such that (1.5.1) is fulfilled. Hence, we have that there exist $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(x)), \xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$ such that

$$0 = \xi_f + \xi_\delta + \xi_{\hat{\mu}g}, \text{ since } 0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}, \quad (1.5.19)$$

and

$$(\hat{\mu}g)(\hat{x}, \hat{v}) = 0. \quad (1.5.20)$$

Since $\xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$,

$$\delta_C(x) - \delta_C(\hat{x}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle \xi_\delta + \xi_{\hat{\mu}g}, x - \hat{x} \rangle. \quad (1.5.21)$$

By the same fashion in the proof of Theorem 56, we have

$$(\hat{\mu}g)(x, \hat{v}) = (\hat{\mu}g)(\hat{x}, \hat{v}) = 0,$$

and

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i).$$

Therefore, it follows from (1.5.21) that

$$\eta d_{\tilde{K}}(x) = 0 \geq \langle b + c, x - \hat{x} \rangle,$$

and so by (1.5.19), we obtain

$$\eta d_{\tilde{K}}(x) \geq \langle \xi_f, \hat{x} - x \rangle.$$

Simultaneously, since $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) = \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{x})$, we have

$$\langle \xi_f, \hat{x} - x \rangle \geq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i).$$

By (1.5.12) in the proof of Theorem 56, we obtain

$$\langle \xi_f, \hat{x} - x \rangle \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geq 0 = \eta d_{\tilde{K}}(x). \quad (1.5.22)$$

Hence, we have that $\langle \xi_f, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x)$. Next, we shall prove that there is $\varepsilon > \eta d_{\tilde{K}}(x) \geq 0$ such that

$$\xi_f \in \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (x).$$

In fact, we can show that $\xi_f \in \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (x)$. For any $y \in \mathbb{R}^n$,

$$\langle \xi_f, y - x \rangle = \langle \xi_f, y - \hat{x} \rangle + \langle \xi_f, \hat{x} - x \rangle \leq \langle \xi_f, y - \hat{x} \rangle$$

as $\langle \xi_f, \hat{x} - x \rangle \leq 0$. Since $a \in \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{x})$ and $f_i(x, \hat{u}_i) = f_i(\hat{x}, \hat{u}_i), i \in I$,

$$\begin{aligned} \langle \xi_f, y - x \rangle &\leq \sum_{i \in I} \hat{\sigma}_i f_i(y, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ &= \sum_{i \in I} \hat{\sigma}_i f_i(y, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i), \end{aligned}$$

which means $\xi_f \in \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (x)$.

($\supseteq$) Let

$$\begin{aligned} x \in \left\{ x \in K : \exists \eta > 0, \exists \xi_f \in \bigcap_{y \in \{x, \hat{x}\}} \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{x}), \exists \varepsilon > \eta d_{\tilde{K}}(x), \right. \\ \left. \langle \xi_f, x - \hat{x} \rangle = \eta d_{\tilde{K}}(x), (\mu g)(x, \hat{v}) = 0, \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = f_i(x, \hat{u}_i) \right\}. \end{aligned}$$

Since $\hat{x} \in \tilde{S}$, $\eta d_{\tilde{K}}(\hat{x}) = 0$ and so the assumption dealing with ξ_f lead to

$$\begin{aligned}
 -\eta d_{\tilde{K}}(\hat{x}) &= 0 \\
 &= \langle \xi_f, \hat{x} - x \rangle - \eta d_{\tilde{K}}(x) \\
 &\leq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \eta d_{\tilde{K}}(x) + \varepsilon \\
 &= \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \eta d_{\tilde{K}}(x) + \eta d_{\tilde{K}}(x) \\
 &= \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i),
 \end{aligned} \tag{1.5.23}$$

for any $\hat{\sigma}_i \geq 0, i \in I$, all nonzero. Therefore, we obtain

$$\begin{aligned}
 \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) &\leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + \eta d_{\tilde{K}}(\hat{x}) \\
 &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i).
 \end{aligned}$$

Since $\hat{x} \in \tilde{S}$ and $x \in K$, the conclusion that $x \in \tilde{S}$ is satisfied. $\square$

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Chapter 2

Output

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Chapter 3

Appendix

On the Existence of Solutions of Symmetric Vector Equilibrium Problems via Nonlinear Scalarization

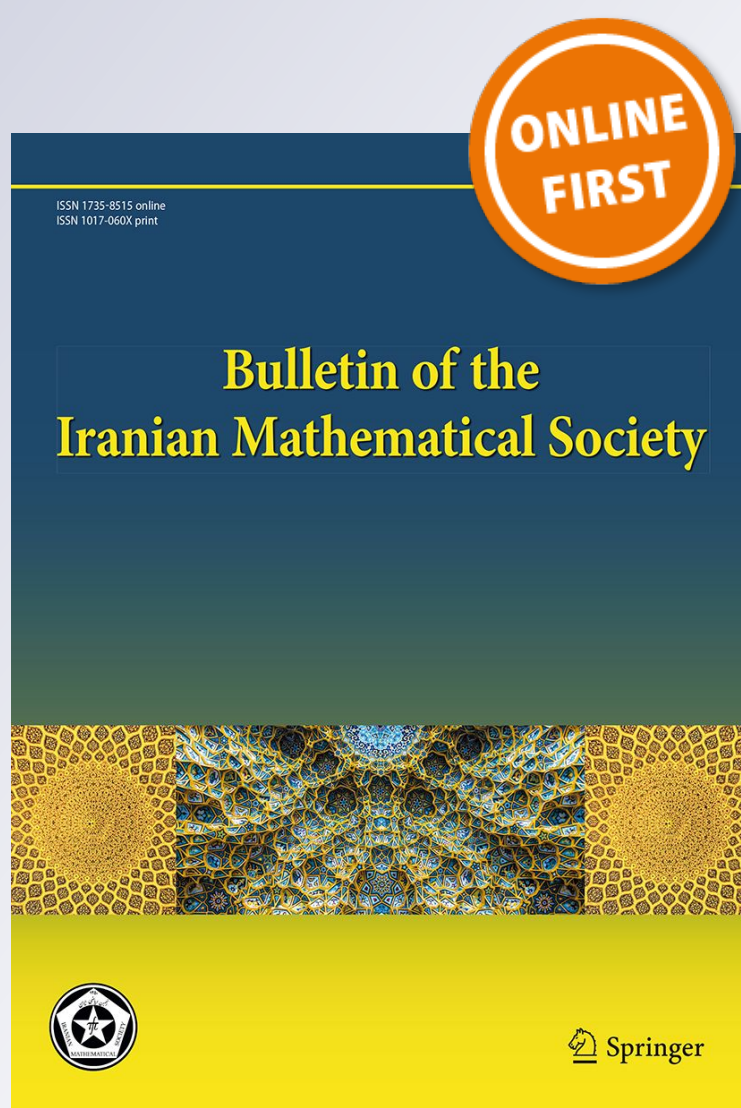
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On the Existence of Solutions of Symmetric Vector Equilibrium Problems via Nonlinear Scalarization

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Abstract

In this paper, by proposing a new type of generalized C -quasiconvexity for the set-valued mappings and using the nonlinear scalarization function ξ_q and its properties, without assumption of monotonicity and boundedness, some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems are established. Moreover, the convexity of solution sets is also investigated. Finally, some examples to support our results are provided.

Keywords Symmetric vector equilibrium problem · Nonlinear scalarization function · Generalized C -quasiconvexity · Upper and lower semicontinuity

Mathematics Subject Classification Primary 49K40; Secondary 90C33 · 91B50

1 Introduction

In 1994, the equilibrium problem was proposed in Blum and Oettli [8]. Then it has been intensively studied and extended. After that, more general equilibrium problems (see [7,8]) have been extended to the case of vector-valued bifunctions, namely vector

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equilibrium problems, which provides a unified model of several classes of problems including vector variational inequality problems, vector complementarity problems, vector optimization problems and vector saddle point problems (see, for example, [4,6,12,22,24,25]).

The system of vector equilibrium problems, which is a family of equilibrium problems for vector-valued bifunctions defined on a product set was introduced in 2000, by Ansari et al. [5]. Furthermore, its applications in vector optimization problems and Nash equilibrium problem for vector-valued functions were presented by the authors. Nowadays, it is well known that the system of equilibrium problems, systems of vector variational inequalities, system of vector variational-like inequalities, system of optimization problems, fixed point problems and several related topics as special cases (see more in [2,3,5,13,15,26,27,29–31]) contained in the system of vector equilibrium problems.

On the other hand, the symmetric vector equilibrium problem which is a generalization of the equilibrium problem has been studied by many authors. A main topic of current research is to establish existence theorems (see, for example, [16,18,20,23]). Another important topic is to study the topological properties of the solution sets, as it provides the possibility of continuously moving from one solution to any other solution.

Recently, reducing a vector optimization problem to a scalar optimization problem is a useful approach for analyzing it. The classical scalarization approaches using linear functionals have been already used for studying the existence of solutions of symmetric vector equilibrium problems (see [19,34]). On the other hand, nonlinear scalarization functions play a vital role in this reduction. The nonlinear scalarization function ξ_q , which was commonly known as the Gerstewitz function in the theory of vector optimization [11,32], has been used to studying many vector optimization problems. It is well known that the nonlinear scalarization function ξ_q has many good properties, such as continuity, sublinearity, convexity and (strict) monotonicity. These properties have been fully exploited in the literature to deal with various nonconvex problems with vector objectives, such as existence of solutions, gap functions, duality, vector variational principles, well-posedness, vector minimax inequalities and vector network equilibrium problems. However, to the best of our knowledge, there is no paper dealing with the existence theorems for the symmetric vector equilibrium problem using a nonlinear scalarization method. So, it is natural to raise and give an answer to the following question.

Question Can one establish existence theorems for the symmetric vector equilibrium problem using a nonlinear scalarization method?

Motivated by the works mentioned above, by proposing a new type of C -quasiconvexity for a set-valued mapping together with using a nonlinear scalarization function and its properties, without assumption of monotonicity and boundedness, some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems are established. Moreover, the convexity of solution sets are investigated. Finally, some examples to support our results are provided.

2 Preliminaries

Throughout this paper, let X, Y, E and Z be real Hausdorff topological vector spaces. Let $A \subseteq X$ and $B \subseteq E$ be nonempty closed convex subsets, $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ be two set-valued mappings. Let $C \subseteq Y$ and $P \subseteq Z$ be two closed convex pointed cones with $\text{int } C \neq \emptyset$ and $\text{int } P \neq \emptyset$. Let Y^* and Z^* be the topological dual spaces of Y and Z , respectively. Let C^* and P^* be the dual cones of C and P , respectively, that is,

$$C^* = \{f \in Y^* : \langle f, y \rangle \geq 0, \text{ for all } y \in C\}$$

and

$$P^* = \{g \in Z^* : \langle g, y \rangle \geq 0, \text{ for all } y \in P\}.$$

The two symmetric vector equilibrium problems under our consideration are as follows: (SVEP₁): find $(x, y) \in A \times B$ such that

$$\begin{cases} F(x, y, u) \not\subseteq (-\text{int } C), & \forall u \in A, \\ G(x, y, v) \not\subseteq (-\text{int } P), & \forall v \in B, \end{cases} \quad (\text{SVEP}_1)$$

and (SVEP₂): find $(x, y) \in A \times B$ such that

$$\begin{cases} F(x, y, u) \cap (-\text{int } C) = \emptyset, & \forall u \in A, \\ G(x, y, v) \cap (-\text{int } P) = \emptyset, & \forall v \in B. \end{cases} \quad (\text{SVEP}_2)$$

It is clear that the solution set of (SVEP₂) is a subset of (SVEP₁). It is remark that (SVEP₁) is a special problem of the symmetric multivalued vector quasiequilibrium problems studied by Anh and Khan [1]. They obtained some sufficient conditions for the solution existence in topological vector spaces. However, in this paper, we will discuss for the solution existence by utilizing the nonlinear scalarization method.

Remark 2.1 (Special cases)

- (i) If $C = P$, $f: A \times B \rightarrow Y$ and $g: A \times B \rightarrow Z$ are two single-valued mappings,

$$F(x, y, u) = \{f(u, y) - f(x, y)\}, \quad \forall (x, y, u) \in A \times B \times A$$

and

$$G(x, y, v) = \{g(x, v) - g(x, y)\}, \quad \forall (x, y, v) \in A \times B \times B,$$

then the problem (SVEP₂) reduces to the single-valued symmetric vector equilibrium problem considered by [16,18,20];

- (ii) If $G \equiv 0$ and $F(x, y, u) = \{f(x, u)\}$ for any $(x, y, u) \in A \times B \times A$, then the problem (SVEP₂) is the equilibrium problem which was considered and studied by many authors (for example, [5,7,10,21]);

- (iii) If $G \equiv 0$ and T is a mapping from A to $L(X, Y)$ where $L(X, Y)$ denotes the space of all continuous linear operators from X to Y , and $F(x, y, u) = \{\langle Tx, u - x \rangle\}$ for any $(x, y, u) \in A \times B \times A$, then the problem (SVEP₂) is the classic vector variational inequality problem which was introduced by Giannessi [21].

Now, we are going to recall the nonlinear scalarization function $\xi_q : Y \rightarrow \mathbb{R}$, where $q \in \text{int } C$, as follows.

Definition 2.2 [12,32] Given a fixed point $q \in \text{int } C$, the nonlinear scalarization function $\xi_q : Y \rightarrow \mathbb{R}$ is defined by

$$\xi_q(y) = \min\{t \in \mathbb{R} : y \in tq - C\}.$$

In the special case of $Y = \mathbb{R}^l$, $C = \mathbb{R}_+^l$ and $q = (1, 1, \dots, 1) \in \text{int } \mathbb{R}_+^l$, the nonlinear scalarization function can be expressed in the following equivalent form [12, Corollary 1.46]:

$$\xi_q(y) = \max_{1 \leq i \leq l} \{y_i\}, \quad \forall y = (y_1, y_2, \dots, y_l) \in \mathbb{R}^l.$$

The following results express some useful properties of the nonlinear scalarization function ξ_q .

Lemma 2.3 [12, Proposition 1.43] For any fixed $q \in \text{int } C$, $y \in Y$ and $r \in \mathbb{R}$. Then

- (i) $\xi_q(y) < r \Leftrightarrow y \in rq - \text{int } C$ (i.e., $\xi_q(y) \geq r \Leftrightarrow y \notin rq - \text{int } C$);
- (ii) $\xi_q(y) \leq r \Leftrightarrow y \in rq - C$;
- (iii) $\xi_q(y) = r \Leftrightarrow y \in rq - \partial C$, where ∂C denotes the boundary of C ;
- (iv) $\xi_q(rq) = r$;
- (v) ξ_q is continuous, positive homogeneous, subadditive and convex on Y ;
- (vi) ξ_q is monotone (i.e., $y_2 - y_1 \in C \Rightarrow \xi_q(y_1) \leq \xi_q(y_2)$) and strictly monotone (i.e., $y_2 - y_1 \in -\text{int } C \Rightarrow \xi_q(y_1) < \xi_q(y_2)$) (see [12,32]).

The property (i) of Lemma 2.3 will play a vital role in scalarization. In fact, as the definition of ξ_q , the property (iv) of Lemma 2.3 could be strengthened to that

$$\xi_q(y + rq) = \xi_q(y) + r, \quad \forall y \in Y, r \in \mathbb{R}. \quad (2.1)$$

For any $q \in \text{int } C$, the set C^q defined by

$$C^q := \{y^* \in C^* : \langle y^*, q \rangle = 1\}$$

is a weak*-compact set of Y^* (see [12]). In addition, for the forms of ξ_q which were used in [28, Proposition 2.2] and [12, Corollary 2.1], the following equivalent form of ξ_q can be deduced from both of them.

Proposition 2.4 [9, Proposition 2.2] Let $q \in \text{int } C$. Then for $y \in Y$, $\xi_q(y) = \max_{y^* \in C^q} \langle y^*, y \rangle$.

Proposition 2.5 [9, Proposition 2.3] ξ_q is Lipschitz on Y , and its Lipschitz constant is

$$L := \sup_{y^* \in C^q} \|y^*\| \in \left[\frac{1}{\|q\|}, +\infty \right).$$

The following example can be found in [9, Example 2.1].

Example 2.6 (i) In the scalar case of $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the Lipschitz constant of ξ_q is $L = \frac{1}{q}$ ($q > 0$). Then,

$$|\xi_q(x) - \xi_q(y)| = \frac{1}{q}|x - y|.$$

for all $x, y \in \mathbb{R}$ and $q > 0$.

(ii) If $Y = \mathbb{R}^2$ and $C = \{(y_1, y_2) \in \mathbb{R}^2 : \frac{1}{4}y_1 \leq y_2 \leq 2y_1\}$. Take $q = (2, 3) \in \text{int } C$. Then,

$$C^q = \{(y_1, y_2) \in \mathbb{R} : 2y_1 + 3y_2 = 1, y_1 \in [-0.1, 2]\}.$$

Then, Lipschitz constant is $L = \sup_{y^* \in C^q} \|y^*\| = \|(-2, 1)\| = \sqrt{5}$. Hence,

$$|\xi_q(y) - \xi_q(y')| = \sqrt{5}|y - y'|,$$

for all $y, y' \in \mathbb{R}$.

Definition 2.7 Let X and Y be real Hausdorff topological vector spaces. A set-valued mapping $T : X \rightarrow 2^Y$ is said to be

(i) *closed* if its graph

$$\text{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}$$

is closed in $X \times Y$;

(ii) *upper semicontinuous* (u.s.c) if, for every $x \in X$ and every open set V satisfying $T(x) \subseteq V$, there exists a neighborhood U of x such that

$$T(U) = \bigcup_{y \in U} T(y) \subseteq V;$$

(iii) *lower semicontinuous* (l.s.c) if, for any $x \in X$, $y \in T(x)$ and any neighborhood V of y , there exists a neighborhood U of x such that

$$T(z) \cap V \neq \emptyset$$

for all $z \in U$.

Lemma 2.8 [33] *A set-valued mapping $T: X \rightarrow 2^Y$ is lower semicontinuous at $x \in X$ if and only if, for any net $\{x_i\}$ such that $x_i \rightarrow x$ and $y \in T(x)$, there exists a net $\{y_i\}$ with $y_i \in T(x_i)$ such that $y_i \rightarrow y$.*

Now we recall some concepts related to the C -convexity for the set-valued mapping.

Definition 2.9 [34] Let $T: A \rightarrow 2^Y$ be a set-valued mapping, where A is a nonempty convex subset of X . T is said to be

- (i) C -convex if for every $z_1, z_2 \in A$ and $t \in [0, 1]$,

$$tT(z_1) + (1-t)T(z_2) \subseteq T(tz_1 + (1-t)z_2) + C.$$

- (ii) C -quasiconvex if for every $z_1, z_2 \in A$ and $t \in [0, 1]$, either

$$T(z_1) \subseteq T(tz_1 + (1-t)z_2) + C;$$

or

$$T(z_2) \subseteq T(tz_1 + (1-t)z_2) + C.$$

In this paper, we introduce a new type of C -quasiconvexity for the given set-valued mapping which is a generalization of both C -convexity and C -quasiconvexity.

Definition 2.10 Let $T: A \rightarrow 2^Y$ be a set-valued mapping, where A is a nonempty convex subset of X . Then T is said to be *generalized C -quasiconvex* if for every $z_1, z_2 \in D$ and $t \in [0, 1]$, either

$$T(z_1) \cap (T(tz_1 + (1-t)z_2) + C) \neq \emptyset;$$

or

$$T(z_2) \cap (T(tz_1 + (1-t)z_2) + C) \neq \emptyset.$$

Remark 2.11 It can be seen from the above definition that every C -quasiconvex mapping is a generalized C -quasiconvex mapping. However, the converse does not hold in general which can be found in Example 3.12 in Sect. 3.

The following lemma plays a key role in results reported in many works (for example, [12,34]). Furthermore, we need it in the sequel.

Lemma 2.12 [14] *Let $\{X_i\}_{i \in I}$ be a family of nonempty convex sets where each X_i is contained in a Hausdorff topological vector space E_i . Let $X = \prod_{i \in I} X_i$. For each $i \in I$, let $P_i: X \rightarrow 2^{X_i}$ be a set-valued mapping such that*

- (i) *for each $i \in I$, $P_i(x)$ is convex for all $x = (x_i)_{i \in I}$;*
- (ii) *for each $x \in X$, $x_i \notin P_i(x)$;*
- (iii) *for each $y_i \in X_i$, $P_i^{-1}(y_i) = \{x \in X: P_i(x) \supseteq \{y_i\}\}$ is open in X ;*

- (iv) for each $i \in I$, there exists a nonempty compact subset N of X and a nonempty compact convex subset B_i of X_i such that for each $x \in X \setminus N$, there is an $i \in I$ satisfying $P_i(x) \cap B_i \neq \emptyset$.

Then there exists $x \in X$ such that $P_i(x) = \emptyset$ for all $i \in I$.

3 Symmetric Vector Equilibrium Problems

In this section, we present the scalar symmetric equilibrium problems which are equivalent to the symmetric vector equilibrium problems (SVEP₁) and (SVEP₂). The relationships between the solution sets and the existence results for them were established.

For any $q \in \text{int } C$ and $q' \in \text{int } P$, we also consider the following scalar symmetric equilibrium problems: (SSEP₁(ξ)): find $(x, y) \in A \times B$, such that

$$\begin{cases} \forall u \in A, \exists z \in F(x, y, u) : \xi_q(z) \geq 0, \\ \forall v \in B, \exists w \in G(x, y, v) : \xi_{q'}(w) \geq 0 \end{cases} \quad (\text{SSEP}_1(\xi))$$

and (SSEP₂(ξ)): find $(x, y) \in A \times B$, such that

$$\begin{cases} \xi_q(F(x, y, u)) \subseteq \mathbb{R}_+, \quad \forall u \in A, \\ \xi_{q'}(G(x, y, v)) \subseteq \mathbb{R}_+, \quad \forall v \in B. \end{cases} \quad (\text{SSEP}_2(\xi))$$

We denote the solution sets of (SVEP₁), (SVEP₂), (SSEP₁(ξ)) and (SSEP₂(ξ)) by S_1 , S_2 , $S_1(\xi)$ and $S_2(\xi)$, respectively.

Before we give the existence of solutions for (SVEP₁) and (SVEP₂), we first need the following simple fact which illustrates the relationship between the solution sets S_1 and $S_1(\xi)$.

Lemma 3.1 For any fixed $q \in \text{int } C$ and $q' \in \text{int } P$, the following assertion is valid:

$$S_1 = S_1(\xi).$$

Proof First, we assume that $(x', y') \in S_1$. Hence, for any $u \in A$, there exists $z \in F(x', y', u)$ such that

$$z \notin -\text{int } C.$$

Similarly, for any $v \in B$, there exists $w \in G(x', y', v)$ such that

$$w \notin -\text{int } P.$$

So, it follows from Lemma 2.3(i) that for any $(u, v) \in A \times B$, there exists (z, w) such that

$$\xi_q(z) \geq 0 \quad \text{and} \quad \xi_{q'}(w) \geq 0.$$

Therefore, we immediately get that $(x', y') \in S_1(\xi)$. Conversely, assume that $(x', y') \in S_1(\xi)$, then we can prove that $(x', y') \in S_1$ using Lemma 2.3 with the reverse way of the above part. $\square$

Theorem 3.2 Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex pointed cone with $q \in \text{int } C \neq \emptyset$ and $q' \in \text{int } P \neq \emptyset$. Suppose $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ are two set-valued mappings satisfying the following conditions:

- (i) for each $(x, y) \in A \times B$, $F(x, y, x) \cap C \neq \emptyset$, and $G(x, y, y) \cap P \neq \emptyset$;
- (ii) for each $(x, y) \in A \times B$, $F(x, y, \cdot)$ is C -quasiconvex on A as well as $G(x, y, \cdot)$ is P -quasiconvex on B ;
- (iii) for each $u \in A$, $F(\cdot, \cdot, u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot, \cdot, v)$ is lower semicontinuous on $A \times B$;
- (iv) there exists nonempty compact convex sets $D_1 \subset A$ and $D_2 \subset B$ such that for each $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$, there exist $x' \in D_1$ such that $F(x, y, x') \subseteq -\text{int } C$ or $y' \in D_2$ such that $G(x, y, y') \subseteq -\text{int } P$.

Then the set S_1 is nonempty.

Proof For each $(x, y) \in A \times B$, define $P_1: A \times B \rightarrow 2^A$ and $P_2: A \times B \rightarrow 2^B$ as follows:

$$P_1(x, y) = \{u \in A : \forall z \in F(x, y, u), \xi_q(z) \notin \mathbb{R}_+\}$$

and

$$P_2(x, y) = \{v \in B : \forall w \in G(x, y, v), \xi_{q'}(w) \notin \mathbb{R}_+\}.$$

We will show that P_1 and P_2 satisfy all conditions of Lemma 2.12. First, we prove that $P_1(x, y)$ and $P_2(x, y)$ are convex for all $(x, y) \in A \times B$. Suppose on the contrary that for some $(x, y) \in A \times B$, $P_1(x, y)$ is not convex. Then there exists $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 = 1$ and $u_1, u_2 \in P_1(x, y)$ such that $t_1 u_1 + t_2 u_2 \notin P_1(x, y)$. This means that

$$\xi_q(z) \in \mathbb{R}_+, \quad \exists z \in F(x, y, t_1 u_1 + t_2 u_2).$$

By assumption (ii), we have either

$$F(x, y, u_1) \subseteq F(x, y, t_1 u_1 + t_2 u_2) + C,$$

or

$$F(x, y, u_2) \subseteq F(x, y, t_1 u_1 + t_2 u_2) + C.$$

Hence, we get either

$$\xi_q(F(x, y, u_1)) \subseteq \xi_q(F(x, y, t_1 u_1 + t_2 u_2)) + \xi_q(C) \subseteq \mathbb{R}_+,$$

or

$$\xi_q(F(x, y, u_2)) \subseteq \xi_q(F(x, y, t_1 u_1 + t_2 u_2)) + \xi_q(C) \subseteq \mathbb{R}_+,$$

which contradicts $u_1, u_2 \in P_1(x, y)$. Similarly, we can show that $P_2(x, y)$ is convex.

Next, we want to verify condition (ii) of Lemma 2.12, in fact we have to show that for each $(x, y) \in A \times B$, $x \notin P_1(x, y)$ and $y \notin P_2(x, y)$. For each $(x, y) \in A \times B$, it follows from assumption (i) that $F(x, y, x) \cap C \neq \emptyset$ and $G(x, y, y) \cap P \neq \emptyset$. Thus, there exists $(z, w) \in F(x, y, x) \times G(x, y, y)$ such that

$$\xi_q(z) \in \mathbb{R}_+ \quad \text{and} \quad \xi_{q'}(w) \in \mathbb{R}_+.$$

Invoking the definitions of $P_1(x, y)$ and $P_2(x, y)$, we have

$$x \notin P_1(x, y) \quad \text{and} \quad y \notin P_2(x, y).$$

To prove condition (iii) of Lemma 2.12, assume that $(u, v) \in A \times B$. Note that

$$(P_1^{-1}(u))^c = \{(x, y) \in A \times B : \exists z \in F(x, y, u) \text{ s.t. } \xi_q(z) \in \mathbb{R}_+\}. \quad (3.1)$$

Let $\{(x_i, y_i)\} \subseteq (P_1^{-1}(u))^c$ with $(x_i, y_i) \rightarrow (x_0, y_0)$. As $F(x_0, y_0, u) \neq \emptyset$, we choose $z_0 \in F(x_0, y_0, u)$. By Lemma 2.8, there exists a net $\{z_i\} \subseteq F(x_i, y_i, u)$ such that $z_i \rightarrow z_0$. Hence, using the continuity of ξ_q we get

$$\xi_q(z_i) \rightarrow \xi_q(z_0).$$

The condition (3.1) yields that $\xi_q(z_0) \geq 0$. Therefore, $(x_0, y_0) \in (P_1^{-1}(u))^c$ and so $(P_1^{-1}(u))^c$ is closed. Thus, we have that $P_1^{-1}(u)$ is open on A . Similarly, we can prove that $P_2^{-1}(v)$ is open on B . This completes the proof of condition (iii) of Lemma 2.12.

Finally, we have to show that condition (iv) of Lemma 2.12 holds. By assumption (iv), there exists nonempty compact set $D_1 \times D_2 \subseteq A \times B$ such that for any $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$, there exists $x' \in D_1$ such that $F(x, y, x') \subseteq -\text{int } C$ or $y' \in D_2$ such that $G(x, y, y') \subseteq -\text{int } P$. Therefore, for each $(z, w) \in F(x, y, x') \times G(x, y, y')$, $\xi_q(z) \notin \mathbb{R}_+$, or $\xi_{q'}(w) \notin \mathbb{R}_+$. So, we immediately obtain, by the definitions of $P_1(x, y)$ and $P_2(x, y)$, that $x' \in P_1(x, y)$ or $y' \in P_2(x, y)$. This completes the proof of the condition (iv) of Lemma 2.12.

Consequently, the set-valued mappings P_1 and P_2 satisfy all conditions given in Lemma 2.12. So, there exists $(\bar{x}, \bar{y}) \in A \times B$ such that

$$P_1(\bar{x}, \bar{y}) = \emptyset \quad \text{and} \quad P_2(\bar{x}, \bar{y}) = \emptyset.$$

Then, for each $(u, v) \in A \times B$, there exists $(z, w) \in F(\bar{x}, \bar{y}, u) \times G(\bar{x}, \bar{y}, v)$ such that

$$\xi_q(z) \in \mathbb{R}_+ \quad \text{and} \quad \xi_{q'}(w) \in \mathbb{R}_+.$$

Therefore, we have $(\bar{x}, \bar{y}) \in S_1(\xi)$. Using Lemma 3.1, we conclude that S_1 is nonempty. $\square$

Remark 3.3 Comparing Theorem 3.2 and the results obtained in Anh and Khan [1], we can see that the main difference is that our techniques are based on the utilizing the nonlinear scalarization method while the mentioned work employed the relaxed quasiconvexities of the multivalued mappings $F(\cdot, y, \cdot)$ and $G(\cdot, x, \cdot)$ as the main tools.

Now, we give the following example to illustrate Theorem 3.2.

Example 3.4 Let $X = Y = Z = \mathbb{R}$, $A = B = [0, 1]$, $C = P = \mathbb{R}_+$ and define the mappings $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ by, for any $(x, y, u) \in A \times B \times A$ and $(x, y, v) \in A \times B \times B$,

$$F(x, y, u) = [x - u, u] \quad \text{and} \quad G(x, y, v) = [y - v, v].$$

It is clear that (i) given in Theorem 3.2 is satisfied. To establish the assumption (ii) of Theorem 3.2, let $u_1, u_2 \in A$ and $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 = 1$. Assume that $u_1 \leq u_2$, then for each $z \in F(x, y, u_1)$

$$x - u_1 \leq z \leq u_1.$$

Then, we can get that

$$x - t_1 u_1 - t_2 u_2 \leq z \leq t_1 u_1 + t_2 u_2,$$

which means

$$F(x, y, u_1) \subseteq F(x, y, t_1 u_1 + t_2 u_2) \subseteq F(x, y, t_1 u_1 + t_2 u_2) + C$$

and so $F(x, y, \cdot)$ is C -quasiconvex on A . By the same fashion, we can show that $G(x, y, \cdot)$ also is P -quasiconvex on B .

Next, we prove the assumption (iii) of Theorem 3.2. Let $u \in A$ be arbitrarily fixed. Let $(x', y') \in A \times B$, $z \in F(x', y', u)$ and U be any neighborhood of z . Then, for each (x, y) in a neighborhood $[x', 1] \times B$ of (x', y') , we have

$$F(x, y, u) = [x - u, u] \supseteq [x' - u, u].$$

Thus, $F(x, y, u) \cap U \supseteq \{z\} \neq \emptyset$, $\forall (x, y) \in [x', 1] \times B$ and so the first statement of assumption (iii) of Theorem 3.2 is true. Similarly, we can check that the second one is also true.

Finally, take $D_1 = [\frac{1}{2}, 1] \subseteq A$ and $D_2 = [\frac{1}{2}, 1] \subseteq B$. Then, for each $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$, there exist $x' = 1 \in D_1$ and $y' = 1 \in D_2$ such that

$$F(x, y, x') = [x - 1, 1] \quad \text{and} \quad G(x, y, y') = [y - 1, 1].$$

Thus, we have

$$F(x, y, x') \subseteq [-1, 0) \subset -\text{int } C \quad \text{and} \quad P(x, y, y') \subseteq [-1, 0) \subset -\text{int } P,$$

for all $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$. The assumption (iv) of Theorem 3.2 is proved.

Now, we will show that $S_1 \neq \emptyset$. Taking $(x, y) = (1, 1) \in A \times B$ leads to

$$F(x, y, u) = F(1, 1, u) = [1 - u, u],$$

and

$$G(x, y, v) = G(1, 1, v) = [1 - v, v],$$

which, respectively, follows that

$$F(1, 1, u) = [1 - u, u] \not\subseteq -\text{int } \mathbb{R}_+ = -\text{int } C, \quad \forall u \in A$$

and

$$G(1, 1, v) = [1 - v, v] \not\subseteq -\text{int } \mathbb{R}_+ = -\text{int } P, \quad \forall v \in B.$$

This yields $(1, 1) \in S_1$. □

We give the following examples to show that all of the assumptions of Theorem 3.2 are essential and cannot be dropped.

Example 3.5 (Assumption (i) of Theorem 3.2 is essential) Let $X = Y = Z = \mathbb{R}$, $A = B = [0, 1]$, $C = P = \mathbb{R}_+$ and define the mappings $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ as

$$F(x, y, u) = \left(-u - \frac{1}{2}, u\right) \quad \text{and} \quad G(x, y, v) = \left(-v - \frac{1}{2}, v\right).$$

First, to show that assumption (i) does not hold, take $x = y = 0$. So, we have that

$$F(x, y, x) \cap C = F(0, 0, 0) \cap \mathbb{R}_+ = \left(-\frac{1}{2}, 0\right) \cap \mathbb{R}_+ = \emptyset$$

and

$$G(x, y, y) \cap P = G(0, 0, 0) \cap \mathbb{R}_+ = \left(-\frac{1}{2}, 0\right) \cap \mathbb{R}_+ = \emptyset.$$

We can verify all of the other assumptions of Theorem 3.2. However, the problem SVEP_1 has no solution, i.e., $S_1(F, G) = \emptyset$ since for each $(x, y) \in A \times B$, there exists $(u, v) = (0, 0) \in A \times B$ such that

$$F(x, y, u) = F(x, y, 0) = \left(-\frac{1}{2}, 0\right) \subseteq -\text{int } \mathbb{R}_+ = -\text{int } C$$

and

$$G(x, y, v) = G(x, y, 0) = \left(-\frac{1}{2}, 0\right) \subseteq -\text{int } \mathbb{R}_+ = -\text{int } P.$$

The reason is assumption (i) of Theorem 3.2 is violated.

Example 3.6 (Assumption (ii) of Theorem 3.2 is essential) Let $X = Y = Z = \mathbb{R}$, $A = B = [0, 1]$, $C = P = \mathbb{R}_+$ and define the mappings $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ by

$$F(x, y, u) = \begin{cases} \left\{\frac{1}{2}\right\}, & u = x, \\ (-u - 1, u - 1], & \text{otherwise,} \end{cases}$$

and

$$G(x, y, v) = \begin{cases} \left\{\frac{1}{2}\right\}, & v = y, \\ (-v - 1, v - 1], & \text{otherwise.} \end{cases}$$

It is clear that assumptions (i), (iii) and (iv) of Theorem 3.2 are satisfied. However, assumption (ii) of Theorem 3.2 is violated. Indeed, let $x = y = \frac{1}{2}$, $t = \frac{1}{2}$, $u_1 = 1$ and $u_2 = 0$. So, we have that

$$\begin{aligned} F(x, y, u_1) &= F\left(\frac{1}{2}, \frac{1}{2}, 1\right) = (-2, 1], \\ F(x, y, u_2) &= F\left(\frac{1}{2}, \frac{1}{2}, 0\right) = (-1, 0] \end{aligned}$$

and

$$F(x, y, t_1 u_1 + t_2 u_2) = F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \left\{\frac{1}{2}\right\}.$$

Thus, we have that

$$F(x, y, u_1) \not\subseteq F(x, y, t_1 u_1 + t_2 u_2) + C$$

and

$$F(x, y, u_2) \not\subseteq F(x, y, t_1 u_1 + t_2 u_2) + C.$$

Note that $S_1(F, G) = \emptyset$. Since for each $(x, y) \in A \times B$, there exists $(u, v) = (0, 0) \in A \times B$ such that

$$F(x, y, u) = (-2, -1] \subseteq -\text{int } \mathbb{R}_+ = -\text{int } C$$

and

$$G(x, y, v) = (-2, -1] \subseteq -\text{int } \mathbb{R}_+ = -\text{int } P.$$

Thus, assumption (ii) of Theorem 3.2 cannot be dropped. $\square$

Example 3.7 (Assumption (iii) of Theorem 3.2 is essential) Let $X = Y = Z = \mathbb{R}$, $A = B = [-1, 1]$, $C = P = \mathbb{R}_+$ and define the mappings $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ as

$$F(x, y, u) = \begin{cases} \{x - u\}, & x \leq 0, \\ [-\frac{1}{2}, \frac{u}{2}), & \text{otherwise,} \end{cases}$$

and

$$G(x, y, v) = \begin{cases} \{y - v\}, & y \leq 0, \\ [-\frac{1}{2}, \frac{v}{2}), & \text{otherwise.} \end{cases}$$

To show that assumption (iii) of Theorem 3.2 is not satisfied, take $x' = y' = 0, u = 1$. Then, we have $F(x', y', u) = \{-1\}$. Let $z \in F(x', y', u)$, then $(-\frac{3}{2}, -\frac{1}{2})$ is a neighborhood of z . Thus, for each neighborhood V of (x', y') we have

$$\left(-\frac{3}{2}, -\frac{1}{2}\right) \cap V = \left(-\frac{3}{2}, -\frac{1}{2}\right) \cap \left[-\frac{1}{2}, \frac{1}{2}\right) = \emptyset,$$

for all $(x, y) \in V$ with $x > x' = 0$. In fact, it is not hard to show that all of other assumptions in Theorem 3.2 are satisfied, especially assumption (i) and (ii), which are clear by the definitions of F and G . However, $S_1(F, G) = \emptyset$. For each $(x, y) \in A \times B$, consider the following two cases:

if $x \leq 0$, then $F(x, y, u) = \{x - u\} \subseteq -\text{int } \mathbb{R}_+, \forall u \in (0, -1]$,

if $x > 0$, then $F(x, y, u) = [-\frac{1}{2}, \frac{u}{2}) \subseteq -\text{int } \mathbb{R}_+, \forall u \in [-1, 0]$. The reason is assumption (iii) of Theorem 3.2 is dropped. $\square$

Example 3.8 (Assumption (iv) of Theorem 3.2 is essential) Let $X = Y = Z = \mathbb{R}$, $A = B = [0, 1]$, $C = P = \mathbb{R}_+$ and define the mappings $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ as

$$F(x, y, u) = \begin{cases} (-xu - x, xu), & x = y \neq 0, \\ [-1, xu), & \text{otherwise} \end{cases}$$

and

$$G(x, y, v) = \begin{cases} (-yv - y, yv), & x = y \neq 0, \\ [-1, yv), & \text{otherwise.} \end{cases}$$

We can show that almost all of the assumptions of Theorem 3.2 are satisfied, except assumption (iv). To show that assumption (iv) of Theorem 3.2 is violated, for any nonempty compact set $D_1 \times D_2 \subseteq A \times B$, we take $(x, y) = (1, 1) \in (A \times B) \setminus (D_1 \times D_2)$. Then, for each $(x', y') \in (D_1 \times D_2)$, we have

$$\begin{aligned} F(x, y, x') &= (-x' - 1, x') \not\subseteq -\text{int } \mathbb{R}_+, \\ G(x, y, y') &= (-y' - 1, y') \not\subseteq -\text{int } \mathbb{R}_+. \end{aligned}$$

Then, the problem (SVEP₁) has no solution since for each $(x, y) \in A \times B$, there exists $(u, v) = (0, 0) \in A \times B$, such that

$$F(x, y, u) = \begin{cases} (-x, 0) \subseteq -\text{int } \mathbb{R}_+, & x = y \neq 0, \\ [-1, 0) \subseteq -\text{int } \mathbb{R}_+, & \text{otherwise} \end{cases}$$

and

$$G(x, y, v) = \begin{cases} (-y, 0) \subseteq -\text{int } \mathbb{R}_+, & x = y \neq 0, \\ [-1, 0) \subseteq -\text{int } \mathbb{R}_+, & \text{otherwise.} \end{cases}$$

Hence, assumption (iii) of Theorem 3.2 is essential. $\square$

Now we shall discuss about a link between the solution sets S_2 and $S_2(\xi)$ for (SVEP₂).

Lemma 3.9 For any fixed $q \in -\text{int } C$ and $q' \in -\text{int } P$,

$$S_2 = S_2(\xi).$$

Proof First, we assume that $(x', y') \in S_2(F, G)$, which means

$$F(x', y', u) \cap (-\text{int } C) = \emptyset, \quad \text{for all } u \in A$$

and

$$G(x', y', v) \cap (-\text{int } P) = \emptyset, \quad \text{for all } v \in B.$$

So, by Lemma 2.3 we obtain that for any $(u, v) \in A \times B$,

$$z \notin -\text{int } C \quad \text{and} \quad w \notin -\text{int } P$$

for all $(z, w) \in F(x', y', u) \times G(x', y', v)$. So, it follows that, for any pair $(u, v) \in A \times B$,

$$\xi_q(z) \in \mathbb{R}_+ \quad \text{and} \quad \xi_{q'}(w) \in \mathbb{R}_+$$

for all $(z, w) \in F(x', y', u) \times G(x', y', v)$. Therefore, we get by the definition of ξ_q and $\xi_{q'}$ that

$$\xi_q(F(x', y', u)) \subseteq \mathbb{R}_+, \quad \forall u \in A$$

and

$$\xi_{q'}(G(x', y', v)) \subseteq \mathbb{R}_+, \quad \forall v \in B.$$

Hence, $(x', y') \in S_2(\xi)$. Conversely, assume that $(x', y') \in S_2(\xi)$, then we can prove that $(x', y') \in S_2$ using the same argument given in the proof of Lemma 2.3. $\square$

Now a result on existence of solutions of the (SVEP₂) is verified by making use of the nonlinear scalarization function.

Theorem 3.10 *Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex cones with $q \in \text{int} C \neq \emptyset$ and $q' \in \text{int} P \neq \emptyset$. Suppose $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ are two set-valued mappings which satisfy the following conditions:*

- (i) *for each $(x, y) \in A \times B$, $F(x, y, x) \subseteq C$ and $G(x, y, y) \subseteq P$;*
- (ii) *for each $(x, y) \in A \times B$, $F(x, y, \cdot)$ is generalized C -quasiconvex on A as well as $G(x, y, \cdot)$ is generalized C -quasiconvex on B ;*
- (iii) *for each $(x, y, u) \in A \times B \times A$ with $F(x, y, u) \cap -\text{int } C \neq \emptyset$,*

$$z \in F(x, y, u) \Rightarrow z - C \subseteq -\text{int } C$$

and also for each $(x, y, v) \in A \times B \times B$ with $G(x, y, v) \cap -\text{int } P \neq \emptyset$,

$$w \in G(x, y, v) \Rightarrow w - P \subseteq -\text{int } P;$$

- (iv) *for each $u \in A$, $F(\cdot, \cdot, u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot, \cdot, v)$ is lower semicontinuous on $A \times B$;*
- (v) *there exist nonempty compact convex sets $D_1 \subseteq A$ and $D_2 \subseteq B$ such that for each $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$, there exists $x' \in D_1$ such that $F(x, y, x') \cap -\text{int } C \neq \emptyset$ or $y' \in D_2$ such that $G(x, y, y') \cap -\text{int } P \neq \emptyset$.*

Then the solution set S_2 is nonempty.

Proof Let the set-valued mappings $P_1: A \times B \rightarrow 2^A$ and $P_2: A \times B \rightarrow 2^B$ be defined by, for any $(x, y) \in A \times B$,

$$P_1(x, y) = \{u \in A : \xi_q(F(x, y, u)) \not\subseteq \mathbb{R}_+\}$$

and

$$P_2(x, y) = \{v \in B : \xi_{q'}(G(x, y, v)) \not\subseteq \mathbb{R}_+\}.$$

We first show that P_1 and P_2 satisfy all the conditions given in Lemma 2.12. First, we prove that $P_1(x, y)$, $P_2(x, y)$ are convex for all $(x, y) \in A \times B$. Assume on the contrary that $P_1(x, y)$ is not convex. Then there exists $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 = 1$ and $u_1, u_2 \in P_1(x, y)$ such that $t_1 u_1 + t_2 u_2 \notin P_1(x, y)$, which gives that

$$\xi_q(F(x, y, t_1 u_1 + t_2 u_2)) \subseteq \mathbb{R}_+.$$

By assumption (ii), we have either

$$F(x, y, u_1) \cap (F(x, y, t_1 u_1 + t_2 u_2) + C) \neq \emptyset,$$

or

$$F(x, y, u_2) \cap (F(x, y, t_1 u_1 + t_2 u_2) + C) \neq \emptyset.$$

It follows that there is $z \in F(x, y, t_1 u_1 + t_2 u_2)$ such that either

$$z = z_1 - c, \quad \exists z_1 \in F(x, y, u_1), \quad \exists c \in C$$

or

$$z = z_2 - c', \quad \exists z_2 \in F(x, y, u_2), \quad \exists c' \in C.$$

Thus, by assumption (iii), we have either

$$\xi_q(z) = \xi_q(z_1 - c) < 0,$$

or

$$\xi_q(z) = \xi_q(z_2 - c') < 0.$$

This contradicts to $t_1 u_1 + t_2 u_2 \notin P_1(x, y)$. Similarly, we can show that $P_2(x, y)$ is convex.

Next, we verify condition (ii) of Lemma 2.12. In fact, we have to show that $x \notin P_1(x, y)$ and $y \notin P_2(x, y)$. Let $(x, y) \in A \times B$. By assumption (i), for each $(z, w) \in F(x, y, x) \times G(x, y, y)$. This says $z \in C$ and $w \in P$, and so

$$z \notin -\text{int } C \quad \text{and} \quad w \notin -\text{int } P.$$

Hence, by Lemma 2.3(i), we get that

$$\xi_q(z) \in \mathbb{R}_+ \quad \text{and} \quad \xi_{q'}(w) \in \mathbb{R}_+$$

for all $(z, w) \in F(x, y, x) \times G(x, y, y)$, which means

$$\xi_q(F(x, y, x)) \subseteq \mathbb{R}_+ \quad \text{and} \quad \xi_{q'}(F(x, y, x)) \subseteq \mathbb{R}_+.$$

It follows that, for all $(x, y) \in A \times B$,

$$x \notin P_1(x, y) \quad \text{and} \quad y \notin P_2(x, y).$$

To verify condition (iii) of Lemma 2.12, assume that $(u, v) \in A \times B$. Note that

$$(P_1^{-1}(u))^c = \{(x, y) \in A \times B : \xi_q(F(x, y, u)) \subseteq \mathbb{R}_+\}.$$

Let $\{(x_i, y_i)\} \in (P_1^{-1}(u))^c$ with $(x_i, y_i) \rightarrow (x_0, y_0)$. By assumption (iv), for each $z_0 \in F(x_0, y_0, u)$, there exist $z_i \in F(x_i, y_i, u)$ such that $z_i \rightarrow z_0$. Since $\xi_q(F(x_i, y_i, u)) \subseteq \mathbb{R}_+$, $\xi_q(z_i) \in \mathbb{R}_+$. By the continuity of ξ_q , we get $\xi_q(z_0) \in \mathbb{R}_+$. As z_0 is an arbitrary, we obtain $\xi_q(F(x_0, y_0, u)) \subseteq \mathbb{R}_+$. Thus, $(x_0, y_0) \in (P_1^{-1}(u))^c$, and so $(P_1^{-1}(u))^c$ is closed. Hence, we have that $P_1^{-1}(u)$ is open on A . Similarly, we can prove that $P_2^{-1}(v)$ is open on B . Finally, we have to show that condition (iv) of Lemma 2.12 is satisfied. By assumption (v), there exist nonempty compact sets $D_1 \times D_2 \subseteq A \times B$ such that for any $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$, there exists $x' \in D_1$ such that $F(x, y, x') \cap -\text{int } C \neq \emptyset$ or there exists $y' \in D_2$ such that $G(x, y, y') \cap -\text{int } P \neq \emptyset$. Thus, for any $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$, we obtain that $\xi_q(F(x, y, x')) \not\subseteq \mathbb{R}_+$, for some $x' \in D_1$ or $\xi'_q(G(x, y, y')) \not\subseteq \mathbb{R}_+$, for some $y' \in D_2$. So, we immediately obtain by the definition of $P_1(x, y)$ that

$$x' \in P_1(x, y), \quad \text{for some } x' \in D_1$$

or

$$y' \in P_2(x, y), \quad \text{for some } y' \in D_2.$$

Therefore, we proved condition (iv) of Lemma 2.12 and so P_1 and P_2 satisfy all conditions of Lemma 2.12. Hence, we can conclude that there exists $(\bar{x}, \bar{y}) \in A \times B$ such that

$$P_1(\bar{x}, \bar{y}) = \emptyset \quad \text{and} \quad P_2(\bar{x}, \bar{y}) = \emptyset.$$

This means there exists $(\bar{x}, \bar{y}) \in A \times B$ such that

$$\xi_q(F(\bar{x}, \bar{y}, u)) \subseteq \mathbb{R}_+, \quad \forall u \in A$$

and

$$\xi_{q'}(G(\bar{x}, \bar{y}, v)) \subseteq \mathbb{R}_+, \quad \forall v \in B.$$

Therefore, $(\bar{x}, \bar{y}) \in S_2(\xi)$ and so by Lemma 3.1 we complete the proof that S_2 is nonempty. $\square$

Remark 3.11 Comparing Theorem 3.10 and the results obtained in Anh and Khan [1] and Lemma 2.3 in Zhong et al. [34], we can see that the main difference is that our techniques are based on the utilizing the nonlinear scalarization method. Further, the C -quasiconvexity of the mapping $F(x, y, \cdot)$ and $G(x, y, \cdot)$ are weakened by generalized C -quasiconvexity. Hence, Theorem 3.10 can be applicable in the following situation while the afore-cited results do not work as in the following example.

Example 3.12 Let $X = Y = Z = \mathbb{R}$, $A = B = [0, 1]$, $C = P = \mathbb{R}_+$ and define the mappings $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ as

$$F(x, y, u) = \begin{cases} (u, u + 1), & u \leq x; \\ [-u, 1), & x < u; \end{cases}$$

and

$$G(x, y, v) = \begin{cases} (v, v + 1), & v \leq y; \\ [-v, 1), & y < v. \end{cases}$$

First, we show that F is not C -quasiconvex. Taking $x = \frac{1}{2}$, $u_1 = 1$, $u_2 = 0$, and $t_1 = t_2 = \frac{1}{2}$, we have the following relations:

$$\begin{aligned} F(x, y, u_2) &= (0, 1) \not\subseteq \left(\frac{1}{2}, +\infty\right) \\ &= \left(\frac{1}{2}, \frac{3}{2}\right) + \mathbb{R}_+ = F(x, y, t_1 u_2 + t_2 u_2) + C \end{aligned}$$

and

$$F(x, y, u_1) = [-1, 1) \not\subseteq \left(\frac{1}{2}, +\infty\right) = F(x, y, t_1 u_2 + t_2 u_2) + C.$$

Hence, F is not C -quasiconvex. However, all assumptions given in Theorem 3.10 are satisfied. First, it is clear that the assumption (i) given in Theorem 3.10 is satisfied. Next, we shall establish the assumption (ii). To this end, for fixed $(x, y) \in A \times B$, let $u_1, u_2 \in A$ and $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 = 1$. Assume that $u_1 \leq u_2$. Then, we have the following three cases.

Case I: If $u_1 \leq u_2 \leq x$, then $t_1 u_1 + t_2 u_2 \leq u_2 \leq x$ and

$$\begin{aligned} F(x, y, u_2) \cap (F(x, y, t_1 u_1 + t_2 u_2) + C) \\ = (u_2, u_2 + 1) \cap (t_1 u_1 + t_2 u_2, +\infty) \neq \emptyset. \end{aligned}$$

Case II: If $u_1 \leq x < u_2$, then we have either $t_1 u_1 + t_2 u_2 > x$ or $t_1 u_1 + t_2 u_2 \leq x$. Thus, we have either

$$\begin{aligned} F(x, y, u_2) \cap (F(x, y, t_1 u_1 + t_2 u_2) + C) \\ = [-u_2, 1) \cap [-t_1 u_1 - t_2 u_2, +\infty) \neq \emptyset \end{aligned}$$

or

$$\begin{aligned} F(x, y, u_2) \cap (F(x, y, t_1 u_1 + t_2 u_2) + C) \\ = [-u_2, 1) \cap (t_1 u_1 + t_2 u_2, +\infty) \neq \emptyset. \end{aligned}$$

Case III: If $x < u_1 \leq u_2$, then $t_1 u_1 + t_2 u_2 > x$, and hence

$$F(x, y, u_2) \cap (F(x, y, t_1 u_1 + t_2 u_2) + C) = [-u_2, 1) \cap [-t_1 u_1 - t_2 u_2, +\infty) \neq \emptyset.$$

Hence, we have that F is generalized C -quasiconvex. Similarly, we can show that G is generalized C -quasiconvex.

To verify assumption (iii), notice that for each element $u \in A$, $F(x, y, u) \cap -\text{int } C \neq \emptyset$ if $u > x$. Assume that $z \in F(x, y, u)$, then z also belongs $[-u, 1) \subseteq [-1, 1)$. It is not hard to see that $z - C \subseteq -\text{int } C$. Similarly, we can show that G also satisfies this assumption.

Next, to verify assumption (iv) of Theorem 3.10, let $(x', y') \in A \times B$ and $z \in F(x', y', u)$.

Case I: If $u \leq x'$, then $z \in (x', x' + 1)$. Let U be arbitrary neighborhood of z . For each (x, y) belonging to neighborhood $(u, x'] \times B$ of (x', y') , we have

$$F(x, y, u) \supseteq (x', x' + 1), \quad \forall x \in (u, x'].$$

Hence, $F(x, y, u) \cap U \neq \emptyset$, $\forall (x, y) \in (u, x'] \times B$.

