รายงานวิจัยฉบับสมบูรณ์

โครงการกราฟและไฮเพอร์กราฟเหนือริงสลับที่จำกัด

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และต้นสังกัด

Abstract (บทคัดย่อ)

There are many graphs and hypergraphs defined algebraically over a finite ring *R*. For example, unitary Cayley graphs/hypergraphs, integral circulant graphs, gcd-graphs, zero divisor graphs, and bilinear form graphs. We develop some tools in number theory, linear algebra, character theory and finite commutative ring theory to study more deeply on these graphs and hypergraphs over a finite ring or a finite commutative ring. The study includes graph/hypergraph structures and their subgraphs, spectra of the graphs/hypergraphs and their energy. We also analyze their hyperenegeticity, Ramanujan property or other important parameters of the graphs.

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Objectives

1. To develop some tools in number theory, linear algebra, character theory and finite commutative ring theory to study more deeply on graphs and hypergraphs algebraically defined over a finite ring.
2. To define and study algebraic properties of t-Cayley hypergraphs over finite commutative rings.
3. To study graph/hypergraph structures, spectra of the graphs/hypergraphs and their energy and the behavior/structure of the lifting from the residue field when R is a finite local ring. This

includes the analysis of hyperenegeticity, Ramanujan property, or other important parameters of the graphs.

Methodology and Results

We work on four algebraically defined graphs.

1. For a finite commutative ring R with unit group R^x and the set of zero divisors Z(R), we know that R = {0} $\cup$ R^x $\cup$ Z(R). The *zero divisor graph* of R is a graph whose vertex set is the set of all zero divisor of R, and two zero divisors are adjacent if and only if their product is zero. We first study zero divisor graphs over finite chain rings. We determine their rank, determinant, and eigenvalues using reduction graphs.

Theorem The determinant of the zero divisor graph of a finite chain ring R is 0 unless R is isomorphic to $\mathbf{Z}_2[x]/(x^2)$ or $\mathbf{Z}_4$ where the determinant equals 1.

Theorem For any finite chain ring R with nilpotency t and residue field k, the rank of the zero divisor graph is t – 1 and the multiplicity of the eigenvalue 0 is $|k|^t - 1$.

Moreover, we extend the work to zero divisor graphs over finite commutative principal ideal rings by using a combinatorial method, find the number of positive eigenvalues and the number of negative eigenvalues, and find upper and lower bounds for the largest eigenvalue (§4 of the paper). We also characterize all finite commutative principal ideal rings such that their zero divisor graphs are complete and compute the Wiener index of the zero divisor graphs over finite commutative principal ideal rings (Theorem 5.2 of the paper).

Write $R = R_1 \times \ldots \times R_k$ where R_i is a finite chain ring of nilpotency t_i for all i.

Theorem The rank of the zero divisor graph of R is $(t_1+1) \ldots (t_k+1) - 2$, and the determinant is 1 if R is isomorphic to $\mathbf{Z}_2[x]/(x^2)$ or $\mathbf{Z}_4$, –1 if R is isomorphic to $(\mathbf{Z}_2)^k$ for some $k \geq 2$, and 0 otherwise.

Theorem If $N = (t_1+1) \ldots (t_k+1) - 2$, the number of positive eigenvalues and the number of negative eigenvalues of the zero divisor graph of R are $\lceil N/2 \rceil$ and $\lfloor N/2 \rfloor$.

This work may be extended to study the zero divisor graphs over any finite commutative rings in the future. The paper appears in Rattanakangwanwong J. and Meemark Y., Eigenvalues of zero divisor graphs of principal ideal rings, *Linear Multilinear Algebra* 2021. DOI:10.1080/03081087.2021.1917501.

2. Suppose that R is a finite commutative ring and m, n, d are positive integers such that $2 \leq d \leq \min\{m,n\}$. The *matrix graph* of type (m,n,d) over R is the graph whose vertices are $m \times n$ matrices over R, and two $m \times n$ matrices A and B are adjacent if and only if $0 < \text{rank}(A-B) < d$. We show that this matrix graph is a connected vertex transitive graph. We determine the distance, diameter, independence number, clique number and chromatic number of this graph over finite principal ideal rings (§3 of the paper). The matrix graph can be applied to study MRD codes over a finite commutative ring R. We prove that if R is a finite principal ideal ring, then the MRD codes coincide with the maximal independent sets of the matrix graph (§4 of the paper). Consequently, we have the existence of linear MRD codes over finite principal ideal rings in our last theorem. For future work, we can propose to study the matrix graph over any finite commutative rings. This work is published in Sirisuk S. and Meemark Y., Matrix graphs and MRD codes over finite principal ideal rings, *Finite Fields Appl.* 2020; 66: #101705.

3. For a finite ring R with identity, the *unitary Cayley graph* of R, C(R), is the graph with vertex set R and for each x, y in R, x and y are adjacent if and only if $x - y$ is a unit of R. Let R be a finite commutative ring and n a positive integer. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R. we study the unitary Cayley graph $C(M_n(R))$ of the matrix ring over R. If F is a field, we use the additive characters of $M_n(F)$ to determine three eigenvalues of $C(M_n(F))$ and use them to analyze strong regularity and hyperenegetic graphs. We find conditions on R and n such that $C(M_n(R))$ is strongly regular. Without explicitly having the spectrum of the graph, we can show that $C(M_n(R))$ is hyperenergetic and characterize R and n such that $C(M_n(R))$ is Ramanujan. Moreover, we compute the clique number, the chromatic number and the independence number of the graph. This work appears in Rattanakangwanwong J. and Meemark Y., Unitary Cayley graphs of matrix rings over finite commutative rings, Finite Fields Appl. 2020; 65: #101689.

Let G be a graph and x a vertex of G. The *first subconstituent* of G at x is the subgraph of G induced by the set of neighborhoods of x and the *second subconstituent* of G at x is the subgraph of G induced by the set of vertices which is non-adjacent to x except x itself. Now, we discuss the subconstituents of the unitary Cayley graph of $M_n(R)$. Let R be a finite ring with identity. The set of *neighborhood* of a vertex x of the graph C(R) is denoted by N(x). For x in R, the maps f from N(0) to N(x) and g from $R - (N(0) \cup \{0\})$ to $R - (N(x) \cup \{x\})$ which both send y to $x-y$ are graph isomorphisms. Hence, we may only study the subconstituents at $x = 0$ and we write $C^{(1)}(R)$ and $C^{(2)}(R)$ for the first subconstituent and the second subconstituent of C(R) at $x = 0$ in R, respectively. Let F be a finite field. We study $C^{(1)}(M_n(F))$ and $C^{(2)}(M_n(F))$. The graph $C^{(1)}(M_n(F))$ is defined on the group of invertible $n \times n$ matrices over F and the graph $C^{(2)}(M_n(F))$

is defined on the set of nonzero non-invertible matrices over F. We have the structure of $C^{(1)}(M_n(F))$ and $C^{(2)}(M_n(F))$. We can determine the spectra of $C^{(1)}(M_2(F))$ and $C^{(2)}(M_2(F))$ and conclude hyperenergeticity and Ramanujan property for both graphs. In addition, we compute the clique number, the chromatic number and the independence number of $C^{(1)}(M_2(F))$ and $C^{(2)}(M_2(F))$. This work is published in Rattanakangwanwong J. and Meemark Y., Subconstituents of unitary Cayley graph of matrix algebras, Finite Fields Appl. 2022; 80: #102004.

4. A *hypergraph* H is a pair (V(H),E(H)) where V(H) is a finite set, called the *vertex set* of H, and E(H) is a family of subsets of V(H), called the *edge set* of H. Let $(G,\cdot)$ be a finite group with the identity e and S a subset of G – {e} such that $S = S^{-1}$. For a positive integer t and $2 \leq t \leq \max\{o(x) : x \text{ in } S\}$, the t-Cayley hypergraph of G over S is a hypergraph H with vertex set V(H) = G and $E(H) = \{\{yx^i : 0 \leq i \leq t-1\} : x \text{ in } S \text{ and } y \text{ in } G\}$. It is denoted by t-Cay(G,S). We study spectral properties of this graph. We characterize integral 2-Cayley hypergraphs of G when G is abelian.

Theorem Let G be a finite abelian group and S a subset of G–{e} such that $S = S^{-1}$. Suppose $G = \mathbf{Z}_{n_1} \times \ldots \times \mathbf{Z}_{n_r}$ and $S = S_1 \times \ldots \times S_r$. The Cayley graph 2-Cay(G, S) is integral if and only if for any i in {1, … , r} such that $S_i \neq \{0\}$, the 2-Cay(G_i, S_i) is integral.

In addition, we obtain the algebraic degree of t-Cayley hypergraphs of $\mathbf{Z}_n$.

Theorem Let H = t-Cay($\mathbf{Z}_n$,S) and $C = S \cup 2S \cup \ldots \cup (t-1)S - \{0\}$. Let m be the number of y in {0,1, … , n-1} such that gcd(y,n) = 1 and there is a positive integer n_y with $C = C_1 \cup \ldots \cup C_{n_y}$, $yC_l \equiv C_l \bmod n$ and $a_{0,k} = a_{0,yk}$ for all k in C_l and l in $\{1,2, \ldots ,n_y\}$. Then

$$\deg H = \phi(n)/m \leq \phi(n)/2.$$

This work appears in Sripaisan N. and Meemark Y., Algebraic degree of spectra of Cayley hypergraphs, Discret. Appl. Math. **316**, 2022, 87–94.

Keywords (คำหลัก) : Cayley hypergraph, Eigenvalue, Matrix graph, Unitary Cayley graph, Zero divisor graph.

Output จากโครงการวิจัยที่ได้รับทุนจาก สกว.

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ ทั้งหมด 5 เรื่อง
    1.1 Sripaisan N. and Meemark Y., Algebraic degree of spectra of Cayley hypergraphs, Discret. Appl. Math. **316**, 2022, 87–94. (SJR Q2)
    1.2 Rattanakangwanwong J. and Meemark Y., Subconstituents of unitary Cayley graph of matrix algebras, Finite Fields Appl. 2022; 80: #102004. (JCR Q1)
    1.3 Rattanakangwanwong J. and Meemark Y., Eigenvalues of zero divisor graphs of principal ideal rings, Linear Multilinear Algebra 2021. DOI:10.1080/03081087.2021.1917501. (JCR Q1)
    1.4 Sirisuk S. and Meemark Y., Matrix graphs and MRD codes over finite principal ideal rings, Finite Fields Appl. 2020; 66: #101705. (JCR Q1)
    1.5 Rattanakangwanwong J. and Meemark Y., Unitary Cayley graphs of matrix rings over finite commutative rings, Finite Fields Appl. 2020; 65: #101689. (JCR Q1)
2. การนำผลงานวิจัยไปใช้ประโยชน์
    เชิงวิชาการ พัฒนาการเรียนการสอน พีชคณิตนามธรรม ทฤษฎีจำนวน ทฤษฎีกราฟเชิงพีชคณิตในระดับอุดมศึกษา สร้างและพัฒนาศักยภาพนักวิจัยไทย 3 ท่าน
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3. อื่นๆ (เช่น ผลงานตีพิมพ์ในวารสารวิชาการในประเทศ การเสนอผลงานในที่ประชุมวิชาการ หนังสือ การจดสิทธิบัตร)
    ไม่มี

สารบัญ

Chapter 1

Eigenvalues of Zero Divisor Graphs of Principal Ideal Rings

Throughout this chapter, a ring always contains the multiplicative identity $1 \neq 0$.

1.1 Zero divisor graphs of finite chain rings

Let R be a finite commutative ring. We denote its group of units by $R^\times$ and write $Z(R)$ for the set of its zero divisors. Recall that we have the disjoint union $R = \{0\} \cup R^\times \cup Z(R)$. The set $Z(R)$ can be empty if R is a field. Note that if u is a unit of R and z is a zero divisor of R, then uz is a zero divisor of R. Thus, the left multiplication induces an action of the group of units of R on the set of zero divisors of R. The *zero divisor graph* of R, $\mathcal{Z}_R$, is a graph whose vertex set is the set of all zero divisors of R and two zero divisors are adjacent if and only if their product is zero.

A *local ring* is a commutative ring with unique maximal ideal. A finite commutative ring R is called a *finite chain ring* if for any ideals I and J of R, we have $I \subseteq J$ or $J \subseteq I$. It is clear that a finite chain ring must be a local ring and every finite field and the ring of integers modulo a prime power are finite chain rings. Also, we can show that if R is a finite chain ring with maximal ideal M and $\theta \in M \setminus M^2$, then $M = R\theta$. In other words, the maximal ideal of a finite chain ring is principal. It is also known that a ring is a finite chain ring if and only if it is a finite principal ideal ring. In particular, the unique maximal ideal of a finite chain ring is a principal ideal generated by a nilpotent element.

Now, let R be a finite chain ring with unique principal maximal ideal $M = R\theta$ for some $\theta \in M \setminus M^2$ and $\Bbbk \cong R/M$ its residue field. Then $R^\times = R \setminus R\theta$ and $Z(R) = R\theta \setminus \{0\}$. We shall repeatedly use basic properties of a finite chain ring taken from [3, 5] and recorded in the next proposition.

Proposition 1.1.1. 1. *There is the smallest positive integer t such that $\theta^t = 0$, called the nilpotency of R.*

2. *For any non-zero element r in R, there is a unique integer i, $0 \leq i < t$ such that $r = u\theta^i$ for some unit u in R.*

3. *Assume that $1 \leq i < j \leq t$ and $r \in R$. If $r\theta^i \in R\theta^j$, then $r \in R\theta^{j-i}$. In particular, if $r\theta^i = 0$, then $r \in R\theta^{t-i}$.*

4. *If $\{v_1, \dots, v_q\}$ is a system of coset representatives of M in R where $q = |\Bbbk|$, then for each r in R, there are unique $r_0, \dots, r_{t-1}$ in $\{v_1, \dots, v_q\}$ such that*

$$r = r_0 + r_1\theta + \cdots + r_{t-1}\theta^{t-1}.$$

5. $|R\theta^i| = |\mathbb{k}|^{t-i}$ *for all* $i \in \{0, 1, \ldots, t-1\}$.

6. *For each* $i \in \{0, 1, \ldots, t-1\}$, $|R\theta^i / R\theta^{i+1}| = |\mathbb{k}|$.

The orbits under action of the unit groups are $R^\times \cdot \theta^i, 1 \leq i \leq t$. The size of the stabilizers and the size of the orbits are determined in the following propositions.

Proposition 1.1.2. $|\operatorname{Stab}_{R^\times}(\theta^i)| = |\mathbb{k}|^i$ *and* $|R^\times \cdot \theta^i| = |\mathbb{k}|^{t-i} - |\mathbb{k}|^{t-i-1} = |\mathbb{k}|^{t-i-1}(|\mathbb{k}| - 1)$ *for all* $i \in \{1, 2, \ldots, t-1\}$.

Proof. Let $i \in \{1, 2, \ldots, t-1\}$. Note that for $a \in R$, we have $a \in \operatorname{Stab}_{R^\times}(\theta^i) \Leftrightarrow (a-1)\theta^i = 0$. It follows from Proposition 1.1.1 (3) that $\operatorname{Stab}_{R^\times}(\theta^i) = \{1 + d\theta^{t-i} : d \in R\}$. Since

$$1 + d_1\theta^{t-i} = 1 + d_2\theta^{t-i} \Leftrightarrow d_1 - d_2 \in R\theta^i,$$

the size of $\operatorname{Stab}_{R^\times}(\theta^i)$ is $|R/R\theta| = |\mathbb{k}|^i$. The orbit-stabilizer theorem implies that the size of the orbit

$$|R^\times \cdot \theta^i| = \frac{|R^\times|}{\operatorname{Stab}_{R^\times}(\theta^i)} = \frac{|\mathbb{k}|^t - |\mathbb{k}|^{t-1}}{|\mathbb{k}|^i} = |\mathbb{k}|^{t-i} - |\mathbb{k}|^{t-i-1}.$$

This completes the proof. □

To study the zero divisor graph of R, we may assume that R is not a field. So we have $t \geq 2$. Furthermore, our definition allows the zero divisor graph to have loops. Note that if a and b are zero divisors in the same orbit $R^\times \cdot \theta^i$ for some $1 \leq i < t$, then $a = u\theta^i$ and $b = v\theta^i$ for some units u and v, for any zero divisor z of R, we have

$$az = 0 \Leftrightarrow u\theta^i z = 0 \Leftrightarrow \theta^i z = 0 \Leftrightarrow v\theta^i z = 0 \Leftrightarrow bz = 0.$$

Next, assume that a is in the orbit $R^\times \cdot \theta^i$ and b is in the orbit $R^\times \cdot \theta^j$ for some $1 \leq i, j < t$. Then $a = u\theta^i$ and $b = v\theta^j$ for some units u and v in R. If $ab = 0$, then $i + j$ must be at least t, so $aw\theta^j = uw\theta^{i+j} = 0$ for any unit w in R. Hence, we have the following lemma.

Lemma 1.1.3. *Let a and b be zero divisors of R.*

1. *If a and b are in the same orbit of the action of units by left multiplication, then a and b have the same neighbors in $\mathcal{Z}_R$.*

2. *If a is adjacent to b in the zero divisor graph, then a is adjacent to all zero divisors in the same orbit of b.*

For each $1 \leq i < t$, let H_i be the subgraph of $\mathcal{Z}_R$ induced by $R^\times \cdot \theta^i$. Then there are $t-1$ such subgraphs. It is easy to see that these subgraphs are either complete or empty (having no edges) and H_i is complete if and only if $2i \geq t$. Moreover, if $1 \leq i < j < t$ such that $i + j \geq t$ and H_i and H_j are empty, then the subgraph induced by $R^\times \cdot \theta^i \cup R^\times \cdot \theta^j$ is a complete bipartite graph by Lemma 1.1.3. We record this observation in the next theorem.

Theorem 1.1.4.

1. *There are $t - \lceil \frac{t}{2} \rceil$ induced subgraphs which are complete.*

2. *There are $\lceil \frac{t}{2} \rceil - 1$ induced subgraphs which have no edges.*

3. *If i and j are two integers such that $1 \leq i < j < t$ and $i + j \geq t$ and H_i and H_j have no edges, then the subgraph induced by $R^\times \cdot \theta^i \cup R^\times \cdot \theta^j$ is a complete bipartite graph.*

The determinant, rank, nullity and eigenvalues of the adjacency matrix of a graph are called the *determinant, rank, nullity and eigenvalues* of a graph. First, we find the determinant of the zero divisor graph of R. Note that if there is an orbit containing more than one element, then each element in the same orbit has the same neighborhood by Lemma 1.1.3, so the rows corresponding to them are identical and force that its determinant becomes zero. Next, we consider the case that every orbit contains exactly one element. Since $|R^\times \cdot \theta| = |\mathbb{k}|^{t-2}(|\mathbb{k}| - 1)$, we have $t = 2$ and $|\mathbb{k}| = 2$. Then $|R| = |\mathbb{k}|^2 = 4$. Hence, R is a finite chain ring of order 4 with maximal ideal of size 2, so $Z(R) = \{a\}$ is a singleton and $a^2 = 0$. Therefore, the determinant is 1. Finally, we remark from [5] that a finite chain ring R of order 4 with maximal ideal of size 2 is $\mathbb{Z}_2[x]/(x^2)$ of characteristic two or $\mathbb{Z}_4$ of characteristic four. We conclude the result of the zero divisor graph of a finite chain ring in the next proposition.

Proposition 1.1.5. *The determinant of the zero divisor graph of a finite chain ring of R is 0 unless R is isomorphic to $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_4$ where the determinant equals 1.*

Assume that R is a finite chain ring in which the determinant of the zero divisor graph $\mathcal{Z}_R$ is 0. It follows that 0 is an eigenvalue of $\mathcal{Z}_R$ with multiplicity being the nullity of $\mathcal{Z}_R$. From the rank theorem, we also know that the sum of the nullity of $\mathcal{Z}_R$ and the rank of $\mathcal{Z}_R$ is the number of zero divisors of R which equals $|R\theta| - 1 = |\mathbb{k}|^{t-1} - 1$. Hence, to determine the multiplicity of the eigenvalue 0, we may compute the rank of $\mathcal{Z}_R$. We eliminate the redundant of the repeated rows by considering the reduction graph $\pi\mathcal{Z}_R$ whose vertices are the orbits: $R^\times \cdot \theta, R^\times \cdot \theta^2, \ldots, R^\times \cdot \theta^{t-1}$ and the vertices $R^\times \cdot \theta^i$ and $R^\times \cdot \theta^j$ are adjacent if and only if $i + j \geq t$. This reduction graph is also called the compressed zero divisor graphs studied in [4]. Write $A(\mathcal{Z}_R)$ and $A(\pi\mathcal{Z}_R)$ for the adjacency matrix of $\mathcal{Z}_R$ and $\pi\mathcal{Z}_R$, respectively. Since for each element in the orbit $R^\times \cdot \theta^i$, its row in $A(\mathcal{Z}_R)$ is identical, we have $\operatorname{rank} A(\mathcal{Z}_R) \leq t - 1$. Also, $\operatorname{rank} A(\pi\mathcal{Z}_R) \geq \operatorname{rank} A(\mathcal{Z}_R)$ because $A(\pi\mathcal{Z}_R)$ is obtained by deleting repeated rows in $A(\mathcal{Z}_R)$. We proceed to show that:

Proposition 1.1.6. $\operatorname{rank}(A(\mathcal{Z}_R)) = t - 1$.

Proof. From the above inequalities, it suffices to show that $\operatorname{rank}(A(\pi\mathcal{Z}_R)) = t - 1$. Since $\pi\mathcal{Z}_R$ has $t - 1$ vertices and

$$A(\pi\mathcal{Z}_R) = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 1 \\ 0 & 0 & \ldots & 1 & 1 & 1 \\ \vdots & & \ddots & \vdots & & \\ 1 & 1 & \ldots & 1 & 1 & 1 \end{bmatrix}$$

directly from its definition, $\operatorname{rank} A(\pi\mathcal{Z}_R) = t - 1$. $\square$

Observe that if R is isomorphic to $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_4$, then the rank of $A(\mathcal{Z}_R)$ is 1 which also equals $t - 1$. Hence, we have shown:

Theorem 1.1.7. *For any finite chain ring R with nilpotency t, the rank of the graph $\mathcal{Z}_R$ is $t - 1$ and the multiplicity of the eigenvalue 0 is $|\mathbb{k}|^t - t$.*

For $i \in \{1, 2, \ldots, t - 1\}$, let $m_i = |R^\times \cdot \theta^i| = |\mathbb{k}|^{t-i-1}(|\mathbb{k}| - 1)$. Then

$$A(\mathcal{Z}_R) = \left[\begin{array}{r} \boxed{J_1} \\ \boxed{\quad J_2 \quad} \\ \boxed{\qquad\quad J_3 \qquad\quad} \\ \vdots \qquad\qquad \\ \boxed{\qquad\qquad\quad J_{t-1} \qquad\qquad\quad} \end{array}\right] \begin{array}{l} \}\, m_1 \\ \}\, m_2 \\ \}\, m_3 \\ \\ \}\, m_{t-1} \end{array}$$

where J_i is the all-one matrix of dimension $m_i \times (m_{t-i} + \cdots + m_{t-2} + m_{t-1})$ for all $i \in \{1, 2, \ldots, t-1\}$. Thus, the eigenvectors of $\mathcal{Z}_R$ corresponding to the eigenvalue 0 are the ones coming from the nullspace of the echelon matrix

$$\begin{bmatrix} \vec{J}_1 & & & & \\ & \vec{J}_2 & & & \\ & & \vec{J}_3 & & \\ & & & \ddots & \\ & & & & \vec{J}_{t-1} \end{bmatrix}$$

where $\vec{J}_i$ is the all-one row vector of size m_i for all $i \in \{1, 2, \ldots, t-1\}$.

Assume that λ is a nonzero eigenvalue of $A(\mathcal{Z}_R)$ with an eigenvector $\vec{V}$. Then $\vec{V}$ can be divided into a block vector

$$\vec{V} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_{t-2} \\ \vec{v}_{t-1} \end{bmatrix} \text{ where } \vec{v}_i = \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{i,m_i} \end{bmatrix} \text{ for all } i \in \{1, 2, \ldots, t-1\}$$

such that

$$J_1 \vec{v}_{t-1} = \lambda \vec{v}_1,\ J_2 \begin{bmatrix} \vec{v}_{t-2} \\ \vec{v}_{t-1} \end{bmatrix} = \lambda \vec{v}_2, \ldots,\ J_{t-1} \vec{V} = \lambda \vec{v}_{t-1}.$$

Since $\lambda \neq 0$, we have $v_{i1} = v_{i2} = \ldots = v_{im_i}$ for all $i \in \{1, 2, \ldots, t-1\}$. It follows that

$$\begin{aligned} m_{t-1} v_{t-1,1} &= \lambda v_{1,1} \\ m_{t-2} v_{t-2,1} + m_{t-1} v_{t-1,1} &= \lambda v_{2,1} \\ &\vdots \\ m_1 v_{1,1} + \cdots + m_{t-2} v_{t-2,1} + m_{t-1} v_{t-1,1} &= \lambda v_{t-1,1} \end{aligned}$$

and so λ is an eigenvalue of $\overline{A} = \begin{bmatrix} 0 & \cdots & 0 & m_{t-1} \\ 0 & \cdots & m_{t-2} & m_{t-1} \\ \vdots & & \vdots & \vdots \\ m_1 & \cdots & m_{t-2} & m_{t-1} \end{bmatrix}$ with an eigenvector $\begin{bmatrix} \vec{v}_{1,1} \\ \vec{v}_{2,1} \\ \vdots \\ \vec{v}_{t-1,1} \end{bmatrix}$. Moreover, the remaining $t-1$ independent eigenvectors of $\mathcal{Z}_R$ corresponding to nonzero eigenvalues can be obtained from the ones of $\overline{A}$. This completes the study of the eigenvalues and eigenvectors of the zero divisor graph $\mathcal{Z}_R$ where R is a finite chain ring.

1.2 Zero divisor graphs of principal ideal rings

Let R be a finite commutative principal ideal ring. Then every ideal of R is principal. Recall that a finite commutative ring is a direct product of finite local rings. Since every ideal of R is principal, so are its factors. Therefore, R is a direct product of finite chain rings.

Write $R \cong R_1 \times R_2 \times \cdots \times R_k$ where R_i is a finite chain ring with maximal ideal $R_i\theta_i$ of nilpotency t_i and residue field $\Bbbk_i = R_i / R_i\theta_i$ for all $i \in \{1, 2, \ldots, k\}$. Note that the set of zero divisors of R is the union of the direct product of orbits of the form

$$R_1^\times \cdot \theta_1^{s_1} \times R_2^\times \cdot \theta_2^{s_2} \times \cdots \times R_k^\times \cdot \theta_k^{s_k}$$

where $0 \leq s_i \leq t_i$ for all $i \in \{1, 2, \ldots, k\}$ except $R_1^\times \times R_2^\times \times \cdots \times R_k^\times$ and $\{(0, 0, \ldots, 0)\}$. Now, we consider the reduction graph $\pi\mathcal{Z}_R$ of $\mathcal{Z}_R$ whose vertices are

$$z(s_1, s_2, \ldots, s_k) = R_1^\times \cdot \theta_1^{s_1} \times R_2^\times \cdot \theta_2^{s_2} \times \cdots \times R_k^\times \cdot \theta_k^{s_k}$$

where $0 \leq s_i \leq t_i$ for all $i \in \{1, 2, \ldots, k\}$ except $s_1 = s_2 = \cdots = s_k = 0$ or $(s_1 = t_1, s_2 = t_2, \ldots, s_k = t_k)$ and $z(s_1, s_2, \ldots, s_k)$ and $z(s_1', s_2', \ldots, s_k')$ are adjacent if and only if $s_i + s_i' \geq t_i$ for all $i \in \{1, 2, \ldots, k\}$. Then, the graph $\pi\mathcal{Z}_R$ has $\displaystyle\prod_{i=1}^{k}(t_i + 1) - 2$ vertices.

Remark. For $\mathbb{Z}_n \cong \mathbb{Z}_{p_1{}^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_k{}^{\alpha_k}}$ where $n = p_1{}^{\alpha_1} p_2{}^{\alpha_2} \ldots p_k{}^{\alpha_k}$, $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$, $z(s_1, \ldots, s_k)$ can be considered as the set $S(d)$ in Young [6] where $d = p_1{}^{s_1} p_2{}^{s_2} \ldots p_k{}^{s_k}$ is a divisor of n.

We order them by the lexicographical order, namely, $z(s_1, s_2, \ldots, s_k) < z(s_1', s_2', \ldots, s_k')$ if and only if

$$(s_1 < s_1') \text{ or } (s_1 = s_1' \text{ and } s_2 < s_2') \text{ or } \ldots \text{ or } (s_1 = s_1', \ldots, s_{k-1} = s_{k-1}' \text{ and } s_k < s_k').$$

Thus, the first vertex is $z(0, 0, \ldots, 0, 1)$ and the last one is $z(t_1, t_2, \ldots, t_{k-1}, t_k - 1)$. Under this order of vertices, we have the adjacency matrix being in the form

$$A(\pi\mathcal{Z}_R) = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \ldots & * & * \\ 1 & * & \ldots & * & * \end{bmatrix}. \tag{1.2.1}$$

To see this, we determine the position of $z(s_1, s_2, \ldots, s_k)$ in $A(\pi\mathcal{Z}_R)$ by counting the number of vertices before it. From the definition of $<$ excluding $(0, 0, \ldots, 0)$, this number equals

$$\sum_{i=1}^{k} s_i \prod_{j=i+1}^{k} (t_j + 1) - 1,$$

so the position of $z(s_1, s_2, \ldots, s_k)$ in $A(\pi\mathcal{Z}_R)$ is $\displaystyle\sum_{i=1}^{k} s_i \prod_{j=i+1}^{k} (t_j + 1)$.

Now, let $r_1, r_2, \ldots, r_k$ be such that $0 \leq r_i \leq t_i$ and $r_i + s_i \geq t_i$ for all $i \in \{1, 2, \ldots, k\}$. In other words, the vertices $z(r_1, r_2, \ldots, r_k)$ and $z(s_1, s_2, \ldots, s_k)$ are adjacent. Then

$$\sum_{i=1}^{k}(r_i + s_i) \prod_{j=i+1}^{k} (t_j + 1) \geq \sum_{i=1}^{k} t_i \prod_{j=i+1}^{k} (t_j + 1).$$

The sum on the right hand side can be simplified to

$$\begin{aligned} \sum_{i=1}^{k} t_i \prod_{j=i+1}^{k} (t_j + 1) &= \sum_{i=1}^{k-2} t_i \prod_{j=i+1}^{k} (t_j + 1) + t_{k-1}(t_k + 1) + t_k + 1 - 1 \\ &= \sum_{i=1}^{k-2} t_i \prod_{j=i+1}^{k} (t_j + 1) + (t_{k-1} + 1)(t_k + 1) + t_k + 1 \\ &\vdots \\ &= (t_1 + 1)\prod_{j=2}^{k}(t_j + 1) - 1 = \prod_{j=1}^{k}(t_j + 1) - 1 = |Z(R)| + 1 \end{aligned}$$

Thus,

$$\sum_{i=1}^{k} r_i \prod_{j=i+1}^{k} (t_j+1) + \sum_{i=1}^{k} s_i \prod_{j=i+1}^{k} (t_j+1) \geq |Z(R)|+1$$

and equality holds if and only if $r_i + s_i = t_i$ for all $\{1, 2, \ldots, k\}$. This proves (1.2.1) and it follows from (1.2.1) that

$$\operatorname{rank} A(\pi \mathcal{Z}_R) = \prod_{i=1}^{k} (t_i+1) - 2.$$

Since $\operatorname{rank} A(\mathcal{Z}_R) = \operatorname{rank} A(\pi \mathcal{Z}_R)$, we have shown:

Proposition 1.2.1. $\operatorname{rank} A(\mathcal{Z}_R) = \prod_{i=1}^{k} (t_i+1) - 2.$

Remark. The entries of $A(\pi\mathcal{Z}_R)$ below the diagonal from bottom-left corner to top-right corner may not always be 1 when R is not local. For example, if $R = \mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$, then

$$A(\pi \mathcal{Z}_R) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Next, we compute the determinant of $A(\mathcal{Z}_R)$. From the reduction graph $\pi\mathcal{Z}_R$, if a vertex $z(s_1, s_2, \ldots, s_k)$ contains more than one element, then $A(\mathcal{Z}_R)$ has some repeated rows, so $\det A(\mathcal{Z}_R) = 0$. Now, we consider the case that every vertex of $\pi\mathcal{Z}_R$ is a singleton. It follows that $|{R_i}^{\times} \cdot {\theta_i}^{s_i}| = 1$ for all $0 \leq s_i \leq t_i$ and $i \in \{1, 2, \ldots, k\}$. Since R_i is a local ring, R_i is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ for $i \in \{1, 2, \ldots, k\}$. If $k = 1$, then R_i must be $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ presented in Proposition 1.1.5. Assume that $k \geq 2$. If for some i, $R_i \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, then $|{R_i}^{\times}| = 2$ and so $|z(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_k)| > 1$. Hence, $R_i \cong \mathbb{Z}_2$ for all $i \in \{1, 2, \ldots, k\}$, so $|A(\mathcal{Z}_R)| = 2^k - 2$ and

$$\det(A(\mathcal{Z}_R)) = (-1)^{2^k-1}(-1)^{2^k-2}(-1)^{2^k-3}\ldots(-1)^3(-1)^2 = -1$$

because $k \geq 2$. We record the determinant of $A(\mathcal{Z}_R)$ in:

Proposition 1.2.2. $\det(A(\mathcal{Z}_R)) = \begin{cases} 1 & \textit{if } R \cong \mathbb{Z}_4 \textit{ or } \mathbb{Z}_2[x]/(x^2), \\ -1 & \textit{if } R \cong (\mathbb{Z}_2)^k \textit{ for some } k \geq 2, \\ 0 & \textit{otherwise.} \end{cases}$

If the determinant of $\mathcal{Z}_R$ is 0, then $\mathcal{Z}_R$ has 0 as an eigenvalue with multiplicity being the nullity of $A(\mathcal{Z}_R)$ because $A(\mathcal{Z}_R)$ is diagonalizable. Thus, the rank theorem gives that the nullity is $|Z(R)| - \operatorname{rank} A(\mathcal{Z}_R)$. Since $|Z(R)| = |R| - |R^{\times}| - 1$, using this fact and proposition 1.2.1 gives the next proposition.

Proposition 1.2.3. *If 0 is an eigenvalue of the graph $\mathcal{Z}_R$, then its multiplicity is given by*

$$\prod_{i=1}^{k} |\mathbb{k}_i|^{t_i} - \prod_{i=1}^{k} (|\mathbb{k}_i|^{t_i} - |\mathbb{k}_i|^{t_i-1}) - \prod_{i=1}^{k} (t_i+1) + 1.$$

Recall that we order the vertices of the reduction graph $\pi\mathcal{Z}_R$ by the lexicographical order. With this order, we may write the vertex set as $\{z_1, z_2, \ldots, z_N\}$ where $N = \prod_{i=1}^{k}(t_i+1) - 2$ and we denote by n_j the number of elements in z_j for all $j \in \{1, 2, \ldots, N\}$. The $(0,1)$-adjacency matrix $A = [a_{ij}]$ of $\pi\mathcal{Z}_R$ of size N in (1.2.1) lifts to the adjacency matrix $A(\mathcal{Z}_R) = [A_{ij}]$ of $\mathcal{Z}_R$ where A_{ij} is a block matrix of dimension

$m_i \times m_j$ with all-zero or all-one entries depending on the entry a_{ij} of $A(\pi\mathcal{Z}_R)$ is 0 or 1, respectively. Thus, $A(\mathcal{Z}_R)$ is a matrix of the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & J_1 \\ \mathbf{0} & \mathbf{0} & \dots & J_2 & * \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & J_{N-1} & \dots & * & * \\ J_N & * & \dots & * & * \end{bmatrix}$$

where J_j is the all-one matrix of dimension $n_j \times n_j$ for all $j \in \{1, 2, \dots, N\}$. Hence, the eigenvectors of $\mathcal{Z}_R$ corresponding to the eigenvalue 0 are the ones coming from the nullspace of the echelon matrix

$$\begin{bmatrix} \vec{J}_1 & & & & \\ & \vec{J}_2 & & & \\ & & \ddots & & \\ & & & \vec{J}_{N-1} & \\ & & & & \vec{J}_N \end{bmatrix}$$

where $\vec{J}_j$ is the all-one row vector of size n_j for all $j \in \{1, 2, \dots, N\}$.

The *independence number* of a graph Γ is the size of the largest set of pairwise nonadjacent vertices. We denote the independence number of Γ by $\alpha(\Gamma)$. Brouwer and Haemers [2] showed that for a graph Γ,

$$\alpha(\Gamma) \leq r(\Gamma) - r_+(\Gamma) \quad \text{and} \quad \alpha(\Gamma) \leq r(\Gamma) - r_-(\Gamma)$$

where $r(\Gamma)$, $r_+(\Gamma)$ and $r_-(\Gamma)$ are the number of eigenvalues, number of positive eigenvalues and number of negative eigenvalues of Γ, respectively.

Recall that $N = \operatorname{rank} A(\mathcal{Z}_R) = \operatorname{rank} A(\pi\mathcal{Z}_R)$. It follows from the adjacency in Eq. (1.2.1) that $\alpha(\pi\mathcal{Z}_R) = \lfloor \frac{N}{2} \rfloor$ and the reduction graph $\pi\mathcal{Z}_R$ has a nonzero determinant, so its eigenvalues are positive or negative. Then N is the number of nonzero eigenvalues of $\pi\mathcal{Z}_R$. We can calculate $r_+(\pi\mathcal{Z}_R)$ and $r_-(\pi\mathcal{Z}_R)$ as follows. Since

$$\lfloor \tfrac{N}{2} \rfloor \leq r(\Gamma) - r_+(\Gamma) \quad \text{and} \quad \lfloor \tfrac{N}{2} \rfloor \leq r(\Gamma) - r_-(\Gamma),$$

we have

$$r_+(\pi\mathcal{Z}_R) \leq N - \lfloor \tfrac{N}{2} \rfloor, r_-(\pi\mathcal{Z}_R) \leq N - \lfloor \tfrac{N}{2} \rfloor \quad \text{and} \quad N = r_+(\pi\mathcal{Z}_R) + r_-(\pi\mathcal{Z}_R).$$

If N is even, they force that $r_+(\pi\mathcal{Z}_R) = r_-(\pi\mathcal{Z}_R) = \frac{N}{2}$. Assume that N is odd. Then $r_+(\pi\mathcal{Z}_R)$ and $r_-(\pi\mathcal{Z}_R)$ are less than or equal to $\frac{N+1}{2}$. Since their sum is N, we get $\{r_+(\pi\mathcal{Z}_R), r_-(\pi\mathcal{Z}_R)\} = \{\frac{N+1}{2}, \frac{N-1}{2}\}$. But the determinant of $\pi\mathcal{Z}_R$ is $(-1)^{\frac{N-1}{2}}$ and the minus sign depends on $r_-(\pi\mathcal{Z}_R)$, so we must have $r_+(\pi\mathcal{Z}_R) = \frac{N+1}{2}$ and $r_-(\pi\mathcal{Z}_R) = \frac{N-1}{2}$. Proposition 1 of [1] implies that $r_+(\pi\mathcal{Z}_R) = r_+(\mathcal{Z}_R)$ and $r_-(\pi\mathcal{Z}_R) = r_-(\mathcal{Z}_R)$. Since $N = \operatorname{rank} \pi\mathcal{Z}_R = \operatorname{rank} \mathcal{Z}_R$ is also the number of nonzero eigenvalues of $\mathcal{Z}_R$, we obtain the number of positive and negative eigenvalues of $\mathcal{Z}_R$ as follows.

Theorem 1.2.4. *$r_+(\mathcal{Z}_R) = \lceil \frac{N}{2} \rceil$ and $r_-(\mathcal{Z}_R) = \lfloor \frac{N}{2} \rfloor$.*

Now, assume that λ is a nonzero eigenvalue of $A(\mathcal{Z}_R)$ with an eigenvector $\vec{V}$. Then $\vec{W}$ can be divided into a block vector

$$\vec{W} = \begin{bmatrix} \vec{w}_N \\ \vec{w}_{N-1} \\ \vdots \\ \vec{w}_2 \\ \vec{w}_1 \end{bmatrix} \text{ where } \vec{w}_i = \begin{bmatrix} w_{i1} \\ w_{i2} \\ \vdots \\ w_{i,m_i} \end{bmatrix} \text{ for all } i \in \{1, 2, \dots, N\}.$$

Note that $J_1\vec{w}_1 = \lambda\vec{w}_1$ implies $w_{11} = w_{12} = \cdots = w_{1n_1}$ because of $\lambda \neq 0$. Since $*$ in $A(\mathcal{Z}_R)$ is all-zero or all-one block, we may inductively deduce that $w_{i1} = w_{i2} = \ldots = w_{in_i}$ for all $i \in \{1, 2, \ldots, N\}$. It follows that λ is an eigenvalue of $B = \begin{bmatrix} 0 & \cdots & 0 & n_1 \\ 0 & \cdots & n_2 & b_{2N} \\ \vdots & & \vdots & \vdots \\ n_N & \cdots & b_{N,N-1} & b_{NN} \end{bmatrix}$ where for $i < j$, $b_{ij} = 0$ if $a_{ij} = 0$ and $b_{ij} = n_j$ if $a_{ij} = 1$, with an eigenvector $\begin{bmatrix} w_{N,1} \\ w_{N-1,1} \\ \vdots \\ w_{1,1} \end{bmatrix}$. Hence, the remaining N independent eigenvectors of $\mathcal{Z}_R$ corresponding to nonzero N eigenvalues can be obtained from the ones of B.

1.3 Bounds for eigenvalues

Let R be a finite commutative principal ideal ring. Write $R \cong R_1 \times R_2 \times \cdots \times R_k$ where R_i is a finite chain ring with maximal ideal $R_i\theta_i$ of nilpotency t_i and residue field $\Bbbk_i = R_i/R_i\theta_i$ for all $i \in \{1, 2, \ldots, k\}$. We proceed to find upper and lower bounds for the zero divisor graph of R in this section. Recall that the set of zero divisors of R is the union of the direct product of orbits of the form

$$z(s_1, s_2, \ldots, s_k) = R_1^\times \cdot \theta_1^{s_1} \times R_2^\times \cdot \theta_2^{s_2} \times \cdots \times R_k^\times \cdot \theta_k^{s_k}$$

where $0 \leq s_i \leq t_i$ for all $i \in \{1, 2, \ldots, k\}$ except $R_1^\times \times R_2^\times \times \cdots \times R_k^\times$ and $\{(0, 0, \ldots, 0)\}$. Consider the vertex $(u_1\theta_1^{s_1}, u_2\theta_2^{s_2}, \ldots, u_k\theta_k^{s_k})$. It is adjacent to vertices $(v_1\theta_1^{r_1}, v_2\theta_2^{r_2}, \ldots, v_k\theta_k^{r_k})$ where $v_i \in R_i^\times$ and $r_i + s_i \geq t_i$ for all $i \in \{1, 2, \ldots, k\}$ except $(0, 0, \ldots, 0)$, so the degree of the vertex $(u_1\theta_1^{s_1}, u_2\theta_2^{s_2}, \ldots, u_k\theta_k^{s_k})$ is

$$\left(\prod_{i=1}^{k} \sum_{r_i+s_i \geq t_i} |R_i^\times \cdot \theta_i^{r_i}| \right) - 1.$$

Suppose that we order the eigenvalues of $\mathcal{Z}_R$ as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. It follows from Propostion 3.1.2 of Brouwer and Haemers [2] that

$$\overline{\deg \mathcal{Z}_R} \leq \lambda_1 \leq \max \deg(\mathcal{Z}_\mathcal{R})$$

where $\overline{\deg \mathcal{Z}_R}$ is the average of degree of vertices of $\mathcal{Z}_R$ given by

$$\frac{\sum_{v \in Z(R)} \deg v}{|Z(R)|} = \frac{\sum_{v \in Z(R)} \deg v}{|R| - |R^\times| - 1}$$

since R is a finite commutative ring. Next, we determine the maximum degree and the average of degree of vertices of $\mathcal{Z}_R$. We shall assume further that $|\Bbbk_1| \leq |\Bbbk_2| \leq \cdots \leq |\Bbbk_k|$. Note that for each $i \in \{1, 2, \ldots, k\}$, we have by Proposition 1.1.2 that

$$\sum_{r_i+s_i \geq t_i} |R_i^\times \cdot \theta_i^{r_i}| = 1 + \sum_{t_i - s_i \leq r_i \leq t_i - 1} |R_i^\times \cdot \theta_i^{r_i}| = 1 + \sum_{t_i - s_i \leq r_i \leq t_i - 1} |\Bbbk_i|^{t_i - r_i - 1}(|\Bbbk_i| - 1),$$

so the geometric sum simplifies the right hand side to

$$1 + (|\Bbbk_i| - 1) \sum_{1 \leq r_i' \leq s_i} |\Bbbk_i|^{r_i' - 1} = 1 + (|\Bbbk_i|^{s_i} - 1) = |\Bbbk_i|^{s_i}.$$

Therefore, the degree of the vertex $(u_1\theta_1^{s_1}, u_2\theta_2^{s_2}, \dots, u_k\theta_k^{s_k})$ is $|\mathbb{k}_1|^{s_1}|\mathbb{k}_2|^{s_2}\dots|\mathbb{k}_k|^{s_k} - 1$ and the maximum degree attains when $s_1 = t_1 - 1$ and $s_i = t_i$ for all $i \geq 2$ and equals

$$\max\deg(\mathcal{Z}_R) = |\mathbb{k}_1|^{t_1-1}|\mathbb{k}_2|^{t_2}\dots|\mathbb{k}_k|^{t_k} - 1.$$

From the set-up at the beginning of Section 2 and the above calculation of the degree of a vertex, we obtain the average of degree of vertices of the zero divisor graph $\mathcal{Z}_R$ as

$$\frac{\displaystyle\sum_{\substack{0\leq s_1\leq t_1,\\ 0\leq s_2\leq t_2,\\ \vdots\\ 0\leq s_k\leq t_k}} (|\mathbb{k}_1|^{s_1}|\mathbb{k}_2|^{s_2}\dots|\mathbb{k}_k|^{s_k} - 1)|z(s_1, s_2, \dots, s_k)| - (|\mathbb{k}_1|^{t_1}|\mathbb{k}_2|^{t_2}\dots|\mathbb{k}_k|^{t_k} - 1)}{\displaystyle\prod_{i=1}^{k}|\mathbb{k}_i|^{t_i} - \prod_{i=1}^{k}(|\mathbb{k}_i|^{t_i} - |\mathbb{k}_i|^{t_i-1}) - 1}$$

where

$$|z(s_1, s_2, \dots, s_k)| = \prod_{i=1}^{k}|R_i^{\times}\cdot\theta_i^{s_i}| = \prod_{j,\, s_j\leq t_j-1}|\mathbb{k}_j|^{t_j-s_j}\left(1 - \frac{1}{|\mathbb{k}_j|}\right)$$

for all $0 \leq s_i \leq t_i$ and $i \in \{1, 2, \dots, k\}$. Hence, we have an upper bound and a lower bound for the largest eigenvalue of $\mathcal{Z}_R$.

1.4 Wiener index

Througout this section, R is a finite commutative principal ideal ring. Write $R \cong R_1 \times R_2 \times \dots \times R_k$ where R_i is a finite chain ring with maximal ideal $R_i\theta_i$ of nilpotency t_i and residue field $\mathbb{k}_i = R_i/R_i\theta_i$ for all $i \in \{1, 2, \dots, k\}$. The *Wiener index* of a connected graph G is the sum $\sum_{u,v\in V(G)} d_G(u, v)$ where $d_G(u, v)$ is the distance of u and v in the graph G. We will compute the Wiener index of $\mathcal{Z}_R$. First, we characterize all finite commutative principal ideal rings such that their zero divisor graphs are complete. It is clear that if R is a finite chain ring with nilpotency 2 or $R = F_1 \times F_2$, where F_1 and F_2 is a finite field, then $\mathcal{Z}_R$ is a complete graph. Next, assume $k \geq 3$. Thus, elements in $R_1^{\times} \times \{0\} \times R_3^{\times} \times \dots \times R_k^{\times}$ are not adjacent to elements in $R_1^{\times} \times R_2^{\times} \times \{0\} \times \dots \times R_k^{\times}$. Now, assume that $R = R_1 \times R_2$. Suppose $t_1 \geq 2$ or $t_2 \geq 2$, say $t_1 \geq 2$. It follows that elements in $R_1^{\times} \times \{0\}$ are not adjcent to elements in $R_1^{\times}\theta_1 \times R_2^{\times}$. Hence, we can conclude that R_1 and R_2 must be fields. Finally, we assume that R is a finite chain ring such that $\mathcal{Z}_R$ is a complete graph. If R has nilpotency $t \geq 3$, then the elements in $R^{\times}\theta$ are not adjacent, so R must have nilpotency 2. We record this result in the following theorem.

Theorem 1.4.1. *Let R be a finite principal ideal ring. Then $\mathcal{Z}_R$ is a complete graph if and only if R is a finite chain ring with nilpotency 2 or $R = F_1 \times F_2$ where F_1 and F_2 are finite fields. In this case, its Wiener index is given by $\binom{|Z(R)|}{2}$.*

Theorem 1.4.2. *Let $R = R_1 \times \dots \times R_k$ where $R_1, \dots, R_k$ are finite chain rings. Assume that $\mathcal{Z}_R$ is not a complete graph. For a proper subset X of $\{1, 2, \dots, k\}$, we define*

$$z(X) = \{z(s_1, \dots, s_k) \in V(\pi\mathcal{Z}_R) : 0 < s_i \leq t_i \text{ for all } i \in X \text{ and } s_i = 0 \text{ for all } i \notin X\}.$$

Under the set-up at the beginning of this section, we have the following statements.

(1) If $k = 1$, that is, R is a finite chain ring with nilpotency t, then the Wiener index of $\mathcal{Z}_R$ is given by

$$\sum_{\substack{0<s,s'<t \\ s+s'\geq t}} |R^\times \cdot \theta^s||R^\times \cdot \theta^{s'}| + 2 \sum_{\substack{0<s,s'<t \\ s+s'<t}} |R^\times \cdot \theta^s||R^\times \cdot \theta^{s'}|.$$

(2) If $k \geq 2$, then the Wiener index of $\mathcal{Z}_R$ is given by

$$\sum_{z(s_1,\ldots,s_k)\sim z(s'_1,\ldots,s'_k)} |z(s_1,\ldots,s_k)||z(s'_1,\ldots,s'_k)| \\ + 2{\sum}' |z(s_1,\ldots,s_k)||z(s'_1,\ldots,s'_k)| + 3{\sum}'' |z(s_1,\ldots,s_k)||z(s'_1,\ldots,s'_k)|$$

where ${\sum}'$ is the sum over $z(s_1,\ldots,s_k) \in z(X)$ and $z(s'_1,\ldots,s'_k) \in z(Y)$ which are non-adjacent in $\pi\mathcal{Z}_R$ and $X \cap Y \neq \emptyset$ and ${\sum}''$ is the sum over $z(s_1,\ldots,s_k) \in z(X)$ and $z(s'_1,\ldots,s'_k) \in z(Y)$ which are non-adjacent in $\pi\mathcal{Z}_R$ and $X \cap Y = \emptyset$.

Proof. Recall that the set $V(\pi\mathcal{Z}_R)$ is a partition of the set of zero divisors of R. Then for any $u \in Z(R)$, there exists a unique $z_u \in V(\pi\mathcal{Z}_R)$ containing u. It follows that $d_{\mathcal{Z}_R}(u,v) = d_{\pi\mathcal{Z}_R}(z_u, z_v)$. We use this observation to calculate the Wiener index of $\mathcal{Z}_R$.

First, we handle the case R being a finite chain ring with nilpotency t. Let s, s' be such that $0 < s, s' < t$ and $s + s' < t$. Let $k = \max\{t - s, t - s'\}$. We have $0 < k < t$, $k + s \geq t$ and $k + s' \geq t$. Then $R^\times \cdot \theta^k$ is adjacent to both $R^\times \cdot \theta^s$ and $R^\times \cdot \theta^{s'}$ in $\pi\mathcal{Z}_R$, so $d_{\pi\mathcal{Z}_R}(R^\times \cdot \theta^s, R^\times \cdot \theta^{s'}) = 2$ whenever $s + s' < t$. Hence, its Wiener index is given by

$$\sum_{\substack{0<s,s'<t \\ s+s'\geq t}} |R^\times \cdot \theta^s||R^\times \cdot \theta^{s'}| + 2 \sum_{\substack{0<s,s'<t \\ s+s'<t}} |R^\times \cdot \theta^s||R^\times \cdot \theta^{s'}|.$$

Second, we assume that $k \geq 2$ and let X, Y be proper subsets of $\{1, 2, \ldots, k\}$. Let $z(s_1,\ldots,s_k) \in z(X)$ and $z(s'_1,\ldots,s'_k) \in z(Y)$ be nonadjacent vertices in $\pi\mathcal{Z}_R$. Suppose that $X \cap Y \neq \emptyset$. There are two cases to consider.

Case 1. There exists $i \in X \cap Y$ such that $s_i, s'_i < t_i$. Then $d_{\pi\mathcal{Z}_R}(z(s_1,\ldots,s_k), z(s'_1,\ldots,s'_k)) = 2$ by the same method as in the case where R was a finite chain ring above.

Case 2. $s_i = s'_i = t_i$ for all $i \in X \cap Y$. For simplicity, we assume $X \cap Y = \{1, 2, \ldots, m\}$. Then $R_1^\times \times \cdots \times R_m^\times \times \{0\} \times \cdots \times \{0\}$ is adjacent to both $z(s_1,\ldots,s_k)$ and $z(s'_1,\ldots,s'_k)$, so we also have $d_{\pi\mathcal{Z}_R}(z(s_1,\ldots,s_k), z(s'_1,\ldots,s'_k)) = 2$.

Next, we assume that X and Y are disjoint. We may write $X = \{1,\ldots,p\}$ and $Y = \{p+1,\ldots,q\}$ where $q \leq k$. We can see that $z(s_1,\ldots,s_k)$ and $z(s'_1,\ldots,s'_k)$ have no common neighbors. However,

$$\begin{aligned} z(s_1,\ldots,s_k) &\sim z(t_1 - s_1,\ldots,t_p - s_p, t_{p+1},\ldots,t_k) \\ &\sim z(t_1,\ldots,t_p, t_{p+1} - s'_{p+1},\ldots,t_q - s'_q, t_{q+1},\ldots,t_k) \\ &\sim z(s'_1,\ldots,s'_k). \end{aligned}$$

where $\sim$ means adjacency in $\pi\mathcal{Z}_R$. We can conclude that $d_{\pi\mathcal{Z}_R}(z(s_1,\ldots,s_k), z(s'_1,\ldots,s'_k)) = 3$.

From the above calculations, the Wiener index can be obtained from the sum

$$\sum_{z(s_1,\ldots,s_k)\sim z(s'_1,\ldots,s'_k)} |z(s_1,\ldots,s_k)||z(s'_1,\ldots,s'_k)| \\ + 2{\sum}' |z(s_1,\ldots,s_k)||z(s'_1,\ldots,s'_k)| + 3{\sum}'' |z(s_1,\ldots,s_k)||z(s'_1,\ldots,s'_k)|$$

where $\sum'$ is the sum over $z(s_1,\dots,s_k)\in z(X)$ and $z(s'_1,\dots,s'_k)\in z(Y)$ which are non-adjacent in $\pi\mathcal{Z}_R$ and $X\cap Y\neq\emptyset$ and $\sum''$ is the sum over $z(s_1,\dots,s_k)\in z(X)$ and $z(s'_1,\dots,s'_k)\in z(Y)$ which are non-adjacent in $\pi\mathcal{Z}_R$ and $X\cap Y=\emptyset$. $\square$

Finally, we deduce the Wiener index of $\mathcal{Z}_{\mathbb{Z}_n}$ for all $n\in\mathbb{N}$ and $n\geq 3$. Let $n={p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots {p_k}^{\alpha_k}$ where $p_1,p_2,\dots,p_k$ are distinct primes and $\alpha_1,\alpha_2,\dots\alpha_k\in\mathbb{N}$, and let $0\leq s_i\leq\alpha_i$ for all $i\in\{1,\dots,k\}$. According to the first remark in Section 3, we have $z(s_1,\dots,s_k)$ is the set $S({p_1}^{s_1}{p_2}^{s_2}\dots {p_k}^{s_k})$ so we know from Young [6] that

$$|z(s_1,\dots,s_k)|=\phi\left(\frac{n}{{p_1}^{s_1}{p_2}^{s_2}\dots {p_k}^{s_k}}\right)$$

where ϕ is the Euler phi-function. In other words, if $d={p_1}^{s_1}{p_2}^{s_2}\dots {p_k}^{s_k}$ is a divisor of n, then $|z(s_1,\dots,s_k)|=\phi\left(\frac{n}{d}\right)$. Moreover, let d_i and d_j be nonadjacent vertices in $\mathcal{Z}_{\mathbb{Z}_n}$ corresponding to the vertex $z(s_1,s_2,\dots,s_k)\in z(X)$ and $z(s'_1,s'_2,\dots,s'_k)\in z(Y)$ where X and Y are proper subsets of $\{1,2,\dots,n\}$, respectively. Note that d_i and d_j are relatively prime if $X\cap Y=\emptyset$ and they have a common divisor otherwise. Using this observation, Theorem 1.4.2 (2) gives us the Wiener index of $\mathcal{Z}_{\mathbb{Z}_n}$.

Corollary 1.4.3. *Let n be a positive integer greater than 3. Let $d_1,\dots,d_l$ be all proper divisors of n. Then the Wiener index of $\mathcal{Z}_{\mathbb{Z}_n}$ is given by*

$$\sum_{d_i\sim d_j}\phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right)+2\sum_{\substack{d_i\not\sim d_j\\ \gcd(d_i,d_j)\neq 1}}\phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right)+3\sum_{\substack{d_i\not\sim d_j\\ \gcd(d_i,d_j)=1}}\phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right).$$

Here, ϕ is the Euler phi-function.

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Chapter 2

Matrix Graphs and MRD Codes over Finite Principal Ideal Rings

Throughout the chapter, our rings always contain the identity $1 \neq 0$.

2.1 Ranks and background from graph theory

In this section, we first discuss the rank and the McCoy rank of matrices. Then we define the matrix graph and recall some terminologies and results from graph theory. We divide them into two subsections.

2.1.1 Rank of matrices

Let R be a commutative ring. We write $R^\times$ for the set of unit in R and the set of $m \times n$ matrices with entries in R is denoted by $R^{m\times n}$. Cohn [5] introduced the concept of rank of matrices over commutative rings which generalizes the usual rank of matrices over fields.

For a nonzero matrix A in $R^{m\times n}$, the *rank* of A, denoted by $\operatorname{rank} A$, is the least positive integer t such that $A = BC$ where $B \in R^{m\times t}$ and $C \in R^{t\times n}$. The rank of the zero matrix is defined to be 0.

This rank of matrices has some basic properties as the usual rank over fields. For instance, if $A, B \in R^{m\times n}$, then $\operatorname{rank} A \leq \min\{m, n\}$, $\operatorname{rank} A = 0$ if and only if $A = 0$, $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$, and $\operatorname{rank} A = \operatorname{rank} PAQ$ where $P \in GL_m(R)$ and $Q \in GL_n(R)$, see [5],[6],[11] for more properties.

Now, we assume that R is a finite commutative ring. It is well known that R can be decomposed as $R \cong R_1 \times R_2 \times \cdots \times R_\ell$ where $R_1, R_2, \ldots R_\ell$ are finite local rings. Let ρ_j be the projection map from R to R_i for all $i \in \{1, 2, \ldots, \ell\}$. Here, a *local ring* is a commutative ring with unique maximal ideal. Recall that if R is a local ring with unique maximal ideal M, then $R^\times = R \setminus M$ and the field R/M is called the *residue field* equipped with the canonical map $\pi : R \to R/M$ given by $\pi(r) = r + M$ for all $r \in R$.

Proposition 2.1.1. *If $A \in R^{m\times n}$, then*

$$\operatorname{rank} A = \max_{1\leq i\leq \ell}\{\operatorname{rank} \rho_i(A)\}.$$

Proof. Suppose that $\operatorname{rank} A = t$. Then $A = BC$ for some $B \in R^{m\times t}$ and $C \in R^{t\times n}$. For each $i \in \{1, 2, \ldots, \ell\}$, we have $\rho_i(A) = \rho_i(B)\rho_i(C)$, so that $\operatorname{rank} \rho_i(A) \leq t$. On the other hand, let $\operatorname{rank} \rho_i(A) = t_i$ for all $i \in \{1, 2, \ldots, \ell\}$. Then for each $i \in \{1, 2, \ldots, \ell\}$, we have t_i is the least integer such that $\rho_i(A) = B'_iC'_i$ where $B'_i \in R_i^{m\times t_i}$ and $C'_i \in R_i^{t_i\times n}$. Without loss of generality, suppose that $\max_{1\leq i\leq \ell}\{\operatorname{rank} \rho_i(A)\} = t_1$.

Set $B_i = (B_i', 0) \in R^{m\times t_1}$ and $C_i = \begin{pmatrix} C_i' \\ 0 \end{pmatrix} \in R^{t_1\times n}$. Then $A = BC$ where $B = (B_1, B_2, \ldots, B_\ell) \in R^{m\times t_1}$ and $C = (C_1, C_2, \ldots, C_\ell) \in R^{t_1\times n}$. Thus, $\operatorname{rank} A \le t_1$. Therefore, $\operatorname{rank} A = \max_{1\le i\le \ell}\{\operatorname{rank}\rho_i(A)\}$. $\square$

Later, McCoy [13] gave another definition of rank of matrices over commutative rings which also generalizes the usual rank of matrices over fields. This rank is described by the annihilators of ideals as follows.

Let R be a commutative ring and $A \in R^{m\times n}$. We define $I_0 = R$ and $I_t(A)$ to be the ideal of R generated by the $t \times t$ minors of A for $1 \le t \le \min\{m, n\}$. Note that

$$R = I_0(A) \supseteq I_1(A) \supseteq \cdots \supseteq I_{\min\{m,n\}}(A)$$

and so

$$\{0\} = \operatorname{Ann}_R I_0(A) \subseteq \operatorname{Ann}_R I_1(A) \subseteq \cdots \subseteq \operatorname{Ann}_R I_{\min\{m,n\}}(A)$$

where the *annihilator of I* is given by $\operatorname{Ann}_R I = \{r \in R : ra = 0 \text{ for all } a \in I\}$. The *Mc-rank of A*, $\text{Mc-rank}\, A$, is the largest integer r such that $\operatorname{Ann}_R I_r(A) = \{0\}$. If R is a field, then $\text{Mc-rank}\, A$ coincides with the maximal number of linearly independent columns of A, so it is the usual rank. To compute the Mc-rank of matrices over finite commutative rings, we have the following propositions.

Proposition 2.1.2. *[3] Let R be a finite local ring with maximal ideal M and $\pi : R \to R/M$ a canonical map. Then for each $A \in R^{m\times n}$, $\text{Mc-rank}\, A = \operatorname{rank}\pi(A)$.*

Proposition 2.1.3. [2] *Let R be a finite commutative ring decomposed as $R = R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite local ring with the projection map $\rho_i : (r_1, r_2, \ldots, r_\ell) \mapsto r_i$ for all $i \in \{1, 2, \ldots, \ell\}$. If $A \in R^{m\times n}$, then*

$$\text{Mc-rank}\, A = \min_{1\le i\le \ell}\{\text{Mc-rank}\,\rho_i(A)\}.$$

2.1.2 Matrix graphs

Suppose that R is a finite commutative ring and m, n, d are positive integers such that $2 \le d \le \min\{m, n\}$. The *matrix graph of type (m, n, d)* over R, denoted by $\Gamma_d(R^{m\times n})$, is the graph whose vertices are $m \times n$ matrices over R, and two matrices $A, B \in R^{m\times n}$ are adjacent if and only if $0 < \operatorname{rank}(A - B) < d$. We write $A \sim B$ when A and B are adjacent.

The graph $\Gamma_2(\mathbb{F}_q^{m\times n})$ is the matrix graph studied in [10]. Besides, the graphs $\Gamma_2(\mathbb{Z}_{p^s}^{m\times n})$ and $\Gamma_d(\mathbb{Z}_{p^s}^{m\times n})$ are the bilinear form graphs in [11] and the generalized bilinear form graphs in [12], respectively.

We next recall some terminologies and properties of graphs. Let G be a graph. An *automorphism* of a graph G is a bijection σ from G to G such that g_1 is adjacent to g_2 if and only if $\sigma(g_1)$ is adjacent to $\sigma(g_2)$. A graph G is said to be *vertex transitive* if for any two vertices of G, there is an automorphism carrying one to the other. An *independent set* of G is a set I of vertices of G in which no two distinct vertices of I are adjacent. An independent set of G with the largest size of vertices is called a *maximal independent set*. We write $\alpha(G)$ for the size of a maximal independent set of G and call it the *independence number* of G. A *clique* C of G is a complete subgraph of G, that is, any two vertices of C are adjacent and a *maximal clique* of G is a clique of G which has the largest size of vertices. Denoted by $\omega(G)$, the number of vertices in a maximal clique is called the *clique number* of G. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G in which no adjacent vertices have the same color. If G is vertex transitive, we have

$$\omega(G) \le \frac{|V(G)|}{\alpha(G)} \le \chi(G).$$

Let $G_1, G_2, \ldots, G_\ell$ be graphs. The *strong product* of graphs $G_1, G_2, \ldots G_\ell$, denoted by $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_\ell$, is the graph whose vertex set is $V(G_1) \times V(G_2) \times \cdots \times V(G_\ell)$, and $g = (g_1, g_2, \ldots, g_\ell)$ is adjacent to $g' = (g_1', g_2', \ldots, g_\ell')$ if $g \ne g'$ and g_i is either equal or adjacent to g_i' in G_i for all $i \in \{1, 2, \ldots, \ell\}$.

2.2 Matrix graphs over finite principal ideal rings

In this section, we study the matrix graphs over finite principal ideal rings. We show that our graph is connected and vertex transitive. We determine the distance between any two vertices of the graph. Moreover, the independence number, the clique number and the chromatic number of the graph are computed.

A finite commutative ring R is called a *finite chain ring* if for any ideal I, J of R, either $I \subseteq J$ or $J \subseteq I$. Clearly, a finite chain ring is a local ring. One can show that if R is a finite chain ring, then its maximal ideal M is principal and generated by θ for some $\theta \in M \setminus M^2$. The smallest positive integer e such that $\theta^e = 0$ is called the *nilpotency* of R. A *principal ideal ring (PIR)* is a commutative ring in which all of its ideals are principal. Recall that a finite commutative ring is a direct product of finite local rings. If every ideal of a ring is principal, so are its factors. Thus, a finite PIR can be decomposed as a direct product of finite chain rings. With this nice relation of PIRs and finite chain rings, we first study some properties of matrices over finite chain rings. Some properties of finite chain rings are recorded in the following proposition.

Proposition 2.2.1. *[14] Let R be a finite chain ring with maximal ideal $M = R\theta$, residue field $\mathbb{F}_q$, nilpotency e and $V = \{v_1, v_2, \ldots, v_q\}$ a system of coset representatives of M in R.*

1. *For any nonzero element r in R, there exists a unique integer i with $0 \leq i \leq e$ such that $r = u\theta^i$ for some $u \in R^\times$.*
2. *For each $r \in R$, r can be uniquely written as*
$$r = r_0 + r_1\theta + r_2\theta^2 + \cdots + r_{e-1}\theta^{e-1}$$
where $r_0, r_1, \ldots, r_{e-1} \in V$.
3. *The ideals of R are in the chain*
$$\{0\} = R\theta^e \subsetneq R\theta^{e-1} \subsetneq R\theta^{e-2} \subsetneq \cdots \subsetneq R\theta^2 \subsetneq R\theta \subsetneq R.$$
4. *$|R\theta^i| = q^{e-i}$ for all $i \in \{0, 1, \ldots, e\}$.*
5. *$R/R\theta^i$ is a finite chain ring with nilpotency i and $|R/R\theta^i| = q^i$ for all $i \in \{1, \ldots, e\}$.*
6. *For each $i \in \{1, 2, \ldots, e\}$, we have*
$$R/R\theta^i = \{r_0 + r_1\theta + r_2\theta^2 + \cdots + r_{i-1}\theta^{i-1} + R\theta^i : r_0, r_1, \ldots, r_{i-1} \in V\}.$$
Thus, an element $r = r_0 + r_1\theta + r_2\theta^2 + \cdots + r_{i-1}\theta^{i-1} + R\theta^i$ in $R/R\theta^i$ can be viewed as an element $r = r_0 + r_1\theta + r_2\theta^2 + \cdots + r_{i-1}\theta^{i-1} + R\theta^{i+1}$ in $R/R\theta^{i+1}$. Moreover, a unit in $R/R\theta^i$ is a unit in $R/R\theta^{i+1}$.

There is a useful property in computing the rank and Mc-rank of matrices over finite chain rings.

Lemma 2.2.2. *[4] Let R be a finite chain ring with maximal ideal $R\theta$ and nilpotency e. If A is a nonzero matrix in $R^{m\times n}$, then there exist $P \in GL_m(R)$ and $Q \in GL_n(R)$ such that*

$$A = P \begin{pmatrix} I_{t_0} & & & & & \\ & \theta I_{t_1} & & & & \\ & & \theta^2 I_{t_2} & & & \\ & & & \ddots & & \\ & & & & \theta^{e-1} I_{t_{e-1}} & \\ & & & & & 0 \end{pmatrix} Q \tag{2.2.1}$$

where $t_0, t_1, \ldots, t_{e-1}$ are non-negative integers. Moreover, this form is unique when θ is fixed.

Proposition 2.2.3. *Let R be a finite chain ring with maximal ideal $R\theta$ and nilpotency e and A a nonzero matrix in $R^{m\times n}$ of the form (2.2.1). Then*

$$\operatorname{rank} A = t_0 + t_1 + \cdots + t_{e-1} \textit{ and } \text{Mc-rank}\, A = t_0.$$

Proof. Let $t = t_0 + t_1 + \cdots + t_{e-1}$. From (2.2.1), we can write

$$A = P \operatorname{diag}(D, 0)Q \text{ where } D = \operatorname{diag}(I_{t_0}, \theta I_{t_1}, \ldots, \theta^{e-1} I_{t_{e-1}}) \in R^{t\times t}.$$

Write $P = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$ and $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, where $P_1 \in R^{m\times t}$ and $Q_1 \in R^{t\times n}$. We have $A = P_1 D Q_1$. Therefore, $\operatorname{rank} A \le t$.

On the other hand, suppose that $\operatorname{rank} A = s$. Then $A = BC$ where $B \in R^{m\times s}$ and $C \in R^{s\times n}$. By Lemma 2.2.2, there exist $P_1 \in GL_m(R), Q_1 \in GL_s(R), P_2 \in GL_s(R)$ and $Q_2 \in GL_n(R)$ such that $B = P_1 \begin{pmatrix} D_1 \\ 0 \end{pmatrix} Q_1$ and $C = P_2(D_2, 0)Q_2$ where D_1 and D_2 are diagonal matrices in $R^{s\times s}$. Hence, $A = P_1 \operatorname{diag}(D_1Q_1P_2D_2, 0)Q_2$ where $D_1Q_1P_2D_2 \in R^{s\times s}$. Since the form of A is unique, $s \ge t$. Thus, $\operatorname{rank} A = t$.

Next, let $\pi : R \to R/R\theta$ be the canonical map. Then $\pi(A) = \pi(P) \operatorname{diag}(\pi(D), 0)\pi(Q)$. It is obvious that $\operatorname{rank} \pi(A) = t_0$. By Proposition 2.1.2, we have $\text{Mc-rank}\, A = \operatorname{rank} \pi(A) = t_0$. □

By Proposition 2.2.1 (6), we note that a matrix A over $R/R\theta^i$ can be viewed as a matrix A over $R/R\theta^{i+1}$ and if A is invertible over $R/R\theta^i$, then A is invertible over $R/R\theta^{i+1}$. We apply Proposition 2.2.3 to prove the next proposition.

Proposition 2.2.4. *Let A be an $m \times n$ matrix of rank t over $R/R\theta^i$. Then A and $A\theta$ are $m \times n$ matrices of rank t over $R/R\theta^{i+1}$.*

Proof. From Proposition 2.2.1 and Lemma 2.2.2, we can write $A = P \operatorname{diag}(I_{t_0}, \theta I_{t_1}, \ldots, \theta^{i-1} I_{t_{i-1}}, 0)Q$ where $P \in GL_m(R/R\theta^i)$ and $Q \in GL_n(R/R\theta^i)$ with $t = t_0 + t_1 + \ldots t_{i-1}$. It follows that both A and $A\theta = P \operatorname{diag}(\theta I_{t_0}, \theta^2 I_{t_1}, \ldots, \theta^i I_{t_{i-1}}, 0)Q$ are $m \times n$ matrices over $R/R\theta^{i+1}$. Since P and Q are invertible over $R/R\theta^i$, they are invertible over $R/R\theta^{i+1}$. Hence, A and $A\theta$ are of rank t over $R/R\theta^{i+1}$. □

Let R be a finite PIR decomposed as $R \overset{\varphi}{\cong} R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring for all $i \in \{1, 2, \ldots, \ell\}$. Let $\rho_i : (r_1, r_2, \ldots, r_\ell) \mapsto r_i$ be a projection map for all $i \in \{1, 2, \ldots, \ell\}$. The isomorphism φ gives $R^{m\times n} \cong R_1^{m\times n} \times R_2^{m\times n} \times \cdots \times R_\ell^{m\times n}$. Thus, we can view the vertex set of $\Gamma_d(R^{m\times n})$ as $\{(\rho_1(A), \rho_2(A), \ldots, \rho_\ell(A)) : A \in R^{m\times n}\}$. By Proposition 2.1.1, if $A = (\rho_1(A), \rho_2(A), \ldots, \rho_\ell(A))$ and $B = (\rho_1(B), \rho_2(B), \ldots, \rho_\ell(B))$ are two vertices of $\Gamma_d(R^{m\times n})$, then

$$A \sim B \iff 0 < \max_{1\le i\le \ell}\{\operatorname{rank}(\rho_i(A) - \rho_i(B))\} < d.$$

With this relation, we proceed to prove the following strong product of graphs theorem.

Theorem 2.2.5. *Let R be a finite PIR decomposed as $R = R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring. Then*

$$\Gamma_d(R^{m\times n}) = \Gamma_d(R_1^{m\times n}) \boxtimes \Gamma_d(R_2^{m\times n}) \boxtimes \cdots \boxtimes \Gamma_d(R_\ell^{m\times n}).$$

Proof. Let $\mathcal{G} = \Gamma_d(R_1^{m\times n}) \boxtimes \Gamma_d(R_2^{m\times n}) \boxtimes \cdots \boxtimes \Gamma_d(R_\ell^{m\times n})$. As mentioned, the vertex sets of graphs $\mathcal{G}$ and $\Gamma_d(R^{m\times n})$ are the same. Let $A = (\rho_1(A), \rho_2(A), \ldots, \rho_\ell(A))$ and $B = (\rho_1(B), \rho_2(B), \ldots, \rho_\ell(B))$ be two vertices. Then

$$\begin{aligned}
A \sim B \text{ in } \Gamma_d(R^{m\times n}) &\Leftrightarrow 0 < \max_{1\le i\le \ell}\{\operatorname{rank}(\rho_i(A)-\rho_i(B))\} < d \\
&\Leftrightarrow A \ne B \text{ and } \operatorname{rank}(\rho_i(A)-\rho_i(B)) < d \text{ for all } i\in\{1,2,\dots,\ell\} \\
&\Leftrightarrow A \ne B \text{ and either } \rho_i(A)=\rho_i(B) \text{ or } \rho_i(A)\sim\rho_i(B) \text{ in } \Gamma_d(R_i^{m\times n}) \\
&\qquad \text{for all } i\in\{1,2,\dots,\ell\} \\
&\Leftrightarrow A\sim B \text{ in } \mathcal{G}.
\end{aligned}$$

This completes the proof. □

Theorem 2.2.6. *Let R be a finite PIR. Then the graph $\Gamma_d(R^{m\times n})$ is connected. Moreover, for two vertices $A, B \in R^{m\times n}$, the distance between A and B is*

$$\mathrm{d_G}(A,B) = \left\lceil \frac{\operatorname{rank}(A-B)}{d-1} \right\rceil.$$

Consequently, the diameter of $\Gamma_d(R^{m\times n})$ is equal to $\lceil \frac{\min\{m,n\}}{d-1} \rceil$.

Proof. We first prove the desired result in the case that R is a finite chain ring. Assume that R is a finite chain ring with maximal ideal $R\theta$ and nilpotency e. Let $A, B \in R^{m\times n}$ with $\operatorname{rank}(B-A) = t$. By Lemma 2.2.2, there exist $P \in GL_m(R)$ and $Q \in GL_n(R)$ such that

$$B - A = P \begin{pmatrix} \theta^{k_1} & & & & \\ & \theta^{k_2} & & & \\ & & \ddots & & \\ & & & \theta^{k_t} & \\ & & & & 0 \end{pmatrix} Q$$

where $0 \le k_1 \le \cdots \le k_t \le e-1$. If $t \le d-1$, then $A \sim B$, and so $\mathrm{d_G}(A,B) = 1$. We assume that $t \ge d$. Write $t = (d-1)q + r$ where q, r are integers with $q \ge 1$ and $0 \le r < d-1$. Let $A_0 = A$ and $A_i = A + P\operatorname{diag}(\theta^{k_1}, \theta^{k_2}, \dots, \theta^{k_{(d-1)i}}, 0)Q$ for all $i \in \{1, \dots, q\}$. Then for each $i \in \{0, 1, \dots, q-1\}$, $A_{i+1} - A_i = P\operatorname{diag}(0, \theta^{k_{(d-1)i+1}}, \dots, \theta^{k_{(d-1)(i+1)}}, 0)Q$, so $\operatorname{rank}(A_{i+1} - A_i) < d$ and thus $A_{i+1} \sim A_i$. Now, we have $A_0 \sim A_1 \sim A_2 \sim \cdots \sim A_q$. Note that $B - A_q = P\operatorname{diag}(0, \theta^{k_{(d-1)q+1}}, \dots, \theta^{k_{(d-1)q+r}}, 0)Q$. So $\operatorname{rank}(B - A_q) \le r < d$. This implies that $B = A_q$ if $r = 0$ or $B \sim A_q$ if $r > 0$. Thus, $\Gamma_d(R^{m\times n})$ is connected. Moreover, $\mathrm{d_G}(A,B)$ equals either q or $q+1$, that is, $\mathrm{d_G}(A,B) \le \lceil \frac{t}{d-1} \rceil$.

On the other hand, let $\mathrm{d_G}(A,B) = s$. Then there exist $C_1, C_2, \dots, C_{s-1} \in R^{m\times n}$ such that $A \sim C_1 \sim C_2 \sim \cdots \sim C_{s-1} \sim B$. By properties of the rank of matrices, we have

$$\begin{aligned}
t = \operatorname{rank}(A-B) &\le \operatorname{rank}(A-C_1) + \operatorname{rank}(C_1-C_2) + \cdots + \operatorname{rank}(C_{s-1}-B) \\
&\le s(d-1).
\end{aligned}$$

Thus, $\mathrm{d_G}(A,B) = s \ge \lceil \frac{t}{d-1} \rceil$. Therefore, $\mathrm{d_G}(A,B) = \lceil \frac{\operatorname{rank}(A-B)}{d-1} \rceil$.

Next suppose that R is decomposed as $R \cong R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring. By Theorem 2.2.5, we have

$$\Gamma_d(R^{m\times n}) = \Gamma_d(R_1^{m\times n}) \boxtimes \Gamma_d(R_2^{m\times n}) \boxtimes \cdots \boxtimes \Gamma_d(R_\ell^{m\times n}).$$

Let $A = (\rho_1(A), \rho_2(A), \dots, \rho_\ell(A))$ and $B = (\rho_1(B), \rho_2(B), \dots, \rho_\ell(B))$ be two vertices in $\Gamma_d(R^{m\times n})$. Since $\Gamma_d(R_i^{m\times n})$ is connected for all $i \in \{1, 2, \dots, \ell\}$, we can suppose that $\mathrm{d_G}(\rho_i(A), \rho_i(B)) = t_i$ for all $i \in$

$\{1, 2, \ldots, \ell\}$. For convenience, we write $\rho_i(A) = X_{i0}$ and $\rho_i(B) = X_{it_i}$. Then for each $i \in \{1, 2, \ldots, \ell\}$, there exist $X_{i1}, X_{i2}, \ldots, X_{i(t_i-1)}$ such that

$$\rho_i(A) = X_{i0} \sim X_{i1} \sim X_{i2} \sim \cdots \sim X_{it_i} = \rho_i(B).$$

Without loss of generality, we assume that $t_1 \leq t_2 \leq \cdots \leq t_\ell$. For each $j \in \{0, 1, \ldots, t_\ell\}$, we set $X_j = (X_{1j}, X_{2j}, \ldots, X_{\ell j})$ where $X_{ij} = \rho_i(B)$ if $t_i \leq j \leq t_\ell$. Then

$$A = X_0 \sim X_1 \sim X_2 \sim \cdots \sim X_{t_\ell} = B.$$

This implies that $\Gamma_d(R^{m\times n})$ is connected and $\mathrm{d_G}(A, B) \leq t_\ell = \max_{1\leq i\leq \ell}\{\mathrm{d_G}(\rho_i(A), \rho_i(B))\}$.

Conversely, assume that $\mathrm{d_G}(A, B) = t$. Then there exist $X_1, X_2, \ldots, X_{t-1}$ such that $A := X_0 \sim X_1 \sim X_2 \sim \cdots \sim X_{t-1} \sim X_t = B$. Let $i \in \{1, 2, \ldots, \ell\}$. Since $X_j \sim X_{j+1}$, we have $\rho_i(X_j) = \rho_i(X_{j+1})$ or $\rho_i(X_j) \sim \rho_i(X_{j+1})$ in $\Gamma_d(R_i^{m\times n})$ for all $j \in \{0, 1, \ldots, t-1\}$. Thus, $\mathrm{d_G}(\rho_i(A), \rho_i(B)) \leq t$. It follows that $\max_{1\leq i\leq \ell}\{\mathrm{d_G}(\rho_i(A), \rho_i(B))\} \leq t = \mathrm{d_G}(A, B)$.

Finally, the distance over finite chain rings implies

$$\mathrm{d_G}(A, B) = \max_{1\leq i\leq \ell}\{\mathrm{d_G}(\rho_i(A), \rho_i(B))\} = \max_{1\leq i\leq \ell}\Big\{\Big\lceil\frac{\mathrm{rank}(\rho_i(A) - \rho_i(B))}{d-1}\Big\rceil\Big\}.$$

By Proposition 2.1.1, we have $\mathrm{d_G}(A, B) = \lceil\frac{\mathrm{rank}(A-B)}{d-1}\rceil$. The diameter of $\Gamma_d(R^{m\times n})$ is obtained from Lemma 2.2.2 together with choosing $A = 0$ and $B = \begin{pmatrix} I_m & 0\end{pmatrix}$ if $m \leq n$ or $B = \begin{pmatrix} I_n \\ 0\end{pmatrix}$ if $n \leq m$. Hence, $\mathrm{rank}(A - B) = \min\{m, n\}$. □

Proposition 2.2.7. *If R is a finite PIR, then the matrix graph $\Gamma_d(R^{m\times n})$ is vertex transitive.*

Proof. Let $A, B \in R^{m\times n}$. Define $\sigma : R^{m\times n} \to R^{m\times n}$ by $\sigma(X) = X - (A - B)$ for all $X \in R^{m\times n}$. For $X, Y \in R^{m\times n}$, we have $\mathrm{rank}(\sigma(X) - \sigma(Y)) = \mathrm{rank}((X - (A - B)) - (Y - (A - B))) = \mathrm{rank}(X - Y)$. Then $X \sim Y$ if and only if $\sigma(X) \sim \sigma(Y)$ in $\Gamma_d(R^{m\times n})$. Thus, σ is a graph automorphism which maps A to B. Therefore, $\Gamma_d(R^{m\times n})$ is vertex transitive. □

Remark 2.2.8. It is well known that a vertex transitive graph is regular, that is, every vertex has the same degree. Thus the matrix graph $\Gamma_d(R^{m\times n})$ is regular. For the degree of this regular graph, we can determine the degree of the zero matrix. Then the degree of $\Gamma_d(R^{m\times n})$ is the number of all nonzero $m \times n$ matrices over R of rank less than d.

We next compute the independence numbers and clique numbers of the matrix graphs. The results over finite fields are given in [12] as follows.

Lemma 2.2.9. *[12] If $\mathbb{F}_q$ is the finite field of q elements, then*

$$\alpha(\Gamma_d(\mathbb{F}_q^{m\times n})) = q^{\max\{m,n\}(\min\{m,n\}-d+1)} \quad \text{and} \quad \omega(\Gamma_d(\mathbb{F}_q^{m\times n})) = q^{\max\{m,n\}(d-1)}.$$

For the case of finite PIRs, we first consider the sets

$$\mathcal{C}_1 := \left\{ \begin{pmatrix} A \\ 0\end{pmatrix} : A \in R^{(d-1)\times n}\right\} \text{ and } \mathcal{C}_2 := \left\{ \begin{pmatrix} A & 0\end{pmatrix} : A \in R^{m\times(d-1)}\right\}.$$

Since $\mathrm{rank}\, A \leq \min\{m, n\}$ for $A \in R^{m\times n}$, it follows that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are cliques of $\Gamma_d(R^{m\times n})$. Thus, $\omega(\Gamma_d(R^{m\times n})) \geq |R|^{\max\{m,n\}(d-1)}$. This provides the lower bound of the clique number. We shall apply it to compute both clique number and independence number.

Theorem 2.2.10. *Let R be a finite PIR. Then*

$$\alpha(\Gamma_d(R^{m\times n})) = |R|^{\max\{m,n\}(\min\{m,n\}-d+1)}$$

and

$$\omega(\Gamma_d(R^{m\times n})) = |R|^{\max\{m,n\}(d-1)}.$$

Proof. We first suppose that R is a finite chain ring with maximal ideal $R\theta$, nilpotency e and a canonical map $\pi : R \to R/R\theta$. Let $m \leq n$. Then Lemma 2.2.9 implies that $\alpha(\Gamma_d((R/R\theta)^{m\times n})) = q^{n(m-d+1)} := \alpha$. Let $\mathcal{A}$ be a maximal independent set of $\Gamma_d((R/R\theta)^{m\times n})$. So $\mathrm{rank}(A-B) \geq d$ over $R/R\theta$ for all distinct A, B in $\mathcal{A}$. By Proposition 2.2.4, we have that a matrix A over $R/R\theta$ can be considered as a matrix A over $R/R\theta^i$ with the same rank for all $i \in \{1, 2, \ldots, e\}$. Thus, $\mathrm{rank}(A-B) \geq d$ over R for all distinct A, B in $\mathcal{A}$. Next, let

$$\mathcal{I} = \mathcal{A} + \mathcal{A}\theta + \mathcal{A}\theta^2 + \cdots + \mathcal{A}\theta^{e-1} = \{A_0 + A_1\theta + A_2\theta^2 + \cdots + A_{e-1}\theta^{e-1} : A_i \in \mathcal{A}\}.$$

By Proposition 2.2.1 (2), it is easy to see that $\mathcal{I}$ is a set of size α^e. We show that $\mathcal{I}$ is an independent set of $\Gamma_d(R^{m\times n})$. Let $A, B \in \mathcal{I}$ with $A \neq B$. Then $A = A_0 + A_1\theta + A_2\theta^2 + \cdots + A_{e-1}\theta^{e-1}$ and $B = B_0 + B_1\theta + B_2\theta^2 + \cdots + B_{e-1}\theta^{e-1}$ where $A_i, B_i \in \mathcal{A}$ and $A_j \neq B_j$ for some $j \in \{0, 1, \ldots, e-1\}$. Hence, $A - B = (A_0 - B_0) + (A_1 - B_1)\theta + (A_2 - B_2)\theta^2 + \cdots + (A_{e-1} - B_{e-1})\theta^{e-1}$. We apply Propositions 2.1.2 and 2.2.3 to show that $\mathrm{rank}(A-B) \geq d$.

First, if $A_0 \neq B_0$, then $\mathrm{rank}(A-B) \geq \text{Mc-rank}(A-B) = \text{Mc-rank}\,\pi(A-B) = \text{Mc-rank}(A_0 - B_0) = \mathrm{rank}(A_0 - B_0) \geq d$. So we suppose that $A_0 \neq B_0$. Let $j \in \{1, 2, \ldots, e-1\}$ be the first index such that $A_j \neq B_j$. Then $A - B = \big((A_j - B_j) + (A_{j+1} - B_{j+1})\theta + \cdots + (A_{e-1} - B_{e-1})\theta^{e-(j+1)}\big)\theta^j$. Write $C := (A_j - B_j) + (A_{j+1} - B_{j+1})\theta + \cdots + (A_{e-1} - B_{e-1})\theta^{e-(j+1)}$. Then $A - B = C\theta^j$. Note that C can also be viewed as a matrix over $R/R\theta^{e-j}$. By Proposition 2.2.4, both C and $C\theta^j$ are matrices over $R/R\theta^e \cong R$ with the same rank as considering them over $R/R\theta^{e-j}$. Therefore, $\mathrm{rank}(A-B) = \mathrm{rank}(C\theta^j) = \mathrm{rank}(C) \geq \text{Mc-rank}(C) = \text{Mc-rank}\,\pi(C) = \text{Mc-rank}(A_j - B_j) = \mathrm{rank}(A_j - B_j) \geq d$. This implies that $\mathcal{I}$ is an independent set of $\Gamma_d(R^{m\times n})$ of size $\alpha^e = q^{en(m-d+1)}$. It follows that $\alpha(\Gamma_d(R^{m\times n})) \geq q^{en(m-d+1)}$.

Recall that $\omega(\Gamma_d(R^{m\times n})) \geq q^{en(d-1)}$. Since $\Gamma_d(R^{m\times n})$ is vertex transitive,

$$\alpha(\Gamma_d(R^{m\times n})) \leq \frac{|V(\Gamma_d(R^{m\times n}))|}{\omega(\Gamma_d(R^{m\times n}))} \leq \frac{q^{emn}}{q^{en(d-1)}} = q^{en(m-d+1)}.$$

Therefore, $\alpha(\Gamma_d(R^{m\times n})) = q^{en(m-d+1)}$. Again,

$$\omega(\Gamma_d(R^{m\times n})) \leq \frac{|V(\Gamma_d(R^{m\times n}))|}{\alpha(\Gamma_d(R^{m\times n}))} = \frac{q^{emn}}{q^{en(m-d+1)}} = q^{en(d-1)}.$$

Thus, $\omega(\Gamma_d(R^{m\times n})) = q^{en(d-1)}$. So we obtain the result over finite chain rings.

Next, assume that the PIR R is decomposed as $R = R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite local ring with nilpotency e_i and residue field $\mathbb{F}_{q_i}$. By Theorem 2.2.5, $\Gamma_d(R^{m\times n}) = \Gamma_d(R_1^{m\times n}) \boxtimes \Gamma_d(R_2^{m\times n}) \boxtimes \cdots \boxtimes \Gamma_d(R_\ell^{m\times n})$. Note that if $\mathcal{I}_i$ is an independent set of $\Gamma_d(R_i^{m\times n})$ for all $i \in \{1, 2, \ldots, \ell\}$, then it is easy to see that

$$\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_\ell = \{(A_1, A_2, \ldots, A_\ell) : A_i \in \mathcal{I}_i\}$$

is an independent set of $\Gamma_d(R^{m\times n})$. Hence,

$$\alpha(\Gamma_d(R^{m\times n})) \geq \alpha(\Gamma_d(R_1^{m\times n}))\alpha(\Gamma_d(R_2^{m\times n}))\ldots\alpha(\Gamma_d(R_\ell^{m\times n})).$$

The previous result on finite chain rings implies that $\alpha(\Gamma_d(R_i^{m\times n})) = \frac{|V(\Gamma_d(R_i^{m\times n}))|}{\omega(\Gamma_d(R_i^{m\times n}))}$ for all $i \in \{1, 2, \ldots, \ell\}$. Moreover, $\Gamma_d(R_i^{m\times n})$ is vertex transitive for all $i \in \{1, 2, \ldots, \ell\}$ by Proposition 2.2.7. Thus, it follows from [16] Corollary 1 that

$$\alpha(\Gamma_d(R^{m\times n})) \leq \alpha(\Gamma_d(R_1^{m\times n}))\alpha(\Gamma_d(R_2^{m\times n}))\ldots\alpha(\Gamma_d(R_\ell^{m\times n})).$$

Therefore,

$$\begin{aligned}\alpha(\Gamma_d(R^{m\times n})) &= \alpha(\Gamma_d(R_1^{m\times n}))\alpha(\Gamma_d(R_2^{m\times n}))\ldots\alpha(\Gamma_d(R_\ell^{m\times n}))\\ &= q_1^{e_1 n(m-d+1)} q_2^{e_2 n(m-d+1)} \ldots q_\ell^{e_\ell n(m-d+1)}\\ &= |R|^{n(m-d+1)}.\end{aligned}$$

Finally, we determine the clique number of the graph. It is proved in [1] that $\omega(G \boxtimes H) = \omega(G)\omega(H)$. Consequently,

$$\begin{aligned}\omega(\Gamma_d(R^{m\times n})) &= \omega(\Gamma_d(R_1^{m\times n}))\omega(\Gamma_d(R_2^{m\times n}))\ldots\omega(\Gamma_d(R_\ell^{m\times n}))\\ &= q_1^{e_1 n(d-1)} q_2^{e_2 n(d-1)} \ldots q_\ell^{e_\ell n(d-1)}\\ &= |R|^{n(d-1)}.\end{aligned}$$

The case $n \leq m$ can be proved in a similar way. □

Remark 2.2.11.

1. The cliques $\mathcal{C}_1$ and $\mathcal{C}_2$ mentioned earlier are maximal cliques.
2. Let R be a finite chain ring with maximal ideal $R\theta$ and nilpotency e. If $\mathcal{A}$ is a maximal independent set of $\Gamma_d((R/R\theta)^{m\times n})$, then
$$\mathcal{I} = \mathcal{A} + \mathcal{A}\theta + \mathcal{A}\theta^2 + \cdots + \mathcal{A}\theta^{e-1}$$
is a maximal independent set of $\Gamma_d(R^{m\times n})$.
3. For a finite PIR $R \cong R_1 \times R_2 \times \cdots \times R_\ell$, if $\mathcal{I}_i$ is a maximal independent set of $\Gamma_d(R_i^{m\times n})$ for all $i \in \{1, 2, \ldots, \ell\}$, then
$$\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_\ell = \{(A_1, A_2, \ldots, A_\ell) : A_i \in \mathcal{I}_i\}$$
is a maximal independent set of $\Gamma_d(R^{m\times n})$.

Let G be a finite group and S a subset of G which does not contain the identity and is closed under taking inverses. The *Cayley graph* $Cay(G, S)$ is an undirected graph with vertex set G and for two vertices $g_1, g_2 \in G$, g_1 and g_2 are adjacent if $g_1 g_2^{-1}$ is in S. A Cayley graph $Cay(G, S)$ is *normal* if $gCg^{-1} = C$ for all $g \in G$.

To determine the chromatic number of the matrix graph, we use the following property of a normal Cayley graph.

Lemma 2.2.12. *[9] If G is a normal Cayley graph with $\alpha(G) = \frac{|V(G)|}{\omega(G)}$, then $\chi(G) = \omega(G)$.*

Note that $R^{m\times n}$ is an additive group. Let S be the set of nonzero matrices of rank less than d. It is easy to see that S does not contain the zero matrix and is closed under taking additive inverses. For $A, B \in \Gamma_d(R^{m\times n})$, we have

$$A \sim B \iff 0 < \operatorname{rank}(A - B) < d \iff A - B \in S.$$

Thus, $\Gamma_d(R^{m\times n})$ is a Cayley graph. Moreover, it is a normal Cayley graph since $R^{m\times n}$ is an abelian group. By Theorem 2.2.10, we have $\alpha(\Gamma_d(R^{m\times n})) = \frac{|V(\Gamma_d(R^{m\times n}))|}{\omega(\Gamma_d(R^{m\times n}))}$. It follows from the above lemma that $\omega(\Gamma_d(R^{m\times n})) = \chi(\Gamma_d(R^{m\times n}))$. Hence, we have shown:

Proposition 2.2.13. *If R is a finite PIR, then $\chi(\Gamma_d(R^{m\times n})) = |R|^{\max\{m,n\}(d-1)}$.*

2.3 MRD codes

This section is devoted to study MRD codes over PIRs. We give the concepts of matrix codes and rank distance of matrix codes. We shall see that matrix codes relate to matrix graphs. Indeed, maximal independent sets of matrix graphs are MRD codes and vice versa. Finally, we show the existence of linear MRD codes over a PIR by lifting linear MRD codes over a direct product of finite fields.

Let R be a finite commutative ring. A *(matrix) code* of size $m \times n$ is defined to be a subset $\mathcal{C}$ of $R^{m\times n}$. For two matrices $A, B \in R^{m\times n}$, we define the *rank distance between A and B*, denoted by $\mathrm{d_{rk}}(A,B)$, to be $\mathrm{rank}(A-B)$. Note that the rank distance is a metric on $R^{m\times n}$. Indeed, $\mathrm{d_{rk}}(A,B) \geq 0$, $\mathrm{d_{rk}}(A,B) = 0$ if and only if $A = B$, $\mathrm{d_{rk}}(A,B) = \mathrm{d_{rk}}(B,A)$ and $\mathrm{d_{rk}}(A,C) \leq \mathrm{d_{rk}}(A,B) + \mathrm{d_{rk}}(B,C)$ for all $A,B,C \in R^{m\times n}$. For a code $\mathcal{C}$ of size $m \times n$ over R, the *rank distance of $\mathcal{C}$* is defined to be

$$\mathrm{d_{rk}}(\mathcal{C}) = \min\{\mathrm{d_{rk}}(A,B) : A, B \in \mathcal{C} \text{ with } A \neq B\}.$$

We call a code $\mathcal{C}$ of size $m \times n$ with rank distance d an *$(m \times n, d)$-code*. If $\mathcal{C} \subseteq R^{m\times n}$ is a submodule of $R^{m\times n}$ over R, we call $\mathcal{C}$ a *linear code*.

Suppose $m \leq n$. Let $\mathcal{C}$ be an $(m\times n, d)$-code. We can consider a matrix A in $\mathcal{C}$ as $A = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m)$ where $\vec{x}_i \in R^n$ is an i-th row of A. This means we can study $\mathcal{C} \subseteq (R^n)^m$ as a code of length m over a set of alphabet R^n and find the Hamming distance of $\mathcal{C}$. Hence, a code $\mathcal{C}$ with the Hamming distance $\mathrm{d_H}(\mathcal{C})$ agrees with the *Singleton bound* $\mathrm{d_H}(\mathcal{C}) \leq m - \log_{|R|^n} |\mathcal{C}| + 1$. That is, $|\mathcal{C}| \leq |R|^{n(m-\mathrm{d_H}(\mathcal{C})+1)}$.

Over the finite field $\mathbb{F}_q$, it is shown in [8] that a matrix code $\mathcal{C}$ of size $m \times n$ with rank distance $\mathrm{d_{rk}}(\mathcal{C})$ has a *Singleton like bound* which satisfies $|\mathcal{C}| \leq q^{n(m-\mathrm{d_{rk}}(\mathcal{C})+1)}$. We show that matrix codes over finite PIRs have a similar bound by using independent sets of the matrix graphs.

Let R be a finite PIR and $\mathcal{C} \subseteq R^{m\times n}$. Then $\mathcal{C}$ is both a matrix code and a set of vertices in the matrix graph $\Gamma_d(R^{m\times n})$. Moreover, if $d \geq 2$, then we have that for any $A, B \in \mathcal{C}$ with $A \neq B$, $\mathrm{d_{rk}}(A,B) = \mathrm{rank}(A-B) \geq d$ if and only if A is not adjacent to B in $\Gamma_d(R^{m\times n})$. This implies the next proposition.

Proposition 2.3.1. *Let R be a finite PIR and $2 \leq d \leq m \leq n$. For a code $\mathcal{C} \subseteq R^{m\times n}$, $\mathrm{d_{rk}}(\mathcal{C}) \geq d$ if and only if $\mathcal{C}$ is an independent set of the graph $\Gamma_d(R^{m\times n})$.*

This proposition and the independence number in Theorem 2.2.10 implies that if $\mathcal{C}$ is a code with $\mathrm{d_{rk}}(\mathcal{C}) = d$ where $d \geq 2$, then $|\mathcal{C}| \leq \alpha(\Gamma_d(R^{m\times n})) = |R|^{n(m-d+1)}$. For the case $\mathrm{d_{rk}}(\mathcal{C}) = 1$, it is obvious that $|C| \leq |R|^{nm}$. Thus, we have the Singleton like bound for the matrix codes over finite PIRs as follows.

Corollary 2.3.2. *Let R be a finite PIR and $m \leq n$. For a code $\mathcal{C} \subseteq R^{m\times n}$, we have $|\mathcal{C}| \leq |R|^{n(m-\mathrm{d_{rk}}(\mathcal{C})+1)}$.*

An $(m\times n, d)$-code $\mathcal{C}$ over a PIR R is called a *maximum rank distance code (MRD code)* if $|\mathcal{C}| = |R|^{n(m-d+1)}$. Obviously, the only $(m \times n, 1)$-MRD code is $R^{m\times n}$. So we may assume $d \geq 2$ to study MRD codes.

Next, suppose that R is a PIR and $d \leq m \leq n$. Let $\mathcal{C} \subseteq R^{m\times n}$. Note that if $\mathcal{C}$ is either a maximal independent set of $\Gamma_d(R^{m\times n})$ or an $(m \times n, d)$-MRD code, then $|\mathcal{C}| = |R|^{n(m-d+1)} = \alpha(\Gamma_d(R^{m\times n}))$. Moreover, $|\mathcal{C}| = |R|^{n(m-d+1)}$ implies $|R|^{n(m-d+1)} = |\mathcal{C}| \leq |R|^{n(m-\mathrm{d_{rk}}(\mathcal{C})+1)}$ by Corollary 2.3.2, so we have $d \geq \mathrm{d_{rk}}(\mathcal{C})$. Applying Proposition 2.3.1 results in

$$\begin{aligned}\mathcal{C} \text{ is an } (m\times n, d)\text{-MRD code } &\Leftrightarrow \mathrm{d_{rk}}(\mathcal{C}) = d \text{ and } |\mathcal{C}| = |R|^{n(m-d+1)}\\ &\Leftrightarrow \mathcal{C} \text{ is an independent set of } \Gamma_d(R^{m\times n}) \text{ and } |\mathcal{C}| = |R|^{n(m-d+1)}\\ &\Leftrightarrow \mathcal{C} \text{ is a maximal independent set of } \Gamma_d(R^{m\times n}).\end{aligned}$$

Therefore, we have shown:

Theorem 2.3.3. *Let R be a finite PIR, $2 \leq d \leq m \leq n$ and $\mathcal{C} \subseteq R^{m\times n}$. Then $\mathcal{C}$ is an $(m \times n, d)$-MRD code if and only if $\mathcal{C}$ is a maximal independent set of $\Gamma_d(R^{m\times n})$.*

We have seen that $(m\times n, d)$-MRD codes coincide with maximal independent sets of the matrix graphs. We next construct linear MRD codes over PIRs by using maximal independent sets of the graphs.

Theorem 2.3.4. *Let R be a finite PIR decomposed as $R = R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring with maximal ideal $R\theta_i$, nilpotency e_i and residue field $\mathbb{F}_{q_i}$ for all $i \in \{1, 2, \ldots, \ell\}$. For any m, n, d with $2 \leq d \leq \min\{m, n\}$, there exists a linear $(m \times n, d)$-MRD code over R. Moreover, this linear $(m \times n, d)$-MRD code is of the form $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_\ell$ where each $\mathcal{C}_i$ is a linear $(m \times n, d)$-MRD code over R_i which is of the form $\mathcal{C}_i = \overline{\mathcal{C}}_i + \overline{\mathcal{C}}_i\theta_i + \overline{\mathcal{C}}_i\theta_i^2 + \cdots + \overline{\mathcal{C}}_i\theta_i^{e_i-1}$ where $\overline{\mathcal{C}}_i$ is a linear $(m \times n, d)$-MRD code over $\mathbb{F}_{q_i}$.*

Proof. Let m, n, d be positive integers with $2 \leq d \leq m \leq n$. Suppose that R is a finite chain ring with maximal ideal $R\theta$, nilpotency e and residue field $R/R\theta \cong \mathbb{F}_q$. It is shown in [8] that there exists a linear $(m \times n, d)$-MRD code over $\mathbb{F}_q$. We shall lift this linear MRD code $\overline{\mathcal{C}}$ over $\mathbb{F}_q$ to obtain a linear MRD code over R.

Theorem 2.3.3 implies that $\overline{\mathcal{C}}$ is a maximal independent set of $\Gamma_d(\mathbb{F}_q^{m\times n})$. Remark 2.2.11 (2) shows that

$$\mathcal{C} := \overline{\mathcal{C}} + \overline{\mathcal{C}}\theta + \overline{\mathcal{C}}\theta^2 + \cdots + \overline{\mathcal{C}}\theta^{e-1} = \{A_0 + A_1\theta + A_2\theta^2 + \cdots + A_{e-1}\theta^{e-1} : A_i \in \overline{\mathcal{C}}\}$$

is a maximal independent set of $\Gamma_d(R^{m\times n})$. From another direction of Theorem 2.3.3, $\mathcal{C}$ is an $(m \times n, d)$-MRD code over R. Since $\overline{\mathcal{C}}$ is a linear code over $\mathbb{F}_q$, we can employ Proposition 2.2.1 (2) to obtain a linear code $\mathcal{C}$ over R.

Finally, suppose that R is a PIR decomposed as $R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring. Then there exists a linear $(m \times n, d)$-MRD code $\mathcal{C}_i$ over R_i for all $i \in \{1, 2, \ldots, \ell\}$. By Theorem 2.3.3, $\mathcal{C}_i$ is a linear independent set of $\Gamma_d(R_i^{m\times n})$. Again, by Remark 2.2.11 (3), we have

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_\ell = \{(A_1, A_2, \ldots, A_\ell) : A_i \in \mathcal{C}_i\}$$

is a maximal independent set of $\Gamma_d(R^{m\times n})$. Thus, $\mathcal{C}$ is an $(m \times n, d)$-MRD code over R. Since $\mathcal{C}_i$ is a linear $(m \times n, d)$-MRD code over R_i for all $i \in \{1, 2, \ldots, \ell\}$, $\mathcal{C}$ is also a linear $(m \times n, d)$-MRD code over R. This completes the proof. □

Remark 2.3.5. Linear MRD codes over finite fields have been intensively applied to linear network coding, and also connected to many areas such as McEliece like public key cryptosystems, semifields, linearized polynomials, see [15] for details. From the above theorem, we obtain linear $(m \times n, d)$-MRD codes for any parameters m, n, d not only over the field alphabet $\mathbb{F}_q$ but also the ring alphabet of any sizes (finite PIRs). Indeed, the ring alphabets are more optimal than field alphabets in some cases to study network coding, see [7]. Moreover, these linear MRD codes over PIRs generalize those over $\mathbb{Z}_{p^s}$ in [12].

2.4 References

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Chapter 3

Unitary Cayley Graphs of Matrix Rings over Finite Commutative Rings

3.1 Unitary Cayley graphs and

For a finite ring R with identity, the *unitary Cayley graph of* R, C_R, is the graph with vertex set R and for each $x, y \in R$, x and y are adjacent if and only if $x - y$ is a unit of R. The unitary Cayley graphs have been widely studied by many authors (see, for example, [2, 8, 4, 1, 5]). As discovered in [1, 5], if R is a finite commutative ring, then R can be decomposed as a direct product of finite local rings $R_1, \ldots, R_s$ and C_R is the tensor product of the graphs $\mathrm{C}_{R_1}, \ldots, \mathrm{C}_{R_s}$ where the *tensor product* of graphs G and H, $G \otimes H$, is the graph defined on $V(G) \times V(H)$ where (a, b) is adjacent to (c, d) if and only if a is adjacent to c in G and b is adjacent to d in H. In addition, if R is a finite local ring with maximal ideal M, then C_R is a complete multi-partite graph whose partite sets are the cosets of M. Thus, the unitary Cayley graphs of finite commutative rings are well studied. Their spectral properties including the energies are also well known (see [5]).

Let G be a graph and $V(G)$ the vertex set of G. We give some terminologies from graph theory as follows. A *clique* is a subgraph that is a complete graph and *clique number* of G is the size of largest clique in G, denoted by $\omega(G)$. A set I of vertices of G is called an *independent set* if no distinct vertices of I are adjacent. The *independence number* of G is the size of a maximal independent set, denoted by $\alpha(G)$. The *chromatic number of* G is the least number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. We write $\chi(G)$ for the chromatic number of G. If every vertex of G is adjacent to k vertices, then G is a k-regular graph. Finally, we say that a k-regular graph G is *edge regular* if there exists a parameter λ such that for any two adjacent vertices, there are exactly λ vertices adjacent to both of them. If an edge regular graph with parameters k, λ also satisfies an additional property that for any two non-adjacent vertices, there are exactly μ vertices adjacent to both of them, then it is called a *strongly regular graph with parameters* k, λ, μ.

Let R be a ring and $n \in \mathbb{N}$. Let $R^\times$ denote the group of units of R. Let $\mathrm{M}_n(R)$ denote the ring of $n \times n$ matrices over R and the group of all invertible matrices over R is denoted by $\mathrm{GL}_n(R)$. Throughout this work, we denote I_n is the $n \times n$ identity matrices and denote $\mathbf{0}_{m\times n}$ is the $m \times n$ zero matrix.

An *eigenvalue* of a graph G is an eigenvalue of the adjacency matrix of a graph G. The *energy* of a graph G, $E(G)$, is the sum of absolute value of its eigenvalues. The *spectrum* of a graph G is the list of its eigenvalues together with their multiplicities. If $\lambda_1, \ldots, \lambda_r$ are eigenvalues of a graph G with multiplicities $m_1, \ldots, m_r$, respectively, we write $\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \ldots & \lambda_r \\ m_1 & \ldots & m_r \end{pmatrix}$ to describe the spectrum of G and so

$E(G) = m_1|\lambda_1| + \cdots + m_r|\lambda_r|$. A graph G on n vertices is said to be *hyperenergytic* if its energy exceeds the energy of the complete graph K_n, that is, $E(G) > 2(n-1)$. A k-regular graph G is a *Ramanujan graph* if $|\lambda| \leq 2\sqrt{k-1}$ for all eigenvalues λ of G other than $\pm k$. A Ramanujan graph is a regular graph whose spectral gap is almost as large as possible. It is an excellent spectral expander. Its name comes from Lubotzky, Phillips and Sarnak [10] who used the Ramanujan conjecture to construct an infinite family of such graphs.

To introduce our methodology, we recall some results on character of finite abelian groups. For more detail, see [9]. Let G be a finite abelian group. A map $\chi : G \to (\mathbb{C} \setminus \{0\}, \cdot)$ is a *character* if χ is a group homomorphism. The set of all characters of G, denoted by $\widehat{G}$, forms an abelian group under point-wise multiplication, that is, for any characters χ_1, χ_2 of G, we define $\chi_1 \cdot \chi_2 : G \to (\mathbb{C} \setminus \{0\}, \cdot)$ where $(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g)$ for all $g \in G$.

Let F be the finite field extension of $\mathbb{Z}_p$ which has order p^r for some $r \in \mathbb{N}$ and a prime p. The *trace map* from F to $\mathbb{Z}_p$ is the $\mathbb{Z}_p$-linear map $\mathrm{Tr} : x \mapsto x + x^p + \cdots + x^{p^{r-1}}$. According to [9], each character of the group $(F, +)$ is given by $\chi_a(x) = e^{\frac{2\pi i}{p}\mathrm{Tr}(ax)}$ for all $x \in F$ where $a \in F$ is fixed. Note that $(\mathrm{M}_n(F), +) \cong (F, +) \times (F, +) \times \cdots \times (F, +)$ (n^2 copies). Recall that if we have G_1, G_2 are finite abelian groups, then there is a canonical isomorphism $\widehat{G_1} \times \widehat{G_2} \to \widehat{G_1 \times G_2}$ given by $(\chi_1, \chi_2) \mapsto \chi_1\chi_2$. Hence, we may identify a character of $\mathrm{M}_n(F)$ as $\chi_A = \prod_{1 \leq i,j \leq n} \chi_{a_{ij}}$ where $A = [a_{ij}]_{n \times n}$ is in $\mathrm{M}_n(F)$ and so it follows from Theorem 2 of [11] that the eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ are given by

$$\rho_A = \sum_{S \in \mathrm{GL}_n(F)} \chi_A(S)$$

as A ranges over all matrices in $\mathrm{M}_n(F)$.