Case II: If $u > x'$, then $z \in [-u, 1)$. Let U be arbitrary neighborhood of z . For each (x, y) belonging to neighborhood $[x', 1] \times B$ of (x', y') , we have

$$F(x, y, u) = [-u, 1) \ni z.$$

Hence, $F(x, y, u) \cap U \neq \emptyset$, $\forall (x, y) \in [x', 1] \times B$. Therefore, $F(\cdot, \cdot, u)$ satisfies the condition (iv) on A . Similarly, $G(\cdot, \cdot, v)$ satisfies the condition (iv) on B .

Finally, we show that the assumption (iv) of Theorem 3.10 holds, take $D_1 = [\frac{1}{2}, 1] \subset A$ and $D_2 = [\frac{1}{2}, 1] \subset B$. Then, for each element (x, y) belongs $(A \times B) \setminus (D_1 \times D_2)$, there exist $x' = 1 \in D_1$ and $y' = 1 \in D_2$ such that

$$F(x, y, x') \cap -\text{int } C = [-1, \infty) \cap -\text{int } \mathbb{R}_+ = [-1, 0) \neq \emptyset.$$

Therefore, all assumptions in Theorem 3.10 are satisfied. In fact, it is easy to see that $(1, 1) \in S_2$. $\square$

4 Convexity of the Solution Set of Symmetric Vector Equilibrium Problem

In this section, we study the convexity of the solution set S_2 . The sufficient conditions for the convexity of S_2 were established. Now, we recall the following useful features, which lead us to obtain our results in the sequel.

Definition 4.1 [17] Let K be a subset of a topological vector space E . A set-valued mapping $F: K \rightarrow 2^E \setminus \{\emptyset\}$ is said to be a *KKM mapping* if for any $\{x_1, x_2, \dots, x_n\} \subset K$,

$$\text{co}\{x_1, x_2, \dots, x_n\} \subseteq \bigcup_{i=1}^n F(x_i),$$

where $2^E \setminus \{\emptyset\}$ stands for the family of all nonempty subsets of E , while the notion $\text{co}\{x_1, x_2, \dots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \dots, x_n\}$.

The following well-known lemma plays vital role in our results in this section.

Lemma 4.2 [17] Let K be a subset of a topological vector space E . A set-valued mapping $F: K \rightarrow 2^X$ be a KKM mapping with closed values in K . Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x \in B} F(x)$ is compact. Then,

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

Theorem 4.3 Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex pointed cone with $q \in \text{int } C \neq \emptyset$ and $q' \in \text{int } P \neq \emptyset$. Suppose $F: A \times B \times A \rightarrow 2^Y$ and $G: A \times B \times B \rightarrow 2^Z$ are two set-valued mappings which satisfy the following conditions:

- (i) for each $(x, y) \in A \times B$, $F(x, y, x) \subseteq C$ and $G(x, y, y) \subseteq P$;
- (ii) for each $(x, y) \in A \times B$, $F(x, y, \cdot)$ is C -convex on A as well as $G(x, y, \cdot)$ is P -convex on B ;
- (iii) for each $u \in A$, $F(\cdot, \cdot, u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot, \cdot, v)$ is lower semicontinuous on $A \times B$;
- (iv) there exist nonempty compact convex set $D_1 \times D_2 \subseteq A \times B$ and compact set $M_1 \times M_2 \subseteq A \times B$ such that for each $(x, y) \in (A \times B) \setminus (M_1 \times M_2)$, there exists $(x', y') \in D_1 \times D_2$ such that $F(x, y, x') \cap -\text{int } C \neq \emptyset$ or $y' \in D_2$ such that $G(x, y, y') \cap -\text{int } P \neq \emptyset$.

Then, the solution set $S_2(\xi)$ is a nonempty compact subset of $A \times B$. Furthermore, S_2 is convex.

Proof Let $q \in -\text{int } C$, and $q' \in -\text{int } P$. Define a set-valued mapping $T: A \times B \rightarrow A \times B$ by

$$T(z, w) = \{(x, y) \in A \times B : \xi_q(F(x, y, z)) \subseteq \mathbb{R}_+, \xi_{q'}(G(x, y, w)) \subseteq \mathbb{R}_+\}.$$

Note that $S_2(\xi) = \bigcap_{(z,w) \in A \times B} T(z, w)$. We assert that the set-valued mapping T fulfills all the assumptions of Lemma 4.2. First, we will show that is a KKM mapping. Suppose on the contrary, then there exists a subset $\{(x_1, y_1), \dots, (x_n, y_n)\}$ of $A \times B$ and $(z, w) \in A \times B$ such that

$$(z, w) \in \text{co}\{(x_1, y_1), \dots, (x_n, y_n)\} \setminus \bigcup_{i=1}^n T(x_i, y_i).$$

Hence, there exists $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$ such that

$$\sum_{i=1}^n \alpha_i = 1 \quad \text{and} \quad (z, w) = \sum_{i=1}^n \alpha_i (x_i, y_i).$$

Thus, for all $i = 1, 2, \dots, n$, we have

$$\xi_q(F(z, w, x_i)) \not\subseteq \mathbb{R}_+ \quad \text{or} \quad \xi_{q'}(G(z, w, y_i)) \not\subseteq \mathbb{R}_+. \quad (4.1)$$

By assumption (ii), the C -quasiconvexity and P -quasiconvexity of F and G are fulfilled, respectively, and so

$$F(z, w, x_i) \subseteq F(z, w, z) + C, \quad \text{for some } i = 1, 2, \dots, n$$

and

$$G(z, w, y_i) \subseteq G(z, w, w) + C, \quad \text{for some } i = 1, 2, \dots, n.$$

Hence, there is $i \in \{1, 2, \dots, n\}$ such that

$$\xi_q(F(z, w, x_i)) \subseteq \xi_q(F(z, w, z)) + \xi_q(C) \subseteq \mathbb{R}_+$$

and

$$\xi_{q'}G((z, w, y_i)) \subseteq \xi_{q'}(G(z, w, w)) + \xi_{q'}(C) \subseteq \mathbb{R}_+.$$

This contradicts (4.1), and so T is a KKM mapping. Next, we will show that for each $(z, w) \in A \times B$, the set $T(z, w)$ is closed. Let $(z, w) \in A \times B$ and $\{(z_i, w_i)\} \subseteq T(z, w)$ be a net converges to (z_1, w_2) . Since $(z_i, w_i) \in T(z, w)$ for all i , we have

$$\xi_q(F(z_i, w_i, z)) \subseteq \mathbb{R}_+ \quad \text{and} \quad \xi_{q'}(G(z_i, w_i, w)) \subseteq \mathbb{R}_+, \quad \forall i.$$

Let $(h_1, h_2) \in \xi_q(F(z_1, w_2, z)) \times \xi_{q'}(G(z_1, w_2, w))$. Then there exists the pair $(z_2, w_3) \in F(z_1, w_2, z) \times G(z_1, w_2, w)$ such that

$$(h_1, h_2) = (\xi_q(z_2), \xi_{q'}(w_3)).$$

By assumption (iii), there is $(t_i, s_i) \in F(z_i, w_i, z) \times G(z_i, w_i, w)$ such that

$$(t_i, s_i) \rightarrow (z_2, w_3).$$

Since $(t_i, s_i) \in F(z_i, w_i, z) \times G(z_i, w_i, w)$ for all i , we have

$$\xi_q(t_i) \geq 0 \quad \text{and} \quad \xi_{q'}(s_i) \geq 0 \quad \text{for all } i.$$

Therefore, by the continuity of ξ_q and $\xi_{q'}$, we get

$$h_1 \geq 0 \quad \text{and} \quad h_2 \geq 0.$$

Since (h_1, h_2) is arbitrary element belongs to $\xi_q(F(z_1, w_2, z)) \times \xi_{q'}(G(z_1, w_2, w))$, we get

$$\xi_q(F(z_1, w_2, z)) \subseteq \mathbb{R}_+ \quad \text{and} \quad \xi_{q'}(G(z_1, w_2, w)) \subseteq \mathbb{R}_+.$$

Hence, $(z_1, w_2) \in T(z, w)$ and so $T(z, w)$ is closed for any $(z, w) \in A \times B$. Now, all the assumptions of Lemma 4.2 are fulfilled and so $S_2(\xi)$ is nonempty. Further, it follows from assumption (iv) that

$$S_2(\xi) \subseteq M_1 \times M_2,$$

and so it completes the proof that $S_2(\xi)$ is a nonempty compact subset of $A \times B$. By Lemma 3.9, S_2 is also nonempty and compact. Finally, the C -convexity of $F(x, y, \cdot)$ on A and the P -convexity of $G(x, y, \cdot)$ on B imply the set $T(z, w)$ is convex for all $(z, w) \in A \times B$. Hence, the set $S_2(\xi)$ is convex (The intersection of the convex sets is convex.). Therefore, by Lemma 3.9, S_2 is also convex. This completes the proof. $\square$

5 Conclusions

In this paper, we considered the problems (SVEP_1) , (SVEP_2) , $(\text{SSEP}_1(\xi))$ and $(\text{SSEP}_2(\xi))$. By introducing the new type of C -quasiconvexity for a set-valued mapping and using a nonlinear scalarization function ξ_q and its properties, we obtained some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems. In fact, our studying is without assumption of monotonicity and boundedness. Moreover, the convexity of solution sets are investigated. Finally, some examples to support our results are provided.

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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Levitin-Polyak well-posedness for parametric quasivariational inclusion and disclusion problems

PANATDA BOONMAN and RABIAN WANGKEEREE

ABSTRACT. In this paper, we aim to suggest the new concept of Levitin-Polyak (for short, LP) well-posedness for the parametric quasivariational inclusion and disclusion problems (for short, (QVIP) (resp. (QVDP))). Necessary and sufficient conditions for LP well-posedness of these problems are proved. As applications, we obtained immediately some results of LP well-posedness for the quasiequilibrium problems and for a scalar equilibrium problem.

1. INTRODUCTION

Well-posedness is very important concept in optimization theory, for well-posed optimization problems, which guarantees that, for every approximating solution sequence, there is a subsequence which converges to a solution. In 1966, well-posedness of unconstrained and constrained scalar optimization problems was first introduced and studied by Tykhonov [24] and Levitin and Polyak [15], respectively. Well-posedness for various problems related to optimization has been recently intensively considered, see e.g: for optimization problems [11, 12, 13, 21, 23, 31, 32], for variational inequalities [5, 7, 9, 10, 17, 25], for Nash equilibria [18, 20], for inclusion problems [10, 26, 27, 28], for equilibrium problems [2, 8, 16, 30] and for fixed point problems [6, 10, 22].

Lin and Chuang [19] studied and extended the well-posedness to variational inclusion and disclusion problems and optimization problems with variational inclusion and disclusion problems as constraints. They proved some results concerned with the well-posedness in the generalized sense, the well-posedness for optimization problems for variational inclusion problems and variational disclusion problems and scalar equilibrium problems as constraint. Recently, Wang and Huang [26] introduced and studied LP well-posedness for generalized quasivariational inclusion and disclusion problems. Necessary and sufficient conditions for LP well-posedness of these problems are proved.

On the other hand, in [3], Anh, Khanh and Quy introduced and studied the parametric generalized quasivariational inclusion problem (QVIP) which contains many kinds of problems such as generalized quasivariational inclusion problems, quasioptimization problems, quasiequilibrium problems, quasivariational inequalities, complementarity problems, vector minimization problems, Nash equilibria, fixed-point and coincidence-point problems, traffic networks, etc. It is well known that a quasioptimization problem is more general than an optimization one as constraint sets depend on the decision variable as well. It is investigated in [3] the semicontinuity properties of solution maps to (QVIP). In 2016, Wangkeeree, Anh and Boonman [29] studied the new concept of well-posedness for

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the general parametric quasivariational inclusion problems (QVIP). The corresponding concepts of well-posedness in the generalized sense are also introduced and investigated for (QVIP). Some metric characterizations of well-posedness for (QVIP) are also studied.

Motivated and inspired by the works mentioned above [3, 19, 26, 29], there is no work to provide the concept of LP well-posedness for (QVIP) (resp. (QVDP)). In this paper, our main aim is to suggest the new concept of LP well-posedness for (QVIP) (resp. (QVDP)). Necessary and sufficient conditions for LP well-posedness of these problems are proved. As applications, we obtained immediately some results of LP well-posedness for the quasiequilibrium problems and for a scalar equilibrium problem.

2. PRELIMINARIES

Let X and Y be two metric spaces, $T : X \rightarrow 2^Y$ be a multivalued map. T is said to be upper semicontinuous (u.s.c., shortly) (resp. lower semicontinuous (l.s.c., shortly)) at $x_0 \in X$ if for any open set $V \subseteq Y$, where $T(x_0) \subseteq V$ (resp. $T(x_0) \cap V \neq \emptyset$), there exists a neighborhood $U \subseteq X$ of x_0 such that $T(x) \subseteq V$ (resp. $T(x) \cap V \neq \emptyset$), $\forall x \in U$; $T(\cdot)$ is said to be u.s.c. (resp. l.s.c.) on X if it is u.s.c. (resp. l.s.c.) at every $x \in X$; T is continuous on X if it is both u.s.c. and l.s.c. on X ; T is closed if $\text{gr}(T) := \{(x, y) \in X \times Y \mid y \in T(x)\}$ is a closed set $X \times Y$; T is open if graph of T is open in $X \times Y$.

Lemma 2.1. [4] *Let X and Y be two metric spaces, $T : X \rightarrow 2^Y$ a multivalued mapping.*

- (i) *If T is u.s.c. and closed-valued, then T is closed.*
- (ii) *If T is u.s.c. at $\bar{x}$ and $T(\bar{x})$ is compact, then for any sequence $\{x_n\}$ converging to $\bar{x}$, every sequence $\{y_n\}$ with $y_n \in T(x_n)$ has a subsequence converging to some point in $T(\bar{x})$. If, in addition, $T(\bar{x}) = \{\bar{y}\}$ is a singleton, then such a sequence $\{y_n\}$ must converge to $\bar{y}$.*
- (iii) *T is l.s.c. at $\bar{x}$ if and only if for any sequence $\{x_n\}$ with $x_n \rightarrow \bar{x}$ and any point $y \in T(\bar{x})$, there is a sequence $\{y_n\}$ with $y_n \in T(x_n)$ converging to y .*

Definition 2.1. [14] Let (E, d) be a complete metric space. The Kuratowski measure of noncompactness of subset M of E is defined by

$$\mu(M) = \inf \left\{ \varepsilon > 0 : M \subseteq \bigcup_{i=1}^n M_i \text{ and } \text{diam} M_i < \varepsilon, i = 1, 2, \dots, n \right\},$$

where $\text{diam} M_i$ denotes the diameter of M_i and is defined by $\text{diam} M_i = \sup\{d(x_1, x_2) : x_1, x_2 \in M_i\}$.

Definition 2.2. Let A and B be nonempty subset of a metric space (E, d) . The Hausdorff distance $\mathcal{H}(\cdot, \cdot)$ between A and B is defined by $\mathcal{H}(A, B) := \max\{H^*(A, B), H^*(B, A)\}$, where $H^*(A, B) := \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} d(a, b)$.

Lemma 2.2. [14] *Let (X, d) be a complete metric space. If (F_n) is a decreasing sequence of nonempty, closed and bounded subsets of X such that $\lim_{n \rightarrow \infty} \mu(F_n) = 0$, then the intersection $F_\infty = \bigcap_{n=1}^\infty F_n$ is a nonempty and compact subset of X .*

3. LP WELL-POSEDNESS FOR PARAMETRIC QUASIVARIATIONAL INCLUSION AND DISCUSSION PROBLEMS

Throughout this article, unless otherwise specified, we use the following notations. Let (E, d) and (E', d') be two metric spaces and X and Λ be nonempty closed subsets of E and E' , respectively. Let Z be a Hausdorff topological vector space. Let $K_1, K_2 : X \times \Lambda \rightarrow 2^X$ and $F_1, F_2 : X \times X \times \Lambda \rightarrow 2^Z$ be multivalued mappings. Let $e : X \rightarrow Z$ be a continuous mapping. We consider the following *parametric quasivariational inclusion and discussion problems*, for each $\lambda \in \Lambda$,

$(\text{QVIP})_\lambda$: Finding $\bar{x} \in K_1(\bar{x}, \lambda)$ such that $0 \in F_1(\bar{x}, y, \lambda)$, for each $y \in K_2(\bar{x}, \lambda)$;

$(\text{QVDP})_\lambda$: Finding $\bar{x} \in K_1(\bar{x}, \lambda)$ such that $0 \notin F_2(\bar{x}, y, \lambda)$, for each $y \in K_2(\bar{x}, \lambda)$.

Denote by (QVIP) (resp. (QVDP)) the families $\{(\text{QVIP})_\lambda : \lambda \in \Lambda\}$ (resp. $\{(\text{QVDP})_\lambda : \lambda \in \Lambda\}$). For each $\lambda \in \Lambda$, let $S_{(\text{QVIP})_\lambda}$ (resp. $S_{(\text{QVDP})_\lambda}$) be solution sets of $(\text{QVIP})_\lambda$ (resp. $(\text{QVDP})_\lambda$).

For each $a \in E$ and each $r > 0$, we denote by $B(a, r)$ the closed ball centered at a with radius r . When $E = \mathbb{R}$, we denote by $B^+(0, r)$ the closed interval $[0, r]$.

Definition 3.3. Let $\lambda \in \Lambda$ and let $\{\lambda_n\} \subseteq \Lambda$ be any sequence such that $\lambda_n \rightarrow \lambda$. A sequence $\{x_n\} \subseteq X$ is called a *LP approximating solution sequence* for $(\text{QVIP})_\lambda$ if there exists a sequence $\{\varepsilon_n\}$ of positive real numbers with $\varepsilon_n \rightarrow 0$ such that, for each $n \in \mathbb{N}$, $d(x_n, K_1(x_n, \lambda_n)) \leq \varepsilon_n$ and $0 \in F_1(x_n, y, \lambda_n) + B^+(0, \varepsilon_n)e(x_n)$, $\forall y \in K_2(x_n, \lambda_n)$.

Definition 3.4. Let $\lambda \in \Lambda$ and let $\{\lambda_n\} \subseteq \Lambda$ be any sequence such that $\lambda_n \rightarrow \lambda$. A sequence $\{x_n\} \subseteq X$ is called a *LP approximating solution sequence* for $(\text{QVDP})_\lambda$ if there exists a sequence $\{\varepsilon_n\}$ of positive real numbers with $\varepsilon_n \rightarrow 0$ such that, for each $n \in \mathbb{N}$, $d(x_n, K_1(x_n, \lambda_n)) \leq \varepsilon_n$ and $0 \notin F_2(x_n, y, \lambda_n) + B^+(0, \varepsilon_n)e(x_n)$, $\forall y \in K_2(x_n, \lambda_n)$.

Definition 3.5. (i) (QVIP) is said to be *LP well-posed* if for every $\lambda \in \Lambda$, $(\text{QVIP})_\lambda$ has a unique solution x_λ , and for every sequence $\{\lambda_n\} \subseteq \Lambda$ with $\lambda_n \rightarrow \lambda$, every approximating solution sequence for $(\text{QVIP})_\lambda$ corresponding to $\{\lambda_n\}$ converges to x_λ , and (QVIP) is said to be *LP well-posed in the generalized sense* if for every $\lambda \in \Lambda$, $(\text{QVIP})_\lambda$ has a nonempty solution set $S_{(\text{QVIP})_\lambda}$, and for every sequence $\{\lambda_n\} \subseteq \Lambda$ with $\lambda_n \rightarrow \lambda$, every approximating solution sequence for $(\text{QVIP})_\lambda$ corresponding to $\{\lambda_n\}$ has a subsequence which converges to a point of $S_{(\text{QVIP})_\lambda}$.
(ii) (QVDP) is said to be *LP well-posed* if for every $\lambda \in \Lambda$, $(\text{QVDP})_\lambda$ has a unique solution x_λ , and for every sequence $\{\lambda_n\} \subseteq \Lambda$ with $\lambda_n \rightarrow \lambda$, every approximating solution sequence for $(\text{QVDP})_\lambda$ corresponding to $\{\lambda_n\}$ converges to x_λ , and (QVDP) is said to be *LP well-posed in the generalized sense* if for every $\lambda \in \Lambda$, $(\text{QVDP})_\lambda$ has a nonempty solution set $S_{(\text{QVDP})_\lambda}$, and for every sequence $\{\lambda_n\} \subseteq \Lambda$ with $\lambda_n \rightarrow \lambda$, every approximating solution sequence for $(\text{QVDP})_\lambda$ corresponding to $\{\lambda_n\}$ has a subsequence which converges to a point of $S_{(\text{QVDP})_\lambda}$.

Remark 3.1. Definition 3.3 generalizes Definition 3.1 of [29]. Indeed, the condition (i) of Definition 3.1 in [29] " $x_n \in K_1(x_n, \lambda_n)$ ", implies that $d(x_n, K_1(x_n, \lambda_n)) = 0$. So, Definition 3.3 generalizes Definition 3.1 of [29].

For each $\lambda \in \Lambda$, the approximating solution set for $(\text{QVIP})_\lambda$ and $(\text{QVDP})_\lambda$, respectively, are defined by, for all $\delta, \varepsilon > 0$, $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) = \bigcup_{\lambda' \in B(\lambda, \delta)} \tilde{S}_{(\text{QVIP})_\lambda}(\lambda', \varepsilon)$,

where $\tilde{S}_{(\text{QVIP})_\lambda} : \Lambda \times \mathbb{R}^+$ is defined by, for all $\lambda' \in \Lambda, \varepsilon \in \mathbb{R}^+$,

$$(3.1) \quad \tilde{S}_{(\text{QVIP})_\lambda}(\lambda', \varepsilon) := \left\{ x \in X \mid \begin{array}{l} d(x, K_1(x, \lambda')) \leq \varepsilon \text{ and} \\ 0 \in F_1(x, y, \lambda') + B^+(0, \varepsilon)e(x), \forall y \in K_2(x, \lambda') \end{array} \right\},$$

and $\Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon) = \bigcup_{\lambda' \in B(\lambda, \delta)} \tilde{S}_{(\text{QVDP})_\lambda}(\lambda', \varepsilon)$, where $\tilde{S}_{(\text{QVDP})_\lambda} : \Lambda \times \mathbb{R}^+$ is defined by, for all $\lambda' \in \Lambda, \varepsilon \in \mathbb{R}^+$,

$$(3.2) \quad \tilde{S}_{(\text{QVDP})_\lambda}(\lambda', \varepsilon) := \left\{ x \in X \mid \begin{array}{l} d(x, K_1(x, \lambda')) \leq \varepsilon \text{ and} \\ 0 \notin F_2(x, y, \lambda') + B^+(0, \varepsilon)e(x), \forall y \in K_2(x, \lambda') \end{array} \right\}.$$

Clearly, we have, for every $\lambda \in \Lambda$, (i) $S_{(\text{QVIP})_\lambda} \equiv \tilde{S}_{(\text{QVIP})_\lambda}(\lambda, 0) \subseteq \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$, $\forall \delta, \varepsilon > 0$ and $S_{(\text{QVDP})_\lambda} \equiv \tilde{S}_{(\text{QVDP})_\lambda}(\lambda, 0) \subseteq \Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)$, $\forall \delta, \varepsilon > 0$, (ii) if $0 < \delta_1 \leq \delta_2$ and $0 < \varepsilon_1 \leq \varepsilon_2$, then $\Omega_{(\text{QVIP})_\lambda}(\delta_1, \varepsilon_1) \subseteq \Omega_{(\text{QVIP})_\lambda}(\delta_2, \varepsilon_2)$ and $\Omega_{(\text{QVDP})_\lambda}(\delta_2, \varepsilon_2) \subseteq \Omega_{(\text{QVDP})_\lambda}(\delta_1, \varepsilon_1)$.

Lemma 3.3. Assume that K_1 is closed-valued and u.s.c. and K_2 is l.s.c..

- (i) If, for each $x \in X$, $F_1(x, \cdot, \cdot)$ is closed, then $S_{(\text{QVIP})_\lambda} = \bigcap_{\delta>0, \varepsilon>0} \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$ for each $\lambda \in \Lambda$.
- (ii) If, for each $x \in X$, $F_2(x, \cdot, \cdot)$ is open, then $S_{(\text{QVDP})_\lambda} = \bigcap_{\delta>0, \varepsilon>0} \Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)$ for each $\lambda \in \Lambda$.

Proof. (i) For any given $\lambda \in \Lambda$, it is clear that $S_{(\text{QVIP})_\lambda} \subseteq \bigcap_{\delta>0, \varepsilon>0} \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$.

Thus, we only need to show that $\bigcap_{\delta>0, \varepsilon>0} \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) \subseteq S_{(\text{QVIP})_\lambda}$. Suppose on the contrary that there exists $x^* \in \bigcap_{\delta>0, \varepsilon>0} \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$ such that $x^* \notin S_{(\text{QVIP})_\lambda}$. Then, for each $\delta > 0$ and each $\varepsilon > 0$, $x^* \in \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) \setminus S_{(\text{QVIP})_\lambda}$. In particular, for each $n \in \mathbb{N}$, we have $x^* \in \Omega_{(\text{QVIP})_\lambda}(\frac{1}{n}, \frac{1}{n}) \setminus S_{(\text{QVIP})_\lambda}$, and so there exists $\lambda_n \in B(\lambda, \frac{1}{n})$ such that

$$(3.3) \quad d(x^*, K_1(x^*, \lambda_n)) \leq \frac{1}{n}, \text{ and}$$

$$(3.4) \quad 0 \in F_1(x^*, y, \lambda_n) + B^+\left(0, \frac{1}{n}\right)e(x^*), \quad \forall y \in K_2(x^*, \lambda_n).$$

Obviously, $\lambda_n \rightarrow \lambda$. Since K_1 is closed-valued, it follows from (3.3) that we can choose $x_n \in K_1(x^*, \lambda_n)$ such that $d(x^*, x_n) \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$. Thus, $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Since K_1 is closed-valued and u.s.c., we have K_1 is closed, it follows that $x^* \in K_1(x^*, \lambda)$. We observe that for each $y \in K_2(x^*, \lambda)$, since K_2 is l.s.c. at (x^*, λ) and $(x^*, \lambda_n) \rightarrow (x^*, \lambda)$, there exists $y_n \in K_2(x^*, \lambda_n)$ such that $y_n \rightarrow y$. Applying (3.4), we have that $0 \in F_1(x^*, y_n, \lambda_n) + B^+\left(0, \frac{1}{n}\right)e(x^*)$. Thus, there exists a sequence $\{\gamma_n\} \subseteq B^+(0, \frac{1}{n})$ such that, for each $n \in \mathbb{N}$, $0 \in F_1(x^*, y_n, \lambda_n) + \gamma_n e(x^*)$, which gives that $-\gamma_n e(x^*) \in F_1(x^*, y_n, \lambda_n)$ that is $((y_n, \lambda_n), -\gamma_n e(x^*)) \in \text{Gr}(F_1(x^*, \cdot, \cdot))$. It is clear that $\{((x^*, y_n, \lambda_n), -\gamma_n e(x^*))\} \rightarrow ((x^*, y, \lambda), 0)$. The closedness of the mapping $F_1(x, \cdot, \cdot)$ implies that $((y, \lambda), 0) \in \text{Gr}(F_1(x^*, \cdot, \cdot))$. That is $0 \in F_1(x^*, y, \lambda)$ and so $x^* \in S_{(\text{QVIP})_\lambda}$, which is a contradiction. Hence

$\bigcap_{\delta>0, \varepsilon>0} \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) \subseteq S_{(\text{QVIP})_\lambda}$. (ii) For any given $\lambda \in \Lambda$ and let $F_1 : X \times X \times \Lambda \rightarrow 2^Z$ be defined by $F_1(x, y, \lambda) = Z \setminus F_2(x, y, \lambda)$ for each $(x, y, \lambda) \in X \times X \times \Lambda$. Then $S_{(\text{QVIP})_\lambda} = S_{(\text{QVDP})_\lambda}$. For each $\delta > 0$ and $\varepsilon > 0$ we have $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) = \Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)$. Since $F_2(x, \cdot, \cdot)$ is open, we have $F_1(x, y, \lambda)$ is closed. By (i), the proof is complete. $\square$

The following example is given to illustrate the case that Lemma 3.3 is applicable.

Example 3.1. Let $E = Z = \mathbb{R}$, $X = [0, +\infty)$ and $\Lambda = [0, 1]$. For every $(x, y, \lambda) \in X \times X \times \Lambda$, let $e(x) = x^2$, $K_1(x, \lambda) = [\lambda^2, +\infty)$ and $K_2(x, \lambda) = [x^2 + \lambda^2, x^2 + 1]$. Define a set-valued mapping $F_1, F_2 : X \times X \times \Lambda \rightarrow 2^Z$ by $F_1(x, y, \lambda) = (-\infty, 2x - y + \lambda]$, $F_2(x, y, \lambda) = (2x - y + \lambda, +\infty)$. Obviously, it is to see that all assumptions of Lemma 3.3 are satisfied. Hence, $S_{(\text{QVIP})_\lambda} = \bigcap_{\delta>0, \varepsilon>0} \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$ and $S_{(\text{QVDP})_\lambda} = \bigcap_{\delta>0, \varepsilon>0} \Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)$ for each $\lambda \in \Lambda$.

Lemma 3.4. For (QVIP) and (QVDP), assume that K_1 is closed-valued and u.s.c. and K_2 is l.s.c..

- (i) If, for each $\lambda \in \Lambda$, $F_1(\cdot, \cdot, \lambda)$ is closed and K_1 is also compact-valued, then for each $(\lambda, \varepsilon) \in \Lambda \times \mathbb{R}^+$, $\tilde{S}_{(\text{QVIP})_\lambda}(\lambda, \varepsilon)$ is closed subset of X , where $\tilde{S}_{(\text{QVIP})_\lambda}$ is defined by (3.1) and so is $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$.
- (ii) If, for each $\lambda \in \Lambda$, $F_2(\cdot, \cdot, \lambda)$ is open and K_1 is also compact-valued, then for each $(\lambda, \varepsilon) \in \Lambda \times \mathbb{R}^+$, $\tilde{S}_{(\text{QVDP})_\lambda}(\lambda, \varepsilon)$ is closed subset of X , where $\tilde{S}_{(\text{QVDP})_\lambda}$ is defined by (3.2) and so is $\Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)$.

Proof. Let $(\lambda, \varepsilon) \in \Lambda \times \mathbb{R}^+$ be fixed and suppose that K_1 is also compact-valued. If $x \in \text{cl}\tilde{S}_{(\text{QVIP})_\lambda}(\lambda, \varepsilon)$, then there exists a sequence $\{x_n\} \subseteq \tilde{S}_{(\text{QVIP})_\lambda}(\lambda, \varepsilon)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. It follows that, for each $n \in \mathbb{N}$, $x_n \in X$ such that for each $y \in K_2(x_n, \lambda)$,

$$(3.5) \quad d(x_n, K_1(x_n, \lambda)) \leq \varepsilon, \text{ and}$$

$$(3.6) \quad 0 \in F_1(x_n, y, \lambda) + B^+(0, \varepsilon)e(x_n).$$

By (3.5), for each $n \in \mathbb{N}$, there exists $u_n \in K_1(x_n, \lambda)$ such that

$$(3.7) \quad d(x_n, u_n) \leq \varepsilon + \frac{1}{n}.$$

Since K_1 is u.s.c. and compact-valued, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightarrow u$ as $k \rightarrow \infty$. It follows that $d(x, u) = \lim_{k \rightarrow \infty} d(x_{n_k}, u_{n_k}) \leq \varepsilon$. Since K_1 is u.s.c. and closed-valued, we have K_1 is closed. Thus $u \in K_1(x, \lambda)$. This implies that

$$(3.8) \quad d(x, K_1(x, \lambda)) \leq \varepsilon.$$

For each $y \in K_2(x, \lambda)$, since K_2 is l.s.c., there exists a sequence $\{y_n\}$ with $y_n \in K_2(x_n, \lambda)$ such that $y_n \rightarrow y$ as $n \rightarrow \infty$. By (3.6), we have $0 \in F_1(x_n, y_n, \lambda) + B^+(0, \varepsilon)e(x_n)$, $\forall n \in \mathbb{N}$. Thus there exists a sequence $\{\alpha_n\} \subseteq B^+(0, \varepsilon)$ such that $0 \in F_1(x_n, y_n, \lambda) + \alpha_n e(x_n)$, $\forall n \in \mathbb{N}$. Observe that $B^+(0, \varepsilon) := [0, \varepsilon] \subseteq \mathbb{R}$ is compact. Assume that $\alpha_n \rightarrow \alpha \in B^+(0, \varepsilon)$ as $n \rightarrow \infty$. Since $F_1(\cdot, \cdot, \lambda)$ is closed, one has $0 \in F_1(x, y, \lambda) + \alpha e(x) \subseteq F_1(x, y, \lambda) + B^+(0, \varepsilon)e(x)$. Therefore $x \in \tilde{S}_{(\text{QVIP})_\lambda}(\lambda, \varepsilon)$, and this implies that $\tilde{S}_{(\text{QVIP})_\lambda}(\lambda, \varepsilon)$ is a closed subset of X . Now it follows $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$ is a closed subset of X . (ii) Let $F_1 : X \times X \times \Lambda \rightarrow 2^Z$ be defined by $F_1(x, y, \lambda) = Z \setminus F_2(x, y, \lambda)$ for each $(x, y, \lambda) \in X \times X \times \Lambda$. Then $\tilde{S}_{(\text{QVIP})_\lambda}(\lambda, \varepsilon) = \tilde{S}_{(\text{QVDP})_\lambda}(\lambda, \varepsilon)$ and $S_{(\text{QVIP})_\lambda} = S_{(\text{QVDP})_\lambda}$, and so $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) = \Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)$. Since $F_2(\cdot, \cdot, \lambda)$ is open, we have $F_1(x, y, \lambda)$ is closed. By (i), the proof is complete. $\square$

If E is finite-dimension normed space, then the assumption that “ K_1 is also compact-valued in Lemma 3.4 ” can be removed

Lemma 3.5. *Let E be finite-dimensional normed space. For (QVIP) and (QVDP), assume that K_1 is closed-valued and u.s.c. and K_2 is l.s.c..*

- (i) *If, for each $\lambda \in \Lambda$, $F_1(\cdot, \cdot, \lambda)$ is closed, then $S_{(\text{QVIP})_\lambda}$, $\tilde{S}_{(\text{QVIP})_\lambda}(\lambda, \varepsilon)$ and $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$ are closed subset of X .*
- (ii) *If, for each $\lambda \in \Lambda$, $F_2(\cdot, \cdot, \lambda)$ is open, then $S_{(\text{QVDP})_\lambda}$, $\tilde{S}_{(\text{QVDP})_\lambda}(\lambda, \varepsilon)$ and $\Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)$ are closed subset of X .*

Proof. We can proceed the proof exactly as that of Lemma 3.4 except for using the Assumption that E is finite-dimension normed space to get $d(x, K_1(x, \lambda)) \leq \varepsilon$. In fact, since $x_n \rightarrow x$, it follows that $\{x_n\}$ is bounded. By (3.7), we have $\{u_n\}$ is also bounded. Thus there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges to some $u \in X$ as $k \rightarrow \infty$. Since K_1 is closed-valued and u.s.c., we have K_1 is closed, it follows that $u \in K_1(x, \lambda)$. It follows that $d(x, u) = \lim_{k \rightarrow \infty} d(x_{n_k}, u_{n_k}) \leq \varepsilon$ and so $d(x, K_1(x, \lambda)) \leq \varepsilon$. This complete the proof. $\square$

Remark 3.2. If $K_1(x, \lambda) \equiv K_2(x, \lambda) \equiv X$, then our problem (QVIP) reduces to (VIP) in Lin and Chuang [19].

Now, we are in a position to state and prove our main results.

Theorem 3.1. *For (QVIP), assume that E is complete, K_1 is closed-valued and u.s.c., K_2 is l.s.c. and F_1 is closed. Then (QVIP) is LP well-posed if and only if for every $\lambda \in \Lambda$,*

$$(3.9) \quad \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) \neq \emptyset, \forall \delta, \varepsilon > 0, \text{ and } \text{diam}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0).$$

Proof. Supposed that (QVIP) is LP well-posed. Then, for every $\lambda \in \Lambda$, $(\text{QVIP})_\lambda$ has a unique solution x_λ , $S_{(\text{QVIP})_\lambda} \neq \emptyset$, and so $\Omega_\lambda(\delta, \varepsilon) \neq \emptyset$, for all $\delta, \varepsilon > 0$. Now we shall show that

$$(3.10) \quad \text{diam}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0).$$

Suppose to the contrary the existences of (3.10), there exist $l > 0$, sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of positive real numbers with $(\delta_n, \varepsilon_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$ and sequence $\{x_n\}$ and $\{x'_n\}$ with $x_n, x'_n \in \Omega_{(\text{QVIP})_\lambda}(\delta_n, \varepsilon_n)$ for each $n \in \mathbb{N}$ such that

$$(3.11) \quad d(x_n, x'_n) > l, \quad \forall n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, since $x_n \in \Omega_{(\text{QVIP})_\lambda}(\delta_n, \varepsilon_n)$, there exists $\lambda_n \in B^+(0, \varepsilon_n)$ such that $d(x_n, K_1(x_n, \lambda_n)) \leq \varepsilon_n$ and $0 \in F_1(x_n, y, \lambda_n) + B^+(0, \varepsilon_n)e(x_n) \quad \forall y \in K_2(x_n, \lambda_n)$, and since $x'_n \in \Omega_{(\text{QVIP})_\lambda}(\delta_n, \varepsilon_n)$, there exists $\lambda'_n \in B^+(0, \varepsilon_n)$ such that $d(x'_n, K_1(x_n, \lambda'_n)) \leq \varepsilon_n$ and $0 \in F_1(x'_n, y, \lambda'_n) + B^+(0, \varepsilon_n)e(x'_n) \quad \forall y \in K_2(x_n, \lambda'_n)$. Clearly, $\lambda_n \rightarrow \lambda$ and $\lambda'_n \rightarrow \lambda$ as $n \rightarrow \infty$. Hence, $\{x_n\}$ and $\{x'_n\}$ are LP approximating solution sequences for $(\text{QVIP})_\lambda$ corresponding to λ_n, λ'_n , respectively. By the LP well-posed of $(\text{QVIP})_\lambda$, $\{x_n\}$ and $\{x'_n\}$ converge to the unique solution x_λ of $(\text{QVIP})_\lambda$, which is a contradiction to (3.11). This implies that (3.10). Conversely, suppose that condition (3.9) holds. Let $\lambda \in \Lambda$ be fixed. Let $\{\lambda_n\}$ be any sequence in Λ with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. Suppose that $\{x_n\}$ is a LP approximating solution sequence for $(\text{QVIP})_\lambda$ corresponding to $\{\lambda_n\}$, then there exists a nonnegative sequence $\{\varepsilon_n\} \downarrow 0$ such that for each $n \in \mathbb{N}$, $d(x_n, K_1(x_n, \lambda)) \leq \varepsilon_n$, and $0 \in F_1(x_n, y, \lambda_n) + B^+(0, \varepsilon_n)e(x_n) \quad \forall y \in K_2(x_n, \lambda)$. For each $n \in \mathbb{N}$, let $\delta_n = d'(\lambda_n, \lambda)$. Then, $\lambda_n \in B(\lambda, \delta_n)$ and $x_n \in \Omega_{(\text{QVIP})_\lambda}(\delta_n, \varepsilon_n)$ for each $n \in \mathbb{N}$, and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. It follows from (3.9) that $\{x_n\}$ is a Cauchy sequence and so it converges to a point $x \in X$. By similar arguments as in the proof of Lemma 3.4, we also deduce that x belongs to $S_{(\text{QVIP})_\lambda}$. Next, we will show that $(\text{QVIP})_\lambda$ has a unique solution. Suppose to the contrary, if $(\text{QVIP})_\lambda$ has two distinct solutions x_1 and x_2 , it is easy to see that $x_1, x_2 \in \Omega_{(\text{QVIP})_\lambda}$ for all $\delta, \varepsilon > 0$. It follows that $0 < d(x_1, x_2) \leq \text{diam}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon))$ which gives a contradiction to (3.9). This implies that $(\text{QVIP})_\lambda$ has a unique solution. This completes the proof. $\square$

The following example is given to illustrate the case that Theorem 3.1 is applicable.

Example 3.2. Let $E = Z = \mathbb{R}$, $X = [0, 1]$ and $\Lambda = [0, 1]$. For every $(x, y, \lambda) \in X \times X \times \Lambda$, let

$$e(x) = 1, K_1(x, \lambda) = \begin{cases} \left[0, \frac{1}{2}\right], & \text{if } \lambda \neq \frac{1}{2}, \\ [0, 1], & \text{if } \lambda = \frac{1}{2}, \end{cases} \quad \text{and } K_2(x, \lambda) = \begin{cases} [0, 1], & \text{if } \lambda \neq \frac{1}{2}, \\ \left[0, \frac{1}{2}\right], & \text{if } \lambda = \frac{1}{2}. \end{cases}$$

Define a set-valued mapping $F_1 : X \times X \times \Lambda \rightarrow 2^Z$ by $F_1(x, y, \lambda) = (-\infty, (\lambda + 2)(y - x)]$. Obviously, it is to see that conditions of Theorem 3.1 are satisfied. For every $\lambda \in \Lambda$, $\text{diam}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$. By Theorem 3.1, $(\text{QVIP})_\lambda$ is well-posed. $\square$

Remark 3.3. We can not the supposed LP well-posedness in Theorem 3.1 by generalized LP well-posedness. Therefore, we have to employ the Kuratowski measure of noncompactness to study characterizations of the LP well-posedness in the generalized sense for (QVIP).

Theorem 3.2. For (QVIP), assume that E is complete and Λ is finite dimensional, K_1 is closed-valued and u.s.c., K_2 is l.s.c. and F_1 is closed. Then (QVIP) is LP well-posed in generalized sense if and only if for every $\lambda \in \Lambda$, $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) \neq \emptyset$, $\forall \delta, \varepsilon > 0$, and $\mu(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$.

Proof. Suppose that (QVIP) LP well-posed in the generalized sense. Let $\lambda \in \Lambda$ be fixed. Then $S_{(\text{QVIP})_\lambda}$ is nonempty. Now we show that $S_{(\text{QVIP})_\lambda}$ is compact. Indeed, let $\{x_n\}$

be any sequence in $S_{(\text{QVIP})_\lambda}$. Then $\{x_n\}$ is a LP approximating solution sequence for $(\text{QVIP})_\lambda$. By the LP well-posedness in the generalized sense of (QVIP) , $\{x_n\}$ has a subsequence which converges to a point of $S_{(\text{QVIP})_\lambda}$. Thus $S_{(\text{QVIP})_\lambda}$ is compact. Clearly, for each $\delta, \varepsilon > 0$, $S_{(\text{QVIP})_\lambda} \subseteq \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$, and so $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) \neq \emptyset$. Now we will show that

$$(3.12) \quad \mu(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0).$$

Observe that for every $\delta, \varepsilon > 0$, $\mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda}) = H^*(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda})$, and $S_{(\text{QVIP})_\lambda}$ is compact. Indeed, let $\{x_n\} := \{(x_n, \lambda_n)\}$ be arbitrary sequence in $S_{(\text{QVIP})_\lambda}$. Then, it is clear that $\{x_n\}$ is a LP approximating sequence of (QVIP) . Thus, it has a subsequence converging to a point in $S_{(\text{QVIP})_\lambda}$. Therefore, $\mu(S_{(\text{QVIP})_\lambda}) = 0$. Now for any $\alpha > 0$, there are finite sets $A_1^\alpha, A_2^\alpha, \dots, A_{n_\alpha}^\alpha$ for some $n_\alpha \in \mathbb{N}$ such that $S_{(\text{QVIP})_\lambda} \subseteq \bigcup_{k=1}^{n_\alpha} A_k^\alpha$ and $\text{diam } A_k^\alpha \leq \alpha$, for all $k = 1, 2, \dots, n_\alpha$. Next, for each $k \in \{1, 2, \dots, n_\alpha\}$, we define the set $M_k^\alpha = \{z \in X : d(z, A_k^\alpha) \leq \mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda})\}$. We show that $\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon) \subseteq \bigcup_{k=1}^{n_\alpha} M_k^\alpha$. To this end, let $x \in \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$ be given. Thus, we have $d(x, S_{(\text{QVIP})_\lambda}) \leq \mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda})$. As $S_{(\text{QVIP})_\lambda} \subseteq \bigcup_{k=1}^{n_\alpha} A_k^\alpha$, we also get $d(x, \bigcup_{k=1}^{n_\alpha} A_k^\alpha) \leq d(x, S_{(\text{QVIP})_\lambda}) \leq \mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda})$. Therefore, there exists $k_0 \in \{1, 2, \dots, n_\alpha\}$ such that $d(x, A_{k_0}^\alpha) \leq \mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda})$, thereby yielding $x \in M_{k_0}^\alpha$. Therefore, we get the desired inclusion. Furthermore, we see that, for any $k \in \{1, 2, \dots, n_\alpha\}$,

$$(3.13) \quad \text{diam } M_k^\alpha \leq \alpha + 2\mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda}).$$

Indeed, for any $y, y' \in M_k^\alpha$ and $m, m' \in A_k^\alpha$, $d(y, y') \leq d(y, m) + d(m, m') + d(m', y')$, which gives that $d(y, y') \leq \alpha + 2\mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda})$, which leads to the desired result (3.13). It follows from the definition of μ that $\mu(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)) \leq 2\mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda}) + \alpha$, for all $\alpha > 0$. Therefore, we can conclude that

$$\mu(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)) \leq 2\mathcal{H}(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda}) = 2H^*(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda}).$$

To prove (3.8), it is sufficient to show that

$$(3.14) \quad H^*(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda}) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0).$$

If (3.14) does not hold, then there exist $r > 0$, sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of positive real numbers with $(\delta_n, \varepsilon_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$ and sequence $\{x_n\}$ with $x_n \in \Omega_{(\text{QVIP})_\lambda}(\delta_n, \varepsilon_n)$ for every $n \in \mathbb{N}$ such that

$$(3.15) \quad d(x_n, S_{(\text{QVIP})_\lambda}) > r, \quad \forall n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, since $x_n \in \Omega_\lambda(\delta_n, \varepsilon_n)$, there exists $\lambda_n \in B(\lambda, \delta_n)$ such that $d(x_n, K_1(x_n, \lambda_n)) \leq \varepsilon_n$ and $0 \in F(x_n, y, \lambda_n) + B^+(0, \varepsilon_n)e(x_n)$, $\forall y \in K_2(x_n, \lambda_n)$. Clearly $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. Hence $\{x_n\}$ is a LP approximating solution sequence for $(\text{QVIP})_\lambda$ corresponding to $\{\lambda_n\}$. Then, by the LP well-posedness in the generalized sense of (QVIP) , $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to some point of $S_{(\text{QVIP})_\lambda}$. This contradicts (3.15), and so (3.14) holds. Therefore, (3.8) is proved. Conversely, suppose that condition (3.7) holds. We will show that (QVIP) is LP well-posed in generalized sense. Let $\lambda \in \Lambda$ be fixed. Thus, by Lemma 3.3 and Lemma 3.4, we have $\Omega_{(\text{QVIP})_\lambda}$ is closed. Further, $S_{(\text{QVIP})_\lambda} = \bigcap_{\delta, \varepsilon > 0} \Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)$. Since $\mu(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon)) \rightarrow 0$ as $(\delta, \varepsilon) \rightarrow (0, 0)$, by Lemma 2.2, $S_{(\text{QVIP})_\lambda}$ is a nonempty compact subset of X and

$$(3.16) \quad H^*(\Omega_{(\text{QVIP})_\lambda}(\delta, \varepsilon), S_{(\text{QVIP})_\lambda}) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0).$$

Let $\{\lambda_n\}$ be any sequence in Λ with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. Suppose that $\{x_n\}$ is a LP approximating solution sequence for $(\text{QVIP})_\lambda$ corresponding to $\{\lambda_n\}$, then there exists a sequence $\{\varepsilon_n\}$ of positive real numbers with $\varepsilon_n \rightarrow 0$ such that, for each $n \in \mathbb{N}$, $d(x_n, K_1(x_n, \lambda_n)) \leq \varepsilon_n$ and $0 \in F_1(x_n, y, \lambda_n) + B^+(0, \varepsilon_n)e(x_n)$, $\forall y \in K_2(x_n, \lambda_n)$. For each $n \in \mathbb{N}$, let $\delta_n = d(\lambda_n, \lambda)$. Then, $\lambda_n \in B(\lambda, \delta_n)$ and $x_n \in \Omega_{(\text{QVIP})_\lambda}(\delta_n, \varepsilon_n)$ for every $n \in \mathbb{N}$, and $\delta_n \rightarrow 0$

as $n \rightarrow \infty$. It follows from (3.10) that $d(x_n, S_{(\text{QVIP})_\lambda}) \leq H^*(\Omega_{(\text{QVIP})_\lambda}(\delta_n, \varepsilon_n), S_{(\text{QVIP})_\lambda}) \rightarrow 0$ as $n \rightarrow \infty$. Since $S_{(\text{QVIP})_\lambda}$ is compact, for each $n \in \mathbb{N}$, there exists $\bar{x}_n \in S_{(\text{QVIP})_\lambda}$ such that $d(x_n, \bar{x}_n) = d(x_n, S_{(\text{QVIP})_\lambda}) \rightarrow 0$ as $n \rightarrow \infty$. By the compactness of $S_{(\text{QVIP})_\lambda}$, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ which converges to a point $\bar{x} \in S_{(\text{QVIP})_\lambda}$. Hence, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$. This implies that (QVIP) is LP well-posed in the generalized sense. This completes the proof. $\square$

Remark 3.4. Theorems 3.1, Theorems 3.2 generalizes Theorem 3.8, Theorems 3.11 of [29], respectively.

By Theorems 3.1 and 3.2, we can get the following results.

Theorem 3.3. For (QVDP), assume that E, K_1, K_2 as in Theorem 3.1 and F_2 is closed. Then (QVDP) is LP well-posed if and only if for every $\lambda \in \Lambda$,

$$\Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon) \neq \emptyset, \forall \delta, \varepsilon > 0, \text{ and } \text{diam}(\Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0).$$

Theorem 3.4. For (QVDP), assume that E, K_1, K_2 as in Theorem 3.2 and F_2 is closed. Then (QVDP) is LP well-posed in generalized the sense if and only if for every $\lambda \in \Lambda$,

$$\Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon) \neq \emptyset, \forall \delta, \varepsilon > 0, \text{ and } \mu(\Omega_{(\text{QVDP})_\lambda}(\delta, \varepsilon)) \rightarrow 0 \text{ as } (\delta, \varepsilon) \rightarrow (0, 0).$$

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SOME CHARACTERIZATIONS OF ROBUST SOLUTION SETS FOR UNCERTAIN CONVEX OPTIMIZATION PROBLEMS WITH LOCALLY LIPSCHITZ INEQUALITY CONSTRAINTS

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ABSTRACT. In this paper, we consider an uncertain convex optimization problem with a robust convex feasible set described by locally Lipschitz constraints. Using robust optimization approach, we give some new characterizations of robust solution sets of the problem. Such characterizations are expressed in terms of convex subdifferentials, Clarke subdifferentials, and Lagrange multipliers. In order to characterize the solution set, we first introduce the so-called pseudo Lagrangian function and establish constant pseudo Lagrangian-type property for the robust solution set. We then used to derive Lagrange multiplier-based characterizations of robust solution set. By means of linear scalarization, the results are applied to derive characterizations of weakly and properly robust efficient solution sets of convex multi-objective optimization problems with data uncertainty. Some examples are given to illustrate the significance of the results.

1. Introduction. The study of characterizations of solution sets has become an important research direction for many mathematical programming problems. Based on understanding characterizations of solution sets, solution methods for solving mathematical programs that have multiple solutions can be developed. The notion of characterizations of solution sets was first introduced and studied by Mangasarian

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for a convex extrema problem with differentiable function [29]. Some useful examples clarifying such characterizations of solution sets can be found in [7] for characterizing the problems that have weak sharp minimum. This being a reason why several characterizations of solution sets for some classes of constrained optimization problems have appeared in the literature (see [6, 8, 13, 14, 19, 23, 32, 33, 36, 38, 39] and other references therein).

However, dealing with real-world optimization problems, the input data associated with the objective function and the constraints of programs are uncertain due to prediction error or measurement errors (see [1, 2, 3, 4]). Moreover, in many situations often we need to make decisions now before we can know the true values or have better estimations of the parameters. Robust optimization is one of the basic methodologies to protect the optimal solution that it is no longer feasible after realization of actual values of parameters. This means that any feasible points must satisfy all constraints including each set of constraints corresponding to a possible realization of the uncertain parameters from the uncertainty sets. Precisely stated, let us first consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} \{f(x) : g_i(x) \leq 0, i = 1, \dots, m\}, \quad (\text{P})$$

where $f, g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are functions. The problem (P) in the face of data uncertainty both in the objective and constraints can be written by the following optimization problems:

$$\min_{x \in \mathbb{R}^n} \{f(x, u) : g_i(x, v_i) \leq 0, i = 1, \dots, m\}, \quad (\text{UP})$$

where $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \rightarrow \mathbb{R}$, and $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are functions, u and v_i are uncertain parameters and they belong to the specified nonempty convex and compact uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^{q_0}$ and $\mathcal{V}_i \subseteq \mathbb{R}^{q_i}$, respectively. The robust (worst case) counterpart of (UP), by construction in [3], is obtained by solving the single problem:

$$\min_{x \in \mathbb{R}^n} \{\max_{u \in \mathcal{U}} f(x, u) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\}, \quad (\text{RP})$$

where the objective and constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets $\mathcal{U}$ and $\mathcal{V}_i$. The set of feasible solutions of problem (RP),

$$F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\},$$

refer to *robust feasible set* of the problem (UP). The optimal solution to the problem (RP) is known as a *robust optimal solution* of (UP). A successful treatment of the robust optimization approaches for treating convex optimization programs with data uncertainty to derive characterizations of robust optimal solution sets was given in [15, 27, 34, 35]. For issues related to optimality conditions and duality properties, see [5, 11, 16, 17, 18, 25, 26] and other references therein.

This paper is an attempt to investigate optimality conditions and to derive characterizations of robust solution sets of (UP). Unlike various related works in the literature mentioned above, in the present paper, appearing constraint functions are not convex necessarily while the robust feasible set F is convex. In this way, we refer to *convex problems without convex representation* in the sense that the constraint functions to represent the convex feasible set are non necessarily convex. Optimality conditions and characterizations of convexity of feasible set for such

problems in the absence of data uncertainty can be found in [24] for differentiable case, and in [10, 30, 21] for non-differentiable case.