In the next section, we shall use the additive characters discussed in the previous paragraph to compute some eigenvalues (namely, ρ_{A_1}, ρ_{A_2} and ρ_{A_3}) and use them to study strong regularity of the unitary Cayley graph $\mathrm{C}_{\mathrm{M}_n(F)}$ of a matrix algebra over a finite field F of q elements. This new approach also allows us to conclude that the multiplicities of eigenvalues are at least the number of matrices of the same rank (Theorem 3.2.1). Without completely having the spectrum of the graph, we work on the eigenvalue ρ_{A_3} and show that $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic and characterize n and q such that $\mathrm{C}_{\mathrm{M}_n(F)}$ is Ramanujan in Section 3.

The final section presents the study of the unitary Cayley graph of product of matrix rings over finite local rings. We start by working on a finite local ring R with unique maximal ideal M and residue field $\Bbbk$. We determine the canonical graph isomorphism from the graph $\mathrm{C}_{\mathrm{M}_n(\Bbbk)} \otimes \mathring{\mathrm{M}}_n(M)$ onto the graph $\mathrm{C}_{\mathrm{M}_n(R)}$ induced from lifting elements of $\Bbbk$ to R via M (Theorem 3.4.2). This isomorphism allows us to obtain the clique number, the chromatic number and the independence number of the unitary Cayley graph of product of matrix rings over finite local rings. Since every finite commutative ring is isomorphic to a direct product of finite local rings, we have these numbers for unitary Cayley graphs of a matrix ring over a finite commutative ring. Moreover, the work in Sections 2 and 3 is generalized to matrix rings over finite local rings and finite commutative rings in Section 4.

3.2 Strong regularity of $\mathrm{M}_n(F)$

Throughout this section, let F be the finite field of q elements and $n \in \mathbb{N}$. Our main work is to show that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular if and only if $n = 2$. We begin by determining some eigenvalues of the graph by considering three matrices in $\mathrm{M}_n(F)$, namely,

$$A_1 = \mathbf{0}_{n\times n}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Clearly, we have

$$\rho_{A_1} = |\operatorname{GL}_n(F)| = (q^n - 1)(q^n - q)\dots(q^n - q^{n-1}).$$

Note that

$$\rho_{A_2} = \sum_{m\in F} N_m e^{\frac{2\pi i}{p}\operatorname{Tr}(m)}$$

where N_m is the number of invertible matrices with m at the left-top corner for all $m \in F$. If an invertible matrix has the left-top corner being 0, then the other $n-1$ elements in the first column cannot be all zeros, so there are $q^n - 1$ choices for the first column. Thus,

$$N_0 = (q^{n-1} - 1)(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1})$$

because the second column must not be multiple of the first column, and the jth column must not be a linear combination of the previous $j-1$ columns for all $j \in \{2, \dots, n\}$. Now, we have

$$(q^n - q^{n-1})(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1})$$

invertible matrices with the top$-$left corner being nonzero. Since $N_m = N_1$ for all $m \neq 0$, we have

$$(q-1)N_1 = (q^n - q^{n-1})(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1})$$

so

$$N_1 = q^{n-1}(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}).$$

It follows that

$$\begin{aligned}
\rho_{A_2} &= N_0 e^{\frac{2\pi i}{p}\operatorname{Tr}(0)} + N_1 \sum_{m\neq 0} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} \\
&= -(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}) + N_1 \sum_{m\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)}.
\end{aligned}$$

By Hilbert's theorem 90, we know that the trace map is surjective, so we get

$$\sum_{m\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} = |\ker \operatorname{Tr}| \sum_{m\in\mathbb{Z}_p} e^{\frac{2\pi i}{p} m} = 0.$$

Therefore,

$$\rho_{A_2} = -(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}).$$

Finally, we determine ρ_{A_3}. Since

$$\rho_{A_3} = N(m_1, m_2, \dots, m_{n+1}) \sum_{m_1, m_2, \dots, m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_1 + m_2 + \dots + m_n + m_{n+1})}$$

where $N(m_1, m_2, \dots, m_{n+1})$ is the number of invertible matrices of the form

$$\begin{bmatrix} m_1 & m_{n+1} & \cdots & * \\ m_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ m_n & * & \cdots & * \end{bmatrix}$$

and $m_1, m_2, \ldots, m_{n+1} \in F$. For $m_1 = 0$, we can determine $N(0, m_2, \ldots, m_{n+1})$ according to m_{n+1} as follows. If $m_{n+1} \neq 0$, then the first column and the second column are linearly independent, so the second column can be arbitrarily chosen. If $m_{n+1} = 0$, then the second column must not be multiple of the first column and the jth column must not be a linear combination of the previous $j-1$ columns for all $j \in \{2, \ldots, n\}$. Thus, $N(0, m_2, \ldots, 0) = (q^{n-1})(q^n - q^2) \ldots (q^n - q^{n-1})$ and $N(0, m_2, \ldots, m_{n+1}) = (q^{n-1})(q^n - q^2) \ldots (q^n - q^{n-1})$ if $m_{n+1} \neq 0$. Now, assume that $m_1 \neq 0$. Then $N(m_1, m_2, \ldots, m_{n+1}) = N(1, m_2, \ldots, m_{n+1})$ for all $m_2, \ldots, m_{n+1} \in F$. To find $N(1, m_2, \ldots, m_{n+1})$, we note that the second column cannot be m_{n+1}-multiple of the first column and similarly the jth column must not be a linear combination of the previous $j-1$ columns for all $j \in \{2, \ldots, n\}$, so

$$N(1, m_2, \ldots, m_{n+1}) = (q^{n-1} - 1)(q^n - q^2) \ldots (q^n - q^{n-1}).$$

Now, we compute

$$\begin{aligned}
\rho_{A_3} &= (q^{n-1} - q)(q^n - q^2) \ldots (q^n - q^{n-1})(q^n + 1) \sum{}' e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n)} \\
&\quad + q^{n-1}(q^n - q^2) \ldots (q^n - q^{n-1}) \sum{}' \sum_{m_{n+1} \neq 0} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n + m_{n+1})} \\
&\quad + (q^{n-1} - 1)(q^n - q^2) \ldots (q^n - q^{n-1}) \sum_{m_1 \neq 0} \sum{}' \sum_{m_{n+1} \in F} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_1 + m_2 + \ldots m_n + m_{n+1})}
\end{aligned}$$

where $\sum{}'$ denotes the sum over $m_2, \ldots, m_n \in F$ such that $\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}$ is the first column of an invertible matrix. Since $\displaystyle\sum_{m_{n+1} \in F} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_{n+1})} = 0$, the last sum is 0, so we can rewrite ρ_{A_3} as

$$\begin{aligned}
\rho_{A_3} = q^{n-1}(q^n - q^2) \ldots (q^n - q^{n-1}) \sum{}' \sum_{m_{n+1} \in F} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n + m_{n+1})} \\
- q(q^n - q^2) \ldots (q^n - q^{n-1}) \sum{}' e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n)}
\end{aligned}$$

The first sum is again zero because m_{n+1} varies over F. Now, since $m_1 = 0$, $m_2, \ldots, m_n$ cannot be all zeros and so

$$\sum{}' e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n)} = \sum_{\{m_2, \ldots, m_n\} \neq \{0\}} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n)} = \sum_{m_2, \ldots, m_n \in F} e^{\frac{2\pi i}{p} \operatorname{Tr}(m_2 + \ldots m_n)} - 1 = -1.$$

Hence, $\rho_{A_3} = q(q^n - q^2) \ldots (q^n - q^{n-1})$.

Let A and B be $n \times n$ matrices over F. Assume that $\operatorname{rank} A = \operatorname{rank} B$. Then there exist invertible matrices P and Q such that $A = PBQ$. Consider $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n}$, $P = [p_{ij}]_{n \times n}$ and $Q = [q_{ij}]_{n \times n}$. For $S = [s_{ij}]_{n \times n} \in \mathrm{GL}_n(F)$, we have

$$\chi_A(S) = e^{\frac{2\pi i}{p} \operatorname{Tr}\left(\sum_{1 \leq i,j \leq n} a_{ij} s_{ij}\right)}$$

From

$$\begin{aligned}
\sum_{1 \leq i,j \leq n} a_{ij} s_{ij} &= \sum_{1 \leq i,j \leq n} \Big(\sum_{1 \leq k,l \leq n} p_{il} b_{lk} q_{kj} \Big) s_{ij} \\
&= \sum_{1 \leq i,j \leq n} \sum_{1 \leq k,l \leq n} b_{lk} (p_{il} s_{ij} q_{kj}) \\
&= \sum_{1 \leq k,l \leq n} b_{lk} \sum_{1 \leq i,j \leq n} (p_{il} s_{ij} q_{kj}).
\end{aligned}$$

and $\sum_{1\le i,j\le n} p_{il}s_{ij}q_{kj} = \left(P^tSQ^t\right)_{lk}$, it follows that $\chi_A(S) = \chi_B(P^tSQ^t)$. Since P and Q are invertible, $\mathrm{GL}_n(F) = P^t\,\mathrm{GL}_n(F)Q^t$, so

$$\sum_{S\in\mathrm{GL}_n(F)} \chi_A(S) = \sum_{S\in\mathrm{GL}_n(F)} \chi_B(S).$$

Hence, we have shown:

Theorem 3.2.1. *If A and B are $n\times n$ matrices over F of the same rank, then $\rho_A = \rho_B$.*

Since $\mathrm{C}_{\mathrm{M}_n(F)}$ is connected and $|\mathrm{GL}_n(F)|$-regular, ρ_{A_1} induced from the zero matrix has multiplicity 1. Observe that ρ_{A_2} and ρ_{A_3} are induced by matrices of rank 1 and 2, respectively. Since the set of characters are linearly independent, the multiplicities of them are the number of matrices of such rank. Suppose $n=2$. The number of matrices of rank 1 is $\frac{(q^2-1)^2}{q-1} = (q-1)(q+1)^2$ and the number of matrices of rank 2 is $(q^2-1)(q^2-q)$. We record this result in:

Theorem 3.2.2. $\operatorname{Spec}\mathrm{C}_{\mathrm{M}_2(F)} = \begin{pmatrix} (q^2-1)(q^2-q) & -(q^2-q) & q \\ 1 & (q-1)(q+1)^2 & (q^2-1)(q^2-q)\end{pmatrix}$ *and* $E(\mathrm{C}_{\mathrm{M}_2(F)}) = 2q(q^2-1)^2$.

If $n=3$, then $\rho_{A_1}(q^3-1)(q^3-q)(q^3-q^2)$, $\rho_{A_2} = -(q^3-q)(q^3-q^2)$ and $\rho_{A_3} = q(q^3-q^2)$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_3(F)}$ induced from matrices of rank 0, 1 and 2, respectively. Let λ be the eigenvalue induced from matrices of rank 3. Counting the number of matrices of each rank gives

$$\begin{aligned}(q^3-1)(q^3-q)(q^3-q^2) - (q^3-q)(q^3-q^2)\frac{(q^3-1)^2}{q-1}& \\ + q(q^3-q^2)\frac{(q^3-1)^2(q^3-q)^2}{(q^2-1)(q^2-q)} &+ (q^3-1)(q^3-q)(q^3-q^2)\lambda = 0.\end{aligned}$$

Dividing by $(q^3-1)(q^3-q)(q^3-q^2)$ implies $\lambda = -q^3$. This proves the following theorem.

Theorem 3.2.3. $\operatorname{Spec}\mathrm{C}_{\mathrm{M}_3(F)} = \begin{pmatrix} (q^3-1)(q^3-q)(q^3-q^2) & -(q^3-q)(q^3-q^2) \\ 1 & (q^3-1)(q^2+q+1) \end{pmatrix.$

$$\left. \begin{matrix} q(q^3-q^2) & -q^3 \\ (q^3-1)(q^3-q)(q^2+q+1) & (q^3-1)(q^3-q)(q^3-q^2)\end{matrix}\right).$$

Recall from Chapter 10 of [3] that a connected regular graph is strongly regular if and only if it has exactly three distinct eigenvalues. So, we can conclude from Theorem 3.2.2 that $\mathrm{C}_{\mathrm{M}_2(F)}$ is strongly regular. Next, we assume that $n\ge 3$ and $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular. Then $\mathrm{C}_{\mathrm{M}_n(F)}$ has only three eigenvalues. From our computation, they must be ρ_{A_1}, ρ_{A_2} and ρ_{A_3}. Suppose the multiplicities of ρ_{A_2} and ρ_{A_3} are m_2 and m_3, respectively. Since the sum of eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ is 0, we have

$$(q^n-1)(q^n-q)\dots(q^n-q^{n-1}) - (q^n-q)\dots(q^n-q^{n-1})m_2 + q(q^n-q^2)\dots(q^n-q^{n-1})m_3 = 0.$$

Dividing by $(q^n-q^2)\dots(q^n-q^{n-1})$ gives

$$(q^n-1)(q^n-q) - (q^n-q)m_2 + qm_3 = 0.$$

Note that $1+m_2+m_3 = q^{n^2}$, so $m_3 = q^{n^2} - m_2 - 1$. Putting m_3 in the previous equation gives $m_2 = q(q^{n-1}-1)(q^{n^2-n}-1)$. Recall from Corollary 8.1.3 of [3] that the sum of square of eigenvalues of the adjacency matrix A is the trace of A^2 which is twice of the number of edges of the graph. Since our graph is $|\mathrm{GL}_n(F)|$-regular, if E_n is the number of edges, then

$$2E_n = q^{n^2}(q^n-1)\dots(q^n-q^{n-1}).$$

This yields another relation on m_2 and m_3 given by

$$((q^n-1)(q^n-q)\dots(q^n-q^{n-1}))^2 + ((q^n-q)\dots(q^n-q^{n-1}))^2 m_2 \\ + (q(q^n-q^2)\dots(q^n-q^{n-1}))^2 m_3 = q^{n^2}(q^n-1)\dots(q^n-q^{n-1}).$$

Dividing by $(q^n-q^2)\dots(q^n-q^{n-1})$ and substituting $m_2 = q(q^{n-1}-1)(q^{n^2-n}-1)$ give

$$\begin{aligned}(q^n-1)^2(q^n-q)^2\dots(q^n-q^{n-1}) &+ q^2(q^n-q)^2(q^n-q^2)\dots(q^n-q^{n-1})m_3 \\ &+ q(q^n-q^2)\dots(q^n-q^{n-1})(q^{n-1}-1)(q^{n^2-n}-1) \\ &= q^{n^2}(q^n-1)(q^n-q)\end{aligned}$$

Since $q^{n^2-n}-1 = \left(q^{n-1}\right)^n - 1$, the left hand side is divisible by $(q^{n-1}-1)^2$, so $(q^{n-1}-1)^2$ divides $q^{n^2}(q^n-1)(q^n-q)$. It follows that $q^{n-1}-1$ divides $q^{n^2+1}(q^n-1)$. Since q and q^n-1 are relatively prime, we have $q^{n-1}-1$ divides $q^n-1 = q^n-q+(q-1)$, so $q^{n-1}-1$ divides $q-1$ which is a contradiction because $n \geq 3$. Therefore, we have our desired result.

Theorem 3.2.4. *The graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular if and only if $n = 2$.*

From the above theorem, we learn that $\mathrm{C}_{\mathrm{M}_n(F)}$ is not strongly regular for $n \geq 3$. Since it is edge regular with $\lambda = e_n$, there are more than one value of the number of common neighborhoods of non-adjacent vertices in $\mathrm{C}_{\mathrm{M}_n(F)}$. If $A, B \in \mathrm{M}_n(F)$ and $\mathrm{rank}(A-B) = r$ for some $0 < r \leq n$, then there exist invertible matrices P, Q such that

$$P(A-B)Q = \begin{bmatrix} I_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{0}_{(n-r)\times(n-r)} \end{bmatrix}$$

For $A \in \mathrm{M}_n(F)$, let $N(A)$ be the set of neighbors of A. According to Kiani (Lemma 2.1 of [7]), we have

$$|N(A)\cap N(B)| = \left| \left(\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathrm{GL}_n(F) \right) \cap \mathrm{GL}_n(F) \right|$$

for all $A, B \in \mathrm{M}_n(F)$ with $A \neq B$. It gives the number of common neighbors of any pair of two vertices A and B in $\mathrm{M}_n(F)$. For $1 \leq r \leq n$, we define

$$d(n,r) = \left| \left(\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathrm{GL}_n(F) \right) \cap \mathrm{GL}_n(F) \right|.$$

Since two matrices A and B are adjacent if and only if $\mathrm{rank}(A-B) = n$, we have $d(n,n) = e_n$ where e_n is mentioned in Section 1. Observe that $d(n,r)$ is the number of invertible matrices A such that $A - \begin{bmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{bmatrix}$ is also invertible. Now, let $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be the standard basis of F^n. Consider the set $\mathcal{X}$ of vectors given by

$$\mathcal{X} = \left\{ A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} \in \mathrm{GL}_n(F) : \vec{a}_1 \in \vec{e}_1 + \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\} \right\}.$$

Note that if $A \in \mathcal{X}$, then A is invertible but $A - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ is not invertible. We proceed to compute $d(n,1)$. Since $d(n,1) = |GL_n(F)| - |\mathcal{X}|$, we shall determine the cardinality of $\mathcal{X}$. Let $A = [a_{ij}]_{n\times n}$ be in $\mathcal{X}$. Then $\mathrm{rank}\, A = n$ and $\mathrm{rank}\left(A - \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = n-1$. It follows that $\vec{a}_1 \notin \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$ but $\vec{a}_1 \in \vec{e}_1 + \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$. This forces that $\vec{e}_1 \notin \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$. Also, $\{\vec{a}_2, \dots, \vec{a}_n\}$ must be linearly independent. Thus, there are $(q^n-q)\dots(q^n-q^{n-1})$ choices for $\{\vec{a}_2, \dots, \vec{a}_n\}$. As for $\vec{a}_1$, it suffices to count under a condition $\vec{a}_1 \in \vec{e}_1 + \operatorname{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$ because if $\vec{a}_1 \in \operatorname{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$, then $\vec{e}_1 \in \operatorname{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$, which is absurd, so there are q^{n-1} choices for $\vec{a}_1$. Hence,

$$|\mathcal{X}| = q^{n-1}(q^n - q) \dots (q^n - q^{n-1}).$$

Then

Theorem 3.2.5. $d(n,1) = |\operatorname{GL}_n(F)| - |\mathcal{X}| = (q^n - q^{n-1} - 1)(q^n - q) \dots (q^n - q^{n-1})$.

Remark. For $r \geq 2$, we can find a lower bound for $d(n,r)$. Consider a matrix of the form $Y = \begin{bmatrix} A & \mathbf{0} \\ B & C \end{bmatrix}$ where A, B and C are $r \times r$, $(n-r) \times r$ and $(n-r) \times (n-r)$ matrices, respectively. It is easy to see that $\det Y = \det A \det C$, and $\det \left(X - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = \det(A - I_r) \det C$. If we choose A to be a derangement matrix and C is an invertible matrix, then Y and $Y - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ are invertible. Since there are e_r choices for A , $q^{r(n-r)}$ choices for B, and $(q^{n-r} - 1) \dots (q^{n-r} - q^{n-r-1})$ choices for C, we have $d(n,r) \geq e_r q^{r(n-r)}(q^{n-r} - 1) \dots (q^{n-r} - q^{n-r-1}) = e_r(q^n - q^r) \dots (q^n - q^{n-1})$.

3.3 Hyperenegetic graphs and Ramanujan graphs

Let F be a finite field of q elements. In this section, without explicitly computing the spectrum of the graph, we show that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic for all $n \geq 2$ and characterize n and q such that $\mathrm{C}_{\mathrm{M}_n(F)}$ is Ramanujan.

Since $q^3 - 1 = (q-1)(q^2 + q + 1) > q^2 + q$, we get $q(q^2 - 1) = q^3 - q > q^2 + 1$, so $E(\mathrm{C}_{\mathrm{M}_2(F)}) = 2q(q^2 - 1)^2 > 2(q^4 - 1)$. Then $\mathrm{C}_{\mathrm{M}_2(F)}$ is hyperenergetic. Next, we assume that $n \geq 3$. Recall that $\rho_{A_3} = q(q^n - q^2) \dots (q^n - q^{n-1})$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_n(F)}$ with multiplicities at least $\dfrac{(q^n - 1)^2 (q^n - q)^2}{(q^2 - 1)(q^2 - q)}$. It follows that

$$E(\mathrm{C}_{\mathrm{M}_n(F)}) > q(q^n - q^2) \dots (q^n - q^{n-1}) \frac{(q^n - 1)^2 (q^n - q)^2}{(q^2 - 1)(q^2 - q)}.$$

Thus, to show that $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic, it suffices to prove

$$q(q^n - q^2) \dots (q^n - q^{n-1}) \frac{(q^n - 1)^2 (q^n - q)^2}{(q^2 - 1)(q^2 - q)} > 2(q^{n^2} - 1).$$

Since $|\operatorname{GL}_n(F)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$, the above inequality is equivalent to

$$|\operatorname{GL}_n(F)| > \frac{2(q^2 - 1)(q^2 - q)(q^{n^2} - 1)}{q(q^n - 1)(q^n - q)}.$$

We shall use induction on $n \geq 3$ to show that this inequality holds and conclude that $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic. If $n = 3$, then the right-hand side becomes

$$\frac{2(q^2 - 1)(q^2 - q)(q^9 - 1)}{q(q^3 - 1)(q^3 - q)} = \frac{2(q-1)}{q}(q^6 + q^3 + 1)$$

and

$$|\operatorname{GL}_3(F)| = (q-1)^3(q^6 + 2q^5 + 2q^4 + q^3) > (q-1)^3(q^6 + q^3 + 1).$$

Since $q \geq 2$, we have $q(q-1)^2 \geq 2$. Then $(q-1)^3 \geq \dfrac{2(q-1)}{q}$ and the inequality is valid for $n = 3$. Now, let $n \geq 4$ and assume that

$$\begin{aligned}
|\operatorname{GL}_{n-1}(F)| &\geq \frac{2(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^{n-1}-1)(q^{n-1}-q)} \\
&= \frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^{n-1}-q)} \\
&\geq \frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^n-1)}
\end{aligned}$$

where the last inequality comes from $q^n - 1 - (q^{n-1} - q) = (q^{n-1}+1)(q-1) \geq 0$. Since $|\operatorname{GL}_n(F)| = (q^n-1)(q^n-q)\dots(q^n-q^{n-1}) = q^{n-1}(q^n-1)|GL_{n-1}(F)|$, it follows from the previous inequality that

$$|\operatorname{GL}_n(F)| \geq q^{n-1}(q^n-1)\frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^n-1)}$$

and so it remains to show that $q^n(q^n-1)(q^{(n-1)^2}-1) \geq q^{n^2}-1$. Rewrite

$$\begin{aligned}
q^n(q^n-1)(q^{(n-1)^2}-1) - q^{n^2} + 1 &= q^n(q^{n^2-n+1} - q^{n^2-2n+1} - q^n + 1) - q^{n^2} + 1 \\
&= q^{n^2+1} - q^{n^2-n+1} - q^{n^2} - q^{2n} + q^n + 1 \\
&= q^{n^2-n+1}\big(q^{n-1}(q-1)-1\big) - q^{2n} + q^n + 1.
\end{aligned}$$

Since $n \geq 4$ and $q \geq 2$,

$$q^{n^2-n+1}\big(q^{n-1}(q-1)-1\big) - q^{2n} \geq q^{n^2-n+1} - q^{2n} = q^{2n}(q^{n^2-3n+1}-1) \geq 0.$$

This completes the proof of the next theorem.

Theorem 3.3.1. $\mathrm{C}_{\mathrm{M}_n(F)}$ *is hyperenergetic for all* $n \geq 2$.

Recall that a k-regular graph is Ramanujan if $|\lambda| \leq 2\sqrt{k-1}$ for all eigenvalues λ other than $\pm k$. Since eigenvalues of a graph are real numbers, this inequality is equivalent to $\lambda^2 - 4(k-1) \leq 0$. We know that $\mathrm{C}_{\mathrm{M}_n(F)}$ is regular with parameter $k = (q^n-1)(q^n-q)\dots(q^n-q^{n-1})$. If $n = 2$, then its eigenvalues are $(q^2-1)(q^2-q), -(q^2-q)$ and q. Since $q \geq 2$, we have $q^2 - q \geq 2$, so

$$q^2 + 4 \leq 4q^2 \quad \text{and} \quad (q^2-q)^2 + 4 \leq 4(q^2-q).$$

The first inequality gives $q^2+4 \leq 4q(q+1)(q-1)^2$ which is equivalent to $q^2 - 4(q^2-1)(q^2-q)+4 \leq 0$ and the second inequality directly proves $(q^2-q)^2 < 4(q^2-1)(q^2-q) - 4$. Thus, $\mathrm{C}_{\mathrm{M}_2(F)}$ is Ramanujan. Now suppose that $n \geq 3$ and $\mathrm{C}_{\mathrm{M}_n(F)}$ is a Ramanujan graph. From the computation in the previous section, $\rho_{A_3} = (q^n-q)(q^n-q^2)\dots(q^n-q^{n-1})$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_n(F)}$, so

$$0 \geq \rho_{A_3}^2 - 4(q^n-1)(q^n-q)\dots(q^n-q^{n-1}) + 4 = \rho_{A_3}^2 - 4(q^n-1)\rho_{A_3} + 4 = (\rho_{A_3}+2)^2 - 4q^n\rho_{A_3}.$$

It follows that $4q^n\rho_{A_3} \geq (\rho_{A_3}+2)^2 > \rho_{A_3}^2$, so $4q^n > \rho_{A_3}$. For $n = 3$, this must imply that $q = 2$ and for $n \geq 4$, we have $n + 2 \leq \frac{(n-1)n}{2}$ and so

$$4q^n > \rho_{A_3} = q^{\frac{(n-1)n}{2}}(q^{n-1}-1)(q^{n-2}-1)\dots(q-1) > q^{\frac{(n-1)n}{2}}$$

which leads to a contradiction for all $q \geq 2$. Finally, if $n = 3$ and $q = 2$, by Theorem 3.2.3, we have $-(2^3-2)(2^3-2^2) = -24, 2(2^3-2^2) = 8$ and $-2^3 = -8$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_3(\mathbb{Z}_2)}$ and $4((2^3-1)(2^3-2)(2^3-2^2)-1) = 668$ is greater than 24^2 and 8^2. Hence, $\mathrm{C}_{\mathrm{M}_3(\mathbb{Z}_2)}$ is also Ramanujan.

We record this result in the following theorem.

Theorem 3.3.2. *The graph* $\mathrm{C}_{\mathrm{M}_n(F)}$ *is Ramanujan if and only if* $n = 2$ *or* ($n = 3$ *and* $F = \mathbb{Z}_2$).

3.4 The unitary Cayley graph of product of matrix rings over finite local rings

Let R be a local ring with unique maximal ideal M and residue field $\Bbbk$. Recall that $R/M \cong \Bbbk$ results in $\mathrm{M}_n(R)/\mathrm{M}_n(M) \cong \mathrm{M}_n(\Bbbk)$. Then elements in R can be partitioned into cosets of M and can be viewed as lifting from elements of $\Bbbk$. Suppose $|M| = m$ and $|\Bbbk| = q$. We fix $A_1, \ldots, A_{q^{n^2}}$ to be coset representatives of $\mathrm{M}_n(M)$ in $\mathrm{M}_n(R)$.

Lemma 3.4.1. *Let $A \in \mathrm{M}_n(R)$ and $X \in \mathrm{M}_n(M)$. Then*

$$\det(A + X) = (\det A) + m \textit{ for some } m \in M.$$

In particular, A is invertible if and only if $A + X$ is invertible.

Proof. Write $A = [a_{ij}]_{n\times n}$ and $X = [m_{ij}]_{n\times n}$. Then

$$\begin{aligned}\det(A + X) &= \sum_{\sigma\in S_n} (\operatorname{sgn} \sigma)(a_{1\sigma(1)} + m_{1\sigma(1)}) \ldots (a_{n\sigma(n)} + m_{n\sigma(n)})\\ &= \sum_{\sigma\in S_n} (\operatorname{sgn} \sigma)(a_{1\sigma(1)} \ldots a_{n\sigma(n)}) + m = (\det A) + m\end{aligned}$$

for some $m \in M$. □

The above lemma directly implies the following theorem.

Theorem 3.4.2. *1. For $A, B \in \mathrm{M}_n(R)$, A and B are adjacent in $\mathrm{C}_{\mathrm{M}_n(R)}$ if and only if $A + \mathrm{M}_n(M)$ and $B + \mathrm{M}_n(M)$ are adjacent in $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$.*

2. The set $\mathrm{M}_n(R)/\mathrm{M}_n(M) = \{A_1 + \mathrm{M}_n(M), \ldots, A_{q^{n^2}} + \mathrm{M}_n(M)\}$ is a partition of the vertex set of $\mathrm{C}_{\mathrm{M}_n(R)}$ such that

(a) for each $i \in \{1, \ldots, q^{n^2}\}$, any two distinct vertices in $A_i + \mathrm{M}_n(M)$ are nonadjacent vertices, and

(b) for $i, j \in \{1, \ldots, q^{n^2}\}$, A_i and A_j are adjacent in $\mathrm{C}_{\mathrm{M}_n(R)}$ if and only if $A_i + \mathrm{M}_n(M)$ and $A_j + \mathrm{M}_n(M)$ are adjacent in $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$.

3. Let $\mathring{\mathrm{M}}_n(M)$ be the complete graph of $|\mathrm{M}_n(M)|$ vertices with a loop on every vertex. Define $f : \mathrm{M}_n(\Bbbk) \times \mathrm{M}_n(M) \to \mathrm{M}_n(R)$ by $f(A_i + \mathrm{M}_n(M), X) = A_i + X$ for all $i \in \{1, \ldots, q^{n^2}\}$ and $X \in \mathrm{M}_n(M)$. Then f is an isomorphism from the graph $\mathrm{C}_{\mathrm{M}_n(\Bbbk)} \otimes \mathring{\mathrm{M}}_n(M)$ onto the graph $\mathrm{C}_{\mathrm{M}_n(R)}$.

Proof. The above discussion implies (1) and (2) For (3), we first show that f is an injection. Let $i, j \in \{1, \ldots, q^{n^2}\}$ and $X, Y \in \mathrm{M}_n(M)$ such that $A_i + X = A_j + Y$. Then $A_i - A_j = Y - X \in \mathrm{M}_n(M)$. This forces that $A_i + \mathrm{M}_n(M) = A_j + \mathrm{M}_n(M)$ in $\mathrm{M}_n(\Bbbk)$, so $i = j$ and $X = Y$. Since $|\mathrm{M}_n(\Bbbk) \times \mathrm{M}_n(M)| = |\mathrm{M}_n(R)|$, f is a bijection. Finally, for $i, j \in \{1, \ldots, q^{n^2}\}$ and $X, Y \in \mathrm{M}_n(M)$, we have $(A_i + \mathrm{M}_n(M), X)$ and $(A_j + \mathrm{M}_n(M), Y)$ are adjacent in $\mathrm{C}_{\mathrm{M}_n(\Bbbk)} \otimes \mathring{\mathrm{M}}_n(M)$ if and only if $A_i + \mathrm{M}_n(M)$ and $A_j + \mathrm{M}_n(M)$ are adjacent if and only if A_i and A_j are adjacent by (2). Hence, f is a graph isomorphism. □

Next, we assume that R is a finite local ring which is not a field with unique maximal ideal M and residue field $\Bbbk$. Let $|M| = m$ and $|\Bbbk| = q$. Since the adjacency matrix of $\mathring{\mathrm{M}}_n(M)$ is the all-ones matrix of size m^{n^2}, we have $\operatorname{Spec}\left(\mathring{\mathrm{M}}_n(M)\right) = \begin{pmatrix} m^{n^2} & 0 \\ 1 & m^{n^2} - 1 \end{pmatrix}$ and $(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1}), -(q^n - q) \ldots (q^n - q^{n-1})$ and $q(q^n - q^2) \ldots (q^n - q^{n-1})$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$. Since the eigenvalues of $G \otimes H$ are $\lambda_i\mu_j$ where λ_i's and μ_j's are eigenvalues of G and H, respectively, we can conclude from the isomorphism in Theorem 3.4.2 (3) that $0, m^{n^2}(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1}), -m^{n^2}(q^n - q) \ldots (q^n - q^{n-1})$ and $m^{n^2} q(q^n - q^2) \ldots (q^n - q^{n-1})$ are distinct eigenvalues of $\mathrm{C}_{\mathrm{M}_n(R)}$. Then we have shown the following theorem.

Theorem 3.4.3. *If R is a local ring which is not a field and $n \geq 2$, then $\mathrm{C}_{\mathrm{M}_n(R)}$ is not strongly regular.*

However, it turns out that the graph $\mathrm{C}_{\mathrm{M}_n(R)}$ is hyperenergetic.

Theorem 3.4.4. *If R is a local ring, then $\mathrm{C}_{\mathrm{M}_n(R)}$ is hyperenergetic for all $n \geq 2$.*

Proof. Let $\Bbbk$ be the residue field of R and assume that $|\Bbbk| = q$. Recall that $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$ is hyperenergetic and $\mathrm{C}_{\mathrm{M}_n(R)}$ has $-m^{n^2}q(q^n - q^2)\dots(q^n - q^{n-1})$ as an eigenvalue with multiplicities at least $\dfrac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)}$. The proof of Theorem 3.3.1 tells us that

$$q(q^n - q^2)\dots(q^n - q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} > 2(q^{n^2} - 1).$$

Note that the left-hand side is a multiple of q. It follows that

$$q(q^n - q^2)\dots(q^n - q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} \geq 2q^{n^2}$$

Multiplying by m^{n^2} both sides gives

$$m^{n^2}q(q^n - q^2)\dots(q^n - q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} \geq 2(mq)^{n^2} > 2((mq)^{n^2} - 1)$$

which completes the proof. □

Theorem 3.4.5. *If R is a local ring which is not a field, then $\mathrm{C}_{\mathrm{M}_n(R)}$ is not Ramanujan for all $n \geq 2$*

Proof. For simplicity, let $r = |\operatorname{GL}_n(\Bbbk)|$. We first handle case $n \geq 3$ and $q \geq 3$. Then $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$ is not Ramanujan by Theorem 3.3.2. From the proof of Theorem 3.3.2, we have $(q^n - q)\dots(q^n - q^{n-1}) \geq 2\sqrt{r-1}$. Thus,

$$m^{n^2}(q^n - q)\dots(q^n - q^{n-1}) \geq 2m^{n^2}\sqrt{r-1},$$

so we must show that $m^{n^2}\sqrt{r-1} > \sqrt{m^{n^2}r - 1}$. Rewrite

$$m^{2n^2}(r-1) - (m^{n^2}r - 1) = (m^{n^2} - 1)(m^{n^2}r - m^{n^2} - 1).$$

Since R is not a field, we have $m \geq 2$, so $(m^{n^2}-1)(m^{n^2}r - m^{n^2} - 1) > 0$ and the desired inequality follows. Next, we assume that $n = 3$ and $q = 2$. Then $-m^9(2^3-2)(2^3-2^2) = -24m^9$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_3(R)}$. Moreover, $r = m^9(2^3-1)(2^3-2)(2^3-2^2) = 168m^9$. We have $576m^{18} - 4(168m^9 - 1) = m^9(576m^9 - 672) + 4$. Since $m \geq 2$, we get $24m^9 > 2\sqrt{168m^9 - 1}$. Finally, if $n = 2$, then $-m^4(q^2-q)$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_2(R)}$ and $r = m^4(q^2-1)(q^2-q)$, so

$$\begin{aligned} m^8(q^2-q)^2 - 4(m^4(q^2-1)(q^2-q) - 1) &= m^8(q^2-q)^2 - 4m^4(q^2-1)(q^2-q) + 4 \\ &\geq m^8(q^2-q)^2 - 4m^4(q^2-q)^2 + 4 \\ &= (m^8 - 4m^4)(q^2-q)^2 + 4 > 0 \end{aligned}$$

because $m \geq 2$. Hence, $\mathrm{C}_{\mathrm{M}_2(R)}$ is not Ramanujan. □

Let $R_1, \dots, R_s$ be finite local rings with maximal ideals $M_1, \dots, M_s$ and residue fields $\Bbbk_1, \dots, \Bbbk_s$, respectively. Let $\mathcal{R} = \mathrm{M}_{n_1}(R_1) \times \cdots \times \mathrm{M}_{n_s}(R_s)$ where $n_1, \dots n_s \in \mathbb{N}$. By Theorem 3.8 of [6], we have

$$\chi(\mathrm{C}_{\mathcal{R}}) = \omega(\mathrm{C}_{\mathcal{R}}) = \omega(\mathrm{C}_{\mathrm{M}_{n_1}(\Bbbk_1)\times\cdots\times\mathrm{M}_{n_k}(\Bbbk_k)}) = \min_{1\leq i\leq s}\{|\Bbbk_i|^{n_i}\}$$

Finally, we compute $\alpha(C_{\mathcal{R}})$. Theorem 3.4.2 (3) gives

$$C_{\mathcal{R}} \cong \big(C_{M_{n_1}(\Bbbk_1)} \otimes \cdots \otimes C_{M_{n_s}(\Bbbk_s)}\big) \otimes \big(\mathring{M}_{n_1}(M_1) \otimes \cdots \otimes \mathring{M}_{n_s}(M_s)\big).$$

Since the second product is a complete graph with a loop on each vertex, we can see that

$$\begin{aligned}\alpha(C_{\mathcal{R}}) &= \alpha(C_{M_{n_1}(\Bbbk_1)} \otimes \cdots \otimes C_{M_{n_s}(\Bbbk_s)}) \prod_{i=1}^{s} |M_{n_i}(M_i)| \\ &= \frac{\prod_{i=1}^{s} |M_{n_i}(\Bbbk_i)|}{\min\limits_{1\le i\le s}\{|\Bbbk_i|^{n_i}\}} \prod_{i=1}^{s} |M_{n_i}(M_i)| = \frac{|\mathcal{R}|}{\min\limits_{1\le i\le s}\{|\Bbbk_i|^{n_i}\}}.\end{aligned}$$

Thus, we prove:

Theorem 3.4.6. $\omega(C_{\mathcal{R}}) = \chi(C_{\mathcal{R}}) = \min\limits_{1\le i\le s}\{|\Bbbk_i|^{n_i}\}$ *and* $\alpha(C_{\mathcal{R}}) = \dfrac{|\mathcal{R}|}{\min\limits_{1\le i\le s}\{|\Bbbk_i|^{n_i}\}}$.

For each $1 \le i \le s$, let $|M_i| = m_i$ and $|\Bbbk_i| = q_i$. Recall that $\rho_i = -{m_i}^{{n_i}^2} q_i({q_i}^n - {q_i}^2)\dots({q_i}^n - {q_i}^{n-1})$ is an eigenvalue of $C_{M_{n_i}(R_i)}$ with multiplicities at least t_i where $t_i = \dfrac{({q_i}^n-1)^2({q_i}^n-q_i)^2}{({q_i}^2-1)({q_i}^2-q_i)}$ for all i. Hence, $\prod_{i=1}^{s}\rho_i$ is an eigenvalue of $C_{\mathcal{R}}$ with multiplicities at least $\prod_{i=1}^{s} t_i$. By Theorem 3.4.3, we have $\rho_i t_i > 2(|M_{n_i}(R_i)| - 1)$ for all $1 \le i \le s$. Note that the left-hand side is a multiple of q_i. We can conclude that $\rho_i t_i \ge 2|R_i|^{{n_i}^2}$. It follows that

$$\prod_{i=1}^{s}\rho_i \prod_{i=1}^{s} t_i = \prod_{i=1}^{s}\rho_i t_i \ge \prod_{i=1}^{s} 2|M_{n_i}(R_i)| = 2^s \prod_{i=1}^{s} |M_{n_i}(R_i)| > 2\left(\prod_{i=1}^{s} |M_{n_i}(R_i)| - 1\right).$$

This shows that:

Theorem 3.4.7. *The graph $C_{\mathcal{R}}$ is hyperenergetic. In particular, if R is a finite commutative ring, then $C_{M_n(R)}$ is hypergeometric for all $n \ge 2$.*

Remark. The later statement comes from the fact that every finite commutative ring is isomorphic to a direct product of finite local rings. Indeed, we can use this fact and Theorem 3.4.6 to compute the clique number, chromatic number and independence number for the unitary Cayley graph of a matrix ring over a finite commutative ring.

3.5 References

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Chapter 4

Subconstituents of Unitary Cayley Graph of Matrix Algebras

4.1 Introduction

Let G be a finite abelian group and S be a subset of G not containing the identity and $S = S^{-1}$ where $S^{-1} = \{s^{-1} : s \in S\}$. The *Cayley graph of G associated to S* is the undirected graph $\mathrm{Cay}(G, S)$ whose vertex set is G and for each $g, h \in G$, g is adjacent to h if and only if $g = hs$ for some $s \in S$. We say that a Cayley graph is *normal* if S is a union of conjugacy classes of G.

For a finite ring R with identity $1 \neq 0$, we know that $(R, +)$ is an abelian group and we denote its group of units by $R^\times$. The *unitary Cayley graph of R*, C_R, is the graph $\mathrm{Cay}(R, R^\times)$, that is, its vertex set is R and for each $x, y \in R$, x is adjacent to y if and only if $x - y \in R^\times$. Since a finite commutative ring R can be decomposed as a direct product of finite local rings $R_1, \ldots, R_s$, the graph C_R is the tensor product of the graphs $\mathrm{C}_{R_1}, \ldots, \mathrm{C}_{R_s}$. Here, for graphs G and H with vertex sets $V(G)$ and $V(H)$, the *tensor product G and H*, $G \otimes H$, is the graph with vertex set $V(G) \times V(H)$ such that (a, b) is adjacent to (c, d) if and only if a is adjacent to c in G and b is adjacent to d in H for all $a, c \in V(G)$ and $b, d \in V(H)$. In addition, if R is a finite local ring with maximal ideal M, it follows from Proposition 2.2 of [1] that C_R is a complete multi-partite graph whose partite sets are the cosets of M. Thus, the unitary Cayley graphs of finite commutative rings are well studied.

Let G be a graph and $V(G)$ the vertex set of G. We give some terminologies from graph theory as follows. A *clique* is a subgraph that is a complete graph and *clique number* of G is the size of largest clique in G, denoted by $\omega(G)$. A set I of vertices of G is called an *independent set* if no distinct vertices of I are adjacent. The *independence number* of G is the size of a maximal independent set, denoted by $\alpha(G)$. The *chromatic number of G* is the least number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. We write $\chi(G)$ for the chromatic number of G. If every vertex of G is adjacent to k vertices, then G is a k-regular graph. Clearly, the above Cayley graph associated to a set S is a $|S|$-regular graph. Finally, we say that a k-regular graph G is *edge regular* if there exists a parameter λ such that for any two adjacent vertices, there are exactly λ vertices adjacent to both of them. If an edge regular graph with parameters k, λ also satisfies an additional property that for any two non-adjacent vertices, there are exactly μ vertices adjacent to both of them, then it is called a *strongly regular graph with parameters k, λ, μ*.

An *eigenvalue* of a graph G is an eigenvalue of the adjacency matrix of a graph G. The *energy* of a graph G, $E(G)$, is the sum of absolute value of its eigenvalues. The *spectrum* of a graph G is the list of its eigenvalues together with their multiplicities. If $\lambda_1, \ldots, \lambda_r$ are eigenvalues of a graph G with multiplicities $m_1, \dots, m_r$, respectively, we write $\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \dots & \lambda_r \\ m_1 & \dots & m_r \end{pmatrix}$ to describe the spectrum of G and so $E(G) = m_1|\lambda_1| + \dots + m_r|\lambda_r|$. A graph G on n vertices is *hyperenergetic* if its energy exceeds the energy of the complete graph K_n, that is, $E(G) > 2(n-1)$. A k-regular connected graph G is a *Ramanujan graph* if $|\lambda| \leq 2\sqrt{k-1}$ for all eigenvalues λ of G other than $\pm k$.

For a ring R with identity $1 \neq 0$ and $n \in \mathbb{N}$, $\mathrm{M}_n(R)$ is the ring of $n \times n$ matrices over R and the group of all invertible matrices over R is denoted by $\mathrm{GL}_n(R)$. Throughout this work, I_n is the $n \times n$ identity matrix and $\mathbf{0}_{m \times n}$ is the $m \times n$ zero matrix for all $m, n \in \mathbb{N}$.

In the previous chapter, we used additive characters of $\mathrm{M}_n(F)$ where F is a finite field and $n \in \mathbb{N}$ to determine three distinct eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ and use them to conclude that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular if and only if $n = 2$. We also showed that $\mathrm{C}_{\mathrm{M}_n(F)}$ is always hyperenergetic and gave a criterion for being a Ramanujan graph. Chen et al. [3] obtained all eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ using Bruhat's decomposition.

Let G be a graph and x a vertex of G. The *first subconstituent of G at x* is the subgraph of G induced by the set of neighborhoods of x and the *second subconstituent of G at x* is the subgraph of G induced by the set of vertices which is non-adjacent to x except x itself. Subconstituents of strongly regular graphs are studied in many graphs and have many interesting properties. The second subconstituent of the Hoffman-Singleton graph is determined by its spectrum in [5]. Moreover, the discovery of which graph has strongly regular subconstituents interests mathematicians. For example, Cameron et al. [4] used the Bose-Mesner algebra of a strongly regular graph to classify strongly regular graphs whose subconstituents are strongly regular, and Kasikova [8] used the same tools to classify distance-regular graph which has strongly regular subconstituents. In addition, we can use eigenvalues of subconstituents to prove the uniqueness of strongly regular of some parameter, e.g., Clebsch graph is a unique strongly regular graph with parameters $(16, 5, 0, 2)$ (see [7] p.230).

Now, we turn to the subconstituents of the unitary Cayley graph. Let R be a finite ring with identity $1 \neq 0$. The set of neighborhood of a vertex x of the graph C_R is denoted by $N(x)$. For $x \in R$, the maps $f : N(0) \to N(x)$ and $g : R \smallsetminus (N(0) \cup \{0\}) \to R \smallsetminus (N(x) \cup \{x\})$ which both send y to $x - y$ are graph isomorphisms. Hence, we may only study the subconstituents at $x = 0$ and we write $\mathrm{C}_R^{(1)}$ and $\mathrm{C}_R^{(2)}$ for the first subconstituent and the second subconstituent of C_R at $x = 0 \in R$, respectively. Let F be a finite field and $n \in \mathbb{N}$. In this work, we study $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ and $\mathrm{C}^{(2)}_{\mathrm{M}_n(F)}$. The graph $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is defined on the group $\mathrm{GL}_n(F)$ and the graph $\mathrm{C}^{(2)}_{\mathrm{M}_n(F)}$ is defined on the set of nonzero non-invertible matrices over F. We have the structure of $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ and $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$. We can determine the spectra of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ and $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ and conclude hyperenergeticity and Ramanujan property for both graphs. In addition, we compute the clique number, the chromatic number and the independence number of $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ and $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Next, we recall some results from representation theory used in this work. We refer the reader to [6] for more detail. Let G be a finite group and V a finite-dimensional complex vector space. A *representation* of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ denotes the group of automorphisms on V. For a representation ρ of G on V, a subspace W of V is *ρ-invariant* under G if $\rho(g)(W) \subseteq W$ for all $g \in G$. If ρ has no proper invariant subspace of V, then we say that ρ is an *irreducible representation*. Next, we define a character of a representation. Let ρ be a representation of G on V. Then for each $g \in G$, $\rho(g)$ is a linear transformation on V. A *character* χ corresponding to ρ is the complex-valued function on G defined by $\chi(g) = \mathrm{tr}(\rho(g))$ for all $g \in G$ where $\mathrm{tr}(\rho(g))$ is the trace of the matrix representation of $\rho(g)$ on V. A character is said to be *irreducible* if they are induced from an irreducible representation. The *dimension* of a character is the dimension of vector space V. It is easy to see that $\chi(1) = \dim V$ where 1 is the identity of the group G, and $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$. Thus, a character is a constant on a conjugacy class of G. Moreover, we have known from [10] that if S is a union of conjugacy classes of G and $\chi_1, \dots, \chi_r$ are irreducible characters of G, then the eigenvalues of $\mathrm{Cay}(G,S)$ are

$$\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s)$$

with multiplicity $m_j = \sum_{\substack{k=1 \\ \lambda_k = \lambda_j}}^{r} \chi_k(1)^2$ for all $j \in \{1, \ldots, r\}$.

Let F be the finite field of order q. Recall that the multiplicative group of nonzero elements of F is cyclic. Write $F^\times = \langle a \rangle$ for some $a \in F^\times$. The irreducible characters of the group $(F^\times, \cdot)$ are $\chi_k(x) = e^{\frac{2\pi i m k}{q-1}}$, where $x = a^m \in F^\times$ and $k \in \{0, 1, 2, \ldots, q-2\}$. In addition, we have

Theorem 4.1.1. *For $k \in \{0, 1, \ldots, q-2\}$,* $\sum_{x \in F^\times} \chi_k(x) = \begin{cases} q-1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$

The conjugacy classes of $\mathrm{GL}_2(F)$ are given in the following table.

Representatives	Number of elements	Number of classes
$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, x \neq 0$	1	$q-1$
$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, x \neq 0$	q^2-1	$q-1$
$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, $x \neq y$ and $x, y \neq 0$	q^2+q	$\dfrac{(q-1)(q-2)}{2}$
$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$, $y \neq 0$ (q is odd) $d_z = \begin{pmatrix} 0 & z^{q+1} \\ 1 & z+z^q \end{pmatrix}$, $z \in E \setminus F$ (q is even)	q^2-q	$\dfrac{q(q-1)}{2}$

where $\varepsilon \in F \setminus F^2$. Here, $c_{x,y}$ and $c_{y,x}$ are conjugate, $d_{x,y}$ and $d_{x,-y}$ are conjugate, and d_z and d_{z^q} are conjugate. Moreover, let $E = F[\sqrt{\varepsilon}]$ an extension of F of degree two. We can identify the matrices $d_{x,y}$ as $\zeta = x + y\sqrt{\varepsilon}$ and the matrices d_z as z in $E \setminus F$. Now, let α, β be distinct irreducible character of $F^\times$ and φ an irreducible characters of $E^\times$ such that $\varphi^q \neq \varphi$ and φ is not an irreducible character of $F^\times$. The next table presents all irreducible characters of $\mathrm{GL}_2(F)$. As mentioned earlier, it suffices to specify their values on each conjugacy class of $\mathrm{GL}_2(F)$.

	$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	q is odd $d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \zeta$	q is even $d_z = \begin{pmatrix} 0 & z^{q+1} \\ 1 & z+z^q \end{pmatrix} = z$
U_α	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(\zeta^q)$	$\alpha(z^q)$
V_α	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(\zeta^q)$	$-\alpha(z^q)$
$W_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	0	0
X_φ	$(q-1)\varphi(x)$	$-\varphi(x)$	0	$-(\varphi(\zeta) + \varphi(\zeta^q))$	$-(\varphi(z) + \varphi(z^q))$

Moreover, $U_\alpha, V_\alpha, W_{\alpha,\beta}$ and X_φ are of dimension $1, q, q+1$ and $q-1$, respectively.

The paper is organized as follows. In the next section, we prove that the graph $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is a normal Cayley graph and we determine all eigenvalues of the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ by using the two tables above. We show further that it is hyperenergetic and Ramanujan if $q \geq 3$. In section 3, we show that the graph

$\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ is the tensor product between a complete graph and a complete multi-partite graph, and obtain its spectrum. We apply this result to conclude that $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is hyperenergetic but it is not Ramanujan if $q \geq 5$. We compute the clique number, chromatic number and the independence number of the subconstituents of the graph $\mathrm{C}_{\mathrm{M}_2(F)}$ in the final section.

4.2 Spectral properties of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$

In this section, we study spectral properties of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$. We start by showing that $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is $\mathrm{Cay}\big(\mathrm{GL}_n(F), (\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)\big)$. To see this, let $A, B \in \mathrm{GL}_n(F)$. Then $AB^{-1} \in \mathrm{GL}_n(F)$ and

$$\begin{aligned} A - B \in GL_n(F) &\iff (AB^{-1} - \mathrm{I}_n)B^{-1} \in \mathrm{GL}_n(F) \\ &\iff (AB^{-1} - \mathrm{I}_n) \in \mathrm{GL}_n(F) \\ &\iff AB^{-1} \in (\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F). \end{aligned}$$

It also follows that the graph $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is regular of degree $|(\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)| = e_n$, defined in the previous section. Moreover, for $A, B \in \mathrm{GL}_n(F)$, we have

$$\begin{aligned} ABA^{-1} \in (\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F) &\iff ABA^{-1} - \mathrm{I}_n \in \mathrm{GL}_n(F) \\ &\iff A(B - \mathrm{I}_n)A^{-1} \in \mathrm{GL}_n(F) \\ &\iff (B - \mathrm{I}_n) \in \mathrm{GL}_n(F) \\ &\iff B \in (\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F). \end{aligned}$$

Thus, $(\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)$ is a union of conjugacy classes, so $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is a normal Cayley graph. We record this result in

Theorem 4.2.1. *The graph $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is the normal Cayley graph of $\mathrm{GL}_n(F)$ associated with $(\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)$ and it is regular of degree e_n.*

Next, we determine all eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$. Let $k \in \{0, 1, \ldots, q-2\}$ and consider χ_k an irreducible character of $F^\times$. We first handle the case q is odd by showing some lemmas on sums of characters of $F^\times$.

Lemma 4.2.2. *If q is odd, then for $k \in \{0, 1, \ldots, q-2\}$,*

$$\sum_{x \in F^\times} \chi_k(x^2) = \begin{cases} q-1 & \text{if } k \in \left\{0, \dfrac{q-1}{2}\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We know that

$$\sum_{x \in F^\times} \chi_k(x^2) = \sum_{m=0}^{q-2} \chi_k(a^{2m}) = \sum_{m=0}^{q-2} e^{\frac{4\pi imk}{q-1}} = \sum_{m=0}^{q-2} \left(e^{\frac{4\pi ik}{q-1}}\right)^m.$$

Note that $e^{\frac{4\pi ik}{q-1}} = 1$ if and only if $k = 0$ or $k = \dfrac{q-1}{2}$. If $k \in \left\{0, \dfrac{q-1}{2}\right\}$, then $\sum\limits_{x \in F^\times} \chi_k(x^2) = q - 1$. Finally, if $k \notin \left\{0, \dfrac{q-1}{2}\right\}$, then

$$\sum_{x \in F^\times} \chi_k(x^2) = \frac{1 - \left(e^{\frac{4\pi ik}{q-1}}\right)^{q-1}}{1 - \left(e^{\frac{4\pi ik}{q-1}}\right)} = 0,$$

and the proof completes. $\square$

Lemma 4.2.3. *If q is odd, then for $k \in \{0, 1, \dots, q-2\}$ and $\varepsilon \in F \setminus F^2$, we have*

(a) $$\sum_{\substack{x,y \in F^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) = \begin{cases} q^2 - 5q + 6 & \text{if } k = 0, \\ -q + 3 & \text{if } k = \dfrac{q-1}{2}, \text{ and} \\ 2 & \text{otherwise,} \end{cases}$$

(b) $$\sum_{(x,y) \in F \times F^\times} \chi_k(x^2 - \varepsilon y^2) = \begin{cases} q^2 - q & \text{if } k = 0, \\ -q + 1 & \text{if } k = \dfrac{q-1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We note that

$$\begin{aligned} \sum_{\substack{x,y \in F^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) &= \left(\sum_{x \in F^\times} \chi_k(x) \right) \left(\sum_{y \in F^\times} \chi_k(y) \right) - \sum_{x \in F^\times} \chi_k(x^2) \\ &\qquad - \sum_{x \in F^\times \setminus \{1\}} \chi_k(x) - \sum_{y \in F^\times \setminus \{1\}} \chi_k(y) \\ &= \left(\sum_{x \in F^\times} \chi_k(x) \right)^2 - \left(\sum_{x \in F^\times} \chi_k(x^2) \right) - 2 \left(\sum_{x \in F^\times} \chi_k(x) \right) + 2. \end{aligned}$$

If $k = 0$, then applying Lemma 4.2.2 gives the right-hand side equals $q^2 - 5q + 6$. If $k = \dfrac{q-1}{2}$, then the right-hand side is $-q+3$. Finally, if $k \notin \{0, \dfrac{q-1}{2}\}$, then the summands on the right-hand side are all gone and we get 2 left. This proves (a).