To the best of our knowledge, complete characterizations of robust solutions for uncertain scalar and multi-objective optimization problems over a robust convex feasible set described by non necessarily convex functions within the framework of robust optimization approach are not available in the literature. So, in this paper we examine a robust optimization framework for studying characterizations of the robust optimal solution set for uncertain convex optimization problems with a robust convex feasible set described by locally Lipschitz constraints. First, complete optimality conditions for uncertain convex optimization problems are given. In order to characterize the robust optimal solution set of a given problem, we introduce the so-called *pseudo-Lagrange function* and then, we show that pseudo-Lagrange function is constant on the robust optimal solution set. Afterwards, we then use this property to derive various characterizations of the robust optimal solution set that these are expressed in terms of convex subdifferentials, Clarke subdifferentials and Lagrange multipliers. Finally, the results are then applied to derive characterizations of weakly robust efficient solution set and properly robust efficient solution set of uncertain convex multi-objective optimization problems without convexity assumption on constraint functions.

The remainder of the present paper is organized as follows. In Sect. 2, we give some notations, definitions and preliminary results. In Sect. 3, we establish a multiplier characterization for the robust optimal solution of uncertain convex optimization problem. Sect. 4 provides characterizations of robust solution set of uncertain convex optimization without convexity assumption on constraint functions. In Sect. 5, we give a sufficient condition that a robust efficient solution of uncertain multi-objective convex optimization problems can be a properly robust efficient solution. Moreover, characterizations of weakly robust efficient solution set and properly robust efficient solution set of such problem are given.

2. Preliminaries. We begin this section by fixing certain notations, definitions and preliminary results that will be used throughout the paper. We denote by $\mathbb{R}^n$ the Euclidean space with dimension n whose norm is denoted by $\|\cdot\|$ and $\langle x, y \rangle$ denotes the usual inner product between two vectors x, y in $\mathbb{R}^n$, that is, $\langle x, y \rangle = x^T y$. Let $\mathbb{R}_+^n := \{x := (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$ be non-negative orthant of $\mathbb{R}^n$. Note also that the interior non-negative orthant of $\mathbb{R}^n$ is denoted by $\text{int}\mathbb{R}_+^n$ and is defined by $\text{int}\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}$. Given a set $A \subseteq \mathbb{R}^n$, we recall that a set A is *convex* whenever $\lambda x + (1 - \lambda)y \in A$ for all $\lambda \in [0, 1]$, $x, y \in A$. A set A is said to be a *cone* if $\lambda A \subseteq A$ for all $\lambda \geq 0$. We denote the *convex hull* and the *conical hull* generated by A , by $\text{conv}A$ and $\text{cone}A$, respectively. The *normal cone* at x to a closed convex set A , denoted by $N(A, x)$, is defined by

$$N(A, x) := \{\xi \in \mathbb{R}^n : \langle \xi, y - x \rangle \leq 0, \forall y \in A\}.$$

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *convex* if for all $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}^n$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

It is a well known fact that a convex function need not be differentiable everywhere. However if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function then the one-sided or rather right-sided directional derivative always exists and is finite. The right-sided directional derivative of f at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is denoted by $f'(x; d)$, is defined as

$$f'(x; d) := \lim_{t \rightarrow 0^+} \frac{f(x + td) - f(x)}{t}.$$

It is important to note that for every fixed x the function $f'(x; \cdot)$ is a positively homogeneous convex function. The subdifferential of convex function f at x is defined as

$$\partial f(x) := \{\xi \in \mathbb{R}^n : f(y) \geq f(x) + \langle \xi, y - x \rangle, \text{ for all } y \in \mathbb{R}^n\}.$$

We now recall the following useful result, which is a subdifferential max-function rule of convex functions over a compact set, that will be used later in the paper.

Lemma 2.1. [15, Lemma 2.1] *Let $\mathcal{U} \subseteq \mathbb{R}^p$ be a convex compact set, and let $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \rightarrow \mathbb{R}$ be a function such that for each fixed $u \in \mathcal{U}$, $f(\cdot, u)$ is a convex function on $\mathbb{R}^n$ and for each fixed $x \in \mathbb{R}^n$, $f(x, \cdot)$ is a concave function on $\mathbb{R}^{q_0}$. Then,*

$$\partial \left(\max_{u \in \mathcal{U}} f(\cdot, u) \right) (\bar{x}) = \bigcup_{u \in \mathcal{U}(\bar{x})} \partial f(\cdot, u)(\bar{x}),$$

where $\mathcal{U}(\bar{x}) := \{\bar{u} \in \mathcal{U} : f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u)\}$.

Definition 2.2. A function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *locally Lipschitz* at $x \in \mathbb{R}^n$, if there exists a positive scalar L and a neighborhood N of x such that, for all $y, z \in N$, one has

$$|h(y) - h(z)| \leq L\|y - z\|.$$

Definition 2.3. [9] Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The *Clarke generalized directional derivative* of h at x in the direction $d \in \mathbb{R}^n$, denoted $h^o(x; d)$, is defined as

$$h^o(x; d) := \limsup_{\substack{y \rightarrow x \\ t \rightarrow 0^+}} \frac{h(y + td) - h(y)}{t},$$

Definition 2.4. [9] Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The *Clarke generalized subdifferential* of h at x , denoted by $\partial^o h(x)$, is defined as

$$\partial^o h(x) := \{\xi \in \mathbb{R}^n : h^o(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in \mathbb{R}^n\}.$$

From the definition of the Clarke generalized subdifferential, it follows that

$$h^o(x; d) = \max_{\xi \in \partial^o h(x)} \langle \xi, d \rangle, \quad \forall d \in \mathbb{R}^n.$$

Definition 2.5. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz at a given point $x \in \mathbb{R}^n$. The function h is said to be *regular* at $x \in \mathbb{R}^n$ if, for each $d \in \mathbb{R}^n$, the directional derivative $h'(x; d)$ exists and coincides with $h^o(x; d)$.

For a given compact subset $\mathcal{V}$ of $\mathbb{R}^q$ and a given function $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$, the following conditions will be considered in this paper.

- (C1) for every $x \in \mathbb{R}^n$ the function $\mathcal{V} \ni v \mapsto g(x, v)$ is upper semicontinuous;
- (C2) g is locally Lipschitz in x , uniformly for v in $\mathcal{V}$, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U of x and a constant $L > 0$ such that for all y and z in U , and $v \in \mathcal{V}$, one has

$$|g(y, v) - g(z, v)| \leq L\|y - z\|;$$

(C3) for each $(x, v) \in \mathbb{R}^n \times \mathcal{V}$, the function $g(\cdot, v)$ is regular at x , that is,

$$g_x^o(x, v; \cdot) = g'_x(x, v; \cdot)$$

(the derivatives being with respect to x);

(C4) set-valued map $\mathbb{R}^n \times \mathcal{V} \ni (x, v) \mapsto \partial^o g(\cdot, v)(x)$ is upper semicontinuous where $\partial^o g(\cdot, v)(x)$ denotes the Clarke subdifferential of g with respect to x .

Remark 1. In a suitable setting, if the function g is convex in x and continuous in v , the conditions (C2), (C3), and (C4) are then automatically satisfied. These conditions also hold whenever the derivative $\nabla_x g(x, v)$ with respect to x exists and is continuous in (x, v) .

Remark 2. [25] Under the conditions (C1) and (C2) the function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\psi(x) := \max\{g(x, v) : v \in \mathcal{V}\},$$

is defined and finite. Further, ψ is locally Lipschitz on $\mathbb{R}^n$, and hence for each $x \in \mathbb{R}^n$ the set $\mathcal{V}(x)$ defined as

$$\mathcal{V}(x) := \{v \in \mathcal{V} : g(x, v) = \psi(x)\},$$

is a nonempty closed subset of $\mathbb{R}^q$.

We conclude this section by the following lemmas which will be useful in our later analysis.

Lemma 2.6. [9] *Let the function ψ be defined in Remark 2. Suppose that the conditions (C1) - (C4) are fulfilled. Then the usual one-sided directional derivative $\psi'(x; d)$ exists, and satisfies the following: for each $x, d \in \mathbb{R}^n$,*

$$\begin{aligned} \psi'(x; d) &= \psi^o(x; d) = \max\{g_x^o(x, v; d) : v \in \mathcal{V}(x)\} \\ &= \max\{\langle \xi, d \rangle : \xi \in \partial^o g(\cdot, v)(x), v \in \mathcal{V}(x)\}. \end{aligned}$$

Lemma 2.7. [26] *For a given compact convex subset $\mathcal{V}$ of $\mathbb{R}^q$ and a given function $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$, suppose that the basic conditions (C1) - (C4) are fulfilled. Further, suppose that $g(x, \cdot)$ is concave on $\mathcal{V}$, for each $x \in \mathbb{R}^n$. Then*

$$\partial^o \psi(x) = \{\xi \in \mathbb{R}^n : \exists v \in \mathcal{V}(x) \text{ such that } \xi \in \partial^o g(\cdot, v)(x)\}.$$

3. Multiplier characterization for the robust solution. In this section, we give a multiplier characterization for the robust optimal solution of (UP), which will play an important role in deriving characterizations of the robust optimal solution sets in the next section. Let us recall the following robust (worst case) counterpart optimization problem of (UP):

$$\min_{x \in \mathbb{R}^n} \left\{ \max_{u \in \mathcal{U}} f(x, u) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m \right\}, \quad (\text{RP})$$

where $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \rightarrow \mathbb{R}$, and $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are given functions and for each $i = 1, 2, \dots, m$, $(u, v_i) \in \mathcal{U} \times \mathcal{V}_i \subseteq \mathbb{R}^{q_0} \times \mathbb{R}^{q_i}$, where $\mathcal{U}$ and $\mathcal{V}_i$ are the specified nonempty convex and compact uncertainty sets. The robust feasible set of (UP) is defined by

$$F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\}.$$

Assumption 3.1. *Throughout this paper, we always assume that $F \neq \emptyset$, $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ is a convex-concave in the sense that $f(\cdot, u)$ is a convex function for any $u \in \mathcal{U}$, and $f(x, \cdot)$ is a concave function for any $x \in \mathbb{R}^n$ while $g_i(x, \cdot)$, $i = 1, \dots, m$, are concave functions for any $x \in \mathbb{R}^n$. Further, let the functions g_i , $i = 1, \dots, m$, be satisfied the conditions (C1) and (C2).*

Definition 3.1. We say that $\bar{x} \in F$ is a robust optimal solution of (UP) if and only if $\bar{x}$ is an optimal solution of (RP).

By using Proposition 2.2 in [10], we can derive the following characterization of convexity for robust feasible set of (UP) in terms of the Clarke directional derivative. Before doing so let us denote, for each $x \in F$,

$$I(x) := \left\{ i \in \{1, \dots, m\} : \max_{v_i \in \mathcal{V}_i} g_i(x, v_i) = 0 \right\},$$

and for all $i = 1, \dots, m$,

$$\mathcal{V}_i(x) := \left\{ \bar{v}_i \in \mathcal{V}_i : g_i(x, \bar{v}_i) = \max_{v_i \in \mathcal{V}_i} g_i(x, v_i) \right\}.$$

Proposition 1. *Let the system $g_i(x, v_i) \leq 0$, $\forall v_i \in \mathcal{V}_i$, $i = 1, \dots, m$, be satisfied the robust Slater constraint qualification, that is, there exists $x_0 \in \mathbb{R}^n$ such that*

$$g_i(x_0, v_i) < 0, \text{ for any } v_i \in \mathcal{V}_i, i = 1, \dots, m.$$

For each $x \in F$ and $i \in I(x)$, let the function g_i be satisfied the conditions (C3), (C4), and $0 \notin \partial^o g_i(\cdot, v_i)(x)$ whenever $v_i \in \mathcal{V}_i(x)$. Then F is convex if and only if

$$F = \{y \in \mathbb{R}^n : g_{ix}^o(x, v_i; y - x) \leq 0, \forall x \in F, \forall i \in I(x), \forall v_i \in \mathcal{V}_i(x)\}.$$

Proof. For each $i = 1, \dots, m$, define a function $\psi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\psi_i(x) := \max_{v_i \in \mathcal{V}_i} g_i(x, v_i) \text{ for all } x \in \mathbb{R}^n.$$

Applying the conditions (C1) and (C2), we have, for each $i = 1, \dots, m$, ψ_i is locally Lipschitz on $\mathbb{R}^n$. To achieve the result, we will use Proposition 2.2 in [10] and then we need to justify that for any $x \in F$, ψ_i , $i \in I(x)$, are regular in the sense of Clarke and $0 \notin \partial^o \psi_i(x)$, and the system $\psi_i(x) \leq 0$, $i = 1, \dots, m$, satisfies the Slater condition. The first and the second requirements will follow from Lemma 2.6 and Lemma 2.7 that for any $x \in F$,

$$\psi'_i(x; d) = \psi_i^o(x; d) = \max\{g_{ix}^o(x, v_i; d) : v_i \in \mathcal{V}_i(x)\}, \forall i \in I(x), \quad (1)$$

and for each $i \in I(x)$

$$\begin{aligned} 0 \in \bigcap_{\substack{v_i \in \mathcal{V}_i \\ g_i(x, v_i) = 0}} \mathbb{R}^n \setminus \left(\partial^o g_i(\cdot, v_i)(x) \right) &= \mathbb{R}^n \setminus \left(\bigcup_{v_i \in \mathcal{V}_i, g_i(x, v_i) = \psi_i(x)} \partial^o g_i(\cdot, v_i)(x) \right) \\ &= \mathbb{R}^n \setminus \partial^o \psi_i(x). \end{aligned}$$

Finally, the robust Slater constraint qualification leads us to the following strict inequality

$$\psi_i(x_0) = \max\{g_i(x_0, v_i) : v_i \in \mathcal{V}_i\} < 0, \forall i = 1, \dots, m,$$

which means that the system $x \in \mathbb{R}^n$, $\psi_i(x) \leq 0$ ($i = 1, \dots, m$) satisfies the Slater's condition¹. Now applying [10, Proposition 2.2] and taking (1) into consideration, we obtain the desired results. $\square$

¹ the system $x \in \mathbb{R}^n$, $g_i(x) \leq 0$ ($i = 1, \dots, m$) satisfies the Slater's condition if there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$ for all $i = 1, \dots, m$.

Remark 3. It should be noted that in Proposition 1 without robust Slater constraint qualification and $0 \notin \partial^o g_i(\cdot, v_i)(x)$ whenever $x \in F$, $i \in I(x)$, and $v_i \in \mathcal{V}_i(x)$, we easily obtain that if F is convex then

$$F \subseteq \{y \in \mathbb{R}^n : g_{ix}^o(x, v_i; y - x) \leq 0, \forall x \in F, \forall i \in I(x), \forall v_i \in \mathcal{V}_i(x)\}.$$

Furthermore, for every $x \in F$ one has

$$\partial^o g_i(\cdot, v_i)(x) \subseteq N(F, x) \text{ whenever } i \in I(x) \text{ and } v_i \in \mathcal{V}_i(x).$$

In order to establish a multiplier characterization for the robust optimal solution of (UP), we first recall a robust basic constraint qualification which was introduced in [5], where the constraint data uncertainty $g_i(\cdot, v_i)$, $i = 1, \dots, m$, are assumed to be convex for each $v_i \in \mathcal{V}_i$.

Definition 3.2. Let $x \in F$ be a robust feasible solution of (UP). The *robust basic constraint qualification* is satisfied at x if

$$N(F, x) = \bigcup_{\substack{\lambda_i \geq 0, v_i \in \mathcal{V}_i \\ \lambda_i g_i(x, v_i) = 0, i=1, \dots, m}} \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, v_i)(x).$$

Now the following theorem declares a result that the robust basic constraint qualification defined in Definition 3.2 is a necessary and sufficient constraint qualification of a robust optimal solution for the given problem, that is, the robust basic constraint qualification holds if and only if the Lagrange multiplier conditions are satisfied for a robust optimal solution.

Theorem 3.3 (Characterizing the robust basic constraint qualification). Suppose that for each $x \in F$ and $i \in I(x)$, the function g_i satisfies the conditions (C3) and (C4). Then, the following statements are equivalent:

- (i) the robust basic constraint qualification holds at $\bar{x} \in F$;
- (ii) for each real-valued convex-concave function f on $\mathbb{R}^n \times \mathcal{U}$, the following statements are equivalent:
 - (a) $\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(\bar{x}, u)$ for all $x \in F$,
 - (b) there exist $\bar{u} \in \mathcal{U}$, $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ such that

$$0 \in \partial f(\cdot, \bar{u})(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}), \quad \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \quad \forall i = 1, \dots, m, \quad (2)$$

and

$$f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u). \quad (3)$$

Proof. [(i) $\Rightarrow$ (ii)] Suppose that (i) holds. Let f be a real-valued convex-concave function on $\mathbb{R}^n \times \mathcal{U}$. Firstly, we assume that (a) holds. Then, $\bar{x}$ is a solution of the following constrained convex optimization problem:

$$\text{Minimize } \max_{u \in \mathcal{U}} f(x, u) \text{ subject to } x \in F,$$

which can be equivalently expressed as,

$$0 \in \partial(\max_{u \in \mathcal{U}} f(\cdot, u))(\bar{x}) + N(F, \bar{x}).$$

By (i), there are $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ such that

$$0 \in \partial(\max_{u \in \mathcal{U}} f(\cdot, u))(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}) \text{ and } \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \quad \forall i = 1, \dots, m.$$

Then, it follows from Lemma 2.1 that there exists $\bar{u} \in \mathcal{U}$ such that (2) and (3) hold.

To prove sufficiency, assume that there exist $\bar{u} \in \mathcal{U}$, $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ such that (2) and (3) hold. According to (2), we can find $\xi \in \partial f(\cdot, \bar{u})(\bar{x})$ and $\eta_i \in \partial^\circ g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \dots, m$, such that

$$\xi + \sum_{i=1}^m \bar{\lambda}_i \eta_i = 0. \quad (4)$$

It stems from $\xi \in \partial f(\cdot, \bar{u})(\bar{x})$ and $\eta_i \in \partial^\circ g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \dots, m$, we get

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) \geq \langle \xi, x - \bar{x} \rangle \quad (5)$$

and

$$g_{ix}^\circ(\bar{x}, \bar{v}_i; x - \bar{x}) \geq \langle \eta_i, x - \bar{x} \rangle \quad \forall i = 1, \dots, m, \quad (6)$$

for any $x \in \mathbb{R}^n$. Multiplying each of inequalities in (6) by $\bar{\lambda}_i$ and summing up the obtained inequalities with (5), we obtain that, for all $x \in \mathbb{R}^n$,

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) + \sum_{i=1}^m \bar{\lambda}_i g_{ix}^\circ(\bar{x}, \bar{v}_i; x - \bar{x}) \geq \langle \xi + \sum_{i=1}^m \bar{\lambda}_i \eta_i, x - \bar{x} \rangle.$$

Taking (4) into account together with the condition $\bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$, $i = 1, \dots, m$, we deduce

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g_{ix}^\circ(\bar{x}, \bar{v}_i; x - \bar{x}) \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Note that for each $i \in I(\bar{x})$ with $g_i(\bar{x}, \bar{v}_i) \neq 0$, $\bar{\lambda}_i = 0$. So, we consider in the case of $g_i(\bar{x}, \bar{v}_i) = 0$ for $i \in I(\bar{x})$, and hence $\bar{v}_i \in \mathcal{V}_i(\bar{x})$. By Remark 3, the last inequality becomes

$$f(x, \bar{u}) - f(\bar{x}, \bar{u}) \geq 0 \quad \text{for all } x \in F.$$

Thus, together with $\max_{u \in \mathcal{U}} f(x, u) \geq f(x, \bar{u})$ for all $x \in \mathbb{R}^n$ and (3), we obtain

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\bar{x}, u) \geq 0, \quad \forall x \in F.$$

It means that $\bar{x}$ is a robust optimal solution of problem (UP).

[(ii) $\Rightarrow$ (i)] The proof is similar to the one in [35, Theorem 3.1], and so is omitted. $\square$

In the uncertainty free case, we can easily obtain the following result, which was obtained by Yamamoto and Kuroiwa in [37].

Corollary 1. [37, Theorem 3.2] *Let $\bar{x} \in F' := \{x \in \mathbb{R}^n : g_i(x) \leq 0, \forall i = 1, \dots, m\}$ be a feasible solution, $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, m$, be locally Lipschitz on $\mathbb{R}^n$. Assume further that for any $x \in F'$ and any $i = 1, \dots, m$ such that $g_i(x) = 0$, the function g_i is regular, and F' is convex. Then the following statement are equivalent:*

- (i) $N(F', \bar{x}) = \bigcup_{\substack{\lambda_i \geq 0 \\ \lambda_i g_i(\bar{x}) = 0, i=1, \dots, m}} \sum_{i=1}^m \lambda_i \partial^\circ g_i(\bar{x});$
- (ii) *for each real-valued convex function f on $\mathbb{R}^n$, the following statements are equivalent:*
 - (a) $f(x) \geq f(\bar{x})$ for all $x \in F'$;
 - (b) *there exist $\bar{\lambda}_i \geq 0$, $i = 1, \dots, m$ such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^\circ g_i(\bar{x}) \text{ and } \bar{\lambda}_i g_i(\bar{x}) = 0, \quad \forall i = 1, \dots, m.$$

Remark 4. Both the robust Slater constraint qualification condition and robust non-degeneracy at $\bar{x}$, i.e.,

$$0 \notin \partial^o g_i(\cdot, v_i)(\bar{x})$$

whenever $i = 1, \dots, m$ and $v_i \in \mathcal{V}_i$ such that $g_i(\bar{x}, v_i) = 0$, is a sufficient condition for the robust basic constraint qualification holds at $\bar{x}$. Indeed, according to Remark 3, we only have to show that

$$N(F, \bar{x}) \subseteq \bigcup_{\substack{\lambda_i \geq 0, v_i \in \mathcal{V}_i \\ \lambda_i g_i(\bar{x}, v_i) = 0, i=1, \dots, m}} \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, v_i)(\bar{x}).$$

Let $\eta \in N(F, \bar{x})$ be arbitrarily. Since the robust Slater constraint qualification condition and robust non-degeneracy are satisfied at $\bar{x}$, by Theorem 2.4 in [10] with $f := \langle -\eta, \cdot \rangle$, and $g_i := \psi_i$, $i = 1, \dots, m$, there exist $\bar{\lambda}_i \geq 0$, $i = 1, \dots, m$, such that

$$0 \in -\eta + \sum_{i=1}^m \bar{\lambda}_i \partial^o \psi_i(\bar{x}) \text{ and } \bar{\lambda}_i \psi_i(\bar{x}) = 0, \forall i = 1, \dots, m.$$

For $i \notin I(\bar{x})$, we get $\bar{\lambda}_i = 0$. In the case of $i \in I(\bar{x})$, Lemma 2.7 together with $\bar{\lambda}_i \psi_i(\bar{x}) = 0$, $\forall i = 1, \dots, m$, implies there exist $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$, such that

$$\eta \in \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}) \text{ and } \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0.$$

This shows that

$$\eta \in \bigcup_{\substack{\lambda_i \geq 0, v_i \in \mathcal{V}_i \\ \lambda_i g_i(\bar{x}, v_i) = 0, i=1, \dots, m}} \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, v_i)(\bar{x}),$$

the result as require.

The following example is given to illustrate the condition (i) of Theorem 3.3 is essential.

Example 3.2. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $v_1 := (v_{1,1}, v_{1,2})$, $v_2 := (v_{2,1}, v_{2,2})$, $v_3 := (v_{3,1}, v_{3,2})$, $\mathcal{V}_1 := \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2 \leq 1\}$, $\mathcal{V}_2 := [0, 1] \times [1, 2]$, $\mathcal{V}_3 := [2, 3] \times [0, 1]$,

$$g_1(x, v_1) := v_{1,1}x_1 + v_{1,2}x_2 - x_1^3 - 2,$$

$$g_2(x, v_2) = -v_{2,1}x_1^3 + v_{2,2} \max\{-x_2, -x_2^3\},$$

$$g_3(x, v_3) = v_{3,1}x_1 + v_{3,2}x_2,$$

$$F := \{x \in \mathbb{R}^2 : g_1(x, v_1) \leq 0, g_2(x, v_2) \leq 0, g_3(x, v_3) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, 2, 3\}$$

and $\bar{x} := (0, 0)$. Then $F = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} - x_1^3 - 2 \leq 0, 2x_1 + x_2 \leq 0, -x_1 - x_2 \leq 0\}$, $I(\bar{x}) = \{2, 3\}$, $\partial^o g_2(\cdot, v_2)(\bar{x}) = \{0\} \times [-v_{2,2}, 0]$ and $\partial^o g_3(\cdot, v_3)(\bar{x}) = \{(v_{3,1}, v_{3,2})\}$. It can be observed that

$$N(F, \bar{x}) = \text{cone}\{(-1, -1), (2, 1)\}$$

and

$$\bigcup_{\substack{\lambda_i \geq 0, v_i \in \mathcal{V}_i \\ \lambda_i g_i(\bar{x}, v_i) = 0, i=1, 2, 3}} \sum_{i=1}^3 \lambda_i \partial^o g_i(\cdot, v_i)(\bar{x}) = \text{cone}\{(0, -1), (2, 1)\}.$$

Hence, we have the condition (i) of Theorem 3.3 does not hold. Thus for some convex-concave function $f : \mathbb{R}^2 \times \mathcal{U} \rightarrow \mathbb{R}$, it is impossible to characterize a sufficient

condition for robust optimal solution for the following uncertain problem by using Theorem 3.3,

$$\text{minimize } f(x, u) \text{ subject to } x \in \mathbb{R}^2, g_i(x, v_i) \leq 0, i = 1, 2, 3.$$

Actually, let $u := (u_1, u_2)$ be an uncertain parameter belong to uncertainty set $\mathcal{U} := \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}$, and $f(x, u) := e^{x_1} - u_1 x_1 - u_2 x_2$. Selecting $\bar{u} := (1, 0)$, $\bar{v}_1 := (1, 0)$, $\bar{v}_2 := (1, 1)$, $\bar{v}_3 := (2, 0)$, $\bar{\lambda}_1 := 0$, $\bar{\lambda}_2 := 1$ and $\bar{\lambda}_3 := 1$ we obtain $\bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$ for all $i = 1, 2, 3$,

$$f(\bar{x}, \bar{u}) = 1 = \max_{u \in \mathcal{U}} f(\bar{x}, u)$$

and

$$(0, 0) \in \{(-2, 0)\} + \{0\} \times [-1, 0] + \{(2, 0)\} = \partial f(\cdot, \bar{u})(\bar{x}) + \sum_{i=1}^3 \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}).$$

However, by taking $x := (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \in F$, $\max_{u \in \mathcal{U}} f(\bar{x}, u) = e^{1/2} - 1 < 1 = f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u)$ which shows $\bar{x}$ is not a minimizer of $\max_{u \in \mathcal{U}} f(\cdot, u)$ on F . $\square$

Remark 5. According to Remark 4, Example 3.2 demonstrates that only robust Slater constraint qualification condition is not sufficient to ensure the robust basic constraint qualification holds at consideration point. The reason is that the robust non-degeneracy condition at such a point is destroyed.

4. Characterizations of the robust solution sets. In this section, we will establish some characterizations of robust optimal solution set in terms of a given robust solution point of the given problem.

We begin by recalling the following constrained convex optimization problem in the face of data uncertainty (UP):

$$\min_{x \in \mathbb{R}^n} \{f(x, u) : g_i(x, v_i) \leq 0, i = 1, \dots, m\}, \quad (\text{UP})$$

where $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ is a convex-concave function, the functions $g_i, i \in I$, satisfy the condition (C1) and (C2), $g_i(x, \cdot) : \mathcal{V}_i \rightarrow \mathbb{R}, i \in I$, are concave functions for any $x \in \mathbb{R}^n$, and the robust feasible set F is convex. Assume that the robust solution set of the problem (UP), denoted by

$$S := \{a \in F : \max_{u \in \mathcal{U}} f(a, u) \leq \max_{u \in \mathcal{U}} f(x, u), \forall x \in F\},$$

is nonempty. In what follows, for any given $y \in \mathbb{R}^n$, $\lambda := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m$, $u \in \mathcal{U}$, $v_i \in \mathcal{V}_i, i \in I$ and $v := (v_1, \dots, v_m)$, we introduce the so-called *pseudo Lagrangian-type function* $L^P(\cdot, y, \lambda, u, v)$ by, for all $x \in \mathbb{R}^n$,

$$L^P(x, y, \lambda, u, v) := f(x, u) + \sum_{i \in I(y)} \lambda_i g_{ix}^o(y, v_i; x - y).$$

Now, we show that the pseudo Lagrangian-type function associated with a Lagrange multiplier vector and uncertainty parameters according to a solution is constant on S .

Proposition 2. Assume all conditions of Theorem 3.3 hold. Let $a \in S$ be a robust optimal solution fulfilling the robust basic constraint qualification. Then there exist a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$, and uncertainty parameters $u^a \in \mathcal{U}$, $v_i^a \in \mathcal{V}_i, i \in I$, such that for any $x \in S$, $\lambda_i^a g_{ix}^o(a, v_i^a; x - a) = 0, \forall i \in I(a)$, $f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)$, and $L^P(\cdot, a, \lambda^a, u^a, v^a)$ is constant on S .

Proof. It follows from $a \in S$ and Theorem 3.3 that there exist a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$, and uncertainty parameters $u^a \in \mathcal{U}$, $v_i^a \in \mathcal{V}_i$, $i \in I$, satisfying the conditions (2) and (3). Then, it stems from the fact that $\partial^o g_i(\cdot, v_i^a)(a) = \partial g_{ix}^o(a, v_i^a; \cdot - a)(a)$ for all $i \in I(a)$ and (2), we get

$$0 \in \partial f(\cdot, u^a)(a) + \sum_{i \in I(a)} \lambda_i^a \partial g_{ix}^o(a, v_i^a; \cdot - a)(a) \subseteq \partial L^P(\cdot, a, \lambda^a, u^a, v^a)(a),$$

which is noting else than

$$f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \geq f(a, u^a) = \max_{u \in \mathcal{U}} f(a, u) \text{ for all } x \in \mathbb{R}^n. \quad (7)$$

Notice that

$$\max_{u \in \mathcal{U}} f(x, u) = \max_{u \in \mathcal{U}} f(a, u), \text{ for any } a \in S \text{ and } x \in S, \quad (8)$$

and taking this into account, (7) deduces $\sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \geq 0$, for any $x \in S$. Let us notice that for indices $i \in I(a)$ such that $\lambda_i^a > 0$, we have $g_i(a, v_i^a) = 0$, and consequently, $v_i^a \in \mathcal{V}_i(a)$. This in turn, by Remark 3, implies that

$$\lambda_i^a g_{ix}^o(a, v_i^a; x - a) = 0, \quad \forall i \in I(a). \quad (9)$$

Now, we prove that

$$f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u), \text{ for any } x \in S. \quad (10)$$

In fact, by (7) and (9), we get the assertion

$$\max_{u \in \mathcal{U}} f(x, u) \geq f(x, u^a) \geq \max_{u \in \mathcal{U}} f(a, u).$$

This together with (8), (10) holds. Therefore, for any $x \in S$, (3), (8), (9) and (10) entail

$$\begin{aligned} L^P(x, a, \lambda^a, u^a, v^a) &= f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \\ &= f(x, a) = \max_{u \in \mathcal{U}} f(x, u) = \max_{u \in \mathcal{U}} f(a, u) = f(a, u^a), \end{aligned}$$

showing that $L^P(\cdot, a, \lambda^a, u^a, v^a)$ is constant on S , and this completes the proof. $\square$

Remark 6. It is worth noting that if $g_i(\cdot, v_i)$, $i \in I$, are convex functions for any $v_i \in \mathcal{V}_i$ then, for each $i \in I$, Proposition 2 gives

$$\lambda_i^a g_i(x, v_i^a) - \lambda_i^a g_i(a, v_i^a) \geq \lambda_i^a g'_i(a, v_i^a; x - a) = \lambda_i^a g_{ix}^o(a, v_i^a; x - a) = 0 \text{ for any } x \in S.$$

This together with $x \in F$ and $\lambda_i^a g_i(a, v_i^a) = 0$, $i \in I$, arrives $\lambda_i^a g_i(x, v_i^a) = 0$, $i \in I$. Furthermore, it yields

$$\begin{aligned} L^P(x, a, \lambda^a, u^a, v^a) &= f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \\ &= f(x, u^a) \\ &= f(x, u^a) + \sum_{i=1}^m \lambda_i^a g_i(x, v_i^a), \quad \forall x \in S. \end{aligned}$$

This shows that pseudo Lagrangian-type function collapses to the well-known Lagrangian-type function on the robust solution set S .

In the sequel, we are now in a position to establish the characterizations of the robust solution set for problem (UP) in terms of convex subdifferentials, Clarke subdifferentials and Lagrange multipliers. But before doing so it will thus be convenient to denote the following:

$$\begin{aligned}\tilde{I}(a) &:= \{i \in I(a) : \lambda_i^a > 0\}, \\ C(x) &:= \{\xi \in \partial f(\cdot, u^a)(a) : \langle \xi, x - a \rangle \geq 0\} \text{ for any given } x \in F.\end{aligned}$$

Theorem 4.1 (Characterizing the robust solution set). *Assume all conditions of Theorem 3.3 hold. Let $a \in S$ be a robust optimal solution fulfilling the robust basic constraint qualification. Then there exist a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$, and uncertainty parameters $u^a \in \mathcal{U}$, $v_i^a \in \mathcal{V}_i$, $i \in I$, such that the robust solution set for the problem (UP) is characterized by*

$$S = S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = S_7,$$

where

$$\begin{aligned}S_1 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \cap \partial f(\cdot, u^a)(a); \\ &\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\ S_2 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad \langle \zeta, a - x \rangle \geq 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \cap \partial f(\cdot, u^a)(a); \\ &\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\ S_3 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad \langle \xi, x - a \rangle = \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); \\ &\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\ S_4 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad \langle \xi, x - a \rangle = \langle \zeta, a - x \rangle \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); \\ &\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\ S_5 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad \langle \xi, x - a \rangle \leq \langle \zeta, a - x \rangle \text{ for some } \zeta \in \partial f(\cdot, u^a)(x) \text{ and } \xi \in C(x); \\ &\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\ S_6 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x); \\ &\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}, \\ S_7 &:= \{x \in F : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad \langle \zeta, a - x \rangle \geq 0 \text{ for some } \zeta \in \partial f(\cdot, u^a)(x); \\ &\quad f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}.\end{aligned}$$

Proof. Evidently, the following containments hold:

$$S_1 \subseteq S_2 \subseteq S_7,$$

$$\begin{aligned} S_1 &\subseteq S_6 \subseteq S_7, \\ S_1 &\subseteq S_3 \subseteq S_4 \subseteq S_5 \subseteq S_7. \end{aligned}$$

Hence, we only have to show that $S \subseteq S_1$ and $S_7 \subseteq S$. In order to establish $S \subseteq S_1$, let $x \in S$ be arbitrarily given. It follows from (2), we therefore obtain vectors $\zeta \in \partial f(\cdot, u^a)(a)$ and $\xi_i \in \partial^\circ g_i(\cdot, v_i^a)(a)$, $i \in I(a)$, such that

$$\zeta + \sum_{i \in I(a)} \lambda_i^a \xi_i = 0 \quad (11)$$

(since $\lambda_i^a = 0$ for $i \notin I(a)$). According to $\zeta \in \partial f(\cdot, u^a)(a)$, $\xi_i \in \partial^\circ g_i(\cdot, v_i^a)(a)$, $i \in I(a)$, and $x, a \in S$, one has

$$f(x, u^a) - f(a, u^a) \geq \langle \zeta, x - a \rangle \quad (12)$$

and

$$g_{ix}^o(a, v_i^a; x - a) \geq \langle \xi_i, x - a \rangle, \quad \forall i \in I(a). \quad (13)$$

Once we have shown, in Proposition 2, that $\lambda_i^a g_{ix}^o(a, v_i^a; x - a) = 0$, $\forall i \in I(a)$, after multiplying both sides of (13) by λ_i^a , $i \in I(a)$ we get

$$0 \geq \langle \lambda_i^a \xi_i, x - a \rangle, \quad \forall i \in \tilde{I}(a).$$

Summing up these inequalities and using (11) we obtain that

$$0 \geq \left\langle \sum_{i \in I(a)} \lambda_i^a \xi_i, x - a \right\rangle = \langle -\zeta, x - a \rangle. \quad (14)$$

Again, it follows from Proposition 2 that

$$f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u), \quad (15)$$

and for each $i \in \tilde{I}(a)$, $\max_{\eta_i \in \partial^\circ g_i(\cdot, v_i^a)(a)} \langle \eta_i, x - a \rangle = g_{ix}^o(a, v_i^a; x - a) = 0$, the latter which in turn leads to there exists $\eta_i \in \partial^\circ g_i(\cdot, v_i^a)(a)$ such that

$$\langle \eta_i, x - a \rangle = 0.$$

On the one hand, taking (3) and (15) into account (12) we obtain

$$\langle \zeta, x - a \rangle \leq f(x, u^a) - f(a, u^a) = \max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(a, u) = 0.$$

This together with (14) arrives at

$$\langle \zeta, x - a \rangle = 0.$$

Now, we only need to prove that $\zeta \in \partial f(\cdot, u^a)(x)$. In fact, for any $y \in \mathbb{R}^n$,

$$\begin{aligned} f(y, u^a) - f(x, u^a) &= f(y, u^a) - f(a, u^a) \\ &\geq \langle \zeta, y - a \rangle \\ &= \langle \zeta, y - x \rangle + \langle \zeta, x - a \rangle = \langle \zeta, y - x \rangle, \end{aligned}$$

which means $\zeta \in \partial f(\cdot, u^a)(x)$ and so, $x \in S_1$. This proves $S \subseteq S_1$.

To obtain $S_7 \subseteq S$, we now let x be arbitrary point of S_7 . It follows that $x \in F$, and it is easy to see that

$$\max_{u \in \mathcal{U}} f(a, u) - \max_{u \in \mathcal{U}} f(x, u) = f(a, u^a) - f(x, u^a) \geq \langle \zeta, a - x \rangle \geq 0.$$

The last inequality together with the fact that $a \in S$ gives $x \in S$, and the proof is complete. $\square$

Now, we give the following example to illustrate the significance of Theorem 4.1 that at least one of the constraint functions $g_i(\cdot, v_i)$ for some $v_i \in \mathcal{V}_i$, is not convex while the robust feasible set is convex. Then the results in [15, 35, 34, 27] may not be relevant to this example.

Example 4.1. Let us denote $x := (x_1, x_2) \in \mathbb{R}^2$, $u := (u_1, u_2)$, $v_1 := (v_{1,1}, v_{1,2})$, $v_2 := (v_{2,1}, v_{2,2})$, $v_3 := (v_{3,1}, v_{3,2})$, $\mathcal{U} := \{(u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}$, $\mathcal{V}_1 := \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2 \leq 1\}$, $\mathcal{V}_2 := [0, 1] \times [1, 2]$ and $\mathcal{V}_3 := [0, 1] \times [0, 1]$. Consider the following constrained optimization problem with uncertainty data (UP):

$$\begin{aligned} & \text{Minimize } f(x, u) \\ & \text{subject to } x \in \mathbb{R}^2, \quad g_1(x, v_1) \leq 0, \quad g_2(x, v_2) \leq 0, \quad g_3(x, v_3) \leq 0. \end{aligned} \quad (\text{UP})$$

where $u \in \mathcal{U}$, $v_i \in \mathcal{V}_i$, $i = 1, 2, 3$,

$$\begin{aligned} f(x, u) &:= u_1 x_1 + u_2 x_2 - x_1 - x_2, \\ g_1(x, v_1) &:= v_{1,1} x_1 + v_{1,2} x_2 - x_1^3 - 2, \\ g_2(x, v_2) &:= v_{2,1} \max\{-x_1, -x_1^3\} - v_{2,2} x_2, \\ g_3(x, v_3) &:= v_{3,1} x_1 - v_{3,2} x_2^2. \end{aligned}$$

A robust solution of (UP) is obtained by solving its robust (worst-case) counterpart (RP)

$$\begin{aligned} & \text{Minimize } \max_{u \in \mathcal{U}} f(x, u) \\ & \text{subject to } x \in F := \left\{ \begin{array}{l} g_1(x, v_1) \leq 0, \quad \forall v_1 \in \mathcal{V}_1, \\ g_2(x, v_2) \leq 0, \quad \forall v_2 \in \mathcal{V}_2, \\ g_3(x, v_3) \leq 0, \quad \forall v_3 \in \mathcal{V}_3, \\ x \in \mathbb{R}^2 \end{array} \right\}. \end{aligned} \quad (\text{RP})$$

Then $F = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} - x_1^3 - 2 \leq 0, -x_1 - x_2 \leq 0, x_1 \leq 0\}$. Evidently, the function $f : \mathbb{R}^2 \times \mathcal{U} \rightarrow \mathbb{R}$ is a convex-concave function. Let us notice that

$$\max_{u \in \mathcal{U}} f(x, u) = \sqrt{x_1^2 + x_2^2} - x_1 - x_2, \quad \text{for all } x \in \mathbb{R}^2,$$

and

$$\max_{u \in \mathcal{U}} f(x, u) \geq |x_2| - x_2 = 0 = \max_{u \in \mathcal{U}} f((0, 0), u), \quad \text{for all } x \in F.$$

Thus $a := (a_1, a_2) = (0, 0) \in S$, $I(a) = \{2, 3\}$, $\partial^o g_2(\cdot, v_2)(a) = \{(r, -v_{2,2}) : -v_{2,1} \leq r \leq 0\}$ for each $v_2 \in \mathcal{V}_2$ and $\partial^o g_3(\cdot, v_3)(a) = \{(v_{3,1}, 0)\}$ for each $v_3 \in \mathcal{V}_3$. So,

$$N(F, a) = \text{cone}\{(-1, -1), (1, 0)\} = \bigcup_{\substack{\lambda_i \geq 0, v_i \in \mathcal{V}_i \\ \lambda_i g_i(a, v_i) = 0, i \in I}} \sum_{i=1}^3 \lambda_i \partial^o g_i(\cdot, v_i)(a),$$

which means that the robust basic constraint qualification holds at a . Also, for each $u \in \mathcal{U}$, the convex subdifferential of $f(\cdot, u)$ at any point x is given by

$$\partial f(\cdot, u)(x) = (u_1 - 1, u_2 - 1).$$

Let us select $\lambda^a := (\lambda_1^a, \lambda_2^a, \lambda_3^a) = (0, 0, 1)$, $u^a := (0, 1)$, $v_2^a := (1, 1)$ and $v_3^a := (1, 0)$. Therefore, $\tilde{I}(a) = \{3\}$ and by solving the following system, for $x \in \mathbb{R}^2$

$$\begin{cases} x_1 = \langle (1, 0), (x_1, x_2) \rangle = 0, \\ \sqrt{x_2^2} - 2 \leq 0, \\ -x_2 \leq 0, \\ \partial f(\cdot, u^a)(x) \cap \partial f(\cdot, u^a)(a) = \{(-1, 0)\}, \\ \langle (-1, 0), (0, x_2) \rangle = 0, \\ -x_1 = \max_{u \in \mathcal{U}} f(x, u), \end{cases}$$

the robust solution set can be described simply as

$$S = S_1 = \{x \in \mathbb{R}^2 : x_1 = 0, 0 \leq x_2 \leq 2\}.$$

□

With the help of Proposition 2, we see now how the robust solution set can be characterized in terms of pseudo Lagrangian-type function.

Proposition 3. *Assume all conditions of Theorem 3.3 hold. Let $a \in S$ be a robust optimal solution fulfilling the robust basic constraint qualification. Then there exist a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$, and uncertainty parameters $u^a \in \mathcal{U}$, $v_i^a \in \mathcal{V}_i$, $i \in I$, such that*

$$\begin{aligned} S &= \{x \in F : \langle \eta_i, x - a \rangle = 0, \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad 0 \in \partial L^P(\cdot, a, \lambda^a, u^a, v^a)(x) \text{ and } f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}. \end{aligned}$$

Proof. It will thus be convenient to denote

$$\begin{aligned} S^* &:= \{x \in F : \langle \eta_i, x - a \rangle = 0, \text{ for some } \eta_i \in \partial^o g_i(\cdot, v_i^a)(a), \forall i \in \tilde{I}(a); \\ &\quad 0 \in \partial L^P(\cdot, a, \lambda^a, u^a, v^a)(x) \text{ and } f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)\}. \end{aligned}$$

By Proposition 2, we have that for each $x \in S$, $\lambda_i^a g_{ix}^o(a, v_i^a; x - a) = 0$, $\forall i \in I(a)$, $f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u)$, and $L^P(\cdot, a, \lambda^a, u^a, v^a)$ is constant on S . The latter means that

$$\partial L^P(\cdot, a, \lambda^a, u^a, v^a)(x) = \{0\},$$

and so, $S \subseteq S^*$. To obtain the converse inclusion, let $x \in S^*$ be given. Then, by the definition of S^* , $x \in F$, there exist $\eta_i \in \partial^o g_i(\cdot, v_i^a)(a)$, $\forall i \in \tilde{I}(a)$, such that $\langle \eta_i, x - a \rangle = 0$, $\forall i \in \tilde{I}(a)$,

$$f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u) \quad (16)$$

and

$$\begin{aligned} f(y, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; y - a) &= L^P(y, a, \lambda^a, u^a, v^a) \\ &\leq L^P(x, a, \lambda^a, u^a, v^a) \\ &= f(x, u^a) + \sum_{i \in I(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \\ &= f(x, u^a) + \sum_{i \in \tilde{I}(a)} \lambda_i^a g_{ix}^o(a, v_i^a; x - a) \\ &= f(x, u^a) \text{ for all } y \in \mathbb{R}^n. \end{aligned}$$

Using (16) and taking $y = a$ in the last inequality, we get that

$$\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(a, u) \geq f(a, u^a) \geq f(x, u^a) = \max_{u \in \mathcal{U}} f(x, u).$$

Hence,

$$\max_{u \in \mathcal{U}} f(a, u) = \max_{u \in \mathcal{U}} f(x, u),$$

which is noting else than $x \in S$. $\square$

In the special case when $\mathcal{U}$ and $\mathcal{V}_i$, $i \in I$, are singletons, we can easily obtain the following results.

Corollary 2. *For the problem (P), let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex function and $F' := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}$ be convex. Assume that for any $x \in F'$ and $i \in I'(x) := \{i \in I : g_i(x) = 0\}$ the functions g_i are locally Lipschitz and regular in the sense of Clarke, $a \in S'$ is an optimal solution fulfilling $N(F', a) = \text{cone} \bigcup_{i \in I'(a)} \partial^o g_i(a)$, and there exists a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$ such that*

$$0 \in \partial f(a) + \sum_{i=1}^m \lambda_i^a \partial^o g_i(a) \text{ and } \lambda_i^a g_i(a) = 0, \forall i = 1, \dots, m. \quad (17)$$

Let further $\tilde{I}(a)' := \{i \in I'(a) : \lambda_i^a > 0\}$ and $C(x)' := \{\xi \in \partial f(a) : \langle \xi, x - a \rangle \geq 0\}$ for any given $x \in F'$. Then, the solution set S' of the problem (P) is characterized by

$$S' = S'_1 = S'_2 = S'_3 = S'_4 = S'_5 = S'_6 = S'_7,$$

where

$$\begin{aligned} S'_1 &:= \{x \in F' : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^0 g_i(a), \forall i \in \tilde{I}(a)'; \\ &\quad \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(x) \cap \partial f(a)\}, \\ S'_2 &:= \{x \in F' : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^0 g_i(a), \forall i \in \tilde{I}(a)'; \\ &\quad \langle \zeta, a - x \rangle \geq 0 \text{ for some } \zeta \in \partial f(x) \cap \partial f(a)\}, \\ S'_3 &:= \{x \in F' : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^0 g_i(a), \forall i \in \tilde{I}(a)'; \\ &\quad \langle \xi, x - a \rangle = \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(x) \text{ and } \xi \in C(x)'\}, \\ S'_4 &:= \{x \in F' : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^0 g_i(a), \forall i \in \tilde{I}(a)'; \\ &\quad \langle \xi, x - a \rangle = \langle \zeta, a - x \rangle \text{ for some } \zeta \in \partial f(x) \text{ and } \xi \in C(x)'\}, \\ S'_5 &:= \{x \in F' : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^0 g_i(a), \forall i \in \tilde{I}(a)'; \\ &\quad \langle \xi, x - a \rangle \leq \langle \zeta, a - x \rangle \text{ for some } \zeta \in \partial f(x) \text{ and } \xi \in C(x)'\}, \\ S'_6 &:= \{x \in F' : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^0 g_i(a), \forall i \in \tilde{I}(a)'; \\ &\quad \langle \zeta, a - x \rangle = 0 \text{ for some } \zeta \in \partial f(x)\}, \\ S'_7 &:= \{x \in F' : \langle \eta_i, x - a \rangle = 0 \text{ for some } \eta_i \in \partial^0 g_i(a), \forall i \in \tilde{I}(a)'; \\ &\quad \langle \zeta, a - x \rangle \geq 0 \text{ for some } \zeta \in \partial f(x)\}. \end{aligned}$$

Corollary 3. *For the problem (P), let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex function and $F' := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}$ be convex. Assume that for any $x \in F'$ and $i \in I'(x)$ the functions g_i are locally Lipschitz and regular in the sense of Clarke, $a \in S'$ is an optimal solution fulfilling $N(F', a) = \text{cone} \bigcup_{i \in I'(a)} \partial^o g_i(a)$, and the optimality conditions (17) hold with a Lagrange multiplier vector $\lambda^a := (\lambda_1^a, \dots, \lambda_m^a) \in \mathbb{R}_+^m$. Then,*

$$S' = \{x \in F' : \langle \eta_i, x - a \rangle = 0, \exists \eta_i \in \partial^o g_i(a), \forall i \in \tilde{I}(a)' \text{ and } 0 \in \partial L^P(\cdot, a, \lambda^a)(x)\},$$

where $L^P(x, a, \lambda^a) := f(x) + \sum_{i \in I'(a)} \lambda_i^a g_i^o(a; x - a)$.

5. Application to robust multi-objective optimization problems. In this section, as an application of the general results of the previous section, we examine the class of multiple-objective programs in the face of data uncertainty both in the objective and constraints that can be written by the following multi-objective optimization problem:

$$\min_{x \in \mathbb{R}^n} \{(f_1(x, u_1), \dots, f_p(x, u_p)) : g_i(x, v_i) \leq 0, i = 1, \dots, m\}, \quad (\text{UMP})$$

where $f_k : \mathbb{R}^n \times \mathbb{R}^{q_k} \rightarrow \mathbb{R}$, $k = 1, \dots, p$, are convex-concave functions, $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are functions satisfying the condition (C1) and (C2), $g_i(x, \cdot)$ are concave functions for any $x \in \mathbb{R}^n$, and u_k and v_i are uncertain parameters and they belong to nonempty convex compact sets $\mathcal{U}_k \subseteq \mathbb{R}^{q_k}$ and $\mathcal{V}_i \subseteq \mathbb{R}^{q_i}$, respectively.

We associate with (UMP) its robust counterpart, which is the worst case of (UMP),

$$\min_{x \in \mathbb{R}^n} \left\{ \left(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \dots, \max_{u_p \in \mathcal{U}_p} f_p(x, u_p) \right) : x \in F \right\}, \quad (\text{RMP})$$

where F stands for the robust feasible set of (UMP), defined by

$$F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\}.$$

In the same way, we will give three kind robust solutions for the problems (UMP) which has been introduced in [22].

$\bar{x} \in F$ is said to be a *robust efficient solution* of (UMP) if there does not exist a robust feasible solution x of (UMP) such that

$$\max_{u_k \in \mathcal{U}_k} f_k(x, u_k) \leq \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k) \text{ for all } k = 1, \dots, p,$$

and

$$\max_{u_l \in \mathcal{U}_l} f_l(x, u_l) < \max_{u_l \in \mathcal{U}_l} f_l(\bar{x}, u_l) \text{ for some } l.$$

$\bar{x} \in F$ is called a *weakly robust efficient solution* of (UMP) if there does not exist a robust feasible solution x of (UMP) such that

$$\max_{u_k \in \mathcal{U}_k} f_k(x, u_k) < \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k) \text{ for all } k = 1, \dots, p.$$

$\bar{x} \in F$ is said to be a *properly robust efficient solution* of (UMP) if it is a robust efficient solution of (UMP) and there is a number $M > 0$ such that for all $k \in \{1, \dots, p\}$ and $x \in F$ satisfying $\max_{u_k \in \mathcal{U}_k} f_k(x, u_k) < \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k)$, there exists an index $l \in \{1, \dots, p\}$ such that $\max_{u_l \in \mathcal{U}_l} f_l(\bar{x}, u_l) < \max_{u_l \in \mathcal{U}_l} f_l(x, u_l)$ and

$$\frac{\max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k) - \max_{u_k \in \mathcal{U}_k} f_k(x, u_k)}{\max_{u_l \in \mathcal{U}_l} f_l(x, u_l) - \max_{u_l \in \mathcal{U}_l} f_l(\bar{x}, u_l)} \leq M.$$

According to these definitions, it is evidently that $\bar{x} \in F$ is a robust efficient solution (resp. weakly, properly robust efficient solution) of (UMP) if and only if $\bar{x} \in F$ is a efficient solution (resp. weakly, properly efficient solution) of (RMP). The search for an efficient solution (resp. weakly, properly efficient solution) to multi-objective optimization problem has been carried out through solving a single (scalar) or a family of single objective optimization problems, possibly depending on some appropriate parameters. We refer the reader to [28, 12, 31, 20] and other references therein for necessary and sufficient conditions for (weakly, properly) efficient solutions to a multiobjective optimization by parameterization and linear scalarization (weighted sum approach).

In this section, we present characterizations of weakly robust efficient solution set ($WR(F)$) and properly robust efficient solution set ($PR(F)$) of the problem

(UMP) by using linear scalarization approach. Before presenting, in the cases of study, let us consider the following scalar convex problem of (RMP) depending on a parameter $\theta := (\theta_1, \dots, \theta_p) \in \mathbb{R}_+^p$:

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{k=1}^p \theta_k \max_{u_k \in \mathcal{U}_k} f_k(x, u_k) : x \in F \right\}. \quad (RP_\theta)$$

Suppose that the solution set of problem (RP_θ) , denoted by S_θ is nonempty. It is well-known, in the literature, that weakly efficient solutions and properly efficient solutions of (RMP) can be characterized by solving some scalar parameterized convex problems (RP_θ) . More precisely,

- (i) $\bar{x} \in WR(F)$ if and only if there exists $\theta \in \mathbb{R}_+^p \setminus \{0\}$ such that $\bar{x} \in S_\theta$.
- (ii) $\bar{x} \in PR(F)$ if and only if there exists $\theta \in \text{int} \mathbb{R}_+^p$ such that $\bar{x} \in S_\theta$.