For (b), since $\varepsilon \in F \setminus F^2$, $E = F[\sqrt{\varepsilon}]$ an extension of degree two of F. Thus, $E = \{x + y\sqrt{\varepsilon} : x, y \in F\}$. Moreover, let $\mathrm{N}_{E/F}$ be the norm map. Recall that for $x, y \in F$, $\mathrm{N}_{E/F}(x + y\sqrt{\varepsilon}) = x^2 - \varepsilon y^2$ and by Hilbert's Theorem 90, $\mathrm{N}_{E/F}$ is surjective with kernel of size $q + 1$. Consider the sum

$$\begin{aligned} \sum_{(x,y) \in F \times F^\times} \chi_k(x^2 - \varepsilon y^2) &= \sum_{(x,y) \in F \times F \setminus \{(0,0)\}} \chi_k(x^2 - \varepsilon y^2) - \sum_{x \in F^\times} \chi_k(x^2) \\ &= \sum_{(x,y) \in F \times F \setminus \{(0,0)\}} \chi_k(\mathrm{N}_{E/F}(x + y\sqrt{\varepsilon})) - \sum_{x \in F^\times} \chi_k(x^2) \\ &= \left| \ker \mathrm{N}_{E/F} \right| \sum_{x \in F^\times} \chi_k(x) - \sum_{x \in F^\times} \chi_k(x^2) \\ &= (q+1) \sum_{x \in F^\times} \chi_k(x) - \sum_{x \in F^\times} \chi_k(x^2). \end{aligned}$$

If $k = 0$, then the right-hand side becomes $q^2 - q$, and if $k = \dfrac{q-1}{2}$, then the right-hand side is $-(q-1)$ by Lemma 4.2.2. Finally, for $k \notin \left\{0, \dfrac{q-1}{2}\right\}$, it also follows that each summand on the right-hand side is 0. $\square$

Lemma 4.2.4. *For $k, l \in \{0, 1, \dots, q-2\}$ such that $k \neq l$, we have*

$$\sum_{\substack{x,y \in F^\times \setminus \{1\} \\ \text{and } x \neq y}} [\chi_k(x)\chi_l(y) + \chi_k(y)\chi_l(x)] = \begin{cases} 4 & \text{if } 0 < k + l < q - 1, k, l \neq 0, \\ 2(3 - q) & \text{otherwise.} \end{cases}$$

Proof. We consider the sum

$$\sum_{\substack{x,y\in F^\times\smallsetminus\{1\}\\ \text{and } x\neq y}} [\chi_k(x)\chi_l(y)+\chi_k(y)\chi_l(x)] = 2\sum_{\substack{x,y\in F^\times\smallsetminus\{1\}\\ \text{and } x\neq y}} \chi_k(x)\chi_l(y)$$

$$= 2\left[\left(\sum_{x\in F^\times}\chi_k(x)\right)\left(\sum_{y\in F^\times}\chi_l(y)\right) - \sum_{x\in F^\times}\chi_k(x)\chi_l(x) - \sum_{x\in F^\times\smallsetminus\{1\}}\chi_k(x) - \sum_{y\in F^\times\smallsetminus\{1\}}\chi_l(y)\right].$$

Recall that

$$\sum_{x\in F^\times}\chi_k(x)\chi_l(x) = \begin{cases} q-1 & \text{if } k+l=q-1,\\ 0 & \text{otherwise.}\end{cases}$$

Since $k\neq l$, $k+l\neq 0$. If $k+l=q-1$, then $k,l\neq 0$ because $0\le k,l\le q-2$. It follows that

$$\sum_{\substack{x,y\in F^\times\smallsetminus\{1\}\\ \text{and } x\neq y}} [\chi_k(x)\chi_l(y)+\chi_k(y)\chi_l(x)] = 2\left(-(q-1)+2\right) = 2(3-q).$$

Assume that $k+l\neq q-1$. We distinguish two cases.
Case 1. $k=0$ or $l=0$, say $k=0$. Then $l\neq 0$ and so

$$\sum_{\substack{x,y\in F^\times\smallsetminus\{1\}\\ \text{and } x\neq y}} [\chi_k(x)\chi_l(y)+\chi_k(y)\chi_l(x)] = 2\left(-(q-1)+2\right) = 2(3-q).$$

Case 2. $k,l\neq 0$. Then we conclude that

$$\sum_{\substack{x,y\in F^\times\smallsetminus\{1\}\\ \text{and } x\neq y}} [\chi_k(x)\chi_l(y)+\chi_k(y)\chi_l(x)] = 2.$$

This completes the proof. □

Remark. Assume that q is odd. Before computing the eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$, we note that for each $x,y\in F$,

(a) $a_x\in \mathrm{GL}_2(F)\cap(\mathrm{I}_2+\mathrm{GL}_2(F))$ if and only if $x\neq 1$

(b) $b_x\in \mathrm{GL}_2(F)\cap(\mathrm{I}_2+\mathrm{GL}_2(F))$ if and only if $x\neq 1$

(c) $c_{x,y}\in \mathrm{GL}_2(F)\cap(\mathrm{I}_2+\mathrm{GL}_2(F))$ if and only if $x,y\neq 1$

(d) $d_{x,y}\in \mathrm{GL}_2(F)\cap(\mathrm{I}_2+\mathrm{GL}_2(F))$ for all $x\in F$ and $y\neq 0$.

To verify (d), we suppose that there exist $x\in F$ and $y\in F^\times$ such that $\det\begin{pmatrix} x-1 & \varepsilon y\\ y & x-1\end{pmatrix} = 0$, so $(x-1)^2-\varepsilon y^2=0$ in F. Thus, $x+y\sqrt{\varepsilon}=1$ in E. Since $\{1,\sqrt{\varepsilon}\}$ is an F-basis of E, we have $y=0$ which is absurd.

From the character table of $\mathrm{GL}_2(F)$ mentioned at the introduction, let λ_χ denote an eigenvalue induced from an irreducible character χ. Since the character U_{χ_k} has dimension one, the above remark gives

$$\lambda_{U_{\chi_k}} = \sum_{x\in F^\times\smallsetminus\{1\}}\chi_k(x^2) + (q^2-1)\sum_{x\in F^\times\smallsetminus\{1\}}\chi_k(x^2)$$
$$+\frac{q^2+q}{2}\sum_{\substack{x,y\in F^\times\smallsetminus\{1\}\\ \text{and } x\neq y}}\chi_k(xy) + \frac{q^2-q}{2}\sum_{(x,y)\in F\times F^\times}\chi_k(x^2-\varepsilon y^2).$$

According to Lemmas 4.2.2 and 4.2.3, we have $\lambda_{U_{\chi_0}} = q^4 - 2q^3 - q^2 + 3q$, $\lambda_{U_{\chi_{\frac{q-1}{2}}}} = q$ and

$$\lambda_{U_{\chi_k}} = (-1) + (q^2-1)(-1) + \frac{q^2+q}{2}(1+1) = q$$

if $k \notin \left\{0, \dfrac{q-1}{2}\right\}$. It follows that the eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ obtained from U_{χ_k} are $q^4 - 2q^3 - q^2 + 3q$ and q with multiplicities 1 and $q-2$, respectively.

Now, we work on V_{χ_k}. Since V_{χ_k} has dimension q, we have

$$\lambda_{V_{\chi_k}} = \frac{1}{q}\left(q \sum_{x \in F^\times \smallsetminus \{1\}} \chi_k(x^2) + \frac{q^2+q}{2} \sum_{\substack{x,y \in F^\times \smallsetminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) - \frac{q^2-q}{2} \sum_{(x,y) \in F \times F^\times} \chi_k(x^2 - \varepsilon y^2)\right).$$

Again, applying Lemmas 4.2.2 and 4.2.3 gives $\lambda_{V_{\chi_0}} = -q^2 + q + 1$, $\lambda_{V_{\chi_{\frac{q-1}{2}}}} = q$ and

$$\lambda_{V_{\chi_k}} = \frac{1}{q}\left(q(-1) + \frac{q^2+q}{2}(1+1)\right) = q$$

if $k \notin \{0, \dfrac{q-1}{2}\}$. Thus, the eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ obtained from V_{χ_k} are $-q^2+q+1$ and q with multiplicities q^2 and $q^2 + q^2(q-3) = q^3 - 2q^2$, respectively.

Next, we consider the eigenvalues induced from the character W_{χ_k,χ_l} with $k \neq l$. Since W_{χ_k,χ_l} has dimension $q+1$, we have

$$\lambda_{W_{\chi_k,\chi_l}} = \frac{1}{q+1}\Bigg((q+1) \sum_{x \in F^\times \smallsetminus \{1\}} \chi_k(x)\chi_l(x) + (q^2-1) \sum_{x \in F^\times \smallsetminus \{1\}} \chi_k(x)\chi_l(x)$$
$$+ \frac{q^2+q}{2} \sum_{\substack{x,y \in F^\times \smallsetminus \{1\} \\ \text{and } x \neq y}} (\chi_k(x)\chi_l(y) + \chi_k(y)\chi_l(x))\Bigg).$$

First, we assume that $k + l = q - 1$. Thus, $k, l \neq 0$. Note that there are $\dfrac{q-3}{2}$ choices of such k, l. It follows from Lemma 4.2.4 that

$$\lambda_{W_{\chi_k,\chi_l}} = \frac{1}{q+1}\left((q+1)(q-2) + (q^2-1)(q-2) + 2\left(\frac{q^2+q}{2}\right)(3-q)\right) = q.$$

If $0 < k + l < q - 1$, then we have two cases to consider. If $k = 0$ or $l = 0$, then there are $q-2$ choices of k and l, and

$$\lambda_{W_{\chi_k,\chi_l}} = \frac{1}{q+1}\left((q+1)(-1) + (q^2-1)(-1) + 2\left(\frac{q^2+q}{2}\right)(3-q)\right) = -q(q-2).$$

If $k, l \neq 0$, then there are $\dfrac{(q-3)^2}{2}$ choices of k and l, and

$$\lambda_{W_{\chi_k,\chi_l}} = \frac{1}{q+1}\left((q+1)(-1) + (q^2-1)(-1) + \left(\frac{q^2+q}{2}\right)(4)\right) = q.$$

Thus, the eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ obtained from W_{χ_k,χ_k} are $-q(q-2)$ and q with multiplicities $(q+1)^2(q-2)$ and $\dfrac{(q+1)^2(q-2)(q-3)}{2}$, respectively.

Finally, let φ be an irreducible character of $E^\times$ such that $\varphi^q \neq \varphi$. Hence, φ is a non-trivial character and there are $\dfrac{q^2-q}{2}$ choices of φ. Since X_φ has dimension $q-1$, we have

$$\begin{aligned}
\lambda_{X_\varphi} &= \frac{1}{q-1}\Bigg((q-1)\sum_{x\in F^\times\smallsetminus\{1\}}\varphi(x) - (q^2-1)\sum_{x\in F^\times\smallsetminus\{1\}}\varphi(x) \\
&\qquad - \frac{q^2-q}{2}\sum_{(x,y)\in F\times F^\times}\left(\varphi(x+y\sqrt{\varepsilon})+\varphi(x-y\sqrt{\varepsilon})\right)\Bigg) \\
&= \frac{1}{q-1}\Bigg(-(q^2-q)\sum_{x\in F^\times}\varphi(x) + (q^2-q) - (q^2-q)\sum_{(x,y)\in F\times F^\times}\varphi(x+y\sqrt{\varepsilon})\Bigg) \\
&= \frac{1}{q-1}\Bigg(-(q^2-q)\sum_{x\in E^\times}\varphi(x) + (q^2-q)\Bigg) = q.
\end{aligned}$$

Hence, the eigenvalue from this case is q with multiplicity $\dfrac{(q-1)^2(q^2-q)}{2}$.

Summing all multiplicities of the eigenvalue q from each character gives its total multiplicity $q^4 - 2q^3 - 2q^2 + 4q + 1$. Therefore, we obtain the spectrum of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ in the case that q is odd. For q even and $q \geq 4$, we can find all eigenvalues corresponding to each U_χ, V_χ and X_φ in the similar manner without the case $k = \dfrac{q-1}{2}$. Note that the eigenvalue obtained from the case $k = \dfrac{q-1}{2}$ when q is odd is always q. Hence, the eigenvalues corresponding to those characters of the case q is even and $q \geq 4$ are equal to the eigenvalues in the case q is odd. As for eigenvalues corresponding to W_{χ_k,χ_l}, we have multiplicities of q become $\dfrac{(q+1)^2(q-2)}{2}$ and $\dfrac{(q+1)^2(q-4)(q-2)}{2}$ whose sum is again $\dfrac{(q+1)^2(q-2)(q-3)}{2}$, so the multiplicities of q when q is even stays same.

Finally, if $q = 2$, then the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ has $(2^2-1)(2^2-2) = 6$ vertices and is two copies of K_3, so its spectra are 2 of multiplicity 2 and -1 of multiplicity 4. Thus, we completely determine the spectrum for the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$.

Theorem 4.2.5. *(a) If $q = 2$, then* $\operatorname{Spec}\mathrm{C}^{(1)}_{\mathrm{M}_2(F)} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$.

(b) If $q \geq 3$, then

$$\operatorname{Spec}\mathrm{C}^{(1)}_{\mathrm{M}_2(F)} = \begin{pmatrix} q^4-2q^3-q^2+3q & q & -q^2+q+1 & -q^2+2q \\ 1 & q^4-2q^3-2q^2+4q+1 & q^2 & (q+1)^2(q-2) \end{pmatrix}.$$

Moreover, $E(\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}) = 2q^5 - 2q^4 - 8q^3 + 6q^2 + 8q$ for all $q \geq 2$.

Furthermore, for all $q \geq 3$, we have

$$\begin{aligned}
E(\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}) &= 2q^5 - 2q^4 - 8q^3 + 6q^2 + 8q - 2\left((q^2-1)(q^2-q)-1\right) \\
&= 2q^5 - 4q^4 - 6q^3 + 8q^2 + 6q + 2 > 2q^5 - 4q^4 - 6q^3 = 2q^3(q-3)(q+1) \geq 0.
\end{aligned}$$

This proves hyperenergeticity of the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ when $q \geq 3$, while $\mathrm{C}^{(1)}_{\mathrm{M}_2(\mathbb{Z}_2)}$ is not hyperenergetic because its energy is $8 < 2(6-1)$.

Since $\mathrm{C}^{(1)}_{\mathrm{M}_2(\mathbb{Z}_2)}$ is disconnected, it is not Ramanujan. To show that the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ is Ramanujan for $q \geq 3$. Since $|-q^2+q+1| > |-q(q-2)| > q$, it suffices to show that $2\sqrt{(q^4-2q^3-q^2+3q-1} \geq q^2-q-1$

which is equivalent to $4(q^4 - 2q^3 - q^2 + 3q - 1) \geq (q^2 - q - 1)^2$, and we have

$$\begin{aligned} 4(q^4 - 2q^3 - q^2 + 3q - 1) - (q^2 - q - 1)^2 &= 3q^4 - 6q^3 - 3q^2 + 10q - 5 \\ &\geq 3q^4 - 6q^3 - 3q^2 = 3q^2((q-1)^2 - 2) \geq 0. \end{aligned}$$

We record this work in

Theorem 4.2.6. *The graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ is hyperenergetic and Ramanujan. Moreover, $\mathrm{C}^{(1)}_{\mathrm{M}_2(\mathbb{Z}_2)}$ is neither hyperenergetic nor Ramanujan.*

4.3 Spectral properties of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$

We study the second subconstituent of $\mathrm{C}_{\mathrm{M}_2(F)}$ in this section. We first show that the graph is a tensor product of a complete graph and a complete multi-partite graph and so we can calculate its eigenvalues. Let $F^{2\times 1}$ denote the set of column vectors of size 2×1 over F. Since a 2×2 matrix is non-invertible if and only if its column vectors are parallel, we can conclude that

$$\mathrm{M}_2(F) \smallsetminus (\mathrm{GL}_2(F) \cup \{\mathbf{0}_{2\times 2}\}) = \left(\dot{\bigcup_{\vec{v} \in F^{2\times 1} \smallsetminus \{\vec{0}\}}} \left\{ \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix} : a \in F \right\} \right) \cup \left\{ \begin{pmatrix} \vec{v} & \vec{0} \end{pmatrix} : \vec{v} \in F^{2\times 1} \smallsetminus \{\vec{0}\} \right\}$$

where $\vec{0}$ denotes the zero vector of $F^{2\times 1}$. Before giving a structure of the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$, we need the next lemma.

Lemma 4.3.1. *Let A, B be non-invertible matrices in $\mathrm{M}_2(F)$, $a, b \in F$ and $\vec{v}, \vec{w} \in F^{2\times 1} \smallsetminus \{\vec{0}\}$.*

(a) If $A = \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix}$ and $B = \begin{pmatrix} b\vec{w} & \vec{w} \end{pmatrix}$, then $A - B$ is non-invertible if and only if $a = b$ or $\vec{v}, \vec{w}$ are linearly dependent, or equivalently, $A - B$ is invertible if and only if $a \neq b$ and $\vec{v}, \vec{w}$ are linearly independent.

(b) If $A = \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix}$ and $B = \begin{pmatrix} \vec{w} & \vec{0} \end{pmatrix}$, then $A - B$ is non-invertible if and only if $\vec{v}$ and $\vec{w}$ are linearly dependent.

Proof. Observe that

$$A - B \text{ is non-invertible} \iff (a\vec{v} - b\vec{w}) = c(\vec{v} - \vec{w}) \text{ for some } c \in F.$$

Assume that $A - B$ is non-invertible and $\vec{v}, \vec{w}$ are linearly independent. Then $a = c$ and $b = c$, so $a = b$. Conversely, the case $a = b$ is clear. If $\vec{w} = c\vec{v}$ for some $c \in F$, then $A - B = \begin{pmatrix} (a - bc)\vec{v} & (1 - c)\vec{v} \end{pmatrix}$ is non-invertible. This proves (a). For (b), we have

$$\begin{aligned} A - B \text{ is non-invertible} &\iff a\vec{v} - \vec{w} = c\vec{v} \text{ for some } c \in F \\ &\iff (a - c)\vec{v} = \vec{w} \text{ for some } c \in F, \end{aligned}$$

which is equivalent to $\vec{v}$ and $\vec{w}$ are linearly dependent. □

In the next step, we define two graphs G and H as follows: G is the complete graph on $q + 1$ vertices parametrized by the set of projective lines $\mathbb{P}^1(F) = \{[a, 1] : a \in F\} \cup \{[1, 0]\}$ and the vertex set of H is $F^{2\times 1} \smallsetminus \{\vec{0}\}$ and for any $\vec{v}, \vec{w} \in F^{2\times 1} \smallsetminus \{\vec{0}\}$, $\vec{v}$ and $\vec{w}$ are adjacent if and only if $\vec{v}$ and $\vec{w}$ are not parallel. Note that H is the complete $(q + 1)$-partite graph such that each partite has $q - 1$ vertices.

Let $f : \mathrm{C}^{(2)}_{\mathrm{M}_2(F)} \to G \otimes H$ defined by $\begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix} \mapsto ([a,1],\vec{v})$ and $\begin{pmatrix} \vec{v} & \vec{0} \end{pmatrix} \mapsto ([1,0],\vec{v})$ for any $a \in F$ and $\vec{v} \in F^{2\times 1} \setminus \{\vec{0}\}$. Thus, f is bijective. Now, let A, B be nonzero non-invertible matrices in $\mathrm{M}_2(F)$, $a, b \in F$ and $\vec{v}, \vec{w} \in F^{2\times 1}$, $A = \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix}$ and $B = \begin{pmatrix} b\vec{w} & \vec{w} \end{pmatrix}$. Lemma 4.3.1 (a) implies

$$\begin{aligned} A - B \in \mathrm{GL}_2(F) &\iff a \neq b \text{ and } \vec{v}, \vec{w} \text{ are linearly independent} \\ &\iff ([a,1],\vec{v}) \text{ is adjacent to } ([b,1],\vec{w})\,. \end{aligned}$$

Next, we assume that $A = \begin{pmatrix} a\vec{v} & \vec{v} \end{pmatrix}$ and $B = \begin{pmatrix} \vec{w} & \vec{0} \end{pmatrix}$. From Lemma 4.3.1 (b), we have

$$\begin{aligned} A - B \in \mathrm{GL}_2(F) &\iff \vec{v} \text{ and } \vec{w} \text{ are linearly independent} \\ &\iff ([a,1],\vec{v}) \text{ is adjacent to } ([1,0],\vec{w})\,. \end{aligned}$$

Hence, f is a graph isomorphism, so we have the structure of the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Theorem 4.3.2. *The graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is the tensor product of the complete graph on $q+1$ vertices and the complete $(q+1)$-partite graph such that each partite has $q-1$ vertices, and it is a (q^3-q^2)-regular graph.*

Recall from [7] that if $\lambda_1, \ldots, \lambda_k$ are eigenvalues of a graph G_1 and $\mu_1, \ldots, \mu_l$ are eigenvalues of a graph G_2, then the eigenvalues of the tensor product $G_1 \otimes G_2$ are $\lambda_i \mu_j$ where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$. Since the eigenvalues of G are q of multiplicity 1 and -1 of multiplicity q and the eigenvalues of H are $q^2-q, -q+1$ and 0 of multiplicities $1, q$ and q^2-q-2, respectively, we obtain the spectrum and energy of the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Theorem 4.3.3. *We have*

$$\operatorname{Spec} \mathrm{C}^{(2)}_{\mathrm{M}_2(F)} = \begin{pmatrix} q^3-q^2 & -q^2+q & q-1 & 0 \\ 1 & 2q & q^2 & q^3-3q-2 \end{pmatrix}.$$

Moreover, $E(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) = 4q^3 - 4q^2$.

Since the number of vertices of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is $|\,\mathrm{M}_2(F) \smallsetminus (\mathrm{GL}_2(F) \cup \{\underline{0}\})\,| = q^3 + q^2 - q - 1$ and

$$4q^3 - 4q^2 - 2(q^3 + q^2 - q - 2) = 2q^3 - 6q^2 + 2q + 4 = 2(q-2)(q^2-q-1) \geq 0,$$

Thus, $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is hyperenergetic unless $q = 2$. Finally, we show that the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is not Ramanujan if $q \geq 5$. Since q^2-q is an eigenvalue of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$, we claim that $(q^2-q)^2 > 4(q^3-q^2-1)$, which is equivalent to the inequality $q^4 - 6q^3 + 5q^2 + 4 > 0$. This holds for $q \geq 5$ because $q^4 - 6q^3 + 5q^2 + 4 = q^2(q-1)(q-5) + 4 > 0$. For $q = 2, 3$ or 4, it is easily seen that $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is Ramanujan. We record both results in

Theorem 4.3.4. *The graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is hyperenergetic if and only if $q \geq 3$, and it is Ramanujan if and only if $q \leq 4$.*

4.4 Clique number, chromatic number and independence number

In this section, we compute the clique number, the chromatic number and the independence number of subconstituents of $\mathrm{C}_{\mathrm{M}_2(F)}$. Recall from the proof of Theorem 3.4 of [9] that the ring $\mathrm{M}_n(F)$ contains a subfield K of order q^n. We start with the first subconstituent. Note that $\mathbf{0}_{n\times n} \in K$ and so $K \smallsetminus \{\mathbf{0}_{n\times n}\}$ forms a complete subgraph in $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$. Hence, $\omega(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \geq q^n - 1$. On the other hand, let J be the set of matrices in $\mathrm{M}_n(F)$ whose all entries in the first row are zero. We can see that J is an ideal of $\mathrm{M}_n(F)$ of

q^{n^2-n} elements. Write $\mathrm{M}_n(F) = \bigcup_{i=1}^{q^n}(B_i + J)$ as a union of cosets of J where the coset $B_1 + J = J$. Note that each coset forms an independent set and $\mathbf{0}_{n\times n} \in J$. It follows that $\mathrm{GL}_n(F)$ is a subset of $\bigcup_{i=2}^{q^n}(B_i + J)$ and hence $\chi(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \leq q^n - 1$. Since $\omega(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \leq \chi(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)})$, we have the following theorem.

Theorem 4.4.1. $\omega(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) = \chi(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) = q^n - 1$.

Recall from [2] p.147 that if G is a graph, then $\alpha(G) \geq \dfrac{|V(G)|}{\chi(G)}$. Theorem 4.4.1 gives

$$\alpha(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \geq \frac{|\,\mathrm{GL}_n(F)|}{\chi(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)})} = (q^n - q)\dots(q^n - q^{n-1}).$$

Consider the group $K^\times$ as a multiplicative subgroup of $\mathrm{GL}_n(F)$. Let $X = AM$ and $Y = AN$ where $M, N \in K^\times$ such that $M \neq N$ and $A \in \mathrm{GL}_n(F)$. Then $X - Y = A(M - N)$ is invertible because $M, N \in K^\times$. It follows that each coset forms a complete graph. This implies that $\alpha(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \leq (q^n - q)\dots(q^n - q^{n-1})$. Hence, we have shown

Theorem 4.4.2. $\alpha(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) = (q^n - q)\dots(q^n - q^{n-1})$.

By Theorem 4.3.2, we have the second subconstituent of $\mathrm{C}_{\mathrm{M}_2(F)}$ is the tensor product of the complete graph on $q+1$ vertices G and the complete $q+1$-partite graph H such that each partite has $q-1$ vertices. Since $\chi(G) = \chi(H) = q+1$, we can conclude that $\chi(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) \leq q+1$. Moreover, let $V(G) = \{a_1, \dots, a_{q+1}\}$ and $V_1, \dots, V_{q+1}$ be the partites of H. Choose $v_i \in V_i$ for all $i \in \{1, \dots, q+1\}$. We can see that the subgraph of $G \otimes H$ induced by $\{(a_1, v_1), \dots, (a_{q+1}, v_{q+1})\}$ is a complete graph, so $\omega(G \otimes H) \geq q+1$. Thus, we obtain the clique number and the chromatic number of the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Theorem 4.4.3. $\omega(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) = \chi(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) = q + 1$.

Our final theorem gives the independence number of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Theorem 4.4.4. $\alpha(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) = q^2 - 1$.

Proof. Similar to the proof of Theorem 4.4.2, we know from Theorem 4.4.3 that

$$\alpha(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) \geq \frac{|\,\mathrm{M}_2(F) \smallsetminus (\mathrm{GL}_2(F) \cup \{\mathbf{0}_{n\times n}\})\,|}{\chi(\mathrm{C}^{(2)}_{\mathrm{M}_n(F)})} = \frac{q^3 + q^2 - q - 1}{q + 1} = q^2 - 1.$$

Write $\mathrm{M}_2(F) = \bigcup_{i=1}^{q^2}(A_i + K)$ as a union of cosets of K. Then an independent set of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is contained in $\bigcup_{i=2}^{q^2}(A_i + K)$. Since each coset forms a complete subgraph, we have $\alpha(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) \leq q^2 - 1$ and the result follows. $\square$

4.5 References

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Chapter 5

Algebraic Degree of Spectra of Cayley Hypergraphs

5.1 Hypergraphs

A *hypergraph* H is a pair $(V(\mathrm{H}), E(\mathrm{H}))$ where $V(\mathrm{H})$ is a finite set, called the *vertex set* of H, and $E(\mathrm{H})$ is a family of subsets of $V(\mathrm{H})$, called the *edge set* of H. The elements in $V(\mathrm{H})$ are called *vertices* and the elements in $E(\mathrm{H})$ are called *hyperedges*. In particular, if $E(\mathrm{H})$ consists only of 2-subsets of $V(\mathrm{H})$, then H is a simple graph. For $v \in V(\mathrm{H})$, we write $\mathfrak{D}(v)$ for the set of all hyperedges containing the vertex v and the number of elements in $\mathfrak{D}(v)$ is the *degree* of the vertex v, denoted by $\deg v$. A hypergraph in which all vertices have the same degree $k \geq 0$ is called *k-regular* and it is said to be *regular* if it is k-regular for some $k \geq 0$. A hypergraph in which all hyperedges have the same cardinality $l \geq 0$ is an *l-uniform* hypergraph. A *path* of length s in H is an alternating sequence $v_1E_1v_2E_2v_3 \dots v_sE_sv_{s+1}$ of distinct vertices $v_1, v_2, \dots, v_{s+1} \in V(\mathrm{H})$ and distinct hyperedges $E_1, E_2, \dots, E_s \in E(\mathrm{H})$ satisfying $v_i, v_{i+1} \in E_i$ for any $i = 1, 2, \dots, s$. The *distance* between two vertices v and w, denoted by $d(v, w)$, is the smallest length of a path from v to w. If there is no path from v to w, we define $d(v, w) = \infty$. The *diameter* of H is $\mathrm{diam}(\mathrm{H}) = \max\{d(v, w) : v, w \in V(\mathrm{H})\}$. A hypergraph H is *connected* if $\mathrm{diam}(\mathrm{H}) < \infty$.

For a hypergraph H with vertex set $\{v_1, \dots, v_n\}$, the *adjacency matrix* of H, denoted by $A(\mathrm{H})$, is the $n \times n$ matrix whose entry a_{ij}, $i \neq j$, is the number of hyperedges that contain both of v_i and v_j and $a_{ii} = 0$ for all $1 \leq i, j \leq n$. This concept was investigated by Bretto [1]. Evidently, it is a generalization of the adjacency matrix of a graph. An equivalent definition of the adjacency matrix is given in [4] by using the bipartite graph associated to H which is the graph whose vertex set is the union of two independent sets $V(\mathrm{H})$ and $E(\mathrm{H})$ and for any $v \in V(\mathrm{H})$ and $E \in E(\mathrm{H})$, they are adjacent whenever $v \in E$. In particular, if H is an l-uniform hypergraph, there is another way to define an adjacency matrix by using hypermatrix, see [3] and [5]. In this work, our hypergraphs may not be l-uniform, so we follow Bretto's. The *Laplacian matrix* of H, denoted by $L(\mathrm{H})$, is the $n \times n$ matrix defined by $L(\mathrm{H}) = E(\mathrm{H}) - A(\mathrm{H})$ where $E(\mathrm{H})$ is the diagonal matrix $\left[\deg v_i\right]_{1 \leq i \leq n}$. This version of Laplacian matrix was introduced by Rodríguez [10]. The *distance matrix* of a connected hypergraph H, denoted by $D(\mathrm{H})$, is the $n \times n$ matrix in which entry $d_{ij} = d(v_i, v_j)$ for all $1 \leq i, j \leq n$.

The *spectrum* of H, denoted by $\mathrm{Spec}(\mathrm{H})$, is the set of all eigenvalues of $A(\mathrm{H})$ including multiplicity. Observe that $A(\mathrm{H})$ is a real symmetric matrix, so $\mathrm{Spec}(\mathrm{H})$ contains only real eigenvalues. Since the characteristic polynomial of $A(\mathrm{H})$ is monic with integral coefficients, its rational roots are integers. A hypergraph is *integral* if all eigenvalues of this hypergraph are integers. Similarly, we can define $\mathrm{Lspec}(\mathrm{H})$ and $\mathrm{Dspec}(\mathrm{H})$ as the sets of all eigenvalues of $L(\mathrm{H})$ and $D(\mathrm{H})$, respectively. Also, an L-*integral hypergraph* is a hypergraph with integral Laplacian eigenvalues and a D-*integral hypergraph* is a hypergraph with integral distance eigenvalues.

For hypergraphs H_1 and H_2, the *Cartesian product* of H_1 and H_2, denoted by $\mathrm{H}_1 \square \mathrm{H}_2$, is the hypergraph with $V(\mathrm{H}_1 \square \mathrm{H}_2) = V(\mathrm{H}_1) \times V(\mathrm{H}_2)$ and $E(\mathrm{H}_1 \square \mathrm{H}_2) = \{\{x\} \times E' : x \in V(\mathrm{H}_1), E' \in E(\mathrm{H}_2)\} \cup \{E \times \{y\} : E \in E(\mathrm{H}_1) \text{ and } y \in V(\mathrm{H}_2)\}$. Observe that $A(\mathrm{H}_1 \square \mathrm{H}_2) = (A(\mathrm{H}_1) \otimes I_{|V(\mathrm{H}_2)|}) + (I_{|V(\mathrm{H}_1)|} \otimes A(\mathrm{H}_2))$ where $A \otimes B$ denotes the Kronecker product of matrices A and B. Therefore,

$$\mathrm{Spec}(\mathrm{H}_1 \square \mathrm{H}_2) = \{\lambda + \beta : \lambda \in \mathrm{Spec}(\mathrm{H}_1) \text{ and } \beta \in \mathrm{Spec}(\mathrm{H}_2)\}. \tag{A}$$

Let H_1 and H_2 be t-uniform hypergraphs. Following Pearson [9], the *tensor product* of H_1 and H_2, denoted by $\mathrm{H}_1 \otimes \mathrm{H}_2$, is the t-uniform hypergraph with $V(\mathrm{H}_1 \otimes \mathrm{H}_2) = V(\mathrm{H}_1) \times V(\mathrm{H}_2)$ and $E(\mathrm{H}_1 \otimes \mathrm{H}_2) = \{\{(x_{i_1}, y_{j_1}), \dots, (x_{i_t}, y_{j_t})\} : \{x_{i_1}, \dots, x_{i_t}\} \in E(\mathrm{H}_1), \{y_{j_1}, \dots, y_{j_t}\} \in E(\mathrm{H}_2)\}$. It follows that the number of hyperedges containing both of two vertices (x_i, y_l) and (x_j, y_m) in $\mathrm{H}_1 \otimes \mathrm{H}_2$ is $(t-2)! a_{ij} b_{lm}$. Hence, $A(\mathrm{H}_1 \otimes \mathrm{H}_2) = (t-2)! A(\mathrm{H}_1) \otimes A(\mathrm{H}_2)$. Consequently,

$$\mathrm{Spec}(\mathrm{H}_1 \otimes \mathrm{H}_2) = \{(t-2)! \lambda\beta : \lambda \in \mathrm{Spec}(\mathrm{H}_1) \text{ and } \beta \in \mathrm{Spec}(\mathrm{H}_2)\}. \tag{B}$$

5.2 t-Cayley hypergraphs

Throughout this section, we let $(G, \cdot)$ be a finite group with the identity e and S a subset of $G \smallsetminus \{e\}$ such that $S = S^{-1}$. For $t \in \mathbb{N}$ and $2 \le t \le \max\{o(x) : x \in S\}$, the *$t$-Cayley hypergraph* $\mathcal{H} = t\text{-}\mathrm{Cay}(G, S)$ of G over S is a hypergraph with vertex set $V(\mathcal{H}) = G$ and $E(\mathcal{H}) = \{\{yx^i : 0 \le i \le t-1\} : x \in S \text{ and } y \in G\}$. Here, $o(x)$ denotes the order of x in G.

Example 5.2.1. For $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{n} = (n_1, \dots, n_r)$ in $\mathbb{Z}^r$, we define the greatest common divisor of $\mathbf{m}$ and $\mathbf{n}$ to be the vector $\mathbf{d} = (d_1, \dots, d_r)$ where $d_i = \gcd(m_i, n_i)$ for all $i \in \{1, \dots, r\}$. Now, let $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$ and a divisor tuple $\mathbf{d} = (d_1, \dots, d_r)$ of $\mathbf{n}$, i.e., $d_i \mid n_i$ for all $i \in \{1, \dots, r\}$. Define

$$G_{\mathbf{n}}(\mathbf{d}) = \{\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r} : \gcd(\mathbf{x}, \mathbf{n}) = \mathbf{d}\}.$$

Let D be a set of divisor tuples of $\mathbf{n}$ not containing the zero vector of $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ and $S = \bigcup_{\mathbf{d} \in D} G_{\mathbf{n}}(\mathbf{d})$. For $t \in \mathbb{N}$ and $2 \le t \le \max\{o(\mathbf{x}) : \mathbf{x} \in S\}$, the t-Cayley hypergraph of $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ over S is called a $\gcd$-*hypergraph*.

Some properties of t-Cayley hypergraphs quoted from [2] are as follows.

Proposition 5.2.2. *Let $\mathcal{H} = t\text{-}\mathrm{Cay}(G, S)$.*

1. *$\mathcal{H}$ is connected if and only if $\langle S \rangle = G$.*
2. *For any $x \in S, y \in G$,* $\left|\{yx^i : 0 \le i \le t-1\}\right| = \begin{cases} t & \text{if } t \le o(x), \\ o(x) & \text{if } t > o(x). \end{cases}$
3. *$\mathcal{H}$ is t-uniform if and only if $t \le o(x)$ for any $x \in S$.*

Clearly, a Cayley graph $2\text{-}\mathrm{Cay}(G, S)$ is $|S|$-regular. We study a Cayley hypergraph $t\text{-}\mathrm{Cay}(G, S)$ in the next proposition.

Proposition 5.2.3. *A t-Cayley hypergraph of G over S is regular.*

Proof. We prove that t-$\mathrm{Cay}(G,S)$ is regular for any $2 \le t \le \max\{o(x) : x \in S\}$ by induction on t. For $t = 2$, we have known that 2-$\mathrm{Cay}(G,S)$ is regular. Now, let $t \in \mathbb{N}$ and $2 \le t < \max\{o(x) : x \in S\}$. Suppose that the hypergraph t-$\mathrm{Cay}(G,S)$ is regular. To prove the regularity of $(t+1)$-$\mathrm{Cay}(G,S)$, we consider the edge set $\{\{yx^i : 0 \le i \le t\} : x \in S, y \in G\}$ as a multi-set. Let $x \in S$ and $y \in G$. It follows that if $t \ge o(x)$, then $\{yx^i : 0 \le i \le t\} = \{yx^i : 0 \le i \le t-1\}$ and if $t < o(x)$, then $\{yx^i : 0 \le i \le t\} = \{yx^i : 0 \le i \le t-1\} \cup \{yx^t\}$. Note that for each $x \in S$ such that $t < o(x)$, we have $\{yx^t : y \in G\} = G$. By the above proposition and induction hypothesis, any vertex in a multi-hypergraph t-$\mathrm{Cay}(G,S)$ have the same degree. Thus, the multi-hypergraph t-$\mathrm{Cay}(G,S)$ is regular.

Now, we delete all multiple hyperedges (if it exists). Suppose that there are multiple hyperedges, say $\{y_1x_1^i : 0 \le i \le t\} = \{y_2x_2^i : 0 \le i \le t\}$. Then $\{\{yx_1^i : 0 \le i \le t\} : y \in G\} = \{\{yx_2^i : 0 \le i \le t\} : y \in G\}$. By deleting a collection of hyperedeges $\{\{yx_2^i : 0 \le i \le t-1\} : y \in G\}$, we have the number of each vertex in the deleted hyperedges are equal. We continue this process until there is no multiple hyperedges. Since the multi-hypergraph t-$\mathrm{Cay}(G,S)$ is regular, the hypergraph t-$\mathrm{Cay}(G,S)$ is regular by the previous paragraph. $\square$

Theorem 5.2.4. *The Cayley graph* 2-$\mathrm{Cay}(\mathbb{Z}_n, S)$ *is integral if and only if S is a union of some $G_n(d)$'s, where $d \mid n$ and $G_n(d) = \{k \in \{1,2,\ldots,n-1\} : \gcd(k,n) = d\}$.*

To characterize integral Cayley graphs of finite abelian groups, we first discuss the Cayley graph of the group $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, +)$. Let $S = S_1 \times S_2$ be a subset of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \setminus \{(0,0)\}$ such that $S = -S$. The Cayley graph 2-$\mathrm{Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$ can be distinguished into three cases.

1. $\overline{K}_{n_1} \square$ 2-$\mathrm{Cay}(\mathbb{Z}_{n_2}, S_2)$ if $S_1 = \{0\}$ and $S_2 \ne \{0\}$, where $\overline{K}_n$ denotes the empty graph on n vertices.

2. 2-$\mathrm{Cay}(\mathbb{Z}_{n_1}, S) \square\, \overline{K}_{n_2}$ if $S_1 \ne \{0\}$ and $S_2 = \{0\}$.

3. 2-$\mathrm{Cay}(\mathbb{Z}_{n_1}, S_1) \otimes$ 2-$\mathrm{Cay}(\mathbb{Z}_{n_2}, S_2)$ if $S_1 \ne \{0\}$ and $S_2 \ne \{0\}$.

It is clear that the eigenvalues of an empty graph are zero. By Eqs. (A), (B) and the fact that the Cayley graph always has an integral eigenvalue, the Cayley graph 2-$\mathrm{Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$ is integral if and only if for any $i \in \{1,2\}$ such that $S_i \ne \{0\}$, the 2-$\mathrm{Cay}(\mathbb{Z}_{n_i}, S_i)$ is integral. By the fundamental theorem of finite abelian groups, a finite abelian group is a direct product of finite cyclic groups. We can obtain a characterization of the integral Cayley graphs of finite abelian groups similar to the above discussion.

Theorem 5.2.5. *Let G be a finite abelian group and S a subset of $G \setminus \{e\}$ such that $S = S^{-1}$. Suppose $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ and $S = S_1 \times \cdots \times S_r$. The Cayley graph* 2-$\mathrm{Cay}(G,S)$ *is integral if and only if for any $i \in \{1,\ldots,r\}$ such that $S_i \ne \{0\}$, the* 2-$\mathrm{Cay}(\mathbb{Z}_{n_i}, S_i)$ *is integral.*

For non-integral graphs, Mönius et al. [7] defined the *algebraic degree* of a graph G to be the degree of extension of the splitting field of the characteristic polynomial of $A(\mathrm{G})$ over $\mathbb{Q}$. Recently, Mönius [8] determined the algebraic degree of Cayley graphs of $\mathbb{Z}_p$ where p is a prime number.

Our purposes are to characterize integral t-Cayley hypergraphs of $\mathbb{Z}_n$ and compute the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$. The paper is organized as follows. In Section 5.3, we study the spectrum of t-Cayley hypergraphs of $\mathbb{Z}_n$. We obtain the characterization of integral t-Cayley hypergraphs of $\mathbb{Z}_n$ similar to So [11]. We use this result to study integral t-Cayley hypergraphs of finite abelian groups. Moreover, we show that a $\gcd$-hypergraph is integral. We study non-integral hypergraphs in Section 5.4. We determine the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$ for all $n \ge 3$ which generalizes Mönius' results and provides an answer to his outlook. Our combinatorial approach is different from him and presented in Lemma 5.4.1.

5.3 Integral t-Cayley hypergraphs

An $n \times n$ matrix is *circulant* if it is of the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix}.$$

From now on, we let $n \geq 2$ and $\mathcal{H} = t\text{-}Cay(\mathbb{Z}_n, S)$. By the natural labeling $\{0, 1, \ldots, n-1\}$, it is easy to see that $A(\mathcal{H}) = [a_{ij}]_{0 \leq i,j \leq n-1}$ is circulant. To work on the adjacency matrix $A(\mathcal{H})$, it suffices to compute the first row of $A(\mathcal{H})$, i.e., $a_{0,k}$ where $0 \leq k \leq n-1$. Let $C = \{k : a_{0,k} \neq 0\} \subseteq \{1, 2, \ldots, n-1\}$ be the set of all vertices that adjacent to the vertex 0. It follows that $C = S \cup 2S \cup \cdots \cup (t-1)S \smallsetminus \{0\}$. Since $A(\mathcal{H})$ is circulant, we have $\mathrm{Spec}(\mathcal{H}) = \{\lambda_j : j = 0, 1, \ldots, n-1\}$ where

$$\lambda_j = \sum_{k \in C} a_{0,k}(e^{2\pi j i/n})^k.$$

We recall some useful properties taken from [11].

Proposition 5.3.1. *1. If d is a proper divisor of n and x is an nth root of unity, then $\sum_{k \in G_n(d)} x^k$ is an integer.*

2. Let $\omega = e^{2\pi i/n}$ and

$$F = \begin{bmatrix} \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot(n-1)} \\ \omega^{2\cdot 1} & \omega^{2\cdot 2} & \cdots & \omega^{2\cdot(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{(n-1)\cdot 1} & \omega^{(n-1)\cdot 2} & \cdots & \omega^{(n-1)\cdot(n-1)} \end{bmatrix}.$$

If $\mathcal{A} = \{v \in \mathbb{Q}^{n-1} : Fv \in \mathbb{Q}^{n-1}\}$, then $\mathcal{A}$ is a vector space over $\mathbb{Q}$. Moreover, $\mathcal{A} = \mathrm{Span}\{v_d : d \mid n \text{ and } d < n\}$ where v_d is the $(n-1)$-vector with 1 at the kth entry for all $k \in G_n(d)$ and 0 elsewhere.

Now, we prove a criterion for integral t-Cayley hypergraphs.

Theorem 5.3.2. *Let $\mathcal{H} = t\text{-}\mathrm{Cay}(\mathbb{Z}_n, S)$. Then $\mathcal{H}$ is integral if and only if C is a union of some $G_n(d)$'s where for each d, there is $c_d \in \left\{1, 2, \ldots, \binom{n}{t-2}\right\}$ such that $a_{0,k} = c_d$ for all $k \in G_n(d)$.*

Proof. Let $d_1, \ldots, d_s$ be all proper divisors of n. Without loss of generality, we assume that $C = G_n(d_1) \cup \cdots \cup G_n(d_l)$ for some $l \in \{1, \ldots, s\}$. Clearly, $\lambda_0 = \sum_{k \in C} a_{0,k} \in \mathbb{Z}$. For any $1 \leq j \leq n-1$, by the assumption and Proposition 5.3.1 (1),

$$\begin{aligned} \lambda_j &= \sum_{k \in C} a_{0,k}(e^{2\pi j i/n})^k \\ &= \sum_{k \in G_n(d_1)} a_{0,k}(e^{2\pi j i/n})^k + \cdots + \sum_{k \in G_n(d_l)} a_{0,k}(e^{2\pi j i/n})^k \\ &= c_{d_1} \sum_{k \in G_n(d_1)} (e^{2\pi j i/n})^k + \cdots + c_{d_l} \sum_{k \in G_n(d_l)} (e^{2\pi j i/n})^k \in \mathbb{Z}. \end{aligned}$$

Conversely, suppose that $\mathcal{H}$ is integral. Then $\lambda_j \in \mathbb{Z}$ for any $0 \leq j \leq n-1$. We consider the vector $v \in \mathbb{Q}^{n-1}$ with $a_{0,k}$ for the kth entry for any $k \in C$ and 0 elsewhere. Then

$$Fv = \begin{bmatrix} \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{(n-1)\cdot 1} & \omega^{(n-1)\cdot 2} & \cdots & \omega^{(n-1)\cdot(n-1)} \end{bmatrix} \begin{bmatrix} a_{0,1} \\ a_{0,2} \\ \vdots \\ a_{0,n-1} \end{bmatrix} = \begin{bmatrix} \sum_{k\in C} a_{0,k}\omega^{1\cdot k} \\ \sum_{k\in C} a_{0,k}\omega^{2\cdot k} \\ \vdots \\ \sum_{k\in C} a_{0,k}\omega^{(n-1)\cdot k} \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} \in \mathbb{Z}^{n-1}.$$

It follows that $v \in \mathcal{A}$ in Proposition 5.3.1 (2), and hence $v = \sum_{d|n, d<n} c_d v_d$ for some rational coefficients c_d's. The definition of v implies that the coefficient $c_d \in \left\{0, 1, \dots, \binom{n}{t-2}\right\}$. Therefore, C is a union of some $G_n(d)$'s where for each such d, we have $a_{0,k} = c_d$ for all $k \in G_n(d)$. □

For $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_n, S)$, it is clear that $\emptyset \neq S \subseteq C$. In particular, for $t = 2$, we have $S = C$. Theorem 5.3.2 implies that $\mathcal{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$ is integral if and only if S is a union of some $G_n(d)$'s and for which d, $a_{0,k} = 1$ for all $k \in G_n(d)$. This coincides So's result recalled in Theorem 5.2.4.

Theorem 5.3.3. *A* gcd*-hypergraph of* $\mathbb{Z}_n$ *is integral.*

Proof. Let $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ be a gcd-hypergraph. Assume that S is a union of some $G_n(d)$'s. This implies $C = \{k : a_{0,k} \neq 0\} = S \cup 2S \cup \cdots \cup (t-1)S \smallsetminus \{0\}$ is also a union of some $G_n(d)$'s. All hyperedges in $\mathcal{H}$ containing 0 must be in the following forms

$$\{-(t-1)x, -(t-2)x, \dots, -x, 0\}, \{-(t-2)x, -(t-3)x, \dots, 0, x\}, \dots,$$
$$\{-x, 0, \dots, (t-3)x, (t-2)x\}, \{0, x, \dots, (t-2)x, (t-1)x\}$$

where $x \in S$. For each d, there is $c_d \in \left\{1, 2, \dots, \binom{n}{t-2}\right\}$ such that $a_{0,k} = c_d$ for any $k \in G_n(d)$. Therefore, $\mathcal{H}$ is integral by Theorem 5.3.2. □

Note that the converse of Theorem 5.3.3 may not be true. For example, if $\mathcal{H} = 9\text{-Cay}(\mathbb{Z}_9, \{\pm 1\})$ which is not a gcd-hypergraph of $\mathbb{Z}_9$, then $E(\mathcal{H}) = \{\{0,1,2,3,4,5,6,7,8\}\}$. Hence, $C = \mathbb{Z}_9 \smallsetminus \{0\} = G_9(1) \cup G_9(3) \cup G_9(6)$ and $a_{0,k} = 1$ for any $k \in C$, but $\mathcal{H}$ is integral by Theorem 5.3.2.

We next characterize integral t-Cayley hypergraphs of finite abelian groups. First, let us consider the t-Cayley hypergraph of the group $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, +)$. Let $S = S_1 \times S_2$ be a subset of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \smallsetminus \{(0,0)\}$ such that $S = -S$ and $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$. To express $\mathcal{H}$ as a product of two hypergraphs, we need to assume that for any $i \in \{1,2\}$ such that $S_i \neq \{0\}$, $t \leq \min\{o(x) : x \in S_i\}$. From this assumption, the hypergraph $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$ can be distinguished into the following three cases.

1. $\overline{K}_{n_1} \square\, t\text{-Cay}(\mathbb{Z}_{n_2}, S_2)$ if $S_1 = \{0\}$ and $S_2 \neq \{0\}$.
2. $t\text{-Cay}(\mathbb{Z}_{n_1}, S) \,\square\, \overline{K}_{n_2}$ if $S_1 \neq \{0\}$ and $S_2 = \{0\}$.
3. $t\text{-Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S_1 \times S_2)$ is a subgraph of $t\text{-Cay}(\mathbb{Z}_{n_1}, S_1) \otimes t\text{-Cay}(\mathbb{Z}_{n_2}, S_2)$ if $S_1 \neq \{0\}$ and $S_2 \neq \{0\}$. In addition, its adjacency matrix is $A(t\text{-Cay}(\mathbb{Z}_{n_1}, S_1)) \otimes A(t\text{-Cay}(\mathbb{Z}_{n_2}, S_2)$.

Extend this argument to a finite product of finite cyclic groups, we obtain the next theorem.

Theorem 5.3.4. *Let G be a finite abelian group. Suppose $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ and $S = S_1 \times \cdots \times S_r$. Let $t \in \mathbb{N}$ and $2 \leq t \leq \min\{o(x) : x \in S_i\}$ for all $S_i \neq \{0\}$ and $i \in \{1, \dots, r\}$.The t-Cayley hypergraph $t\text{-Cay}(G, S)$ is integral if and only if for any $i \in \{1, \dots, r\}$ such that $S_i \neq \{0\}$, $t\text{-Cay}(\mathbb{Z}_{n_i}, S_i)$ is integral.*

Finally, we study L-integral t-Cayley hypergraphs. Let $\mathcal{H} = t\text{-Cay}(G, S)$ with $V(\mathcal{H}) = \{v_1, \ldots, v_n\}$. By Proposition 5.2.3, $\mathcal{H}$ is regular, so there exists $d \in \mathbb{N}$ such that $\deg v_i = d$ for any $i \in \{1, \ldots, n\}$. It follows that

$$L(\mathcal{H}) = \mathcal{D}(\mathcal{H}) - A(\mathcal{H}) = dI_n - A(\mathcal{H}).$$

Hence,

$$\text{Lspec}(\mathcal{H}) = \{\lambda - d : \lambda \in \text{Spec}(\mathcal{H})\}.$$

Corollary 5.3.5. *Let $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_n, S)$. Then $\mathcal{H}$ is* L*-integral if and only if $\mathcal{H}$ is integral. In particular, a* gcd*-hypergraph of $\mathbb{Z}_n$ is* L*-integral.*

Now, we consider D-integral t-Cayley hypergraphs. For $t = 2$, Ilić [6] showed that if S is a union of some $G_n(d)$'s, then $2\text{-Cay}(\mathbb{Z}_n, S)$ has integral D-spectra. Assume that $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ is connected. That is, $\langle S \rangle = G$ by Proposition 5.2.2 (1). By the natural labeling in $D(\mathcal{H})$, it is clear that $D(\mathcal{H})$ is circulant. Thus, it suffices to consider the first row of $D(\mathcal{H})$. Since $\mathcal{H}$ is connected, the set $\{k : d(0, k) \neq 0\} = \{1, 2, \ldots, n-1\}$. Hence, we get the characterization of D-integral t-Cayley hypergraphs similar to Theorem 5.3.2.

Theorem 5.3.6. *Assume that $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ is connected. Then $\mathcal{H}$ is* D*-integral if and only if for each $d \mid n$, there is $c_d \in \{1, 2, \ldots, \text{diam}(\mathcal{H})\}$ such that $d(0, k) = c_d$ for all $k \in G_n(d)$.*

5.4 Algebraic degree of spectra of t-Cayley hypergraphs of $\mathbb{Z}_n$

Let H be a hypergraph on m vertices and $f(x) = \det(xI_m - A(\text{H}))$ the characteristic polynomial of H. Let E_f be the splitting field of $f(x)$ over $\mathbb{Q}$. The *algebraic degree* of H is $[E_f : \mathbb{Q}]$ and denoted by $\deg \text{H}$. In Section 5.3, we have the characterization of integral t-Cayley hypergraphs. They are hypergraphs of algebraic degree one. We study the algebraic degree of t-Cayley hypergraphs in this section.

Let $n \geq 3$ and $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_n, S)$. Recall from the previous section that the eigenvalues of $\mathcal{H}$ are

$$\lambda_j = \sum_{k \in C} a_{0,k} (e^{2\pi j i/n})^k$$

where $C = \{k : a_{0,k} \neq 0\} = S \cup 2S \cup \cdots \cup (t-1)S \smallsetminus \{0\}$. Let $\omega = e^{2\pi i/n}$ be a primitive nth root of unity. By the fundamental theorem of Galois theory,

$$\deg \mathcal{H} = [\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) : \mathbb{Q}] = \frac{\phi(n)}{|\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}))|}, \tag{C}$$

where $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})) = \{\sigma \in \text{Aut}(\mathbb{Q}(\omega)) : \sigma \text{ is a } \mathbb{Q}\text{-automorphism and } \sigma(\lambda_j) = \lambda_j \text{ for all } j \in \{0, 1, \ldots, n-1\}\}$. We shall determine the size of this group and obtain $\deg \mathcal{H}$.

Lemma 5.4.1. *Let $y \in \{0, 1, \ldots, n-1\}$ be such that $\gcd(y, n) = 1$ and $\sigma_y \in \text{Aut}(\mathbb{Q}(\omega))$ be the $\mathbb{Q}$-automorphism defined by $\omega \mapsto \omega^y$. Then $\sigma_y(\lambda_j) = \lambda_j$ for all $j \in \{0, 1, \ldots, n-1\}$ if and only if there is $n_y \in \mathbb{N}$ with $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ and $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$.*

Proof. If there is $n_y \in \mathbb{N}$ with $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ and $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$, then

$$\begin{aligned}\sigma_y(\lambda_j) = \sigma_y\left(\sum_{k\in C} a_{0,k}\omega^{jk}\right) &= \sum_{l=1}^{n_y}\sum_{k\in C_l} a_{0,k}\sigma_y\left(\omega^{jk}\right) = \sum_{l=1}^{n_y}\sum_{k\in C_l} a_{0,k}\omega^{jky} \\ &= \sum_{l=1}^{n_y}\sum_{k\in C_l} a_{0,yk}\omega^{jky} = \sum_{k\in C} a_{0,yk}\omega^{jyk} = \sum_{yk\in C} a_{0,yk}\omega^{jyk} = \lambda_j\end{aligned}$$

for all $j \in \{0, 1, \ldots, n-1\}$. On the other hand, suppose that $\sigma_y(\lambda_j) = \lambda_j$ for all $j \in \{0, 1, \ldots, n-1\}$. Then $\sum_{k\in C} a_{0,k}\left(\omega^j\right)^{yk} = \sum_{k\in C} a_{0,k}\left(\omega^j\right)^k$ for all $j \in \{0, 1, \ldots, n-1\}$. Let $p(x) = \sum_{k\in C} a_{0,k}x^{yk} - \sum_{k\in C} a_{0,k}x^k$. It is a polynomial of degree at most $n-1$. Since $1, \omega, \ldots, \omega^{n-1}$ are distinct roots of $p(x)$, we have $p(x) = 0$. Define an equivalence relation on C by $k \sim k'$ whenever $a_{0,k} = a_{0,k'}$. Let $C_1, \ldots, C_{n_y}$ be all equivalence classes of $\sim$. Then $C = C_1 \cup \cdots \cup C_{n_y}$. Since $p(x) = 0$, we have $yC_l \equiv C_l \mod n$ and so $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$. ☐

Theorem 5.4.2. *Let $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ and $C = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$. Let m be the number of y in $\{0, 1, \ldots, n-1\}$ such that $\gcd(y, n) = 1$ and there is $n_y \in \mathbb{N}$ with $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ and $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$. Then*

$$\deg \mathcal{H} = \frac{\phi(n)}{m}.$$

Moreover, $\deg \mathcal{H} \le \frac{\phi(n)}{2}$.

Proof. By Lemma 5.4.1, m is the size of $\mathrm{Gal}\left(\mathbb{Q}(\omega)/\mathbb{Q}\left(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\right)\right)$. It follows from Eq. (C) that $\deg \mathcal{H} = \frac{\phi(n)}{m}$. From $S \equiv -S \mod n$, we have $C = -C \mod n$. Since $\{\pm k\} = -\{\pm k\}$ and $a_{0,k} = a_{0,-k}$ for any $k \in C$, 1 and -1 are such y. Hence, $m \ge 2$, so $\frac{\phi(n)}{m} \le \frac{\phi(n)}{2}$. ☐

Consider $\mathcal{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$. Then $C = S$ and $a_{0,k} = 1$ for any $k \in S$ and $a_{0,k} = 0$ otherwise. The assumption of Theorem 5.4.2 can be reduced to $yS \equiv S \mod n$. In addition, if $n = p$ is a prime number, Mönius showed in the proof of Theorem 2.5 of [8] that m in Theorem 5.4.2 is the maximum number of $M \in \{1, 2, \ldots, |S|\}$ such that M divides $\gcd(|S|, p-1)$ and

$$S = \bigcup_{l=1}^{|S|/M} S_l$$

where $|S_l| = M$ and for each $l \in \{1, \ldots, |S|/M\}$, $k^M = (k')^M \mod p$ for all $k, k' \in S_l$. The next corollary gives the algebraic degree of Cayley graph of $\mathbb{Z}_n$ over S which generalizes Theorem 2.5 of [8].

Corollary 5.4.3. *Let $\mathcal{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$. If m is the number of y in $\{0, 1, \ldots, n-1\}$ such that $yS \equiv S \mod n$, then*

$$\deg \mathcal{H} = \frac{\phi(n)}{m}.$$

Example 5.4.4. Consider $\mathcal{H} = 2\text{-Cay}(\mathbb{Z}_{31}, S)$ where $S = \{\pm 2, \pm 3, \pm 10, \pm 12, \pm 13, \pm 15\} = C$. Since $\pm 1, \pm 5, \pm 6$ are all elements of y such that $\gcd(y, 31) = 1$ and $yC \equiv C \mod 31$, by Corollary 5.4.3, $\deg \mathcal{H} = \frac{\phi(31)}{6} = 5$. This coincides Example 2.10 of [8].

In the proof of Theorem 5.4.2, we have known that 1 and -1 are always such y satisfying $yC \equiv C \mod n$. If only they satisfy this congruence, we have a special case of Theorem 5.4.2 as follows.

Corollary 5.4.5. *Let $\mathcal{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ and $C = S \cup 2S \cup \cdots \cup (t-1)S \smallsetminus \{0\}$. If $y = 1$ and $y = -1$ are the only elements in $\mathbb{Z}_n$ such that $\gcd(y, n) = 1$ and $yC \equiv C \mod n$, then*

$$\deg \mathcal{H} = \frac{\phi(n)}{2}.$$

We provide some numerical examples using Theorem 5.4.2 and Corollary 5.4.5 as follows.

Example 5.4.6. Consider $\mathcal{H} = 3\text{-Cay}(\mathbb{Z}_{12}, \{\pm 1\})$. We have $C = \{\pm 1, \pm 2\}$. In addition, $a_{0,\pm 1} = 2$ and $a_{0,\pm 2} = 1$. The characteristic polynomial of $A(\mathcal{H})$ is

$$(x-1)^2(x+2)^3(x+3)^2(x-6)(x^2-2x-11)^2$$

and hence $\deg \mathcal{H} = 2$. Since 1 and -1 are the only elements y in $\mathbb{Z}_{12}$ such that $\gcd(y, 12) = 1$ and $yC \equiv C \mod 12$, by Corollary 5.4.5, $\deg \mathcal{H} = \frac{\phi(12)}{2} = 2$.

Example 5.4.7. Let $S = \{\pm 1\}$ be a subset of $(\mathbb{Z}_9, +)$. Them $\max\{o(x) : x \in S\} = 9$, so $2 \leq t \leq 9$. The algebraic degree of t-Cayley hypergraph of $\mathbb{Z}_9$ over S for all t are presented in the following table. The cases $t \in \{2, 3, 4\}$ are computed by Corollary 5.4.5 and the others are obtained from Theorem 5.4.2.

t	$a_{0,\pm 1}$	$a_{0,\pm 2}$	$a_{0,\pm 3}$	$a_{0,\pm 4}$	y with $yC \equiv C \mod 9$	$\deg t\text{-Cay}(\mathbb{Z}_9, S)$
2	1				± 1	3
3	2	1			± 1	3
4	3	2	1		± 1	3
5	4	3	2	1	$\pm 1, \pm 2, \pm 4$	3
6	5	4	3	3	$\pm 1, \pm 2, \pm 4$	3
7	6	5	5	5	$\pm 1, \pm 2, \pm 4$	3
8	7	7	7	7	$\pm 1, \pm 2, \pm 4$	1
9	1	1	1	1	$\pm 1, \pm 2, \pm 4$	1

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Eigenvalues of zero divisor graphs of principal ideal rings

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ABSTRACT

In this paper, we first study zero divisor graphs over finite chain rings. We determine their rank, determinant, and eigenvalues using reduction graphs. Moreover, we extend the work to zero divisor graphs over finite commutative principal ideal rings using a combinatorial method, finding the number of positive eigenvalues and the number of negative eigenvalues, and finding upper and lower bounds for the largest eigenvalue. Finally, we characterize all finite commutative principal ideal rings such that their zero divisor graphs are complete and compute the Wiener index of the zero divisor graphs over finite commutative principal ideal rings.

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1. Introduction

Throughout the paper, a ring always contains the multiplicative identity $1 \neq 0$.

Let R be a finite commutative ring. We denote its group of units by $R^{\times}$ and write $Z(R)$ for the set of its zero divisors. Recall that we have the disjoint union $R = \{0\} \cup R^{\times} \cup Z(R)$. The set $Z(R)$ can be empty if R is a field. Note that if u is a unit of R and z is a zero divisor of R, then uz is a zero divisor of R. Thus, the left multiplication induces an action of the group of units of R on the set of zero divisors of R.

The *zero divisor graph* of R, $\mathcal{Z}_R$, is a graph whose vertex set is the set of all zero divisors of R, and two zero divisors are adjacent if and only if their product is zero. A zero divisor graph was introduced by Beck [1] and was later modified by Anderson and Livingston [2]. Sharma et al. [3] analyzed the adjacency matrices of zero-divisor graphs of $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$, where p is a prime number and $\mathbb{Z}_p[i] = \mathbb{Z}_p[x]/(x^2+1)$ by studying the neighbourhood set of zero divisors. He observed properties of a zero divisor graph and its adjacency matrix of some rings such as $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2[i]$, and $\mathbb{Z}_2[i] \times \mathbb{Z}_2[i]$ before concluding results to $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$. In addition, he showed that, in $\mathbb{Z}_p \times \mathbb{Z}_p$, the maximum degree of this graph is at least $|Z(\mathbb{Z}_p \times \mathbb{Z}_p)|/2$. Later, Young [4] worked on the adjacency matrix of the zero-divisor graph of $\mathbb{Z}_n$. He divided $\mathbb{Z}_n$ into the set $S(d) = \{k \in \mathbb{Z}_n | (k,n) = d\}$ where d is a divisor of n. This shows that the zero divisor graph of $\mathbb{Z}_n$ is a multipartite graph with classes $S(d)$ where d is a proper divisor of n. He showed that

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the determinant of the adjacency matrix is zero. In addition, he obtained the rank for all n and some non-zero eigenvalues for the case $n = p^3$ and $n = p^2q$ where p and q are distinct prime numbers and upper and lower bounds for the largest eigenvalues.

The *Wiener index* of a connected graph G is the sum $\sum_{u,v\in V(G)} d_G(u, v)$ where $d_G(u, v)$ is the distance of u and v in the graph G. It was introduced by Wiener [5]. This index is studied extensively as distance-based topological indices in chemical graph theory. In 2011, Ahmadi and Nezhad [6] calculated the Wiener index of the zero divisor graphs of $\mathbb{Z}_{pq}$ and $\mathbb{Z}_{p^2}$ where p and q are distinct primes and provided a computer code to find the Wiener index of the zero divisor graph of $\mathbb{Z}_n$ where $n \in \mathbb{N}$. Reddy et al. [7] obtained the Wiener index of the zero divisor graphs of $\mathbb{Z}_{p^3}$ and $\mathbb{Z}_{p^2q}$, where p and q are distinct primes. Mohammad and Authman [8] used Hosoya polynomials to determine the Wiener index of zero divisor graphs of $\mathbb{Z}_{p^m}$ and $\mathbb{Z}_{p^mq}$ where p, q are distinct primes and $m \in \mathbb{N}$. Later, in 2019, Shuker et al. [9] also applied Hosoya polynomials and obtained the Wiener index of zero divisor graph of $\mathbb{Z}_{p^mq^2}$ where p, q are distinct primes and $m \in \mathbb{N}$.

In this paper, our main purpose is to study eigenvalues and the Wiener index of zero divisor graphs of finite chain rings. The action of the group of units of the set of zero divisor sets when R is a finite chain ring is studied in the next section. We determine their rank, determinant, and eigenvalues using reduction graphs and basic properties of finite chain rings and the size of the orbits discussed in the next section. Following Young [4], in Section 3, we work on zero divisor graphs of commutative principal ideal rings. It turns out that every principal ideal ring is a finite direct product of finite chain rings. We order the vertices by the lexicographical order and have a nice adjacency matrix of the reduction graph. We are able to determine the rank and the independence number and we use them to find the number of positive eigenvalues and the number of negative eigenvalues, and the eigenvalues and eigenvectors can be obtained from a smaller matrix which completes the study of the eigenvalues and eigenvectors of the zero divisor graphs of finite direct products of finite chain rings. The combinatorial approach is different from Young's and can be used to answer his problems deeper and can be done over any commutative principal ideal rings. We present all cardinalities in terms of the residue fields. By using the set-up of R in Section 3, we can find upper and lower bounds for the largest eigenvalue of the zero divisor graph $\mathcal{Z}_R$ in Section 4. Finally, we use the relation between the zero divisor graph and its reduction graph to compute the distance of any two vertices. This leads us to determine the Wiener index of our zero divisor graphs in Section 5.

2. Zero divisor graphs of finite chain rings

A *local ring* is a commutative ring with unique maximal ideal. A finite commutative ring R is called a *finite chain ring* if for any ideals I and J of R, we have $I \subseteq J$ or $J \subseteq I$. It is clear that a finite chain ring must be a local ring and every finite field and the ring of integers modulo a prime power are finite chain rings. Also, we can show that if R is a finite chain ring with maximal ideal M and $\theta \in M \setminus M^2$, then $M = R\theta$. In other words, the maximal ideal of a finite chain ring is principal. It is also known that a ring is a finite chain ring if and only if it is a finite principal ideal ring. In particular, the unique maximal ideal of a finite chain ring is a principal ideal generated by a nilpotent element.

Now, let R be a finite chain ring with unique principal maximal ideal $M = R\theta$ for some $\theta \in M \setminus M^2$ and $\Bbbk \cong R/M$, its residue field. Then, $R^\times = R \setminus R\theta$ and $Z(R) = R\theta \setminus \{0\}$. We shall repeatedly use basic properties of a finite chain ring taken from Refs. [10,11] and recorded in the next proposition.

Proposition 2.1: (1) *There is the smallest positive integer t such that $\theta^t = 0$, called the nilpotency of R.*

(2) *For any non-zero element r in R, there is a unique integer i, $0 \le i < t$, such that $r = u\theta^i$ for some unit u in R.*

(3) *Assume that $1 \le i < j \le t$ and $r \in R$. If $r\theta^i \in R\theta^j$, then $r \in R\theta^{j-i}$. In particular, if $r\theta^i = 0$, then $r \in R\theta^{t-i}$.*

(4) *If $\{v_1, \dots, v_q\}$ is a system of coset representatives of M in R where $q = |\mathbb{k}|$, then for each r in R, there are unique $r_0, \dots, r_{t-1}$ in $\{v_1, \dots, v_q\}$ such that*

$$r = r_0 + r_1\theta + \cdots + r_{t-1}\theta^{t-1}.$$

(5) *$|R\theta^i| = |\mathbb{k}|^{t-i}$ for all $i \in \{0, 1, \dots, t-1\}$.*

(6) *For each $i \in \{0, 1, \dots, t-1\}$, $|R\theta^i / R\theta^{i+1}| = |\mathbb{k}|$.*

The orbits under action of the unit groups are $R^\times \cdot \theta^i$, $1 \le i \le t$. The size of the stabilizers and the size of the orbits are determined in the following propositions.

Proposition 2.2: $|\mathrm{Stab}_{R^\times}(\theta^i)| = |\mathbb{k}|^i$ *and* $|R^\times \cdot \theta^i| = |\mathbb{k}|^{t-i} - |\mathbb{k}|^{t-i-1} = |\mathbb{k}|^{t-i-1}(|\mathbb{k}| - 1)$ *for all $i \in \{1, 2, \dots, t-1\}$.*

Proof: Let $i \in \{1, 2, \dots, t-1\}$. Note that for $a \in R$, we have $a \in \mathrm{Stab}_{R^\times}(\theta^i) \Leftrightarrow (a-1)\theta^i = 0$. It follows from Proposition 2.1 (3) that $\mathrm{Stab}_{R^\times}(\theta^i) = \{1 + d\theta^{t-i} : d \in R\}$. Since

$$1 + d_1\theta^{t-i} = 1 + d_2\theta^{t-i} \Leftrightarrow d_1 - d_2 \in R\theta^i,$$

the size of $\mathrm{Stab}_{R^\times}(\theta^i)$ is $|R/R\theta| = |\mathbb{k}|^i$. The orbit-stabilizer theorem implies that the size of the orbit

$$|R^\times \cdot \theta^i| = \frac{|R^\times|}{\mathrm{Stab}_{R^\times}(\theta^i)} = \frac{|\mathbb{k}|^t - |\mathbb{k}|^{t-1}}{|\mathbb{k}|^i} = |\mathbb{k}|^{t-i} - |\mathbb{k}|^{t-i-1}.$$

This completes the proof. ∎

To study the zero divisor graph of R, we may assume that R is not a field. So we have $t \ge 2$. Furthermore, our definition allows the zero divisor graph to have loops. Note that if a and b are zero divisors in the same orbit $R^\times \cdot \theta^i$ for some $1 \le i < t$, then $a = u\theta^i$ and $b = v\theta^i$ for some units u and v, for any zero divisor z of R, we have

$$az = 0 \Leftrightarrow u\theta^i z = 0 \Leftrightarrow \theta^i z = 0 \Leftrightarrow v\theta^i z = 0 \Leftrightarrow bz = 0.$$

Next, assume that a is in the orbit $R^\times \cdot \theta^i$ and b is in the orbit $R^\times \cdot \theta^j$ for some $1 \le i, j < t$. Then, $a = u\theta^i$ and $b = v\theta^j$ for some units u and v in R. If $ab = 0$, then $i + j$ must be at least t, so $aw\theta^j = uw\theta^{i+j} = 0$ for any unit w in R. Hence, we have the following lemma.

Lemma 2.3: *Let a and b be zero divisors of R.*

(1) *If a and b are in the same orbit of the action of units by left multiplication, then a and b have the same neighbours in $\mathcal{Z}_R$.*
(2) *If a is adjacent to b in the zero divisor graph, then a is adjacent to all zero divisors in the same orbit of b.*

For each $1 \le i < t$, let H_i be the subgraph of $\mathcal{Z}_R$ induced by $R^\times \cdot \theta^i$. Then there are $t-1$ such subgraphs. It is easy to see that these subgraphs are either complete or empty (having no edges) and H_i is complete if and only if $2i \ge t$. Moreover, if $1 \le i < j < t$ such that $i + j \ge t$ and H_i and H_j are empty, then the subgraph induced by $R^\times \cdot \theta^i \cup R^\times \cdot \theta^j$ is a complete bipartite graph by Lemma 2.3. We record this observation in the next theorem.

Theorem 2.4: (1) *There are $t - \lceil t/2 \rceil$ induced subgraphs which are complete.*
(2) *There are $\lceil t/2 \rceil - 1$ induced subgraphs which have no edges.*
(3) *If i and j are two integers such that $1 \le i < j < t$ and $i + j \ge t$ and H_i and H_j have no edges, then the subgraph induced by $R^\times \cdot \theta^i \cup R^\times \cdot \theta^j$ is a complete bipartite graph.*

The determinant, rank, nullity, and eigenvalues of the adjacency matrix of a graph are called the *determinant, rank, nullity, and eigenvalues* of a graph. First, we find the determinant of the zero divisor graph of R. Note that if there is an orbit containing more than one element, then each element in the same orbit has the same neighbourhood by Lemma 2.3, so the rows corresponding to them are identical and force that its determinant becomes zero. Next, we consider the case that every orbit contains exactly one element. Since $|R^\times \cdot \theta| = |\mathbb{k}|^{t-2}(|\mathbb{k}| - 1)$, we have $t = 2$ and $|\mathbb{k}| = 2$. Then, $|R| = |\mathbb{k}|^2 = 4$. Hence, R is a finite chain ring of order 4 with maximal ideal of size 2, so $Z(R) = \{a\}$ is a singleton and $a^2 = 0$. Therefore, the determinant is 1. Finally, we remark from [5] that a finite chain ring R of order 4 with maximal ideal of size 2 is $\mathbb{Z}_2[x]/(x^2)$ of characteristic two or $\mathbb{Z}_4$ of characteristic four. We conclude the result of the zero divisor graph of a finite chain ring in the next proposition.

Proposition 2.5: *The determinant of the zero divisor graph of a finite chain ring of R is* 0 *unless R is isomorphic to $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_4$ where the determinant equals* 1.

Assume that R is a finite chain ring in which the determinant of the zero divisor graph $\mathcal{Z}_R$ is 0. It follows that 0 is an eigenvalue of $\mathcal{Z}_R$ with multiplicity being the nullity of $\mathcal{Z}_R$. From the rank theorem, we also know that the sum of the nullity of $\mathcal{Z}_R$ and the rank of $\mathcal{Z}_R$ is the number of zero divisors of R which equals $|R\theta| - 1 = |\mathbb{k}|^{t-1} - 1$. Hence, to determine the multiplicity of the eigenvalue 0, we may compute the rank of $\mathcal{Z}_R$. We eliminate the redundant of the repeated rows by considering the reduction graph $\pi\mathcal{Z}_R$ whose vertices are the orbits: $R^\times \cdot \theta, R^\times \cdot \theta^2, \ldots, R^\times \cdot \theta^{t-1}$ and the vertices $R^\times \cdot \theta^i$ and $R^\times \cdot \theta^j$ are adjacent if and only if $i + j \ge t$. This reduction graph is also called the compressed zero divisor graphs studied in Ref. [12]. Write $A(\mathcal{Z}_R)$ and $A(\pi\mathcal{Z}_R)$ for the adjacency matrix of $\mathcal{Z}_R$ and $\pi\mathcal{Z}_R$, respectively. Since for each element in the orbit $R^\times \cdot \theta^i$, its row in $A(\mathcal{Z}_R)$ is identical, we have $\operatorname{rank} A(\mathcal{Z}_R) \le t - 1$. Also, $\operatorname{rank} A(\pi\mathcal{Z}_R) \ge \operatorname{rank} A(\mathcal{Z}_R)$ because $A(\pi\mathcal{Z}_R)$ is obtained by deleting repeated rows in $A(\mathcal{Z}_R)$. We proceed to show that

Proposition 2.6: rank $(A(\mathcal{Z}_R)) = t - 1$.

Proof: From the above inequalities, it suffices to show that rank $(A(\pi\,\mathcal{Z}_R)) = t - 1$. Since $\pi\,\mathcal{Z}_R$ has $t{-}1$ vertices and

$$A(\pi\,\mathcal{Z}_R) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & \dots & 1 & 1 & 1 \\ \vdots & & \ddots & \vdots & & \\ 1 & 1 & \dots & 1 & 1 & 1 \end{bmatrix}$$

directly from its definition, rank $A(\pi\,\mathcal{Z}_R) = t - 1$. ■

Observe that if R is isomorphic to $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_4$, then the rank of $A(\mathcal{Z}_R)$ is 1 which also equals $t{-}1$. Hence, we have shown

Theorem 2.7: *For any finite chain ring R with nilpotency t, the rank of the graph $\mathcal{Z}_R$ is $t{-}1$ and the multiplicity of the eigenvalue 0 is $|\mathbb{k}|^t - t$.*

For $i \in \{1, 2, \dots, t-1\}$, let $m_i = |R^\times \cdot \theta^i| = |\mathbb{k}|^{t-i-1}(|\mathbb{k}| - 1)$. Then,

$$A(\mathcal{Z}_R) = \begin{array}{c} \begin{bmatrix} & & & J_1 \\ & & J_2 & \\ & J_3 & & \\ & \vdots & & \\ & J_{t-1} & & \end{bmatrix} \end{array} \begin{array}{l} \}\, m_1 \\ \}\, m_2 \\ \}\, m_3 \\ \\ \}\, m_{t-1} \end{array}$$

where J_i is the all-one matrix of dimension $m_i \times (m_{t-i} + \cdots + m_{t-2} + m_{t-1})$ for all $i \in \{1, 2, \dots, t-1\}$. Thus, the eigenvectors of $\mathcal{Z}_R$ corresponding to the eigenvalue 0 are the ones coming from the nullspace of the echelon matrix

$$\begin{bmatrix} \vec{J}_1 & & & & \\ & \vec{J}_2 & & & \\ & & \vec{J}_3 & & \\ & & & \ddots & \\ & & & & \vec{J}_{t-1} \end{bmatrix},$$

where $\vec{J}_i$ is the all-one row vector of size m_i for all $i \in \{1, 2, \dots, t-1\}$.

Assume that λ is a nonzero eigenvalue of $A(\mathcal{Z}_R)$ with an eigenvector $\vec{V}$. Then, $\vec{V}$ can be divided into a block vector

$$\vec{V} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_{t-2} \\ \vec{v}_{t-1} \end{bmatrix}, \quad \text{where } \vec{v}_i = \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{i,m_i} \end{bmatrix} \text{ for all } i \in \{1, 2, \dots, t-1\}$$

such that

$$J_1\vec{v}_{t-1} = \lambda\vec{v}_1, \quad J_2\begin{bmatrix}\vec{v}_{t-2}\\ \vec{v}_{t-1}\end{bmatrix} = \lambda\vec{v}_2, \ldots, J_{t-1}\vec{V} = \lambda\vec{v}_{t-1}.$$

Since $\lambda \neq 0$, we have $v_{i1} = v_{i2} = \ldots = v_{im_i}$ for all $i \in \{1, 2, \ldots, t-1\}$. It follows that

$$\begin{aligned} m_{t-1}v_{t-1,1} &= \lambda v_{1,1}, \\ m_{t-2}v_{t-2,1} + m_{t-1}v_{t-1,1} &= \lambda v_{2,1}, \\ &\vdots \\ m_1v_{1,1} + \cdots + m_{t-2}v_{t-2,1} + m_{t-1}v_{t-1,1} &= \lambda v_{t-1,1}, \end{aligned}$$

and so λ is an eigenvalue of

$$\overline{A} = \begin{bmatrix} 0 & \cdots & 0 & m_{t-1} \\ 0 & \cdots & m_{t-2} & m_{t-1} \\ \vdots & & \vdots & \vdots \\ m_1 & \cdots & m_{t-2} & m_{t-1} \end{bmatrix}$$

with an eigenvector

$$\begin{bmatrix} \vec{v}_{1,1} \\ \vec{v}_{2,1} \\ \vdots \\ \vec{v}_{t-1,1} \end{bmatrix}.$$

Moreover, the remaining $t-1$ independent eigenvectors of $\mathcal{Z}_R$ corresponding to nonzero eigenvalues can be obtained from the ones of $\overline{A}$. This completes the study of the eigenvalues and eigenvectors of the zero divisor graph $\mathcal{Z}_R$ where R is a finite chain ring.

3. Zero divisor graphs of principal ideal rings

Let R be a finite commutative principal ideal ring. Then every ideal of R is principal. Recall that a finite commutative ring is a direct product of finite local rings. Since every ideal of R is principal, so are its factors. Therefore, R is a direct product of finite chain rings.

Write $R \cong R_1 \times R_2 \times \cdots \times R_k$ where R_i is a finite chain ring with maximal ideal $R_i\theta_i$ of nilpotency t_i and residue field $\Bbbk_i = R_i/R_i\theta_i$ for all $i \in \{1, 2, \ldots, k\}$. Note that the set of zero divisors of R is the union of the direct product of orbits of the form

$$R_1^\times \cdot \theta_1^{s_1} \times R_2^\times \cdot \theta_2^{s_2} \times \cdots \times R_k^\times \cdot \theta_k^{s_k},$$

where $0 \leq s_i \leq t_i$ for all $i \in \{1, 2, \ldots, k\}$ except $R_1^\times \times R_2^\times \times \cdots \times R_k^\times$ and $\{(0, 0, \ldots, 0)\}$. Now, we consider the reduction graph $\pi\mathcal{Z}_R$ of $\mathcal{Z}_R$ whose vertices are

$$z(s_1, s_2, \ldots, s_k) = R_1^\times \cdot \theta_1^{s_1} \times R_2^\times \cdot \theta_2^{s_2} \times \cdots \times R_k^\times \cdot \theta_k^{s_k},$$

where $0 \leq s_i \leq t_i$ for all $i \in \{1, 2, \ldots, k\}$ except $s_1 = s_2 = \cdots = s_k = 0$ or $(s_1 = t_1, s_2 = t_2, \ldots, s_k = t_k)$ and $z(s_1, s_2, \ldots, s_k)$ and $z(s_1', s_2', \ldots, s_k')$ are adjacent if and only if $s_i + s_i' \geq t_i$ for all $i \in \{1, 2, \ldots, k\}$. Then, the graph $\pi\mathcal{Z}_R$ has $\prod_{i=1}^k (t_i + 1) - 2$ vertices.

Remark 3.1: For $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $p_1, p_2, \dots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{N}$, $z(s_1, \dots, s_k)$ can be considered as the set $S(d)$ in Ref. [4] where $d = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$ is a divisor of n.

We order them by the lexicographical order, namely, $z(s_1, s_2, \dots, s_k) < z(s_1', s_2', \dots, s_k')$, if and only if

$$(s_1 < s_1') \text{ or } (s_1 = s_1' \text{ and } s_2 < s_2') \text{ or } \cdots \text{ or } (s_1 = s_1', \dots, s_{k-1} = s_{k-1}' \text{ and } s_k < s_k').$$

Thus, the first vertex is $z(0, 0, \dots, 0, 1)$ and the last one is $z(t_1, t_2, \dots, t_{k-1}, t_k - 1)$. Under this order of vertices, we have the adjacency matrix being in the form

$$A(\pi \mathcal{Z}_R) = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & * & * \\ 1 & * & \dots & * & * \end{bmatrix}. \tag{1}$$

To see this, we determine the position of $z(s_1, s_2, \dots, s_k)$ in $A(\pi \mathcal{Z}_R)$ by counting the number of vertices before it. From the definition of $<$ excluding $(0, 0, \dots, 0)$, this number equals

$$\sum_{i=1}^{k} s_i \prod_{j=i+1}^{k} (t_j + 1) - 1,$$

so the position of $z(s_1, s_2, \dots, s_k)$ in $A(\pi \mathcal{Z}_R)$ is $\sum_{i=1}^{k} s_i \prod_{j=i+1}^{k} (t_j + 1)$.

Now, let $r_1, r_2, \dots, r_k$ be such that $0 \le r_i \le t_i$ and $r_i + s_i \ge t_i$ for all $i \in \{1, 2, \dots, k\}$. In other words, the vertices $z(r_1, r_2, \dots, r_k)$ and $z(s_1, s_2, \dots, s_k)$ are adjacent. Then

$$\sum_{i=1}^{k} (r_i + s_i) \prod_{j=i+1}^{k} (t_j + 1) \ge \sum_{i=1}^{k} t_i \prod_{j=i+1}^{k} (t_j + 1).$$

The sum on the right-hand side can be simplified to

$$\begin{aligned} \sum_{i=1}^{k} t_i \prod_{j=i+1}^{k} (t_j + 1) &= \sum_{i=1}^{k-2} t_i \prod_{j=i+1}^{k} (t_j + 1) + t_{k-1}(t_k + 1) + t_k + 1 - 1 \\ &= \sum_{i=1}^{k-2} t_i \prod_{j=i+1}^{k} (t_j + 1) + (t_{k-1} + 1)(t_k + 1) + t_k + 1 \\ &\vdots \\ &= (t_1 + 1) \prod_{j=2}^{k} (t_j + 1) - 1 = \prod_{j=1}^{k} (t_j + 1) - 1 = |Z(R)| + 1. \end{aligned}$$

Thus,

$$\sum_{i=1}^{k} r_i \prod_{j=i+1}^{k} (t_j+1) + \sum_{i=1}^{k} s_i \prod_{j=i+1}^{k} (t_j+1) \geq |Z(R)| + 1$$

and equality holds if and only if $r_i + s_i = t_i$ for all $\{1, 2, \ldots, k\}$. This proves (1) and it follows from (1) that

$$\operatorname{rank} A(\pi \mathcal{Z}_R) = \prod_{i=1}^{k} (t_i+1) - 2.$$

Since $\operatorname{rank} A(\mathcal{Z}_R) = \operatorname{rank} A(\pi \mathcal{Z}_R)$, we have shown

Proposition 3.1: $\operatorname{rank} A(\mathcal{Z}_R) = \prod_{i=1}^{k}(t_i+1) - 2$.

Remark 3.2: The entries of $A(\pi \mathcal{Z}_R)$ below the diagonal from bottom-left corner to top-right corner may not always be 1 when R is not local. For example, if $R = \mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$, then

$$A(\pi \mathcal{Z}_R) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Next, we compute the determinant of $A(\mathcal{Z}_R)$. From the reduction graph $\pi \mathcal{Z}_R$, if some vertex $z(s_1, s_2, \ldots, s_k)$ contains more than one element, then $A(\mathcal{Z}_R)$ has some repeated rows, so $\det A(\mathcal{Z}_R) = 0$. Now, we consider the case that every vertex of $\pi \mathcal{Z}_R$ is a singleton. It follows that $|R_i^{\times} \cdot \theta_i^{s_i}| = 1$ for all $0 \leq s_i \leq t_i$ and $i \in \{1, 2, \ldots, k\}$. Since R_i is a local ring, R_i is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ for $i \in \{1, 2, \ldots, k\}$. If $k = 1$, then R_i must be $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ presented in Proposition 2.5. Assume that $k \geq 2$. If for some i, $R_i \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, then $|R_i^{\times}| = 2$ and so $|z(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_k)| > 1$. Hence, $R_i \cong \mathbb{Z}_2$ for all $i \in \{1, 2, \ldots, k\}$, so $|A(\mathcal{Z}_R)| = 2^k - 2$ and

$$\det(A(\mathcal{Z}_R)) = (-1)^{2^k-1}(-1)^{2^k-2}(-1)^{2^k-3} \cdots (-1)^3(-1)^2 = -1$$

because $k \geq 2$. We record the determinant of $A(\mathcal{Z}_R)$ in

Proposition 3.2: $\det(A(\mathcal{Z}_R)) = \begin{cases} 1 & \textit{if } R \cong \mathbb{Z}_4 \textit{ or } \mathbb{Z}_2[x]/(x^2), \\ -1 & \textit{if } R \cong (\mathbb{Z}_2)^k \textit{ for some } k \geq 2, \\ 0 & \textit{otherwise.} \end{cases}$

If the determinant of $\mathcal{Z}_R$ is 0, then $\mathcal{Z}_R$ has 0 as an eigenvalue with multiplicity being the nullity of $A(\mathcal{Z}_R)$ because $A(\mathcal{Z}_R)$ is diagonalizable. Thus, the rank theorem gives that the nullity is $|Z(R)| - \operatorname{rank} A(\mathcal{Z}_R)$. Since $|Z(R)| = |R| - |R^{\times}| - 1$, using this fact and proposition 3.1 gives the next proposition.

Proposition 3.3: *If 0 is an eigenvalue of the graph $\mathcal{Z}_R$, then its multiplicity is given by*

$$\prod_{i=1}^{k} |\mathbb{k}_i|^{t_i} - \prod_{i=1}^{k} (|\mathbb{k}_i|^{t_i} - |\mathbb{k}_i|^{t_i-1}) - \prod_{i=1}^{k} (t_i+1) + 1.$$

Recall that we order the vertices of the reduction graph $\pi\mathcal{Z}_R$ by the lexicographical order. With this order, we may write the vertex set as $\{z_1, z_2, \ldots, z_N\}$ where $N = \prod_{i=1}^{k}(t_i + 1) - 2$ and we denote by n_j the number of elements in z_j for all $j \in \{1, 2, \ldots, N\}$. The $(0, 1)$-adjacency matrix $A = [a_{ij}]$ of $\pi\mathcal{Z}_R$ of size N in (1) lifts to the adjacency matrix $A(\mathcal{Z}_R) = [A_{ij}]$ of $\mathcal{Z}_R$ where A_{ij} is a block matrix of dimension $m_i \times m_j$ with all-zero or all-one entries depending on the entry a_{ij} of $A(\pi\mathcal{Z}_R)$ is 0 or 1, respectively. Thus, $A(\mathcal{Z}_R)$ is a matrix of the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & J_1 \\ \mathbf{0} & \mathbf{0} & \ldots & J_2 & * \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & J_{N-1} & \ldots & * & * \\ J_N & * & \ldots & * & * \end{bmatrix},$$

where J_j is the all-one matrix of dimension $n_j \times n_j$ for all $j \in \{1, 2, \ldots, N\}$. Hence, the eigenvectors of $\mathcal{Z}_R$ corresponding to the eigenvalue 0 are the ones coming from the nullspace of the echelon matrix

$$\begin{bmatrix} \vec{J}_1 & & & & \\ & \vec{J}_2 & & & \\ & & \ddots & & \\ & & & \vec{J}_{N-1} & \\ & & & & \vec{J}_N \end{bmatrix},$$

where $\vec{J}_j$ is the all-one row vector of size n_j for all $j \in \{1, 2, \ldots, N\}$.

The *independence number* of a graph Γ is the size of the largest set of pairwise nonadjacent vertices. We denote the independence number of Γ by $\alpha(\Gamma)$. Brouwer and Haemers [13] showed that for a graph Γ,

$$\alpha(\Gamma) \le r(\Gamma) - r_+(\Gamma) \quad \text{and} \quad \alpha(\Gamma) \le r(\Gamma) - r_-(\Gamma),$$

where $r(\Gamma)$, $r_+(\Gamma)$, and $r_-(\Gamma)$ are the number of eigenvalues, number of positive eigenvalues, and number of negative eigenvalues of Γ, respectively.

Recall that $N = \operatorname{rank} A(\mathcal{Z}_R) = \operatorname{rank} A(\pi\mathcal{Z}_R)$. It follows from the adjacency in Equation (1) that $\alpha(\pi\mathcal{Z}_R) = \lfloor N/2 \rfloor$ and the reduction graph $\pi\mathcal{Z}_R$ has a nonzero determinant, so its eigenvalues are positive or negative. Then, N is the number of nonzero eigenvalues of $\pi\mathcal{Z}_R$. We can calculate $r_+(\pi\mathcal{Z}_R)$ and $r_-(\pi\mathcal{Z}_R)$ as follows. Since

$$\left\lfloor \frac{N}{2} \right\rfloor \le r(\Gamma) - r_+(\Gamma) \quad \text{and} \quad \left\lfloor \frac{N}{2} \right\rfloor \le r(\Gamma) - r_-(\Gamma),$$

we have

$$r_+(\pi\mathcal{Z}_R) \le N - \left\lfloor \frac{N}{2} \right\rfloor, \quad r_-(\pi\mathcal{Z}_R) \le N - \left\lfloor \frac{N}{2} \right\rfloor, \quad \text{and} \quad N = r_+(\pi\mathcal{Z}_R) + r_-(\pi\mathcal{Z}_R).$$

If N is even, they force that $r_+(\pi\mathcal{Z}_R) = r_-(\pi\mathcal{Z}_R) = N/2$. Assume that N is odd. Then $r_+(\pi\mathcal{Z}_R)$ and $r_-(\pi\mathcal{Z}_R)$ are less than or equal to $(N+1)/2$. Since their sum is N, we get $\{r_+(\pi\mathcal{Z}_R), r_-(\pi\mathcal{Z}_R)\} = \{(N+1)/2, (N-1)/2\}$. But the determinant of $\pi\mathcal{Z}_R$ is

$(-1)^{(N-1)/2}$ and the minus sign depends on $r_-(\pi \mathcal{Z}_R)$, so we must have $r_+(\pi \mathcal{Z}_R) = (N+1)/2$ and $r_-(\pi \mathcal{Z}_R) = (N-1)/2$. Proposition 1 of Ref. [14] implies that $r_+(\pi \mathcal{Z}_R) = r_+(\mathcal{Z}_R)$ and $r_-(\pi \mathcal{Z}_R) = r_-(\mathcal{Z}_R)$. Since $N = \operatorname{rank} \pi \mathcal{Z}_R = \operatorname{rank} \mathcal{Z}_R$ is also the number of nonzero eigenvalues of $\mathcal{Z}_R$, we obtain the number of positive and negative eigenvalues of $\mathcal{Z}_R$ as follows.

Theorem 3.4: *$r_+(\mathcal{Z}_R) = \lceil N/2 \rceil$ and $r_-(\mathcal{Z}_R) = \lfloor N/2 \rfloor$.*

Now, assume that λ is a nonzero eigenvalue of $A(\mathcal{Z}_R)$ with an eigenvector $\vec{V}$. Then, $\vec{W}$ can be divided into a block vector

$$\vec{W} = \begin{bmatrix} \vec{w}_N \\ \vec{w}_{N-1} \\ \vdots \\ \vec{w}_2 \\ \vec{w}_1 \end{bmatrix}, \quad \text{where } \vec{w}_i = \begin{bmatrix} w_{i1} \\ w_{i2} \\ \vdots \\ w_{i,m_i} \end{bmatrix} \text{ for all } i \in \{1, 2, \ldots, N\}.$$

Note that $J_1 \vec{w}_1 = \lambda \vec{w}_1$ implies $w_{11} = w_{12} = \cdots = w_{1n_1}$ because of $\lambda \neq 0$. Since $*$ in $A(\mathcal{Z}_R)$ is all-zero or all-one block, we may inductively deduce that $w_{i1} = w_{i2} = \ldots = w_{in_i}$ for all $i \in \{1, 2, \ldots, N\}$. It follows that λ is an eigenvalue of

$$B = \begin{bmatrix} 0 & \cdots & 0 & n_1 \\ 0 & \cdots & n_2 & b_{2N} \\ \vdots & & \vdots & \vdots \\ n_N & \cdots & b_{N,N-1} & b_{NN} \end{bmatrix},$$

where for $i < j$, $b_{ij} = 0$ if $a_{ij} = 0$ and $b_{ij} = n_j$ if $a_{ij} = 1$, with an eigenvector

$$\begin{bmatrix} w_{N,1} \\ w_{N-1,1} \\ \vdots \\ w_{1,1} \end{bmatrix}.$$

Hence, the remaining N independent eigenvectors of $\mathcal{Z}_R$ corresponding to nonzero N eigenvalues can be obtained from the ones of B.

4. Bounds for eigenvalues

Let R be a finite commutative principal ideal ring. Write $R \cong R_1 \times R_2 \times \cdots \times R_k$ where R_i is a finite chain ring with maximal ideal $R_i\theta_i$ of nilpotency t_i and residue field $\Bbbk_i = R_i/R_i\theta_i$ for all $i \in \{1, 2, \ldots, k\}$. We proceed to find upper and lower bounds for the zero divisor graph of R in this section. Recall that the set of zero divisors of R is the union of the direct product of orbits of the form

$$z(s_1, s_2, \ldots, s_k) = R_1^\times \cdot \theta_1^{s_1} \times R_2^\times \cdot \theta_2^{s_2} \times \cdots \times R_k^\times \cdot \theta_k^{s_k},$$

where $0 \leq s_i \leq t_i$ for all $i \in \{1, 2, \ldots, k\}$ except $R_1^\times \times R_2^\times \times \cdots \times R_k^\times$ and $\{(0, 0, \ldots, 0)\}$. Consider the vertex $(u_1\theta_1^{s_1}, u_2\theta_2^{s_2}, \ldots, u_k\theta_k^{s_k})$. It is adjacent to vertices $(v_1\theta_1^{r_1}, v_2\theta_2^{r_2}, \ldots,$

$v_k\theta_k^{r_k}$) where $v_i \in R_i^\times$ and $r_i + s_i \geq t_i$ for all $i \in \{1, 2, \ldots, k\}$ except $(0, 0, \ldots, 0)$, so the degree of the vertex $(u_1\theta_1^{s_1}, u_2\theta_2^{s_2}, \ldots, u_k\theta_k^{s_k})$ is

$$\left(\prod_{i=1}^{k} \sum_{r_i+s_i\geq t_i} |R_i^\times \cdot \theta_i^{r_i}|\right) - 1.$$

Suppose that we order the eigenvalues of $\mathcal{Z}_R$ as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. It follows from Proposition 3.1.2 of Brouwer and Haemers [13] that

$$\overline{\deg \mathcal{Z}_R} \leq \lambda_1 \leq \max \deg(\mathcal{Z}_{\mathcal{R}}),$$

where $\overline{\deg \mathcal{Z}_R}$ is the average of degree of vertices of $\mathcal{Z}_R$ given by

$$\frac{\sum_{v\in Z(R)} \deg v}{|Z(R)|} = \frac{\sum_{v\in Z(R)} \deg v}{|R| - |R^\times| - 1},$$

since R is a finite commutative ring. Next, we determine the maximum degree and the average of degree of vertices of $\mathcal{Z}_R$. We shall assume further that $|\mathbb{k}_1| \leq |\mathbb{k}_2| \leq \cdots \leq |\mathbb{k}_k|$. Note that for each $i \in \{1, 2, \ldots, k\}$, we have by Proposition 2.2 that

$$\sum_{r_i+s_i\geq t_i} |R_i^\times \cdot \theta_i^{r_i}| = 1 + \sum_{t_i-s_i\leq r_i\leq t_i-1} |R_i^\times \cdot \theta_i^{r_i}| = 1 + \sum_{t_i-s_i\leq r_i\leq t_i-1} |\mathbb{k}_i|^{t_i-r_i-1}(|\mathbb{k}_i| - 1),$$

so the geometric sum simplifies the right-hand side to

$$1 + (|\mathbb{k}_i| - 1) \sum_{1\leq r_i'\leq s_i} |\mathbb{k}_i|^{r_i'-1} = 1 + (|\mathbb{k}_i|^{s_i} - 1) = |\mathbb{k}_i|^{s_i}.$$

Therefore, the degree of the vertex $(u_1\theta_1^{s_1}, u_2\theta_2^{s_2}, \ldots, u_k\theta_k^{s_k})$ is $|\mathbb{k}_1|^{s_1}|\mathbb{k}_2|^{s_2}\ldots|\mathbb{k}_k|^{s_k} - 1$ and the maximum degree attains when $s_1 = t_1 - 1$ and $s_i = t_i$ for all $i \geq 2$ and equals

$$\max \deg(\mathcal{Z}_R) = |\mathbb{k}_1|^{t_1-1}|\mathbb{k}_2|^{t_2}\ldots|\mathbb{k}_k|^{t_k} - 1.$$

From the set-up at the beginning of Section 2 and the above calculation of the degree of a vertex, we obtain the average of degree of vertices of the zero divisor graph $\mathcal{Z}_R$ as

$$\frac{\displaystyle\sum_{\substack{0\leq s_1\leq t_1,\\ 0\leq s_2\leq t_2,\\ \vdots\\ 0\leq s_k\leq t_k}} (|\mathbb{k}_1|^{s_1}|\mathbb{k}_2|^{s_2}\ldots|\mathbb{k}_k|^{s_k} - 1)|z(s_1, s_2, \ldots, s_k)| - (|\mathbb{k}_1|^{t_1}|\mathbb{k}_2|^{t_2}\ldots|\mathbb{k}_k|^{t_k} - 1)}{\displaystyle\prod_{i=1}^{k} |\mathbb{k}_i|^{t_i} - \prod_{i=1}^{k}(|\mathbb{k}_i|^{t_i} - |\mathbb{k}_i|^{t_i-1}) - 1},$$

where

$$|z(s_1, s_2, \ldots, s_k)| = \prod_{i=1}^{k} |R_i^\times \cdot \theta_i^{s_i}| = \prod_{j,\, s_j\leq t_j-1} |\mathbb{k}_j|^{t_j-s_j}\left(1 - \frac{1}{|\mathbb{k}_j|}\right)$$

for all $0 \leq s_i \leq t_i$ and $i \in \{1, 2, \ldots, k\}$. Hence, we have an upper bound and a lower bound for the largest eigenvalue of $\mathcal{Z}_R$.

5. Wiener index

Throughout this section, R is a finite commutative principal ideal ring. Write $R \cong R_1 \times R_2 \times \cdots \times R_k$ where R_i is a finite chain ring with maximal ideal $R_i\theta_i$ of nilpotency t_i and residue field $\mathbb{k}_i = R_i/R_i\theta_i$ for all $i \in \{1, 2, \ldots, k\}$. We will compute the Wiener index of $\mathcal{Z}_R$. First, we characterize all finite commutative principal ideal rings such that their zero divisor graphs are complete. It is clear that if R is a finite chain ring with nilpotency 2 or $R = F_1 \times F_2$, where F_1 and F_2 are finite fields, then $\mathcal{Z}_R$ is a complete graph. Next, assume $k \geq 3$. Thus, elements in $R_1^\times \times \{0\} \times R_3^\times \times \cdots \times R_k^\times$ are not adjacent to elements in $R_1^\times \times R_2^\times \times \{0\} \times \cdots \times R_k^\times$. Now, assume that $R = R_1 \times R_2$. Suppose $t_1 \geq 2$ or $t_2 \geq 2$, say $t_1 \geq 2$. It follows that elements in $R_1^\times \times \{0\}$ are not adjacent to elements in $R_1^\times \cdot \theta_1 \times R_2^\times$. Hence, we can conclude that R_1 and R_2 must be fields. Finally, we assume that R is a finite chain ring such that $\mathcal{Z}_R$ is a complete graph. If R has nilpotency $t \geq 3$, then the elements in $R^\times\theta$ are not adjacent, so R must have nilpotency 2. We record this result in the following theorem.

Theorem 5.1: *Let R be a finite principal ideal ring. Then $\mathcal{Z}_R$ is a complete graph if and only if R is a finite chain ring with nilpotency 2 or $R = F_1 \times F_2$ where F_1 and F_2 are finite fields. In this case, its Wiener index is given by*

$$\binom{|Z(R)|}{2}.$$

Theorem 5.2: *Let $R = R_1 \times \cdots \times R_k$ where $R_1, \ldots, R_k$ are finite chain rings. Assume that $\mathcal{Z}_R$ is not a complete graph. For a proper subset X of $\{1, 2, \ldots, k\}$, we define*

$$z(X) = \{z(s_1, \ldots, s_k) \in V(\pi\mathcal{Z}_R) : 0 < s_i \leq t_i \text{ for all } i \in X \text{ and } s_i = 0 \text{ for all } i \notin X\}.$$

Under the set-up at the beginning of this section, we have the following statements.

(1) *If $k = 1$, that is, R is a finite chain ring with nilpotency t, then the Wiener index of $\mathcal{Z}_R$ is given by*

$$\sum_{\substack{0<s,s'<t\\ s+s'\geq t}} |R^\times \cdot \theta^s||R^\times \cdot \theta^{s'}| + 2\sum_{\substack{0<s,s'<t\\ s+s'<t}} |R^\times \cdot \theta^s||R^\times \cdot \theta^{s'}|.$$

(2) *If $k \geq 2$, then the Wiener index of $\mathcal{Z}_R$ is given by*

$$\sum_{z(s_1,\ldots,s_k)\sim z(s_1',\ldots,s_k')} |z(s_1, \ldots, s_k)||z(s_1', \ldots, s_k')| + 2{\sum}' |z(s_1, \ldots, s_k)||z(s_1', \ldots, s_k')| + 3{\sum}'' |z(s_1, \ldots, s_k)||z(s_1', \ldots, s_k')|,$$

where ${\sum}'$ is the sum over $z(s_1, \ldots, s_k) \in z(X)$ and $z(s_1', \ldots, s_k') \in z(Y)$ which are non-adjacent in $\pi\mathcal{Z}_R$ and $X \cap Y \neq \emptyset$ and ${\sum}''$ is the sum over $z(s_1, \ldots, s_k) \in z(X)$ and $z(s_1', \ldots, s_k') \in z(Y)$ which are non-adjacent in $\pi\mathcal{Z}_R$ and $X \cap Y = \emptyset$.

Proof: Recall that the set $V(\pi\mathcal{Z}_R)$ is a partition of the set of zero divisors of R. Then for any $u \in Z(R)$, there exists a unique $z_u \in V(\pi\mathcal{Z}_R)$ containing u. It follows that $d_{\mathcal{Z}_R}(u,v) = d_{\pi\mathcal{Z}_R}(z_u, z_v)$. We use this observation to calculate the Wiener index of $\mathcal{Z}_R$.

First, we handle the case R being a finite chain ring with nilpotency t. Let s, s' be such that $0 < s, s' < t$ and $s + s' < t$. Let $k = \max\{t - s, t - s'\}$. We have $0 < k < t$, $k + s \geq t$, and $k + s' \geq t$. Then $R^\times \cdot \theta^k$ is adjacent to both $R^\times \cdot \theta^s$ and $R^\times \cdot \theta^{s'}$ in $\pi\mathcal{Z}_R$, so $d_{\pi\mathcal{Z}_R}(R^\times \cdot \theta^s, R^\times \cdot \theta^{s'}) = 2$ whenever $s + s' < t$. Hence, its Wiener index is given by

$$\sum_{\substack{0<s,s'<t\\ s+s'\geq t}} |R^\times \cdot \theta^s||R^\times \cdot \theta^{s'}| + 2 \sum_{\substack{0<s,s'<t\\ s+s'<t}} |R^\times \cdot \theta^s||R^\times \cdot \theta^{s'}|.$$

Second, we assume that $k \geq 2$ and let X, Y be proper subsets of $\{1, 2, \ldots, k\}$. Let $z(s_1, \ldots, s_k) \in z(X)$ and $z(s'_1, \ldots, s'_k) \in z(Y)$ be nonadjacent vertices in $\pi\mathcal{Z}_R$. Suppose that $X \cap Y \neq \emptyset$. There are two cases to consider.

Case 1. There exists $i \in X \cap Y$ such that $s_i, s'_i < t_i$. Then $d_{\pi\mathcal{Z}_R}(z(s_1, \ldots, s_k), z(s'_1, \ldots, s'_k)) = 2$ by the same method as in the case where R was a finite chain ring above.

Case 2. $s_i = s'_i = t_i$ for all $i \in X \cap Y$. For simplicity, we assume $X \cap Y = \{1, 2, \ldots, m\}$. Then $R_1^\times \times \cdots \times R_m^\times \times \{0\} \times \cdots \times \{0\}$ is adjacent to both $z(s_1, \ldots, s_k)$ and $z(s'_1, \ldots, s'_k)$, so we also have $d_{\pi\mathcal{Z}_R}(z(s_1, \ldots, s_k), z(s'_1, \ldots, s'_k)) = 2$.

Next, we assume that X and Y are disjoint. We may write $X = \{1, \ldots, p\}$ and $Y = \{p + 1, \ldots, q\}$ where $q \leq k$. We can see that $z(s_1, \ldots, s_k)$ and $z(s'_1, \ldots, s'_k)$ have no common neighbours. However,

$$\begin{aligned} z(s_1, \ldots, s_k) &\sim z(t_1 - s_1, \ldots, t_p - s_p, t_{p+1}, \ldots, t_k) \\ &\sim z(t_1, \ldots, t_p, t_{p+1} - s'_{p+1}, \ldots, t_q - s'_q, t_{q+1}, \ldots, t_k) \\ &\sim z(s'_1, \ldots, s'_k), \end{aligned}$$

where $\sim$ means adjacency in $\pi\mathcal{Z}_R$. We can conclude that $d_{\pi\mathcal{Z}_R}(z(s_1, \ldots, s_k), z(s'_1, \ldots, s'_k)) = 3$.

From the above calculations, the Wiener index can be obtained from the sum

$$\begin{aligned} &\sum_{z(s_1,\ldots,s_k)\sim z(s'_1,\ldots,s'_k)} |z(s_1, \ldots, s_k)||z(s'_1, \ldots, s'_k)| \\ &\quad + 2\sum{}' |z(s_1, \ldots, s_k)||z(s'_1, \ldots, s'_k)| \\ &\quad + 3\sum{}'' |z(s_1, \ldots, s_k)||z(s'_1, \ldots, s'_k)|, \end{aligned}$$

where $\sum'$ is the sum over $z(s_1, \ldots, s_k) \in z(X)$ and $z(s'_1, \ldots, s'_k) \in z(Y)$ which are non-adjacent in $\pi\mathcal{Z}_R$ and $X \cap Y \neq \emptyset$ and $\sum''$ is the sum over $z(s_1, \ldots, s_k) \in z(X)$ and $z(s'_1, \ldots, s'_k) \in z(Y)$ which are non-adjacent in $\pi\mathcal{Z}_R$ and $X \cap Y = \emptyset$. ∎

Finally, we deduce the Wiener index of $\mathcal{Z}_{\mathbb{Z}_n}$ for all $n \in \mathbb{N}$ and $n \geq 3$. Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \ldots {p_k}^{\alpha_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots \alpha_k \in \mathbb{N}$, and let $0 \le s_i \le \alpha_i$ for all $i \in \{1, \dots, k\}$. According to the first remark in Section 3, we have $z(s_1, \dots, s_k)$ is the set $S({p_1}^{s_1}{p_2}^{s_2} \dots {p_k}^{s_k})$ so we know from Ref. [4] that

$$|z(s_1, \dots, s_k)| = \phi\left(\frac{n}{{p_1}^{s_1}{p_2}^{s_2} \dots {p_k}^{s_k}}\right),$$

where ϕ is the Euler phi-function. In other words, if $d = {p_1}^{s_1}{p_2}^{s_2} \dots {p_k}^{s_k}$ is a divisor of n, then $|z(s_1, \dots, s_k)| = \phi(n/d)$. Moreover, let d_i and d_j be nonadjacent vertices in $\mathcal{Z}_{\mathbb{Z}_n}$ corresponding to $z(s_1, s_2, \dots, s_k) \in z(X)$ and $z(s'_1, s'_2, \dots, s'_k) \in z(Y)$ where X and Y are proper subsets of $\{1, 2, \dots, n\}$, respectively. Note that d_i and d_j are relatively prime if $X \cap Y = \emptyset$ and they have a common divisor otherwise. Using this observation, Theorem 5.2 (2) gives us the Wiener index of $\mathcal{Z}_{\mathbb{Z}_n}$.