Thus, by using Theorem 3.3, we can obtain immediately the following necessary and sufficient optimality conditions for weakly robust efficient solution as well as properly robust efficient solution of (UMP).

Theorem 5.1. *For the problem (UMP), suppose all conditions of Theorem 3.3 hold and $\bar{x} \in F := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i \in I\}$ fulfilling the robust basic constraint qualification. Assume further that the set F is convex. Then,*

- (i) $\bar{x} \in F$ is a weakly robust efficient solution of (UMP) if and only if there exist $\theta_k \geq 0, k = 1, \dots, p$, not all zero, $\lambda_i \geq 0, i = 1, \dots, m$, $\bar{u}_k \in \mathcal{U}_k, k = 1, \dots, p$ and $\bar{v}_i \in \mathcal{V}_i, i = 1, \dots, m$ such that

$$\begin{aligned} 0 &\in \sum_{k=1}^p \theta_k \partial f_k(\cdot, \bar{u}_k)(\bar{x}) + \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}), \\ \lambda_i g_i(\bar{x}, \bar{v}_i) &= 0, \quad \forall i = 1, \dots, m, \text{ and} \\ f_k(\bar{x}, \bar{u}_k) &= \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k), \quad k = 1, \dots, p. \end{aligned}$$

- (ii) $\bar{x} \in F$ is a properly robust efficient solution of (UMP) if and only if there exist $\theta_k > 0, k = 1, \dots, p$, $\lambda_i \geq 0, i = 1, \dots, m$, $\bar{u}_k \in \mathcal{U}_k, k = 1, \dots, p$ and $\bar{v}_i \in \mathcal{V}_i, i = 1, \dots, m$ such that

$$\begin{aligned} 0 &\in \sum_{k=1}^p \theta_k \partial f_k(\cdot, \bar{u}_k)(\bar{x}) + \sum_{i=1}^m \lambda_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}), \\ \lambda_i g_i(\bar{x}, \bar{v}_i) &= 0, \quad \forall i = 1, \dots, m, \text{ and} \\ f_k(\bar{x}, \bar{u}_k) &= \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k), \quad k = 1, \dots, p. \end{aligned}$$

Proof. (i) As $\bar{x} \in F$ is a weakly robust efficient solution of (UMP) if and only if $\bar{x} \in F$ is a weakly efficient solution of (RMP), there exist $\theta_k \geq 0, k = 1, \dots, p$, not all zero, such that $\bar{x} \in F$ is a solution of (RP_θ) . In the other word, $\bar{x} \in F$ is a robust solution of the following uncertain (only in the constraints) convex optimization problem:

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{k=1}^p \theta_k \max_{u_k \in \mathcal{U}_k} f_k(x, u_k) : g_i(x, v_i) \leq 0, i \in I \right\}.$$

Applying Theorem 3.3, we get that there exist $\bar{\lambda}_i \geq 0$, and $\bar{v}_i \in \mathcal{V}_i, i \in I$ such that

$$0 \in \partial \left(\sum_{k=1}^p \theta_k \max_{u_k \in \mathcal{U}_k} f_k(\cdot, u_k) \right) (\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x}), \quad \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \quad \forall i \in I.$$

By employing the summation, positively homogeneous and max-function of convex subdifferential rule, the result as required.

(ii) The proof of (ii) is quite similar to that of (i) and so is omitted. $\square$

In the following proposition, we give a sufficient condition that a robust efficient solution of (UMP) can be a properly robust efficient solution of (UMP).

Proposition 4. *For the problem (UMP), let $\bar{x} \in F$ be a robust feasible solution for (UMP). Assume all conditions of Theorem 3.3 hold. Assume further that the set F is convex, $F \cap F(\bar{x}) \neq \emptyset$ and*

$$N(F \cap F(\bar{x}), \bar{x}) = \text{cone} \left\{ \left(\bigcup_{k=1, \dots, p} \bigcup_{u_k \in \mathcal{U}_k(\bar{x})} \partial f_k(\cdot, u_k)(\bar{x}) \right) \bigcup \left(\bigcup_{i \in I(\bar{x})} \bigcup_{v_i \in \mathcal{V}_i(\bar{x})} \partial^o g_i(\cdot, v_i)(\bar{x}) \right) \right\}, \quad (18)$$

where

$$\begin{aligned} F(\bar{x}) &:= \{x \in \mathbb{R}^n : \max_{u_k \in \mathcal{U}_k} f_k(x, u_k) \leq \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k), \forall k = 1, \dots, p\}, \text{ and} \\ \mathcal{U}_k(\bar{x}) &:= \{\bar{u}_k \in \mathcal{U}_k : f_k(\bar{x}, \bar{u}_k) = \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k)\}, \quad k = 1, \dots, p, \\ \mathcal{V}_i(\bar{x}) &:= \{\bar{v}_i \in \mathcal{V}_i : g_i(\bar{x}, \bar{v}_i) = \max_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i)\}, \quad i \in I(\bar{x}). \end{aligned}$$

If $\bar{x}$ is a robust efficient solution of (UMP), then $\bar{x}$ is a properly robust efficient solution of (UMP).

Proof. Let $\bar{x}$ be a robust efficient solution of (UMP). Then $\bar{x}$ is a minimizer of the following scalar convex problem:

$$\min_{x \in \mathbb{R}^n} \left\{ \sum_{k=1}^p \max_{u_k \in \mathcal{U}_k} f_k(x, u_k) : x \in F \cap F(\bar{x}) \right\},$$

or equivalently,

$$0 \in \sum_{k=1}^p \partial(\max_{u_k \in \mathcal{U}_k} f_k(\cdot, u_k))(\bar{x}) + N(F \cap F(\bar{x}), \bar{x}).$$

It follows that there exists $\eta \in N(F \cap F(\bar{x}), \bar{x})$ such that

$$-\eta \in \sum_{k=1}^p \partial(\max_{u_k \in \mathcal{U}_k} f_k(\cdot, u_k))(\bar{x}).$$

Then, by the condition (18), there exist $\theta_k \geq 0$, $\bar{u}_k \in \mathcal{U}_k$, $\xi_k \in \partial f_k(\cdot, \bar{u}_k)(\bar{x})$, $k = 1, \dots, p$, $\lambda_i \geq 0$, $\bar{v}_i \in \mathcal{V}_i(\bar{x})$ and $\zeta_i \in \partial^o g_i(\cdot, \bar{v}_i)(\bar{x})$, $i \in I(\bar{x})$, such that

$$\eta = \sum_{k=1}^p \theta_k \xi_k + \sum_{i \in I(\bar{x})} \lambda_i \zeta_i$$

and

$$f_k(\bar{x}, \bar{u}) = \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k), \quad \forall k = 1, \dots, p.$$

which implies that

$$0 = -\eta + \eta \in \sum_{k=1}^p \partial(\max_{u_k \in \mathcal{U}_k} f_k(\cdot, u_k))(\bar{x}) + \sum_{k=1}^p \theta_k \partial f_k(\cdot, \bar{u}_k)(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial^o g_i(\cdot, \bar{v}_i)(\bar{x})$$

$$\begin{aligned}
&\subseteq \sum_{k=1}^p \partial(\max_{u_k \in \mathcal{U}_k} f_k(\cdot, u_k))(\bar{x}) + \sum_{k=1}^p \theta_k \left(\bigcup_{u_k \in \mathcal{U}_k(\bar{x})} \partial f_k(\cdot, u_k)(\bar{x}) \right) + N(F, \bar{x}) \\
&= \sum_{k=1}^p \partial(\max_{u_k \in \mathcal{U}_k} f_k(\cdot, u_k))(\bar{x}) + \sum_{k=1}^p \theta_k \partial(\max_{u_k \in \mathcal{U}_k} f_k(\cdot, u_k))(\bar{x}) + N(F, \bar{x}) \\
&= \sum_{k=1}^p (1 + \theta_k) \partial(\max_{u_k \in \mathcal{U}_k} f_k(\cdot, u_k))(\bar{x}) + N(F, \bar{x}).
\end{aligned}$$

Therefore,

$$\sum_{k=1}^p (1 + \theta_k) \max_{u_k \in \mathcal{U}_k} f_k(x, u_k) \geq \sum_{k=1}^p (1 + \theta_k) \max_{u_k \in \mathcal{U}_k} f_k(\bar{x}, u_k) \text{ for all } x \in F,$$

which gives that $\bar{x} \in S_{\tilde{\theta}}$ with $\tilde{\theta} := (1 + \theta_1, \dots, 1 + \theta_p) \in \text{int} \mathbb{R}_+^p$, and so $\bar{x}$ is a properly robust efficient solution of (UMP). $\square$

Let $\theta \in \mathbb{R}_+^p \setminus \{0\}$ (resp. $\text{int} \mathbb{R}_+^p$) and $a^\theta \in S_\theta$. We have seen already that if the robust basic constraint qualification holds at a^θ , the set of Lagrange multiplier and uncertain parameters $M(a^\theta)$ for (RP $_\theta$) corresponding to a^θ , given as

$$\begin{aligned}
M(a^\theta) := & \left\{ (\lambda^\theta, u^\theta, v^\theta) \in \mathbb{R}_+^m \times \prod_{k=1}^p \mathbb{R}^{q_k} \times \prod_{i=1}^m \mathbb{R}^{q_i} : \right. \\
& 0 \in \sum_{k=1}^p \theta_k \partial f_k(\cdot, u_k^\theta)(a^\theta) + \sum_{i=1}^m \lambda_i^\theta \partial^\circ g_i(\cdot, v_i^\theta)(a^\theta), \\
& \left. \lambda_i^\theta g_i(a^\theta, v_i^\theta) = 0, \forall i \in I \text{ and } f_k(a^\theta, u^\theta) = \max_{u_k \in \mathcal{U}_k} f_k(a^\theta, u_k), \forall k = 1, \dots, p \right\},
\end{aligned}$$

is nonempty where $\lambda^\theta := (\lambda_1^\theta, \dots, \lambda_m^\theta)$, $u^\theta := (u_1^\theta, \dots, u_p^\theta)$ and $v^\theta := (v_1^\theta, \dots, v_m^\theta)$. Let further $I(a^\theta) := \{i \in I : \exists v_i^\theta \in \mathcal{V}_i \text{ such that } g_i(a^\theta, v_i^\theta) = 0\}$ and $\tilde{I}(a^\theta) := \{i \in I(a^\theta) : \lambda_i^\theta > 0\}$.

By means of linear scalarization applied in Theorem 4.1, we can get characterizations of the weakly robust efficient solution sets $WR(F)$ and properly robust efficient solution set $PR(F)$ of the problem (UMP) immediately.

Theorem 5.2. *For the problem (UMP), assume all conditions of Theorem 3.3 hold, and the set F is convex.*

- (i) *Suppose further that for each $\theta \in \mathbb{R}_+^p \setminus \{0\}$, S_θ is non-empty. Let $a^\theta \in S_\theta$ and the robust basic constraint qualification holds at a^θ . Let $(\lambda^\theta, u^\theta, v^\theta) \in M(a^\theta)$. Then*

$$\begin{aligned}
&WR(F) \\
&= \bigcup_{\theta \in \mathbb{R}_+^p \setminus \{0\}} \left\{ x \in F : \langle \eta_i^\theta, x - a^\theta \rangle = 0 \text{ for some } \eta_i^\theta \in \partial^\circ g_i(\cdot, v_i^\theta)(a^\theta), \forall i \in \tilde{I}(a^\theta); \right. \\
&\quad \langle \zeta^\theta, x - a^\theta \rangle = 0 \text{ for some } \zeta^\theta \in \sum_{k=1}^p \theta_k \partial f_k(\cdot, u_k^\theta)(a^\theta) \cap \sum_{k=1}^p \theta_k \partial f_k(\cdot, u_k^\theta)(a^\theta) \text{ and} \\
&\quad \left. f_k(x, u^\theta) = \max_{u_k \in \mathcal{U}_k} f_k(x, u_k), \forall k = 1, \dots, p \right\}.
\end{aligned}$$

(ii) If for each $\theta \in \text{int}\mathbb{R}_+^p$, S_θ is non-empty, $a^\theta \in S_\theta$ is fulfilled the robust basic constraint qualification, $(\lambda^\theta, u^\theta, v^\theta) \in M(a^\theta)$, then

$PR(F)$

$$= \bigcup_{\theta \in \text{int}\mathbb{R}_+^p} \left\{ x \in F : \langle \eta_i^\theta, x - a^\theta \rangle = 0 \text{ for some } \eta_i^\theta \in \partial^\circ g_i(\cdot, v_i^\theta)(a^\theta), \forall i \in \tilde{I}(a^\theta); \right. \\ \left. \langle \zeta^\theta, x - a^\theta \rangle = 0 \text{ for some } \zeta^\theta \in \sum_{k=1}^p \theta_k \partial f_k(\cdot, u^\theta)(x) \cap \sum_{k=1}^p \theta_k \partial f_k(\cdot, u^\theta)(a^\theta) \text{ and } \right. \\ \left. f_k(x, u^\theta) = \max_{u_k \in \mathcal{U}_k} f_k(x, u_k), \forall k = 1, \dots, p \right\}.$$

□

To close this section, we give an example illustrating Theorem 5.2 which is indicated to be conveniently applied is applicable while the aforementioned result, due to Sun et al. [34, Theorem 4.7], are not. It means that at least one of the constraint functions $g_i(\cdot, v_i)$ for some $v_i \in \mathcal{V}_i$, is not convex while the robust feasible set is convex.

Example 5.1. Let $x := (x_1, x_2) \in \mathbb{R}^2$, $u_1 := (u_{1,1}, u_{1,2})$, $u_2 := (u_{2,1}, u_{2,2})$, $v_1 := (v_{1,1}, v_{1,2})$, $v_2 := (v_{2,1}, v_{2,2})$, $v_3 := (v_{3,1}, v_{3,2})$, $v_4 := (v_{4,1}, v_{4,2})$, $\mathcal{U}_1 = \mathcal{U}_2 := [0, 1]$, $\mathcal{V}_1 := \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2 \leq 1\}$, $\mathcal{V}_2 := [-2, -1] \times [-2, -1]$, $\mathcal{V}_3 := [4, 5] \times [2, 3]$ and $\mathcal{V}_4 := [0, 1] \times [0, 1]$.

We now consider the following constrained multiobjective optimization problem with uncertainty data :

$$\text{Minimize } (f_1(x, u_1), f_2(x, u_2))$$

$$\text{subject to } x \in \mathbb{R}^2, g_1(x, v_1) \leq 0, g_2(x, v_2) \leq 0, g_3(x, v_3) \leq 0, g_4(x, v_4) \leq 0,$$

where

$$\begin{aligned} f_1(x, u_1) &:= u_{1,1}x_1, \\ f_2(x, u_2) &:= u_{2,2}x_2, \\ g_1(x, v_1) &:= v_{1,1}x_1 + v_{1,2}x_2 + x_1^3 - 2, \\ g_2(x, v_2) &:= v_{2,1}x_1 + v_{2,2}x_2 + 1, \\ g_3(x, v_3) &:= -v_{3,1}x_1 - v_{3,2}x_2 + 3, \\ g_4(x, v_4) &:= -v_{4,1}x_1 - v_{4,2}x_2^2, \end{aligned}$$

and its robust counterpart

$$\begin{aligned} &\text{Minimize } (\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \max_{u_2 \in \mathcal{U}_2} f_2(x, u_2)) \\ &\text{subject to } x \in F := \left\{ \begin{array}{l} g_1(x, v_1) \leq 0, \forall v_1 \in \mathcal{V}_1, \\ g_2(x, v_2) \leq 0, \forall v_2 \in \mathcal{V}_2, \\ g_3(x, v_3) \leq 0, \forall v_3 \in \mathcal{V}_3, \\ g_4(x, v_4) \leq 0, \forall v_4 \in \mathcal{V}_4, \\ x \in \mathbb{R}^2 \end{array} \right\}. \end{aligned}$$

We obtain that for every $x \in \mathbb{R}^2$,

$$\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1) = x_1,$$

$$\begin{aligned} \max_{u_2 \in \mathcal{U}_2} f_2(x, u_2) &= x_2, \\ F &= \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} + x_1^3 - 2 \leq 0, -x_1 - x_2 + 1 \leq 0, -4x_1 - 2x_2 + 3 \leq 0, \\ &\quad -x_1 \leq 0\}, \end{aligned}$$

as a straightforward calculation shows. Let us denote

$$\begin{aligned} \Lambda &:= \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 + \theta_2 = 1\}, \\ \text{int}\Lambda &:= \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 > 0, \theta_2 > 0, \theta_1 + \theta_2 = 1\}. \end{aligned}$$

We now consider the following possibilities:

- (i) If $\theta := (\theta_1, \theta_2) = (1, 0)$ then (RP_θ) reads as follows:

$$\begin{aligned} \text{Minimize} \quad & x_1 \\ \text{subject to} \quad & x \in F. \end{aligned}$$

As $\theta_1 x_1 + \theta_2 x_2 = x_1 \geq 0, \forall x \in F$, we can take $a^\theta := (0, 2) \in S_\theta$ and so, $I(a^\theta) = \{1, 4\}$. Let us choose $\lambda^\theta := (0, 0, 0, 1)$, $u^\theta := (1, 1)$, $v^\theta := (v_1^\theta, v_2^\theta, v_3^\theta, v_4^\theta) = ((0, 1), (-1, -1), (4, 2), (1, 0))$, and we also have $\tilde{I}(a^\theta) = \{4\}$. Let

$$\begin{aligned} A_\theta &:= \{x \in F : \langle \eta_i^\theta, x - a^\theta \rangle = 0 \text{ for some } \eta_i^\theta \in \partial^\circ g_i(\cdot, v_i^\theta)(a^\theta), \forall i \in \tilde{I}(a^\theta); \\ &\quad \exists \zeta^\theta \in (\theta_1 \partial f_1(\cdot, u_1^\theta)(x) + \theta_2 \partial f_2(\cdot, u_2^\theta)(x)) \\ &\quad \cap (\theta_1 \partial f_1(\cdot, u_1^\theta)(a^\theta) + \theta_2 \partial f_2(\cdot, u_2^\theta)(a^\theta)); \\ &\quad \langle \zeta^\theta, x - a^\theta \rangle = 0, \max_{u_1 \in \mathcal{U}_1} f_1(x, u_1) = f_1(x, u_1^\theta), \max_{u_2 \in \mathcal{U}_2} f_2(x, u_2) = f_2(x, u_2^\theta)\}. \end{aligned}$$

Then A_θ can be easily calculated $A_\theta = \{x \in \mathbb{R}^2 : x_1 = 0, \frac{3}{2} \leq x_2 \leq 2\}$.

- (ii) Similarly, if $\theta = (0, 1)$ then we can take $a^\theta := (1, 0) \in S_\theta$ and so, $I(a^\theta) = \{1, 2, 4\}$. Let us choose $\lambda^\theta := (\frac{1}{4}, 1, 0, 0)$, $u^\theta := (1, 1)$, $v^\theta := (v_1^\theta, v_2^\theta, v_3^\theta, v_4^\theta) = ((1, 0), (-1, -1), (4, 2), (0, 1))$, and we also have $\tilde{I}(a^\theta) = \{1, 2\}$. Thus $A_\theta = \{(1, 0)\}$.

- (iii) If $\theta \in \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \frac{2}{3} < \theta_1 < 1, \theta_2 > 0, \theta_1 + \theta_2 = 1\}$ then (RP_θ) becomes

$$\begin{aligned} \text{Minimize} \quad & \theta_1 x_1 + \theta_2 x_2 \\ \text{subject to} \quad & x \in F. \end{aligned}$$

As $\theta_1 x_1 + \theta_2 x_2 \geq \theta_1 x_1 - 2\theta_2 x_1 + \frac{3}{2}\theta_2 = (3\theta_1 - 2)x_1 + \frac{3}{2}\theta_2 \geq \frac{3}{2}\theta_2, \forall x \in F$, then we can take $a^\theta := (0, \frac{3}{2}) \in S_\theta$ and so, $I(a^\theta) = \{2, 4\}$. Let us choose $\lambda^\theta := (0, 0, \frac{\theta_2}{2}, \theta_1 - 2\theta_2)$ (note that $\theta_1 > \frac{2}{3}$ and $\theta_1 + \theta_2 = 1$ imply $\theta_1 - 2\theta_2 > 0$), $u^\theta := (1, 1)$, $v^\theta := (v_1^\theta, v_2^\theta, v_3^\theta, v_4^\theta) = ((1, 0), (-1, -1), (4, 2), (1, 0))$, and we also have $\tilde{I}(a^\theta) = \{2, 4\}$. In this case, it is easy to see that $A_\theta = \{(0, \frac{3}{2})\}$.

- (iv) Similarly, if $\theta \in \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \frac{1}{2} < \theta_1 < \frac{2}{3}, \theta_2 > 0, \theta_1 + \theta_2 = 1\}$ then $A_\theta = \{(0, 1)\}$ and if $\theta \in \{(\theta_1, \theta_2) \in \mathbb{R}^2 : 0 < \theta_1 < \frac{1}{2}, \theta_2 > 0, \theta_1 + \theta_2 = 1\} = \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \frac{1}{2} < \theta_2 < 1, \theta_1 > 0, \theta_1 + \theta_2 = 1\}$ then $A_\theta = \{(1, 0)\}$.

- (v) If $\theta := (\frac{1}{2}, \frac{1}{2})$, then $\theta_1 x_1 + \theta_2 x_2 \geq \frac{1}{2}, \forall x \in F$. Take $a^\theta := (\frac{1}{2}, \frac{1}{2}) \in S_\theta$ and so, $I(a^\theta) = \{2, 3, 4\}$. Let us choose $\lambda^\theta := (0, \frac{1}{2}, 0, 0)$, $u^\theta := (1, 1)$, $v^\theta := (v_1^\theta, v_2^\theta, v_3^\theta, v_4^\theta) = ((1, 0), (-1, -1), (4, 2), (1, 0))$, and we also have $\tilde{I}(a^\theta) = \{2\}$. Then elementary calculations give us $A_\theta = \{x \in \mathbb{R}^2 : -x_1 - x_2 + 1 = 0, x_1 \geq 0, x_2 \geq 0\}$.

(vi) Similarly, if $\theta = (\frac{2}{3}, \frac{1}{3})$ then we can take $a^\theta = (\frac{1}{4}, 1)^T \in S_\theta$, $\lambda^\theta = (0, 0, \frac{1}{6}, 0)$ and so,

$$A_\theta = \{x \in \mathbb{R}^2 : -4x_1 - 2x_2 + 3 = 0, x_1 \geq 0, x_2 \geq 0\}.$$

Therefore, by Theorem 5.2, weakly and properly robust efficient solution sets of (UMP) look like

$$\begin{aligned} WR(F) &= \bigcup_{\theta \in \mathbb{R}_+^2 \setminus \{0\}} A_\theta = \bigcup_{\theta \in \Lambda} A_\theta \\ &= \{x \in \mathbb{R}^2 : x_1 = 0, 1 \leq x_2 \leq 2\} \\ &\quad \cup \{x \in \mathbb{R}^2 : -4x_1 - 2x_2 + 3 = 0, x_1 \geq 0, x_2 \geq 0\} \\ &\quad \cup \{x \in \mathbb{R}^2 : -x_1 - x_2 + 1 = 0, x_1 \geq 0, x_2 \geq 0\} \end{aligned}$$

and

$$\begin{aligned} PR(F) &= \bigcup_{\theta \in \text{int}\mathbb{R}_+^2} A_\theta = \bigcup_{\theta \in \text{int}\Lambda} A_\theta \\ &= \{x \in \mathbb{R}^2 : -4x_1 - 2x_2 + 3 = 0, x_1 \geq 0, x_2 \geq 0\} \\ &\quad \cup \{x \in \mathbb{R}^2 : -x_1 - x_2 + 1 = 0, x_1 \geq 0, x_2 \geq 0\}. \end{aligned}$$

6. Conclusions. In this paper, following the framework of robust optimization, we consider an uncertain convex optimization problem without convexity assumption on constraint functions. We provide a new pseudo Lagrangian-type function which is constant on the robust optimal solution set. We also obtain some characterizations of the robust optimal solution set of all robust optimal solutions of a given problem. Furthermore, as applications, we obtain some characterizations of both weakly robust efficient solution set and properly robust efficient solution set for a convex multi-objective optimization problem with data uncertainty.

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Characterizing the solution set of convex optimization problems without convexity of constraints

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Abstract

Some characterizations of solution sets of a convex optimization problem with a convex feasible set described by tangentially convex constraints are given. The results are expressed in terms of convex subdifferentials, tangential subdifferentials, and Lagrange multipliers. In order to characterize the solution set, we first introduce the so-called pseudo Lagrangian-type function and establish a constant pseudo Lagrangian-type property for the solution set. This property is still valid in the case of a pseudoconvex locally Lipschitz objective function, and then used to derive Lagrange multiplier-based characterizations of the solution set. Some examples are given to illustrate the significances of our theoretical results.

Keywords Convex optimization problems · Pseudo Lagrangian functions · Tangentially convex functions · Solution sets

1 Introduction

Characterizations and properties of the solution sets play an important role for understanding the behavior of solution methods for mathematical programs that have multiple optimal solutions, see [1–3]. Mangasarian [4] initially presented several characterizations of the solution set for convex programs when one solution is known. This study was further extended to convex/nonconvex optimization problems, see, e.g., [5–13]. In addition, Lagrange multipliers and its properties are employed to charac-

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terize the solution set of some classes of constrained optimization problems published such as [14–23].

In a recent paper, under Slater’s condition together with an additional condition on the constraints, Lagrange multipliers for differentiable convex problems without the convexity of the constraint functions have been obtained by Lasserre [24]. Further study has been done for non-differentiability case, see [25,26]. The primary aim of this paper is to investigate the characterization of solution sets of the following convex optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) \text{ s.t. } x \in K, \quad (\mathbf{P})$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function, the feasible set K , defined by

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\},$$

is a nonempty convex subset of the Euclidean space $\mathbb{R}^n$ and the functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I := \{1, 2, \dots, m\}$, are continuous, but they are *not assumed to be convex functions*. Assume that the solution set of problem (P), denoted by

$$S := \{x \in K : f(x) \leq f(y), \forall y \in K\},$$

is nonempty.

It is remarkable that the characterization of solution sets of (P) is done by applying [21, Corollary 3.10.] if the functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I$, are restricted to be locally Lipschitz and regular in the sense of Clarke [27] and additionally the pseudoconvexity in the first argument of the Lagrange function,

$$\mathcal{L}(\cdot, \lambda) := f(\cdot) + \sum_{i \in I} \lambda_i g_i(\cdot), \quad \forall \lambda := (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}_+^m,$$

is satisfied. However, the pseudoconvexity assumption of $\mathcal{L}(\cdot, \lambda)$ for every $\lambda \in \mathbb{R}_+^m$ often fails (see Remark 7 in Sect. 4). Further, regularity requirements of g_i s may fail even if g_i s are differentiable functions due to the fact that differentiable functions are not necessarily regular unless they are continuously differentiable.

Motivated and inspired by the facts mentioned above, we aim to give characterizations of the solution set of (P) without the pseudoconvexity assumption of Lagrange function. In order to make use of the obtained results for both the differentiable setting and the regular locally Lipschitz setting, we deal with the problem (P) with continuous tangentially convex constraint functions (see [26]). First, we give the weakest constraint qualification for guaranteeing the Lagrange multiplier conditions to be necessary and sufficient for optimality of (P). After introducing the so-called *pseudo-Lagrange function*, we then establish the constant pseudo-Lagrange property and employ it to derive a characterization of the solution set of (P). These are expressed in terms of convex subdifferentials, tangential subdifferentials and Lagrange multipliers. Moreover, Lagrange multiplier characterizations of the solution set for optimization

problems with a pseudoconvex locally Lipschitz objective function, without convexity of the constraint functions are given.

The paper is organized as follows: Sect. 2 gives some notations, definitions and preliminary results. In Sect. 3, we establish a multiplier characterization for the optimal solution of (P). Section 4 provides characterizations of the solution set of a convex optimization problem. Finally, characterizations of the solution set for optimization problems with a pseudoconvex locally Lipschitz objective function, without convexity of the constraint functions are given in Sect. 5.

2 Preliminaries

We begin this section by defining notation and preliminary results which shall be used later in this paper. All spaces under consideration are the n -dimensional Euclidean space $\mathbb{R}^n$ with the inner product $\langle \cdot, \cdot \rangle$. The norm of $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. The closed (resp. open, left closed right open) interval between $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ is denoted by $[\alpha, \beta]$ (resp. $] \alpha, \beta[$, $[\alpha, \beta[$). The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}_+^n$ and is defined by $\mathbb{R}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$. Given a set $A \subseteq \mathbb{R}^n$, the set A is a cone if $\mu A \subseteq A$ for all $\mu \geq 0$. We denote the *conical hull* generated by A , by $\text{cone} A$. The set A is convex whenever $\mu a_1 + (1 - \mu)a_2 \in A$ for all $\mu \in [0, 1]$, $a_1, a_2 \in A$. The *normal cone* at a to a closed convex set A , denoted by $N(A, a)$, is defined by $N(A, a) := \{u \in \mathbb{R}^n : \langle u, x - a \rangle \leq 0, \forall x \in A\}$. Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}$. A function f is said to be *convex* if for all $\mu \in [0, 1]$, $f(\mu a_1 + (1 - \mu)a_2) \leq \mu f(a_1) + (1 - \mu)f(a_2)$ for all $a_1, a_2 \in \mathbb{R}^n$. The subdifferential of a convex function f at a is defined as $\partial f(a) := \{u \in \mathbb{R}^n : \forall x \in \mathbb{R}^n, f(x) \geq f(a) + \langle u, x - a \rangle\}$. The function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *tangentially convex* at $x \in \mathbb{R}^n$ (see [28,29]) if for every $d \in \mathbb{R}^n$ the right-sided directional derivative of f at x

$$f'(x, d) := \lim_{t \rightarrow 0^+} \frac{f(x + td) - f(x)}{t}$$

exists, is finite, and is a convex function of d .

Example 1 [Classes of tangentially convex functions] The following points are taken from [26].

- Every convex function which has an open domain is tangentially convex at each point of its domain.
- Every function which is Gateaux differentiable at a point x is tangentially convex at x by the linearity of the directional derivative $f'(x, \cdot)$.
- Every locally Lipschitz function which is regular (in the sense of Clarke) at a point x is tangentially convex at x , since in such a case the classical one-sided directional derivative is convex because it coincides with the Clarke directional derivative.

- (d) The class of tangentially convex functions at a given point is a real vector space, and hence some tangentially convex functions (not necessarily convex and differentiable) will follow from the sum of a convex function with a differentiable function.

The concept of subdifferential for tangentially convex function is implicitly given in [28]. The *tangential subdifferential* of (a tangentially convex function) f at $x \in \mathbb{R}^n$ is the set $\partial_T f(x)$ given as

$$\partial_T f(x) := \{u \in \mathbb{R}^n : \langle u, d \rangle \leq f'(x, d), \forall d \in \mathbb{R}^n\},$$

which is a nonempty compact convex set. It is important to note that if f is tangentially convex at $x \in \mathbb{R}^n$ such that $f(x) \in \mathbb{R}$, the function $f'(x, \cdot)$ is the support function of the tangential subdifferential, that is,

$$f'(x, d) = \max_{u \in \partial_T f(x)} \langle u, d \rangle, \text{ for all } d \in \mathbb{R}^n. \quad (1)$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be tangentially convex at $x \in \mathbb{R}^n$. The tangential subdifferentials enjoy nice calculus properties including the positive homogeneous rule and the sum rule, i.e.,

- (i) for every $\mu \geq 0$, $\partial_T(\mu f)(x) = \mu \partial_T f(x)$;
- (ii) if f and g is tangentially convex at the same point x , one has

$$\partial_T(f + g)(x) = \partial_T f(x) + \partial_T g(x).$$

Remark 1 For a given tangentially convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $x \in \mathbb{R}^n$, it is easily to verify that the function $y \mapsto f'(x, y - x)$ is convex and

$$\partial_T f(x) = \partial f'(x, \cdot - x)(x) = \partial f'(x, \cdot)(0).$$

Example 2 Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as $f(x_1, x_2) := \sqrt{x_1^2 + x_2^2} - x_1^3 - x_2^3$. Then, for every $(d_1, d_2) \in \mathbb{R}^2$, we can verify that

$$\frac{f((0, 0) + t(d_1, d_2)) - f(0, 0)}{t} = \sqrt{d_1^2 + d_2^2} - t^2 d_1^3 - t^2 d_2^3 \quad \text{for all } t > 0.$$

Letting $t \rightarrow 0^+$, we get $f'((0, 0), (d_1, d_2)) = \sqrt{d_1^2 + d_2^2}$, from which it follows that $(d_1, d_2) \mapsto f'((0, 0), (d_1, d_2))$ is convex. So, f is tangentially convex at $(0, 0)$ and its tangential subdifferential at $(0, 0)$ is

$$\partial_T f(0, 0) = \partial f'((0, 0), \cdot)((0, 0)) = [-1, 1] \times [-1, 1].$$

□

We conclude this section with the following useful result which will be used in the proofs of the main results.

Lemma 1 *Let $x \in K$ and $I(x) := \{i \in I : g_i(x) = 0\}$. Assume that for every $i \in I(x)$ the function g_i is tangentially convex at x . If the set K is convex, then for any $y \in K$, one has*

$$g'_i(x, y - x) \leq 0, \quad \forall i \in I(x). \quad (2)$$

Moreover, for each $i \in I(x)$, $\partial_T g_i(x) \subseteq N(K, x)$.

Proof It is proved in [26, Proposition 1] that

$$K \subseteq \{y \in \mathbb{R}^n : g'_i(x, y - x) \leq 0, \quad \forall (x, i) \in K \times I(x)\},$$

without the Slater's condition together with an additional condition on the constraints. Thus, for any $y \in K$, (2) holds.

Furthermore, (1) and (2) yield, for any $u \in \partial_T g_i(x)$, $\langle u, y - x \rangle \leq 0, \forall y \in K$, which gives that $u \in N(K, x)$, thereby establishing the desired result. $\square$

3 Multiplier characterization for the optimal solution

In this section, we give some constraint qualifications for guaranteeing the Lagrange multiplier conditions to be necessary and sufficient for optimality of (P).

It should be noted that, in convex programs, Slater's condition is usually used to obtain the Lagrange multiplier conditions which characterize optimality (see [14, 16, 17, 19–21] and other references therein). However, the Lagrange multiplier conditions for the convex optimization problems without convexity of the constraint functions may fail under the Slater's condition. Recently, Lagrange multiplier conditions have been obtained under the Slater's condition together with an additional condition on the constraints. Some constraint qualifications, which are also necessary for the existence of Lagrange multipliers for convex optimization problems without convexity of the constraint functions, has been introduced in [30, 31]. In an analogous manner as [30], we introduce the following constraint qualification in terms of tangential subdifferentials and show that it is the weakest constraint qualification for guaranteeing the Lagrange multiplier conditions to be necessary and sufficient for optimality of (P).

Definition 1 Let $x \in K$ and $g_i, i \in I(x)$, be tangentially convex at x . The *normal cone condition* is satisfied at x if

$$N(K, x) = \text{cone} \bigcup_{i \in I(x)} \partial_T g_i(x).$$

Theorem 1 [Weakest CQ for Lagrange multiplier conditions] *Let $\bar{x} \in K$ be given, and for every $i \in I(\bar{x})$ the functions g_i be tangentially convex at $\bar{x}$. Then, the following assertions are equivalent:*

- (i) The normal cone condition is satisfied at $\bar{x}$;
(ii) For each convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ attaining its global minimizer over K at $\bar{x}$, there exist $\lambda_i \geq 0$, $i \in I$, such that

$$0 \in \partial f(\bar{x}) + \sum_{i \in I} \bar{\lambda}_i \partial_T g_i(\bar{x}), \quad (3)$$

and

$$\bar{\lambda}_i g_i(\bar{x}) = 0, \quad \forall i \in I. \quad (4)$$

Proof [(i) $\Rightarrow$ (ii)]. Suppose that (i) holds. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be any convex function such that $\bar{x} \in K$ is a global minimizer of (P). The convexity of K implies that, for each $t \in [0, 1]$, $f(\bar{x}) \leq f(\bar{x} + t(x - \bar{x}))$, $\forall x \in K$, which gives

$$f'(\bar{x}, \bar{x} - \bar{x}) = 0 \leq f'(\bar{x}, x - \bar{x}), \quad \forall x \in K. \quad (5)$$

It means that $\bar{x}$ is a minimizer of the convex function $f'(\bar{x}, \cdot - \bar{x})$ over K and it can be equivalently expressed as, by Remark 1,

$$0 \in \partial f'(\bar{x}, \cdot - \bar{x})(\bar{x}) + N(K, \bar{x}) = \partial f(\bar{x}) + N(K, \bar{x}). \quad (6)$$

The condition (i) yields that there exists $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that

$$0 \in \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial_T g_i(\bar{x}).$$

Setting $\lambda_i = 0$ for $i \notin I(\bar{x})$, the above expression can be rewritten as

$$0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial_T g_i(\bar{x}) \text{ and } \lambda_i g_i(\bar{x}) = 0, \quad \forall i \in I,$$

and hence (ii) has been justified.

[(ii) $\Rightarrow$ (i)]. Suppose that (ii) holds. By the virtue of Lemma 1, we only need to prove that

$$N(K, \bar{x}) \subseteq \text{cone} \bigcup_{i \in I(\bar{x})} \partial_T g_i(\bar{x}).$$

In fact, let $u \in N(K, \bar{x})$ be given. The definition of $N(K, \bar{x})$ yields that $\langle -u, \bar{x} \rangle \leq \langle -u, x \rangle$ for all $x \in K$. It can be seen that $f(x) := \langle -u, x \rangle$, $x \in \mathbb{R}^n$, is a convex function attaining its global minimizer over K at $\bar{x}$. So, from (ii) and $\partial f(\bar{x}) = \{-u\}$, there exist $\lambda_i \geq 0$, $i \in I$, such that

$$0 \in \{-u\} + \sum_{i \in I} \lambda_i \partial_T g_i(\bar{x}) \text{ and } \lambda_i g_i(\bar{x}) = 0, \quad \forall i \in I,$$

which follows that

$$u \in \sum_{i \in I(\bar{x})} \lambda_i \partial_T g_i(\bar{x}) \subseteq \text{cone} \bigcup_{i \in I(\bar{x})} \partial_T g_i(\bar{x}),$$

thereby leading to the desired result. $\square$

Remark 2 [Sufficient condition for the normal cone condition] As seen before, for each $u \in N(K, \bar{x})$, $f(x) := \langle -u, x \rangle$, $x \in \mathbb{R}^n$, is a convex function attaining its global minimizer over K at $\bar{x}$. Thus, if the system $g_i(x) \leq 0$, $i \in I$, satisfies the Slater's condition and the *non-degeneracy condition* at $\bar{x}$, i.e., for every $i \in I(\bar{x})$,

$$0 \notin \partial_T g_i(\bar{x}),$$

then [26, Theorem 9] guarantees the existence of multipliers $\lambda_i \geq 0$, $i \in I$, such that

$$0 \in \{-u\} + \sum_{i \in I} \lambda_i \partial_T g_i(\bar{x}) \text{ and } \lambda_{g_i}(\bar{x}) = 0, \quad \forall i \in I,$$

consequently, $u \in \text{cone} \bigcup_{i \in I(\bar{x})} \partial_T g_i(\bar{x})$. Therefore, the normal cone condition holds at $\bar{x}$.

Remark 3 In view of the proof of (5) and (6) in the proof of Theorem 1, one can notice that if “For each convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ attaining its global minimizer over K at $\bar{x}$ ” is replaced by “For each tangentially convex and *pseudoconvex*¹ function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $\bar{x}$ attaining its global minimizer over K at $\bar{x}$ ”, then its conclusions hold also true when the convex subdifferential $\partial f(\bar{x})$ is replaced by the tangential subdifferential $\partial_T f(\bar{x})$.

The following example illustrates that if the normal cone condition, (i), in Theorem 1 does not hold, then the optimality condition in Theorem 1 is not derived for a convex objective function.

Example 3 [Failure of Multiplier Characterization] Let us denote $x := (x_1, x_2) \in \mathbb{R}^2$, $g_1(x) = \sqrt{x_1^2 + x_2^2} - x_1^3 - 2$, $g_2(x) = -x_1^3 + \max\{-x_2, -x_2^3\}$, $g_3(x) = 2x_1 + x_2$, $K := \{x \in \mathbb{R}^2 : g_i(x) \leq 0, i \in I := \{1, 2, 3\}\}$ and $\bar{x} := (0, 0)$. It is easy to verify that $K = \{x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} - x_1^3 - 2 \leq 0, 2x_1 + x_2 \leq 0, -x_1 - x_2 \leq 0\}$, $I(\bar{x}) = \{2, 3\}$, $\partial_T g_2(\bar{x}) = \{0\} \times [-1, 0]$ and $\partial_T g_3(\bar{x}) = \{(2, 1)\}$. It can be observed that

$$N(K, \bar{x}) = \text{cone}\{(-1, -1), (2, 1)\}$$

and

$$\text{cone}(\partial_T g_2(\bar{x}) \cup \partial_T g_3(\bar{x})) = \text{cone}\{(0, -1), (2, 1)\}.$$

¹ A tangentially convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $x \in \mathbb{R}^n$ is said to be *pseudoconvex* at x (see [26]) if $\forall y \in \mathbb{R}^n$, $f'(x, y - x) \geq 0 \implies f(y) \geq f(x)$.

Hence, we have that condition (i) of Theorem 1 does not hold. Thus for some pseudoconvex function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ at $\bar{x}$, it is impossible to characterize a sufficient condition for global optimality for the following problem by using Theorem 1,

$$\min_{x \in \mathbb{R}^2} f(x) \text{ s.t. } x \in K.$$

Actually, let $f(x) := e^{-x_1} - x_1 - 2x_2$. Put $\bar{\lambda}_1 := 0$, $\bar{\lambda}_2 := 1$ and $\bar{\lambda}_3 := 2$, then

$$(0, 0) \in \{(-2, -2)\} + (\{0\} \times [-1, 0]) + 2\{(1, 1)\} = \partial f(\bar{x}) + \sum_{i \in I} \bar{\lambda}_i \partial_T g_i(\bar{x}).$$

However, by taking $x := \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in K$, $f(x) := \sqrt{e} + \frac{1}{2} - \sqrt{3} < 1 = f(\bar{x})$, which shows that $\bar{x}$ is not a global minimizer of f over K . $\square$

Remark 4 In the case that for any $x \in K$ and $i \in I(x)$, g_i are locally Lipschitz and regular in the sense of Clarke, Theorem 3.2 in [30] can be obtained immediately by Theorem 1.

4 Characterizations of the solution sets of convex optimization problems

In this section, we will present some characterizations of the solution sets in terms of a given solution point of the convex minimization problem (P).

Let $\bar{x} \in S$ be a given solution point fulfilling the normal cone condition and for every $i \in I(\bar{x})$, the functions g_i be tangentially convex at $\bar{x}$. Let $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_m) \in \mathbb{R}_+^m$ be a Lagrange multiplier vector corresponding to $\bar{x}$ such that (3) and (4) hold.

It is important to note that the constant Lagrangian-type property for the solution sets are commonly used to establish characterizations of solution sets for constrained optimization problems involving convex/pseudoconvex functions (see [14, 16, 17, 19, 21] and other references therein). However, the constant Lagrangian-type property for the solution sets may fail when some g_i are not convex even if the objective function is convex, for instance, let us define $f(x) := \max\{-x - 1, 0\}$, $g_1(x) := \max\{x, x^3\}$ for $x \in \mathbb{R}$. We can see that f is a convex function while g_1 is not a convex function. Moreover, $\bar{x} := 0$ is a minimizer of f on a convex set $K :=]-\infty, 0]$ with Lagrange multiplier $\bar{\lambda} := 1$, and the solution set is $S = [-1, 0]$. However, the standard Lagrangian-type function $\mathcal{L}(x, \bar{\lambda}) := f(x) + \bar{\lambda}g_1(x)$ is not constant on the solution set S . In fact, $\mathcal{L}(\bar{x}, \bar{\lambda}) = 0 \neq -1 = \mathcal{L}(-1, \bar{\lambda})$. This situation motivates us to consider the so-called *pseudo Lagrangian-type function* $\mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})$, defined by

$$\mathcal{L}^P(x, \bar{x}, \bar{\lambda}) := f(x) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}), \text{ for all } x \in \mathbb{R}^n,$$

instead of the standard Lagrangian-type function. It can be seen that $\mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})$ is constant on S , since $\mathcal{L}^P(x, \bar{x}, \bar{\lambda}) = \max\{-x - 1, 0\} + \max\{x, 0\}$ for any $x \in \mathbb{R}$ and $\mathcal{L}^P(x, \bar{x}, \bar{\lambda}) = 0$ for any $x \in S$.

Next, we are in a position to prove that the pseudo Lagrangian function associated with a Lagrange multiplier corresponding to a solution is constant on S .

Proposition 1 [Constant pseudo Lagrangian-type property] For the problem (P), assume that $\bar{x} \in S$ satisfies the normal cone condition and the optimality conditions (3) and (4) hold with a Lagrange multiplier vector $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_m) \in \mathbb{R}_+^m$. Then for any $x \in S$,

$$\bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) = 0, \quad i \in I(\bar{x})$$

and $\mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})$ is constant on S .

Proof By Remark 1, $\partial_T g_i(\bar{x}) = \partial g'_i(\bar{x}, \cdot - \bar{x})(\bar{x})$ for all $i \in I(\bar{x})$. It follows from (3) and (4) that

$$0 \in \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i \partial g'_i(\bar{x}, \cdot - \bar{x})(\bar{x}) \subseteq \partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(\bar{x}),$$

and so,

$$f(x) + \sum_{i=1}^m \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \geq f(\bar{x}) \quad \text{for all } x \in \mathbb{R}^n.$$

By (4), it is easy to observe that $\bar{\lambda}_i = 0$ for all $i \notin I(\bar{x})$. Therefore, by the fact that $f(x) = f(\bar{x})$, $\forall x \in S$, the above expression can be rewritten as

$$\sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \geq 0 \quad \text{for all } x \in S.$$

Applying Lemma 1, we obtain

$$\bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) = 0, \quad \forall i \in I(\bar{x}). \quad \text{Therefore, for any } x \in S,$$

$$\mathcal{L}^P(x, \bar{x}, \bar{\lambda}) = f(x) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, \cdot - \bar{x})(\bar{x}) = f(x) = f(\bar{x}),$$

thus yielding the desired results. $\square$

Remark 5 [Pseudo Lagrangian-type function coincides with standard Lagrangian-type function] It is worth noting that if g_i , $i \in I(\bar{x})$, are pseudoconvex functions at $\bar{x}$ then, by Proposition 1, for any $x \in S$,

$$(\bar{\lambda}_i g_i)'(\bar{x}, x - \bar{x}) = \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) = 0 \implies \bar{\lambda}_i g_i(x) \geq \bar{\lambda}_i g_i(\bar{x}) = 0.$$

This together with $x \in K$ yields, $\bar{\lambda}_i g_i(x) = 0$, $i \in I(\bar{x})$. Furthermore,

$$\mathcal{L}^P(x, \bar{x}, \bar{\lambda}) = f(x) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) = f(x) + \sum_{i=1}^m \bar{\lambda}_i g_i(x), \quad \forall x \in S.$$

It means that the pseudo Lagrangian-type function is the standard Lagrangian-type function on the solution set S . $\square$

Remark 6 In Proposition 1, if f is Locally Lipschitz, regular in the sense of Clarke and pseudoconvex, it is proved in [12, Lemma 3] that $\forall x, y \in \mathbb{R}^n$, one has

$$f(y) \leq f(x) \implies f^o(x, y - x) \leq 0. \quad (7)$$

Applying (7), we can show that the conclusions given in Proposition 1 are still valid. Indeed, as $\bar{x} \in S$, Theorem 1 asserts that there exists a Lagrange multiplier vector $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_m) \in \mathbb{R}_+^m$ such that (3) and (4) hold. The fact that $\partial^o f(\bar{x}) = \partial_T f(\bar{x})$ along with tangential subdifferential calculus rules at $\bar{x}$ imply that

$$f'(\bar{x}, x - \bar{x}) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Using (7) with the fact that $f(x) = f(\bar{x})$, $\forall x \in S$ and regularity of f , we deduce $\sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \geq 0$, $\forall x \in S$, and hence, by Lemma 1, we obtain $\bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) = 0$, $\forall i \in I(\bar{x})$, and for any $x \in S$, $\mathcal{L}^P(x, \bar{x}, \bar{\lambda}) = f(\bar{x})$. $\square$

In the sequel, we present characterizations of the solution set for problem (P) in terms of convex subdifferentials, tangential subdifferentials and Lagrange multipliers.

Theorem 2 [Characterization of the solution set] For the problem (P), assume all conditions of Proposition 1. Then, the solution set S is characterized by

$$S = \{x \in K : \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial_T g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0; \\ \exists u \in \partial f(x) \cap \partial f(\bar{x}), \langle u, x - \bar{x} \rangle = 0\},$$

where $\tilde{I}(\bar{x}) := \{i \in I(\bar{x}) : \bar{\lambda}_i > 0\}$.

Proof [$\subseteq$]. Let $x \in S$ be arbitrarily given. Then, x belongs to K . Furthermore, by (1) and Proposition 1, we have that for each $i \in \tilde{I}(\bar{x})$, $\max_{v_i \in \partial_T g_i(\bar{x})} \langle v_i, x - \bar{x} \rangle = g'_i(\bar{x}, x - \bar{x}) = 0$. Therefore, for each $i \in \tilde{I}(\bar{x})$, there exists $v_i \in \partial_T g_i(\bar{x})$ such that

$$\langle v_i, x - \bar{x} \rangle = 0.$$

On the other hand, it follows from (3) and (4) that there exist $u \in \partial f(\bar{x})$ such that $-u \in \sum_{i \in I} \bar{\lambda}_i \partial_T g_i(\bar{x}) = \partial_T(\sum_{i \in I} \bar{\lambda}_i g_i)(\bar{x})$. That is, for any $d \in \mathbb{R}^n$,

$$\sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, d) = \left(\sum_{i \in I(\bar{x})} \bar{\lambda}_i g_i \right)'(\bar{x}, d) = \left(\sum_{i \in I} \bar{\lambda}_i g_i \right)'(\bar{x}, d) \geq \langle -u, d \rangle, \quad (8)$$

where the second equality follows from (4). Note from $x, \bar{x} \in S$ that $f(x) = f(\bar{x})$. Letting $d := x - \bar{x}$ in (8), one has $\sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \geq \langle -u, x - \bar{x} \rangle$, which together with Lemma 1 and $u \in \partial f(\bar{x})$ implies that

$$0 \geq \sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \geq \langle -u, x - \bar{x} \rangle \geq f(\bar{x}) - f(x) = 0.$$

So, $\langle u, x - \bar{x} \rangle = 0$. It remains to prove that $u \in \partial f(x)$. Now, for any $y \in \mathbb{R}^n$, we have

$$f(y) - f(x) = f(y) - f(\bar{x}) \geq \langle u, y - \bar{x} \rangle = \langle u, y - x \rangle + \langle u, x - \bar{x} \rangle = \langle u, y - x \rangle.$$

Therefore, $u \in \partial f(x)$.

[$\supseteq$]. Conversely, let x be an arbitrary point of $S_1 := \{x \in K : \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial_T g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0; \exists u \in \partial f(x) \cap \partial f(\bar{x}), \langle u, x - \bar{x} \rangle = 0\}$. Then, $x \in K$ and there exists $u \in \partial f(x) \cap \partial f(\bar{x})$ such that $\langle u, \bar{x} - x \rangle = 0$. So

$$f(\bar{x}) - f(x) \geq \langle u, \bar{x} - x \rangle = 0,$$

which together with the fact that $\bar{x} \in S$ yields $f(x) = f(\bar{x})$, and so $x \in S$. $\square$

As tangential convexity collapses to regularly locally Lipschitz setting and differentiability, the following corollaries are immediately direct consequences as special cases of Theorem 2.

Corollary 1 For the problem (P), let for any $x \in K$ and $i \in I(x)$ the functions g_i be locally Lipschitz and regular in the sense of Clarke, $\bar{x} \in S$ be an optimal solution fulfilling the condition: $N(K, \bar{x}) = \text{cone} \bigcup_{i \in I(\bar{x})} \partial^o g_i(\bar{x})$, where $\partial^o g_i(\bar{x})$ denotes the Clarke subdifferential of g_i at $\bar{x}$. Assume that the optimality conditions (3) and (4) hold with a Lagrange multiplier vector $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_m) \in \mathbb{R}_+^m$. Then, the solution set is characterized by

$$S = \{x \in K : \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial^o g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0; \\ \exists u \in \partial f(x) \cap \partial f(\bar{x}), \langle u, x - \bar{x} \rangle = 0\}.$$

Proof The desired results can be obtained immediately by Theorem 2, since every locally Lipschitz regular function g_i is tangentially convex at every point x , with $\partial_T g_i(\bar{x}) = \partial^o g_i(\bar{x})$, $\forall i \in \tilde{I}(\bar{x})$. $\square$

Corollary 2 For the problem (P), let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex differentiable function and $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, m$) be differentiable functions, $\bar{x} \in S$ be an optimal solution of (P) fulfilling the condition: $N(K, \bar{x}) = \text{cone} \bigcup_{i \in I(\bar{x})} \{\nabla g_i(\bar{x})\}$. Assume that the optimality conditions (3) and (4) hold with a Lagrange multiplier vector $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_m) \in \mathbb{R}_+^m$. Then,

$$S = \{x \in K : \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle = 0, \forall i \in \tilde{I}(\bar{x}), \nabla f(x) = \nabla f(\bar{x})\}.$$

Proof It is clear that every differentiable functions g_i are tangentially convex at every point x , with $\partial_T g_i(\bar{x}) = \{\nabla g_i(\bar{x})\}$, $\forall i \in \tilde{I}(\bar{x})$. It follows from Theorem 2 with $\partial f(x) = \{\nabla f(x)\}$ for every point $x \in \mathbb{R}^n$ that

$$S = \{x \in K : \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle = 0, \forall i \in \tilde{I}(\bar{x}), \nabla f(x) = \nabla f(\bar{x}), \langle \nabla f(x), x - \bar{x} \rangle = 0\}.$$

Further, since $\bar{x}$ satisfies optimality condition (3), we have

$$0 = \left\langle \sum_{i=1}^m \bar{\lambda}_i \nabla g_i(\bar{x}), x - \bar{x} \right\rangle = \langle \nabla f(\bar{x}), x - \bar{x} \rangle = \langle -\nabla f(x), x - \bar{x} \rangle.$$

Thus, the condition $\langle \nabla f(x), x - \bar{x} \rangle = 0$ is superfluous. Therefore, $S = \{x \in K : \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle = 0, \forall i \in \tilde{I}(\bar{x}), \nabla f(x) = \nabla f(\bar{x})\}$. $\square$

When one solution of the considered problem is known, by using Theorem 2, we can find all of solutions of the convex optimization problem that have multiple solutions, and moreover at least one of the constraint functions g_i is not convex while the constraint set is convex. So, Theorem 2.2 and Corollary 2.1 in [14] cannot be applied in the following example.

Example 4 [Verifying solution set] Let us denote $x := (x_1, x_2) \in \mathbb{R}^2$. Consider the following constrained optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) \text{ s.t. } x \in K := \{x \in \mathbb{R}^2 : g_1(x) \leq 0, g_2(x) \leq 0, g_3(x) \leq 0\},$$

where $f(x) := \sqrt{x_1^2 + x_2^2} - x_1 - x_2$, $g_1(x) := \sqrt{x_1^2 + x_2^2} - x_1^3 - 2$, $g_2(x) := \max\{-x_1, -x_1^3\} - x_2$, $g_3(x) := x_1$. Evidently, the function f is a convex function. Let us notice that

$$f(x) = \sqrt{x_1^2 + x_2^2} - x_1 - x_2 \geq |x_2| - x_2 = 0, \text{ for all } x \in K.$$

Thus $\bar{x} := (\bar{x}_1, \bar{x}_2) = (0, 0) \in S$, $I(\bar{x}) = \{2, 3\}$, $\partial_T g_2(\bar{x}) = \{(r, -1) : r \in [-1, 0]\}$ and $\partial_T g_3(\bar{x}) = \{(1, 0)\}$. It is easy to verify that this problem satisfies the Slater condition and non-degeneracy at $\bar{x}$.

Also, the convex subdifferential of f at any point x is given by

$$\partial f(x) = \begin{cases} \{(-1, -1)\} + \mathbb{B} & \text{if } x = (0, 0), \\ \left\{ \left[\frac{1}{\sqrt{x_1^2 + x_2^2}} \right] (x_1, x_2) + (-1, -1) \right\} & \text{if } x \neq (0, 0), \end{cases}$$

where $\mathbb{B} := \{w \in \mathbb{R}^2 : \|w\| \leq 1\}$. Let us select $\bar{\lambda} := (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) = (0, 0, 1)$. Therefore, by using Theorem 2, the solution set can be described simply as

$$\begin{aligned}
 S &= \{x \in K : \langle (v_{3,1}, v_{3,2}), (x_1, x_2) \rangle = 0 \text{ for some } (v_{3,1}, v_{3,2}) \in \partial_T g_3(\bar{x}), \\
 &\quad \langle (u_1, u_2), (x_1, x_2) \rangle = 0 \text{ for some } (u_1, u_2) \in \partial f(x) \cap \partial f(\bar{x})\} \\
 &= \left\{ x \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} - x_1^3 - 2 \leq 0, \max\{-x_1, -x_1^3\} - x_2 \leq 0, x_1 \leq 0, \right. \\
 &\quad \left. \langle (1, 0), (x_1, x_2) \rangle = 0, \left\langle \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}} - 1, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} - 1 \right), (x_1, x_2) \right\rangle = 0 \right\} \\
 &= \{x \in \mathbb{R}^2 : x_1 = 0, \sqrt{x_2^2} - 2 \leq 0, -x_2 \leq 0\} \\
 &= \{x \in \mathbb{R}^2 : x_1 = 0, 0 \leq x_2 \leq 2\}.
 \end{aligned}$$

□

Next, we give a characterization of S using subdifferentials of the pseudo Lagrangian-type function. To this aim, we need the following lemma.

Lemma 2 *For the problem (P), assume all conditions of Proposition 1. Then for each $x \in S$,*

$$\partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(x) = \partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(\bar{x}).$$

Proof Now take any $u \in \partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(x)$. Then, by the definition of convex subdifferential,

$$\mathcal{L}^P(y, \bar{x}, \bar{\lambda}) - \mathcal{L}^P(x, \bar{x}, \bar{\lambda}) \geq \langle u, y - x \rangle, \quad \forall y \in \mathbb{R}^n. \quad (9)$$

Since $\mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})$ is constant on S (Proposition 1) and $\bar{x} \in S$, it follows from (9) that $\langle u, \bar{x} - x \rangle = 0$ and so, $\langle u, y - x \rangle = \langle u, y - \bar{x} \rangle + \langle u, \bar{x} - x \rangle = \langle u, y - \bar{x} \rangle$ for all $y \in \mathbb{R}^n$. This together with (9) entails

$$\mathcal{L}^P(y, \bar{x}, \bar{\lambda}) - \mathcal{L}^P(\bar{x}, \bar{x}, \bar{\lambda}) \geq \langle u, y - \bar{x} \rangle, \quad \forall y \in \mathbb{R}^n,$$

which shows that $u \in \partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(\bar{x})$. Therefore, $\partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(x) \subseteq \partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(\bar{x})$. The proof of the converse inclusion is quite a similar argument and will be omitted. □

With the help of Proposition 1 and Lemma 2, we see now how the solution set can be characterized in terms of the pseudo Lagrangian-type function.

Proposition 2 *For the problem (P), assume all conditions of Proposition 1. Then,*

$$S = \{x \in K : \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial_T g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0 \text{ and } 0 \in \partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(x)\}.$$

Proof Denote

$$S^* := \{x \in K : \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial_T g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0 \text{ and } 0 \in \partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(x)\}.$$

By Lemma 2 and the optimality condition (3), we get $0 \in \partial \mathcal{L}^P(\cdot, \bar{x}, \bar{\lambda})(x)$ for all $x \in S$. This together with Proposition 1 implies easily that $S \subseteq S^*$. To establish the converse inclusion, let $x \in S^*$ be given. Then, by the definition of S^* , $x \in K$ and for each $i \in \tilde{I}(\bar{x})$ there exist $v_i \in \partial_T g_i(\bar{x})$ such that

$$\langle v_i, x - \bar{x} \rangle = 0,$$

which implies, for every $y \in \mathbb{R}^n$, that

$$\begin{aligned} f(y) &+ \sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, y - \bar{x}) \\ &\geq f(x) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \\ &= f(x) + \sum_{i \in I(\bar{x}) \setminus \tilde{I}(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) + \sum_{i \in \tilde{I}(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \\ &= f(x) + \sum_{i \in \tilde{I}(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, x - \bar{x}) \\ &\geq f(x) + \sum_{i \in \tilde{I}(\bar{x})} \bar{\lambda}_i \langle v_i, x - \bar{x} \rangle = f(x). \end{aligned}$$

Taking $y := \bar{x}$ in the last inequality, we get that $f(\bar{x}) \geq f(x)$, and hence, for all $y \in K$,

$$f(y) \geq f(y) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i g'_i(\bar{x}, y - \bar{x}) \geq f(\bar{x}) \geq f(x),$$

where the first inequality follows from Lemma 1. This proves that $x \in S$. $\square$

It turns out that Theorem 2 and Proposition 2 immediately yield the characterizations of the solution set for convex programs that was proposed in [14, Corollary 2.1 and Corollary 2.6].

Corollary 3 *For the problem (P), let the functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I$ be convex, and $\bar{x} \in S$ an optimal solution fulfilling the normal cone condition and the optimality conditions (3) and (4) hold with a Lagrange multiplier vector $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_m) \in \mathbb{R}_+^m$. Then, the solution set S of (P) is characterized by*

$$\begin{aligned} S = \{x \in \mathbb{R}^n : g_i(x) = 0, \forall i \in \tilde{I}(\bar{x}), g_i(x) \leq 0, \forall i \in I \setminus \tilde{I}(\bar{x}), \\ \exists u \in \partial f(x) \cap \partial f(\bar{x}), \langle u, x - \bar{x} \rangle = 0\}. \end{aligned}$$

Proof By Theorem 2, we have that

$$\begin{aligned} S = \{x \in K : \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial_T g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0; \\ \exists u \in \partial f(x) \cap \partial f(\bar{x}), \langle u, x - \bar{x} \rangle = 0\}. \end{aligned}$$

As $\langle v_i, x - \bar{x} \rangle = 0$ for some $v_i \in \partial_T g_i(\bar{x})$, $\forall i \in \tilde{I}(\bar{x})$, we have $g'_i(\bar{x}, x - \bar{x}) \geq \langle v_i, x - \bar{x} \rangle = 0$ for each $i \in \tilde{I}(\bar{x})$. This together with $x \in K$, by Lemma 1, yields $g'_i(\bar{x}, x - \bar{x}) = 0$, $\forall i \in \tilde{I}(\bar{x})$. Moreover, by Remark 5, we get that

$$\begin{aligned} & [x \in K, \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial_T g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0] \\ & \implies [x \in \mathbb{R}^n, g_i(x) = 0, \forall i \in \tilde{I}(\bar{x}), g_i(x) \leq 0, \forall i \in I \setminus \tilde{I}(\bar{x})]. \end{aligned}$$

□

Corollary 4 For the problem (P), let the functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I$ be convex, and $\bar{x} \in S$ an optimal solution fulfilling the normal cone condition and the optimality conditions (3) and (4) hold with a Lagrange multiplier vector $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_m) \in \mathbb{R}_+^m$. Then,

$$S = \{x \in \mathbb{R}^n : g_i(x) = 0, \forall i \in \tilde{I}(\bar{x}), g_i(x) \leq 0, \forall i \in I \setminus \tilde{I}(\bar{x}), 0 \in \partial \mathcal{L}(\cdot, \bar{\lambda})(x)\}.$$

5 Characterizations of the solution sets of pseudoconvex minimization problems

In this section, we derive characterizations of the solution set of the following pseudoconvex minimization problem over a convex set (P'):

$$\min_{x \in \mathbb{R}^n} f(x) \text{ s.t. } x \in K, \quad (\text{P}')$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz, regular in the sense of Clarke and pseudoconvex, and $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I$, are continuous functions and the feasible set K is a nonempty convex subset of $\mathbb{R}^n$. In view of Remark 6, we can obtain the following results.

Theorem 3 For the problem (P'), let S' be the optimal solution set of (P'), $\bar{x} \in S'$ an optimal solution fulfilling the normal cone condition and the optimality conditions (3) and (4) hold with a Lagrange multiplier vector $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_m) \in \mathbb{R}_+^m$, and the functions g_i , $i \in I(\bar{x})$, be tangentially convex at $\bar{x}$. Then,

$$\begin{aligned} S' = \{x \in K : & \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial_T g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0; \\ & \exists p^x > 0, \exists u \in \partial^o f(x) \cap p^x \partial^o f(\bar{x}), \langle u, x - \bar{x} \rangle = 0\}. \end{aligned}$$

Proof [$\subseteq$]. Let us assume that $x \in S'$. By the same arguments given in the proof of Theorem 2, we can obtain that, for each $i \in \tilde{I}(\bar{x})$, there exists $v_i \in \partial_T g_i(\bar{x})$ such that $\langle v_i, x - \bar{x} \rangle = 0$. Furthermore, by Lee and Yao [23, Lemma 3.4], there exists a real number $p^x > 0$ (depending on x) such that

$$\partial^o f(x) \cap p^x \partial^o f(\bar{x}) \neq \emptyset.$$

It follows that there exists $u \in \mathbb{R}^n$ such that $u \in \partial^o f(x)$ and $\frac{u}{p^x} \in \partial^o f(\bar{x})$. As $x, \bar{x} \in S'$, $f(x) = f(\bar{x})$. It follows from [12, Lemma 3] that $f^o(x, x - \bar{x}) \leq 0$ and $f^o(\bar{x}, \bar{x} - x) \leq 0$. So,

$$\langle u, x - \bar{x} \rangle \leq 0 \text{ and } \left\langle \frac{u}{p^x}, \bar{x} - x \right\rangle \leq 0,$$

and hence $\langle u, x - \bar{x} \rangle = 0$.

[$\supseteq$]. For every $x \in S_2 := \{x \in K : \forall i \in \tilde{I}(\bar{x}), \exists v_i \in \partial_T g_i(\bar{x}), \langle v_i, x - \bar{x} \rangle = 0; \exists p^x > 0, \exists u \in \partial^o f(x) \cap p^x \partial^o f(\bar{x}), \langle u, x - \bar{x} \rangle = 0\}$, we get that $x \in K$, $\frac{u}{p^x} \in \partial^o f(\bar{x})$ and $\langle \frac{u}{p^x}, x - \bar{x} \rangle = 0$ for some $p^x > 0$ and $u \in \partial^o f(x)$. In addition, for any $v \in \mathbb{R}^n$ such that $\langle \frac{u}{p^x}, v \rangle \geq 0$, one has $f^o(x, v) \geq \langle u, v \rangle \geq 0$. Therefore,

$$x \in \{z \in K : \exists \xi \in \partial^o f(\bar{x}), \langle \xi, z - \bar{x} \rangle; \forall v \in \mathbb{R}^n, \langle \xi, v \rangle \geq 0 \Rightarrow f^o(z, v) \geq 0\},$$

and hence, [12, Theorem 9] leads to $x \in S'$, thus yielding the desired results. $\square$

Before we end this section, let us illustrate the usefulness of Theorem 3 via an example.

Example 5 Consider the constrained optimization problem (P') where $f(x) := \max\{0, \frac{1}{2}x^2 - \frac{1}{2}, (x-1)^3 + 1\}$, $g_1(x) := \max\{x, x^3\}$, $g_2(x) := 4x - x^3$, $g_3(x) := |x-1| - 3$. Evidently, the function f is a locally Lipschitz pseudoconvex function. Let us notice that

$$f(x) \geq 0 = \max\left\{0, -\frac{1}{2}, 0\right\} = f(0), \text{ for all } x \in K.$$

Then $\bar{x} := 0 \in S'$, $I(\bar{x}) = \{1, 2\}$ and each $g_i, i \in I(\bar{x})$, is tangentially convex at $\bar{x}$. We can verify that

$$K = [-2, 0], \partial_T g_1(\bar{x}) = [0, 1], \partial_T g_2(\bar{x}) = \{4\}, \partial_T g_3(\bar{x}) = \{-1\}.$$

Also, the Clarke subdifferential of f at any point x is given by

$$\partial^o f(x) = \begin{cases} \{x\} & \text{if } x \in]-\infty, -1[, \\ [-1, 0] & \text{if } x = -1, \\ \{0\} & \text{if } x \in]-1, 0[, \\ [0, 3] & \text{if } x = 0, \\ \{3(x-1)^2\} & \text{if } x \in]0, +\infty[. \end{cases}$$

We can see that this problem does not satisfy non-degeneracy at $\bar{x}$, the normal cone condition is fulfilled. Let us select $\bar{\lambda} := (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) = (1, 0, 0)$. Observe that for any $x \in [-1, 0[$, we can find $p^x > 0$ and $u \in \partial^o f(x) \cap p^x \partial^o f(\bar{x})$ such that $\langle u, x - \bar{x} \rangle = 0$. So, by Theorem 3, we can obtain that the solution set can be described as $S' = [-1, 0]$. $\square$

Remark 7 In Example 5,

- (i) g_1 is not pseudoconvex at $\bar{x} := 0$, i.e., taking $y := -1$ we have $g'_1(\bar{x}, y - \bar{x}) = \max\{0, y\} = 0$, but $g_1(y) = -1 < 0 = g_1(\bar{x})$. Then Theorems 4.1 and 4.2 in [19] may not be relevant to this example.
- (ii) The standard Lagrangian-type function with Lagrange multiplier $\bar{\lambda} := (1, 0, 0)$, $\mathcal{L}(x, \bar{\lambda}) = f(x) + \sum_{i=1}^3 \bar{\lambda}_i g_i(x)$, is not pseudoconvex at $\bar{x}$, i.e., by taking $y := -1$ we get $\mathcal{L}(\cdot, \bar{\lambda})'(\bar{x}, y - \bar{x}) = \max\{4y, 0\} = 0$ while $\mathcal{L}(y, \bar{\lambda}) = -1 < 0 = \mathcal{L}(\bar{x}, \bar{\lambda})$. So, Theorem 3.3 in [21] cannot be applied.
- (iii) Theorem 3.2 and Corollary 3.1 in [16] may not actually be relevant to this example because the constraint functions are not linear, and moreover, f is not pseudoconcave, i.e. $-f$ is not pseudoconvex, by considering $y := 2$ and $\bar{x} = 0$ we have $(-f)'(\bar{x}, y - \bar{x}) = 0$, but $(-f)(y) = -2 < 0 = (-f)(\bar{x})$.

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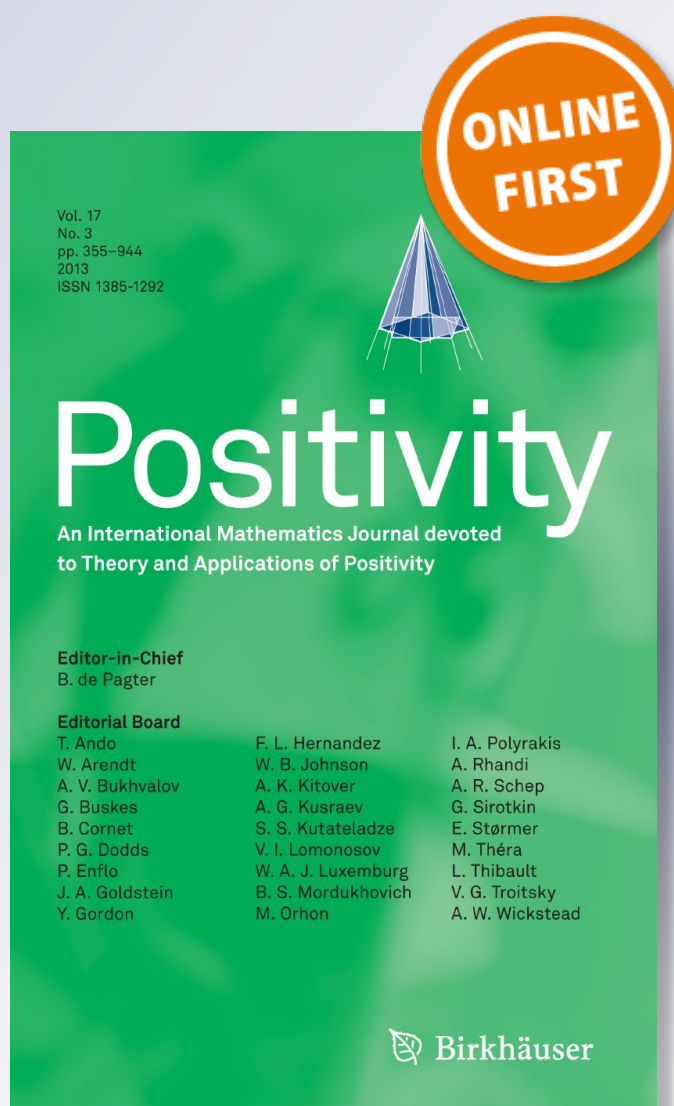
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Levitin–Polyak well-posedness for set optimization problems involving set order relations

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Abstract

In this paper set optimization problems with three types of set order relations are concerned. We introduce various types of Levitin–Polyak (LP) well-posedness for set optimization problems and survey their relationships. After that, sufficient and necessary conditions for the reference problems to be LP well-posed are given. Furthermore, using the Kuratowski measure of noncompactness, we study characterizations of well-posedness for set optimization problems. Moreover, the links between stability and LP well-posedness of such problems are established via the study on approximating solution mappings. Tools and techniques used in this study and our results are different from existing ones in the literature.

Keywords Set order relation · Set optimization problem · Levitin–Polyak well-posedness · Stability

Mathematics Subject Classification 49J53 · 49K40 · 90C31

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1 Introduction

The concept of well-posedness was firstly introduced by Tikhonov [36]. Beside the requirement about the uniqueness of the solution, Tikhonov well-posedness requested the convergence of each minimizing sequence to the unique solution. Therefore, well-posedness plays a vital role to make connections between stability properties and solution methods for problems related to optimization. On this topic, many research results have been devoted to a lot of important problems such as variational inequalities [7], equilibrium problems [1], inclusion problems [37] and the references therein. Several generalizations of Tikhonov well-posedness were introduced and investigated for various kinds of optimization problems [4,10,30–32]. Levitin–Polyak (LP) well-posedness is an extension of Tikhonov well-posedness and was originally proposed in [28]. Every minimizing sequence must belong to the feasible set in Tikhonov well-posedness, whereas it can be outside of the feasible region but the distance between it and this set has to approach zero in LP well-posedness. There have been many studies of LP well-posedness (see, e.g., [18,27] and the references therein).

Kuroiwa et al. [26] proposed set order relations including lower set less relation, upper set less relation and set less relation (combination of the lower and the upper set less relation). This gave a new way to formulate the solution of set-valued optimization problems which is called solutions concept based on the set approach [25], and hence the optimization problems in this approach are called set optimization problems involving set order relations [21,24]. As pointed out in [19], the set less relation is generalized and more appropriate in practical problems than both the lower and upper set less relations. Furthermore, the set less relation plays a center role in relationships with other new order relations for sets proposed in [5,19] which are more useful in set optimization. Although set optimization is a new direction in the field of optimization, it has attracted a great deal of attention of researchers with many important and interesting results [11,13,17,20]. Useful applications of set optimization in practical problems were reported, for example, the application in socio-economic [34] (to manage noise disturbance in the region surrounding the Frankfurt Airport in Germany), the application in finance [14] (to evaluate the risk of a multivariate random outcome). Moreover, relationships between set optimizations and other important problems such as variational inequalities [8], Ky Fan inequality problems (so-called equilibrium problems) [35] were investigated. For further reading and references, we refer to books [15,21].

The first introduction of well-posedness for set optimization problems was presented by Zhang et al. [38]. The authors established both sufficient and necessary conditions for set optimization problems involving the lower set less relation to be well-posed and obtained criteria as well as characterizations of well-posedness for these problems by the scalarization method. This research was generalized by Gutiérrez et al. [12] under assumptions of cone properness. After that, Dhingra and Lalitha [9] introduced a concept of well-setness for such problems and proved that it is an extension of the generalized well-posedness which was considered in [38]. Recently, well-posedness of set optimization problems involving not only the lower but also the upper set less relation have been discussed in [16,22,29]. For LP well-posedness, to the best of our knowledge there is only the paper of Khoshkhabar-amiranloo and

Khorram [22] which studied set optimization problem involving the lower set less relation. Of course such an important topic as LP well-posedness for set optimization problems must be the aim of many works. Moreover, the scalarization method which is the main tool used in papers mentioned above investigates difficultly set optimization problems involving different set order relations, some other approaches to study well-posedness for such problems should be considered.

Motivated and inspired by works mentioned above, in this paper, without using the scalarization method, we investigate different types of LP well-posedness for set optimization problems involving several kinds of set order relations. More precisely, we concern set optimization problems involving three types of set order relations. Then, we introduce concepts of LP well-posedness for such problems and discuss relationships among them. Moreover, necessary and/or sufficient conditions for these concepts of well-posedness are investigated. Applying Kuratowski measure of noncompactness, we study characterizations of such concepts. Finally, approximating solution mappings and their stability are studied to build the connection between stability of approximating problem and LP well-posedness of the set optimization problem.

The outline of this paper is given as follows. In Sect. 2, we recall some definitions and results needed in what follows. Sect. 3 introduces various kinds of LP well-posedness for set optimization problems and investigates their relationships. Furthermore, sufficient and/or necessary conditions of pointwise LP well-posedness for such problems are also obtained. In this section, characterizations of these types of pointwise LP well-posedness are surveyed by using measure of noncompactness. In the last section, Sect. 4, we study sufficient conditions for such problem to be metrically LP well-posed and their relationships.

2 Preliminaries

Let X be a normed space and Y be a real Hausdorff topological linear space. Let K be a closed convex pointed cone in Y with $\text{int}K \neq \emptyset$, where $\text{int}K$ denotes the interior of K . The space Y is endowed with an order relation induced by cone K in the following way

$$\begin{aligned}x \leq_K y &\Leftrightarrow y - x \in K, \\x <_K y &\Leftrightarrow y - x \in \text{int}K.\end{aligned}$$

The cone K induces various set orderings in Y . These such orderings as the following were presented in [19,21,25]. Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y . For $A, B \in \mathcal{P}(Y)$, lower set less relation, upper set less relation and set less relation, respectively, are defined by

$$\begin{aligned}A \leq^l B &\text{ if and only if } B \subset A + K, \\A \leq^u B &\text{ if and only if } A \subset B - K, \\A \leq^s B &\text{ if and only if } A \subset B - K \text{ and } B \subset A + K.\end{aligned}$$

Remark 2.1 The relationship between the lower set less relation $\leq^l$ and the upper set less relation $\leq^u$ was given by Remark 2.6.10 in [21] as the following

$$A \leq^l B \Leftrightarrow -B \leq^u -A.$$

Definition 2.1 [19] The binary relation $\leq$ is said to be

- (i) compatible with the addition if and only if $A \leq B$ and $D \leq E$ imply $A + D \leq B + E$ for all $A, B, D, E \in \mathcal{P}(Y)$;
- (ii) compatible with the multiplication with a nonnegative real number if and only if $A \leq B$ implies $\lambda A \leq \lambda B$ for all scalars $\lambda \geq 0$ and all $A, B \in \mathcal{P}(Y)$;
- (iii) compatible with the conlinear structure of $\mathcal{P}(Y)$ if and only if it is compatible with both the addition and the multiplication with a nonnegative real number.

Proposition 2.1 [19]

- (i) The order relations $\leq^l, \leq^u$ and $\leq^s$ are pre-order (i.e., these relations are reflexive and transitive).
- (ii) The order relations $\leq^l, \leq^u$ and $\leq^s$ are compatible with the conlinear structure of $\mathcal{P}(Y)$.
- (iii) In general, the order relations $\leq^l, \leq^u$ and $\leq^s$ are not antisymmetric; more precisely, for arbitrary sets $A, B \in \mathcal{P}(Y)$ we have

$$\begin{aligned} (A \leq^l B \text{ and } B \leq^l A) &\Leftrightarrow A + K = B + K, \\ (A \leq^u B \text{ and } B \leq^u A) &\Leftrightarrow A - K = B - K, \\ (A \leq^s B \text{ and } B \leq^s A) &\Leftrightarrow (A + K = B + K \text{ and } A - K = B - K). \end{aligned}$$

For $\alpha \in \{l, u, s\}$, we say that

$$A \sim^\alpha B \text{ if and only if } A \leq^\alpha B \text{ and } B \leq^\alpha A.$$

Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values on X . For each $\alpha \in \{l, u, s\}$, we consider the following set optimization problem

$$\begin{aligned} (P_\alpha) \quad & \alpha\text{-Min}F(x) \\ & \text{subject to } x \in M, \end{aligned}$$

where M is a nonempty closed subset of X . A point $\bar{x} \in M$ is said to be an α -minimal solution of (P_α) if and only if for any $x \in M$ such that $F(x) \leq^\alpha F(\bar{x})$ then $F(\bar{x}) \leq^\alpha F(x)$. The set of all α -minimal solutions of (P_α) is denoted by $S_{\alpha\text{-Min}F}$.

Remark 2.2 It can be seen that if $\bar{x} \in S_{\alpha\text{-Min}F}$ and $F(\bar{x}) \sim^\alpha F(\hat{x})$ for some $\hat{x} \in M$, then $\hat{x} \in S_{\alpha\text{-Min}F}$.

We recall the following definitions of semicontinuity for a set-valued mapping and their properties used in the sequel.

Definition 2.2 ([3], p. 38, 39) Let $F : X \rightrightarrows Y$ be a set-valued mapping.

- (i) F is said to be upper semicontinuous at $x_0 \in X$ if and only if for any open subset U of Y with $F(x_0) \subset U$, there is a neighborhood N of x_0 such that $F(x) \subset U$ for every $x \in N$.
- (ii) F is said to be lower semicontinuous at $x_0 \in X$ if and only if for any open subset U of Y with $F(x_0) \cap U \neq \emptyset$, there is a neighborhood N of x_0 such that $F(x) \cap U \neq \emptyset$ for all $x \in N$.
- (iii) F is said to be lower (upper) semicontinuous on a subset S of X if and only if it is lower (upper) semicontinuous at every $x \in S$.

Lemma 2.1 *Let $F : X \rightrightarrows Y$ be a set-valued mapping.*

- (i) ([3], p. 39) *F is lower semicontinuous at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ converging to x_0 and for any $y \in F(x_0)$, there exist $y_\alpha \in F(x_\alpha)$ such that $\{y_\alpha\}$ converges to y .*
- (ii) ([2]) *If $F(x_0)$ is compact, then F is upper semicontinuous at $x_0 \in X$ if and only if for any net $\{x_\alpha\}$ converging to x_0 and for any $y_\alpha \in F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $\{y_\beta\}$ converges to y_0 . If, in addition, $F(x_0) = \{y_0\}$ is a singleton, then for the above nets, $\{y_\beta\}$ converges to y_0 .*

Now we recall the concepts of Hausdorff distance and Hausdorff convergence of sequence of sets. If S is a nonempty subset of X and $x \in X$, then the distance d between x and S is defined by

$$d(x, S) := \inf_{u \in S} \|x - u\|.$$

If S_1 and S_2 are two nonempty subsets of X , then Hausdorff distance between S_1 and S_2 , denoted by $H(S_1, S_2)$, is defined by

$$H(S_1, S_2) := \max\{H^*(S_1, S_2), H^*(S_2, S_1)\},$$

where $H^*(S_1, S_2) := \sup_{x \in S_1} d(x, S_2)$.

Definition 2.3 ([23], p. 359) Let $\{A_n\}$ be a sequence of subsets of X . We say that A_n converge to $A \subset X$ in the sense of the Hausdorff metric, denoted by $A_n \rightarrow A$, if and only if $H(A_n, A) \rightarrow 0$ as $n \rightarrow \infty$.

Next, we recall the concept of Kuratowski measure of noncompactness and its properties used in the sequel.

Definition 2.4 ([33], Definition 2.1) Let M be a nonempty subset of X . The Kuratowski measure of noncompactness μ of the set M is defined by

$$\mu(M) := \inf \left\{ \varepsilon > 0 \mid M \subset \bigcup_{i=1}^n M_i, \text{diam} M_i < \varepsilon, i = 1, \dots, n \text{ for some } n \in \mathbb{N} \right\},$$

where $\text{diam} M_i := \sup\{d(x, y) \mid x, y \in M_i\}$ is the diameter of M_i .

Lemma 2.2 ([33], Proposition 2.3) *The following assertions are true:*

- (i) $\mu(M) = 0$ if M is compact;
- (ii) $\mu(M) \leq \mu(N)$ whenever $M \subset N$;
- (iii) if $\{M_n\}$ is a sequence of closed subsets in X satisfying $M_{n+1} \subset M_n$ for every $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \mu(M_n) = 0$, then $K := \bigcap_{n \in \mathbb{N}} M_n$ is nonempty compact and $\lim_{n \rightarrow \infty} H(M_n, K) = 0$.

Lemma 2.3 *Let X be a normed space and A, B be subsets of X . If A is compact and B is closed, then $A + B$ is closed.*

Proof Assume that $\{a_n + b_n\}$, $a_n \in A, b_n \in B$, converges to c for some $c \in X$. We show that $c \in A + B$. In fact, since A is compact, there exist a subsequence $\{a_{n_k}\}$ of sequence $\{a_n\}$ and $a \in A$ such that $\{a_{n_k}\}$ converges to a . We have

$$\|b_{n_k} - c + a\| = \|(b_{n_k} + a_{n_k}) - c + (a - a_{n_k})\| \leq \|b_{n_k} + a_{n_k} - c\| + \|a - a_{n_k}\|.$$

We obtain that $\{b_{n_k}\}$ converges to $c - a$. Since B is closed, we get $c - a \in B$. Hence, there exists $b \in B$ such that $b = c - a$. Then, $c = a + b \in A + B$. So, $A + B$ is closed. $\square$

Lemma 2.4 *Let M be a nonempty subset of a normed space X . Then, for every $x, y \in X$, $|d(x, M) - d(y, M)| \leq \|x - y\|$.*

Proof Let $x, y \in X$, we have $\|x - y\| + d(y, M) = \|x - y\| + \inf_{z \in M} \|y - z\| = \inf_{z \in M} \{\|x - y\| + \|y - z\|\} \geq \inf_{z \in M} \|x - z\| = d(x, M)$. Hence, $\|x - y\| \geq d(x, M) - d(y, M)$. Similarly, we also get $\|x - y\| \geq d(y, M) - d(x, M)$. We conclude that $|d(x, M) - d(y, M)| \leq \|x - y\|$. $\square$

3 Pointwise LP well-posedness and generalized pointwise LP well-posedness

Motivated by the study [22] on the pointwise LP well-posedness for (P_I) , we are going to establish characterizations of this type of well-posedness for (P_α) without using the scalarization method. Consider the problem (P_α) , for a given $\bar{x} \in S_{\alpha-\text{Min}F}$, the LP approximating solution mapping at $\bar{x}$, $S_{\alpha-\text{Min}F}(\bar{x}, \cdot) : \{\bar{x}\} \times \mathbb{R}_+ \rightrightarrows M$ is defined by

$$S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon) := \{x \in X \mid d(x, M) \leq \varepsilon, F(x) \leq^\alpha F(\bar{x}) + \varepsilon e\},$$

for each $\varepsilon \in \mathbb{R}_+$.

Inspired by ideas in [22] (Definition 5.1), we extend some notions for the problem (P_I) in [22] to the problem (P_α) and propose some new concepts for the problem (P_α) . Let $e \in \text{int}K$.

Definition 3.1 Let $\bar{x} \in S_{\alpha-\text{Min}F}$ be given. A sequence $\{x_n\} \subset X$ is said to be a LP -minimizing sequence for the problem (P_α) at $\bar{x}$ if and only if there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that

$$d(x_n, M) \leq \varepsilon_n, \quad F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e.$$

The equivalence of this concept is given by the following result.

Proposition 3.1 $\{x_n\} \subset X$ is a LP -minimizing sequence for the problem (P_α) at $\bar{x} \in S_{\alpha\text{-Min}F}$ if and only if there exists a sequence $\{d_n\} \subset K \setminus \{0\}$ converging to 0 such that

$$d(x_n, M) \rightarrow 0, \quad F(x_n) \leq^\alpha F(\bar{x}) + d_n.$$

Proof We only prove the assertion for the case $\alpha = s$; the proofs of the assertion for the cases $\alpha = l$ and $\alpha = u$ are similar. Let $\{x_n\} \subset X$ and $\{d_n\} \subset K \setminus \{0\}$ converging to 0 such that $d(x_n, M) \rightarrow 0$ and $F(x_n) \leq^s F(\bar{x}) + d_n$, i.e.,

$$F(\bar{x}) + d_n \subset F(x_n) + K, \quad F(x_n) \subset F(\bar{x}) + d_n - K. \quad (1)$$

Since $e - K$ is a neighborhood of the origin 0 in Y , there exists $\varepsilon > 0$ such that $\varepsilon B(0, 1) \subset e - K$ where $B(x, r)$ is the closed ball centered x with radius r . For a given $n \in \mathbb{N}$, we have $d_n \in \|d_n\| B(0, 1) \subset \|d_n\| \varepsilon^{-1} (e - K) = \|d_n\| \varepsilon^{-1} e - K$. For each $n \in \mathbb{N}$, taking $\varepsilon_n = \|d_n\| \varepsilon^{-1}$, then $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converges to 0 and $\varepsilon_n e - d_n \in K$. It follows from (1) that

$$F(\bar{x}) + \varepsilon_n e \subset F(x_n) + K, \quad F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K,$$

i.e., $F(x_n) \leq^s F(\bar{x}) + \varepsilon_n e$. So, $\{x_n\}$ is a LP -minimizing sequence for (P_s) at $\bar{x}$. Conversely, it is clear that if $\{x_n\} \subset X$ is a LP -minimizing sequence for (P_s) at $\bar{x} \in S_{s\text{-Min}F}$, then the assertion is satisfied by setting $d_n = \varepsilon_n e$. $\square$

Definition 3.2 The problem (P_α) is said to be

- (i) LP well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$ if and only if any LP -minimizing sequence for (P_α) at $\bar{x}$ converges to $\bar{x}$;
- (ii) generalized LP well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$ if and only if any LP -minimizing sequence for (P_α) at $\bar{x}$ has a subsequence converging to an element $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$.

Remark 3.1 When $\alpha = l$, the concept of pointwise well-posedness becomes the corresponding concepts studied in [22] (Definitions 5.1 and 5.2, respectively), even for this special case, the concept of generalized well-posedness is a new one.

The following examples illustrate the above-introduced concepts.

Example 3.1 Let $X = Y = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [0, 1]$ for all $x \in X$. Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. We have $S_{\alpha\text{-Min}F}(\bar{x}, 0) = S_{\alpha\text{-Min}F} = \mathbb{R}$. Setting $x_n = n$, $\{x_n\}$ is a LP -minimizing sequence for (P_α) at $\bar{x} = 0$. Since $\{x_n\}$ admits no convergent subsequence, (P_α) is not both LP well-posed and generalized LP well-posed at 0.

Example 3.2 Let $X = Y = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [0, 1], & \text{if } -1 \leq x \leq 1, \\ [1, 1 + x^2], & \text{otherwise.} \end{cases}$$

Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. By direct computations, we have $S_{\alpha\text{-Min}F}(\bar{x}, 0) = [-1, 1]$. Setting $x_n = 1 - \frac{1}{n}$, $\{x_n\}$ is a LP -minimizing sequence for (P_α) at $\bar{x} = 0$ and converges to 1. Hence, (P_α) is not LP well-posed at 0, but it is generalized LP well-posed at 0. Indeed, if $\{\hat{x}_n\}$ is a LP -minimizing sequence for (P_α) at $\bar{x} = 0$, then there is a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(\hat{x}_n, M) \leq \varepsilon_n$ and $F(\hat{x}_n) \leq^\alpha F(0) + \varepsilon_n e = [\varepsilon_n, 1 + \varepsilon_n]$. This implies that $-1 \leq \hat{x}_n \leq 1$ for n sufficiently large, and hence there exists a subsequence of $\{\hat{x}_n\}$ converging to some point of $S_{\alpha\text{-Min}F}(\bar{x}, 0)$.

Example 3.3 Let $X = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [x^2, 2x^2]$ for all $x \in X$. Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. Direct calculations give us $S_{\alpha\text{-Min}F}(\bar{x}, 0) = \{0\}$. Let $\{x_n\}$ be a LP -minimizing sequence for (P_α) at $\bar{x} = 0$. Then, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(0) + \varepsilon_n e = \{\varepsilon_n\}$. It leads to $x_n^2 \leq \varepsilon_n$, so $\{x_n\}$ converges to 0. Therefore, (P_α) is LP well-posed at 0.

Lemma 3.1 *If (P_α) is generalized LP well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$, then $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact.*

Proof For every sequence $\{x_n\} \subset S_{\alpha\text{-Min}F}(\bar{x}, 0)$, we always have $d(x_n, M) = 0$ and

$$F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$$

for any $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0. This means that $\{x_n\}$ is a LP -minimizing sequence for (P_α) at $\bar{x}$. By the generalized LP well-posedness of (P_α) at $\bar{x}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$. This leads to the compactness of $S_{\alpha\text{-Min}F}(\bar{x}, 0)$. $\square$

The next results give some properties of the mapping $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ which are useful in the sequel.

Proposition 3.2 *Let $\bar{x} \in S_{\alpha\text{-Min}F}$ be given. Then, the following statements are true:*

- (i) $S_{\alpha\text{-Min}F} = \bigcup_{z \in S_{\alpha\text{-Min}F}} S_{\alpha\text{-Min}F}(z, 0)$;
- (ii) if $\varepsilon_1 \leq \varepsilon_2$, then $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_1) \subset S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_2)$;
- (iii) $\bigcap_{\varepsilon > 0} S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon) = S_{\alpha\text{-Min}F}(\bar{x}, 0)$ if F is compact-valued on M .

Proof We only demonstrate the proof of the assertions (i)-(iii) for the case $\alpha = s$; the proofs of these assertions for the cases $\alpha = l$ and $\alpha = u$ are similar.

(i) Let $z \in S_{s\text{-Min}F}$ be given. Since $z \in S_{s\text{-Min}F}(z, 0)$, $S_{s\text{-Min}F} \subset \bigcup_{z \in S_{s\text{-Min}F}} S_{s\text{-Min}F}(z, 0)$. Moreover, let $x \in \bigcup_{z \in S_{s\text{-Min}F}} S_{s\text{-Min}F}(z, 0)$, there exists $z \in S_{s\text{-Min}F}$ such that $x \in S_{s\text{-Min}F}(z, 0)$. Therefore, $d(x, M) = 0$ and $F(x) \leq^s F(z)$. Since $z \in S_{s\text{-Min}F}$, $x \in S_{s\text{-Min}F}$. It implies that $\bigcup_{z \in S_{s\text{-Min}F}} S_{s\text{-Min}F}(z, 0) \subset S_{s\text{-Min}F}$.

(ii) Assume $\varepsilon_1 \leq \varepsilon_2$. Let $x \in S_{s\text{-Min}F}(\bar{x}, \varepsilon_1)$, then $d(x, M) \leq \varepsilon_1$ and $F(x) \leq^s F(\bar{x}) + \varepsilon_1 e$. It follows from the definition of set less relation $\leq^s$ that

$$F(x) \leq^l F(\bar{x}) + \varepsilon_1 e \quad \text{and} \quad F(x) \leq^u F(\bar{x}) + \varepsilon_1 e,$$

i.e.,

$$F(\bar{x}) + \varepsilon_1 e \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) + \varepsilon_1 e - K.$$

We observe that

$$F(\bar{x}) + \varepsilon_2 e - K = F(\bar{x}) + \varepsilon_1 e - K + (\varepsilon_2 - \varepsilon_1)e.$$

Combining the convexity of K with Proposition 2.1, we obtain that

$$F(\bar{x}) + \varepsilon_2 e \subset F(x) + K,$$

and

$$F(x) \subset F(\bar{x}) + \varepsilon_1 e - K \subset F(\bar{x}) + \varepsilon_2 e - K.$$

Thus, $F(x) \leq^l F(\bar{x}) + \varepsilon_2 e$ and $F(x) \leq^u F(\bar{x}) + \varepsilon_2 e$. Since $\bar{x} \in S_{s\text{-Min}F}$, $F(x) \leq^s F(\bar{x}) + \varepsilon_2 e$. Moreover, $d(x, M) \leq \varepsilon_2$ as $d(x, M) \leq \varepsilon_1$. Therefore, $x \in s\text{-Min}F(\bar{x}, \varepsilon_2)$. We conclude that $S_{s\text{-Min}F}(\bar{x}, \varepsilon_1) \subset S_{s\text{-Min}F}(\bar{x}, \varepsilon_2)$.

(iii) Let $x \in S_{s\text{-Min}F}(\bar{x}, 0)$. It is clear that $x \in S_{s\text{-Min}F}(\bar{x}, \varepsilon)$ for any $\varepsilon > 0$. Therefore, $x \in \bigcap_{\varepsilon > 0} S_{s\text{-Min}F}(\bar{x}, \varepsilon)$. For the converse, let $x \in \bigcap_{\varepsilon > 0} S_{s\text{-Min}F}(\bar{x}, \varepsilon)$, we have $x \in S_{s\text{-Min}F}(\bar{x}, \varepsilon)$ for any $\varepsilon > 0$. It follows from definition of $S_{s\text{-Min}F}(\bar{x}, \varepsilon)$ that $d(x, M) \leq \varepsilon$ and $F(x) \leq^s F(\bar{x}) + \varepsilon e$, i.e.,

$$d(x, M) \leq \varepsilon, \quad F(\bar{x}) + \varepsilon e \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) + \varepsilon e - K. \quad (2)$$

By the compact-valuedness of F and Lemma 2.3, $F(x) - K$ and $F(\bar{x}) - K$ are closed. From (2), let $\varepsilon \rightarrow 0$, we obtain that

$$d(x, M) = 0, \quad F(\bar{x}) \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) - K,$$

i.e.,

$$d(x, M) = 0, \quad F(x) \leq^s F(\bar{x}).$$

Hence, $x \in S_{s\text{-Min}F}(\bar{x}, 0)$. We get $\bigcap_{\varepsilon > 0} S_{s\text{-Min}F}(\bar{x}, \varepsilon) \subset S_{s\text{-Min}F}(\bar{x}, 0)$. □

Next, using the Kuratowski measure of noncompactness of LP approximating solution sets, we establish metric characterizations of two types of pointwise LP well-posedness for (P_α) .

Theorem 3.1 (i) If (P_α) is generalized LP well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$, then $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$.
(ii) If (P_α) is LP well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$, then $\text{diam}(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof (i) Suppose that (P_α) is generalized LP well-posed at $\bar{x} \in S_{\alpha-\text{Min}F}$. First of all, we show that $H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Indeed, we observe that, for each $\varepsilon > 0$, $S_{\alpha-\text{Min}F}(\bar{x}, 0) \subset S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon)$, and hence

$$H^*(S_{\alpha-\text{Min}F}(\bar{x}, 0), S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon)) = 0.$$

It is sufficient to show that $H^*(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Suppose by contrary that there exist a real number $r > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0, and for each $n \in \mathbb{N}$ there exists $x_n \in S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon_n)$ such that $d(x_n, S_{\alpha-\text{Min}F}(\bar{x}, 0)) \geq r$. We have $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that $\{x_n\}$ is a LP -minimizing sequence for (P_α) at $\bar{x}$, and hence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point $\hat{x} \in S_{\alpha-\text{Min}F}(\bar{x}, 0)$. Therefore, for n_k sufficiently large, we have $\|x_{n_k} - \hat{x}\| < r$ which is a contradiction.

Next, we prove that $\mu(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. By Lemma 3.1, $S_{\alpha-\text{Min}F}(\bar{x}, 0)$ is compact. Now, for any $\varepsilon > 0$, there are sets $M_1, M_2, \dots, M_n$ for some $n \in \mathbb{N}$ such that $S_{\alpha-\text{Min}F}(\bar{x}, 0) \subset \cup_{i=1}^n M_i$ with $\text{diam} M_i \leq \varepsilon$ for all $i = 1, \dots, n$. For each $i \in \{1, \dots, n\}$, denote

$$N_i := \{x \in X \mid d(x, M_i) \leq H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0))\}.$$

We claim that $S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon) \subset \cup_{i=1}^n N_i$. Indeed, let $x \in S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon)$ be arbitrary, we have

$$d(x, S_{\alpha-\text{Min}F}(\bar{x}, 0)) \leq H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)).$$

Since $S_{\alpha-\text{Min}F}(\bar{x}, 0) \subset \cup_{i=1}^n M_i$, we conclude that

$$d(x, \cup_{i=1}^n M_i) \leq H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)).$$

So, there is $k_0 \in \{1, 2, \dots, n\}$ such that

$$d(x, M_{k_0}) \leq H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)).$$

It means that $x \in N_{k_0}$. Therefore, $S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon) \subset \cup_{i=1}^n N_i$. Notice further that

$$\begin{aligned} \text{diam} N_i &= \text{diam} M_i + 2H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)) \\ &\leq \varepsilon + 2H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)). \end{aligned}$$

Hence, we get

$$\mu(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon)) \leq \mu(S_{\alpha-\text{Min}F}(\bar{x}, 0)) + 2H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)).$$

Since $S_{\alpha-\text{Min}F}(\bar{x}, 0)$ is compact, we have $\mu(S_{\alpha-\text{Min}F}(\bar{x}, 0)) = 0$. Therefore,

$$\mu(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon)) \leq 2H(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon), S_{\alpha-\text{Min}F}(\bar{x}, 0)).$$

It follows that $\mu(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

(ii) Suppose, to the contrary, that there exist a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ and a positive real number r such that $\text{diam}(S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon_n)) > r$. Because $\bar{x} \in S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon_n)$, for each n , there exists $x_n \in S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon_n)$ such that $\|x_n - \bar{x}\| > \frac{r}{2}$. However, since $\{x_n\}$ is a LP -minimizing sequence for (P_α) at $\bar{x}$, $\{x_n\}$ converges to $\bar{x}$ which is a contradiction. $\square$

The next result gives sufficient conditions for the closedness of approximating solution set.

Proposition 3.3 $S_{\alpha-\text{Min}F}(\bar{x}, \varepsilon)$ is closed for each $\varepsilon \geq 0$ if F is continuous and compact-valued on M .

Proof We only prove the assertion for the case $\alpha = s$. Taking $\varepsilon \geq 0$, let $\{x_n\} \subset S_{s-\text{Min}F}(\bar{x}, \varepsilon)$ converge to x , we need to prove that $x \in S_{s-\text{Min}F}(\bar{x}, \varepsilon)$. Since $x_n \in S_{s-\text{Min}F}(\bar{x}, \varepsilon)$, $d(x_n, M) \leq \varepsilon$ and

$$F(x_n) \leq^s F(\bar{x}) + \varepsilon e. \quad (3)$$

By the continuity of the function $d(\cdot, M)$, $d(x, M) \leq \varepsilon$. Next, we show that $F(x) \leq^s F(\bar{x}) + \varepsilon e$. Indeed, from (3), we have

$$F(\bar{x}) + \varepsilon e \subset F(x_n) + K, \quad (4)$$

and

$$F(x_n) \subset F(\bar{x}) + \varepsilon e - K. \quad (5)$$

Let $y \in F(x)$ be arbitrary. Since F is lower semicontinuous and $\{x_n\}$ converges to x , there exist $y_n \in F(x_n)$ such that $\{y_n\}$ converges to y . Combining this with (5), there exist $w_n \in F(\bar{x})$ such that

$$y_n \in w_n + \varepsilon e - K. \quad (6)$$

Since $F(\bar{x})$ is compact, we can assume that $\{w_n\}$ converges to some $w \in F(\bar{x})$. By (6), there exist $k_n \in K$ such that $y_n = w_n + \varepsilon e - k_n$. This leads to $\lim_{n \rightarrow \infty} k_n = w + \varepsilon e - y$. Moreover, we get $w + \varepsilon e - y \in K$ as K is closed. Therefore, there exists $k \in K$ such that $w + \varepsilon e - y = k$. We have $y = w + \varepsilon e - k \in w + \varepsilon e - K$. It yields that $y \in F(\bar{x}) + \varepsilon e - K$ as $w \in F(\bar{x})$. We arrive at the fact that $F(x) \subset F(\bar{x}) + \varepsilon e - K$, i.e., $F(x) \leq^u F(\bar{x}) + \varepsilon e$.

Similarly, let $t \in F(\bar{x})$ be arbitrary, it follows from (4) that, for each $n \in \mathbb{N}$, there exists $v_n \in F(x_n)$ such that

$$t \in v_n - \varepsilon e + K. \quad (7)$$

Since F is upper semicontinuous and compact-valued at x , we can assume that $\{v_n\}$ converges to some element $v \in F(x)$. It implies from (7) that there exist $k_n \in K$ such that $t = v_n - \varepsilon e + k_n$. Hence, $k_n = t + \varepsilon e - v_n$. This leads to $\lim_{n \rightarrow \infty} k_n = t + \varepsilon e - v$. Since K is closed, there exists $k \in K$ such that $t + \varepsilon e - v = k$. We get $t = v - \varepsilon e + k \in v - \varepsilon e + K$. It yields that $t \in F(x) - \varepsilon e + K$ as $v \in F(x)$. We have $F(\bar{x}) \subset F(x) - \varepsilon e + K$. It means that $F(x) \leq^l F(\bar{x}) + \varepsilon e$. So, we obtain $F(x) \leq^s F(\bar{x}) + \varepsilon e$. The proof is complete. $\square$

Theorem 3.2 Suppose that F is continuous and compact-valued on M . Then,

- (i) (P_α) is generalized LP well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$ if $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$.
- (ii) (P_α) is LP well-posed at $\bar{x} \in S_{\alpha\text{-Min}F}$ if $\text{diam}(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof (i) Suppose that $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Let $\{x_n\}$ be a LP -minimizing sequence for (P_α) at $\bar{x}$. Therefore, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$. It is clear that $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)) \rightarrow 0$ as $n \rightarrow \infty$, and hence by Lemma 2.2 (iii), we have $\cap_{n \in \mathbb{N}} S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$ is a nonempty compact set and

$$H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n), \cap_{n \in \mathbb{N}} S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)) \rightarrow 0$$

as $n \rightarrow \infty$. Note further from Proposition 3.2 (iii) that

$$S_{\alpha\text{-Min}F}(\bar{x}, 0) = \cap_{n \in \mathbb{N}} S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n).$$

Hence, we conclude that $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact and

$$H(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n), S_{\alpha\text{-Min}F}(\bar{x}, 0)) \rightarrow 0$$

as $n \rightarrow \infty$. Thus,

$$d(x_n, S_{\alpha\text{-Min}F}(\bar{x}, 0)) \rightarrow 0. \quad (8)$$

Therefore, there exists a sequence $\{\hat{x}_n\} \subset S_{\alpha\text{-Min}F}(\bar{x}, 0)$ such that $d(x_n, \hat{x}_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact, there is a subsequence $\{\hat{x}_{n_k}\}$ of $\{\hat{x}_n\}$ converging to some $\hat{x} \in S_{\alpha\text{-Min}F}$. This implies that $\{x_n\}$ has a corresponding subsequence $\{x_{n_k}\}$ converging to $\hat{x}$. Hence, (P_α) is generalized LP well-posed at $\bar{x}$.

(ii) Assume that $\text{diam}(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then, $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and hence (P_α) is generalized LP well-posed at $\bar{x}$. By Proposition 3.2, $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is a singleton. By Lemma 2.1 (ii), (P_α) is LP well-posed at $\bar{x}$. $\square$

The below example shows that Theorem 3.2 is applicable.