Corollary 5.3: *Let n be a positive integer greater than 3. Let $d_1, \dots, d_l$ be all proper divisors of n. Then the Wiener index of $\mathcal{Z}_{\mathbb{Z}_n}$ is given by*

$$\sum_{d_i \sim d_j} \phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right) + 2 \sum_{\substack{d_i \not\sim d_j \\ \gcd(d_i, d_j) \neq 1}} \phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right) + 3 \sum_{\substack{d_i \not\sim d_j \\ \gcd(d_i, d_j) = 1}} \phi\left(\frac{n}{d_i}\right)\phi\left(\frac{n}{d_j}\right).$$

Here, ϕ is the Euler phi-function.

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Matrix graphs and MRD codes over finite principal ideal rings

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Let R be a finite principal ideal ring and m, n, d positive integers. In this paper, we study the matrix graph over R which is the graph whose vertices are $m \times n$ matrices over R and two matrices A and B are adjacent if and only if $0 < \operatorname{rank}(A - B) < d$. We show that this graph is a connected vertex transitive graph. The distance, diameter, independence number, clique number and chromatic number of this graph are also determined. This graph can be applied to study MRD codes over R. We obtain that a maximal independent set of the matrix graph is a maximum rank distance (MRD) code and vice versa. Moreover, we show the existence of linear MRD codes over R.

1. Introduction

Throughout the paper, our rings always contain the identity $1 \neq 0$.

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Matrices and the concept of rank of matrices have been playing important roles in graph theory and coding theory. They have been applied to define and study many types of graphs.

One famous graph applied by the rank of matrices is the bilinear form graph. The *bilinear form graph* or *matrix graph* is a graph whose vertices are $m \times n$ matrices over a finite field, and two vertices A and B are adjacent if and only if $\operatorname{rank}(A - B) = 1$. This graph has been widely studied, see [34], [25], [23], [5] and [28]. In 2014, Huang et al. [18] also explored many properties of this graph such as the regularity, connectivity, independence number, clique number and chromatic number. In addition, alternating form graphs [19], quadratic form graphs [22] and Hermitian form graphs [5] are defined by the rank of matrices over fields.

Not only graphs over finite fields but also graphs over finite commutative rings have been vastly studied by applying the rank of matrices. The rank of matrices over rings has been investigated in many directions such as the one defined by McCoy [26] and the one defined by Cohn [7]. Recently, McCoy rank was used to define the adjacency conditions in generalized symplectic graphs and generalized orthogonal graphs over finite commutative rings, see [32]. Huang et al. [20] applied the concept of Cohn rank to generalize bilinear from graphs over finite fields by studying the graph over the ring $\mathbb{Z}_{p^s}$ of integers modulo p^s. They also called this graph a bilinear form graph. The *bilinear form graph* over $\mathbb{Z}_{p^s}$ is the graph whose vertex set is the set of $m \times n$ matrices over $\mathbb{Z}_{p^s}$ and two vertices A and B are adjacent if and only if $\operatorname{rank}(A - B) = 1$. They obtained properties of this graph over $\mathbb{Z}_{p^s}$ similar to those over finite fields.

A year later, Huang [21] generalized the bilinear form graphs over $\mathbb{Z}_{p^s}$ by twisting the adjacency condition. Let d, m, n be positive integers where $1 < d \leq \min\{m, n\}$. The *generalized bilinear form graph* over $\mathbb{Z}_{p^s}$ is the graph whose vertex set is the set of $m \times n$ matrices over $\mathbb{Z}_{p^s}$, and two distinct vertices A and B are adjacent if and only if $\operatorname{rank}(A - B) < d$. So when $d = 2$, it is a usual bilinear form graph over $\mathbb{Z}_{p^s}$. This generalized bilinear form graph has applications in the existence of linear MRD codes.

Delsarte [11] considered a collection of matrices over a finite field as a code. He defined the distance of codes using the rank of matrices and called it rank distance. He showed that these codes have a Singleton like bound. A code that meets this bound is called a maximum rank distance (MRD) code. MRD codes over finite fields have been extensively studied such as their various applications in error correcting codes and network codings, see [15], [29], [17], [10]. Codes over finite rings are also active topics. As a generalization of the field $\mathbb{Z}_p$, codes over the ring $\mathbb{Z}_n$ were investigated such as MRD codes over $\mathbb{Z}_{p^k}$ [14], MDS codes over $\mathbb{Z}_{p^m}$ [31] and self-dual codes over $\mathbb{Z}_{p^m}$ [24]. As well, the concept of MRD codes over $\mathbb{Z}_{p^s}$ is another type of codes over rings studied by Huang [21]. These MRD codes over $\mathbb{Z}_{p^s}$ arise from the generalized bilinear form graphs over $\mathbb{Z}_{p^s}$. More generally, many codes over finite chain rings and finite principal ideal rings continue being more interesting, see [12], [13], [1].

The purpose of this paper is to study matrix graphs over finite principal ideal rings which generalize the matrix graphs over finite fields, bilinear form graphs and generalized bilinear form graphs over $\mathbb{Z}_{p^s}$. We use the concept of the rank of matrices over finite commutative rings to define our graphs. Moreover, we apply the graphs to study MRD codes over finite principal ideal rings.

2. Ranks and background from graph theory

In this section, we first discuss the rank and the McCoy rank of matrices. Then we define the matrix graph and recall some terminologies and results from graph theory. We divide them into two subsections.

2.1. Rank of matrices

Let R be a commutative ring. We write $R^\times$ for the set of unit in R and the set of $m \times n$ matrices with entries in R is denoted by $R^{m\times n}$. Cohn [7] introduced the concept of rank of matrices over commutative rings which generalizes the usual rank of matrices over fields.

For a nonzero matrix A in $R^{m\times n}$, the *rank* of A, denoted by $\operatorname{rank} A$, is the least positive integer t such that $A = BC$ where $B \in R^{m\times t}$ and $C \in R^{t\times n}$. The rank of the zero matrix is defined to be 0.

This rank of matrices has some basic properties as the usual rank over fields. For instance, if $A, B \in R^{m\times n}$, then $\operatorname{rank} A \le \min\{m, n\}$, $\operatorname{rank} A = 0$ if and only if $A = 0$, $\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$, and $\operatorname{rank} A = \operatorname{rank} PAQ$ where $P \in GL_m(R)$ and $Q \in GL_n(R)$, see [7], [8], [20] for more properties.

Now, we assume that R is a finite commutative ring. It is well known that R can be decomposed as $R \cong R_1 \times R_2 \times \cdots \times R_\ell$ where $R_1, R_2, \ldots, R_\ell$ are finite local rings. Let ρ_j be the projection map from R to R_i for all $i \in \{1, 2, \ldots, \ell\}$. Here, a *local ring* is a commutative ring with unique maximal ideal. Recall that if R is a local ring with unique maximal ideal M, then $R^\times = R \setminus M$ and the field R/M is called the *residue field* equipped with the canonical map $\pi : R \to R/M$ given by $\pi(r) = r + M$ for all $r \in R$.

Proposition 2.1. *If $A \in R^{m\times n}$, then*

$$\operatorname{rank} A = \max_{1\le i\le \ell}\{\operatorname{rank} \rho_i(A)\}.$$

Proof. Suppose that $\operatorname{rank} A = t$. Then $A = BC$ for some $B \in R^{m\times t}$ and $C \in R^{t\times n}$. For each $i \in \{1, 2, \ldots, \ell\}$, we have $\rho_i(A) = \rho_i(B)\rho_i(C)$, so that $\operatorname{rank} \rho_i(A) \le t$. On the other hand, let $\operatorname{rank} \rho_i(A) = t_i$ for all $i \in \{1, 2, \ldots, \ell\}$. Then for each $i \in \{1, 2, \ldots, \ell\}$, we have t_i is the least integer such that $\rho_i(A) = B_i' C_i'$ where $B_i' \in R_i^{m\times t_i}$ and $C_i' \in R_i^{t_i\times n}$. Without loss of generality, suppose that $\max_{1\le i\le \ell}\{\operatorname{rank} \rho_i(A)\} = t_1$. Set $B_i = (B_i', 0) \in R^{m\times t_1}$ and $C_i = \begin{pmatrix} C_i' \\ 0 \end{pmatrix} \in R^{t_1\times n}$. Then $A = BC$ where $B = (B_1, B_2, \ldots, B_\ell) \in R^{m\times t_1}$ and $C = (C_1, C_2, \ldots, C_\ell) \in R^{t_1\times n}$. Thus, $\operatorname{rank} A \le t_1$. Therefore, $\operatorname{rank} A = \max_{1\le i\le \ell}\{\operatorname{rank} \rho_i(A)\}$. $\square$

Later, McCoy [26] gave another definition of rank of matrices over commutative rings which also generalizes the usual rank of matrices over fields. This rank is described by the annihilators of ideals as follows.

Let R be a commutative ring and $A \in R^{m\times n}$. We define $I_0 = R$ and $I_t(A)$ to be the ideal of R generated by the $t \times t$ minors of A for $1 \le t \le \min\{m, n\}$. Note that

$$R = I_0(A) \supseteq I_1(A) \supseteq \cdots \supseteq I_{\min\{m,n\}}(A)$$

and so

$$\{0\} = \operatorname{Ann}_R I_0(A) \subseteq \operatorname{Ann}_R I_1(A) \subseteq \cdots \subseteq \operatorname{Ann}_R I_{\min\{m,n\}}(A)$$

where the *annihilator of* I is given by $\operatorname{Ann}_R I = \{r \in R : ra = 0 \text{ for all } a \in I\}$. The *Mc-rank of* A, $\text{Mc-rank}\, A$, is the largest integer r such that $\operatorname{Ann}_R I_r(A) = \{0\}$. If R is a field, then $\text{Mc-rank}\, A$ coincides with the maximal number of linearly independent columns of A, so it is the usual rank. To compute the Mc-rank of matrices over finite commutative rings, we have the following propositions.

Proposition 2.2. *[4] Let R be a finite local ring with maximal ideal M and $\pi : R \to R/M$ a canonical map. Then for each $A \in R^{m\times n}$, $\text{Mc-rank}\, A = \operatorname{rank} \pi(A)$.*

Proposition 2.3. *[3] Let R be a finite commutative ring decomposed as $R = R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite local ring with the projection map $\rho_i : (r_1, r_2, \dots, r_\ell) \mapsto r_i$ for all $i \in \{1, 2, \dots, \ell\}$. If $A \in R^{m\times n}$, then*

$$\text{Mc-rank}\, A = \min_{1\le i\le \ell}\{\text{Mc-rank}\, \rho_i(A)\}.$$

2.2. Matrix graphs

Suppose that R is a finite commutative ring and m, n, d are positive integers such that $2 \le d \le \min\{m, n\}$. The *matrix graph of type* (m, n, d) over R, denoted by $\Gamma_d(R^{m\times n})$, is the graph whose vertices are $m \times n$ matrices over R, and two matrices $A, B \in R^{m\times n}$ are adjacent if and only if $0 < \operatorname{rank}(A - B) < d$. We write $A \sim B$ when A and B are adjacent.

The graph $\Gamma_2(\mathbb{F}_q^{m\times n})$ is the matrix graph studied in [18]. Besides, the graphs $\Gamma_2(\mathbb{Z}_{p^s}^{m\times n})$ and $\Gamma_d(\mathbb{Z}_{p^s}^{m\times n})$ are the bilinear form graphs in [20] and the generalized bilinear form graphs in [21], respectively.

We next recall some terminologies and properties of graphs. Let G be a graph with vertex set $V(G)$. An *automorphism* of a graph G is a bijection σ from G to G such that g_1 is adjacent to g_2 if and only if $\sigma(g_1)$ is adjacent to $\sigma(g_2)$. A graph G is said to be *vertex transitive* if for any two vertices of G, there is an automorphism carrying one to the other. An *independent set* of G is a set I of vertices of G in which no two distinct vertices of I are adjacent. An independent set of G with the largest size of vertices is called a *maximal independent set*. We write $\alpha(G)$ for the size of a maximal independent set of G and call it the *independence number* of G. A *clique* C of G is a complete subgraph of G, that is, any two vertices of C are adjacent and a *maximal clique* of G is a clique of G which has the largest size of vertices. Denoted by $\omega(G)$, the number of vertices in a maximal clique is called the *clique number* of G. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G in which no adjacent vertices have the same color. If G is vertex transitive, we have

$$\omega(G) \leq \frac{|V(G)|}{\alpha(G)} \leq \chi(G).$$

Let $G_1, G_2, \ldots, G_\ell$ be graphs. The *strong product* of graphs $G_1, G_2, \ldots G_\ell$, denoted by $G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_\ell$, is the graph whose vertex set is $V(G_1) \times V(G_2) \times \cdots \times V(G_\ell)$, and $g = (g_1, g_2, \ldots, g_\ell)$ is adjacent to $g' = (g'_1, g'_2, \ldots, g'_\ell)$ if $g \neq g'$ and g_i is either equal or adjacent to g'_i in G_i for all $i \in \{1, 2, \ldots, \ell\}$.

In what follows, we show some properties of rank of matrices over finite chain rings in Section 3. We present the results on matrix graph over finite principal ideal rings. We determine distance, connectivity, vertex transitivity, independence number, clique number and chromatic number in this section. In Section 4, we introduce the MRD codes over finite principal ideal rings. We prove that the MRD codes coincide with the maximal independent sets of the matrix graph. Consequently, we have the existence of linear MRD codes over finite principal ideal rings in our last theorem.

3. Matrix graphs over finite principal ideal rings

In this section, we study the matrix graphs over finite principal ideal rings. We show that our graph is connected and vertex transitive. We determine the distance between any two vertices of the graph. Moreover, the independence number, the clique number and the chromatic number of the graph are computed.

A finite commutative ring R is called a *finite chain ring* if for any ideals I, J of R, either $I \subseteq J$ or $J \subseteq I$. Clearly, a finite chain ring is a local ring. One can show that if R is a finite chain ring, then its maximal ideal M is principal and generated by θ for some $\theta \in M \setminus M^2$. The smallest positive integer e such that $\theta^e = 0$ is called the *nilpotency* of R. A *principal ideal ring (PIR)* is a commutative ring in which all of its ideals are principal. Recall that a finite commutative ring is a direct product of finite local rings. If every ideal of a ring is principal, so are its factors. Thus, a finite PIR can be decomposed as a direct product of finite chain rings. With this nice relation of PIRs and finite chain rings, we first study some properties of matrices over finite chain rings. Some properties of finite chain rings are recorded in the following proposition.

Proposition 3.1. *[27] Let R be a finite chain ring with maximal ideal $M = R\theta$, residue field $\mathbb{F}_q$, nilpotency e and $V = \{v_1, v_2, \ldots, v_q\}$ a system of coset representatives of M in R.*

(1) For any nonzero element r in R, there exists a unique integer i with $0 \le i \le e$ such that $r = u\theta^i$ for some $u \in R^\times$.
(2) For each $r \in R$, r can be uniquely written as

$$r = r_0 + r_1\theta + r_2\theta^2 + \cdots + r_{e-1}\theta^{e-1}$$

where $r_0, r_1, \ldots, r_{e-1} \in V$.
(3) The ideals of R are in the chain

$$\{0\} = R\theta^e \subsetneq R\theta^{e-1} \subsetneq R\theta^{e-2} \subsetneq \cdots \subsetneq R\theta^2 \subsetneq R\theta \subsetneq R.$$

(4) $|R\theta^i| = q^{e-i}$ for all $i \in \{0, 1, \ldots, e\}$.
(5) $R/R\theta^i$ is a finite chain ring with nilpotency i and $|R/R\theta^i| = q^i$ for all $i \in \{1, \ldots, e\}$.
(6) For each $i \in \{1, 2, \ldots, e\}$, we have

$$R/R\theta^i = \{r_0 + r_1\theta + r_2\theta^2 + \cdots + r_{i-1}\theta^{i-1} + R\theta^i : r_0, r_1, \ldots, r_{i-1} \in V\}.$$

Thus, an element $r = r_0 + r_1\theta + r_2\theta^2 + \cdots + r_{i-1}\theta^{i-1} + R\theta^i$ in $R/R\theta^i$ can be viewed as an element $r = r_0 + r_1\theta + r_2\theta^2 + \cdots + r_{i-1}\theta^{i-1} + R\theta^{i+1}$ in $R/R\theta^{i+1}$. Moreover, a unit in $R/R\theta^i$ is a unit in $R/R\theta^{i+1}$.

There is a useful property in computing the rank and Mc-rank of matrices over finite chain rings.

Lemma 3.2. *[6] Let R be a finite chain ring with maximal ideal $R\theta$ and nilpotency e. If A is a nonzero matrix in $R^{m\times n}$, then there exist $P \in GL_m(R)$ and $Q \in GL_n(R)$ such that*

$$A = P \begin{pmatrix} I_{t_0} & & & & & \\ & \theta I_{t_1} & & & & \\ & & \theta^2 I_{t_2} & & & \\ & & & \ddots & & \\ & & & & \theta^{e-1} I_{t_{e-1}} & \\ & & & & & 0 \end{pmatrix} Q \tag{3.1}$$

where $t_0, t_1, \ldots, t_{e-1}$ are non-negative integers. Moreover, this form is unique when θ is fixed.

Proposition 3.3. *Let R be a finite chain ring with maximal ideal $R\theta$ and nilpotency e and A a nonzero matrix in $R^{m\times n}$ of the form (3.1). Then*

$$\operatorname{rank} A = t_0 + t_1 + \cdots + t_{e-1} \textit{ and } \operatorname{Mc-rank} A = t_0.$$

Proof. Let $t = t_0 + t_1 + \cdots + t_{e-1}$. From (3.1), we can write $A = P \operatorname{diag}(D, 0) Q$ where $D = \operatorname{diag}(I_{t_0}, \theta I_{t_1}, \ldots, \theta^{e-1} I_{t_{e-1}}) \in R^{t\times t}$. Write $P = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$ and $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, where $P_1 \in R^{m\times t}$ and $Q_1 \in R^{t\times n}$. We have $A = P_1 D Q_1$. Therefore, $\operatorname{rank} A \le t$.

On the other hand, suppose that $\operatorname{rank} A = s$. Then $A = BC$ where $B \in R^{m\times s}$ and $C \in R^{s\times n}$. By Lemma 3.2, there exist $P_1 \in GL_m(R), Q_1 \in GL_s(R), P_2 \in GL_s(R)$ and $Q_2 \in GL_n(R)$ such that $B = P_1 \begin{pmatrix} D_1 \\ 0 \end{pmatrix} Q_1$ and $C = P_2(D_2, 0)Q_2$ where D_1 and D_2 are diagonal matrices in $R^{s\times s}$. Hence, $A = P_1 \operatorname{diag}(D_1Q_1P_2D_2, 0)Q_2$ where $D_1Q_1P_2D_2 \in R^{s\times s}$. Since the form of A is unique, $s \ge t$. Thus, $\operatorname{rank} A = t$.

Next, let $\pi : R \to R/R\theta$ be the canonical map. Then $\pi(A) = \pi(P) \operatorname{diag}(\pi(D), 0)\pi(Q)$. It is obvious that $\operatorname{rank} \pi(A) = t_0$. By Proposition 2.2, we have $\operatorname{Mc-rank} A = \operatorname{rank} \pi(A) = t_0$. $\square$

By Proposition 3.1 (6), we note that a matrix A over $R/R\theta^i$ can be viewed as a matrix A over $R/R\theta^{i+1}$ and if A is invertible over $R/R\theta^i$, then A is invertible over $R/R\theta^{i+1}$. We apply Proposition 3.3 to prove the next proposition.

Proposition 3.4. *Let A be an $m \times n$ matrix of rank t over $R/R\theta^i$. Then A and $A\theta$ are $m \times n$ matrices of rank t over $R/R\theta^{i+1}$.*

Proof. From Proposition 3.1 and Lemma 3.2, we can write $A = P \operatorname{diag}(I_{t_0}, \theta I_{t_1}, \ldots, \theta^{i-1} I_{t_{i-1}}, 0)Q$ where $P \in GL_m(R/R\theta^i)$ and $Q \in GL_n(R/R\theta^i)$ with $t = t_0 + t_1 + \cdots + t_{i-1}$. It follows that both A and $A\theta = P \operatorname{diag}(\theta I_{t_0}, \theta^2 I_{t_1}, \ldots, \theta^i I_{t_{i-1}}, 0)Q$ are $m \times n$ matrices over $R/R\theta^{i+1}$. Since P and Q are invertible over $R/R\theta^i$, they are invertible over $R/R\theta^{i+1}$. Hence, A and $A\theta$ are of rank t over $R/R\theta^{i+1}$. $\square$

Let R be a finite PIR decomposed as $R \overset{\varphi}{\cong} R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring for all $i \in \{1, 2, \ldots, \ell\}$. Let $\rho_i : (r_1, r_2, \ldots, r_\ell) \mapsto r_i$ be a projection map for all $i \in \{1, 2, \ldots, \ell\}$. The isomorphism φ gives $R^{m\times n} \cong R_1^{m\times n} \times R_2^{m\times n} \times \cdots \times R_\ell^{m\times n}$. Thus, we can view the vertex set of $\Gamma_d(R^{m\times n})$ as $\{(\rho_1(A), \rho_2(A), \ldots, \rho_\ell(A)) : A \in R^{m\times n}\}$. By Proposition 2.1, if $A = (\rho_1(A), \rho_2(A), \ldots, \rho_\ell(A))$ and $B = (\rho_1(B), \rho_2(B), \ldots, \rho_\ell(B))$ are two vertices of $\Gamma_d(R^{m\times n})$, then

$$A \sim B \iff 0 < \max_{1\le i\le \ell} \{\operatorname{rank}(\rho_i(A) - \rho_i(B))\} < d.$$

With this relation, we proceed to prove the following strong product of graphs theorem.

Theorem 3.5. *Let R be a finite PIR decomposed as $R = R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring. Then*

$$\Gamma_d(R^{m\times n}) = \Gamma_d(R_1^{m\times n}) \boxtimes \Gamma_d(R_2^{m\times n}) \boxtimes \cdots \boxtimes \Gamma_d(R_\ell^{m\times n}).$$

Proof. Let $\mathcal{G} = \Gamma_d(R_1^{m\times n}) \boxtimes \Gamma_d(R_2^{m\times n}) \boxtimes \cdots \boxtimes \Gamma_d(R_\ell^{m\times n})$. As mentioned, the vertex sets of graphs $\mathcal{G}$ and $\Gamma_d(R^{m\times n})$ are the same. Let $A = (\rho_1(A), \rho_2(A), \dots, \rho_\ell(A))$ and $B = (\rho_1(B), \rho_2(B), \dots, \rho_\ell(B))$ be two vertices. Then

$$\begin{aligned} A \sim B \text{ in } \Gamma_d(R^{m\times n}) &\Leftrightarrow 0 < \max_{1\le i\le \ell}\{\operatorname{rank}(\rho_i(A) - \rho_i(B))\} < d \\ &\Leftrightarrow A \neq B \text{ and } \operatorname{rank}(\rho_i(A) - \rho_i(B)) < d \text{ for all } i \in \{1, 2, \dots, \ell\} \\ &\Leftrightarrow A \neq B \text{ and either } \rho_i(A) = \rho_i(B) \text{ or } \rho_i(A) \sim \rho_i(B) \text{ in } \Gamma_d(R_i^{m\times n}) \\ &\qquad \text{for all } i \in \{1, 2, \dots, \ell\} \\ &\Leftrightarrow A \sim B \text{ in } \mathcal{G}. \end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let R be a finite PIR. Then the graph $\Gamma_d(R^{m\times n})$ is connected. Moreover, for two vertices $A, B \in R^{m\times n}$, the distance between A and B is*

$$\mathrm{d_G}(A, B) = \left\lceil \frac{\operatorname{rank}(A - B)}{d - 1} \right\rceil.$$

Consequently, the diameter of $\Gamma_d(R^{m\times n})$ is equal to $\lceil \frac{\min\{m,n\}}{d-1} \rceil$.

Proof. We first prove the desired result in the case that R is a finite chain ring. Assume that R is a finite chain ring with maximal ideal $R\theta$ and nilpotency e. Let $A, B \in R^{m\times n}$ with $\operatorname{rank}(B - A) = t$. By Lemma 3.2, there exist $P \in GL_m(R)$ and $Q \in GL_n(R)$ such that

$$B - A = P \begin{pmatrix} \theta^{k_1} & & & & \\ & \theta^{k_2} & & & \\ & & \ddots & & \\ & & & \theta^{k_t} & \\ & & & & 0 \end{pmatrix} Q$$

where $0 \le k_1 \le \cdots \le k_t \le e - 1$. If $t \le d - 1$, then $A \sim B$, and so $\mathrm{d_G}(A, B) = 1$. We assume that $t \ge d$. Write $t = (d - 1)q + r$ where q, r are integers with $q \ge 1$ and $0 \le r < d - 1$. Let $A_0 = A$ and $A_i = A + P\operatorname{diag}(\theta^{k_1}, \theta^{k_2}, \dots, \theta^{k_{(d-1)i}}, 0)Q$ for all $i \in \{1, \dots, q\}$. Then for each $i \in \{0, 1, \dots, q - 1\}$, $A_{i+1} - A_i = P\operatorname{diag}(0, \theta^{k_{(d-1)i+1}}, \dots, \theta^{k_{(d-1)(i+1)}}, 0)Q$, so $\operatorname{rank}(A_{i+1} - A_i) < d$ and thus $A_{i+1} \sim A_i$. Now, we have $A_0 \sim A_1 \sim A_2 \sim \cdots \sim A_q$. Note that $B - A_q = P\operatorname{diag}(0, \theta^{k_{(d-1)q+1}}, \dots, \theta^{k_{(d-1)q+r}}, 0)Q$. So $\operatorname{rank}(B - A_q) \le r < d$. This implies that $B = A_q$ if $r = 0$ or $B \sim A_q$ if $r > 0$. Thus, $\Gamma_d(R^{m\times n})$ is connected. Moreover, $\mathrm{d_G}(A, B)$ is at most $q+1$, that is, $\mathrm{d_G}(A, B) \le \lceil \frac{t}{d-1} \rceil$.

On the other hand, let $\mathrm{d_G}(A, B) = s$. Then there exist $C_1, C_2, \dots, C_{s-1} \in R^{m\times n}$ such that $A \sim C_1 \sim C_2 \sim \cdots \sim C_{s-1} \sim B$. By properties of the rank of matrices, we have

$$\begin{aligned} t = \operatorname{rank}(A-B) &\le \operatorname{rank}(A-C_1) + \operatorname{rank}(C_1-C_2) + \cdots + \operatorname{rank}(C_{s-1}-B) \\ &\le s(d-1). \end{aligned}$$

Thus, $\mathrm{d_G}(A,B) = s \ge \lceil \frac{t}{d-1} \rceil$. Therefore, $\mathrm{d_G}(A,B) = \lceil \frac{\operatorname{rank}(A-B)}{d-1} \rceil$.

Next suppose that R is decomposed as $R \cong R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring. By Theorem 3.5, we have

$$\Gamma_d(R^{m\times n}) = \Gamma_d(R_1^{m\times n}) \boxtimes \Gamma_d(R_2^{m\times n}) \boxtimes \cdots \boxtimes \Gamma_d(R_\ell^{m\times n}).$$

Let $A = (\rho_1(A), \rho_2(A), \ldots, \rho_\ell(A))$ and $B = (\rho_1(B), \rho_2(B), \ldots, \rho_\ell(B))$ be two vertices in $\Gamma_d(R^{m\times n})$. Since $\Gamma_d(R_i^{m\times n})$ is connected for all $i \in \{1, 2, \ldots, \ell\}$, we can suppose that $\mathrm{d_G}(\rho_i(A), \rho_i(B)) = t_i$ for all $i \in \{1, 2, \ldots, \ell\}$. For convenience, we write $\rho_i(A) = X_{i0}$ and $\rho_i(B) = X_{it_i}$. Then for each $i \in \{1, 2, \ldots, \ell\}$, there exist $X_{i1}, X_{i2}, \ldots, X_{i(t_i-1)}$ such that

$$\rho_i(A) = X_{i0} \sim X_{i1} \sim X_{i2} \sim \cdots \sim X_{it_i} = \rho_i(B).$$

Without loss of generality, we assume that $t_1 \le t_2 \le \cdots \le t_\ell$. For each $j \in \{0, 1, \ldots, t_\ell\}$, we set $X_j = (X_{1j}, X_{2j}, \ldots, X_{\ell j})$ where $X_{ij} = \rho_i(B)$ if $t_i \le j \le t_\ell$. Then

$$A = X_0 \sim X_1 \sim X_2 \sim \cdots \sim X_{t_\ell} = B.$$

This implies that $\Gamma_d(R^{m\times n})$ is connected and $\mathrm{d_G}(A,B) \le t_\ell = \max_{1\le i\le \ell}\{\mathrm{d_G}(\rho_i(A), \rho_i(B))\}$.

Conversely, assume that $\mathrm{d_G}(A,B) = t$. Then there exist $X_1, X_2, \ldots, X_{t-1}$ such that $A := X_0 \sim X_1 \sim X_2 \sim \cdots \sim X_{t-1} \sim X_t = B$. Let $i \in \{1, 2, \ldots, \ell\}$. Since $X_j \sim X_{j+1}$, we have $\rho_i(X_j) = \rho_i(X_{j+1})$ or $\rho_i(X_j) \sim \rho_i(X_{j+1})$ in $\Gamma_d(R_i^{m\times n})$ for all $j \in \{0, 1, \ldots, t-1\}$. Thus, $\mathrm{d_G}(\rho_i(A), \rho_i(B)) \le t$. It follows that $\max_{1\le i\le \ell}\{\mathrm{d_G}(\rho_i(A), \rho_i(B))\} \le t = \mathrm{d_G}(A,B)$.

Finally, the distance over finite chain rings implies $\mathrm{d_G}(A,B) = \max_{1\le i\le \ell}\{\mathrm{d_G}(\rho_i(A), \rho_i(B))\} = \max_{1\le i\le \ell}\left\{\left\lceil \frac{\operatorname{rank}(\rho_i(A)-\rho_i(B))}{d-1} \right\rceil\right\}$. By Proposition 2.1, we have $\mathrm{d_G}(A,B) = \left\lceil \frac{\operatorname{rank}(A-B)}{d-1} \right\rceil$. The diameter of $\Gamma_d(R^{m\times n})$ is obtained from Lemma 3.2 together with choosing $A = 0$ and $B = \begin{pmatrix} I_m & 0 \end{pmatrix}$ if $m \le n$ or $B = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$ if $n \le m$. Hence, $\operatorname{rank}(A-B) = \min\{m, n\}$. $\square$

Proposition 3.7. *If R is a finite PIR, then the matrix graph $\Gamma_d(R^{m\times n})$ is vertex transitive.*

Proof. Let $A, B \in R^{m\times n}$. Define $\sigma : R^{m\times n} \to R^{m\times n}$ by $\sigma(X) = X - (A-B)$ for all $X \in R^{m\times n}$. For $X, Y \in R^{m\times n}$, we have $\operatorname{rank}(\sigma(X) - \sigma(Y)) = \operatorname{rank}((X-(A-B)) - (Y-(A-B))) = \operatorname{rank}(X-Y)$. Then $X \sim Y$ if and only if $\sigma(X) \sim \sigma(Y)$ in $\Gamma_d(R^{m\times n})$. Thus, σ is a graph automorphism which maps A to B. Therefore, $\Gamma_d(R^{m\times n})$ is vertex transitive. $\square$

Remark 3.8. It is well known that a vertex transitive graph is regular, that is, every vertex has the same degree. Thus the matrix graph $\Gamma_d(R^{m\times n})$ is regular. For the degree of this regular graph, we can determine the degree of the zero matrix. Then the degree of $\Gamma_d(R^{m\times n})$ is the number of all nonzero $m \times n$ matrices over R of rank less than d.

We next compute the independence numbers and clique numbers of the matrix graphs. The results over finite fields are given in [21] as follows.

Lemma 3.9. *[21] If $\mathbb{F}_q$ is the finite field of q elements, then*

$$\alpha(\Gamma_d(\mathbb{F}_q^{m\times n})) = q^{\max\{m,n\}(\min\{m,n\}-d+1)} \quad \textit{and} \quad \omega(\Gamma_d(\mathbb{F}_q^{m\times n})) = q^{\max\{m,n\}(d-1)}.$$

For the case of finite PIRs, we first consider the sets

$$\mathcal{C}_1 := \left\{ \begin{pmatrix} A \\ 0 \end{pmatrix} : A \in R^{(d-1)\times n} \right\} \text{ and } \mathcal{C}_2 := \left\{ \begin{pmatrix} A & 0 \end{pmatrix} : A \in R^{m\times (d-1)} \right\}.$$

Since $\operatorname{rank} A \le \min\{m,n\}$ for $A \in R^{m\times n}$, it follows that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are cliques of $\Gamma_d(R^{m\times n})$. Thus, $\omega(\Gamma_d(R^{m\times n})) \ge |R|^{\max\{m,n\}(d-1)}$. This provides the lower bound of the clique number. We shall apply it to compute both clique number and independence number.

Theorem 3.10. *Let R be a finite PIR. Then*

$$\alpha(\Gamma_d(R^{m\times n})) = |R|^{\max\{m,n\}(\min\{m,n\}-d+1)}$$

and

$$\omega(\Gamma_d(R^{m\times n})) = |R|^{\max\{m,n\}(d-1)}.$$

Proof. We first suppose that R is a finite chain ring with maximal ideal $R\theta$, nilpotency e and a canonical map $\pi : R \to R/R\theta$. Let $m \le n$. Then Lemma 3.9 implies that $\alpha(\Gamma_d((R/R\theta)^{m\times n})) = q^{n(m-d+1)} := \alpha$. Let $\mathcal{A}$ be a maximal independent set of $\Gamma_d((R/R\theta)^{m\times n})$. So $\operatorname{rank}(A - B) \ge d$ over $R/R\theta$ for all distinct A, B in $\mathcal{A}$. By Proposition 3.4, we have that a matrix A over $R/R\theta$ can be considered as a matrix A over $R/R\theta^i$ with the same rank for all $i \in \{1, 2, \dots, e\}$. Thus, $\operatorname{rank}(A - B) \ge d$ over R for all distinct A, B in $\mathcal{A}$. Next, let

$$\mathcal{I} = \mathcal{A} + \mathcal{A}\theta + \mathcal{A}\theta^2 + \cdots + \mathcal{A}\theta^{e-1} = \{A_0 + A_1\theta + A_2\theta^2 + \cdots + A_{e-1}\theta^{e-1} : A_i \in \mathcal{A}\}.$$

By Proposition 3.1 (2), it is easy to see that $\mathcal{I}$ is a set of size α^e. We show that $\mathcal{I}$ is an independent set of $\Gamma_d(R^{m\times n})$. Let $A, B \in \mathcal{I}$ with $A \ne B$. Then $A = A_0 + A_1\theta + A_2\theta^2 + \cdots + A_{e-1}\theta^{e-1}$ and $B = B_0 + B_1\theta + B_2\theta^2 + \cdots + B_{e-1}\theta^{e-1}$ where $A_i, B_i \in \mathcal{A}$

and $A_j \neq B_j$ for some $j \in \{0, 1, \ldots, e-1\}$. Hence, $A - B = (A_0 - B_0) + (A_1 - B_1)\theta + (A_2 - B_2)\theta^2 + \cdots + (A_{e-1} - B_{e-1})\theta^{e-1}$. We apply Propositions 2.2 and 3.3 to show that $\operatorname{rank}(A - B) \geq d$.

First, if $A_0 \neq B_0$, then $\operatorname{rank}(A - B) \geq \text{Mc-rank}(A - B) = \text{Mc-rank}\,\pi(A - B) = \text{Mc-rank}(A_0 - B_0) = \operatorname{rank}(A_0 - B_0) \geq d$. So we suppose that $A_0 \neq B_0$. Let $j \in \{1, 2, \ldots, e-1\}$ be the first index such that $A_j \neq B_j$. Then $A - B = \big((A_j - B_j) + (A_{j+1} - B_{j+1})\theta + \cdots + (A_{e-1} - B_{e-1})\theta^{e-(j+1)}\big)\theta^j$. Write $C := (A_j - B_j) + (A_{j+1} - B_{j+1})\theta + \cdots + (A_{e-1} - B_{e-1})\theta^{e-(j+1)}$. Then $A - B = C\theta^j$. Note that C can also be viewed as a matrix over $R/R\theta^{e-j}$. By Proposition 3.4, both C and $C\theta^j$ are matrices over $R/R\theta^e \cong R$ with the same rank as considering them over $R/R\theta^{e-j}$. Therefore, $\operatorname{rank}(A - B) = \operatorname{rank}(C\theta^j) = \operatorname{rank}(C) \geq \text{Mc-rank}(C) = \text{Mc-rank}\,\pi(C) = \text{Mc-rank}(A_j - B_j) = \operatorname{rank}(A_j - B_j) \geq d$. This implies that $\mathcal{I}$ is an independent set of $\Gamma_d(R^{m\times n})$ of size $\alpha^e = q^{en(m-d+1)}$. It follows that $\alpha(\Gamma_d(R^{m\times n})) \geq q^{en(m-d+1)}$.

Recall that $\omega(\Gamma_d(R^{m\times n})) \geq q^{en(d-1)}$. Since $\Gamma_d(R^{m\times n})$ is vertex transitive,

$$\alpha(\Gamma_d(R^{m\times n})) \leq \frac{|V(\Gamma_d(R^{m\times n}))|}{\omega(\Gamma_d(R^{m\times n}))} \leq \frac{q^{emn}}{q^{en(d-1)}} = q^{en(m-d+1)}.$$

Therefore, $\alpha(\Gamma_d(R^{m\times n})) = q^{en(m-d+1)}$. Again,

$$\omega(\Gamma_d(R^{m\times n})) \leq \frac{|V(\Gamma_d(R^{m\times n}))|}{\alpha(\Gamma_d(R^{m\times n}))} = \frac{q^{emn}}{q^{en(m-d+1)}} = q^{en(d-1)}.$$

Thus, $\omega(\Gamma_d(R^{m\times n})) = q^{en(d-1)}$. So we obtain the result over finite chain rings.

Next, assume that the PIR R is decomposed as $R = R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite local ring with nilpotency e_i and residue field $\mathbb{F}_{q_i}$. By Theorem 3.5, $\Gamma_d(R^{m\times n}) = \Gamma_d(R_1^{m\times n}) \boxtimes \Gamma_d(R_2^{m\times n}) \boxtimes \cdots \boxtimes \Gamma_d(R_\ell^{m\times n})$. Note that if $\mathcal{I}_i$ is an independent set of $\Gamma_d(R_i^{m\times n})$ for all $i \in \{1, 2, \ldots, \ell\}$, then it is easy to see that

$$\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_\ell = \{(A_1, A_2, \ldots, A_\ell) : A_i \in \mathcal{I}_i\}$$

is an independent set of $\Gamma_d(R^{m\times n})$. Hence,

$$\alpha(\Gamma_d(R^{m\times n})) \geq \alpha(\Gamma_d(R_1^{m\times n}))\alpha(\Gamma_d(R_2^{m\times n}))\cdots\alpha(\Gamma_d(R_\ell^{m\times n})).$$

The previous result on finite chain rings implies that $\alpha(\Gamma_d(R_i^{m\times n})) = \frac{|V(\Gamma_d(R_i^{m\times n}))|}{\omega(\Gamma_d(R_i^{m\times n}))}$ for all $i \in \{1, 2, \ldots, \ell\}$. Moreover, $\Gamma_d(R_i^{m\times n})$ is vertex transitive for all $i \in \{1, 2, \ldots, \ell\}$ by Proposition 3.7. Thus, it follows from [33] Corollary 1 that

$$\alpha(\Gamma_d(R^{m\times n})) \leq \alpha(\Gamma_d(R_1^{m\times n}))\alpha(\Gamma_d(R_2^{m\times n}))\cdots\alpha(\Gamma_d(R_\ell^{m\times n})).$$

Therefore,

$$\begin{aligned}\alpha(\Gamma_d(R^{m\times n})) &= \alpha(\Gamma_d(R_1^{m\times n}))\alpha(\Gamma_d(R_2^{m\times n}))\cdots\alpha(\Gamma_d(R_\ell^{m\times n}))\\ &= q_1^{e_1 n(m-d+1)} q_2^{e_2 n(m-d+1)} \cdots q_\ell^{e_\ell n(m-d+1)}\\ &= |R|^{n(m-d+1)}.\end{aligned}$$

Finally, we determine the clique number of the graph. It is proved in [2] that $\omega(G \boxtimes H) = \omega(G)\omega(H)$. Consequently,

$$\begin{aligned}\omega(\Gamma_d(R^{m\times n})) &= \omega(\Gamma_d(R_1^{m\times n}))\omega(\Gamma_d(R_2^{m\times n}))\cdots\omega(\Gamma_d(R_\ell^{m\times n}))\\ &= q_1^{e_1 n(d-1)} q_2^{e_2 n(d-1)} \cdots q_\ell^{e_\ell n(d-1)}\\ &= |R|^{n(d-1)}.\end{aligned}$$

The case $n \le m$ can be proved in a similar way. □

Remark 3.11.

(1) The cliques $\mathcal{C}_1$ and $\mathcal{C}_2$ mentioned earlier are maximal cliques.
(2) Let R be a finite chain ring with maximal ideal $R\theta$ and nilpotency e. If $\mathcal{A}$ is a maximal independent set of $\Gamma_d((R/R\theta)^{m\times n})$, then

$$\mathcal{I} = \mathcal{A} + \mathcal{A}\theta + \mathcal{A}\theta^2 + \cdots + \mathcal{A}\theta^{e-1}$$

is a maximal independent set of $\Gamma_d(R^{m\times n})$.
(3) For a finite PIR $R \cong R_1 \times R_2 \times \cdots \times R_\ell$, if $\mathcal{I}_i$ is a maximal independent set of $\Gamma_d(R_i^{m\times n})$ for all $i \in \{1, 2, \ldots, \ell\}$, then

$$\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_\ell = \{(A_1, A_2, \ldots, A_\ell) : A_i \in \mathcal{I}_i\}$$

is a maximal independent set of $\Gamma_d(R^{m\times n})$.

Let G be a finite group and S a subset of G which does not contain the identity and is closed under taking inverses. The *Cayley graph* $Cay(G, S)$ is an undirected graph with vertex set G and for two vertices $g_1, g_2 \in G$, g_1 and g_2 are adjacent if $g_1 g_2^{-1}$ is in S. A Cayley graph $Cay(G, S)$ is *normal* if $gSg^{-1} = S$ for all $g \in G$.

To determine the chromatic number of the matrix graph, we use the following property of a normal Cayley graph.

Lemma 3.12. *[16] If G is a normal Cayley graph with $\alpha(G) = \frac{|V(G)|}{\omega(G)}$, then $\chi(G) = \omega(G)$.*

Note that $R^{m\times n}$ is an additive group. Let S be the set of nonzero matrices of rank less than d. It is easy to see that S does not contain the zero matrix and is closed under taking additive inverses. For $A, B \in \Gamma_d(R^{m\times n})$, we have

$$A \sim B \Longleftrightarrow 0 < \operatorname{rank}(A - B) < d \Longleftrightarrow A - B \in S.$$

Thus, $\Gamma_d(R^{m\times n})$ is a Cayley graph. Moreover, it is a normal Cayley graph since $R^{m\times n}$ is an abelian group. By Theorem 3.10, we have $\alpha(\Gamma_d(R^{m\times n})) = \frac{|V(\Gamma_d(R^{m\times n}))|}{\omega(\Gamma_d(R^{m\times n}))}$. It follows from the above lemma that $\omega(\Gamma_d(R^{m\times n})) = \chi(\Gamma_d(R^{m\times n}))$. Hence, we have shown:

Proposition 3.13. *If R is a finite PIR, then $\chi(\Gamma_d(R^{m\times n})) = |R|^{\max\{m,n\}(d-1)}$.*

4. MRD codes

This section is devoted to study MRD codes over PIRs. We give the concepts of matrix codes and rank distance of matrix codes. We shall see that matrix codes relate to matrix graphs. Indeed, maximal independent sets of matrix graphs are MRD codes and vice versa. Finally, we show the existence of linear MRD codes over a PIR by lifting linear MRD codes over a direct product of finite fields.

Let R be a finite commutative ring. A *(matrix) code* of size $m \times n$ is defined to be a subset $\mathcal{C}$ of $R^{m\times n}$. For two matrices $A, B \in R^{m\times n}$, we define the *rank distance between A and B*, denoted by $\mathrm{d}_{\mathrm{rk}}(A,B)$, to be $\operatorname{rank}(A-B)$. Note that the rank distance is a metric on $R^{m\times n}$. Indeed, $\mathrm{d}_{\mathrm{rk}}(A,B) \geq 0$, $\mathrm{d}_{\mathrm{rk}}(A,B) = 0$ if and only if $A = B$, $\mathrm{d}_{\mathrm{rk}}(A,B) = \mathrm{d}_{\mathrm{rk}}(B,A)$ and $\mathrm{d}_{\mathrm{rk}}(A,C) \leq \mathrm{d}_{\mathrm{rk}}(A,B) + \mathrm{d}_{\mathrm{rk}}(B,C)$ for all $A, B, C \in R^{m\times n}$. For a code $\mathcal{C}$ of size $m \times n$ over R, the *rank distance of* $\mathcal{C}$ is defined to be

$$\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = \min\{\mathrm{d}_{\mathrm{rk}}(A,B) : A, B \in \mathcal{C} \text{ with } A \neq B\}.$$

We call a code $\mathcal{C}$ of size $m \times n$ with rank distance d an *$(m \times n, d)$-code*. If $\mathcal{C} \subseteq R^{m\times n}$ is a submodule of $R^{m\times n}$ over R, we call $\mathcal{C}$ a *linear code*.

Suppose $m \leq n$. Let $\mathcal{C}$ be an $(m \times n, d)$-code. We can consider a matrix A in $\mathcal{C}$ as $A = (\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m)$ where $\vec{x}_i \in R^n$ is an i-th row of A. This means we can study $\mathcal{C} \subseteq (R^n)^m$ as a code of length m over a set of alphabet R^n and find the Hamming distance of $\mathcal{C}$. Hence, a code $\mathcal{C}$ with the Hamming distance $\mathrm{d}_{\mathrm{H}}(\mathcal{C})$ agrees with the *Singleton bound* $\mathrm{d}_{\mathrm{H}}(\mathcal{C}) \leq m - \log_{|R|^n} |\mathcal{C}| + 1$. That is, $|\mathcal{C}| \leq |R|^{n(m-\mathrm{d}_{\mathrm{H}}(\mathcal{C})+1)}$.

Over the finite field $\mathbb{F}_q$, it is shown in [11] that a matrix code $\mathcal{C}$ of size $m \times n$ with rank distance $\mathrm{d}_{\mathrm{rk}}(\mathcal{C})$ has a *Singleton like bound* which satisfies $|\mathcal{C}| \leq q^{n(m-\mathrm{d}_{\mathrm{rk}}(\mathcal{C})+1)}$. We show that matrix codes over finite PIRs have a similar bound by using independent sets of the matrix graphs.

Let R be a finite PIR and $\mathcal{C} \subseteq R^{m\times n}$. Then $\mathcal{C}$ is both a matrix code and a set of vertices in the matrix graph $\Gamma_d(R^{m\times n})$. Moreover, if $d \geq 2$, then we have that for any $A, B \in \mathcal{C}$ with $A \neq B$, $\mathrm{d}_{\mathrm{rk}}(A,B) = \operatorname{rank}(A-B) \geq d$ if and only if A is not adjacent to B in $\Gamma_d(R^{m\times n})$. This implies the next proposition.

Proposition 4.1. *Let R be a finite PIR and $2 \leq d \leq m \leq n$. For a code $\mathcal{C} \subseteq R^{m\times n}$, $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) \geq d$ if and only if $\mathcal{C}$ is an independent set of the graph $\Gamma_d(R^{m\times n})$.*

This proposition and the independence number in Theorem 3.10 implies that if $\mathcal{C}$ is a code with $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = d$ where $d \geq 2$, then $|\mathcal{C}| \leq \alpha(\Gamma_d(R^{m\times n})) = |R|^{n(m-d+1)}$. For the case $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = 1$, it is obvious that $|\mathcal{C}| \leq |R|^{nm}$. Thus, we have the Singleton like bound for the matrix codes over finite PIRs as follows.

Corollary 4.2. *Let R be a finite PIR and $m \leq n$. For a code $\mathcal{C} \subseteq R^{m\times n}$, we have $|\mathcal{C}| \leq |R|^{n(m-\mathrm{d}_{\mathrm{rk}}(\mathcal{C})+1)}$.*

An $(m \times n, d)$-code $\mathcal{C}$ over a PIR R is called a *maximum rank distance code (MRD code)* if $|\mathcal{C}| = |R|^{n(m-d+1)}$. Obviously, the only $(m \times n, 1)$-MRD code is $R^{m\times n}$. So we may assume $d \geq 2$ to study MRD codes.

Next, suppose that R is a PIR and $d \leq m \leq n$. Let $\mathcal{C} \subseteq R^{m\times n}$. Note that if $\mathcal{C}$ is either a maximal independent set of $\Gamma_d(R^{m\times n})$ or an $(m \times n, d)$-MRD code, then $|\mathcal{C}| = |R|^{n(m-d+1)} = \alpha(\Gamma_d(R^{m\times n}))$. Moreover, $|\mathcal{C}| = |R|^{n(m-d+1)}$ implies $|R|^{n(m-d+1)} = |\mathcal{C}| \leq |R|^{n(m-\mathrm{d}_{\mathrm{rk}}(\mathcal{C})+1)}$ by Corollary 4.2, so we have $d \geq \mathrm{d}_{\mathrm{rk}}(\mathcal{C})$. Applying Proposition 4.1 results in

$$
\begin{aligned}
&\mathcal{C} \text{ is an } (m \times n, d)\text{-MRD code} \\
&\quad \Leftrightarrow \mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = d \text{ and } |\mathcal{C}| = |R|^{n(m-d+1)} \\
&\quad \Leftrightarrow \mathcal{C} \text{ is an independent set of } \Gamma_d(R^{m\times n}) \text{ and } |\mathcal{C}| = |R|^{n(m-d+1)} \\
&\quad \Leftrightarrow \mathcal{C} \text{ is a maximal independent set of } \Gamma_d(R^{m\times n}).
\end{aligned}
$$

Therefore, we have shown:

Theorem 4.3. *Let R be a finite PIR, $2 \leq d \leq m \leq n$ and $\mathcal{C} \subseteq R^{m\times n}$. Then $\mathcal{C}$ is an $(m \times n, d)$-MRD code if and only if $\mathcal{C}$ is a maximal independent set of $\Gamma_d(R^{m\times n})$.*

We have seen that $(m \times n, d)$-MRD codes coincide with maximal independent sets of the matrix graphs. We next construct linear MRD codes over PIRs by using maximal independent sets of the graphs.

Theorem 4.4. *Let R be a finite PIR decomposed as $R = R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring with maximal ideal $R\theta_i$, nilpotency e_i and residue field $\mathbb{F}_{q_i}$ for all $i \in \{1, 2, \ldots, \ell\}$. For any m, n, d with $2 \leq d \leq \min\{m, n\}$, there exists a linear $(m \times n, d)$-MRD code over R. Moreover, this linear $(m \times n, d)$-MRD code is of the form $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_\ell$ where each $\mathcal{C}_i$ is a linear $(m \times n, d)$-MRD code over R_i which is of the form $\mathcal{C}_i = \overline{\mathcal{C}}_i + \overline{\mathcal{C}}_i\theta_i + \overline{\mathcal{C}}_i\theta_i^2 + \cdots + \overline{\mathcal{C}}_i\theta_i^{e_i-1}$ where $\overline{\mathcal{C}}_i$ is a linear $(m \times n, d)$-MRD code over $\mathbb{F}_{q_i}$.*

Proof. Let m, n, d be positive integers with $2 \leq d \leq m \leq n$. Suppose that R is a finite chain ring with maximal ideal $R\theta$, nilpotency e and residue field $R/R\theta \cong \mathbb{F}_q$. It is shown in [11] that there exists a linear $(m \times n, d)$-MRD code over $\mathbb{F}_q$. We shall lift this linear MRD code $\overline{\mathcal{C}}$ over $\mathbb{F}_q$ to obtain a linear MRD code over R.

Theorem 4.3 implies that $\overline{\mathcal{C}}$ is a maximal independent set of $\Gamma_d(\mathbb{F}_q^{m\times n})$. Remark 3.11 (2) shows that

$$\mathcal{C} := \overline{\mathcal{C}} + \overline{\mathcal{C}}\theta + \overline{\mathcal{C}}\theta^2 + \cdots + \overline{\mathcal{C}}\theta^{e-1} = \{A_0 + A_1\theta + A_2\theta^2 + \cdots + A_{e-1}\theta^{e-1} : A_i \in \overline{\mathcal{C}}\}$$

is a maximal independent set of $\Gamma_d(R^{m\times n})$. From another direction of Theorem 4.3, $\mathcal{C}$ is an $(m \times n, d)$-MRD code over R. Since $\overline{\mathcal{C}}$ is a linear code over $\mathbb{F}_q$, we can employ Proposition 3.1 (2) to obtain a linear code $\mathcal{C}$ over R.

Finally, suppose that R is a PIR decomposed as $R_1 \times R_2 \times \cdots \times R_\ell$ where R_i is a finite chain ring. Then there exists a linear $(m \times n, d)$-MRD code $\mathcal{C}_i$ over R_i for all $i \in \{1, 2, \ldots, \ell\}$. By Theorem 4.3, $\mathcal{C}_i$ is a linear independent set of $\Gamma_d(R_i^{m\times n})$. Again, by Remark 3.11 (3), we have

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_\ell = \{(A_1, A_2, \ldots, A_\ell) : A_i \in \mathcal{C}_i\}$$

is a maximal independent set of $\Gamma_d(R^{m\times n})$. Thus, $\mathcal{C}$ is an $(m \times n, d)$-MRD code over R. Since $\mathcal{C}_i$ is a linear $(m \times n, d)$-MRD code over R_i for all $i \in \{1, 2, \ldots, \ell\}$, $\mathcal{C}$ is also a linear $(m \times n, d)$-MRD code over R. This completes the proof. $\square$

Remark 4.5. Linear MRD codes over finite fields have been intensively applied to linear network coding, and also connected to many areas such as McEliece like public key cryptosystems, semifields, linearized polynomials, see [30] for details. From the above theorem, we obtain linear $(m \times n, d)$-MRD codes for any parameters m, n, d not only over the field alphabet $\mathbb{F}_q$ but also the ring alphabet of any sizes (finite PIRs). Indeed, the ring alphabets are more optimal than field alphabets in some cases to study network coding, see [9]. Moreover, these linear MRD codes over PIRs generalize those over $\mathbb{Z}_{p^s}$ in [21].

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Unitary Cayley graphs of matrix rings over finite commutative rings ☆

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Let R be a finite commutative ring and n a positive integer. In this paper, we study the unitary Cayley graph $\mathrm{C}_{\mathrm{M}_n(R)}$ of the matrix ring over R. If F is a field, we use the additive characters of $\mathrm{M}_n(F)$ to determine three eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ and use them to analyze strong regularity and hyperenergetic graphs. We find conditions on R and n such that $\mathrm{C}_{\mathrm{M}_n(R)}$ is strongly regular. Without explicitly having the spectrum of the graph, we can show that $\mathrm{C}_{\mathrm{M}_n(R)}$ is hyperenergetic and characterize R and n such that $\mathrm{C}_{\mathrm{M}_n(R)}$ is Ramanujan. Moreover, we compute the clique number, the chromatic number and the independence number of the graph.