Example 3.4 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $M = [0, 1]$, $K = \mathbb{R}_+^2$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = [x, x + 1] \times [x, x + 1], \forall x \in X.$$

Let $e = (1, 1) \in \text{int}K$, $\bar{x} = 0$. Clearly, all assumptions of Theorem 3.2 hold. By direct calculations, we get $S_{\alpha\text{-Min}F}(0, \varepsilon) = [0, \varepsilon]$ and $S_{\alpha\text{-Min}F} = \{0\}$. So, $\text{diam}(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Applying Theorem 3.2, the problem (P_α) is LP well-posed at $\bar{x} = 0$. In fact, if $\{x_n\}$ is a LP minimizing sequence for (P_α) at $\bar{x}$, then there is a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(0) + \varepsilon_n e = [\varepsilon_n, \varepsilon_n + 1] \times [\varepsilon_n, \varepsilon_n + 1]$. We get $0 \leq x_n \leq \varepsilon_n$, and hence $\{x_n\}$ converges to 0. So, (P_α) is LP well-posed at 0.

The following example shows that the continuity of F in Theorem 3.2 is crucial.

Example 3.5 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $M = [-1, 1]$, $K = \mathbb{R}_+^2$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [0, 1] \times [0, 1], & \text{if } x < 0, \\ [1, 2] \times [1, 2], & \text{if } x \geq 0. \end{cases}$$

Let $e = (1, 1) \in \text{int}K$, $\bar{x} = -\frac{1}{2}$. Then, F is compact-valued on M . Direct computations give us that $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon) = [-1, 0)$, and hence $\mu(S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. However, the problem (P_α) is not generalized LP well-posed at $\bar{x}$. Indeed, setting $x_n = -\frac{1}{n}$, we have $\{x_n\}$ is a LP minimizing sequence for (P_α) at $\bar{x}$ but $\{x_n\}$ converges to $0 \notin S_{\alpha\text{-Min}F}(\bar{x}, 0)$. The reason here is that F is not continuous.

Next, employing properties of the approximating solution mapping of (P_α) , the connection between LP well-posedness of (P_α) and stability of approximating problem is established.

Theorem 3.3 Let $\bar{x} \in S_{\alpha\text{-Min}F}$.

- (i) Problem (P_α) is generalized LP well-posed at $\bar{x}$ if and only if $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous and compact-valued at 0.
- (ii) Problem (P_α) is LP well-posed at $\bar{x}$ if and only if $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous at 0 and $S_{\alpha\text{-Min}F}(\bar{x}, 0) = \{\bar{x}\}$.

Proof (i) Suppose that (P_α) is generalized LP well-posed at $\bar{x}$. By Lemma 3.1, $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is compact. Suppose by contrary that $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is not upper semicontinuous at 0. Then, there exists an open set $N \supset S_{\alpha\text{-Min}F}(\bar{x}, 0)$ such that for any $\delta > 0$, there exists $\varepsilon \in [0, \delta)$, $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon) \not\subset N$. It means that there exists a sequence $\{\varepsilon_n\}$ converging to 0 such that for each $n \in \mathbb{N}$, we have $S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n) \not\subset N$. Thus, for each $n \in \mathbb{N}$, there is $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$, $x_n \notin N$. Then, $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$, which imply that $\{x_n\}$ is a LP -minimizing sequence for (P_α) at $\bar{x}$. Because (P_α) is generalized LP well-posed at $\bar{x}$, there is a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to an element $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0) \subset N$. This is impossible as $x_{n_k} \notin N$ for all k .

Conversely, let $\{x_n\} \subset X$ be a LP -minimizing sequence for (P_α) at $\bar{x}$. Then, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that

$$d(x_n, M) \leq \varepsilon_n, \quad F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e.$$

So, $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n)$. It follows from the upper semicontinuity and compact-valuedness of $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ at 0, Lemma 2.1 (ii) implies that there exist an element $\hat{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to $\hat{x}$. So, (P_α) is generalized LP well-posed at $\bar{x}$.

(ii) Let $\{x_n\} \subset X$ be a LP -minimizing sequence for (P_α) at $\bar{x}$, then there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that, for each $n \in \mathbb{N}$,

$$x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n). \quad (9)$$

Since $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous at 0, for any open set N , $S_{\alpha\text{-Min}F}(\bar{x}, 0) \subset N$, there is a neighborhood U of 0 such that for all $t \in U$, $t \geq 0$, we have

$S_{\alpha\text{-Min}F}(\bar{x}, t) \subset N$. Since $\{\varepsilon_n\}$ converges to 0, there exists $n_0 \in \mathbb{N}$ such that $\varepsilon_n \in B(0, \frac{1}{n_0})$ for all $n \geq n_0$. Combining this with (9), we obtain $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, \varepsilon_n) \subset N$ for all $n \geq n_0$. Therefore, for every neighborhood W of 0, $x_n \in S_{\alpha\text{-Min}F}(\bar{x}, 0) + W$ for all $n \geq n_0$. Since $S_{\alpha\text{-Min}F}(\bar{x}, 0) = \{\bar{x}\}$, $\{x_n\}$ converges to $\bar{x}$. So, (P_α) is LP well-posed at $\bar{x}$.

For the converse, suppose that (P_α) is LP well-posed at $\bar{x}$. Using (i), $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous and compact-valued at 0. We show that $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is a singleton. Suppose by the contrary that there exist $x_1, x_2 \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$ with $x_1 \neq x_2$. Putting $x_{2n+k} = x_k$ where $k = 1$ or $k = 2$. Clearly, $\{x_n\}$ is a LP -minimizing sequence for (P_α) at $\bar{x}$. However, $\{x_n\}$ is not convergent. This is a contradiction. Therefore, $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is a singleton. Moreover, it is obvious that $\bar{x} \in S_{\alpha\text{-Min}F}(\bar{x}, 0)$. So, $S_{\alpha\text{-Min}F}(\bar{x}, 0) = \{\bar{x}\}$. $\square$

The assumption about the upper semicontinuity of approximating solution mapping of (P_α) is used in Theorem 3.3. Next, we give the sufficient conditions for this assumption.

Proposition 3.4 *Suppose that the following conditions hold:*

- (i) M is compact;
- (ii) F is continuous and compact-valued on M .

Then, $S_{\alpha\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous at 0.

Proof By the similarity, we only focus on the proof of the assertion for the case $\alpha = u$. By contradiction, suppose that $S_{u\text{-Min}F}(\bar{x}, \cdot)$ is not upper semicontinuous at 0. Then, there exist an open set $N \supset S_{u\text{-Min}F}(\bar{x}, 0)$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+ \setminus \{0\}$ converging to 0 such that for each n , there exists x_n satisfying

$$x_n \in S_{u\text{-Min}F}(\bar{x}, \varepsilon_n) \setminus W_0. \quad (10)$$

Since $x_n \in S_{u\text{-Min}F}(\bar{x}, \varepsilon_n)$,

$$d(x_n, M) \leq \varepsilon_n \quad (11)$$

and

$$F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K. \quad (12)$$

It implies from (11) that there exist $\hat{x}_n \in M$ such that $d(x_n, \hat{x}_n) \leq \varepsilon_n$. By the compactness of M , we can assume that $\{\hat{x}_n\}$ converges to an element $x_0 \in M$. Hence, $\{x_n\}$ converges to x_0 . Next, we prove that

$$F(x_0) \subset F(\bar{x}) - K. \quad (13)$$

Indeed, by the compact-valuedness of F , the closedness of K and Lemma 2.3, $F(\bar{x}) - K$ is closed. From (12), taking $n \rightarrow \infty$, we obtain (13). It means that $x_0 \in S_{u\text{-Min}F}(\bar{x}, 0)$. Combining this, (10) and the convergence to x_0 of $\{x_n\}$, we get a contradiction. Therefore, $S_{u\text{-Min}F}(\bar{x}, \cdot)$ is upper semicontinuous at 0. $\square$

Corollary 3.1 *Suppose that the following conditions hold:*

- (i) M is compact;
- (ii) F is continuous and compact-valued on M .

Then,

- (a) (P_α) is generalized LP well-posed at $\bar{x} \in S_{\alpha-\text{Min}F}$ if $S_{\alpha-\text{Min}F}(\bar{x}, 0)$ is closed.
- (b) (P_α) is LP well-posed at $\bar{x} \in S_{\alpha-\text{Min}F}$ if $S_{\alpha-\text{Min}F}(\bar{x}, 0) = \{\bar{x}\}$.

4 Metrically LP well-posed set optimization problems

Picking up the ideas in [22], we introduce the following new concepts of LP well-posedness related to metrically approach for the problem (P_α) .

Definition 4.1 A sequence $\{x_n\} \subset X$ is said to be a

- (i) metrically LP -minimizing sequence for problem (P_l) at $\bar{x} \in S_{l-\text{Min}F}$ if and only if $H^*(F(\bar{x}), F(x_n)) \rightarrow 0$ and $d(x_n, M) \rightarrow 0$ as $n \rightarrow \infty$.
- (ii) metrically LP -minimizing sequence for problem (P_u) at $\bar{x} \in S_{u-\text{Min}F}$ if and only if $H^*(F(x_n), F(\bar{x})) \rightarrow 0$ and $d(x_n, M) \rightarrow 0$ as $n \rightarrow \infty$.
- (iii) metrically LP -minimizing sequence for problem (P_s) at $\bar{x} \in S_{s-\text{Min}F}$ if and only if $H(F(\bar{x}), F(x_n)) \rightarrow 0$ and $d(x_n, M) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 4.2 The problem (P_α) is said to be metrically LP well-posed if and only if $S_{\alpha-\text{Min}F} \neq \emptyset$ and for any metrically LP -minimizing sequence $\{x_n\}$ for problem (P_α) at some $\bar{x} \in S_{\alpha-\text{Min}F}$, we have $d(x_n, S_{\alpha-\text{Min}F}) \rightarrow 0$ as $n \rightarrow \infty$.

Remark 4.1 When $\alpha = l$, concepts introduced in Definitions 4.1 (i) and 4.2 are similar to the corresponding ones studied in Definitions 4.5 (ii) and 4.6 (ii) in [22].

Example 4.1 (a) Let $X = Y = \mathbb{R}$, $M = [0, 1]$, $K = \mathbb{R}_+$, and let $F : X \rightrightarrows Y$ be defined by $F(x) = [1, 2]$ for all $x \in X$. Obviously, $S_{\alpha-\text{Min}F} = [0, 1] = M$, and the problem (P_α) is metrically LP well-posed.

(b) Let $X = Y = \mathbb{R}$, $M = K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [x^2, 3x^2]$ for all $x \in X$. Direct calculations give us $S_{\alpha-\text{Min}F} = \{0\}$ and the problem (P_α) is metrically LP well-posed. Indeed, let $\{x_n\}$ be a metrically LP -minimizing sequence for (P_α) at $\bar{x} = 0$, it implies from definition of metrically LP -minimizing sequence for (P_α) at $\bar{x} = 0$ that $\{x_n\}$ converges to $\bar{x}$. Therefore, $d(x_n, S_{\alpha-\text{Min}F}) \rightarrow 0$.

Example 4.2 Let $X = Y = \mathbb{R}$, $M = [-1, 1]$, $K = \mathbb{R}_+$, and $F : X \rightrightarrows Y$ is defined by

$$F(x) = \begin{cases} [0, 1), & \text{if } x \leq 0, \\ (0, 1], & \text{if } x > 0. \end{cases}$$

By direct computations, we get $S_{\alpha-\text{Min}F} = [-1, 0]$. Taking $x_n = 1 + \frac{1}{n}$, then $\{x_n\}$ is a metrically LP -minimizing sequence for the problem (P_α) at $\bar{x} = 0 \in S_{\alpha-\text{Min}F}$, but $d(x_n, S_{\alpha-\text{Min}F}) \rightarrow 1$. Therefore, the problem (P_α) is not metrically LP well-posed.

Next, we introduce a generalized form of the above concept.

Definition 4.3 The problem (P_α) is said to be generalized metrically LP well-posed if and only if $S_{\alpha-\text{Min}F} \neq \emptyset$ and for any metrically LP -minimizing sequence $\{x_n\}$ for (P_α) at some $\bar{x} \in S_{\alpha-\text{Min}F}$, $\{x_n\}$ has a subsequence, denoted by $\{x_{n_k}\}$, such that $d(x_{n_k}, S_{\alpha-\text{Min}F}) \rightarrow 0$ as $k \rightarrow \infty$.

It is clear that if (P_α) is metrically LP well-posed, then it is generalized metrically LP well-posed.

These following results give the relationships between these kinds of LP well-posedness considered in this study.

Theorem 4.1 (i) If (P_α) is LP well-posed at all $\bar{x} \in S_{\alpha-\text{Min}F}$, then (P_α) is metrically LP well-posed.
(ii) If (P_α) is generalized LP well-posed at all $\bar{x} \in S_{\alpha-\text{Min}F}$, then (P_α) is generalized metrically LP well-posed.

Proof (i) By the similarity we verify the assertions (i), (ii) for the case $\alpha = s$ as an example. Let $\{x_n\}$ be a metrically LP -minimizing sequence for problem (P_s) at some $\bar{x} \in S_{s-\text{Min}F}$. We need to prove that $d(x_n, S_{s-\text{Min}F}) \rightarrow 0$. In fact, since $\{x_n\}$ is a metrically LP -minimizing sequence for problem (P_s) at some $\bar{x} \in S_{s-\text{Min}F}$, $d(x_n, M) \rightarrow 0$ and

$$H(F(\bar{x}), F(x_n)) \rightarrow 0. \quad (14)$$

Observe that we can choose a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 satisfying $d(x_n, M) \leq \varepsilon_n$, both $-\varepsilon_n e + K$ and $\varepsilon_n e - K$ are neighborhoods of the origin in Y . By (14), there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$F(\bar{x}) \subset F(x_n) - \varepsilon_n e + K \text{ and } F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K.$$

This implies that $F(x_n) \leq^s F(\bar{x}) + \varepsilon_n e$. Hence, $\{x_n\}$ is a LP -minimizing sequence for (P_s) at $\bar{x}$. By the LP well-posedness of (P_s) at $\bar{x}$, $\{x_n\}$ converges to $\bar{x}$. Moreover, since $\bar{x} \in S_{s-\text{Min}F}$, $d(x_n, S_{s-\text{Min}F}) \leq \|x_n - \bar{x}\| \rightarrow 0$. So, (P_s) is metrically LP well-posed.

(ii) Using a similar argument with one above, we can prove that the statement (ii) is satisfied. $\square$

Remark 4.2 When $\alpha = l$, (P_α) reduces to (P_l) studied in [22]. To obtain the metrically LP well-posedness for (P_l) , the authors used an important assumption about the K -closed values of F on M , i.e., $F(x) + K$ is closed for all $x \in M$. Using another approach, as in Theorem 4.1, we can remove this assumption but also obtain the metrically LP well-posedness for (P_α) .

Combining Theorem 4.1 and Corollary 3.1, we obtain the following results.

Theorem 4.2 Suppose that the following conditions are satisfied:

- (i) M is compact;
- (ii) F is continuous and compact-valued on M .

Then,

- (a) (P_α) is metrically LP well-posed if $S_{\alpha\text{-Min}F}(\bar{x}, 0) = \{\bar{x}\}$ for every $\bar{x} \in S_{\alpha\text{-Min}F}$.
- (b) (P_α) is generalized metrically LP well-posed if $S_{\alpha\text{-Min}F}(\bar{x}, 0)$ is closed for every $\bar{x} \in S_{\alpha\text{-Min}F}$.

Remark 4.3 Very recently, in [6], the authors studied several kinds of well-posedness for set optimization problems via the lower set less relation, including B -well-posedness, L -well-posedness, DH -well-posedness, and they obtained many interesting results related to this topic. In this paper, we consider the Levitin–Polyak well-posedness and the generalized Levitin–Polyak well-posedness for set optimization problems involving various kinds of set less relations, and hence the concepts of well-posedness investigated in this paper are different from those in [6]. Therefore, it could not compare our results with theirs.

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B-well-posedness for set optimization problems involving set order relations

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Abstract : In this paper, both pointwise and global B -well-posedness for set optimization problems involving three kinds of set order relations are investigated. We give characterizations and sufficient and/or necessary conditions of these types of well-posedness. Moreover, pointwise L -well-posedness and relationships between these kinds of pointwise well-posedness are studied.

Keywords : B -well-posedness; generalized B -well-posedness; pointwise B -well-posedness; pointwise L -well-posedness; set optimization problem

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1 Introduction

Set order relations were firstly introduced by Kuroiwa et al. in [1] and then they were generalized in [2]. These concepts gave a new way, so-called set approach, to formulate the optimal of set-valued optimization problems [3]. In this approach, all images of the set-valued objective mapping were compared by set order relations [4,5], and hence, it is a truly natural and practical approach. Therefore, this field

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has attracted a great deal of attention of researchers although it is a young direction in optimization. Many interesting and important results have been obtained in different topics in this area [6–11].

Well-posedness was originally proposed by Tikhonov in [12]. This concept requires two conditions, namely the uniqueness of solution and the convergence of each minimizing sequence to the unique solution. In other words, whenever we are able to compute approximately the optimal value then we automatically do approximate the optimal solution. So, well-posedness plays an important role in both theory results and numerical methods, and hence it has been attracted much attention of researchers (see e.g., [13–15] and the reference therein). Later on, generalizations of Tikhonov well-posedness were proposed and studied widely. One of these extensions is the so-called B -well-posedness proposed by Bednarczuk for vector optimization problems in [16]. After that, this notion has been intensively considered for various problems related to optimization [17–21].

Studying on well-posedness for set optimization problems was initialed by Zhang et al. in [22]. The authors obtained sufficient, necessary conditions and characterizations for set optimization problems involving the lower set less relation to be well-posed by the scalarization method. After that, some different types of well-posedness for these problems introduced and investigated [23–26]. In 2013, as the first authors concerned B -well-posedness for set optimization problems, Long and Peng [24] introduced three types of B -well-posedness for set optimization problems involving upper set less relations $\leq^u$ and established some relations among these kinds of B -well-posedness. Moreover, the authors also provided necessary and sufficient conditions of these notions for set optimization problems. To extend the research in [24], Han and Huang [8] studied B -well-posedness for set optimization problems involving set order relations $\leq^l$ and $\leq^u$. They gave characterizations for the generalized l - B -well-posedness and the generalized u - B -well-posedness and provided the semicontinuity of solution mapping.

As mentioned in [2,27] that among three kinds of set order relations introduced in [1], the set less relation $\leq^s$ is generalized and more appropriate in practical problems than both the lower and upper set less relations; and it also occupies an important role in relationships with other new order relations for sets proposed in [2] which are more useful in real world. Moreover, to the best of our knowledge, there is no paper devoted to well-posedness for set optimization problems involving set less relation, and hence the well-posedness properties for such problems are deserved to study more. Consequently, we aim to investigate both pointwise and global B -well-posedness as well as pointwise L -well-posedness for set optimization problems involving three kinds of set order relations.

The outline of this paper is as follows. In Sect. 2, some concepts and results used in what follows are recalled. Sect. 3 studies global B -well-posedness for set optimization problems, including B -well-posedness and generalized B -well-posedness. Relationships between them are discussed. Moreover, sufficient conditions of B -well-posedness for such problems are provided. In Sect. 4, we pay more attention on pointwise B -well-posedness. Characterizations as well as relationships between pointwise B -well-posedness and global B -well-posedness are

studied. In the last section, Sect. 5, pointwise L -well-posedness is investigated. Then, relationships between it and pointwise B -well-posedness are researched.

2 Preliminaries

Let X and Y be normed spaces. We denote the closed unit ball of Y by B_Y . Let K be a closed convex pointed cone in Y with $\text{int}K \neq \emptyset$, where $\text{int}K$ denotes the interior of K . Orderings induced by cone K in the space Y are defined as the following

$$x \leq_K y \Leftrightarrow y - x \in K,$$

$$x <_K y \Leftrightarrow y - x \in \text{int}K.$$

To compare two subsets of Y , we use set order relations introduced in [2, 5, 28]. We list here three kinds of set order relations used in this paper. Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y . For $A, B \in \mathcal{P}(Y)$, lower set less relation, upper set less relation and set less relation, respectively, are defined by

$$A \leq^l B \text{ if and only if } B \subset A + K,$$

$$A \leq^u B \text{ if and only if } A \subset B - K,$$

$$A \leq^s B \text{ if and only if } A \subset B - K \text{ and } B \subset A + K.$$

Definition 2.1. [2] We say that the binary relation $\leq$ is

- (i) compatible with the addition if and only if $A \leq B$ and $D \leq E$ imply $A + D \leq B + E$ for all $A, B, D, E \in \mathcal{P}(Y)$.
- (ii) compatible with the multiplication with a nonnegative real number if and only if $A \leq B$ implies $\lambda A \leq \lambda B$ for all scalars $\lambda \geq 0$ and all $A, B \in \mathcal{P}(Y)$.
- (iii) compatible with the conlinear structure of $\mathcal{P}(Y)$ if and only if it is compatible with both the addition and the multiplication with a nonnegative real number.

Proposition 2.1. [2]

- (i) The set order relations $\leq^l$, $\leq^u$ and $\leq^s$ are pre-order (i.e., these relations are reflexive and transitive).
- (ii) The set order relations $\leq^l$, $\leq^u$ and $\leq^s$ are compatible with the conlinear structure of $\mathcal{P}(Y)$.
- (iii) In general, the set order relations $\leq^l$, $\leq^u$ and $\leq^s$ are not antisymmetric; more precisely, for arbitrary sets $A, B \in \mathcal{P}(Y)$ we have

$$(A \leq^l B \text{ and } B \leq^l A) \Leftrightarrow A + K = B + K,$$

$$(A \leq^u B \text{ and } B \leq^u A) \Leftrightarrow A - K = B - K,$$

$$(A \leq^s B \text{ and } B \leq^s A) \Leftrightarrow (A + K = B + K \text{ and } A - K = B - K).$$

For $\alpha \in \{u, l, s\}$, we say that

$$A \sim^\alpha B \text{ if and only if } A \leq^\alpha B \text{ and } B \leq^\alpha A.$$

Let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty values on X , we denote $F(M) = \cup_{x \in M} F(x)$. For each $\alpha \in \{u, l, s\}$, we consider the following set optimization problem

$$\begin{aligned} (P_\alpha) \quad & \alpha\text{-Min } F(x) \\ & \text{subject to} \quad x \in M, \end{aligned}$$

where M is a nonempty subset of X . A point $\bar{x} \in M$ is said to be an α -minimal solution of (P_α) if for any $x \in M$ such that $F(x) \leq^\alpha F(\bar{x})$, then $F(\bar{x}) \leq^\alpha F(x)$. The set of all α -minimal solutions of (P_α) is called the solution set of (P_α) and denoted by $S_{\alpha\text{-Min } F}$.

Remark 2.2. It can be seen that if $\bar{x} \in S_{\alpha\text{-Min } F}$ and $F(\bar{x}) \sim^\alpha F(x)$ for some $x \in M$, then $x \in S_{\alpha\text{-Min } F}$.

Next, we recall definitions of semicontinuity for a set-valued mapping and their properties used in the sequel.

Definition 2.2. [29] A set-valued mapping $F : X \rightrightarrows Y$ is said to be

- (i) upper semicontinuous at $x_0 \in \text{Dom} F$ if and only if for any open subset V of Y with $F(x_0) \subset V$ there is a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$;
- (ii) lower semicontinuous at $x_0 \in \text{Dom} F$ if and only if for any open subset V of Y with $F(x_0) \cap V \neq \emptyset$ there is a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;
- (iii) lower (upper) semicontinuous on a subset D of X if it is lower (upper) semicontinuous at every $x \in D$;

where $\text{Dom} F = \{x \in X \mid F(x) \neq \emptyset\}$.

Lemma 2.3. [30] Let $F : X \rightrightarrows Y$ be a set-valued mapping.

- (i) F is lower semicontinuous at $x_0 \in \text{Dom} F$, if for every $\{x_n\}$ converging to x_0 and for every $y \in F(x_0)$ there exists $\{y_n\}$ with $y_n \in F(x_n)$ such that $\{y_n\}$ converges to y .
- (ii) If $F(x_0)$ is compact and F is upper semicontinuous at $x_0 \in \text{Dom} F$, then for every $\{x_n\}$ converging to x_0 and $y_n \in F(x_n)$ there exist $y_0 \in F(x_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ converges to y_0 .

Definition 2.3. [30] A set-valued mapping $F : X \rightrightarrows Y$ is said to be

- (i) Hausdorff upper semicontinuous at $x_0 \in \text{Dom}F$ if and only if for each neighborhood V of the origin in Y , there exists a neighborhood U of x_0 such that $F(x) \subset F(x_0) + V$ for all $x \in U \cap X$.
- (ii) Hausdorff lower semicontinuous at $x_0 \in \text{Dom}F$ if and only if for any neighborhood V of the origin in Y , there exists a neighborhood U of x_0 such that $F(x_0) \subset F(x) + V$ for all $x \in U \cap X$.
- (iii) Hausdorff lower (upper) semicontinuous on a subset D of X if and only if F is Hausdorff lower (upper) semicontinuous at every point of D .

Remark 2.4. [31] If F is upper semicontinuous at $x_0 \in \text{Dom}F$, then F is Hausdorff upper semicontinuous at x_0 ; the converse implication is true when $F(x_0)$ is compact.

Next, we recall concepts of Hausdorff distance and Hausdorff convergence of sequence of sets. Let S be a nonempty subset of X and $x \in X$. The distance d between x and S is defined as

$$d(x, S) = \inf_{u \in S} d(x, u).$$

Let S_1 and S_2 be two nonempty subsets of X . The Hausdorff distance between S_1 and S_2 , denoted by $H(S_1, S_2)$, is defined as

$$H(S_1, S_2) = \max\{H^*(S_1, S_2), H^*(S_2, S_1)\},$$

where $H^*(S_1, S_2) = \sup_{x \in S_1} d(x, S_2)$.

Definition 2.4. [32] Let $\{A_n\}$ be a sequence of subsets of X . We say that

- (i) A_n converge to $A \subset X$ in the sense of the upper Hausdorff set-convergence, denoted by $A_n \rightarrow A$, if and only if $H^*(A_n, A) \rightarrow 0$.
- (ii) A_n converge to $A \subset X$ in the sense of the lower Hausdorff set-convergence, denoted by $A_n \rightarrow A$, if and only if $H^*(A, A_n) \rightarrow 0$.
- (iii) A_n converge to $A \subset X$ in the sense of the Hausdorff set-convergence, denoted by $A_n \rightarrow A$, if and only if $H(A_n, A) \rightarrow 0$.

3 B -well-posedness for set optimization problems

In this section, two kinds of global B -well-posedness for the problem (P_α) are considered and their relationships are discussed. Moreover, we also provide characterizations and sufficient conditions of B -well-posedness for such problems.

We observe from the definitions of set order relations that $\leq^s$ is a combination of $\leq^l$ and $\leq^u$. For relationships between $\leq^l$ and $\leq^u$, they were given in Remark 2.6.10 of [4] as the following

$$A \leq^l B \Leftrightarrow -B \leq^u -A.$$

Some properties about these set order relations are demonstrated in next results.

Proposition 3.1. *The following statements are true:*

- (i) If $A \leq^\alpha B$, then $\lambda A \leq^\alpha \lambda B$, $\forall \lambda > 0$;
- (ii) $A \leq^l B \Leftrightarrow \lambda A \geq^u \lambda B$, $\forall \lambda < 0$;
- (iii) $A \leq^s B \Leftrightarrow \lambda A \geq^s \lambda B$, $\forall \lambda < 0$.

Proof. (i) We give the proof of the assertion (i) for the case $\alpha = s$, proofs of this assertion for other cases $\alpha = l$ and $\alpha = u$ are similar. Since $A \leq^s B$, $B \subset A + K$ and $A \subset B - K$. Clearly, for any $\lambda > 0$, we get $\lambda B \subset \lambda A + K$ and $\lambda A \subset \lambda B - K$. Hence, $\lambda A \leq^s \lambda B$.

(ii) We have $B \subset A + K$ as $A \leq^l B$. For any $\lambda < 0$, this yields $\lambda B \subset \lambda A - K$, i.e., $\lambda B \leq^u \lambda A$.

(iii) Since $A \leq^s B$, $B \subset A + K$ and $A \subset B - K$. For any $\lambda < 0$, this implies that $\lambda B \subset \lambda A - K$ and $\lambda A \subset \lambda B + K$. So, $\lambda A \geq^s \lambda B$. $\square$

We define a set-valued mapping $Q : K \rightrightarrows M$ as follows

$$Q(k) = \bigcup_{y \in S_{\alpha-\text{MinF}}} \{x \in M \mid F(x) \leq^\alpha F(y) + k\}. \quad (3.1)$$

The following results provide some properties of this mapping.

Proposition 3.2. *These assertions hold:*

- (i) If $k_1 \leq_K k_2$, then $Q(k_1) \subset Q(k_2)$;
- (ii) $S_{\alpha-\text{MinF}} \subset Q(0)$;
- (iii) $Q(0) = \bigcap_{k \in K} Q(k)$.

Proof. (i) We only demonstrate the proof of the above assertion for the case $\alpha = s$, proofs of this assertion for other cases are proved similarly. Let $x \in Q(k_1)$ be given, then there exists $y \in S_{\alpha-\text{MinF}}$ such that $F(x) \leq^s F(y) + k_1$, i.e., $F(y) + k_1 \subset F(x) + K$ and $F(x) \subset F(y) + k_1 - K$. Combining this with $k_1 \leq_K k_2$, we get $F(y) + k_2 = F(y) + k_1 + (k_2 - k_1) \subset F(x) + K$ and $F(x) \subset F(y) + k_1 - K = F(y) + k_2 + (k_1 - k_2) - K \subset F(y) + k_2 - K$. This means that $F(x) \leq^l F(y) + k_2$ and $F(x) \leq^u F(y) + k_2$. Hence, $x \in Q(k_2)$.

(ii) It is clear that for every $x \in S_{\alpha-\text{MinF}}$, we have $F(x) \leq^\alpha F(x)$, and hence $x \in Q(0)$. Therefore, $S_{\alpha-\text{MinF}} \subset Q(0)$.

(iii) It is obvious that $Q(0) \subset Q(k)$ for all $k \in K$, and thus $Q(0) \subset \bigcap_{k \in K} Q(k)$. Conversely, suppose that there exists $x \in \bigcap_{k \in K} Q(k)$ but $x \notin Q(0)$, i.e., $x \notin \bigcup_{y \in S_{\alpha-\text{MinF}}} \{z \in M \mid F(z) \leq^\alpha F(y)\}$. Then, $F(x) \not\leq^\alpha F(y)$ for any $y \in S_{\alpha-\text{MinF}}$. On the other hand, since $x \in \bigcap_{k \in K} Q(k)$, $x \in Q(k)$ for all $k \in K$. Therefore, there is $y \in S_{\alpha-\text{MinF}}$ such that $F(x) \leq^\alpha F(y) + k$ for all $k \in K$. Particularly, for $k = 0$, there exists $y \in S_{\alpha-\text{MinF}}$ such that $F(x) \leq^\alpha F(y)$ which is a contradiction. $\square$

Next, we give two concepts related to global B -well-posedness for (P_α) .

Definition 3.1. Problem (P_α) is said to be

- (i) B -well-posed if and only if $S_{\alpha\text{-MinF}} \neq \emptyset$ and Q is upper semicontinuous at $k = 0$.
- (ii) generalized B -well-posed if and only if $S_{\alpha\text{-MinF}} \neq \emptyset$ and Q is Hausdorff upper semicontinuous at $k = 0$.

Remark 3.3. Clearly, if the problem (P_α) is B -well-posedness, then it is generalized B -well-posedness. It follows from Proposition 3.2(ii) and Remark 2.4 that the converse holds if $S_{\alpha\text{-MinF}}$ is compact. In the sequel, we focus on generalized B -well-posedness.

Definition 3.2. A sequence $\{x_n\} \subset M$ is said to be a generalized B -minimizing sequence of (P_α) if and only if there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_{\alpha\text{-MinF}}$ such that $F(x_n) \leq^\alpha F(y_n) + k_n$.

Equivalently, $\{x_n\}$ is a generalized B -minimizing sequence of (P_α) if and only if there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_{\alpha\text{-MinF}}$ such that $x_n \in Q(k_n)$.

Remark 3.4. When $\alpha = u$, concepts in Definitions 3.1 and 3.2 reduce to ones in Definitions 3.1-3.3 in [24], respectively.

Characterizations for B -well-posedness of (P_α) are provided in the next result through the B -minimizing sequence.

Theorem 3.5. *Problem (P_α) is generalized B -well-posed if and only if these following conditions are satisfied*

- (a) $S_{\alpha\text{-MinF}} \neq \emptyset$;
- (b) *for every generalized B -minimizing sequence $\{x_n\} \subset M$ and for every neighborhood U of the origin in X , there exists $n_0 \in \mathbb{N}$ such that $x_n \in Q(0) + U$ for all $n \geq n_0$.*

Proof. Suppose that (P_α) is generalized B -well-posed. Let $\{x_n\} \subset M$ be a generalized B -minimizing sequence of (P_α) , then there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_{\alpha\text{-MinF}}$ such that $x_n \in Q(k_n)$. Since (P_α) is generalized B -well-posed, Q is Hausdorff upper semicontinuous at 0. Let U be a neighborhood of the origin in X , there exists $n_0 \in \mathbb{N}$ such that $Q(k_n) \subset Q(0) + U$ for all $n \geq n_0$. Therefore, we get $x_n \in Q(0) + U$ for all $n \geq n_0$.

Conversely, suppose on the contrary that (P_α) is not generalized B -well-posed. Thus, Q is not Hausdorff upper semicontinuous at 0. Then, there exists a neighborhood U of the origin in X such that $Q(k) \not\subset Q(0) + U$ for some k belongs to a neighborhood of 0. So, we can build a sequence $\{k_n\} \subset K$ converging to 0 such that $Q(k_n) \not\subset Q(0) + U$. It leads to the existence of a sequence $\{x_n\}$ with $x_n \in Q(k_n)$ satisfying $x_n \notin Q(0) + U$ which contradicts the assumption (b). This completes the proof. $\square$

We now give sufficient conditions for (P_α) to be generalized B -well-posed.

Theorem 3.6. *Suppose that $S_{\alpha\text{-Min}F} \neq \emptyset$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(F(M) - F(S_{\alpha\text{-Min}F})) \cap (\delta B_Y - K) \subset \varepsilon B_Y. \quad (3.2)$$

Then,

- (i) (P_l) is generalized B -well-posed if each sequence of sets $\{A_n\} \subset M$ with $F(A_n) \rightarrow F(S_{l\text{-Min}F})$ satisfies $A_n \rightarrow S_{l\text{-Min}F}$.
- (ii) (P_u) is generalized B -well-posed if each sequence of sets $\{A_n\} \subset M$ with $F(A_n) \rightarrow F(S_{u\text{-Min}F})$ satisfies $A_n \rightarrow S_{u\text{-Min}F}$.
- (iii) (P_s) is generalized B -well-posed if each sequence of sets $\{A_n\} \subset M$ with $F(A_n) \rightarrow F(S_{s\text{-Min}F})$ satisfies $A_n \rightarrow S_{s\text{-Min}F}$.

Proof. (i) By contradiction, suppose that (P_l) is not generalized B -well-posed. It follows from Theorem 3.5 that there exist a generalized B -minimizing sequence $\{x_n\}$ and a neighborhood U of the origin in X such that for some $n_0 \in \mathbb{N}$, $x_n \notin Q(0) + U$ for all $n \geq n_0$. Combining this with Proposition 3.2(ii), we get

$$x_n \notin S_{l\text{-Min}F} + U, \quad \forall n \geq n_0. \quad (3.3)$$

Since $\{x_n\}$ is a generalized B -minimizing sequence, there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_{l\text{-Min}F}$ such that

$$F(x_n) \leq^l F(y_n) + k_n. \quad (3.4)$$

We consider two following cases:

Case 1: If $F(x_n) \rightarrow F(S_{l\text{-Min}F})$, then choosing $A_n = \{x_n\}$. It implies from the hypothesis that $\{x_n\} \rightarrow S_{l\text{-Min}F}$, and hence $H^*(S_{l\text{-Min}F}, \{x_n\}) \rightarrow 0$. Therefore, $d(x_n, S_{l\text{-Min}F}) \rightarrow 0$ which contradicts (3.3).

Case 2: If $F(x_n) \not\rightarrow F(S_{l\text{-Min}F})$, then $\sup_{x \in S_{l\text{-Min}F}} d(x, F(x_n)) \not\rightarrow 0$. So, there exists $x \in S_{l\text{-Min}F}$ such that $d(x, F(x_n)) \not\rightarrow 0$, i.e., there exist $n_1 \in \mathbb{N}$ and a neighborhood V of the origin in Y such that

$$x \notin F(x_n) + V, \quad \forall n \geq n_1. \quad (3.5)$$

Take ε such that $\varepsilon B_Y \subset V$. For δ satisfying (3.2), since $\{k_n\} \subset K$ converges to 0, there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$, we have

$$k_n \in \delta B_Y. \quad (3.6)$$

By (3.4), we get $F(y_n) + k_n \subset F(x_n) + K$. Therefore, $F(y_n) \subset F(x_n) - k_n + K \subset F(x_n) + \delta B_Y + K$. This implies that for an arbitrary $z_n \in F(y_n)$, there exists $\bar{z}_n \in F(x_n)$ such that $z_n \in \bar{z}_n + \delta B_Y + K$, and thus $z_n - \bar{z}_n \in \delta B_Y + K$. So, $\bar{z}_n - z_n \in \delta B_Y - K$. On the other hand, we have $\bar{z}_n - z_n \in F(x_n) - F(y_n) \subset F(M) - F(S_{l\text{-Min}F})$. It derives from (3.2) that $\bar{z}_n - z_n \in \varepsilon B_Y \subset V$. So, now we get $z_n \in \bar{z}_n + V \subset F(x_n) + V$ which contradicts (3.5).

For (ii) (iii), the proofs of these assertions are technically similar to that of the assertion (i). $\square$

4 Pointwise B -well-posedness for set optimization problems

In this section, we consider a notion of pointwise B -well-posedness for the problem (P_α) . At a reference point $x_0 \in M$, we define a corresponding set-valued mapping as follows $Q_{x_0} : K \rightrightarrows M$, $Q_{x_0}(k) = \{x \in M \mid F(x) \leq^\alpha F(x_0) + k\}$.

Definition 4.1. Problem (P_α) is said to be B -well-posed at $x_0 \in S_{\alpha\text{-Min}F}$ if and only if Q_{x_0} is upper semicontinuous at $k = 0$.

We observe that $Q_{x_0}(0) = \{x \in M \mid F(x) \sim^\alpha F(x_0)\}$ for each $x_0 \in S_{\alpha\text{-Min}F}$. The next results give some properties of the mapping Q_{x_0} .

Proposition 4.1. *The following statements are true:*

- (i) If $k_1 \leq_K k_2$, then $Q_{x_0}(k_1) \subset Q_{x_0}(k_2)$;
- (ii) $Q_{x_0}(0) \subset S_{\alpha\text{-Min}F}$ with $x_0 \in S_{\alpha\text{-Min}F}$;
- (iii) $x_0 \in Q_{x_0}(0) \subset Q_{x_0}(k)$ for every $k \in K$;
- (iv) $Q(0) = \bigcup_{x_0 \in S_{\alpha\text{-Min}F}} Q_{x_0}(0)$.

Proof. (i) The statement is proved by a similar argument in Proposition 3.2(i).

(ii) By the similarity, we prove the assertion for the case $\alpha = s$. Let $x \in Q_{x_0}(0)$ and $y \in M$ such that

$$F(y) \leq^s F(x), \quad (4.1)$$

we need to show that $F(x) \leq^s F(y)$. Since $x \in Q_{x_0}(0)$,

$$F(x) \leq^s F(x_0). \quad (4.2)$$

Combining (4.1) and (4.2), we get $F(y) \leq^s F(x_0)$. Because $x_0 \in S_{\alpha\text{-Min}F}$, $F(x_0) \leq^s F(y)$. Therefore, $F(x) \leq^s F(y)$.

For (iii) and (iv), these assertions were implied by definitions of mappings Q and Q_{x_0} . $\square$

Now we discuss the converse of (ii) of the above proposition. Because the proof of this assertion is elementary, we would like to omit it.

Lemma 4.2. *If $x_0 \in S_{\alpha\text{-Min}F}$, then $S_{\alpha\text{-Min}F} \subset Q_{x_0}(0)$ if and only if $F(x) \sim^\alpha F(y)$ for all $x, y \in S_{\alpha\text{-Min}F}$.*

Definition 4.3. A sequence $\{x_n\} \subset M$ is said to be an x_0 -minimizing sequence of (P_α) where $x_0 \in S_{\alpha\text{-Min}F}$ if and only if there exists $\{k_n\} \subset K$ converging to 0 such that

$$F(x_n) \leq^\alpha F(x_0) + k_n.$$

Definition 4.3 reduces to Definition 3.3 in [25] when $\alpha = u$.

Clearly, for each $x_0 \in S_{\alpha\text{-MinF}}$, $\{x_n\} \subset M$ is an x_0 -minimizing sequence if and only if there exists $\{k_n\} \subset K$ converging to 0 such that $x_n \in Q_{x_0}(k_n)$.

The next result illustrates the relationship between the pointwise B -well-posedness and B -well-posedness for (P_α) .

Theorem 4.2. *If $S_{\alpha\text{-MinF}}$ is a finite set and (P_α) is pointwise B -well-posed at every $x \in S_{\alpha\text{-MinF}}$, then it is B -well-posed.*

Proof. Let $S_{\alpha\text{-MinF}} = \{x_1, \dots, x_n\}$ and V be an open set in X such that $Q(0) \subset V$. By Proposition 4.1(iv), $Q_{x_i}(0) \subset V$ for all $i = 1, \dots, n$. Since (P_α) is pointwise B -well-posed at every x_i , Q_{x_i} is upper semicontinuous at $k = 0$, and hence for each $i \in \{1, \dots, n\}$, there exists a neighborhood U_{x_i} of 0 such that $Q_{x_i}(U_{x_i}) \subset V$. Let $U = \bigcap_{i=1}^n U_{x_i}$, this finite intersection of neighborhoods U_{x_i} is also a neighborhood of 0. Obviously, we have $Q_{x_i}(U) \subset V$ for all $i = 1, \dots, n$. By definitions of mappings Q_{x_i} and Q , we also get $Q(U) \subset V$. It leads to the upper semicontinuity at $k = 0$ of Q . We conclude that (P_α) is B -well-posed. $\square$

We next investigate characterizations of pointwise B -well-posedness for (P_α) .

Theorem 4.3. *Problem (P_α) is pointwise B -well-posed at $x_0 \in S_{\alpha\text{-MinF}}$ if and only if for a given $e \in \text{int}K$, the set-valued mapping $Q_{x_0}^+ : \mathbb{R}_+ \rightrightarrows M$ defined as*

$$Q_{x_0}^+(t) = \{x \in M \mid F(x) \leq^\alpha F(x_0) + te\}$$

is upper semicontinuous at $t = 0$.

Proof. Assume that (P_α) is pointwise B -well-posed at $x_0 \in S_{\alpha\text{-MinF}}$, then Q_{x_0} is upper semicontinuous at $k = 0$. Let V be an open set in X such that $Q_{x_0}^+(0) \subset V$, we get $Q_{x_0}(0) \subset V$. By the upper semicontinuity of Q_{x_0} , there is a positive number r such that $Q_{x_0}(k) \subset V$ for all $k \in B(0, r) \cap K$, where $B(0, r)$ is the open ball centered at the origin in Y with radius r . Then, there exists a positive number β such that $[0, \beta e) \subset B(0, r)$, where $[0, \beta e) = \{te \mid t \in [0, \beta)\}$. For $t \in [0, \beta)$ and $x \in Q_{x_0}^+(t)$, we have $F(x) \leq^\alpha F(x_0) + te$, which implies that $x \in Q_{x_0}(te)$. This fact, together with $te \in B(0, r)$, yields $x \in V$, and so $Q_{x_0}^+(t) \subset V$. We conclude that $Q_{x_0}^+$ is upper semicontinuous at $t = 0$.

Conversely, suppose that $Q_{x_0}^+$ is upper semicontinuous at $t = 0$. Let V be an open set in X such that $Q_{x_0}(0) \subset V$, then $Q_{x_0}^+(0) \subset V$. It follows from the upper semicontinuity of $Q_{x_0}^+$ that there exists a positive number β such that $Q_{x_0}^+(t) \subset V$ for every $t \in [0, \beta)$. For convenience in writing, we only prove the assertion for case $\alpha = s$ because proofs of this assertion for other cases $\alpha = l$ and $\alpha = u$ are similar. Let $\gamma \in [0, \beta)$, there exists a positive number r such that $B(0, r) \subset \gamma e - K$ and $B(0, r) \subset -\gamma e + K$. Let $k \in B(0, r) \cap K$ and $x \in Q_{x_0}(k)$, it follows from the definition of Q_{x_0} that

$$F(x) \leq^u F(x_0) + k, \quad (4.3)$$

and

$$F(x) \leq^l F(x_0) + k. \quad (4.4)$$

It yields from (4.3) that $F(x) \subset F(x_0) + k - K \subset F(x_0) + \gamma e - K$ as $k \in \gamma e - K$. On other hand, by (4.4), we get $F(x_0) + k \subset F(x) + K$, and hence $F(x_0) \subset F(x) - k + K$. Since $-k \in B(0, r) \subset -\gamma e + K$, we have $F(x_0) \subset F(x) - \gamma e + K$, i.e., $F(x_0) + \gamma e \subset F(x) + K$. So, we now get $F(x) \leq^s F(x_0) + \gamma e$. It implies that $x \in Q_{x_0}^+(\gamma)$, and thus $x \in V$ and $Q_{x_0}(k) \subset V$. The proof is complete. $\square$

Theorem 4.4. *If $S_{\alpha-\text{MinF}}$ is closed and (P_α) is pointwise B -well-posed at $x_0 \in S_{\alpha-\text{MinF}}$, then for every x_0 -minimizing sequence $\{x_n\} \subset M \setminus S_{\alpha-\text{MinF}}$, one can extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some $\bar{x} \in S_{\alpha-\text{MinF}}$.*

Proof. Assume that (P_α) is pointwise B -well-posed at $x_0 \in S_{\alpha-\text{MinF}}$. Then, Q_{x_0} is upper semicontinuous at $k = 0$. By contradiction, suppose that there exists an x_0 -minimizing sequence $\{x_n\} \subset M \setminus S_{\alpha-\text{MinF}}$ which admits no subsequence $\{x_{n_k}\}$ converging to some $\bar{x} \in S_{\alpha-\text{MinF}}$. By the closedness of $S_{\alpha-\text{MinF}}$, we may find an open set $V \subset X$ such that $S_{\alpha-\text{MinF}} \subset V$ and $x_n \notin V$. We have $Q_{x_0}(0) \subset V$ due to $Q_{x_0}(0) \subset S_{\alpha-\text{MinF}}$ and Proposition 4.1(ii). Since $\{x_n\}$ is an x_0 -minimizing sequence, there exists $\{k_n\} \subset K$ converging to 0 such that $x_n \in Q_{x_0}(k_n)$. It follows from the upper semicontinuity of Q_{x_0} at $k = 0$ that $Q_{x_0}(k_n) \subset V$. Hence, $x_n \in V$ which is a contradiction. So, we get the desired result. $\square$

Remark 4.5. Our results extend the corresponding results of Long and Peng [24]. More precisely,

- (i) When $\alpha = l$ or $\alpha = s$, our results here are new. To the best of our knowledge, there is no paper devoted to this type of well-posedness for set optimization problem involving the set less relation $\leq^s$.
- (ii) When $\alpha = u$, the corresponding set optimization problem (P_u) was studied in [24].

5 Pointwise L -well-posedness and relationship with pointwise B -well-posedness

Motivated by the studies in [22, 25], we introduce the concept of pointwise L -well-posedness for the problem (P_α) .

Definition 5.1. Problem (P_α) is said to be L -well-posed at $x_0 \in S_{\alpha-\text{MinF}}$ if and only if every x_0 -minimizing sequence of (P_α) has a subsequence converging to some element $\bar{x} \in S_{\alpha-\text{MinF}}$.

Remark 5.1. When $\alpha = l$, Definition 5.1 reduces to Definition 2.1 in [22].

We are going to study sufficient and necessary conditions of pointwise L -well-posedness for (P_α) .

Theorem 5.2. *Let $x_0 \in S_{\alpha-\text{MinF}}$ be given.*

- (i) If (P_α) is L -well-posed at x_0 and $Q_{x_0}(0) = S_{\alpha\text{-Min}F}$, then Q_{x_0} is upper semicontinuous and compact-valued at 0.
- (ii) If Q_{x_0} is upper semicontinuous and compact-valued at 0, then (P_α) is L -well-posed at x_0 .

Proof. (i) Assume that (P_α) is L -well-posed at x_0 . First of all, we show that Q_{x_0} is upper semicontinuous at 0. By contradiction, suppose that Q_{x_0} is not upper semicontinuous at 0. Then, there exist a neighborhood U of $Q_{x_0}(0)$ and $\{k_n\} \subset K$ converging to 0 such that for each $n \in \mathbb{N}$, there exists $x_n \in Q_{x_0}(0) \setminus U$, i.e.,

$$x_n \notin U \quad (5.1)$$

and

$$F(x_n) \leq^\alpha F(x_0) + k_n. \quad (5.2)$$

It follows from (5.2) that $\{x_n\}$ is an x_0 -minimizing sequence of (P_α) . Because (P_α) is L -well-posed at x_0 , there exists a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to some element $\bar{x} \in S_{\alpha\text{-Min}F}$, and thus we get $\bar{x} \in Q_{x_0}(0)$. Therefore, $\bar{x} \in U$ which contradicts (5.1). So, Q_{x_0} is upper semicontinuous at 0.

Next, we prove that $Q_{x_0}(0)$ is compact. Indeed, for every sequence $\{x_n\} \subset Q_{x_0}(0)$, we have $F(x_n) \leq^\alpha F(x_0) + k_n$ where $\{k_n\} \subset K$ converges to 0. This means that $\{x_n\}$ is an x_0 -minimizing sequence of (P_α) . By the L -well-posedness of (P_α) at x_0 , there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\bar{x} \in S_{\alpha\text{-Min}F}$. Therefore, $\bar{x} \in Q_{x_0}(0)$. This leads to the compactness of $Q_{x_0}(0)$.

(ii) Let $\{x_n\} \subset M$ be an x_0 -minimizing sequence of (P_α) , there exists $\{k_n\} \subset K$ converging to 0 such that $F(x_n) \leq^\alpha F(x_0) + k_n$. Hence, $x_n \in Q_{x_0}(k_n)$. Since Q_{x_0} is upper semicontinuous and compact-valued at 0, there exists a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to some $\bar{x} \in Q_{x_0}(0)$. Combining this with Proposition 4.1(ii), we get $\bar{x} \in S_{\alpha\text{-Min}F}$. So, (P_α) is L -well-posed at x_0 . $\square$

The next results illustrate relationships between the pointwise L -well-posedness and pointwise B -well-posedness for the problem (P_α) .

Theorem 5.3. Let $x_0 \in S_{\alpha\text{-Min}F}$ be given.

- (i) If (P_α) is L -well-posed at x_0 and $Q_{x_0}(0) = S_{\alpha\text{-Min}F}$, then it is B -well-posed at x_0 .
- (ii) If (P_α) is B -well-posed at x_0 and $S_{\alpha\text{-Min}F}$ is compact, then it is L -well-posed at x_0 .

Proof. (i) By contradiction, suppose that (P_α) is not B -well-posed at x_0 . We get that Q_{x_0} is not upper semicontinuous at $k = 0$. Hence, there exist a neighborhood V of $Q_{x_0}(0)$ and $\{k_n\} \subset K$ converging to 0 such that for each $n \in \mathbb{N}$, there exists $x_n \in Q_{x_0}(k_n) \setminus V$. By definition of Q_{x_0} , we have $F(x_n) \leq^\alpha F(x_0) + k_n$, i.e., $\{x_n\}$ is an x_0 -minimizing sequence of (P_α) . It follows from the L -well-posedness at x_0

of (P_α) that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\bar{x} \in S_{\alpha\text{-MinF}}$. Therefore, $\bar{x} \in Q_{x_0}(0)$, and hence we now get

$$\bar{x} \in Q_{x_0}(0) \subset V. \quad (5.3)$$

On the other hand, since $x_n \notin V$, $x_n \in X \setminus V$. By the closedness of $X \setminus V$, we have $\bar{x} \in X \setminus V$ which contradicts (5.3). So, (P_α) is B -well-posed at x_0 .

(ii) Suppose that (P_α) is B -well-posed at x_0 . Let $\{x_n\}$ be an x_0 -minimizing sequence of (P_α) , we consider two cases as follows:

Case 1: $\{x_n\}$ has infinite elements which belong to $S_{\alpha\text{-MinF}}$.

Since $S_{\alpha\text{-MinF}}$ is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some $\bar{x} \in S_{\alpha\text{-MinF}}$. Hence, (P_α) is L -well-posed at x_0 .

Case 2: $\{x_n\}$ has infinite elements which do not belong to $S_{\alpha\text{-MinF}}$. Without lost of generality, we can assume that $\{x_n\} \subset M \setminus S_{\alpha\text{-MinF}}$. By Theorem 4.4, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some $\bar{x} \in S_{\alpha\text{-MinF}}$. Therefore, (P_α) is L -well-posed at x_0 . $\square$

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CHARACTERIZING ROBUST WEAK SHARP SOLUTION SETS OF CONVEX OPTIMIZATION PROBLEMS WITH UNCERTAINTY

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ABSTRACT. We introduce robust weak sharp and robust sharp solution to a convex programming with the objective and constraint functions involved uncertainty. The characterizations of the sets of all the robust weak sharp solutions are obtained by means of subdifferentials of convex functions, DC functions, Fermat rule and the robust-type subdifferential constraint qualification, which was introduced in X.K. Sun, Z.Y. Peng and X. Le Guo, *Some characterizations of robust optimal solutions for uncertain convex optimization problems*, Optim Lett. **10**. (2016), 1463-1478. In addition, some applications to the multi-objective case are presented.

1. Introduction. The notion of a weak sharp minimizer in general mathematical programming problems was first introduced in [1]. It is an extension of a sharp minimizer (or equivalently, strongly unique minimizer) in [2] to include the possibility of non-unique solution set. It has been acknowledged that the weak sharp minimizer plays important roles in stability/sensitivity analysis and convergence analysis of a wide range of numerical algorithms in mathematical programming (see [3, 4, 5, 6, 7, 8] and references therein).