1. Introduction

For a finite ring R with identity, the *unitary Cayley graph of* R, C_R, is the graph with vertex set R and for each $x, y \in R$, x and y are adjacent if and only if $x - y$ is a unit

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of R. The unitary Cayley graphs have been widely studied by many authors (see, for example, [2,8,4,1,5]). As discovered in [1,5], if R is a finite commutative ring, then R can be decomposed as a direct product of finite local rings $R_1, \dots, R_s$ and C_R is the tensor product of the graphs $\mathrm{C}_{R_1}, \dots, \mathrm{C}_{R_s}$ where the *tensor product* of graphs G and H, $G \otimes H$, is the graph defined on $V(G) \times V(H)$ where (a, b) is adjacent to (c, d) if and only if a is adjacent to c in G and b is adjacent to d in H. In addition, if R is a finite local ring with maximal ideal M, then C_R is a complete multi-partite graph whose partite sets are the cosets of M. Thus, the unitary Cayley graphs of finite commutative rings are well studied. Their spectral properties including the energies are also well known (see [5]).

Let G be a graph and $V(G)$ the vertex set of G. We give some terminologies from graph theory as follows. A *clique* is a subgraph that is a complete graph and *clique number* of G is the size of largest clique in G, denoted by $\omega(G)$. A set I of vertices of G is called an *independent set* if no distinct vertices of I are adjacent. The *independence number* of G is the size of a maximal independent set, denoted by $\alpha(G)$. The *chromatic number of* G is the least number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. We write $\chi(G)$ for the chromatic number of G. If every vertex of G is adjacent to k vertices, then G is a k-regular graph. Finally, we say that a k-regular graph G is *edge regular* if there exists a parameter λ such that for any two adjacent vertices, there are exactly λ vertices adjacent to both of them. If an edge regular graph with parameters k, λ also satisfies an additional property that for any two non-adjacent vertices, there are exactly μ vertices adjacent to both of them, then it is called a *strongly regular graph with parameters* k, λ, μ.

Let R be a ring and $n \in \mathbb{N}$. Let $R^\times$ denote the group of units of R. Let $\mathrm{M}_n(R)$ denote the ring of $n \times n$ matrices over R and the group of all invertible matrices over R is denoted by $\mathrm{GL}_n(R)$. Throughout this work, I_n is the $n \times n$ identity matrices and $\mathbf{0}_{m \times n}$ is the $m \times n$ zero matrix.

For non-commutative rings, Kiani et al. [6] worked on unitary Cayley graphs of the ring $\mathrm{M}_{n_1}(F_1) \times \dots \times \mathrm{M}_{n_k}(F_k)$ where $n_1, \dots, n_k \in \mathbb{N}$ and $F_1, \dots, F_k$ are finite fields. They obtained the clique number, the chromatic number and the independence number of the graph. They also studied the role between C_R and the structure of R. Later in [7], they proved that if F is a finite field, then $\mathrm{C}_{\mathrm{M}_n(F)}$ is an edge regular graph with $k = |\,\mathrm{GL}_n(F)|$ and $\lambda = |(I_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)| = e_n$ where e_n is the number of invertible matrices which do not fix any non-zero vector. Such matrices are called *derangement matrices*. We know from [11] that e_n satisfies the recursion $e_n = e_{n-1}(q^n - 1)q^{n-1} + (-1)^n q^{n(n-1)/2}$ and $e_0 = 1$. Kiani showed further that $\mathrm{C}_{\mathrm{M}_2(F)}$ is strongly regular with $\mu = \left| \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mathrm{GL}_2(F) \right) \cap \mathrm{GL}_2(F) \right|$ but $\mathrm{C}_{\mathrm{M}_3(F)}$ is not strongly regular.

An *eigenvalue* of a graph G is an eigenvalue of the adjacency matrix of a graph G. The *energy* of a graph G, $E(G)$, is the sum of absolute value of its eigenvalues. The *spectrum* of a graph G is the list of its eigenvalues together with their multiplicities. If $\lambda_1, \dots, \lambda_r$ are eigenvalues of a graph G with multiplicities $m_1, \dots, m_r$, respectively, we write $\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \dots & \lambda_r \\ m_1 & \dots & m_r \end{pmatrix}$ to describe the spectrum of G and so $E(G) =$

$m_1|\lambda_1| + \cdots + m_r|\lambda_r|$. A graph G on n vertices is said to be *hyperenergetic* if its energy exceeds the energy of the complete graph K_n, that is, $E(G) > 2(n-1)$. A k-regular graph G is a *Ramanujan graph* if $|\lambda| \leq 2\sqrt{k-1}$ for all eigenvalues λ of G other than $\pm k$. A Ramanujan graph is a regular graph whose spectral gap is almost as large as possible. It is an excellent spectral expander. Its name comes from Lubotzky, Phillips and Sarnak [10] who used the Ramanujan conjecture to construct an infinite family of such graphs.

To introduce our methodology, we recall some results on characters of finite abelian groups. For more detail, see [9]. Let G be a finite abelian group. A map $\chi : G \to (\mathbb{C} \setminus \{0\}, \cdot)$ is a *character* if χ is a group homomorphism. The set of all characters of G, denoted by $\widehat{G}$, forms an abelian group under point-wise multiplication, that is, for any characters χ_1, χ_2 of G, we define $\chi_1 \cdot \chi_2 : G \to (\mathbb{C} \setminus \{0\}, \cdot)$ where $(\chi_1 \cdot \chi_2)(g) = \chi_1(g)\chi_2(g)$ for all $g \in G$.

Let F be the finite field extension of $\mathbb{Z}_p$ which has p^r elements for some $r \in \mathbb{N}$ and a prime p. The *trace map* from F to $\mathbb{Z}_p$ is the $\mathbb{Z}_p$-linear map $\mathrm{Tr} : x \mapsto x + x^p + \cdots + x^{p^{r-1}}$. According to [9], each character of the group $(F, +)$ is given by $\chi_a(x) = e^{\frac{2\pi i}{p}\mathrm{Tr}(ax)}$ for all $x \in F$ where $a \in F$ is fixed. Note that $(\mathrm{M}_n(F), +) \cong (F, +) \times (F, +) \times \cdots \times (F, +)$ (n^2 copies). Recall that if we have G_1, G_2 are finite abelian groups, then there is a canonical isomorphism $\widehat{G_1} \times \widehat{G_2} \to \widehat{G_1 \times G_2}$ given by $(\chi_1, \chi_2) \mapsto \chi_1\chi_2$. Hence, we may identify a character of $\mathrm{M}_n(F)$ as $\chi_A = \prod_{1 \leq i,j \leq n} \chi_{a_{ij}}$ where $A = [a_{ij}]_{n \times n}$ is in $\mathrm{M}_n(F)$ and so it follows from Theorem 2 of [12] that the eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ are given by

$$\rho_A = \sum_{S \in \mathrm{GL}_n(F)} \chi_A(S)$$

as A ranges over all matrices in $\mathrm{M}_n(F)$.

In the next section, we shall use the additive characters discussed in the previous paragraph to compute some eigenvalues (namely, ρ_{A_1}, ρ_{A_2} and ρ_{A_3}) and use them to study strong regularity of the unitary Cayley graph $\mathrm{C}_{\mathrm{M}_n(F)}$ of a matrix algebra over a finite field F of q elements. This new approach also allows us to conclude that the multiplicities of eigenvalues are at least the number of matrices of the same rank (Theorem 2.1). Without completely having the spectrum of the graph, we work on the eigenvalue ρ_{A_3} and show that $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic and characterize n and q such that $\mathrm{C}_{\mathrm{M}_n(F)}$ is Ramanujan in Section 3.

The final section presents the study of the unitary Cayley graph of product of matrix rings over finite local rings. We start by working on a finite local ring R with unique maximal ideal M and residue field $\Bbbk$. We determine the canonical graph isomorphism from the graph $\mathrm{C}_{\mathrm{M}_n(\Bbbk)} \otimes \mathring{\mathrm{M}}_n(M)$ onto the graph $\mathrm{C}_{\mathrm{M}_n(R)}$ induced from lifting elements of $\Bbbk$ to R via M (Theorem 4.2). This isomorphism allows us to obtain the clique number, the chromatic number and the independence number of the unitary Cayley graph of product of matrix rings over finite local rings. Since every finite commutative ring is isomorphic to a direct product of finite local rings, we have these numbers for unitary Cayley graphs of a matrix ring over a finite commutative ring. Moreover, the work in Sections 2 and 3 is generalized to matrix rings over finite local rings and finite commutative rings in Section 4.

2. Strong regularity of $\mathrm{M}_n(F)$

Throughout this section, let F be the finite field of q elements and $n \in \mathbb{N}$. Our main work is to show that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular if and only if $n = 2$. We begin by determining some eigenvalues of the graph by considering three matrices in $\mathrm{M}_n(F)$, namely,

$$A_1 = \mathbf{0}_{n\times n}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Clearly, we have

$$\rho_{A_1} = |\,\mathrm{GL}_n(F)| = (q^n - 1)(q^n - q)\ldots(q^n - q^{n-1}).$$

Note that

$$\rho_{A_2} = \sum_{m\in F} N_m e^{\frac{2\pi i}{p}\,\mathrm{Tr}(m)}$$

where N_m is the number of invertible matrices with m at the left-top corner for all $m \in F$. If an invertible matrix has the left-top corner being 0, then the other $n-1$ elements in the first column cannot be all zeros, so there are $q^n - 1$ choices for the first column. Thus,

$$N_0 = (q^{n-1} - 1)(q^n - q)(q^n - q^2)\ldots(q^n - q^{n-1})$$

because the second column must not be multiple of the first column, and the jth column must not be a linear combination of the previous $j-1$ columns for all $j \in \{2, \ldots, n\}$. Now, we have

$$(q^n - q^{n-1})(q^n - q)(q^n - q^2)\ldots(q^n - q^{n-1})$$

invertible matrices with the top-left corner being nonzero. Since $N_m = N_1$ for all $m \neq 0$, we have

$$(q-1)N_1 = (q^n - q^{n-1})(q^n - q)(q^n - q^2)\ldots(q^n - q^{n-1})$$

so

$$N_1 = q^{n-1}(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}).$$

It follows that

$$\begin{aligned}\rho_{A_2} &= N_0 e^{\frac{2\pi i}{p}\operatorname{Tr}(0)} + N_1 \sum_{m\neq 0} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} \\ &= -(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}) + N_1 \sum_{m\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)}.\end{aligned}$$

By Hilbert's theorem 90, we know that the trace map is surjective, so we get

$$\sum_{m\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m)} = |\ker \operatorname{Tr}| \sum_{m\in\mathbb{Z}_p} e^{\frac{2\pi i}{p}m} = 0.$$

Therefore,

$$\rho_{A_2} = -(q^n - q)(q^n - q^2)\dots(q^n - q^{n-1}).$$

Finally, we determine ρ_{A_3}. Since

$$\rho_{A_3} = N(m_1, m_2, \dots, m_{n+1}) \sum_{m_1, m_2, \dots, m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_1+m_2+\dots+m_n+m_{n+1})}$$

where $N(m_1, m_2, \dots, m_{n+1})$ is the number of invertible matrices of the form

$$\begin{bmatrix} m_1 & m_{n+1} & \cdots & * \\ m_2 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ m_n & * & \cdots & * \end{bmatrix}$$

and $m_1, m_2, \dots, m_{n+1} \in F$. For $m_1 = 0$, we can determine $N(0, m_2, \dots, m_{n+1})$ according to m_{n+1} as follows. If $m_{n+1} \neq 0$, then the first column and the second column are linearly independent, so the second column can be arbitrarily chosen. If $m_{n+1} = 0$, then the second column must not be multiple of the first column and the jth column must not be a linear combination of the previous $j-1$ columns for all $j \in \{2, \dots, n\}$. Thus, $N(0, m_2, \dots, 0) = (q^{n-1})(q^n - q^2)\dots(q^n - q^{n-1})$ and $N(0, m_2, \dots, m_{n+1}) = (q^{n-1})(q^n - q^2)\dots(q^n - q^{n-1})$ if $m_{n+1} \neq 0$. Now, assume that $m_1 \neq 0$. Then $N(m_1, m_2, \dots, m_{n+1}) = N(1, m_2, \dots, m_{n+1})$ for all $m_2, \dots, m_{n+1} \in F$. To find $N(1, m_2, \dots, m_{n+1})$, we note that the second column cannot be m_{n+1}-multiple of the first column and similarly the jth column must not be a linear combination of the previous $j-1$ columns for all $j \in \{2, \dots, n\}$, so

$$N(1, m_2, \dots, m_{n+1}) = (q^{n-1} - 1)(q^n - q^2)\dots(q^n - q^{n-1}).$$

Now, we compute

$$
\begin{aligned}
\rho_{A_3} &= (q^{n-1}-q)(q^n-q^2)\dots(q^n-q^{n-1})(q^n+1)\sum{}' e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\dots m_n)}\\
&\quad + q^{n-1}(q^n-q^2)\dots(q^n-q^{n-1})\sum{}' \sum_{m_{n+1}\neq 0} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\dots m_n+m_{n+1})}\\
&\quad + (q^{n-1}-1)(q^n-q^2)\dots(q^n-q^{n-1})\sum_{m_1\neq 0}\sum{}' \sum_{m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_1+m_2+\dots m_n+m_{n+1})}
\end{aligned}
$$

where $\sum{}'$ denotes the sum over $m_2,\dots,m_n \in F$ such that $\begin{bmatrix} m_1\\ m_2\\ \vdots\\ m_n\end{bmatrix}$ is the first column of an invertible matrix. Since $\sum\limits_{m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_{n+1})} = 0$, the last sum is 0, so we can rewrite ρ_{A_3} as

$$
\begin{aligned}
\rho_{A_3} = q^{n-1}(q^n-q^2)\dots(q^n-q^{n-1})\sum{}' \sum_{m_{n+1}\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\dots m_n+m_{n+1})}\\
- q(q^n-q^2)\dots(q^n-q^{n-1})\sum{}' e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\dots m_n)}.
\end{aligned}
$$

The first sum is again zero because m_{n+1} varies over F. Now, since $m_1 = 0$, $m_2,\dots,m_n$ cannot be all zeros and so

$$
\begin{aligned}
\sum{}' e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\dots m_n)} &= \sum_{\{m_2,\dots,m_n\}\neq\{0\}} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\dots m_n)}\\
&= \sum_{m_2,\dots,m_n\in F} e^{\frac{2\pi i}{p}\operatorname{Tr}(m_2+\dots m_n)} - 1 = -1.
\end{aligned}
$$

Hence, $\rho_{A_3} = q(q^n-q^2)\dots(q^n-q^{n-1})$.

Let A and B be $n\times n$ matrices over F. Assume that $\operatorname{rank} A = \operatorname{rank} B$. Then there exist invertible matrices P and Q such that $A = PBQ$. Consider $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $P = [p_{ij}]_{n\times n}$ and $Q = [q_{ij}]_{n\times n}$. For $S = [s_{ij}]_{n\times n} \in \mathrm{GL}_n(F)$, we have

$$
\chi_A(S) = e^{\frac{2\pi i}{p}\operatorname{Tr}\left(\sum_{1\le i,j\le n} a_{ij}s_{ij}\right)}.
$$

From

$$
\begin{aligned}
\sum_{1\le i,j\le n} a_{ij}s_{ij} &= \sum_{1\le i,j\le n}\Big(\sum_{1\le k,l\le n} p_{il}b_{lk}q_{kj}\Big)s_{ij}\\
&= \sum_{1\le i,j\le n}\sum_{1\le k,l\le n} b_{lk}(p_{il}s_{ij}q_{kj})\\
&= \sum_{1\le k,l\le n} b_{lk}\sum_{1\le i,j\le n}(p_{il}s_{ij}q_{kj})
\end{aligned}
$$

and $\sum_{1\le i,j\le n} p_{il}s_{ij}q_{kj} = \left(P^tSQ^t\right)_{lk}$, it follows that $\chi_A(S) = \chi_B(P^tSQ^t)$. Since P and Q are invertible, $\mathrm{GL}_n(F) = P^t\,\mathrm{GL}_n(F)Q^t$, so

$$\sum_{S\in\mathrm{GL}_n(F)} \chi_A(S) = \sum_{S\in\mathrm{GL}_n(F)} \chi_B(S).$$

Hence, we have shown:

Theorem 2.1. *If A and B are $n\times n$ matrices over F of the same rank, then $\rho_A = \rho_B$.*

Since $\mathrm{C}_{\mathrm{M}_n(F)}$ is connected and $|\mathrm{GL}_n(F)|$-regular, ρ_{A_1} induced from the zero matrix has multiplicity 1. Observe that ρ_{A_2} and ρ_{A_3} are induced by matrices of rank 1 and 2, respectively. Since the set of characters are linearly independent, the multiplicities of them are the number of matrices of such rank. Suppose $n = 2$. The number of matrices of rank 1 is $\frac{(q^2-1)^2}{q-1} = (q-1)(q+1)^2$ and the number of matrices of rank 2 is $(q^2-1)(q^2-q)$. We record this result in:

Theorem 2.2. $\operatorname{Spec}\mathrm{C}_{\mathrm{M}_2(F)} = \begin{pmatrix} (q^2-1)(q^2-q) & -(q^2-q) & q \\ 1 & (q-1)(q+1)^2 & (q^2-1)(q^2-q) \end{pmatrix}$ *and* $E(\mathrm{C}_{\mathrm{M}_2(F)}) = 2q(q^2-1)^2$.

If $n = 3$, then $\rho_{A_1} = (q^3-1)(q^3-q)(q^3-q^2)$, $\rho_{A_2} = -(q^3-q)(q^3-q^2)$ and $\rho_{A_3} = q(q^3-q^2)$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_3(F)}$ induced from matrices of rank 0, 1 and 2, respectively. Let λ be the eigenvalue induced from matrices of rank 3. Counting the number of matrices of each rank gives

$$\begin{aligned}(q^3-1)(q^3-q)(q^3-q^2) - (q^3-q)(q^3-q^2)\frac{(q^3-1)^2}{q-1}\\ + q(q^3-q^2)\frac{(q^3-1)^2(q^3-q)^2}{(q^2-1)(q^2-q)} + (q^3-1)(q^3-q)(q^3-q^2)\lambda = 0.\end{aligned}$$

Dividing by $(q^3-1)(q^3-q)(q^3-q^2)$ implies $\lambda = -q^3$. This proves the following theorem.

Theorem 2.3.

$$\operatorname{Spec}\mathrm{C}_{\mathrm{M}_3(F)} = \begin{pmatrix} (q^3-1)(q^3-q)(q^3-q^2) & -(q^3-q)(q^3-q^2) \\ 1 & (q^3-1)(q^2+q+1) \end{pmatrix.$$
$$\begin{pmatrix} q(q^3-q^2) & -q^3 \\ (q^3-1)(q^3-q)(q^2+q+1) & (q^3-1)(q^3-q)(q^3-q^2) \end{pmatrix}.$$

Recall from Chapter 10 of [3] that a connected regular non-complete graph is strongly regular if and only if it has exactly three distinct eigenvalues. So, we can conclude from Theorem 2.2 that $\mathrm{C}_{\mathrm{M}_2(F)}$ is strongly regular. Next, we assume that $n\ge 3$ and $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular. Then $\mathrm{C}_{\mathrm{M}_n(F)}$ has only three eigenvalues. From our computation, they must be ρ_{A_1}, ρ_{A_2} and ρ_{A_3}. Suppose the multiplicities of ρ_{A_2} and ρ_{A_3} are m_2 and m_3, respectively. Since the sum of eigenvalues of $\mathrm{C}_{\mathrm{M}_n(F)}$ is 0, we have

$$(q^n-1)(q^n-q)\dots(q^n-q^{n-1})-(q^n-q)\dots(q^n-q^{n-1})m_2+q(q^n-q^2)\dots(q^n-q^{n-1})m_3=0.$$

Dividing by $(q^n-q^2)\dots(q^n-q^{n-1})$ gives

$$(q^n-1)(q^n-q)-(q^n-q)m_2+qm_3=0.$$

Note that $1+m_2+m_3=q^{n^2}$, so $m_2=q^{n^2}-m_3-1$. Putting this m_2 in the previous equation gives $m_3=q(q^{n-1}-1)(q^{n^2-n}-1)$. Recall from Corollary 8.1.3 of [3] that the sum of square of eigenvalues of the adjacency matrix A is the trace of A^2 which is twice of the number of edges of the graph. Since our graph is $|\mathrm{GL}_n(F)|$-regular, if E_n is the number of edges, then

$$2E_n=q^{n^2}(q^n-1)\dots(q^n-q^{n-1}).$$

This yields another relation on m_2 and m_3 given by

$$\begin{aligned}((q^n-1)(q^n-q)\dots(q^n-q^{n-1}))^2+((q^n-q)\dots(q^n-q^{n-1}))^2m_2&\\ +(q(q^n-q^2)\dots(q^n-q^{n-1}))^2m_3=q^{n^2}(q^n-1)\dots(q^n-q^{n-1}).&\end{aligned}$$

Dividing by $(q^n-q^2)\dots(q^n-q^{n-1})$ and substituting $m_3=q(q^{n-1}-1)(q^{n^2-n}-1)$ give

$$\begin{aligned}(q^n-1)^2(q^n-q)^2\dots(q^n-q^{n-1})+(q^n-q)^2(q^n-q^2)\dots(q^n-q^{n-1})m_2&\\ +q^3(q^n-q^2)\dots(q^n-q^{n-1})(q^{n-1}-1)(q^{n^2-n}-1)&\\ =q^{n^2}(q^n-1)(q^n-q).&\end{aligned}$$

Since $q^{n^2-n}-1=\left(q^{n-1}\right)^n-1$, the left hand side is divisible by $(q^{n-1}-1)^2$, so $(q^{n-1}-1)^2$ divides $q^{n^2}(q^n-1)(q^n-q)$. It follows that $q^{n-1}-1$ divides $q^{n^2+1}(q^n-1)$. Since q and q^n-1 are relatively prime, we have $q^{n-1}-1$ divides $q^n-1=q^n-q+(q-1)$, so $q^{n-1}-1$ divides $q-1$ which is a contradiction because $n\geq 3$. Therefore, we have our desired result.

Theorem 2.4. *The graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is strongly regular if and only if $n=2$.*

From the above theorem, we learn that $\mathrm{C}_{\mathrm{M}_n(F)}$ is not strongly regular for $n\geq 3$. Since it is edge regular with $\lambda=e_n$, there are more than one value of the number of common neighborhoods of non-adjacent vertices in $\mathrm{C}_{\mathrm{M}_n(F)}$. If $A,B\in\mathrm{M}_n(F)$ and $\mathrm{rank}(A-B)=r$ for some $0<r\leq n$, then there exist invertible matrices P,Q such that

$$P(A-B)Q = \begin{bmatrix} I_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{0}_{(n-r)\times(n-r)} \end{bmatrix}.$$

For $A \in \mathrm{M}_n(F)$, let $N(A)$ be the set of neighbors of A. According to Kiani (in the proof of Lemma 2.1 of [7]), we have

$$|N(A)\cap N(B)| = \left| \left(\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathrm{GL}_n(F) \right) \cap \mathrm{GL}_n(F) \right|$$

for all $A, B \in \mathrm{M}_n(F)$ with $A \neq B$. It gives the number of common neighbors of any pair of two vertices A and B in $\mathrm{M}_n(F)$. For $1 \le r \le n$, we define

$$d(n,r) = \left| \left(\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathrm{GL}_n(F) \right) \cap \mathrm{GL}_n(F) \right|.$$

Since two matrices A and B are adjacent if and only if $\operatorname{rank}(A-B) = n$, we have $d(n,n) = e_n$ where e_n is mentioned in Section 1. Observe that $d(n,r)$ is the number of invertible matrices A such that $A - \begin{bmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{bmatrix}$ is also invertible. Now, let $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be the standard basis of F^n. Consider the set $\mathcal{X}$ of vectors given by

$$\mathcal{X} = \{A = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n] \in \mathrm{GL}_n(F) : \vec{a}_1 \in \vec{e}_1 + \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}\}.$$

Note that if $A \in \mathcal{X}$, then A is invertible but $A - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ is not invertible. We proceed to compute $d(n,1)$. Since $d(n,1) = |\mathrm{GL}_n(F)| - |\mathcal{X}|$, we shall determine the cardinality of $\mathcal{X}$. Let $A = [a_{ij}]_{n\times n}$ be in $\mathcal{X}$. Then $\operatorname{rank} A = n$ and $\operatorname{rank}\left(A - \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right) = n-1$. It follows that $\vec{a}_1 \notin \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$ but $\vec{a}_1 \in \vec{e}_1 + \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$. This forces that $\vec{e}_1 \notin \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$. Also, $\{\vec{a}_2, \dots, \vec{a}_n\}$ must be linearly independent. Thus, there are $(q^n - q)\dots(q^n - q^{n-1})$ choices for $\{\vec{a}_2, \dots, \vec{a}_n\}$. As for $\vec{a}_1$, it suffices to count under a condition $\vec{a}_1 \in \vec{e}_1 + \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$ because if $\vec{a}_1 \in \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$, then $\vec{e}_1 \in \mathrm{Span}\{\vec{a}_2, \dots, \vec{a}_n\}$, which is absurd, so there are q^{n-1} choices for $\vec{a}_1$. Hence,

$$|\mathcal{X}| = q^{n-1}(q^n - q)\dots(q^n - q^{n-1}).$$

Then

Theorem 2.5. *$d(n,1) = |\mathrm{GL}_n(F)| - |\mathcal{X}| = (q^n - q^{n-1} - 1)(q^n - q)\dots(q^n - q^{n-1})$.*

Remark. For $r \ge 2$, we can find a lower bound for $d(n,r)$. Consider a matrix of the form $Y = \begin{bmatrix} A & \mathbf{0} \\ B & C \end{bmatrix}$ where A, B and C are $r \times r$, $(n-r)\times r$ and $(n-r)\times(n-r)$ matrices, respectively. It is easy to see that $\det Y = \det A \det C$, and $\det\left(X - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right) =$

$\det(A - I_r)\det C$. If we choose A to be a derangement matrix and C is an invertible matrix, then Y and $Y - \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ are invertible. Since there are e_r choices for A, $q^{r(n-r)}$ choices for B, and $(q^{n-r}-1)\dots(q^{n-r}-q^{n-r-1})$ choices for C, we have $d(n,r) \geq e_r q^{r(n-r)}(q^{n-r}-1)\dots(q^{n-r}-q^{n-r-1}) = e_r(q^n - q^r)\dots(q^n - q^{n-1})$.

3. Hyperenergetic graphs and Ramanujan graphs

Let F be a finite field of q elements. In this section, without explicitly computing the spectrum of the graph, we show that the graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic for all $n \geq 2$ and characterize n and q such that $\mathrm{C}_{\mathrm{M}_n(F)}$ is Ramanujan.

Since $q^3 - 1 = (q-1)(q^2+q+1) > q^2 + q$, we get $q(q^2-1) = q^3 - q > q^2 + 1$, so $E(\mathrm{C}_{\mathrm{M}_2(F)}) = 2q(q^2-1)^2 > 2(q^4-1)$. Then $\mathrm{C}_{\mathrm{M}_2(F)}$ is hyperenergetic. Next, we assume that $n \geq 3$. Recall that $\rho_{A_3} = q(q^n - q^2)\dots(q^n - q^{n-1})$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_n(F)}$ with multiplicities at least $\dfrac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)}$. It follows that

$$E(\mathrm{C}_{\mathrm{M}_n(F)}) > q(q^n - q^2)\dots(q^n - q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)}.$$

Thus, to show that $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic, it suffices to prove

$$q(q^n - q^2)\dots(q^n - q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} > 2(q^{n^2} - 1).$$

Since $|\operatorname{GL}_n(F)| = (q^n-1)(q^n-q)\dots(q^n-q^{n-1})$, the above inequality is equivalent to

$$|\operatorname{GL}_n(F)| > \frac{2(q^2-1)(q^2-q)(q^{n^2}-1)}{q(q^n-1)(q^n-q)}.$$

We shall use induction on $n \geq 3$ to show that this inequality holds and conclude that $\mathrm{C}_{\mathrm{M}_n(F)}$ is hyperenergetic. If $n = 3$, then the right-hand side becomes

$$\frac{2(q^2-1)(q^2-q)(q^9-1)}{q(q^3-1)(q^3-q)} = \frac{2(q-1)}{q}(q^6+q^3+1)$$

and

$$|\operatorname{GL}_3(F)| = (q-1)^3(q^6+2q^5+2q^4+q^3) > (q-1)^3(q^6+q^3+1).$$

Since $q \geq 2$, we have $q(q-1)^2 \geq 2$. Then $(q-1)^3 \geq \dfrac{2(q-1)}{q}$ and the inequality is valid for $n = 3$. Now, let $n \geq 4$ and assume that

$$\begin{aligned}
|\operatorname{GL}_{n-1}(F)| &\geq \frac{2(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^{n-1}-1)(q^{n-1}-q)} \\
&= \frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^{n-1}-q)} \\
&\geq \frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^n-1)}
\end{aligned}$$

where the last inequality comes from $q^n-1-(q^{n-1}-q)=(q^{n-1}+1)(q-1)\geq 0$. Since $|\operatorname{GL}_n(F)|=(q^n-1)(q^n-q)\dots(q^n-q^{n-1})=q^{n-1}(q^n-1)|\operatorname{GL}_{n-1}(F)|$, it follows from the previous inequality that

$$|\operatorname{GL}_n(F)| \geq q^{n-1}(q^n-1)\frac{2q(q^2-1)(q^2-q)(q^{(n-1)^2}-1)}{q(q^n-q)(q^n-1)}$$

and so it remains to show that $q^n(q^n-1)(q^{(n-1)^2}-1)\geq q^{n^2}-1$. Rewrite

$$\begin{aligned}
q^n(q^n-1)(q^{(n-1)^2}-1)-q^{n^2}+1 &= q^n(q^{n^2-n+1}-q^{n^2-2n+1}-q^n+1)-q^{n^2}+1 \\
&= q^{n^2+1}-q^{n^2-n+1}-q^{n^2}-q^{2n}+q^n+1 \\
&= q^{n^2-n+1}\big(q^{n-1}(q-1)-1\big)-q^{2n}+q^n+1.
\end{aligned}$$

Since $n\geq 4$ and $q\geq 2$,

$$q^{n^2-n+1}\big(q^{n-1}(q-1)-1\big)-q^{2n}\geq q^{n^2-n+1}-q^{2n}=q^{2n}(q^{n^2-3n+1}-1)\geq 0.$$

This completes the proof of the next theorem.

Theorem 3.1. $\mathrm{C}_{\mathrm{M}_n(F)}$ *is hyperenergetic for all* $n\geq 2$.

Recall that a k-regular graph is Ramanujan if $|\lambda|\leq 2\sqrt{k-1}$ for all eigenvalues λ other than $\pm k$. Since eigenvalues of a graph are real numbers, this inequality is equivalent to $\lambda^2-4(k-1)\leq 0$. We know that $\mathrm{C}_{\mathrm{M}_n(F)}$ is regular with parameter $k=(q^n-1)(q^n-q)\dots(q^n-q^{n-1})$. If $n=2$, then its eigenvalues are $(q^2-1)(q^2-q), -(q^2-q)$ and q. Since $q\geq 2$, we have $q^2-q\geq 2$, so

$$q^2+4\leq 4q^2 \quad \text{and} \quad (q^2-q)^2+4\leq 4(q^2-q).$$

The first inequality gives $q^2+4\leq 4q(q+1)(q-1)^2$ which is equivalent to $q^2-4(q^2-1)(q^2-q)+4\leq 0$ and the second inequality directly proves $(q^2-q)^2<4(q^2-1)(q^2-q)-4$. Thus, $\mathrm{C}_{\mathrm{M}_2(F)}$ is Ramanujan. Now suppose that $n\geq 3$ and $\mathrm{C}_{\mathrm{M}_n(F)}$ is a Ramanujan graph. From the computation in the previous section, $\rho_{A_3}=(q^n-q)(q^n-q^2)\dots(q^n-q^{n-1})$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_n(F)}$, so

$$0 \geq \rho_{A_3}^2 - 4(q^n-1)(q^n-q)\dots(q^n-q^{n-1})+4 = \rho_{A_3}^2 - 4(q^n-1)\rho_{A_3}+4 = (\rho_{A_3}+2)^2 - 4q^n\rho_{A_3}.$$

It follows that $4q^n\rho_{A_3} \geq (\rho_{A_3}+2)^2 > \rho_{A_3}^2$, so $4q^n > \rho_{A_3}$. For $n = 3$, this must imply that $q = 2$ and for $n \geq 4$, we have $n + 2 \leq \frac{(n-1)n}{2}$ and so

$$4q^n > \rho_{A_3} = q^{\frac{(n-1)n}{2}}(q^{n-1}-1)(q^{n-2}-1)\dots(q-1) > q^{\frac{(n-1)n}{2}}$$

which leads to a contradiction for all $q \geq 2$. Finally, if $n = 3$ and $q = 2$, by Theorem 2.3, we have $-(2^3-2)(2^3-2^2) = -24, 2(2^3-2^2) = 8$ and $-2^3 = -8$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_3(\mathbb{Z}_2)}$ and $4((2^3-1)(2^3-2)(2^3-2^2)-1) = 668$ is greater than 24^2 and 8^2. Hence, $\mathrm{C}_{\mathrm{M}_3(\mathbb{Z}_2)}$ is also Ramanujan.

We record this result in the following theorem.

Theorem 3.2. *The graph $\mathrm{C}_{\mathrm{M}_n(F)}$ is Ramanujan if and only if $n = 2$ or ($n = 3$ and $F = \mathbb{Z}_2$).*

4. The unitary Cayley graph of product of matrix rings over finite local rings

Let R be a local ring with unique maximal ideal M and residue field $\Bbbk$. Recall that $R/M \cong \Bbbk$ results in $\mathrm{M}_n(R)/\mathrm{M}_n(M) \cong \mathrm{M}_n(\Bbbk)$. Then elements in R can be partitioned into cosets of M and can be viewed as lifting from elements of $\Bbbk$. Suppose $|M| = m$ and $|\Bbbk| = q$. We fix $A_1, \dots, A_{q^{n^2}}$ to be coset representatives of $\mathrm{M}_n(M)$ in $\mathrm{M}_n(R)$.

Lemma 4.1. *Let $A \in \mathrm{M}_n(R)$ and $X \in \mathrm{M}_n(M)$. Then*

$$\det(A+X) = (\det A) + m \textit{ for some } m \in M.$$

In particular, A is invertible if and only if $A + X$ is invertible.

Proof. Write $A = [a_{ij}]_{n\times n}$ and $X = [m_{ij}]_{n\times n}$. Then

$$\begin{aligned}\det(A+X) &= \sum_{\sigma\in S_n}(\operatorname{sgn}\sigma)(a_{1\sigma(1)}+m_{1\sigma(1)})\dots(a_{n\sigma(n)}+m_{n\sigma(n)})\\ &= \sum_{\sigma\in S_n}(\operatorname{sgn}\sigma)(a_{1\sigma(1)}\dots a_{n\sigma(n)}) + m = (\det A) + m\end{aligned}$$

for some $m \in M$. $\square$

The above lemma directly implies the following theorem.

Theorem 4.2.

(1) *For $A, B \in \mathrm{M}_n(R)$, A and B are adjacent in $\mathrm{C}_{\mathrm{M}_n(R)}$ if and only if $A + \mathrm{M}_n(M)$ and $B + \mathrm{M}_n(M)$ are adjacent in $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$.*

(2) *The set* $\mathrm{M}_n(R)/\mathrm{M}_n(M) = \{A_1 + \mathrm{M}_n(M), \dots, A_{q^{n^2}} + \mathrm{M}_n(M)\}$ *is a partition of the vertex set of* $\mathrm{C}_{\mathrm{M}_n(R)}$ *such that*
   (a) *for each* $i \in \{1, \dots, q^{n^2}\}$*, any two distinct vertices in* $A_i + \mathrm{M}_n(M)$ *are nonadjacent vertices, and*
   (b) *for* $i, j \in \{1, \dots, q^{n^2}\}$*,* A_i *and* A_j *are adjacent in* $\mathrm{C}_{\mathrm{M}_n(R)}$ *if and only if* $A_i + \mathrm{M}_n(M)$ *and* $A_j + \mathrm{M}_n(M)$ *are adjacent in* $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$*.*

(3) *Let* $\mathring{\mathrm{M}}_n(M)$ *be the complete graph of* $|\mathrm{M}_n(M)|$ *vertices with a loop on every vertex. Define* $f : \mathrm{M}_n(\Bbbk) \times \mathrm{M}_n(M) \to \mathrm{M}_n(R)$ *by* $f(A_i + \mathrm{M}_n(M), X) = A_i + X$ *for all* $i \in \{1, \dots, q^{n^2}\}$ *and* $X \in \mathrm{M}_n(M)$*. Then* f *is an isomorphism from the graph* $\mathrm{C}_{\mathrm{M}_n(\Bbbk)} \otimes \mathring{\mathrm{M}}_n(M)$ *onto the graph* $\mathrm{C}_{\mathrm{M}_n(R)}$*.*

Proof. The above discussion implies (1) and (2) For (3), we first show that f is an injection. Let $i, j \in \{1, \dots, q^{n^2}\}$ and $X, Y \in \mathrm{M}_n(M)$ such that $A_i + X = A_j + Y$. Then $A_i - A_j = Y - X \in \mathrm{M}_n(M)$. This forces that $A_i + \mathrm{M}_n(M) = A_j + \mathrm{M}_n(M)$ in $\mathrm{M}_n(\Bbbk)$, so $i = j$ and $X = Y$. Since $|\mathrm{M}_n(\Bbbk) \times \mathrm{M}_n(M)| = |\mathrm{M}_n(R)|$, f is a bijection. Finally, for $i, j \in \{1, \dots, q^{n^2}\}$ and $X, Y \in \mathrm{M}_n(M)$, we have $(A_i + \mathrm{M}_n(M), X)$ and $(A_j + \mathrm{M}_n(M), Y)$ are adjacent in $\mathrm{C}_{\mathrm{M}_n(\Bbbk)} \otimes \mathring{\mathrm{M}}_n(M)$ if and only if $A_i + \mathrm{M}_n(M)$ and $A_j + \mathrm{M}_n(M)$ are adjacent if and only if A_i and A_j are adjacent by (2). Hence, f is a graph isomorphism. □

Next, we assume that R is a finite local ring which is not a field with unique maximal ideal M and residue field $\Bbbk$. Let $|M| = m$ and $|\Bbbk| = q$. Since the adjacency matrix of $\mathring{\mathrm{M}}_n(M)$ is the all-ones matrix of size m^{n^2}, we have $\mathrm{Spec}\left(\mathring{\mathrm{M}}_n(M)\right) = \begin{pmatrix} m^{n^2} & 0 \\ 1 & m^{n^2} - 1 \end{pmatrix}$ and $(q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$, $-(q^n - q) \dots (q^n - q^{n-1})$ and $q(q^n - q^2) \dots (q^n - q^{n-1})$ are eigenvalues of $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$. Since the eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$ where λ_i's and μ_j's are eigenvalues of G and H, respectively, we can conclude from the isomorphism in Theorem 4.2 (3) that $0, m^{n^2}(q^n - 1)(q^n - q) \dots (q^n - q^{n-1}), -m^{n^2}(q^n - q) \dots (q^n - q^{n-1})$ and $m^{n^2} q(q^n - q^2) \dots (q^n - q^{n-1})$ are distinct eigenvalues of $\mathrm{C}_{\mathrm{M}_n(R)}$. Then we have shown the following theorem.

Theorem 4.3. *If* R *is a local ring which is not a field and* $n \geq 2$*, then* $\mathrm{C}_{\mathrm{M}_n(R)}$ *is not strongly regular.*

However, it turns out that the graph $\mathrm{C}_{\mathrm{M}_n(R)}$ is hyperenergetic.

Theorem 4.4. *If* R *is a local ring, then* $\mathrm{C}_{\mathrm{M}_n(R)}$ *is hyperenergetic for all* $n \geq 2$*.*

Proof. Let $\Bbbk$ be the residue field of R and assume that $|\Bbbk| = q$. Recall that $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$ is hyperenergetic and $\mathrm{C}_{\mathrm{M}_n(R)}$ has $-m^{n^2} q(q^n - q^2) \dots (q^n - q^{n-1})$ as an eigenvalue with multiplicities at least $\dfrac{(q^n - 1)^2 (q^n - q)^2}{(q^2 - 1)(q^2 - q)}$. The proof of Theorem 3.1 tells us that

$$q(q^n - q^2)\dots(q^n - q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} > 2(q^{n^2}-1).$$

Note that the left-hand side is a multiple of q. It follows that

$$q(q^n - q^2)\dots(q^n - q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} \geq 2q^{n^2}.$$

Multiplying by m^{n^2} both sides gives

$$m^{n^2}q(q^n - q^2)\dots(q^n - q^{n-1})\frac{(q^n-1)^2(q^n-q)^2}{(q^2-1)(q^2-q)} \geq 2(mq)^{n^2} > 2((mq)^{n^2}-1)$$

which completes the proof. $\square$

Theorem 4.5. *If R is a local ring which is not a field, then $\mathrm{C}_{\mathrm{M}_n(R)}$ is not Ramanujan for all $n \geq 2$.*

Proof. For simplicity, let $r = |\,\mathrm{GL}_n(\Bbbk)|$. We first handle case $n \geq 3$ and $q \geq 3$. Then $\mathrm{C}_{\mathrm{M}_n(\Bbbk)}$ is not Ramanujan by Theorem 3.2. From the proof of Theorem 3.2, we have $(q^n - q)\dots(q^n - q^{n-1}) \geq 2\sqrt{r-1}$. Thus,

$$m^{n^2}(q^n - q)\dots(q^n - q^{n-1}) \geq 2m^{n^2}\sqrt{r-1},$$

so we must show that $m^{n^2}\sqrt{r-1} > \sqrt{m^{n^2}r-1}$. Rewrite

$$m^{2n^2}(r-1) - (m^{n^2}r-1) = (m^{n^2}-1)(m^{n^2}r - m^{n^2}-1).$$

Since R is not a field, we have $m \geq 2$, so $(m^{n^2}-1)(m^{n^2}r - m^{n^2}-1) > 0$ and the desired inequality follows. Next, we assume that $n = 3$ and $q = 2$. Then $-m^9(2^3-2)(2^3-2^2) = -24m^9$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_3(R)}$. Moreover, $r = m^9(2^3-1)(2^3-2)(2^3-2^2) = 168m^9$. We have $576m^{18} - 4(168m^9-1) = m^9(576m^9 - 672) + 4$. Since $m \geq 2$, we get $24m^9 > 2\sqrt{168m^9-1}$. Finally, if $n = 2$, then $-m^4(q^2-q)$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_2(R)}$ and $r = m^4(q^2-1)(q^2-q)$, so

$$\begin{aligned} m^8(q^2-q)^2 - 4(m^4(q^2-1)(q^2-q)-1) &= m^8(q^2-q)^2 - 4m^4(q^2-1)(q^2-q) + 4 \\ &\geq m^8(q^2-q)^2 - 4m^4(q^2-q)^2 + 4 \\ &= (m^8 - 4m^4)(q^2-q)^2 + 4 > 0 \end{aligned}$$

because $m \geq 2$. Hence, $\mathrm{C}_{\mathrm{M}_2(R)}$ is not Ramanujan. $\square$

Let $R_1, \dots, R_s$ be finite local rings with maximal ideals $M_1, \dots, M_s$ and residue fields $\Bbbk_1, \dots, \Bbbk_s$, respectively. Let $\mathcal{R} = \mathrm{M}_{n_1}(R_1) \times \dots \times \mathrm{M}_{n_s}(R_s)$ where $n_1, \dots n_s \in \mathbb{N}$. By Theorem 3.8 of [6], we have

$$\chi(\mathrm{C}_{\mathcal{R}}) = \omega(\mathrm{C}_{\mathcal{R}}) = \omega(\mathrm{C}_{\mathrm{M}_{n_1}(\mathbb{k}_1)\times\cdots\times\mathrm{M}_{n_k}(\mathbb{k}_k)}) = \min_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}.$$

Finally, we compute $\alpha(\mathrm{C}_{\mathcal{R}})$. Theorem 4.2 (3) gives

$$\mathrm{C}_{\mathcal{R}} \cong \left(\mathrm{C}_{\mathrm{M}_{n_1}(\mathbb{k}_1)} \otimes \cdots \otimes \mathrm{C}_{\mathrm{M}_{n_s}(\mathbb{k}_s)}\right) \otimes \left(\mathring{\mathrm{M}}_{n_1}(M_1) \otimes \cdots \otimes \mathring{\mathrm{M}}_{n_s}(M_s)\right).$$

Since the second product is a complete graph with a loop on each vertex, we can see that

$$\begin{aligned}\alpha(\mathrm{C}_{\mathcal{R}}) &= \alpha(\mathrm{C}_{\mathrm{M}_{n_1}(\mathbb{k}_1)} \otimes \cdots \otimes \mathrm{C}_{\mathrm{M}_{n_s}(\mathbb{k}_s)}) \prod_{i=1}^{s} |\mathrm{M}_{n_i}(M_i)| \\ &= \frac{\prod_{i=1}^{s} |\mathrm{M}_{n_i}(\mathbb{k}_i)|}{\min\limits_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}} \prod_{i=1}^{s} |\mathrm{M}_{n_i}(M_i)| = \frac{|\mathcal{R}|}{\min\limits_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}}.\end{aligned}$$

Thus, we prove:

Theorem 4.6. $\omega(\mathrm{C}_{\mathcal{R}}) = \chi(\mathrm{C}_{\mathcal{R}}) = \min\limits_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}$ *and* $\alpha(\mathrm{C}_{\mathcal{R}}) = \dfrac{|\mathcal{R}|}{\min\limits_{1\le i\le s}\{|\mathbb{k}_i|^{n_i}\}}$.

For each $1 \le i \le s$, let $|M_i| = m_i$ and $|\mathbb{k}_i| = q_i$. Recall that $\rho_i = -{m_i}^{{n_i}^2} q_i({q_i}^n - {q_i}^2)\dots({q_i}^n - {q_i}^{n-1})$ is an eigenvalue of $\mathrm{C}_{\mathrm{M}_{n_i}(R_i)}$ with multiplicities at least t_i where $t_i = \dfrac{({q_i}^n-1)^2({q_i}^n-q_i)^2}{({q_i}^2-1)({q_i}^2-q_i)}$ for all i. Hence, $\prod_{i=1}^{s}\rho_i$ is an eigenvalue of $\mathrm{C}_{\mathcal{R}}$ with multiplicities at least $\prod_{i=1}^{s} t_i$. By Theorem 4.3, we have $\rho_i t_i > 2(|\,\mathrm{M}_{n_i}(R_i)| - 1)$ for all $1 \le i \le s$. Note that the left-hand side is a multiple of q_i. We can conclude that $\rho_i t_i \ge 2|R_i|^{{n_i}^2}$. It follows that

$$\prod_{i=1}^{s}\rho_i \prod_{i=1}^{s} t_i = \prod_{i=1}^{s}\rho_i t_i \ge \prod_{i=1}^{s} 2|\,\mathrm{M}_{n_i}(R_i)| = 2^s \prod_{i=1}^{s} |\,\mathrm{M}_{n_i}(R_i)| > 2\left(\prod_{i=1}^{s} |\,\mathrm{M}_{n_i}(R_i)| - 1\right).$$

This proves

Theorem 4.7. *The graph* $\mathrm{C}_{\mathcal{R}}$ *is hyperenergetic. In particular, if* R *is a finite commutative ring, then* $\mathrm{C}_{\mathrm{M}_n(R)}$ *is hyperenergetic for all* $n \ge 2$.

Remark. The later statement comes from the fact that every finite commutative ring is isomorphic to a direct product of finite local rings. Indeed, we can use this fact and Theorem 4.6 to compute the clique number, chromatic number and independence number for the unitary Cayley graph of a matrix ring over a finite commutative ring.

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Subconstituents of unitary Cayley graph of matrix algebras ☆

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Let F be a finite field, $n \in \mathbb{N}$ and $\mathrm{C}_{\mathrm{M}_n(F)}$ denote the unitary Cayley graph of the matrix algebra $\mathrm{M}_n(F)$. In this paper, we study the first and second subconstituents of $\mathrm{C}_{\mathrm{M}_n(F)}$. We determine the spectra of the subconstituents of $\mathrm{C}_{\mathrm{M}_2(F)}$ by using the character table of $\mathrm{GL}_2(F)$ and elementary linear algebra, and conclude their hyperenergeticity and Ramanujan property. Moreover, we compute the clique number, the chromatic number and the independence number of those subconstituents.

1. Introduction

Let G be a finite abelian group and S be a subset of G not containing the identity and $S = S^{-1}$ where $S^{-1} = \{s^{-1} : s \in S\}$. The *Cayley graph of G associated to S* is the

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undirected graph $\mathrm{Cay}(G, S)$ whose vertex set is G and for each $g, h \in G$, g is adjacent to h if and only if $g = hs$ for some $s \in S$. We say that a Cayley graph is *normal* if S is a union of conjugacy classes of G.

For a finite ring R with identity $1 \neq 0$, we know that $(R, +)$ is an abelian group and we denote its group of units by $R^\times$. The *unitary Cayley graph of* R, C_R, is the graph $\mathrm{Cay}(R, R^\times)$, that is, its vertex set is R and for each $x, y \in R$, x is adjacent to y if and only if $x - y \in R^\times$. Many properties of the unitary Cayley graphs have been extensively studied by many authors such as [1,5,9,11,14]. Since a finite commutative ring R can be decomposed as a direct product of finite local rings $R_1, \dots, R_s$, the graph C_R is the tensor product of the graphs $\mathrm{C}_{R_1}, \dots, \mathrm{C}_{R_s}$. Here, for graphs G and H with vertex sets $V(G)$ and $V(H)$, the *tensor product* G *and* H, $G \otimes H$, is the graph with vertex set $V(G) \times V(H)$ such that (a, b) is adjacent to (c, d) if and only if a is adjacent to c in G and b is adjacent to d in H for all $a, c \in V(G)$ and $b, d \in V(H)$. In addition, if R is a finite local ring with maximal ideal M, it follows from Proposition 2.2 of [1] that C_R is a complete multi-partite graph whose partite sets are the cosets of M. Thus, the unitary Cayley graphs of finite commutative rings are well studied. Their spectral properties including energies are completely determined in [11].

Let G be a graph and $V(G)$ the vertex set of G. We give some terminologies from graph theory as follows. A *clique* is a subgraph that is a complete graph and *clique number* of G is the size of largest clique in G, denoted by $\omega(G)$. A set I of vertices of G is called an *independent set* if no distinct vertices of I are adjacent. The *independence number* of G is the size of a maximal independent set, denoted by $\alpha(G)$. The *chromatic number of* G is the least number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. We write $\chi(G)$ for the chromatic number of G. If every vertex of G is adjacent to k vertices, then G is a k-regular graph. Clearly, the above Cayley graph associated to a set S is a $|S|$-regular graph. Finally, we say that a k-regular graph G is *edge regular* if there exists a parameter λ such that for any two adjacent vertices, there are exactly λ vertices adjacent to both of them. If an edge regular graph with parameters k, λ also satisfies an additional property that for any two non-adjacent vertices, there are exactly μ vertices adjacent to both of them, then it is called a *strongly regular graph with parameters* k, λ, μ.

An *eigenvalue* of a graph G is an eigenvalue of the adjacency matrix of a graph G. The *energy* of a graph G, $E(G)$, is the sum of absolute value of its eigenvalues. The *spectrum* of a graph G is the list of its eigenvalues together with their multiplicities. If $\lambda_1, \dots, \lambda_r$ are eigenvalues of a graph G with multiplicities $m_1, \dots, m_r$, respectively, we write $\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \dots & \lambda_r \\ m_1 & \dots & m_r \end{pmatrix}$ to describe the spectrum of G and so $E(G) = m_1|\lambda_1| + \cdots + m_r|\lambda_r|$. A graph G on n vertices is *hyperenergetic* if its energy exceeds the energy of the complete graph K_n, that is, $E(G) > 2(n-1)$. A k-regular connected graph G is a *Ramanujan graph* if $|\lambda| \leq 2\sqrt{k-1}$ for all eigenvalues λ of G other than

$\pm k$. Spectral gap of Ramanujan graph is almost as large as possible, so it is a great spectral expander. Its name comes from Lubotzky, Phillips and Sarnak [15] who used the Ramanujan conjecture to construct an infinite family of such graphs.

For a ring R with identity $1 \neq 0$ and $n \in \mathbb{N}$, $\mathrm{M}_n(R)$ is the ring of $n \times n$ matrices over R and the group of all invertible matrices over R is denoted by $\mathrm{GL}_n(R)$. Throughout this work, I_n is the $n \times n$ identity matrix and $\mathbf{0}_{m\times n}$ is the $m \times n$ zero matrix for all $m, n \in \mathbb{N}$.

Kiani et al. [12] worked on the unitary Cayley graph of the ring $\mathrm{M}_{n_1}(F_1) \times \cdots \times \mathrm{M}_{n_k}(F_k)$ where $n_1, \ldots, n_k \in \mathbb{N}$ and $F_1, \ldots, F_k$ are finite fields. They computed the clique number, the chromatic number and the independence number of the graph. Later in [13], they proved that $C_{\mathrm{M}_n(F)}$ is an edge regular graph with $k = |\mathrm{GL}_n(F)|$ and $\lambda = |(\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)| = e_n$ where F is a finite field and e_n is the number of derangement matrices. (A *derangement matrix* in $\mathrm{M}_n(F)$ is an invertible matrix that does not fix any nonzero vectors in F^n.) We know from [16] that if $|F| = q$, then e_n satisfies the recursion $e_n = e_{n-1}(q^n - 1)q^{n-1} + (-1)^n q^{n(n-1)/2}$ and $e_0 = 1$. Kiani proved further that $C_{\mathrm{M}_2(F)}$ is strongly regular with $\mu = \left|\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mathrm{GL}_2(F)\right) \cap \mathrm{GL}_2(F)\right|$ but $C_{\mathrm{M}_3(F)}$ is not strongly regular where F is a finite field. Recently, the authors [17] used additive characters of $\mathrm{M}_n(F)$ where F is a finite field and $n \in \mathbb{N}$ to determine three distinct eigenvalues of $C_{\mathrm{M}_n(F)}$ and use them to conclude that the graph $C_{\mathrm{M}_n(F)}$ is strongly regular if and only if $n = 2$. We also showed that $C_{\mathrm{M}_n(F)}$ is always hyperenergetic and gave a criterion for being a Ramanujan graph. Chen et al. [3] obtained all eigenvalues of $C_{\mathrm{M}_n(F)}$ using Bruhat's decomposition.

Let G be a graph and x a vertex of G. The *first subconstituent of G at x* is the subgraph of G induced by the set of neighborhoods of x and the *second subconstituent of G at x* is the subgraph of G induced by the set of vertices which is non-adjacent to x except x itself. Subconstituents of strongly regular graphs are studied in many graphs and have many interesting properties. The second subconstituent of the Hoffman-Singleton graph is determined by its spectrum in [6]. Moreover, the discovery of which graph has strongly regular subconstituents interests mathematicians. For example, Cameron et al. [4] used the Bose-Mesner algebra of a strongly regular graph to classify strongly regular graphs whose subconstituents are strongly regular, and Kasikova [10] used the same tools to classify distance-regular graph which has strongly regular subconstituents. In addition, we can use eigenvalues of subconstituents to prove the uniqueness of strongly regular of some parameter, e.g., Clebsch graph is a unique strongly regular graph with parameters $(16, 5, 0, 2)$ (see [8] p.230).