In the context of optimization, much attention has been paid to concerning sufficient and/or necessary conditions for weak sharp minimizers/solutions and characterizing weak sharp solution sets (of such weak sharp minimizers) in various types of problems. Particularly, the study of characterizations of the weak sharp solution sets covers both single-objective and multi-objective optimization problems (see, [9, 10, 11, 12] and references therein) and, recently, is extended to mathematical

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programs with inequality constraints and semi-infinite programs (see, e.g., [13, 14]). As it might be seen, the study of characterizations of the weak sharp solution sets has been popular in many optimization problems. *How about the issue of this study, particularly, in a robust optimization?*

Robust (convex) optimization has been known as an important class of convex optimization deals with uncertainty in the data of the problems [15, 16]. The goal of robust optimization is to immunize an optimization problem against uncertain parameters in the problem. In the last two decades, it has been through a rapid development owing to the practical requirement and its effective implementation in real-world applications of optimization (see, e.g., [17, 18, 19, 20, 21] and the references therein). A successful treatment of the robust optimization approaches to convex optimization problems under data uncertainty was given in ([15, 16, 22, 23, 24]).

While the characterizations of optimal solution sets have been in the limelight presently, there has been no research concerning the characterizations robust weak sharp solution sets for such problems. Indeed, a robust weak sharp solution of an uncertain optimization problem is the weak sharp minimizer of the robust counterpart of such problem. Our main goal in this paper is to establish characterizations of the robust weak sharp solution set of the convex optimization problem under the data uncertainty.

This paper is organized as follows. In section 2, we recall the basic definitions. In Section 3, we establish necessary conditions for a robust weak sharp solution, constancy of Lagrangian-type function on the robust weak sharp solution set, and some characterizations of robust weak sharp solution sets are established respectively. Some properties of subdifferentials of convex functions and the (RSCQ), which was introduced in [24], are employed in the section. Finally, in section 4, we consider the characterizations of the robust weak sharp weakly efficient solutions for the multi-objective optimization problem under data uncertainty.

2. Preliminary. Throughout the paper, let $\mathbb{R}^n, n \in \mathbb{N}$, be the n -dimensional Euclidean space, and the inner product and the norm of $\mathbb{R}^n$ are denoted respectively by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. The symbol $B(x, r)$ stands for the open ball centered at $x \in \mathbb{R}^n$ with the radius $r > 0$ while the $\mathcal{B}_{\mathbb{R}^n}$ stands for the closed unit ball in $\mathbb{R}^n$. For a nonempty subset $A \subseteq \mathbb{R}^n$, we denote the notations of the closure, boundary and convex hull of A by $\text{cl}A$, $\text{bd}A$, and $\text{co}A$, respectively. In particular, when $\lambda x \in E \subseteq \mathbb{R}^n$ for every $\lambda \geq 0$ and every $x \in E$, the set E in $\mathbb{R}^n$ is said to be a cone. A dual cone E^* of the cone E is given as $E^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in E\}$. Observe that the dual cone E^* is always closed and convex (regardless of E).

In general, for a given nonempty set $A \subseteq \mathbb{R}^n$, the indicator function $\delta_A : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ of A and the support function $\sigma_A : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ of A are, respectively, defined by

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A; \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\sigma_A(x^*) := \sup_{x \in A} \langle x^*, x \rangle.$$

The distance function $d_A : \mathbb{R}^n \rightarrow \mathbb{R}_+ : [0, +\infty)$ is defined by

$$d_A(x) := \inf_{y \in A} \|x - y\|.$$

A normal cone of the set A at the point x is the following set:

$$N_A(x) = \begin{cases} \{y \in \mathbb{R}^n : \langle y, a - x \rangle \leq 0 \text{ for all } a \in A\}, & \text{if } x \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$

The normal cone $N_A(x)$ is always closed and convex for any set A .

For any extended real-valued function $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}} := [-\infty, +\infty]$ the following notations stand, respectively, for its effective domain and epigraph:

$$\text{dom} h := \{x \in \mathbb{R}^n : h(x) < +\infty\},$$

and

$$\text{epi} h := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq r\}.$$

The function h is said to be a proper function if and only if $h(x) > -\infty$ for every $x \in \mathbb{R}^n$ and $\text{dom} h$ is nonempty. Further, it is said to be a convex function if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y),$$

or equivalently, $\text{epi} h$ is convex. On the other hand, the function h is said to be a concave function if and only if $-h$ is a convex function. In the case of vector valued function, let $\tilde{h} : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a given function and $D \subseteq \mathbb{R}^p$ is a convex set. The function $\tilde{h}$ is said to be D -convex if and only if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$\tilde{h}(\lambda x + (1 - \lambda)y) - \lambda \tilde{h}(x) - (1 - \lambda)\tilde{h}(y) \in -D.$$

Simultaneously, the function h is called a lower semicontinuous at $x \in \mathbb{R}^n$ if for every sequence $\{x_k\} \subseteq \mathbb{R}^n$ converging to x ,

$$h(x) \leq \liminf_{k \rightarrow \infty} h(x_k).$$

Equivalently,

$$h(x) \leq \liminf_{y \rightarrow x} h(y),$$

where the term on the right-hand side of the inequality denotes the lower limit of the function h defined as

$$\liminf_{y \rightarrow x} h(y) = \lim_{r \downarrow 0} \inf_{y \in B(x, r)} h(y).$$

For any proper and convex function $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, the subdifferential of h at $\hat{x} \in \text{dom} h$, is defined by

$$\partial h(\hat{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, x - \hat{x} \rangle \leq h(x) - h(\hat{x}), \forall x \in \mathbb{R}^n\}.$$

More specifically, for each $\varepsilon \geq 0$, the ε -subdifferential of h at $\hat{x} \in \text{dom} h$, is defined by

$$\partial_\varepsilon h(\hat{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, x - \hat{x} \rangle \leq h(x) - h(\hat{x}) + \varepsilon, \forall x \in \mathbb{R}^n\}.$$

It is obvious that for $\varepsilon \geq \varepsilon'$, we have $\partial_{\varepsilon'} h(\hat{x}) \subseteq \partial_\varepsilon h(\hat{x})$. In particular, if h is a proper lower semicontinuous convex function, then for every $\hat{x} \in \text{dom} h$, the ε -subdifferential $\partial_\varepsilon h(\hat{x})$ is a nonempty closed convex set and

$$\partial h(\hat{x}) = \bigcap_{\varepsilon > 0} \partial_\varepsilon h(\hat{x}).$$

If $x \notin \text{dom} h$, then we set $\partial h(x) = \emptyset$. Simultaneously, for the nonempty subset A of $\mathbb{R}^n$ we get $\partial \delta_A(x) = N_A(x)$ and $\partial d_A(x) = \mathcal{B}_{\mathbb{R}^n} \cap N_A(x)$.

The conjugate function $h^* : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ of any $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is defined by

$$h^*(x^*) := \sup_{x \in \mathbb{R}^n} \{\langle x^*, x \rangle - h(x)\}$$

for all $x \in \mathbb{R}^n$. The function h^* is lower semicontinuous convex irrespective of the nature of h but for h^* to be proper, we need h to be a proper convex function.

Next, let us recall some basic concepts dealing a DC problem/programming. A DC function is the difference of two convex functions. The minimization (or maximization) problem of a DC function is called a DC problem, i.e., the DC problem concerned about finding

$$\inf_{x \in \mathbb{R}^n} h(x) := f(x) - \phi(x)$$

where $f, \phi : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex. Note that the function h is DC and it is not expected to be convex.

It shall be found later that some DC problems are considered and their properties, in particular the following lemma, are employed.

Lemma 2.1. [25] Let $h_1, h_2 : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be two proper lower semicontinuous convex functions. Then

- (i) A point $\hat{x} \in \text{dom } h_1 \cap \text{dom } h_2$ is a (global) minimizer of the DC problem :

$$\inf_{x \in \mathbb{R}^n} \{h_1(x) - h_2(x)\}$$

if and only if for any $\varepsilon \geq 0$, $\partial_\varepsilon h_2(\hat{x}) \subseteq \partial_\varepsilon h_1(\hat{x})$.

- (ii) If $\hat{x} \in \text{dom } h_1 \cap \text{dom } h_2$ is a local minimizer of the DC problem :

$$\inf_{x \in \mathbb{R}^n} \{h_1(x) - h_2(x)\}$$

then $\partial h_2(\hat{x}) \subseteq \partial h_1(\hat{x})$.

Lemma 2.2. [19] Let $\mathcal{U} \subseteq \mathbb{R}^p$ be a convex compact set, and $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ be a function such that, $f(\cdot, u)$ is a convex function for any $u \in \mathcal{U}$, and $f(x, \cdot)$ is a concave function for any $x \in \mathbb{R}^n$. Then,

$$\partial \left(\max_{u \in \mathcal{U}} f(\cdot, u) \right) (\hat{x}) = \bigcup_{u \in \mathcal{U}(\hat{x})} \partial f(\cdot, u)(\hat{x}),$$

where

$$\mathcal{U}(\hat{x}) := \left\{ \hat{u} \in \mathcal{U} : f(\hat{x}, \hat{u}) = \max_{u \in \mathcal{U}} f(\hat{x}, u) \right\}.$$

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Let $D \subseteq \mathbb{R}^p$ be a nonempty closed convex cone. Consider the following convex optimization problem:

$$\min f(x) \text{ s.t. } x \in C, g(x) \in -D \quad (\text{P})$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a D -convex function. The feasible set of (P) is defined by

$$K_0 := \{x \in C : g(x) \in -D\}.$$

The problem (P) in the face of data uncertainty both in the objective and constraints can be captured by the following *uncertain optimization problem* :

$$\min \{f(x, u) : x \in C, g(x, v) \in -D\}. \quad (\text{UP})$$

where $\mathcal{U} \subseteq \mathbb{R}^p$ and $\mathcal{V} \subseteq \mathbb{R}^q$ are convex and compact uncertainty sets, $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ is a given real-valued function such that, for any uncertain parameter $u \in \mathcal{U}$, $f(\cdot, u)$ is convex as well as $f(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathcal{V} \rightarrow \mathbb{R}^m$ is a vector-valued function such that, for any uncertain parameter $v \in \mathcal{V}$, $g(\cdot, v)$ is D -convex as well as $g(x, \cdot)$ is D -concave for any $x \in \mathbb{R}^n$. The uncertain sets can be apprehended in the sense that the parameter vectors u and v are not known exactly at the time of the decision.

For examining the uncertain optimization problem (UP), one usually associates with its *robust (worst-case) counterpart*, which is the following problem:

$$\min \left\{ \max_{u \in \mathcal{U}} f(x, u) : x \in C, g(x, v) \in -D, \forall v \in \mathcal{V} \right\}. \quad (\text{RUP})$$

It is worth observing here that the robust counterpart, which is termed as the robust optimization problem, finds a worst-case possible solution that can be immunized opposed the data uncertainty.

The problem (RUP) is said to be feasible if the *robust feasible set* K is nonempty where it is denoted by

$$K := \{x \in C : g(x, v) \in -D, \forall v \in \mathcal{V}\}. \quad (1)$$

Now, we recall the following concept of solutions, which was introduced in [26].

Definition 2.3. [26] A point $\hat{x} \in K$ is said to be a *robust optimal solution* for (UP) if it is an optimal solution for (RUP), i.e., for all $x \in K$,

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq 0.$$

The *robust optimal solution set* of (UP) is the set which consists of all robust optimal solutions of (UP) and is given by

$$S := \left\{ x \in K : \max_{u \in \mathcal{U}} f(x, u) \leq \max_{u \in \mathcal{U}} f(y, u), \forall y \in K \right\}.$$

In this paper, using the idea of weak sharp minimizer, and the robust optimal solution, we introduce a new concept of solutions for (UP), which related to the sharpness, namely the robust weak sharp solution.

Definition 2.4. A point $\hat{x} \in K$ is said to be a (or an optimal) *weak sharp solution* for (RUP) if there exist a real number $\eta > 0$ such that for all $x \in K$,

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(x)$$

where $\tilde{K} := \left\{ x \in K : \max_{u \in \mathcal{U}} f(x, u) = \max_{u \in \mathcal{U}} f(\hat{x}, u) \right\}$.

Definition 2.5. A point $\hat{x} \in K$ is said to be a (or an optimal) *robust weak sharp solution* for (UP) if it is a weak sharp solution for (RUP). The *robust weak sharp solution set* of (UP) is given by

$$\tilde{S} := \left\{ \hat{x} \in K : \exists \eta > 0 \text{ s.t. } \max_{u \in \mathcal{U}} f(y, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(y), \forall y \in K \right\}.$$

Throughout the paper, we assume that $\tilde{S}$ is nonempty.

Remark 1. It is worthwhile to be noted that every robust weak sharp solution for (UP) is a robust optimal solution. In general, the reverse implication need not to be valid.

3. Characterizations of robust weak sharp solutions. In this section, we establish some optimality conditions for the robust weak sharp solution in convex uncertain optimization problems as well as obtain characterizations of the robust weak sharp solution sets for the considered problems. For any $\hat{x} \in \mathbb{R}^n$, we use the following notations:

$$\mathcal{U}(\hat{x}) := \left\{ \hat{u} \in \mathcal{U} : f(\hat{x}, \hat{u}) = \max_{u \in \mathcal{U}} f(\hat{x}, u) \right\},$$

and

$$\mathcal{V}(\hat{x}) := \left\{ \hat{v} \in \mathcal{V} : g(\hat{x}, \hat{v}) = \max_{v \in \mathcal{V}} g(\hat{x}, v) \right\}.$$

The following definition, which was introduced in [24], plays a vital role in determining characterizations of robust optimal weak sharp solution sets.

Definition 3.1. [24] The robust type subdifferential constraint qualification (RSCQ) is said to be satisfied at $\hat{x} \in K$ if

$$\partial\delta_K(\hat{x}) \subseteq \partial\delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial(\mu g)(\cdot, v)(\hat{x}).$$

Remark 2. In an excellent work, [24], Sun et. al. introduced the (RSCQ) and then obtained some characterizations of the the robust optimal solution set, for an uncertain convex optimization problem.

Although it has been used as a guideline for dealing with the (UP), our attention is paid to characterizing the sets containing the robust weak sharp solutions of such problem. Furthermore, the presence of the term $d_{\tilde{K}}(x)$ in this paper has led us to deal with some different tools and methods from those in work of Sun et.al.

The following theorem presents that the robust type subdifferential constraint qualification (RSCQ) defined in Definition 3.1 is fulfilled if and only if optimality conditions for a robust weak sharp solution of (UP) are satisfied.

Theorem 3.2. Let $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ satisfy the following properties :

- (i) for any $u \in \mathcal{U}$ and $v \in \mathcal{V}$, $f(\cdot, u)$ is convex and continuous as well as $g(\cdot, v)$ is D -convex on $\mathbb{R}^n$;
- (ii) for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ is concave on $\mathcal{U}$ and $g(x, \cdot)$ is D -concave on $\mathcal{V}$.

Then, the following statements are equivalent:

- (a) The (RSCQ) is fulfilled at $\hat{x} \in K$;
- (b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp solution of (UP) if and only if there exists a positive constant η such that

$$\begin{aligned} & N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n} \\ & \subseteq \bigcup_{u \in \mathcal{U}(\hat{x})} \partial f(\cdot, u)(\hat{x}) + \partial\delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}). \end{aligned} \quad (2)$$

Proof. (a) $\Rightarrow$ (b) Assume that the (RSCQ) is satisfied at $\hat{x} \in K$. Let $\hat{x}$ be a robust weak sharp solution of (UP). Consequently, there exists $\eta > 0$ such that

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(x). \quad (3)$$

By (3), we obtain that for all $x \in K$,

$$\begin{aligned} \max_{u \in \mathcal{U}} f(x, u) + \delta_K(x) - \eta d_{\tilde{K}}(x) &\geq \max_{u \in \mathcal{U}} f(\hat{x}, u) \\ &= \max_{u \in \mathcal{U}} f(\hat{x}, u) + \delta_K(\hat{x}) - \eta d_{\tilde{K}}(\hat{x}), \end{aligned}$$

thereby implying that, for all $\xi_d \in \partial \eta d_{\tilde{K}}(x)$,

$$\begin{aligned} &\left(\max_{u \in \mathcal{U}} f(\cdot, u) + \delta_K \right)(x) - \left(\max_{u \in \mathcal{U}} f(\cdot, u) + \delta_K \right)(\hat{x}) \\ &\geq \eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x}) \\ &\geq \langle \xi_d, x - \hat{x} \rangle. \end{aligned}$$

Thus, $\xi_d \in \partial (\max_{u \in \mathcal{U}} f(\cdot, u) + \delta_K)(\hat{x})$. Hence,

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial \left(\max_{u \in \mathcal{U}} f(\cdot, u) + \delta_K \right)(\hat{x}).$$

As $\max_{u \in \mathcal{U}} f(\cdot, u)$ is continuous on $\mathbb{R}^n$ and δ_K is proper lower semicontinuous convex on $\mathbb{R}^n$, we have

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial(\max_{u \in \mathcal{U}} f(\cdot, u))(\hat{x}) + \partial \delta_K(\hat{x}).$$

It can be noted that $\partial d_{\tilde{K}}(x) = N_{\tilde{K}}(x) \cap \mathcal{B}_{\mathbb{R}^n}$. Since (RSCQ) is satisfied at $\hat{x}$, we have the following:

$$\begin{aligned} N_{\tilde{K}}(x) \cap \mathcal{B}_{\mathbb{R}^n} &= \partial(\eta d_{\tilde{K}})(\hat{x}) \\ &\subseteq \bigcup_{u \in \mathcal{U}(\hat{x})} \partial f(\cdot, u)(\hat{x}) + \partial \delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial(\mu g)(\cdot, v)(\hat{x}), \end{aligned}$$

which implies that (2) holds.

Conversely, assume that there is a positive number η such that (2) holds. Since $N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$ always contains 0, it is a nonempty set and so is $\bigcap_{\varepsilon > 0} \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$. Thus, for any $\varepsilon \geq 0$, $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \neq \emptyset$. Let $\varepsilon > 0$ be arbitrary and let $\xi \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$. Then for any $x \in K$,

$$\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x}) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \quad (4)$$

Note that $0 \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$. It follows that

$$\eta d_{\tilde{K}}(\hat{x}) \leq \inf_{x \in \mathbb{R}^n} \eta d_{\tilde{K}}(x) + \varepsilon \leq \inf_{x \in K} \eta d_{\tilde{K}}(x) + \varepsilon.$$

Above inequality and (4) imply that

$$0 \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \quad (5)$$

Simultaneously, there exist $\hat{u} \in \mathcal{U}(\hat{x})$, $\hat{\mu} \in D^*$, $\hat{v} \in \mathcal{V}(\hat{x})$

$\xi_f \in \partial f(\cdot, \hat{u})(\hat{x})$, $\xi_\delta \in \partial \delta_C(\hat{x})$, and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$ such that

$$\xi_f + \xi_\delta + \xi_{\hat{\mu}g} = 0, \quad (6)$$

and for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} f(x, \hat{u}) - f(\hat{x}, \hat{u}) &\geq \langle \xi_f, x - \hat{x} \rangle, \\ \delta_C(x) - \delta_C(\hat{x}) &\geq \langle \xi_\delta, x - \hat{x} \rangle, \text{ and} \\ (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) &\geq \langle \xi_{\hat{\mu}g}, x - \hat{x} \rangle. \end{aligned}$$

Adding these above inequalities implies that for each $x \in K$

$$f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle 0, x - \hat{x} \rangle = 0.$$

Since $\hat{u}$ belongs to $\mathcal{U}(\hat{x})$, for each $x \in K$, above inequality becomes

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq 0.$$

This along with $(\hat{\mu}g)(x, \hat{v}) \leq 0$, $(\hat{\mu}g)(\hat{x}, \hat{v}) = 0$, and (6) imply

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq 0, \quad (7)$$

for all $x \in K$. Observe that, combining inequalities (5) and (7) leads to

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon, \quad \forall x \in K.$$

This means $\xi \in \partial_\varepsilon(\max_{u \in \mathcal{U}} f(\cdot, u))(\hat{x})$, and so $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial_\varepsilon(\max_{u \in \mathcal{U}} f(\cdot, u))(\hat{x})$. Since the inclusion holds for arbitrary $\varepsilon \geq 0$, it follows from the Lemma 2.1 that $\hat{x}$ is a minimizer of the DC problem: $\inf_{x \in \mathbb{R}^n} \{\max_{u \in \mathcal{U}} f(x, u) - \eta d_{\tilde{K}}(x)\}$ and hence for any $x \in K$

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) - (\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x})) \geq 0.$$

Therefore, for any $x \in K$,

$$\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(x).$$

This means $\hat{x}$ is a robust weak sharp solution of (UP).

(b) $\Rightarrow$ (a) Let $\xi_\delta \in \partial \delta_K(\hat{x})$ be given. Then, we have

$$0 = \delta_K(x) - \delta_K(\hat{x}) \geq \langle \xi_\delta, x - \hat{x} \rangle$$

holds for all $x \in K$. Let $\bar{\eta} > 0$ be given, and then, set $f(x, u) := -\langle \xi_\delta, x \rangle + \bar{\eta} d_{\tilde{K}}(x)$. Thus, for any $x \in K$,

$$\begin{aligned} \max_{u \in \mathcal{U}} f(x, u) - \bar{\eta} d_{\tilde{K}}(x) &= -\langle \xi_\delta, x \rangle \\ &\geq -\langle \xi_\delta, \hat{x} \rangle + \bar{\eta} d_{\tilde{K}}(\hat{x}) \\ &= \max_{u \in \mathcal{U}} f(\hat{x}, u). \end{aligned}$$

Thus, $\hat{x}$ is a robust weak sharp solution of (UP). By hypothesis, there is $\eta := \bar{\eta}$ such that (2) is fulfilled. Since for any $u \in \mathcal{U}$, $\partial f(\cdot, u)(\hat{x}) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x})$, we obtain that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u} \in \mathcal{U}(\hat{x})$, $\hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$x^* \in \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x}) + \partial \delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

As $0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, we obtain

$$\xi_\delta \in \partial \delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

It follows that

$$\xi_\delta \in \partial \delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}),$$

and so we get the desired inclusion. Therefore, the proof is complete. $\square$

Remark 3. In [27], the necessary conditions for weak sharp minima in cone constrained optimization problems, which can be captured by weak sharp minima in cone constrained robust optimization problems, were established by means of upper Studniarski or Dini directional derivatives. With the result in Theorem 3.2, the mentioned necessary conditions are established by an alternative method different from the referred work.

The following result is established easily by means of the basic concepts of variational analysis.

Corollary 1. Let $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^p$ satisfying the following properties:

1. for any $u \in \mathcal{U}$, and $v \in \mathcal{V}$, $f(\cdot, u)$ is convex and continuous as well as $g(\cdot, v)$ is D -convex on $\mathbb{R}^n$;
2. for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ is concave on $\mathcal{U}$ and $g(x, \cdot)$ is D -concave on $\mathcal{V}$, respectively.

The following two below statements are equivalent:

- (a) The (RSCQ) is fulfilled at $\hat{x} \in K$;
- (b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp solution of (UP) if and only if there exists a real number $\eta > 0$ such that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u} \in \mathcal{U}(\hat{x})$, $\hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ yield

$$x^* \in \partial f(\cdot, \hat{u}) + \partial \delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}), \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0. \quad (8)$$

The result, which deals with a special case that $\mathcal{U}$ and $\mathcal{V}$ are singleton sets, can be obtained easily and be presented as follows:

Corollary 2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and continuous and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is D -convex. The following statements are equivalent:

1. The (SCQ) is fulfilled at $\hat{x} \in K$
2. $\hat{x} \in \mathbb{R}^n$ is a weak sharp solution of (P) if and only there exists a real number $\eta > 0$ such that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{\mu} \in D^*$ such that

$$x^* \in \partial f(\hat{x}) + \partial \delta_C(\hat{x}) + \partial(\hat{\mu}g)(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}) = 0. \quad (9)$$

Next, a characterization of robust weak sharp solution sets in terms of a given robust weak sharp solution point of our considered problem is also illustrated in this section. In order to present the mentioned characterization, we first prove that the Lagrangian-type function associated with fixed Lagrange multiplier and uncertainty parameters corresponding to a robust weak sharp solution is constant on the robust weak sharp solution set under suitable conditions. In what follows, let $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $\mu \in D^*$. The Lagrangian-type function $\mathcal{L}(\cdot, \mu, u, v)$ is given by

$$\mathcal{L}(x, \mu, u, v) = f(x, u) + (\mu g)(x, v), \quad \forall x \in \mathbb{R}^n.$$

Now, we denote by

$$\tilde{S} := \left\{ x \in K : \exists \eta > 0 \text{ s.t. } \max_{u \in \mathcal{U}} f(y, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta d_{\tilde{K}}(y), \forall y \in K \right\}.$$

the robust weak sharp solution set of (UP), and then we prove that the Lagrangian-type function associated with a Lagrange multiplier corresponding to a robust weak sharp solution is constant on the robust weak sharp solution set.

Theorem 3.3. Let $\hat{x} \in \tilde{S}$ be given. Suppose that the (RSCQ) is satisfied at $\hat{x}$. Then, there exist uncertainty parameters $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$, and Lagrange multiplier $\hat{\mu} \in D^*$, such that for any $x \in \tilde{S}$,

$$(\hat{\mu}g)(x, \hat{v}) = 0, \hat{u} \in \mathcal{U}(x), \text{ and } \mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) \text{ is a constant on } \tilde{S}.$$

Proof. Since $\hat{x} \in \tilde{S}$ with the real number $\eta_1 > 0$ and the (RSCQ) is satisfied at this point $\hat{x}$, by Theorem 3.2 we have that (2) holds for $\eta := \eta_1$. Clearly $N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$ contains 0, then it is nonempty and so is any $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ where $\varepsilon > 0$. Let $\varepsilon > 0$ and $x^* \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ be arbitrary. Again, we obtain that there exist $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that (2) is fulfilled. Let $x \in \tilde{S}$ be arbitrary, then we have

$$f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle x^*, x - \hat{x} \rangle,$$

and so

$$f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle x^*, x - \hat{x} \rangle - \varepsilon. \quad (10)$$

Since $f(\cdot, u)$ and $g(\cdot, v)$ are convex, for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$ respectively,

$$x^* \in \partial_\varepsilon(f(\cdot, u) + \lambda g(\cdot, v))(\hat{x}).$$

Therefore, we obtain $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial_\varepsilon(f(\cdot, u) + \lambda g(\cdot, v))(\hat{x})$, and so

$$f(x, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - \eta d_{\tilde{K}}(x) \geq f(\hat{x}, \hat{u}) = \max_{u \in \mathcal{U}} f(\hat{x}, u). \quad (11)$$

Note that, as $x \in \tilde{S}$, there exists $\eta_2 > 0$ such that

$$\max_{u \in \mathcal{U}} f(y, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta_2 d_{\tilde{K}}(y), \forall y \in \tilde{S},$$

and so

$$\max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta_2 d_{\tilde{K}}(\hat{x}) = \max_{u \in \mathcal{U}} f(x, u). \quad (12)$$

From $\hat{\mu} \in D^*, g(x, \hat{v}) \in -D$, and (11), it is not hard to see that

$$(\hat{\mu}g)(x, \hat{v}) = 0. \quad (13)$$

Then, by (11) and the positivity of $\eta d_{\tilde{K}}(x)$, we see that

$$\max_{u \in \mathcal{U}} f(x, u) \geq f(x, \hat{u}) \geq \max_{u \in \mathcal{U}} f(\hat{x}, u) + \eta d_{\tilde{K}}(x) \geq \max_{u \in \mathcal{U}} f(\hat{x}, u), \quad (14)$$

which together with (12) leads to

$$\max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}). \quad (15)$$

It follows that $\mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) = f(\hat{x}, \hat{u})$, which is constant. Since $x \in \tilde{S}$ was arbitrary, we finish the proof. $\square$

Theorem 3.4. For the problem (UP), let $\tilde{S}$ be the robust weak sharp solutions set of (UP) and $\hat{x}$ belongs to it. Suppose that the (RSCQ) is satisfied at $\hat{x} \in \tilde{S}$. Then, there exist uncertain parameters $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and Lagrange multiplier $\hat{\mu} \in D^*$ such that

$$\begin{aligned} \tilde{S} = & \left\{ x \in K : \exists \eta > 0, \exists \xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(\hat{x}) \cap \partial_\varepsilon f(\cdot, \hat{u})(x), \exists \varepsilon > \eta d_{\tilde{K}}(x), \right. \\ & \left. \langle \xi_f, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x), (\hat{\mu}g)(x, \hat{v}) = 0, \max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}) \right\}. \end{aligned} \quad (16)$$

Proof. ($\subseteq$) Let $x \in \tilde{S}$ be given. Then there exists $\eta > 0$ such that (2) holds. Hence, there exist $\xi_f \in \partial f(\cdot, \hat{u})(x)$, $\xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$ such that

$$0 = \xi_f + \xi_\delta + \xi_{\hat{\mu}g} \text{ since } 0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}, \quad (17)$$

and

$$(\hat{\mu}g)(\hat{x}, \hat{v}) = 0. \quad (18)$$

Since $\xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$,

$$\delta_C(x) - \delta_C(\hat{x}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle \xi_\delta + \xi_{\hat{\mu}g}, x - \hat{x} \rangle. \quad (19)$$

By the same fashion in the proof of Theorem 3.2, we have

$$(\hat{\mu}g)(x, \hat{v}) = (\hat{\mu}g)(\hat{x}, \hat{v}) = 0,$$

and

$$\max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}).$$

Therefore, it follows from (19) that

$$0 \geq \langle \xi_\delta + \xi_{\hat{\mu}g}, x - \hat{x} \rangle,$$

and so by (17), we obtain

$$\eta d_{\tilde{K}}(x) \geq \langle \xi_f, \hat{x} - x \rangle.$$

Simultaneously, since $\xi_f \in \partial f(\cdot, \hat{u})(\hat{x})$, we have

$$\langle \xi_f, \hat{x} - x \rangle \geq f(\hat{x}, \hat{u}) - f(x, \hat{u}).$$

By (15) in the proof of Theorem 3.2, we obtain

$$\langle \xi_f, \hat{x} - x \rangle \geq \max_{u \in \mathcal{U}} f(\hat{x}, \hat{u}) - \max_{u \in \mathcal{U}} f(x, u) \geq 0 = \eta d_{\tilde{K}}(x). \quad (20)$$

Hence, we have that $\langle \xi_f, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x)$. Now, we prove that for $\xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(x)$, there is an $\varepsilon > \eta d_{\tilde{K}}(x) \geq 0$. In fact, we can show that for any $y \in \mathbb{R}^n$,

$$\langle \xi_f, y - x \rangle = \langle \xi_f, y - \hat{x} \rangle + \langle \xi_f, \hat{x} - x \rangle \leq \langle \xi_f, y - \hat{x} \rangle$$

as $\langle \xi_f, \hat{x} - x \rangle \leq 0$. Since $\xi_f \in \partial f(\cdot, \hat{u})(\hat{x})$ and $f(x, \hat{u}) = f(\hat{x}, \hat{u})$ by (14) and (12),

$$\langle \xi_f, y - x \rangle \leq f(y, \hat{u}) - f(\hat{x}, \hat{u}) = f(y, \hat{u}) - f(x, \hat{u}),$$

which means $\xi_f \in \partial f(\cdot, \hat{u})(x)$.

($\supseteq$) Let

$$x \in \left\{ x \in K : \exists \eta > 0, \exists \xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(\hat{x}) \cap \partial_\varepsilon f(\cdot, \hat{u})(x), \exists \varepsilon > \eta d_{\tilde{K}}(x), \right. \\ \left. \langle \xi_f, x - \hat{x} \rangle = \eta d_{\tilde{K}}(x), (\mu g)(x, \hat{v}) = 0, \max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}) \right\}.$$

Since $\hat{x} \in \tilde{S}$, it is clear that $\eta d_{\tilde{K}}(\hat{x}) = 0$. By assumption and $\xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(x)$ for some $\varepsilon > 0$, we get

$$\begin{aligned} -\eta d_{\tilde{K}}(\hat{x}) &= 0 \\ &= \langle \xi_f, \hat{x} - x \rangle - \eta d_{\tilde{K}}(x) \\ &\leq f(\hat{x}, \hat{u}) - f(x, \hat{u}) + \varepsilon - \eta d_{\tilde{K}}(x) \\ &= f(\hat{x}, \hat{u}) - f(x, \hat{u}) - \eta d_{\tilde{K}}(x) + \eta d_{\tilde{K}}(x) \\ &= f(\hat{x}, \hat{u}) - f(x, \hat{u}). \end{aligned} \quad (21)$$

Therefore, we obtain

$$\max_{u \in \mathcal{U}} f(x, u) \leq \max_{u \in \mathcal{U}} f(x, u) + \eta d_{\tilde{K}}(\hat{x}).$$

Since $\hat{x} \in \tilde{S}$ and $x \in K$, the conclusion that $x \in \tilde{S}$ is satisfied. $\square$

In the case that $D := \mathbb{R}_+$, which is a closed convex (and pointed) cone in $\mathbb{R}$, the problem is reduced to be an inequality constrain problem. Suppose that $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ is a function such that $f(\cdot, u)$ is convex for any $u \in \mathcal{U}$ and $f(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$ as well as $g : \mathbb{R}^n \times \mathcal{V} \rightarrow \mathbb{R}$ is a function such that $g(\cdot, v)$ is convex for any $v \in \mathcal{V}$ and $g(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$. Here, the problem (UP) is represented as

$$\min \{f(x, u) : g(x, v) \leq 0, \forall v \in \mathcal{V}\},$$

and its robust counter part is

$$\min \left\{ \max_{u \in \mathcal{U}} f(x, u) : g(x, v) \leq 0, \forall v \in \mathcal{V} \right\}.$$

In this case, we can see that *robust feasible set* K is denoted by

$$K := \{x \in \mathbb{R}^n : g(x, v) \leq 0, \forall v \in \mathcal{V}\}.$$

Corollary 3. Let $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$ satisfying the following properties:

1. for any $u \in \mathcal{U}$, and $v \in \mathcal{V}$, $f(\cdot, u)$ is convex and continuous as well as $g(\cdot, v)$ is convex on $\mathbb{R}^n$;
2. for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ and $g(x, \cdot)$ are concave on $\mathcal{U}$ and $\mathcal{V}$, respectively.

The following statements are equivalent:

- (a) The (RSCQ) is fulfilled at $\hat{x} \in K$;
- (b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp solution of (UP) if and only if there exists a real number $\eta > 0$ such that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u} \in \mathcal{U}(\hat{x})$, $\hat{v} \in \mathcal{V}$ and $\hat{\mu} \geq 0$ yield

$$x^* \in \partial f(\cdot, \hat{u})(\hat{x}) + \partial \delta_C(\hat{x}) + \partial(\hat{\mu}g)(\cdot, \hat{v})(\hat{x}), \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

Corollary 4. Let $\hat{x} \in \tilde{S}$ be given. Suppose that the (RSCQ) is satisfied at $\hat{x}$. Then, there exist uncertain parameters $\hat{u} \in \mathcal{U}$, $\hat{v} \in \mathcal{V}$, and Lagrange multiplier $\hat{\mu} \geq 0$ such that for any $x \in \tilde{S}$,

$$(\hat{\mu}g)(x, \hat{v}) = 0, \hat{u} \in \mathcal{U}(x), \text{ and } \mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) \text{ is constant on } \tilde{S}.$$

Corollary 5. For the problem (UP), let $\tilde{S}$ be the robust weak sharp solutions set of (UP) and $\hat{x}$ belongs to it. Suppose that the (RSCQ) is satisfied at $\hat{x} \in \tilde{S}$. Then, there exist uncertain parameters $\hat{u} \in \mathcal{U}$, $\hat{v} \in \mathcal{V}$ and Lagrange multiplier $\hat{\mu} \geq 0$ such that

$$\begin{aligned} \tilde{S} = & \left\{ x \in K : \exists \eta > 0, \exists a \in \partial_\varepsilon f(\cdot, \hat{u})(\hat{x}) \cap \partial_\varepsilon f(\cdot, \hat{u})(x), \exists \varepsilon > \eta d_{\tilde{K}}(x), \right. \\ & \left. \langle a, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x), (\hat{\mu}g)(x, \hat{v}) = 0, \max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}) \right\}. \end{aligned} \quad (22)$$

4. Applications to multi-objective optimization. In this section, in order to apply our general results of the previous section, we investigate the class multi-objective optimization problem

$$\min_{x \in \mathbb{R}^n} \{(f_1(x), f_2(x), \dots, f_l(x)) : x \in C, g(x) \in -D\}, \quad (\text{MP})$$

where where $C \subseteq \mathbb{R}^n$ is a nonempty convex set, $D \subseteq \mathbb{R}^m$, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function for any $i \in I$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a D -convex function. The feasible set of (MP) is defined by

$$K_0 := \{x \in C : g(x) \in -D\}.$$

The problem (MP) in the face of data uncertainty both in the objective and constraint can be captured by the following multi-objective optimization problem

$$\min_{x \in \mathbb{R}^n} \{(f_1(x, u_1), f_2(x, u_2), \dots, f_l(x, u_l)) : x \in C, g(x, v) \in -D\}, \quad (\text{UMP})$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$, $i = 1, \dots, l$, and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$. u_i , $i = 1, \dots, l$, and v are uncertain parameters, and they belong to the corresponding convex and compact uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^p$, and $\mathcal{V} \subseteq \mathbb{R}^q$. Suppose that for any $u_i \in \mathcal{U}_i$, $i \in I$, the function $f_i(\cdot, u_i)$ is convex on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, $f_i(x, \cdot)$ is concave on $\mathcal{U}_i$, $i \in I$. Besides, suppose that for any $v \in \mathcal{V}$, the function $g(\cdot, v)$ is D -convex on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, $g(x, \cdot)$ is D -concave on $\mathcal{V}$.

Similarly, we obtain some characterizations of the robust weak sharp weakly efficient solutions of (UMP) by using investigation of its robust (worst case) counterpart:

$$\min_{x \in \mathbb{R}^n} \left\{ \left(\max_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \dots, \max_{u_l \in \mathcal{U}_l} f_l(x, u_l) \right) : x \in C, g(x, v) \in -D \right\} \quad (\text{RUMP})$$

where the *robust feasible set* of (UMP) is also defined by

$$K := \{x \in C : g(x, v) \in -D, \}.$$

Now, we recall the following concepts of robust weak sharp weakly efficient solutions in multi-objective optimization, which can be found in the literature; see e.g., [21] and [12].

Definition 4.1. [21] A point $\hat{x} \in K$ is said to be a *weakly robust efficient solution* of for (UMP) if it is a weakly efficient solution for (RUMP) i.e., there does not exist $x \in K$ such that

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) < \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i), \text{ for all } i \in I.$$

Definition 4.2. [12] A point feasible element $\hat{x}$ is said to be a weak sharp efficient solution for (MP) if there exists a real number $\eta > 0$ such that for any $x \in K$

$$\max_{1 \leq k \leq l} \{f_k(x) - f_k(\hat{x})\} \geq \eta d_{\hat{K}}(x)$$

where $\hat{K} := \{x \in K : f(x) = f(\hat{x})\}$.

Now, we introduce a new concept of solution, which related to the sharpness, namely the robust weak sharp weakly efficient solutions.

Definition 4.3. A point $\hat{x} \in K$ is said to be a *weak sharp weakly efficient solution* for (RUMP) if and only if there exist a real number $\eta > 0$ such that there does not exist $y \in K \setminus \{\hat{x}\}$ satisfying

$$\max_{u_i \in \mathcal{U}_i} f_i(y, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < \eta d_{\hat{K}}(y), \text{ for all } i \in I,$$

or equivalently, for all $x \in K$

$$\max_{i \in I} \left\{ \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_{\hat{K}}(x)$$

where $\tilde{K} := \left\{ x \in K : \max_{u \in \mathcal{U}} f_i(x, u) = \max_{u \in \mathcal{U}} f_i(\hat{x}, u), i \in I \right\}$.

Definition 4.4. A point $\hat{x} \in K$ is said to be a *robust weak sharp weakly efficient solution* for (UMP) if it is a weakly weak sharp weakly efficient solution for (RUMP).

The following lemma is useful for establishing our results in this section.

Lemma 4.5. [30] Let $\mathcal{U}_1, \dots, \mathcal{U}_l$ be nonempty convex and compact sets of $\mathbb{R}^p$ and for any $u_i \in \mathcal{U}_i, i \in I$, the function $f_i(\cdot, u_i) : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex as well as for any $x \in \mathbb{R}^n, f_i(x, \cdot) : \mathcal{U}_i \rightarrow \mathbb{R}$ be concave where $i \in I$. Then, for any $\lambda_i \geq 0, i \in I$,

$$\partial \left(\max_{u \in \prod_{i \in I} \mathcal{U}_i(\hat{x})} \sum_{i \in I} \lambda_i f_i(\cdot, u_i) \right) (\hat{x}) = \bigcup_{u \in \prod_{i \in I} \mathcal{U}_i(\hat{x})} \sum_{i \in I} \lambda_i (f_i(\cdot, u_i)) (\hat{x}),$$

where

$$\begin{aligned} \prod_{i \in I} \mathcal{U}_i(\hat{x}) := & \left\{ (\hat{u}_1, \dots, \hat{u}_i) \in \prod_{i \in I} \mathcal{U}_i \right. \\ & \left. : \sum_{i \in I} \lambda_i f_i(\hat{x}, \hat{u}_i) = \max_{u \in \prod_{i \in I} \mathcal{U}_i} \sum_{i \in I} \lambda_i f_i(\hat{x}, u_i) \right\} \end{aligned}$$

Now, by using the similar methods of Section 3, we can characterize the corresponding robust weak sharp weakly efficient solutions of (UMP).

Theorem 4.6. Let $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^l$ and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$ satisfying the following properties:

1. for any $u_i \in \mathcal{U}_i, i \in I$ and $v_j \in \mathcal{V}_j, j \in J, f_i(\cdot, u_i)$ is convex and continuous as well as $g(\cdot, v)$ is D -convex on $\mathbb{R}^n$;
2. for any $x \in \mathbb{R}^n, f_i(x, \cdot)$ is concave on $\mathcal{U}_i, i \in I$ and $g(x, \cdot)$ is D -concave on $\mathcal{V}$.

Then, the following statements are equivalent:

- (a) The (RSCQ) is fulfilled at $\hat{x} \in K$;
- (b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp weakly efficient solutions of (UMP) if and only if there exists $\eta > 0$ such that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u}_i \in \mathcal{U}_i(\hat{x}), \sigma_i \geq 0, i \in I$, not all zero, $\hat{v} \in \mathcal{V}$, and $\hat{\mu} \geq 0$ such that

$$0 \in \{-x^*\} + \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) + \partial \delta_C(\hat{x}) + \partial ((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \quad (23)$$

$$(\hat{\mu}_j g_j)(\hat{x}, \hat{v}_j) = 0, \quad (24)$$

and

$$\sigma_i f_i(\hat{x}, \hat{u}_i) = \sigma_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i), i \in I. \quad (25)$$

Proof. (a) $\Rightarrow$ (b) Assume that the (RSCQ) is satisfied at $\hat{x} \in \mathbb{R}^n$. Let $\hat{x}$ be a robust weak sharp weakly efficient solutions of (UMP) i.e., there exists $\eta > 0$ such that there does not exist $y \in K \setminus \{\hat{x}\}$ satisfying

$$\max_{u_i \in \mathcal{U}_i} f_i(y, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) < \eta d_{\tilde{K}}(y), \text{ for all } i \in I,$$

or equivalently, for any $x \in K$,

$$\max_{i \in I} \left\{ \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_{\tilde{K}}(x). \quad (26)$$

By (26), there is $s \in I$ such that for all $x \in K$,

$$\begin{aligned} \max_{u_s \in \mathcal{U}_s} f_s(x, u_s) + \delta_K(x) - \eta d_{\tilde{K}}(x) &\geq \max_{u_s \in \mathcal{U}_s} f(\hat{x}, u_s) \\ &= \max_{u_s \in \mathcal{U}_s} f_s(\hat{x}, u_s) + \delta_K(\hat{x}) - \eta d_{\tilde{K}}(\hat{x}). \end{aligned} \quad (27)$$

Besides, according to (27), we follow the techniques used in Theorem 3.2 and obtain that for any $\xi \in \partial \eta d_{\tilde{K}}(x)$,

$$\begin{aligned} &\langle \xi, x - \hat{x} \rangle \\ &\leq \max_{u_s \in \mathcal{U}_s} f_s(x, u_s) + \delta_K(x) - \max_{u_s \in \mathcal{U}_s} f_s(\hat{x}, u_s) - \delta_K(\hat{x}). \end{aligned} \quad (28)$$

Therefore,

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial \left(\max_{u_s \in \mathcal{U}_s} f_s(\cdot, u_s) + \delta_K \right)(\hat{x}), \quad (29)$$

Note that the right hand side term of above inclusion is in the subdifferential of the max function:

$$\phi(x) = \max_{i \in I} \phi_i(x) := \max_{i \in I} \left(\max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) + \delta_K \right)(x).$$

Due to the well-known fact, subdifferential of maximum of functions at x is the convex hull of the union of subdifferentials of the active functions at x , the inclusion (29) becomes

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \text{co}(\cup \{ \partial \phi_i(\hat{x}) : \phi_i(\hat{x}) = \phi(\hat{x}) \}),$$

thereby

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \sum_{i \in I(\hat{x})} \sigma_i \partial \phi_i(\hat{x}),$$

where $\sigma_i \geq 0, i \in I(\hat{x})$ with $\sum_{i \in I(\hat{x})} \sigma_i = 1$ and $I(\hat{x}) := \{k \in I : \phi_k(\hat{x}) = \phi(\hat{x})\}$.

Further, setting $\hat{\sigma}_i = \sigma, i \in I(\hat{x})$, and otherwise equals to 0 leads to

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \sum_{i \in I} \hat{\sigma}_i \partial \phi_i(\hat{x}).$$

By the definition of $\phi_i, i \in I$, the continuity of $\max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i), i \in I$ and the lower semicontinuity and convexity of δ_K , we have

$$\partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq \sum_{i \in I} \hat{\sigma}_i \partial \left(\max_{u_i \in \mathcal{U}_i} f(\cdot, u_i) \right)(\hat{x}) + \sum_{i \in I} \hat{\sigma}_i (\partial \delta_K(\hat{x})).$$

It follows from Lemma 4.5 and the hypothesis such (RSCQ) is satisfied at $\hat{x} \in K$ that

$$\begin{aligned} \partial(\eta d_{\tilde{K}})(\hat{x}) &\subseteq \bigcup_{u \in \prod_{i=1}^l \mathcal{U}_i(\hat{x})} \sum_{i \in I} \hat{\sigma}_i \partial f_i(\cdot, u_i)(\hat{x}) + \sum_{i \in I} \hat{\sigma}_i (\partial \delta_C(\hat{x})) \\ &\quad + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}). \end{aligned}$$

Because $\hat{\sigma}_i \geq 0, i = 1, 2, \dots, l$, all nonzero, thereby

$$\begin{aligned} \partial(\eta d_{\tilde{K}})(\hat{x}) \subseteq & \bigcup_{\substack{u=(u_i)_{i=1}^l, \\ u \in \prod_{i=1}^l \mathcal{U}_i(\hat{x})}} \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, u_i)(\hat{x})) + \partial \delta_C(\hat{x}) \\ & + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}). \end{aligned}$$

As $\partial d_{\tilde{K}}(x) = N_{\tilde{K}}(x) \cap \mathcal{B}_{\mathbb{R}^n}$, we obtain (23) as desired.

Conversely, assume that there is $\eta > 0$ such that (23)-(25) hold. Then, for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u} := (\hat{u}_1, \dots, \hat{u}_l) \in \prod_{i \in I} \mathcal{U}_i(\hat{x}), \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$\begin{aligned} x^* \in & \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) + \partial \delta_C(\hat{x}) + \partial((\hat{\mu} g)(\cdot, \hat{v}))(\hat{x}), \text{ and} \\ & (\hat{\mu} g)(\hat{x}, \hat{v}) = 0. \end{aligned} \quad (30)$$

Since $0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n} = \bigcap_{\varepsilon > 0} \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$, for each positive $\varepsilon, \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ is nonempty. Let $\varepsilon > 0$ and $\xi \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ be arbitrary, then for any $x \in K$

$$\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x}) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \quad (31)$$

Therefore, we obtain

$$\eta d_{\tilde{K}}(\hat{x}) \leq \inf_{x \in \mathbb{R}^n} \eta d_{\tilde{K}}(x) + \varepsilon \leq \inf_{x \in K} \eta d_{\tilde{K}}(x) + \varepsilon.$$

Above inequality and (31) imply that

$$0 \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \quad (32)$$

Further, since $0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, we have that there exist $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})), \xi_\delta \in \partial \delta_C(\hat{x})$, and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$ such that

$$\xi_f + \xi_\delta + \xi_{\hat{\mu}g} = 0. \quad (33)$$

As $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) = \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{x}), \xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$, we have

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) & \geq \langle \xi_f, x - \hat{x} \rangle, \\ \delta_C(x) - \delta_C(\hat{x}) & \geq \langle \xi_\delta, x - \hat{x} \rangle, \text{ and} \\ (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) & \geq \langle \xi_{\hat{\mu}g}, x - \hat{x} \rangle. \end{aligned}$$

Then, adding these inequalities yields

$$\begin{aligned} \langle 0, x - \hat{x} \rangle & \leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ & \quad + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}). \end{aligned}$$

Since $\hat{u}_i$ belongs to $\mathcal{U}_i(\hat{x})$, above inequality becomes the following one:

$$0 \leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \\ + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}).$$

This together with $(\hat{\mu}g)(x, \hat{v}) \leq 0$, $(\hat{\mu}g)(\hat{x}, \hat{v}) = 0$, and (33), for any $x \in K$,

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \langle 0, \cdot \rangle. \quad (34)$$

By summing (34) with (31), for any $x \in K$, we obtain

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon,$$

which means $\xi \in \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) \right) (\hat{x})$, and so $\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \subseteq$

$\partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) \right) (\hat{x})$. As $\varepsilon > 0$ was arbitrary, for each $x \in K$,

$$0 \leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \\ - (\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x})),$$

which is equivalent to the following inequality: for all $x \in K$

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_{\tilde{K}}(x) \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_{\tilde{K}}(\hat{x}).$$

It follows that

$$\sum_{i \in I} \hat{\sigma}_i \left(\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_{\tilde{K}}(x) \right) \geq \sum_{i \in I} \hat{\sigma}_i \left(\max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_{\tilde{K}}(\hat{x}) \right),$$

for any $x \in K$, which yields for any $i \in I$,

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_{\tilde{K}}(x) \geq \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_{\tilde{K}}(\hat{x}), \forall x \in K.$$

Therefore, for any $x \in K$

$$\max_{i \in I} \left\{ \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_{\tilde{K}}(x).$$

This means $\hat{x}$ is a robust weak sharp weakly efficient solutions of (UMP).

(b) $\Rightarrow$ (a) Let $\bar{\eta} > 0$ be given. Consider $f_i(x, u_i) := -\langle \xi_\delta, x \rangle + \bar{\eta} d_{\tilde{K}}(x)$, $i \in I$. Thus, for any $x \in K$,

$$\begin{aligned} \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \bar{\eta} d_{\tilde{K}}(x) &= -\langle \xi_\delta, x \rangle \\ &\geq -\langle \xi_\delta, \hat{x} \rangle + \bar{\eta} d_{\tilde{K}}(\hat{x}) \\ &= \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned}$$

Thus, $\hat{x}$ is a robust weak sharp weakly efficient solutions of (UMP). By hypothesis, there is $\eta := \bar{\eta}$ such that (23) is fulfilled. Since for any $u_i \in \mathcal{U}_i$, $\partial f_i(\cdot, u_i)(\hat{x}) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x})$, one has

$$\sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, u_i)(\hat{x})) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x}),$$

where $\hat{\sigma}_i \geq 0, i \in I$ and all nonzero. Thus, we obtain that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, there exist $\hat{u}_i \in \mathcal{U}_i(\hat{x}), \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$x^* \in \{-\xi_\delta\} + \partial(\eta d_{\tilde{K}})(\hat{x}) + \partial\delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

As $0 \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$, we obtain

$$\xi_\delta \in \partial\delta_C(\hat{x}) + \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

It follows that

$$\xi_\delta \in \partial\delta_C(\hat{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\hat{x}, v) = 0}} \partial((\mu g)(\cdot, v))(\hat{x}),$$

and so we get the desired inclusion. Therefore, the proof is complete. $\square$

Remark 4. (i) In [28] and [29], the authors presented the necessary condition for the local sharp efficiency for the semi-infinite vector optimization problem by using the different method with Theorem 4.6. In fact, they employed the exact sum rule for Fréchet subdifferentials to obtained their results.
(ii) In [31], the exact sum rule for Mordukhovich subdifferentials was used as a vital tool under some regularity and differentiability assumptions for establishing their results. This means Theorem 4.6 use the different method from the mentioned work.

Next, by using the similar methods of section 3, a characterization of robust weak sharp weakly efficient solution sets in terms of a given robust weak sharp weakly efficient solution point of the problem is also illustrated in this section. In order to present the mentioned characterization, we start by deriving constant Lagrangian-type property for robust weak sharp weakly efficient solution sets of (MP). In what follows, let $u = (u_1, \dots, u_l) \in \mathcal{U}_1 \times \dots \times \mathcal{U}_l, \sigma = (\sigma_1, \dots, \sigma_l) \in \mathbb{R}_+^l, v \in \mathcal{V}$ and $\mu \geq 0$. The Lagrangian-type function $\mathcal{L}(\cdot, \sigma, \mu, u, v)$ is given by

$$\mathcal{L}(x, \sigma, \mu, u, v) = \sum_{i \in I} \sigma_i f_i(x, u_i) + (\mu g)(x, v), \quad \forall x \in \mathbb{R}^n.$$

Theorem 4.7. Let $x \in \tilde{S}$ be given. Suppose that the (RSCQ) is fulfilled at $\hat{x}$. Then, there exist a positive valued vector $\hat{\sigma} := (\hat{\sigma}_1, \dots, \hat{\sigma}_l) \in \mathbb{R}_+^l, \hat{\sigma}_i, i \in I$ all nonzero, uncertain parameters $\hat{u} := (u_1, \dots, u_l) \in \mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_l, \hat{v} \in \mathcal{V}$, and Lagrange multiplier $\hat{\mu} \geq 0$ such that for any $x \in \tilde{S}$,

$$(\hat{\mu}g)(x, \hat{v}) = 0, \hat{u} \in \mathcal{U}(x), \text{ and } \mathcal{L}(x, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) \text{ is a constant on } \tilde{S}.$$

Proof. Since $\hat{x} \in \tilde{S}$ with the real number $\eta_1 > 0$ and the (RSCQ) is satisfied at this point $\hat{x}$, by Theorem 4.6, (23) holds for $\eta := \eta_1$. Since $N_{\tilde{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}$ is nonempty we can let $\varepsilon > 0$ be arbitrary and $x^* \in \partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x})$ be given. Besides, there exist $\hat{\sigma} \in \mathbb{R}_+^l$, all nonzero, $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that (23) is fulfilled. Let $x \in \tilde{S}$

be arbitrary. By the same fashion using in the proof of Theorem 3.2 we have

$$\begin{aligned} \langle x^*, x - \hat{x} \rangle &\leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ &\quad + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}), \end{aligned}$$

and so

$$\begin{aligned} \langle x^*, x - \hat{x} \rangle - \varepsilon &\leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ &\quad + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}), \end{aligned} \quad (35)$$

As $f_i(\cdot, u_i), i \in I$ and $g(\cdot, v)$ are convex, for any $u_i \in \mathcal{U}_i$ and $v \in \mathcal{V}$, we have $x^* \in \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i (f_i(\cdot, u_i) + \lambda g(\cdot, v)) \right) (\hat{x})$. Hence, one has

$$\partial_\varepsilon(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i (f_i(\cdot, u_i) + \lambda g(\cdot, v)) \right) (\hat{x}),$$

thereby

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) + (\hat{\mu}g)(x, \hat{v}) - \eta d_{\tilde{K}}(x) &\geq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned} \quad (36)$$

Note that, as $x \in \tilde{S}$, then there exists $\eta_2 > 0$ such that for all $y \in K$,

$$\max_{u_i \in \mathcal{U}_i} f_i(y, u_i) \geq \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\tilde{K}}(y),$$

which implies

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(y, u_i) &\geq \sum_{i \in I} \hat{\sigma}_i \left(\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\tilde{K}}(y) \right) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\tilde{K}}(y) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i), \end{aligned}$$

for all $y \in \tilde{S}$. Since $\hat{x} \in \tilde{S}$,

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i). \quad (37)$$

From $\hat{\mu} \geq 0, g(x, \hat{v}) \leq 0$, and (36), it is not hard to see that

$$(\hat{\mu}g)(x, \hat{v}) = 0. \quad (38)$$

Moreover, by (36) and the positivity of $\eta d_{\tilde{K}}(x)$, we see that

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) &\geq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) \\ &\geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + \eta d_{\tilde{K}}(x) \\ &\geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned} \quad (39)$$

This together with (38) leads to

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i). \quad (40)$$

Thus, $\mathcal{L}(\cdot, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v})$ is constant on $\tilde{S}$ as follows:

$$\begin{aligned} \mathcal{L}(x, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) &= \sum_{i \in I} \hat{\sigma}_i f_i(x, u_i) + (\hat{\mu}g)(x, \hat{v}) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + (\hat{\mu}g)(x, \hat{v}) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + (\hat{\mu}g)(x, \hat{v}) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned}$$

This completes the proof. $\square$

Theorem 4.8. For the problem (UMP), let $\tilde{S}$ be the robust weak sharp weakly efficient solution set of (UMP) and $\hat{x} \in \tilde{S}$. Suppose that the (RSCQ) is fulfilled at $\hat{x} \in \tilde{S}$. Then, there exist $\hat{\sigma}_i \geq 0, i \in I$, all non zero, $\hat{u} := (\hat{u}_1, \dots, \hat{u}_l) \in \mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_l, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \geq 0$ such that

$$\begin{aligned} \tilde{S} = \left\{ x \in K : \exists \eta > 0, \exists a \in \bigcap_{y \in \{x, \hat{x}\}} \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{y}), \right. \\ \left. \exists \varepsilon > \eta d_{\bar{K}}(x), \langle a, \hat{x} - x \rangle = \eta d_{\bar{K}}(x), (\mu g)(x, \hat{v}) = 0, \right. \\ \left. \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = f_i(x, \hat{u}_i), i \in I \right\}. \end{aligned}$$

Proof. ($\subseteq$) Let $x \in \tilde{S}$ be given. Then there exists $\eta > 0$ such that (23) holds. Thus, there exist $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \geq 0$ such that (23) is fulfilled. Hence, we have that there exist $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(x)), \xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$ such that

$$0 = \xi_f + \xi_\delta + \xi_{\hat{\mu}g}, \text{ since } 0 \in N_{\bar{K}}(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n}, \quad (41)$$

and

$$(\hat{\mu}g)(\hat{x}, \hat{v}) = 0. \quad (42)$$

Since $\xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x})$,

$$\delta_C(x) - \delta_C(\hat{x}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq \langle \xi_\delta + \xi_{\hat{\mu}g}, x - \hat{x} \rangle. \quad (43)$$

By the same fashion in the proof of Theorem 4.6, we have

$$(\hat{\mu}g)(x, \hat{v}) = (\hat{\mu}g)(\hat{x}, \hat{v}) = 0,$$

and

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i).$$

Therefore, it follows from (43) that

$$\eta d_{\bar{K}}(x) = 0 \geq \langle b + c, x - \hat{x} \rangle,$$

and so by (41), we obtain

$$\eta d_{\bar{K}}(x) \geq \langle \xi_f, \hat{x} - x \rangle.$$

Simultaneously, since $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) = \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{x})$, we have

$$\langle \xi_f, \hat{x} - x \rangle \geq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i).$$

By (34) in the proof of Theorem 4.6, we obtain

$$\langle \xi_f, \hat{x} - x \rangle \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geq 0 = \eta d_{\tilde{K}}(x). \quad (44)$$

Hence, we have that $\langle \xi_f, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x)$. Next, we shall prove that there is $\varepsilon > \eta d_{\tilde{K}}(x) \geq 0$ such that

$$\xi_f \in \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (x).$$

In fact, we can show that $\xi_f \in \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (x)$. For any $y \in \mathbb{R}^n$,

$$\langle \xi_f, y - x \rangle = \langle \xi_f, y - \hat{x} \rangle + \langle \xi_f, \hat{x} - x \rangle \leq \langle \xi_f, y - \hat{x} \rangle$$

as $\langle \xi_f, \hat{x} - x \rangle \leq 0$. Since $a \in \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{x})$ and $f_i(x, \hat{u}_i) = f_i(\hat{x}, \hat{u}_i), i \in I$,

$$\begin{aligned} \langle \xi_f, y - x \rangle &\leq \sum_{i \in I} \hat{\sigma}_i f_i(y, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \\ &= \sum_{i \in I} \hat{\sigma}_i f_i(y, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i), \end{aligned}$$

which means $\xi_f \in \partial \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (x)$.

($\supseteq$) Let

$$\begin{aligned} x \in \left\{ x \in K : \exists \eta > 0, \exists \xi_f \in \bigcap_{y \in \{x, \hat{x}\}} \partial_\varepsilon \left(\sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right) (\hat{x}), \exists \varepsilon > \eta d_{\tilde{K}}(x), \right. \\ \left. \langle \xi_f, x - \hat{x} \rangle = \eta d_{\tilde{K}}(x), (\mu g)(x, \hat{v}) = 0, \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) = f_i(x, \hat{u}_i) \right\}. \end{aligned}$$

Since $\hat{x} \in \tilde{S}$, $\eta d_{\tilde{K}}(\hat{x}) = 0$ and so the assumption dealing with ξ_f lead to

$$\begin{aligned} -\eta d_{\tilde{K}}(\hat{x}) &= 0 \\ &= \langle \xi_f, \hat{x} - x \rangle - \eta d_{\tilde{K}}(x) \\ &\leq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \eta d_{\tilde{K}}(x) + \varepsilon \\ &= \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \eta d_{\tilde{K}}(x) + \eta d_{\tilde{K}}(x) \\ &= \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i), \end{aligned} \quad (45)$$

for any $\hat{\sigma}_i \geq 0, i \in I$, all nonzero. Therefore, we obtain

$$\begin{aligned} \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) &\leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + \eta d_{\tilde{K}}(\hat{x}) \\ &= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \end{aligned}$$

Since $\hat{x} \in \tilde{S}$ and $x \in K$, the conclusion that $x \in \tilde{S}$ is satisfied. $\square$

Conclusion. In this paper, we examined convex optimization problems with uncertain constraints and have defined a robust weak sharp solution by studying weak sharp solution of robust convex optimization problems where the uncertain constraints are enforced for all possible uncertainties within prescribed uncertainty sets. By employing tools of convex analysis and the valuable of constraint qualifications, we have established the necessary and sufficient conditions of robust weak sharp solutions, and characterizations of robust weak sharp solution set. As an application, we provided the characterization of the robust weak sharp weakly efficient solution sets for multi-objective convex optimization problems with uncertain constraints.

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Necessary and sufficient KKT optimality conditions in non-convex multi-objective optimization problems with cone constraints

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Abstract: This paper deals with a class of differentiable multi-objective optimization problems (MOP) over cone constraints without the convexity of the feasible set, and the cone-convexity of objectives and constraint functions. We present constraint qualifications for these (MOP) problems and establish the relationships between them. We also present necessary and sufficient the Karush-Kuhn-Tucker (KKT) optimality conditions for weak Pareto minimum as well as Pareto minimum. Our main results improve some recent ones in the literature. Illustrative examples are also provided to guarantee the advantages of each of our results.

Keywords: Non-convex multi-objective optimization, Cone-convex functions, Level set, Karush-Kuhn-Tucker optimality conditions

Mathematics Subject Classification: 90C26, 90C29, 90C46

1 Introduction

Multi-objective (vector-valued) optimization is a subject of mathematical programming that extensively studied and applied in various decision-making contexts like economics, human decision making, control engineering, transportation and many others. We refer the reader to [18, 19, 24]. For comprehensive treatment of theoretical issues concerning multi-objective optimization can be found in [2, 9, 13, 17]. In the multi-objective setting, the scalar concept of optimality does not apply directly due to the fact that all the objectives can not be simultaneously optimized with a single solution. To this effect, we must decide which objective to improve, and so compromise solutions must be considered. In this way, we refer to a weak Pareto minimum (resp. Pareto minimum [15]) which usually uses coordinate-wise ordering (induced by the positive orthant as ordering cone) to examine the objective vectors. However, in real-world multi-objective problems concerning especially fractional programming even computational aspects of Pareto optimum, not only the coordinate-wise ordering appears but also the cone defining the lexicographic partial order is of practical interest [7]. This being a reason, study of multi-objective optimization problems involving general ordering cones has gained attention. Precisely stated, in this paper we will be mainly concerned with the multi-objective optimization problem with cone constraint (MOP) given as

$$\begin{aligned} K - \text{Minimize } \mathbf{f}(\mathbf{x}) \\ \text{subject to } \mathbf{x} \in \mathbb{R}^n, -\mathbf{g}(\mathbf{x}) \in Q, \end{aligned} \tag{MOP}$$

where $\mathbf{f} := (f_1, \dots, f_p)^T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $\mathbf{g} := (g_1, \dots, g_m)^T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, are differentiable functions, K and Q are closed convex cones with non-empty interiors in $\mathbb{R}^p$ and $\mathbb{R}^m$, respectively. Let

$$\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{g}(\mathbf{x}) \in Q\} \tag{1.1}$$

be the set of all feasible solutions of (MOP). The notation “ $K - \text{Minimize}$ ” refers to the *weak Pareto minimum* (resp. *Pareto minimum*) with respect to the ordering cone K for the problem (MOP), namely a point $\mathbf{x}^* \in \mathcal{X}$ such that for every $\mathbf{x} \in \mathcal{X}$, $\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \notin \text{int}K$ (resp. $\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \notin K \setminus \{\mathbf{0}\}$).

Recall that a feasible point $\mathbf{x}^* \in \mathcal{X}$ is said to be a *KKT point* if there exist multipliers $\boldsymbol{\lambda} \in K^* \setminus \{\mathbf{0}\}$ and $\boldsymbol{\mu} \in Q^*$ such that the following Karush-Kuhn-Tucker (KKT) optimality conditions hold:

- (i) $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0}$;
- (ii) $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$,

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where K^* , $K^* := \{\mathbf{z} \in \mathbb{R}^p : \mathbf{x}^T \mathbf{z} \geq 0 \text{ for all } \mathbf{x} \in K\}$, denotes the *dual (positive polar) cone* of K . In this paper, the above feasible point $\mathbf{x}^*$ is also called a *non-trivial KKT point* if the corresponding $\boldsymbol{\mu}$ is a non-zero vector.

As far as we know, the search for weak Pareto minimum (resp. Pareto minimum) to (MOP) has been carried out through the study of the KKT optimality conditions provided that some constraint qualifications hold, and of the convexity of the functions $\mathbf{f}$ and $\mathbf{g}$. In the current work, with the introduction of scalar convex optimization without convexity of constraint functions by Lasserre [12], studies have been done on establishing KKT optimality conditions for weak Pareto minimum (resp. Pareto minimum) of some classes of multi-objective convex optimization problems. In particular, the authors have shown in [21] that even if the convex feasible set is not necessarily described by cone-convex constraint, the *Slater-type cone constraint qualification*³ renders the KKT optimality conditions both necessary and sufficient.

The classes of scalar convex optimization problems without convexity of constraint functions have been studied in the literature [6, 12, 14] where apart from [12] in other references inequality constraints are not assumed to be differentiable. A more recent exhaustive treatment of constraint qualifications can be found in [5, 23]. Recently, Ho [8] went further in the case of scalar differentiable problems but moreover without the convexity of the feasible set and of the functions that are involved, and showed that necessary and sufficient KKT optimality conditions are then considered in relation to the presence of convexity of the level sets of objective function. It is therefore of interest to investigate KKT optimality conditions for weak Pareto minimum and Pareto minimum of (MOP) without the convexity of the feasible set $\mathcal{X}$ and of the vector-valued functions $\mathbf{f}, \mathbf{g}$. The main purpose of this paper is to make an effort in this direction. Since we now focus our investigations to (MOP) in which the feasible set $\mathcal{X}$ is not necessarily convex, we are going to consider the feasible point $\mathbf{x}^*$ under the question in which satisfies the following property [8]:

$$\forall \mathbf{x} \in \mathcal{X}, \exists t_n \rightarrow 0^+ \text{ such that } \mathbf{x}^* + t_n(\mathbf{x} - \mathbf{x}^*) \in \mathcal{X}, \quad (1.2)$$

which can be seen as generalized convexity of the feasible set $\mathcal{X}$. Admittedly, some non-convex sets that satisfy the condition (1.2) will illustrate in Example 3.11 in Section 3. It is important to note that Slater's condition together with a mild non-degeneracy conditions on the constraints has been shown to guarantee that the KKT conditions are necessary and sufficient for optimality of the scalar problems ([6, 8, 12, 14]). Now, for the problem (MOP), we will assume only non-degeneracy at the point $\mathbf{x}^*$ under consideration (see Assumption 1 in the next section). In what follows the connections among non-degeneracy condition, Slater-type cone constraint qualification, and *Slater's condition*⁴ for cone-constraint are also investigated ones. Further, illustrative examples are also provided to demonstrate that our results generalize and improve the corresponding known results obtained in [21] for the problem (MOP) in some appropriate situations.

The rest of the paper is organized as follows. In Sect. 2 we recall some basic definitions and points out important results that will be used later in the paper. Section 3 presents relationships among constraint qualifications of multi-objective optimization problem (MOP) over cone constraint (1.1) and establishes necessary and sufficient KKT optimality conditions for a feasible point under the question to be a weak Pareto minimum of (MOP). We finally give sufficient conditions for guaranteeing a weak Pareto minimum to be a Pareto minimum of the problem (MOP).

2 Preliminaries

In this section, we briefly overview some notations, basic definitions, and preliminary results which will be used throughout the paper. All spaces under consideration are n -dimensional Euclidean space $\mathbb{R}^n$. All vectors are considered to be column vectors which can be transposed to a row vector by the superscript T . A nonempty subset K of $\mathbb{R}^p$ is said to be a *cone* if $tK \subseteq K$ for all $t \geq 0$. For a set A in $\mathbb{R}^n$, we say A is *convex* whenever $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in A$ for all $t \in [0, 1]$, $\mathbf{x}_1, \mathbf{x}_2 \in A$. By $\text{int}A$ (resp. $\text{co}A$) we will denote the *interior* (resp. *convex hull*) of the set A . The *normal cone* to a closed convex set A at $\mathbf{x} \in A$, denoted by

$$N(A, \mathbf{x}) := \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}^T(\mathbf{y} - \mathbf{x}) \leq 0, \forall \mathbf{y} \in A\}.$$

A set $A \subseteq \mathbb{R}^n$ is called *strictly convex* at $\mathbf{x} \in A$ if $\mathbf{u}^T(\mathbf{y} - \mathbf{x}) < 0$ for every $\mathbf{y} \in A \setminus \{\mathbf{x}\}$ and $\mathbf{u} \in N(A, \mathbf{x}) \setminus \{\mathbf{0}\}$. It is worth noting that the strict convexity of A at some point $\mathbf{x}$ does not guarantee the convexity of A . For instance, the set $A := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 > 0\} \cup \{(0, 0)^T\}$ is strictly convex at $(0, 0)^T$ while A is not convex.

³The feasible set $\mathcal{X}$ as in (1.1) is said to satisfy *Slater-type cone constraint qualification* [9] at $\mathbf{x} \in \mathcal{X}$ if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{g}(\hat{\mathbf{x}}) + \nabla \mathbf{g}(\hat{\mathbf{x}})(\hat{\mathbf{x}} - \mathbf{x}) \in -\text{int}Q$.

⁴The feasible set $\mathcal{X}$ as in (1.1) is said to satisfy *Slater's condition* if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $-\mathbf{g}(\hat{\mathbf{x}}) \in \text{int}Q$.

For a closed convex cone $K \subseteq \mathbb{R}^p$, a vector valued function $\mathbf{f} := (f_1, \dots, f_p)^T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is said to be *K-convex* (*K-pseudoconvex* [1, 22]) at a point $\mathbf{x}^* \in \mathbb{R}^n$ if for every $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^*) - \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \in K$$

(resp. $-\nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \notin \text{int}K \Rightarrow \mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \in \text{int}K$), where $\nabla \mathbf{f}(\mathbf{x}^*) := (\nabla f_1(\mathbf{x}^*), \dots, \nabla f_p(\mathbf{x}^*))^T$ is the $p \times n$ Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}^*$ and for each $k = 1, 2, \dots, p$, $\nabla f_k(\mathbf{x}^*) = \left(\frac{\partial f_k(\mathbf{x}^*)}{\partial x_1}, \frac{\partial f_k(\mathbf{x}^*)}{\partial x_2}, \dots, \frac{\partial f_k(\mathbf{x}^*)}{\partial x_n} \right)^T$ is the $n \times 1$ gradient vector of f_k at $\mathbf{x}^*$. If $\mathbf{f}$ is *K-convex* (*K-pseudoconvex*) at every point $\mathbf{x}^* \in \mathbb{R}^n$ then $\mathbf{f}$ is said to be *K-convex* (resp. *K-pseudoconvex*) on $\mathbb{R}^n$.

Now, let us recall the following results which will be useful in the sequel.

Lemma 2.1. [9, Lemma 3.21, p. 77] Let K be a convex cone in $\mathbb{R}^p$.

(i) If K is closed, then

$$K = \{\mathbf{x} \in \mathbb{R}^p : \mathbf{x}^T \mathbf{z} \geq 0 \text{ for all } \mathbf{z} \in K^*\}.$$

(ii) If $\text{int}K \neq \emptyset$, then

$$\text{int}K = \{\mathbf{x} \in \mathbb{R}^p : \mathbf{x}^T \mathbf{z} > 0 \text{ for all } \mathbf{z} \in K^* \setminus \{\mathbf{0}\}\}.$$

Lemma 2.2. [20, Lemma 1] Consider the problem (MOP). If $\mathbf{x}^* \in \mathcal{X}$ is a weak Pareto minimum of (MOP), then there exist $\boldsymbol{\lambda} \in K^*$ and $\boldsymbol{\mu} \in Q^*$ not both zero such that

$$\left(\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \right) (\mathbf{x} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n$$

and

$$\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0.$$

Now, we recall the following important result which can be found in [11] and will play a key role in deriving a feasible point to be a weak Pareto minimum as well as a Pareto minimum of (MOP).

Proposition 2.3. [11, Proposition 2.2.] Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable at $\mathbf{x}^*$ with $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$. Then:

(i) $N(L_f^<(\mathbf{x}^*), \mathbf{x}^*) = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} = r \nabla f(\mathbf{x}^*), \text{ for some } r \geq 0\}$ provided that

$$L_f^<(\mathbf{x}^*) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < f(\mathbf{x}^*)\}$$

is convex.

(ii) $N(L_f(\mathbf{x}^*), \mathbf{x}^*) = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} = r \nabla f(\mathbf{x}^*), \text{ for some } r \geq 0\}$ provided that

$$L_f(\mathbf{x}^*) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq f(\mathbf{x}^*)\}$$

is convex.

We conclude this section by the following useful lemma, which will be crucial in the sequel.

Lemma 2.4. Let $\mathcal{X}$ be as in (1.1). Assume that the condition (1.2) is satisfied at a feasible point $\mathbf{x}^* \in \mathcal{X}$. Then for every $\boldsymbol{\mu} \in Q^* \setminus \{\mathbf{0}\}$ for which $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$, one has

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) \leq 0 \text{ for all } \mathbf{v} \in \mathcal{X}.$$

Proof. Suppose on contrary that there exists $\mathbf{v} \in \mathcal{X}$ such that $(\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*))(\mathbf{v} - \mathbf{x}^*) > 0$. Then, by the first order approximation together with the condition (1.2), we can find some t_n small enough such that

$$\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) = \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + t_n \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) + o(t_n) > 0, \quad (2.1)$$

where $\frac{o(t)}{t} \rightarrow 0$ as $t \rightarrow 0^+$, and $\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*) \in \mathcal{X}$. The latter means that $-\mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) \in Q$ and consequently, $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^* + t_n(\mathbf{v} - \mathbf{x}^*)) \leq 0$, which contradicts (2.1). $\square$

3 Main results

In this section, we present the constraint qualifications that are used to derive the KKT conditions for (MOP) and their connections. Afterward, we will establish necessary and sufficient KKT optimality conditions for a weak Pareto minimum of (MOP). In addition, we also establish sufficient conditions for guaranteeing a weak Pareto minimum to be a Pareto minimum of the problem (MOP).

At first, we recall one of constraint qualifications the so-called *non-degeneracy condition* at some feasible point $\mathbf{x}^* \in \mathcal{X}$ in the vector setting, which introduced in [21].

Assumption 1: (Non-degeneracy condition [21]) Consider (MOP), for every $\boldsymbol{\mu} \in Q^* \setminus \{\mathbf{0}\}$,

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \neq \mathbf{0} \text{ whenever } \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0.$$

Remark 3.1 (Sufficient condition for non-degeneracy condition to be valid). Note that if the Slater-type cone constraint qualification at $\mathbf{x}^*$ holds, then the non-degeneracy condition is satisfied at $\mathbf{x}^*$. Indeed, if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q$, then for every $\boldsymbol{\mu} \in Q^* \setminus \{\mathbf{0}\}$ for which $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$, one has $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) = \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) < 0$ which implies that $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) \neq \mathbf{0}$.

Remark 3.2. The Slater's condition can also be guaranteed by the Slater-type cone constraint qualification at some point $\mathbf{x}^*$ as well. To see this, it follows from the Slater-type cone constraint qualification that $\nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q - \mathbf{g}(\mathbf{x}^*)$ for some $\hat{\mathbf{x}} \in \mathbb{R}^n$. This together with the fact that

$$\frac{\mathbf{g}(\mathbf{x}^* + t(\hat{\mathbf{x}} - \mathbf{x}^*)) - \mathbf{g}(\mathbf{x}^*)}{t} = \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) + o(t),$$

where $\frac{o(t)}{t} \rightarrow \mathbf{0}$ as $t \rightarrow 0^+$, for some $t_0 > 0$ sufficiently small, it holds

$$\mathbf{g}(\mathbf{x}^* + t_0(\hat{\mathbf{x}} - \mathbf{x}^*)) \in (1 - t_0)\mathbf{g}(\mathbf{x}^*) - t_0 \text{int}Q \subseteq -\text{int}Q.$$

Hence, the Slater's condition has been justified.

Now, we present some sufficient conditions for the Slater-type cone constraint qualification to be valid.

Theorem 3.3. Let $\mathcal{X}$ be as in (1.1). Assume that the Slater's condition holds and the condition (1.2) is satisfied at a feasible point $\mathbf{x}^* \in \mathcal{X}$. If the non-degeneracy condition holds at $\mathbf{x}^*$, then the Slater-type cone constraint qualification also holds at $\mathbf{x}^*$.

Proof. Suppose that the non-degeneracy condition holds at $\mathbf{x}^*$. Assume on contrary that for every $\mathbf{x} \in \mathbb{R}^n$, one has $\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \notin -\text{int}Q$, equivalently,

$$-[\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbb{R}^n - \mathbf{x}^*)] \cap \text{int}Q = \emptyset.$$

So, by the Eidelheit separation theorem, there exists $\boldsymbol{\mu} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ such that

$$\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + \boldsymbol{\mu}^T \mathbf{y} \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall \mathbf{y} \in Q. \quad (3.1)$$

By taking $\mathbf{x} = \mathbf{x}^*$ and $\mathbf{y} = \mathbf{0}$ in (3.1), we would have $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$. Hence, with regard to (3.1) with $\mathbf{x} = \mathbf{x}^*$, we get $\boldsymbol{\mu} \in Q$. Therefore, in view of (3.1), we find a vector $\boldsymbol{\mu} \in Q^* \setminus \{\mathbf{0}\}$ with $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$ such that

$$\dot{\boldsymbol{\mu}}^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \quad (3.2)$$

On the other hand, by assumption, there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that $-\mathbf{g}(\hat{\mathbf{x}}) \in \text{int}Q$. Then, since $\mathbf{g}$ is continuous at $\hat{\mathbf{x}}$, there exists $r > 0$ such that $\mathbf{g}(\hat{\mathbf{x}} + r\mathbf{u}) \in -Q$ for all $\mathbf{u} \in \mathbb{B} := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$. Consequently, $\hat{\mathbf{x}} + r\mathbf{u} \in \mathcal{X}$ for all $\mathbf{u} \in \mathbb{B}$. So, as $\mathbf{x}^* \in \mathcal{X}$ and $\mathbf{x}^*$ satisfies the condition (1.2), we conclude from Lemma 2.4 that

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} + r\mathbf{u} - \mathbf{x}^*) \leq 0, \quad \forall \mathbf{u} \in \mathbb{B}. \quad (3.3)$$

In particular, put $\mathbf{u} = \mathbf{0} \in \mathbb{B}$, one has $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \leq 0$. Thus, with regard to (3.2), $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) = 0$, and hence we deduce from (3.3) that

$$\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)\mathbf{u} \leq 0, \quad \forall \mathbf{u} \in \mathbb{B}.$$

So, $\boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*)$ must ultimately be zero vector, which contradicts the validity of non-degeneracy condition at $\mathbf{x}^*$. $\square$

Remark 3.4. In the absence of the condition (1.2) at $\mathbf{x}^*$, the validity of both Slater and the non-degeneracy conditions at $\mathbf{x}^*$ does not guarantee the validity of Slater-type cone constraint qualification at $\mathbf{x}^*$, for instance, let $\mathbf{x} := (x_1, x_2)^T \in \mathbb{R}^2$, $Q := \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $\mathbf{g}(\mathbf{x}) := (x_2^3 + x_2 - x_1, x_1 - x_2)^T$. We see that $\mathbf{g}(-3, -2) = (-7, -1)^T \in -\text{int}Q$, that is, Slater's condition holds. Also, one has $\nabla \mathbf{g}(\mathbf{x}) = \begin{pmatrix} -1 & 3x_2^2 + 1 \\ 1 & -1 \end{pmatrix}$ and a short calculation shows that the non-degeneracy holds at $\mathbf{x}^* := (0, 0)^T \in \mathcal{X}$, while the condition (1.2) together with the Slater-type cone constraint qualification is invalid at $\mathbf{x}^*$. In fact, let us consider $x_0 := (-2, -1)^T \in \mathcal{X}$ and arbitrary sequence $\{t_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ such that $t_n \rightarrow 0$ as $n \rightarrow +\infty$. So, $t_{n_0} < 1$ for some $n_0 \in \mathbb{N}$ and $x^* + t_{n_0}(x_0 - x^*) = t_{n_0}x_0 \notin \mathcal{X}$. Otherwise, we have that

$$t_{n_0}(1 - t_{n_0})(1 + t_{n_0}) = (-t_{n_0})^3 + (-t_{n_0}) - (-2t_{n_0}) \leq 0,$$

whence, $1 \leq t_{n_0}$. This contradicts to the fact that $t_{n_0} < 1$. In addition, we can not find out $\hat{\mathbf{x}} := (\hat{x}_1, \hat{x}_2)^T \in \mathbb{R}^2$ such that

$$\begin{pmatrix} -\hat{x}_1 + \hat{x}_2 \\ \hat{x}_1 - \hat{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \in -\text{int}Q.$$

Remark 3.5. (i) It is worth noticing that there is a partial overlapping between Slater's condition and non-degeneracy condition at a given point $\mathbf{x}^*$ in general. For example, it is easy to check that Slater's condition fails to hold for $\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{g}(\mathbf{x}) \in Q\}$, where $Q := \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $\mathbf{g}(\mathbf{x}) := (-x_1 + x_2, x_1 - x_2)^T$ for all $\mathbf{x} \in \mathbb{R}^2$, while non-degeneracy condition holds at $\mathbf{x}^* := (0, 0)^T$. In contrast, redefining $\mathbf{g}(\mathbf{x}) := (x_1^3 - x_2 + 1, -x_1^2 + x_2 - 1)^T$ for all $\mathbf{x} \in \mathbb{R}^2$, we get $-\mathbf{g}(-1, 1) = (1, 1)^T \in \text{int}Q$ and so, Slater's condition holds. Now we see that non-degeneracy does not hold at $\mathbf{x}^*$. Indeed, taking $\boldsymbol{\mu}_0 := (1, 1)^T \in Q^* \setminus \{0\}$ entails that $\boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*) = 0$ and

$$\boldsymbol{\mu}_0^T \nabla \mathbf{g}(\mathbf{x}^*) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

showing that non-degeneracy fails to hold at $\mathbf{x}^*$.

- (ii) In addition to the Q -convexity of $\mathbf{g}$ at a given point $\mathbf{x}^*$, if Slater's condition holds, then non-degeneracy condition is satisfied at $\mathbf{x}^*$. To see this, suppose now by contradiction that there exists $\boldsymbol{\mu}_0 \in Q^* \setminus \{0\}$ satisfying $\boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*) = 0$ and $\boldsymbol{\mu}_0^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0}$. It then follows from Q -convexity of $\mathbf{g}$ at $\mathbf{x}^*$ that $\boldsymbol{\mu}_0^T \mathbf{g}(\hat{\mathbf{x}}) - \boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*) = \boldsymbol{\mu}_0^T \mathbf{g}(\hat{\mathbf{x}}) - \boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*) - \boldsymbol{\mu}_0^T \nabla \mathbf{g}(\mathbf{x}^*)(\hat{\mathbf{x}} - \mathbf{x}^*) \geq 0$ for a Slater's point $\hat{\mathbf{x}}$. This contradicts to the fact that $\boldsymbol{\mu}_0^T \mathbf{g}(\hat{\mathbf{x}}) < 0 = \boldsymbol{\mu}_0^T \mathbf{g}(\mathbf{x}^*)$.

Remark 3.6. In the case of $Q = \mathbb{R}_+^m := \{(x_1, x_2, \dots, x_m)^T \in \mathbb{R}^m : x_i \geq 0, \forall i = 1, \dots, m\}$, non-degeneracy conditions at $\mathbf{x}^*$ can be view as the *Mangasarian-Fromovitz constraint qualification*⁵ at $\mathbf{x}^*$ and non-degeneracy conditions at $\mathbf{x}^*$ in [12, 8] as well. Indeed,

$$\begin{aligned} & \exists \mathbf{v} \in \mathbb{R}^n \text{ such that } \nabla g_i(\mathbf{x}^*)^T \mathbf{v} < 0, \forall i \in I(\mathbf{x}^*) \\ \Leftrightarrow & \mathbf{0} \notin \text{co}\{\nabla g_i(\mathbf{x}^*) : i \in I(\mathbf{x}^*)\} \\ \Leftrightarrow & \forall \boldsymbol{\mu} := (\mu_1, \mu_2, \dots, \mu_m)^T \in \mathbb{R}_+^m \setminus \{0\} \text{ with } \mu_i g_i(\mathbf{x}^*) = 0, i = 1, 2, \dots, m, \text{ one has } \sum_{i=1}^m \mu_i \nabla g_i(\mathbf{x}^*) \neq \mathbf{0}, \end{aligned}$$

and for each $i \in \{1, 2, \dots, m\}$, by taking $\boldsymbol{\mu} := \mathbf{e}_i$, where $\mathbf{e}_i$ is the unit vector in $\mathbb{R}^m$ with the i th component is 1 and the others 0, one has $\nabla g_i(\mathbf{x}^*) \neq \mathbf{0}$ whenever $i \in I(\mathbf{x}^*)$. Note that Slater-type cone constraint qualification at $\mathbf{x}^*$ also is equivalent to the *Robinson constraint qualification*⁶ at $\mathbf{x}^*$ [4, Lemma 2.99, p. 69]. Then, as the considered set $\{\mathbf{x} \in \mathbb{R}^n, g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$ is not necessarily convex, one can notice that Theorem 3.3 extends [5, Theorem 2.1] to non-convex setting on the set $\{\mathbf{x} \in \mathbb{R}^n, g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$.

Now, we are in the position to give necessary and sufficient KKT optimality conditions for a weak Pareto minimum of (MOP).

Theorem 3.7. *Consider the problem (MOP) and let both Assumption 1 and the condition (1.2) be satisfied at a feasible point $\mathbf{x}^*$.*

- (i) *If $\mathbf{x}^*$ is a weak Pareto minimum of (MOP), then $\mathbf{x}^*$ is a KKT point.*
(ii) *Conversely, if $\mathbf{x}^*$ is a non-trivial KKT point with multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, and $L_{\boldsymbol{\lambda}^T \mathbf{f}}^<(\mathbf{x}^*)$ is convex then $\mathbf{x}^*$ is a weak Pareto minimum of (MOP).*

Proof. (i) Let $\mathbf{x}^* \in \mathcal{X}$ be a weak Pareto minimum of (MOP). By Lemma 2.2, there exist $\boldsymbol{\lambda} \in K^*$ and $\boldsymbol{\mu} \in Q^*$ not both zero such that $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0$ and

$$(\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*))(\mathbf{x} - \mathbf{x}^*) \geq 0, \forall \mathbf{x} \in \mathbb{R}^n. \quad (3.4)$$

As the inequality (3.4) holds for every $\mathbf{x} \in \mathbb{R}^n$, we conclude that

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0} \text{ and } \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0.$$

Moreover, we assert that $\boldsymbol{\lambda} = \mathbf{0}$. Otherwise, it follows in turn that $\boldsymbol{\mu} \neq \mathbf{0}$, which stands in contradiction to Assumption 1, and therefore, $\boldsymbol{\lambda} \neq \mathbf{0}$.

- (ii) Let $\mathbf{x}^* \in \mathcal{X}$ be an arbitrary non-trivial KKT point, i.e.,

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\mu}^T \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0}; \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}^*) = 0,$$

⁵The set $\{\mathbf{x} \in \mathbb{R}^n, g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$ is said to satisfy the *Mangasarian-Fromovitz constraint qualification* [4] at $\mathbf{x}^*$ if there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\nabla g_i(\mathbf{x}^*)^T \mathbf{v} < 0$ for each $i \in I(\mathbf{x}^*) := \{i \in \{1, 2, \dots, m\} : g_i(\mathbf{x}^*) = 0\}$.

⁶One says that the set $\{\mathbf{x} \in \mathbb{R}^n, g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$ satisfies the *Robinson constraint qualification* at $\mathbf{x}^*$ if $\mathbf{0} \in \text{int}\{\mathbf{g}(\mathbf{x}^*) + \nabla \mathbf{g}(\mathbf{x}^*)(\mathbb{R}^n - \mathbf{x}^*) + \mathbb{R}_+^m\}$ where $\mathbf{g}(\mathbf{x}) := (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_m(\mathbf{x}))^T$.

for some non-zero vectors $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$. This together with Assumption 1 implies that $\lambda^T \nabla f(\mathbf{x}^*)$ must ultimately be non-zero vector. It can be seen that if the set $L_{\lambda^T f}^<(\mathbf{x}^*)$ is empty, then $\mathbf{x}^*$ actually is a weak Pareto minimum of (MOP). In fact, if $\mathbf{x}^*$ is not a weak Pareto minimum of (MOP), there exists $\mathbf{x} \in \mathcal{X}$ such that $\mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{x}) \in \text{int}K$. So, by the virtue of Lemma 2.1, $\lambda^T \mathbf{f}(\mathbf{x}^*) > \lambda^T \mathbf{f}(\mathbf{x})$, which contradicts to the fact that $L_{\lambda^T f}^<(\mathbf{x}^*) = \emptyset$. Let us consider in the case $L_{\lambda^T f}^<(\mathbf{x}^*) \neq \emptyset$. Applying Proposition 2.3(i) with $f(\mathbf{x}) := \lambda^T \mathbf{f}(\mathbf{x})$, we obtain that

$$\lambda^T \nabla f(\mathbf{x}^*)(\mathbf{u} - \mathbf{x}^*) \leq 0, \quad \forall \mathbf{u} \in L_{\lambda^T f}^<(\mathbf{x}^*). \quad (3.5)$$

Therefore, by Lemma 2.4,

$$\lambda^T \nabla f(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) = -\mu^T \nabla \mathbf{g}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{v} \in \mathcal{X}. \quad (3.6)$$

Note that,

$$\{\mathbf{y} \in \mathbb{R}^n : \mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{y}) \in \text{int}K\} \subseteq L_{\lambda^T f}^<(\mathbf{x}^*).$$

Thus, in order to obtain that $\mathbf{x}^*$ is a weak Pareto minimum of (MOP), it suffices to show that $\mathcal{X} \subseteq \mathbb{R}^n \setminus L_{\lambda^T f}^<(\mathbf{x}^*)$ or consequently, $L_{\lambda^T f}^<(\mathbf{x}^*) \cap \mathcal{X} = \emptyset$. Suppose, ad absurdum, $L_{\lambda^T f}^<(\mathbf{x}^*) \cap \mathcal{X} \neq \emptyset$. Thus, from (3.5) and (3.6) we get the assertion $\lambda^T \nabla f(\mathbf{x}^*)(\mathbf{w} - \mathbf{x}^*) = \mathbf{0}$ for any $\mathbf{w} \in L_{\lambda^T f}^<(\mathbf{x}^*) \cap \mathcal{X}$. Furthermore, as the set $L_{\lambda^T f}^<(\mathbf{x}^*)$ being open, for each $\mathbf{d} \in \mathbb{R}^n$ we can find $t > 0$ small enough such that $\mathbf{w} + t\mathbf{d} \in L_{\lambda^T f}^<(\mathbf{x}^*)$. Hence,

$$t\lambda^T \nabla f(\mathbf{x}^*)\mathbf{d} = \lambda^T \nabla f(\mathbf{x}^*)(\mathbf{w} + t\mathbf{d} - \mathbf{x}^*) - \lambda^T \nabla f(\mathbf{x}^*)(\mathbf{w} - \mathbf{x}^*) \leq 0.$$

This means $\lambda^T \nabla f(\mathbf{x}^*) = \mathbf{0}$, a contradiction. Thus, $L_{\lambda^T f}^<(\mathbf{x}^*) \cap \mathcal{X} = \emptyset$, and $\mathbf{x}^*$ is a weak Pareto minimum of (MOP) as desired. $\square$

Remark 3.8. It is worth mentioning here that Proposition 2.3 plays a significant role in Theorem 3.7(ii) for ensuring a feasible point $\mathbf{x}^*$ to be a weak Pareto minimum of (MOP). Beside, non-degeneracy condition (Assumption 1) at $\mathbf{x}^*$ need to be assumed for guaranteeing $\lambda^T \nabla f(\mathbf{x}^*) \neq \mathbf{0}$ with correspond to multiplier vector $\lambda \in K^* \setminus \{\mathbf{0}\}$. In contrast, it generally does not need constraint qualification to establish the sufficient optimality conditions. Therefore, it might be reasonably assumed the assertion $\lambda^T \nabla f(\mathbf{x}^*) \neq \mathbf{0}$ instead of assuming the non-degeneracy condition at $\mathbf{x}^*$. However, keeping in mind the fact that we need to justify the convexity of $L_{\lambda^T f}^<(\mathbf{x}^*)$ with the same choice λ , and so in this case the multiplier vector λ turn out to be difficult to determine for which satisfying all conditions in Theorem 3.7(ii) simultaneously. This being a reason why non-degeneracy condition make used in Theorem 3.7(ii). Another reason is that non-degeneracy condition is actually justified to check a feasible point that can be a weak Pareto minimum of (MOP) or not before to justify sufficient optimality conditions.

We now demonstrate with the following example to guarantee that Theorem 3.7 is indicated to be conveniently applied in some cases where Theorem 3.1 and Theorem 3.2 of [21] cannot be used even when the feasible set $\mathcal{X}$ is convex.

Example 3.9. Consider the following multi-objective optimization problem (MOP) over cones:

$$\begin{aligned} K - \quad & \text{Minimize } \mathbf{f}(x) := (x + 1, x^3 - 5x^2 + 8x - 3)^T \\ & \text{subject to } x \in \mathcal{X} := \{x \in \mathbb{R} : -\mathbf{g}(x) \in Q\}, \end{aligned}$$

where $\mathbf{g}(x) := (x - 1, x^2 - x - 1)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq x_1\}$. A straightforward calculation shows that:

- $\mathcal{X} = [2, +\infty)$,
- $K^* = K$,
- $Q^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_2 \leq -x_1\}$,
- $x^* := 2$ satisfies the non-trivial KKT conditions by taking $\lambda := (2, 0)^T$ and $\mu := (1, -1)^T$,
- $L_{\lambda^T f}^<(x^*) = (-\infty, 2)$ is convex,
- It is easily to seen that Assumption 1 and the condition (1.2) are satisfied.

Applying Theorem 3.7 (ii), we can conclude that x^* is a weak Pareto minimum of (MOP). However, it can be checked that $\mathbf{g}$ is not Q -convex, i.e.

$$\mathbf{g}(1) - \mathbf{g}(2) - \nabla \mathbf{g}(2)(1 - 2) = (0, 1)^T \notin Q,$$

but the feasible set $\mathcal{X}$ is convex. Furthermore, the function $\mathbf{f}$ is not K -pseudoconvex at $x^* := 2$, because if we take $x = 0$ then

$$-\nabla \mathbf{f}(x^*)(x - x^*) = (2, 0)^T \notin \text{int}K, \text{ but } \mathbf{f}(x^*) - \mathbf{f}(x) = (2, 4)^T \in \text{int}K.$$

Hence, the corresponding results [21] is not applicable. $\square$

Note that the multiplier vector $\boldsymbol{\mu}$ is assumed to be non-zero vector (the non-triviality of the KKT conditions) in order to ensure that $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$ in Theorem 3.7(ii). The following example demonstrates that this assumption cannot be dropped.

Example 3.10. Let $\mathbf{f}(x) := (x + 1, -(x - 2)^3)^T$, $\mathbf{g}(x) := (x^2 - 1, 2x - 1)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \geq -x_1, x_1 \geq 0\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq x_2, x_1 \geq 0\}$. It is not hard to check that $\mathcal{X} = [1, 2]$, $x^* := 2$ is a KKT point with $\boldsymbol{\lambda} := (0, -1)^T$ and $\boldsymbol{\mu} := (0, 0)^T$, and all the conditions in Theorem 3.7 (ii) are fulfilled. However x^* is not even a weak Pareto minimum, i.e., if we take $x := \frac{3}{2}$ then $\mathbf{f}(x^*) - \mathbf{f}(x) = (3, 0)^T - (\frac{5}{2}, \frac{1}{8})^T = (\frac{1}{2}, -\frac{1}{8})^T \in \text{int}K$. The main reason is that x^* is not a non-trivial KKT point. $\square$

To appreciate Theorem 3.7 we present an example that is applicable while the aforementioned result in [21] is not.

Example 3.11. Consider the following multi-objective optimization problem (MOP) over cones :

$$K - \text{ Minimize } \mathbf{f}(x) := (x^2 - 1, -x^3 + 5x^2 - 8x + 5)^T \\ \text{subject to } x \in \mathcal{X} := \{x \in \mathbb{R} : -\mathbf{g}(x) \in Q\},$$

where $\mathbf{g}(x) := (x^3 + x^2 + x, x^3 + 2x^2 - 5x + 8)^T$, $K := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0, x_2 \leq x_1\}$ and $Q := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq x_1\}$. Evidently, $\mathbf{f}$, and $\mathbf{g}$ are not K , and Q -convex, respectively. Indeed, $\mathbf{f}(1) - \mathbf{f}(0) - \nabla \mathbf{f}(0)(1 - 0) = (1, 4)^T \notin K$, and $\mathbf{g}(1) - \mathbf{g}(0) - \nabla \mathbf{g}(0)(1 - 0) = (2, 3)^T \notin Q$. It is easy to verify that $\mathcal{X} = [0, 2] \cup [4, +\infty)$. Then we have already seen that the feasible set $\mathcal{X}$ is not convex. Therefore, the results in [21] cannot be applicable. However, it is not hard to verify that

- $K^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_2 \geq -x_1\}$,
- $Q^* = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_2 \leq -x_1\}$,
- $x^* := 0$ satisfies the non-trivial KKT conditions by taking $\boldsymbol{\lambda} := (1, -1)^T$ and $\boldsymbol{\mu} := (-8, 0)^T$,
- Assumption 1 and the condition (1.2) are satisfied,
- $L_{\boldsymbol{\lambda}^T \mathbf{f}}^<(x^*) = (-\infty, 0)$, which is convex.

Hence, Theorem 3.7 (ii) indicates that x^* is a weak Pareto minimum of (MOP). $\square$

Next, we will see now how the convexity of $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ together with the strict convexity of $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ at a non-trivial KKT point $\mathbf{x}^*$ possess $\mathbf{x}^*$ to be a Pareto minimum of (MOP).

Theorem 3.12. Consider the problem (MOP) and let both Assumption 1 and the condition (1.2) be satisfied at a feasible point $\mathbf{x}^*$. If $\mathbf{x}^*$ is a non-trivial KKT point with multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ is convex, and additionally $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ is strictly convex at $\mathbf{x}^*$, then $\mathbf{x}^*$ is a Pareto minimum of (MOP).

Proof. In a similar manner of the second argument as the proof of Theorem 3.7, by the KKT conditions and Proposition 2.3(ii), we arrive at the following assertion

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{v} - \mathbf{x}^*) \geq 0 \geq \boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{u} - \mathbf{x}^*), \quad \forall \mathbf{v} \in \mathcal{X}, \forall \mathbf{u} \in L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*), \quad (3.7)$$

and $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) \neq \mathbf{0}$. To establish the desired results, we argue first by using Lemma 2.1 that

$$\{\mathbf{y} \in \mathbb{R}^n : \mathbf{f}(\mathbf{x}^*) - \mathbf{f}(\mathbf{y}) \in K \setminus \{\mathbf{0}\}\} \subseteq L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*) \setminus \{\mathbf{x}^*\}.$$

Thus, we only need to justify this containment

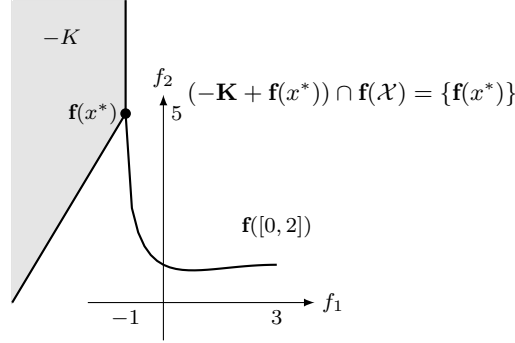
$$\mathcal{X} \subseteq \mathbb{R}^n \setminus (L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*) \setminus \{\mathbf{x}^*\})$$

We argue by contradiction that there exists some $\mathbf{w} \in \mathcal{X}$ such that $\mathbf{w} \neq \mathbf{x}^*$ and $\mathbf{w} \in L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$. Taking (3.7) into account we actually have

$$\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{w} - \mathbf{x}^*) = 0.$$

Furthermore, as $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*) \in N(L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*), \mathbf{x}^*) \setminus \{\mathbf{0}\}$ (by the second inequality in (3.7)) and $L_{\boldsymbol{\lambda}^T \mathbf{f}}(\mathbf{x}^*)$ is strictly convex, then $\boldsymbol{\lambda}^T \nabla \mathbf{f}(\mathbf{x}^*)(\mathbf{w} - \mathbf{x}^*) < 0$. This is a contradiction, and thereby implying that $\mathbf{x}^*$ is a Pareto minimum of (MOP). $\square$

Remark 3.13. In Example 3.11 with $\boldsymbol{\lambda} := (1, -1)^T$, it is evident that $L_{\boldsymbol{\lambda}^T \mathbf{f}}(x^*)$ is strictly convex at $x^* := 0$, by Theorem 3.12, and hence x^* is a Pareto minimum of (MOP) (see the below figure).



In Example 3.11, $x^* := 0$ is a Pareto minimum of (MOP).

Remark 3.14. It should be noted that to obtain a Pareto minimum from a drawback (see [9, 21] and other references therein), the multiplier vector λ in KKT conditions need to be taken from the strict positive dual cone of K , K^{s*} , which defined as

$$K^{s*} := \{z \in \mathbb{R}^n : x^T z > 0 \text{ for all } x \in K \setminus \{0\}\}.$$

However, in this case study the multiplier vector λ is not necessarily to take from the strict positive dual cone. In fact, as K defined in Example 3.11 and $\lambda := (1, -1)^T$, Then elementary calculations give us

$$K^{s*} = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 > 0, x_2 > -x_1\}$$

and so, $\lambda \notin K^{s*}$.

To this end, we now give an example showing that the strict convexity of $L_{\lambda T f}(x^*)$ with corresponding multiplier λ is essential for x^* under the question to be a Pareto minimum of (MOP) in Theorem 3.12.

Example 3.15. Let $x := (x_1, x_2)^T \in \mathbb{R}^2$, $f(x) := (x_1^2, x_2 - x_1)^T$, $g(x) := (-x_1^3 + 3x_1 + x_2, x_1 - x_2)^T$ and $K = Q := \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \geq 0\}$. It is easy to check that the feasible set $\mathcal{X}$ is not convex and the condition (1.2) is valid at $x^* := (1, 1)^T \in \mathcal{X}$. Then elementary calculations give us

- $K^* = Q^* = K$,
- $g(x^*) = (3, 0)^T$, $\nabla g(x^*) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $f(x^*) = (1, 0)^T$, $\nabla f(x^*) = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$,
- x^* satisfies Assumption 1 and the non-trivial KKT conditions by taking $\lambda = \mu := (0, 1)^T$,
- $L_{\lambda T f}^<(x^*) = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 < x_1\}$ and $L_{\lambda T f}(x^*) = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq x_1\}$ are convex sets.

By Theorem 3.7 (ii), we can conclude that x^* is a weak Pareto minimum of (MOP). However, the set $L_{\lambda T f}(x^*)$ is not a strictly convex set at x^* , i.e., it is clear that $N(L_{\lambda T f}(x^*), x^*) = \{(-r, r)^T \in \mathbb{R}^2 : r \geq 0\}$. So, by taking $u := (-1, 1)^T \in N(L_{\lambda T f}(x^*), x^*) \setminus \{(0, 0)^T\}$ and $y := (2, 2)^T \in L_{\lambda T f}(x^*) \setminus \{(0, 0)^T\}$, $u^T(y - x^*) = 0$. Actually, a point x^* is not even a Pareto minimum, i.e., if we take $\bar{x} := (-2, -2)^T \in \mathcal{X}$, one has

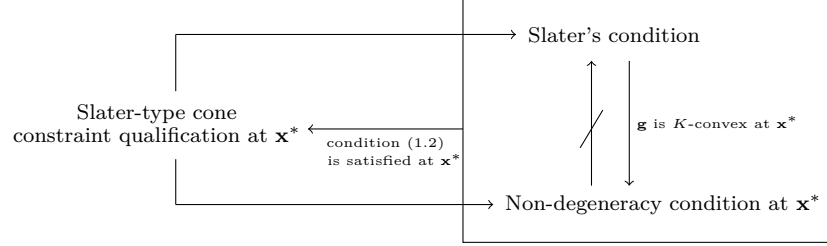
$$f(x^*) - f(\bar{x}) = (-3, 0)^T \in K \setminus \{(0, 0)^T\}.$$

Remark 3.16. It is worth noting that the convexity of $L_{\lambda T f}^<(x^*)$ (resp. $L_{\lambda T f}(x^*)$) in Theorem 3.7 (resp. in Theorem 3.12) can be viewed as a generalized quasiconvexity of f at x^* due to the notion of **-quasiconvexity* [10] in the sense that for each $\lambda \in K^*$ the function $\lambda^T f : \mathbb{R}^n \rightarrow \mathbb{R}$ is quasiconvex⁷. It is quite clear from the definition that **-quasiconvexity* of f guarantees the convexity of the level set $L_{\lambda T f}^<(x^*)$ or of $L_{\lambda T f}(x^*)$. In fact, the function f in Example 3.15 is not **-quasiconvex*, i.e., by taking $\lambda := (-1, 1)^T \in K^*$ and $x := (1, 1)^T$, the sublevel set $L_f(x)$ is non-convex. For related conditions for cone quasiconvex mappings we refer the reader to [13, 16, 3].

⁷A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *quasiconvex* if its sublevel set $L_f(x)$ at x is convex for all $x \in \mathbb{R}^n$ or, equivalently, the strict sublevel set $L_f^<(x)$ at x is convex for all $x \in \mathbb{R}^n$.

4 Conclusions

In this paper, we have established necessary and sufficient the Karush-Kuhn-Tucker optimality conditions for weak Pareto minimum as well as Pareto minimum of a differentiable multi-objective optimization problem (MOP) over cone constraint without the convexity of the feasible set, and the cone-convexity of objective and constraint functions. We also have proposed constraint qualifications, and discussed the relationship between them which can be summarized in following diagram whenever $\mathbf{x}^* \in \mathcal{X}$:



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