Now, we turn to the subconstituents of the unitary Cayley graph. Let R be a finite ring with identity $1 \neq 0$. The set of neighborhood of a vertex x of the graph C_R is denoted by $N(x)$. For $x \in R$, the maps $f : N(0) \to N(x)$ and $g : R \smallsetminus (N(0) \cup \{0\}) \to R \smallsetminus (N(x) \cup \{x\})$ which both send y to $x - y$ are graph isomorphisms. Hence, we may only study the subconstituents at $x = 0$ and we write $\mathrm{C}_R^{(1)}$ and $\mathrm{C}_R^{(2)}$ for the first subconstituent and the second subconstituent of C_R at $x = 0 \in R$, respectively. Let F be a finite field and $n \in \mathbb{N}$. In this work, we study $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ and $\mathrm{C}^{(2)}_{\mathrm{M}_n(F)}$. The graph $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is defined on the group $\mathrm{GL}_n(F)$ and the graph $\mathrm{C}^{(2)}_{\mathrm{M}_n(F)}$ is defined on the set of nonzero non-invertible matrices over F. We have the structure of $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ and $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$. We can determine the spectra of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ and $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ and conclude hyperenergeticity and Ramanujan property for both graphs. In addition, we compute the clique number, the chromatic number and the independence number of $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ and $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Next, we recall some results from representation theory used in this work. We refer the reader to [7] for more detail. Let G be a finite group and V a finite-dimensional complex vector space. A *representation* of G on V is a homomorphism $\rho : G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ denotes the group of automorphisms on V. For a representation ρ of G on V, a subspace W of V is *ρ-invariant* under G if $\rho(g)(W) \subseteq W$ for all $g \in G$. If ρ has no proper invariant subspace of V, then we say that ρ is an *irreducible representation*. Next, we define a character of a representation. Let ρ be a representation of G on V. Then for each $g \in G, \rho(g)$ is a linear transformation on V. A *character* χ corresponding to ρ is the complex-valued function on G defined by $\chi(g) = \mathrm{tr}(\rho(g))$ for all $g \in G$ where $\mathrm{tr}(\rho(g))$ is the trace of the matrix representation of $\rho(g)$ on V. A character is said to be *irreducible* if they are induced from an irreducible representation. The *dimension* of a character is the dimension of vector space V. It is easy to see that $\chi(1) = \dim V$ where 1 is the identity of the group G, and $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$. Thus, a character is a constant on a conjugacy class of G. Moreover, we have known from [18] that if S is a union of conjugacy classes of G and $\chi_1, \dots, \chi_r$ are irreducible characters of G, then the eigenvalues of $\mathrm{Cay}(G, S)$ are

$$\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s)$$

with multiplicity $m_j = \sum_{\substack{k=1 \\ \lambda_k = \lambda_j}}^{r} \chi_k(1)^2$ for all $j \in \{1, \dots, r\}$.

Let F be the finite field of q elements. Recall that the multiplicative group of nonzero elements of F is cyclic. Write $F^\times = \langle a \rangle$ for some $a \in F^\times$. The irreducible characters of the group $(F^\times, \cdot)$ are $\chi_k(x) = e^{\frac{2\pi i m k}{q-1}}$, where $x = a^m \in F^\times$ and $k \in \{0, 1, 2, \dots, q-2\}$. In addition, we have

Theorem 1.1. *For $k \in \{0, 1, \dots, q-2\}$,* $\displaystyle\sum_{x \in F^\times} \chi_k(x) = \begin{cases} q-1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$

The conjugacy classes of $\mathrm{GL}_2(F)$ are given in the following table.

Representatives	Number of elements	Number of classes
$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, x \neq 0$	1	$q-1$
$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, x \neq 0$	q^2-1	$q-1$
$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y$ and $x, y \neq 0$	q^2+q	$\frac{(q-1)(q-2)}{2}$
$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, y \neq 0$ (q is odd) $d_z = \begin{pmatrix} 0 & z^{q+1} \\ 1 & z+z^q \end{pmatrix}, z \in E \smallsetminus F$ (q is even)	q^2-q	$\frac{q(q-1)}{2}$

where $\varepsilon \in F \smallsetminus F^2$. Here, $c_{x,y}$ and $c_{y,x}$ are conjugate, $d_{x,y}$ and $d_{x,-y}$ are conjugate, and d_z and d_{z^q} are conjugate. Moreover, let $E = F[\sqrt{\varepsilon}]$ an extension of F of degree two. We can identify the matrices $d_{x,y}$ as $\zeta = x + y\sqrt{\varepsilon}$ and the matrices d_z as z in $E \smallsetminus F$. Now, let α, β be distinct irreducible character of $F^\times$ and φ an irreducible characters of $E^\times$ such that $\varphi^q \neq \varphi$ and φ is not an irreducible character of $F^\times$. The next table presents all irreducible characters of $\mathrm{GL}_2(F)$. As mentioned earlier, it suffices to specify their values on each conjugacy class of $\mathrm{GL}_2(F)$.

	$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	q is odd $d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \zeta$	q is even $d_z = \begin{pmatrix} 0 & z^{q+1} \\ 1 & z+z^q \end{pmatrix} = z$
U_α	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(\zeta^q)$	$\alpha(z^q)$
V_α	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(\zeta^q)$	$-\alpha(z^q)$
$W_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0	0
X_φ	$(q-1)\varphi(x)$	$-\varphi(x)$	0	$-(\varphi(\zeta)+\varphi(\zeta^q))$	$-(\varphi(z)+\varphi(z^q))$

Moreover, $U_\alpha, V_\alpha, W_{\alpha,\beta}$ and X_φ are of dimension $1, q, q+1$ and $q-1$, respectively.

The paper is organized as follows. In the next section, we prove that the graph $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is a normal Cayley graph and we determine all eigenvalues of the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ by using the two tables above. We show further that it is hyperenergetic and Ramanujan if $q \geq 3$. In Section 3, we show that the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ is the tensor product between a complete graph and a complete multi-partite graph, and obtain its spectrum. We apply this result to conclude that $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is hyperenergetic but it is not Ramanujan if $q \geq 5$. We compute the clique number, chromatic number and the independence number of the subconstituents of the graph $\mathrm{C}_{\mathrm{M}_2(F)}$ in the final section.

2. Spectral properties of $\mathbf{C}^{(1)}_{\mathbf{M_2}(F)}$

In this section, we study spectral properties of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$. We start by showing that $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is $\mathrm{Cay}\left(\mathrm{GL}_n(F), (\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)\right)$. To see this, let $A, B \in \mathrm{GL}_n(F)$. Then $AB^{-1} \in \mathrm{GL}_n(F)$ and

$$\begin{aligned} A - B \in GL_n(F) &\iff (AB^{-1} - \mathrm{I}_n)B^{-1} \in \mathrm{GL}_n(F) \\ &\iff (AB^{-1} - \mathrm{I}_n) \in \mathrm{GL}_n(F) \end{aligned}$$

$$\iff AB^{-1} \in (\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F).$$

It also follows that the graph $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is regular of degree $|(\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)| = e_n$, defined in the previous section. Moreover, for $A, B \in \mathrm{GL}_n(F)$, we have

$$\begin{aligned} ABA^{-1} \in (\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F) &\iff ABA^{-1} - \mathrm{I}_n \in \mathrm{GL}_n(F) \\ &\iff A(B - \mathrm{I}_n)A^{-1} \in \mathrm{GL}_n(F) \\ &\iff (B - \mathrm{I}_n) \in \mathrm{GL}_n(F) \\ &\iff B \in (\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F). \end{aligned}$$

Thus, $(\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)$ is a union of conjugacy classes, so $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is a normal Cayley graph. We record this result in

Theorem 2.1. *The graph $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$ is the normal Cayley graph of $\mathrm{GL}_n(F)$ associated with $(\mathrm{I}_n + \mathrm{GL}_n(F)) \cap \mathrm{GL}_n(F)$ and it is regular of degree e_n.*

Next, we determine all eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$. Let $k \in \{0, 1, \dots, q-2\}$ and consider χ_k an irreducible character of $F^\times$. We first handle the case q is odd by showing some lemmas on sums of characters of $F^\times$.

Lemma 2.2. *If q is odd, then for $k \in \{0, 1, \dots, q-2\}$,*

$$\sum_{x \in F^\times} \chi_k(x^2) = \begin{cases} q-1 & \text{if } k \in \left\{0, \dfrac{q-1}{2}\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We know that

$$\sum_{x \in F^\times} \chi_k(x^2) = \sum_{m=0}^{q-2} \chi_k(a^{2m}) = \sum_{m=0}^{q-2} e^{\frac{4\pi imk}{q-1}} = \sum_{m=0}^{q-2} \left(e^{\frac{4\pi ik}{q-1}}\right)^m.$$

Note that $e^{\frac{4\pi ik}{q-1}} = 1$ if and only if $k = 0$ or $k = \dfrac{q-1}{2}$. If $k \in \left\{0, \dfrac{q-1}{2}\right\}$, then $\sum_{x \in F^\times} \chi_k(x^2) = q-1$. Finally, if $k \notin \left\{0, \dfrac{q-1}{2}\right\}$, then

$$\sum_{x \in F^\times} \chi_k(x^2) = \frac{1 - \left(e^{\frac{4\pi ik}{q-1}}\right)^{q-1}}{1 - \left(e^{\frac{4\pi ik}{q-1}}\right)} = 0,$$

and the proof completes. $\square$

Lemma 2.3. *If q is odd, then for $k \in \{0, 1, \ldots, q-2\}$ and $\varepsilon \in F \setminus F^2$, we have*

(a) $$\sum_{\substack{x,y \in F^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) = \begin{cases} q^2 - 5q + 6 & \text{if } k = 0, \\ -q + 3 & \text{if } k = \dfrac{q-1}{2}, \text{ and} \\ 2 & \text{otherwise,} \end{cases}$$

(b) $$\sum_{(x,y) \in F \times F^\times} \chi_k(x^2 - \varepsilon y^2) = \begin{cases} q^2 - q & \text{if } k = 0, \\ -q + 1 & \text{if } k = \dfrac{q-1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We note that

$$\begin{aligned} \sum_{\substack{x,y \in F^\times \setminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) &= \left(\sum_{x \in F^\times} \chi_k(x) \right) \left(\sum_{y \in F^\times} \chi_k(y) \right) - \sum_{x \in F^\times} \chi_k(x^2) \\ &\qquad - \sum_{x \in F^\times \setminus \{1\}} \chi_k(x) - \sum_{y \in F^\times \setminus \{1\}} \chi_k(y) \\ &= \left(\sum_{x \in F^\times} \chi_k(x) \right)^2 - \left(\sum_{x \in F^\times} \chi_k(x^2) \right) - 2 \left(\sum_{x \in F^\times} \chi_k(x) \right) + 2. \end{aligned}$$

If $k = 0$, then applying Lemma 2.2 gives the right-hand side equals $q^2 - 5q + 6$. If $k = \frac{q-1}{2}$, then the right-hand side is $-q + 3$. Finally, if $k \notin \{0, \frac{q-1}{2}\}$, then the summands on the right-hand side are all gone and we get 2 left. This proves (a).

For (b), since $\varepsilon \in F \setminus F^2$, $E = F[\sqrt{\varepsilon}]$ an extension of degree two of F. Thus, $E = \{x + y\sqrt{\varepsilon} : x, y \in F\}$. Moreover, let $\mathrm{N}_{E/F}$ be the norm map. Recall that for $x, y \in F$, $\mathrm{N}_{E/F}(x + y\sqrt{\varepsilon}) = x^2 - \varepsilon y^2$ and by Hilbert's Theorem 90, $\mathrm{N}_{E/F}$ is surjective with kernel of size $q + 1$. Consider the sum

$$\begin{aligned} \sum_{(x,y) \in F \times F^\times} \chi_k(x^2 - \varepsilon y^2) &= \sum_{(x,y) \in F \times F \setminus \{(0,0)\}} \chi_k(x^2 - \varepsilon y^2) - \sum_{x \in F^\times} \chi_k(x^2) \\ &= \sum_{(x,y) \in F \times F \setminus \{(0,0)\}} \chi_k(\mathrm{N}_{E/F}(x + y\sqrt{\varepsilon})) - \sum_{x \in F^\times} \chi_k(x^2) \\ &= \left| \ker \mathrm{N}_{E/F} \right| \sum_{x \in F^\times} \chi_k(x) - \sum_{x \in F^\times} \chi_k(x^2) \\ &= (q+1) \sum_{x \in F^\times} \chi_k(x) - \sum_{x \in F^\times} \chi_k(x^2). \end{aligned}$$

If $k = 0$, then the right-hand side becomes $q^2 - q$, and if $k = \dfrac{q-1}{2}$, then the right-hand side is $-(q-1)$ by Lemma 2.2. Finally, for $k \notin \left\{0, \dfrac{q-1}{2}\right\}$, it also follows that each summand on the right-hand side is 0. $\square$

Lemma 2.4. *For $k, l \in \{0, 1, \ldots, q-2\}$ such that $k \neq l$, we have*

$$\sum_{\substack{x,y\in F^\times\setminus\{1\}\\ \textit{and } x\neq y}} [\chi_k(x)\chi_l(y) + \chi_k(y)\chi_l(x)] = \begin{cases} 4 & \textit{if } 0 < k+l < q-1, k, l \neq 0, \\ 2(3-q) & \textit{otherwise.} \end{cases}$$

Proof. We consider the sum

$$\sum_{\substack{x,y\in F^\times\setminus\{1\}\\ \text{and } x\neq y}} [\chi_k(x)\chi_l(y) + \chi_k(y)\chi_l(x)] = 2 \sum_{\substack{x,y\in F^\times\setminus\{1\}\\ \text{and } x\neq y}} \chi_k(x)\chi_l(y)$$

$$= 2\left[\left(\sum_{x\in F^\times} \chi_k(x)\right)\left(\sum_{y\in F^\times} \chi_l(y)\right) - \sum_{x\in F^\times} \chi_k(x)\chi_l(x) - \sum_{x\in F^\times\setminus\{1\}} \chi_k(x) - \sum_{y\in F^\times\setminus\{1\}} \chi_l(y)\right].$$

Recall that

$$\sum_{x\in F^\times} \chi_k(x)\chi_l(x) = \begin{cases} q-1 & \text{if } k+l = q-1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $k \neq l, k+l \neq 0$. If $k+l = q-1$, then $k, l \neq 0$ because $0 \leq k, l \leq q-2$. It follows that

$$\sum_{\substack{x,y\in F^\times\setminus\{1\}\\ \text{and } x\neq y}} [\chi_k(x)\chi_l(y) + \chi_k(y)\chi_l(x)] = 2\left(-(q-1) + 2\right) = 2(3-q).$$

Assume that $k + l \neq q - 1$. We distinguish two cases.
Case 1. $k = 0$ or $l = 0$, say $k = 0$. Then $l \neq 0$ and so

$$\sum_{\substack{x,y\in F^\times\setminus\{1\}\\ \text{and } x\neq y}} [\chi_k(x)\chi_l(y) + \chi_k(y)\chi_l(x)] = 2\left(-(q-1) + 2\right) = 2(3-q).$$

Case 2. $k, l \neq 0$. Then we conclude that

$$\sum_{\substack{x,y\in F^\times\setminus\{1\}\\ \text{and } x\neq y}} [\chi_k(x)\chi_l(y) + \chi_k(y)\chi_l(x)] = 2.$$

This completes the proof. $\square$

Remark. Assume that q is odd. Before computing the eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$, we note that for each $x, y \in F$,

(a) $a_x \in \mathrm{GL}_2(F) \cap (\mathrm{I}_2 + \mathrm{GL}_2(F))$ if and only if $x \neq 1$
(b) $b_x \in \mathrm{GL}_2(F) \cap (\mathrm{I}_2 + \mathrm{GL}_2(F))$ if and only if $x \neq 1$
(c) $c_{x,y} \in \mathrm{GL}_2(F) \cap (\mathrm{I}_2 + \mathrm{GL}_2(F))$ if and only if $x, y \neq 1$
(d) $d_{x,y} \in \mathrm{GL}_2(F) \cap (\mathrm{I}_2 + \mathrm{GL}_2(F))$ for all $x \in F$ and $y \neq 0$.

To verify (d), we suppose that there exist $x \in F$ and $y \in F^\times$ such that $\det \begin{pmatrix} x-1 & \varepsilon y \\ y & x-1 \end{pmatrix} = 0$, so $(x-1)^2 - \varepsilon y^2 = 0$ in F. Thus, $x + y\sqrt{\varepsilon} = 1$ in E. Since $\{1, \sqrt{\varepsilon}\}$ is an F-basis of E, we have $y = 0$ which is absurd.

From the character table of $\mathrm{GL}_2(F)$ mentioned at the introduction, let λ_χ denote an eigenvalue induced from an irreducible character χ. Since the character U_{χ_k} has dimension one, the above remark gives

$$\lambda_{U_{\chi_k}} = \sum_{x \in F^\times \smallsetminus \{1\}} \chi_k(x^2) + (q^2-1) \sum_{x \in F^\times \smallsetminus \{1\}} \chi_k(x^2) + \frac{q^2+q}{2} \sum_{\substack{x,y \in F^\times \smallsetminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) + \frac{q^2-q}{2} \sum_{(x,y) \in F \times F^\times} \chi_k(x^2 - \varepsilon y^2).$$

According to Lemmas 2.2 and 2.3, we have $\lambda_{U_{\chi_0}} = q^4 - 2q^3 - q^2 + 3q$, $\lambda_{U_{\chi_{\frac{q-1}{2}}}} = q$ and

$$\lambda_{U_{\chi_k}} = (-1) + (q^2-1)(-1) + \frac{q^2+q}{2}(1+1) = q$$

if $k \notin \left\{0, \frac{q-1}{2}\right\}$. It follows that the eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ obtained from U_{χ_k} are $q^4 - 2q^3 - q^2 + 3q$ and q with multiplicities 1 and $q-2$, respectively.

Now, we work on V_{χ_k}. Since V_{χ_k} has dimension q, we have

$$\lambda_{V_{\chi_k}} = \frac{1}{q}\left(q \sum_{x \in F^\times \smallsetminus \{1\}} \chi_k(x^2) + \frac{q^2+q}{2} \sum_{\substack{x,y \in F^\times \smallsetminus \{1\} \\ \text{and } x \neq y}} \chi_k(xy) - \frac{q^2-q}{2} \sum_{(x,y) \in F \times F^\times} \chi_k(x^2 - \varepsilon y^2) \right).$$

Again, applying Lemmas 2.2 and 2.3 gives $\lambda_{V_{\chi_0}} = -q^2 + q + 1$, $\lambda_{V_{\chi_{\frac{q-1}{2}}}} = q$ and

$$\lambda_{V_{\chi_k}} = \frac{1}{q}\left(q(-1) + \frac{q^2+q}{2}(1+1)\right) = q$$

if $k \notin \{0, \frac{q-1}{2}\}$. Thus, the eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ obtained from V_{χ_k} are $-q^2+q+1$ and q with multiplicities q^2 and $q^2+q^2(q-3) = q^3-2q^2$, respectively.

Next, we consider the eigenvalues induced from the character W_{χ_k,χ_l} with $k \neq l$. Since W_{χ_k,χ_l} has dimension $q+1$, we have

$$\lambda_{W_{\chi_k,\chi_l}} = \frac{1}{q+1}\Bigg((q+1)\sum_{x\in F^\times\smallsetminus\{1\}}\chi_k(x)\chi_l(x) + (q^2-1)\sum_{x\in F^\times\smallsetminus\{1\}}\chi_k(x)\chi_l(x)$$
$$+\frac{q^2+q}{2}\sum_{\substack{x,y\in F^\times\smallsetminus\{1\}\\ \text{and } x\neq y}}(\chi_k(x)\chi_l(y)+\chi_k(y)\chi_l(x))\Bigg).$$

First, we assume that $k+l = q-1$. Thus, $k,l \neq 0$. Note that there are $\frac{q-3}{2}$ choices of such k,l. It follows from Lemma 2.4 that

$$\lambda_{W_{\chi_k,\chi_l}} = \frac{1}{q+1}\left((q+1)(q-2) + (q^2-1)(q-2) + 2\left(\frac{q^2+q}{2}\right)(3-q)\right) = q.$$

If $0 < k+l < q-1$, then we have two cases to consider. If $k=0$ or $l=0$, then there are $q-2$ choices of k and l, and

$$\lambda_{W_{\chi_k,\chi_l}} = \frac{1}{q+1}\left((q+1)(-1) + (q^2-1)(-1) + 2\left(\frac{q^2+q}{2}\right)(3-q)\right) = -q(q-2).$$

If $k,l \neq 0$, then there are $\frac{(q-3)^2}{2}$ choices of k and l, and

$$\lambda_{W_{\chi_k,\chi_l}} = \frac{1}{q+1}\left((q+1)(-1) + (q^2-1)(-1) + \left(\frac{q^2+q}{2}\right)(4)\right) = q.$$

Thus, the eigenvalues of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ obtained from W_{χ_k,χ_k} are $-q(q-2)$ and q with multiplicities $(q+1)^2(q-2)$ and $\frac{(q+1)^2(q-2)(q-3)}{2}$, respectively.

Finally, let φ be an irreducible character of $E^\times$ such that $\varphi^q \neq \varphi$. Hence, φ is a non-trivial character and there are $\frac{q^2-q}{2}$ choices of φ. Since X_φ has dimension $q-1$, we have

$$\lambda_{X_\varphi} = \frac{1}{q-1}\Bigg((q-1)\sum_{x\in F^\times\smallsetminus\{1\}}\varphi(x) - (q^2-1)\sum_{x\in F^\times\smallsetminus\{1\}}\varphi(x)$$
$$- \frac{q^2-q}{2}\sum_{(x,y)\in F\times F^\times}\left(\varphi(x+y\sqrt{\varepsilon})+\varphi(x-y\sqrt{\varepsilon})\right)\Bigg)$$
$$= \frac{1}{q-1}\left(-(q^2-q)\sum_{x\in F^\times}\varphi(x) + (q^2-q) - (q^2-q)\sum_{(x,y)\in F\times F^\times}\varphi(x+y\sqrt{\varepsilon})\right)$$
$$= \frac{1}{q-1}\left(-(q^2-q)\sum_{x\in E^\times}\varphi(x) + (q^2-q)\right) = q.$$

Hence, the eigenvalue from this case is q with multiplicity $\dfrac{(q-1)^2(q^2-q)}{2}$.

Summing all multiplicities of the eigenvalue q from each character gives its total multiplicity $q^4-2q^3-2q^2+4q+1$. Therefore, we obtain the spectrum of $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ in the case that q is odd. For q even and $q \geq 4$, we can find all eigenvalues corresponding to each U_χ, V_χ and X_φ in the similar manner without the case $k = \dfrac{q-1}{2}$. Note that the eigenvalue obtained from the case $k = \dfrac{q-1}{2}$ when q is odd is always q. Hence, the eigenvalues corresponding to those characters of the case q is even and $q \geq 4$ are equal to the eigenvalues in the case q is odd. As for eigenvalues corresponding to W_{χ_k,χ_l}, we have multiplicities of q become $\dfrac{(q+1)^2(q-2)}{2}$ and $\dfrac{(q+1)^2(q-4)(q-2)}{2}$ whose sum is again $\dfrac{(q+1)^2(q-2)(q-3)}{2}$, so the multiplicities of q when q is even stays the same.

Finally, if $q = 2$, then the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ has $(2^2-1)(2^2-2) = 6$ vertices and is two copies of K_3, so its spectra are 2 of multiplicity 2 and -1 of multiplicity 4. Thus, we completely determine the spectrum for the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$.

Theorem 2.5.

(a) *If* $q = 2$, *then* $\operatorname{Spec}\mathrm{C}^{(1)}_{\mathrm{M}_2(F)} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$.

(b) *If* $q \geq 3$, *then*

$$\operatorname{Spec}\mathrm{C}^{(1)}_{\mathrm{M}_2(F)} =$$
$$\begin{pmatrix} q^4-2q^3-q^2+3q & q & -q^2+q+1 & -q^2+2q \\ 1 & q^4-2q^3-2q^2+4q+1 & q^2 & (q+1)^2(q-2) \end{pmatrix}.$$

Moreover, $E(\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}) = 2q^5-2q^4-8q^3+6q^2+8q$ *for all* $q \geq 2$.

Furthermore, for all $q \geq 3$, we have

$$\begin{aligned} E(\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}) &= 2q^5 - 2q^4 - 8q^3 + 6q^2 + 8q - 2\left((q^2-1)(q^2-q) - 1\right) \\ &= 2q^5 - 4q^4 - 6q^3 + 8q^2 + 6q + 2 > 2q^5 - 4q^4 - 6q^3 = 2q^3(q-3)(q+1) \geq 0. \end{aligned}$$

This proves hyperenergeticity of the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ when $q \geq 3$, while $\mathrm{C}^{(1)}_{\mathrm{M}_2(\mathbb{Z}_2)}$ is not hyperenergetic because its energy is $8 < 2(6-1)$.

Since $\mathrm{C}^{(1)}_{\mathrm{M}_2(\mathbb{Z}_2)}$ is disconnected, it is not Ramanujan. To show that the graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ is Ramanujan for $q \geq 3$. Since $|-q^2+q+1| > |-q(q-2)| > q$, it suffices to show that $2\sqrt{(q^4 - 2q^3 - q^2 + 3q - 1} \geq q^2 - q - 1$ which is equivalent to $4(q^4 - 2q^3 - q^2 + 3q - 1) \geq (q^2 - q - 1)^2$, and we have

$$\begin{aligned} 4(q^4 - 2q^3 - q^2 + 3q - 1) - (q^2 - q - 1)^2 &= 3q^4 - 6q^3 - 3q^2 + 10q - 5 \\ &\geq 3q^4 - 6q^3 - 3q^2 = 3q^2((q-1)^2 - 2) \geq 0. \end{aligned}$$

We record this work in

Theorem 2.6. *The graph $\mathrm{C}^{(1)}_{\mathrm{M}_2(F)}$ is hyperenergetic and Ramanujan when $q \geq 3$. Moreover, $\mathrm{C}^{(1)}_{\mathrm{M}_2(\mathbb{Z}_2)}$ is neither hyperenergetic nor Ramanujan.*

3. Spectral properties of $\mathbf{C}^{(2)}_{\mathbf{M}_2(F)}$

We study the second subconstituent of $\mathrm{C}_{\mathrm{M}_2(F)}$ in this section. We first show that the graph is a tensor product of a complete graph and a complete multi-partite graph and so we can calculate its eigenvalues. Let $F^{2\times 1}$ denote the set of column vectors of size 2×1 over F. Since a 2×2 matrix is non-invertible if and only if its column vectors are parallel, we can conclude that

$$\mathrm{M}_2(F) \smallsetminus (\mathrm{GL}_2(F) \cup \{\mathbf{0}_{2\times 2}\}) = \left(\dot{\bigcup_{\vec{v} \in F^{2\times 1} \smallsetminus \{\vec{0}\}}} \{(a\vec{v} \quad \vec{v}) : a \in F\} \right) \cup \left\{ \begin{pmatrix} \vec{v} & \vec{0} \end{pmatrix} : \vec{v} \in F^{2\times 1} \smallsetminus \{\vec{0}\} \right\}$$

where $\vec{0}$ denotes the zero vector of $F^{2\times 1}$. Before giving a structure of the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$, we need the next lemma.

Lemma 3.1. *Let A, B be non-invertible matrices in $\mathrm{M}_2(F), a, b \in F$ and $\vec{v}, \vec{w} \in F^{2\times 1} \smallsetminus \{\vec{0}\}$.*

(a) *If $A = (a\vec{v} \quad \vec{v})$ and $B = (b\vec{w} \quad \vec{w})$, then $A - B$ is non-invertible if and only if $a = b$ or $\vec{v}, \vec{w}$ are linearly dependent, or equivalently, $A - B$ is invertible if and only if $a \neq b$ and $\vec{v}, \vec{w}$ are linearly independent.*

(b) *If* $A = (a\vec{v} \quad \vec{v})$ *and* $B = \begin{pmatrix}\vec{w} & \vec{0}\end{pmatrix}$, *then* $A - B$ *is non-invertible if and only if* $\vec{v}$ *and* $\vec{w}$ *are linearly dependent.*

Proof. Observe that

$$A - B \text{ is non-invertible} \iff (a\vec{v} - b\vec{w}) = c(\vec{v} - \vec{w}) \text{ for some } c \in F.$$

Assume that $A - B$ is non-invertible and $\vec{v}, \vec{w}$ are linearly independent. Then $a = c$ and $b = c$, so $a = b$. Conversely, the case $a = b$ is clear. If $\vec{w} = c\vec{v}$ for some $c \in F$, then $A - B = ((a - bc)\vec{v} \quad (1 - c)\vec{v})$ is non-invertible. This proves (a). For (b), we have

$$\begin{aligned} A - B \text{ is non-invertible} &\iff a\vec{v} - \vec{w} = c\vec{v} \text{ for some } c \in F \\ &\iff (a - c)\vec{v} = \vec{w} \text{ for some } c \in F, \end{aligned}$$

which is equivalent to $\vec{v}$ and $\vec{w}$ are linearly dependent. $\square$

In the next step, we define two graphs G and H as follows: G is the complete graph on $q+1$ vertices parametrized by the set of projective lines $\mathbb{P}^1(F) = \{[a, 1] : a \in F\} \cup \{[1, 0]\}$ and the vertex set of H is $F^{2\times 1} \smallsetminus \{\vec{0}\}$ and for any $\vec{v}, \vec{w} \in F^{2\times 1} \smallsetminus \{\vec{0}\}$, $\vec{v}$ and $\vec{w}$ are adjacent if and only if $\vec{v}$ and $\vec{w}$ are not parallel. Note that H is the complete $(q + 1)$-partite graph such that each partite has $q - 1$ vertices.

Let $f : \mathrm{C}^{(2)}_{\mathrm{M}_2(F)} \to G \otimes H$ defined by $(a\vec{v} \quad \vec{v}) \mapsto ([a, 1], \vec{v})$ and $\begin{pmatrix}\vec{v} & \vec{0}\end{pmatrix} \mapsto ([1, 0], \vec{v})$ for any $a \in F$ and $\vec{v} \in F^{2\times 1} \setminus \{\vec{0}\}$. Thus, f is bijective. Now, let A, B be nonzero non-invertible matrices in $\mathrm{M}_2(F), a, b \in F$ and $\vec{v}, \vec{w} \in F^{2\times 1}$, $A = (a\vec{v} \quad \vec{v})$ and $B = (b\vec{w} \quad \vec{w})$. Lemma 3.1 (a) implies

$$\begin{aligned} A - B \in \mathrm{GL}_2(F) &\iff a \neq b \text{ and } \vec{v}, \vec{w} \text{ are linearly independent} \\ &\iff ([a, 1], \vec{v}) \text{ is adjacent to } ([b, 1], \vec{w}). \end{aligned}$$

Next, we assume that $A = (a\vec{v} \quad \vec{v})$ and $B = \begin{pmatrix}\vec{w} & \vec{0}\end{pmatrix}$. From Lemma 3.1 (b), we have

$$\begin{aligned} A - B \in \mathrm{GL}_2(F) &\iff \vec{v} \text{ and } \vec{w} \text{ are linearly independent} \\ &\iff ([a, 1], \vec{v}) \text{ is adjacent to } ([1, 0], \vec{w}). \end{aligned}$$

Hence, f is a graph isomorphism, so we have the structure of the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Theorem 3.2. *The graph* $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ *is the tensor product of the complete graph on* $q + 1$ *vertices and the complete* $(q + 1)$*-partite graph such that each partite has* $q - 1$ *vertices, and it is a* $(q^3 - q^2)$*-regular graph.*

Recall from [8] that if $\lambda_1, \ldots, \lambda_k$ are eigenvalues of a graph G_1 and $\mu_1, \ldots, \mu_l$ are eigenvalues of a graph G_2, then the eigenvalues of the tensor product $G_1 \otimes G_2$ are $\lambda_i\mu_j$

where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$. Since the eigenvalues of G are q of multiplicity 1 and -1 of multiplicity q and the eigenvalues of H are $q^2 - q, -q+1$ and 0 of multiplicities $1, q$ and $q^2 - q - 2$, respectively, we obtain the spectrum and energy of the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Theorem 3.3. *We have*

$$\operatorname{Spec} \mathrm{C}^{(2)}_{\mathrm{M}_2(F)} = \begin{pmatrix} q^3 - q^2 & -q^2 + q & q - 1 & 0 \\ 1 & 2q & q^2 & q^3 - 3q - 2 \end{pmatrix}.$$

Moreover, $E(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) = 4q^3 - 4q^2$.

Since the number of vertices of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is $|\,\mathrm{M}_2(F) \smallsetminus (\mathrm{GL}_2(F) \cup \{\underline{0}\})\,| = q^3 + q^2 - q - 1$ and

$$4q^3 - 4q^2 - 2(q^3 + q^2 - q - 2) = 2q^3 - 6q^2 + 2q + 4 = 2(q - 2)(q^2 - q - 1) \geq 0.$$

Thus, $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is hyperenergetic unless $q = 2$. Finally, we show that the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is not Ramanujan if $q \geq 5$. Since $q^2 - q$ is an eigenvalue of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$, we claim that $(q^2 - q)^2 > 4(q^3 - q^2 - 1)$, which is equivalent to the inequality $q^4 - 6q^3 + 5q^2 + 4 > 0$. This holds for $q \geq 5$ because $q^4 - 6q^3 + 5q^2 + 4 = q^2(q - 1)(q - 5) + 4 > 0$. For $q = 2, 3$ or 4, it is easily seen that $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is Ramanujan. We record both results in

Theorem 3.4. *The graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is hyperenergetic if and only if $q \geq 3$, and it is Ramanujan if and only if $q \leq 4$.*

4. Clique number, chromatic number and independence number

In this section, we compute the clique number, the chromatic number and the independence number of subconstituents of $\mathrm{C}_{\mathrm{M}_2(F)}$. Recall from the proof of Theorem 3.4 of [12] that the ring $\mathrm{M}_n(F)$ contains a subfield K of order q^n. We start with the first subconstituent. Note that $\mathbf{0}_{n\times n} \in K$ and so $K \smallsetminus \{\mathbf{0}_{n\times n}\}$ forms a complete subgraph in $\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}$. Hence, $\omega(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \geq q^n - 1$. On the other hand, let J be the set of matrices in $\mathrm{M}_n(F)$ whose all entries in the first row are zero. We can see that J is an ideal of $\mathrm{M}_n(F)$ of q^{n^2-n} elements. Write $\mathrm{M}_n(F) = \bigcup_{i=1}^{q^n} (B_i + J)$ as a union of cosets of J where the coset $B_1 + J = J$. Note that each coset forms an independent set and $\mathbf{0}_{n\times n} \in J$. It follows that $\mathrm{GL}_n(F)$ is a subset of $\bigcup_{i=2}^{q^n} (B_i + J)$ and hence $\chi(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \leq q^n - 1$. Since $\omega(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \leq \chi(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)})$, we have the following theorem.

Theorem 4.1. $\omega(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) = \chi(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) = q^n - 1$.

Recall from [2] p.147 that if G is a graph, then $\alpha(G) \geq \dfrac{|V(G)|}{\chi(G)}$. Theorem 4.1 gives

$$\alpha(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \geq \frac{|\,\mathrm{GL}_n(F)|}{\chi(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)})} = (q^n - q)\dots(q^n - q^{n-1}).$$

Consider the group $K^\times$ as a multiplicative subgroup of $\mathrm{GL}_n(F)$. Let $X = AM$ and $Y = AN$ where $M, N \in K^\times$ such that $M \neq N$ and $A \in \mathrm{GL}_n(F)$. Then $X - Y = A(M - N)$ is invertible because $M, N \in K^\times$. It follows that each coset forms a complete graph. This implies that $\alpha(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) \leq (q^n - q)\dots(q^n - q^{n-1})$. Hence, we have shown

Theorem 4.2. $\alpha(\mathrm{C}^{(1)}_{\mathrm{M}_n(F)}) = (q^n - q)\dots(q^n - q^{n-1})$.

By Theorem 3.2, we have the second subconstituent of $\mathrm{C}_{\mathrm{M}_2(F)}$ is the tensor product of the complete graph on $q + 1$ vertices G and the complete $q + 1$-partite graph H such that each partite has $q - 1$ vertices. Since $\chi(G) = \chi(H) = q + 1$, we can conclude that $\chi(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) \leq q+1$. Moreover, let $V(G) = \{a_1, \dots, a_{q+1}\}$ and $V_1, \dots, V_{q+1}$ be the partites of H. Choose $v_i \in V_i$ for all $i \in \{1, \dots, q + 1\}$. We can see that the subgraph of $G \otimes H$ induced by $\{(a_1, v_1), \dots, (a_{q+1}, v_{q+1})\}$ is a complete graph, so $\omega(G \otimes H) \geq q + 1$. Thus, we obtain the clique number and the chromatic number of the graph $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Theorem 4.3. $\omega(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) = \chi(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) = q + 1$.

Our final theorem gives the independence number of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$.

Theorem 4.4. $\alpha(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) = q^2 - 1$.

Proof. Similar to the proof of Theorem 4.2, we know from Theorem 4.3 that

$$\alpha(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) \geq \frac{|\,\mathrm{M}_2(F) \smallsetminus (\mathrm{GL}_2(F) \cup \{\mathbf{0}_{n\times n}\})\,|}{\chi(\mathrm{C}^{(2)}_{\mathrm{M}_n(F)})} = \frac{q^3 + q^2 - q - 1}{q + 1} = q^2 - 1.$$

Write $\mathrm{M}_2(F) = \displaystyle\bigcup_{i=1}^{q^2}(A_i + K)$ as a union of cosets of K. Then an independent set of $\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}$ is contained in $\displaystyle\bigcup_{i=2}^{q^2}(A_i + K)$. Since each coset forms a complete subgraph, we have $\alpha(\mathrm{C}^{(2)}_{\mathrm{M}_2(F)}) \leq q^2 - 1$ and the result follows. □

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Algebraic degree of spectra of Cayley hypergraphs

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ABSTRACT

Let $(G, \cdot)$ be a finite group with the identity e and S a subset of $G \setminus \{e\}$ such that $S = S^{-1}$. For $t \in \mathbb{N}$ and $2 \leq t \leq \max\{o(x) : x \in S\}$, the t-Cayley hypergraph of G over S is the hypergraph whose vertex set is G and edge set is $\{\{yx^i : 0 \leq i \leq t-1\} : x \in S \text{ and } y \in G\}$. In this work, we study spectral properties of this hypergraph. We characterize integral 2-Cayley hypergraphs of G when G is abelian. In addition, we obtain the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$.

1. Introduction

This section contains some terminologies from algebraic graph and hypergraph theory. We also discuss many earlier results on this topic. Our objective is to provide some algebraic properties of eigenvalues of t-Cayley hypergraphs.

1.1. Hypergraphs

A *hypergraph* H is a pair $(V(\mathrm{H}), E(\mathrm{H}))$ where $V(\mathrm{H})$ is a finite set, called the *vertex set* of H, and $E(\mathrm{H})$ is a family of subsets of $V(\mathrm{H})$, called the *edge set* of H. The elements in $V(\mathrm{H})$ are called *vertices* and the elements in $E(\mathrm{H})$ are called *hyperedges*. In particular, if $E(\mathrm{H})$ consists only of 2-subsets of $V(\mathrm{H})$, then H is a simple graph. For $v \in V(\mathrm{H})$, we write $\mathfrak{D}(v)$ for the set of all hyperedges containing the vertex v and the number of elements in $\mathfrak{D}(v)$ is the *degree* of the vertex v, denoted by $\deg v$. A hypergraph in which all vertices have the same degree $k \geq 0$ is called *k-regular* and it is said to be *regular* if it is k-regular for some $k \geq 0$. A hypergraph in which all hyperedges have the same cardinality $l \geq 0$ is an *l-uniform* hypergraph. A *path* of length s in H is an alternating sequence $v_1E_1v_2E_2v_3 \ldots v_sE_sv_{s+1}$ of distinct vertices $v_1, v_2, \ldots, v_{s+1} \in V(\mathrm{H})$ and distinct hyperedges $E_1, E_2, \ldots, E_s \in E(\mathrm{H})$ satisfying $v_i, v_{i+1} \in E_i$ for any $i \in \{1, 2, \ldots, s\}$. The *distance* between two vertices v and w, denoted by $d(v, w)$, is the smallest length of a path from v to w. If there is no path from v to w, we define $d(v, w) = \infty$. The *diameter* of H is $\mathrm{diam}(\mathrm{H}) = \max\{d(v, w) : v, w \in V(\mathrm{H})\}$. A hypergraph H is *connected* if $\mathrm{diam}(\mathrm{H}) < \infty$.

For a hypergraph H with vertex set $\{v_1, \ldots, v_n\}$, the *adjacency matrix* of H, denoted by $A(\mathrm{H})$, is the $n \times n$ matrix whose entry a_{ij}, $i \neq j$, is the number of hyperedges that contain both of v_i and v_j and $a_{ii} = 0$ for all $1 \leq i, j \leq n$. This concept was investigated by Bretto [1]. Evidently, it is a generalization of the adjacency matrix of a graph. An equivalent definition of the adjacency matrix is given in [6] by using the bipartite graph associated to H which is the graph whose vertex set is the union of two independent sets $V(\mathrm{H})$ and $E(\mathrm{H})$ and for any $v \in V(\mathrm{H})$ and $E \in E(\mathrm{H})$, they are adjacent whenever $v \in E$. In particular, if H is an l-uniform hypergraph, there is another way to define an adjacency matrix by using hypermatrix, see [4,8]. In this work, our hypergraphs may not be l-uniform, so we follow Bretto's. The *Laplacian matrix* of H, denoted by $L(\mathrm{H})$, is the $n \times n$ matrix defined by $L(\mathrm{H}) = \mathcal{D}(\mathrm{H}) - A(\mathrm{H})$ where $\mathcal{D}(\mathrm{H})$ is the diagonal matrix $\left[\deg v_i\right]_{1 \leq i \leq n}$. This version

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of Laplacian matrix was introduced by Rodríguez [16]. The *distance matrix* of a connected hypergraph H, denoted by $D(\mathrm{H})$, is the $n \times n$ matrix in which entry $d_{ij} = d(v_i, v_j)$ for all $1 \leq i, j \leq n$.

The *spectrum* of H, denoted by Spec(H), is the multi-set of all eigenvalues of $A(\mathrm{H})$ including multiplicity. Observe that $A(\mathrm{H})$ is a real symmetric matrix, so Spec(H) contains only real eigenvalues. Since the characteristic polynomial of $A(\mathrm{H})$ is monic with integral coefficients, its rational roots are integers. A hypergraph is *integral* if all eigenvalues of this hypergraph are integers. Similarly, we can define Lspec(H) and Dspec(H) as the sets of all eigenvalues of $L(\mathrm{H})$ and $D(\mathrm{H})$, respectively. Also, an L-*integral hypergraph* is a hypergraph with integral Laplacian eigenvalues and a D-*integral hypergraph* is a hypergraph with integral distance eigenvalues.

For hypergraphs H_1 and H_2, the *Cartesian product* of H_1 and H_2, denoted by $\mathrm{H}_1 \square \mathrm{H}_2$, is the hypergraph with $V(\mathrm{H}_1 \square \mathrm{H}_2) = V(\mathrm{H}_1) \times V(\mathrm{H}_2)$ and $E(\mathrm{H}_1 \square \mathrm{H}_2) = \{\{x\} \times E' : x \in V(\mathrm{H}_1), E' \in E(\mathrm{H}_2)\} \cup \{E \times \{y\} : E \in E(\mathrm{H}_1) \text{ and } y \in V(\mathrm{H}_2)\}$. Observe that $A(\mathrm{H}_1 \square \mathrm{H}_2) = (A(\mathrm{H}_1) \otimes I_{|V(\mathrm{H}_2)|}) + (I_{|V(\mathrm{H}_1)|} \otimes A(\mathrm{H}_2))$ where $A \otimes B$ denotes the Kronecker product of matrices A and B. Therefore,

$$\mathrm{Spec}(\mathrm{H}_1 \square \mathrm{H}_2) = \{\lambda + \beta : \lambda \in \mathrm{Spec}(\mathrm{H}_1) \text{ and } \beta \in \mathrm{Spec}(\mathrm{H}_2)\}. \tag{A}$$

Let H_1 and H_2 be t-uniform hypergraphs. Following Pearson [15], the *tensor product* of H_1 and H_2, denoted by $\mathrm{H}_1 \otimes \mathrm{H}_2$, is the t-uniform hypergraph with $V(\mathrm{H}_1 \otimes \mathrm{H}_2) = V(\mathrm{H}_1) \times V(\mathrm{H}_2)$ and $E(\mathrm{H}_1 \otimes \mathrm{H}_2) = \{\{(x_{i_1}, y_{j_1}), \dots, (x_{i_t}, y_{j_t})\} : \{x_{i_1}, \dots, x_{i_t}\} \in E(\mathrm{H}_1), \{y_{j_1}, \dots, y_{j_t}\} \in E(\mathrm{H}_2)\}$. It follows that the number of hyperedges containing both of two vertices (x_i, y_l) and (x_j, y_m) in $\mathrm{H}_1 \otimes \mathrm{H}_2$ is $(t-2)! a_{ij} b_{lm}$. Hence, $A(\mathrm{H}_1 \otimes \mathrm{H}_2) = (t-2)! A(\mathrm{H}_1) \otimes A(\mathrm{H}_2)$. Consequently,

$$\mathrm{Spec}(\mathrm{H}_1 \otimes \mathrm{H}_2) = \{(t-2)! \lambda \beta : \lambda \in \mathrm{Spec}(\mathrm{H}_1) \text{ and } \beta \in \mathrm{Spec}(\mathrm{H}_2)\}. \tag{B}$$

Several properties of hypergraphs have been studied such as diameter, connectivity and chromatic number. Spectral and combinatorial properties of hypergraphs are widely related (see for example [5,6,12,16]). Feng and Li [6] showed the relation between the diameter of H and its eigenvalues. They proved that if $\{\mathrm{H}_n\}_{n \in \mathbb{N}}$ is a collection of k-regular and l-uniform hypergraphs with $\lim_{n\to\infty} |V(\mathrm{H}_n)| = \infty$, then $\lim_{n\to\infty} \mathrm{diam}(\mathrm{H}_n) = \infty$ by using the second largest eigenvalue of H_n. Later, Rodríguez [16] showed that if $b+1$ is the number of distinct Laplacian eigenvalues of a connected hypergraph H, then $\mathrm{diam}(\mathrm{H}) \leq b$.

1.2. t-Cayley Hypergraphs

Throughout this section, we let $(G, \cdot)$ be a finite group with the identity e and S a subset of $G \setminus \{e\}$ such that $S = S^{-1}$. For $t \in \mathbb{N}$ and $2 \leq t \leq \max\{o(x) : x \in S\}$, the *t-Cayley hypergraph* $\mathrm{H} = t\text{-Cay}(G, S)$ of G over S is a hypergraph with vertex set $V(\mathrm{H}) = G$ and $E(\mathrm{H}) = \{\{yx^i : 0 \leq i \leq t-1\} : x \in S \text{ and } y \in G\}$. Here, $o(x)$ denotes the order of x in G.

Example 1.1. For $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{n} = (n_1, \dots, n_r)$ in $\mathbb{Z}^r$, we define the greatest common divisor of $\mathbf{m}$ and $\mathbf{n}$ to be the vector $\mathbf{d} = (d_1, \dots, d_r)$ where $d_i = \gcd(m_i, n_i)$ for all $i \in \{1, \dots, r\}$. Now, let $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$ and a divisor tuple $\mathbf{d} = (d_1, \dots, d_r)$ of $\mathbf{n}$, i.e., $d_i \mid n_i$ for all $i \in \{1, \dots, r\}$. Define

$$G_{\mathbf{n}}(\mathbf{d}) = \{\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r} : \gcd(\mathbf{x}, \mathbf{n}) = \mathbf{d}\}.$$

Let D be a set of divisor tuples of $\mathbf{n}$ not containing the zero vector of $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ and $S = \bigcup_{\mathbf{d} \in D} G_{\mathbf{n}}(\mathbf{d})$. For $t \in \mathbb{N}$ and $2 \leq t \leq \max\{o(\mathbf{x}) : \mathbf{x} \in S\}$, the t-Cayley hypergraph of $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ over S is called a gcd-*hypergraph*.

Some properties of t-Cayley hypergraphs quoted from [2] are as follows.

Proposition 1.2. *Let* $\mathrm{H} = t\text{-Cay}(G, S)$.

(1) H is connected if and only if $\langle S \rangle = G$.

(2) For any $x \in S, y \in G$, $\left|\{yx^i : 0 \leq i \leq t-1\}\right| = \begin{cases} t & \text{if } t \leq o(x), \\ o(x) & \text{if } t > o(x). \end{cases}$

(3) H is t-uniform if and only if $t \leq o(x)$ *for any* $x \in S$.

Clearly, a Cayley graph 2-Cay(G, S) is $|S|$-regular. We study a Cayley hypergraph t-Cay(G, S). For any $y \in G$, we have that all hyperedges (may not be distinct) containing y are

$$\{yx^{-(t-1)}, yx^{-(t-2)}, \dots, yx^{-1}, y\}, \{yx^{-(t-2)}, yx^{-(t-3)}, \dots, y, yx\}, \dots, \{y, yx, \dots, yx^{t-2}, yx^{t-1}\}$$

where $x \in S$. This implies

$$\begin{aligned} \deg y &= \left|\{\{yx^{i-j} : 0 \leq i \leq t-1\} : 0 \leq j \leq t-1, x \in S\}\right| \\ &= \left|\{\{x^{i-j} : 0 \leq i \leq t-1\} : 0 \leq j \leq t-1, x \in S\}\right|. \end{aligned}$$

for all $y \in G$. Hence, we have shown

Proposition 1.3. *A t-Cayley hypergraph of G over S is regular of degree equal to the number of distinct subsets* $\{x^{i-j} : 0 \leq i \leq t-1\}$ *where* $0 \leq j \leq t-1$ *and* $x \in S$.

Cayley graphs, as known as Cayley color graphs or Cayley color diagrams were first introduced by Cayley [3] in 1878. They are regularly studied and have many applications. Harary and Schwenk [7] asked "Which graphs have integral spectra?". From this question, the integral Cayley graphs are widely studied, e.g., [9–12,17]. A well-studied Cayley graph is the 2-unitary Cayley graph of a finite ring. Klotz and Sander [12] studied combinatorial properties of the unitary Cayley graph 2-Cay$(\mathbb{Z}_n, \mathbb{Z}_n^\times)$. They explored the chromatic number, the clique number, the independence number, the diameter and the vertex connectivity of this graph. In addition, they showed that the gcd-graphs are integral. A few year later, Ilić [9] determined the energy of unitary Cayley graph 2-Cay$(\mathbb{Z}_n, \mathbb{Z}_n^\times)$. Kiani et al. [11] worked on the eigenvalues of the unitary Cayley graph of finite local rings and extended the result to finite commutative rings. So [17] completely characterized integral Cayley graphs of $(\mathbb{Z}_n, +)$ as follows.

Theorem 1.4. *The Cayley graph* 2-Cay$(\mathbb{Z}_n, S)$ *is integral if and only if* S *is a union of some* $G_n(d)$*'s, where* $d \mid n$ *and* $G_n(d) = \{k \in \{1, 2, \dots, n-1\} : \gcd(k, n) = d\}$.

To characterize integral Cayley graphs of finite abelian groups, we first discuss the Cayley graph of the group $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, +)$. Let $S = S_1 \times S_2$ be a subset of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \setminus \{(0, 0)\}$ such that $S = -S$. The Cayley graph 2-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$ can be distinguished into three cases.

(1) $\overline{K}_{n_1} \square$ 2-Cay$(\mathbb{Z}_{n_2}, S_2)$ if $S_1 = \{0\}$ and $S_2 \neq \{0\}$, where $\overline{K}_n$ denotes the empty graph on n vertices.
(2) 2-Cay$(\mathbb{Z}_{n_1}, S) \square \overline{K}_{n_2}$ if $S_1 \neq \{0\}$ and $S_2 = \{0\}$.
(3) 2-Cay$(\mathbb{Z}_{n_1}, S_1) \otimes$ 2-Cay$(\mathbb{Z}_{n_2}, S_2)$ if $S_1 \neq \{0\}$ and $S_2 \neq \{0\}$.

It is clear that the eigenvalues of an empty graph are zero. By Eqs. (A), (B) and a Cayley graph always has an integral eigenvalue, the Cayley graph 2-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$ is integral if and only if for any $i \in \{1, 2\}$ such that $S_i \neq \{0\}$, the 2-Cay$(\mathbb{Z}_{n_i}, S_i)$ is integral. By the fundamental theorem of finite abelian groups, a finite abelian group is a direct product of finite cyclic groups. We can obtain a characterization of the integral Cayley graphs of finite abelian groups similar to the above discussion.

Theorem 1.5. *Let* G *be a finite abelian group and* S *a subset of* $G \setminus \{e\}$ *such that* $S = S^{-1}$. *Suppose* $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ *and* $S = S_1 \times \cdots \times S_r$. *The Cayley graph* 2-Cay(G, S) *is integral if and only if for any* $i \in \{1, \dots, r\}$ *such that* $S_i \neq \{0\}$, *the* 2-Cay$(\mathbb{Z}_{n_i}, S_i)$ *is integral.*

For non-integral graphs, Mönius et al. [14] defined the *algebraic degree* of a graph G to be the degree of extension of the splitting field of the characteristic polynomial of $A(\mathrm{G})$ over $\mathbb{Q}$. Recently, Mönius [13] determined the algebraic degree of Cayley graphs of $\mathbb{Z}_p$ where p is a prime number.

Our purposes are to characterize integral t-Cayley hypergraphs of $\mathbb{Z}_n$ and compute the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$. The paper is organized as follows. In Section 2, we study the spectrum of t-Cayley hypergraphs of $\mathbb{Z}_n$. We obtain the characterization of integral t-Cayley hypergraphs of $\mathbb{Z}_n$ similar to So [17]. We use this result to show that a gcd-hypergraph of $\mathbb{Z}_n$ is integral, L-integral and D-integral. In addition, we can determine the first row of the circulant adjacency matrix of a gcd-hypergraph of $\mathbb{Z}_n$ (Theorem 2.3). We study non-integral hypergraphs in Section 3. We compute the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$ for all $n \geq 3$ which generalizes Mönius' results and provides an answer to his outlook. Our combinatorial approach is different from him and presented in Lemma 3.1.

2. Integral t-Cayley hypergraphs of $\mathbb{Z}_n$

A *circulant matrix* is a square matrix in which each row is obtained by a right cyclic shift of the preceding row. From now on, we let $n \geq 2$ and $\mathrm{H} = t$-Cay$(\mathbb{Z}_n, S)$. By the natural labeling $\{0, 1, \dots, n-1\}$ of $\mathbb{Z}_n$, it is easy to see that $A(\mathrm{H}) = [a_{ij}]_{0 \leq i,j \leq n-1}$ is circulant. To work on the adjacency matrix $A(\mathrm{H})$, it suffices to compute the first row of $A(\mathrm{H})$. Let C be the set of vertices adjacent to the vertex 0. Since all hyperedges containing 0 are of the form $\{(i-j)x : 0 \leq i \leq t-1\}$ where $x \in S$ and $0 \leq j \leq t-1$, and $S = -S$, we have the union of all hyperedges containing 0 is

$$\bigcup_{0 \leq i,j \leq t-1} (i-j)S = \bigcup_{-(t-1) \leq k \leq t-1} kS = S \cup 2S \cup \cdots \cup (t-1)S.$$

It follows that $C = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$. Since $A(\mathrm{H})$ is circulant, the eigenvalues of H are

$$\lambda_j = \sum_{k \in C} a_{0,k} (e^{2\pi j i/n})^k$$

where $0 \leq j \leq n-1$. We recall some useful properties taken from [17].

Proposition 2.1.

(1) If d *is a proper divisor of* n *and* x *is an* n*th root of unity, then* $\sum_{k \in G_n(d)} x^k$ *is an integer.*

(2) Let $\omega = e^{2\pi i/n}$ and

$$F = \begin{bmatrix} \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot(n-1)} \\ \omega^{2\cdot 1} & \omega^{2\cdot 2} & \cdots & \omega^{2\cdot(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{(n-1)\cdot 1} & \omega^{(n-1)\cdot 2} & \cdots & \omega^{(n-1)\cdot(n-1)} \end{bmatrix}.$$

If $\mathcal{A} = \{v \in \mathbb{Q}^{n-1} : Fv \in \mathbb{Q}^{n-1}\}$, then $\mathcal{A}$ is a vector space over $\mathbb{Q}$. Moreover, $\mathcal{A} = \mathrm{Span}\{v_d : d \mid n \text{ and } d < n\}$ where v_d is the $(n-1)$-vector with 1 at the kth entry for all $k \in G_n(d)$ and 0 elsewhere.

Now, we prove a criterion for integral t-Cayley hypergraphs.

Theorem 2.2. *Let $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$. Then H is integral if and only if C is a union of some $G_n(d)$'s where for each d, there is $c_d \in \left\{1, 2, \ldots, \binom{n}{t-2}\right\}$ such that $a_{0,k} = c_d$ for all $k \in G_n(d)$.*

Proof. Let $d_1, \ldots, d_s$ be all proper divisors of n. Without loss of generality, we assume that $C = G_n(d_1) \cup \cdots \cup G_n(d_l)$ for some $l \in \{1, \ldots, s\}$. Clearly, $\lambda_0 = \sum_{k\in C} a_{0,k} \in \mathbb{Z}$. For any $1 \le j \le n-1$, by the assumption and Proposition 2.1 (1),

$$\begin{aligned} \lambda_j &= \sum_{k\in C} a_{0,k}(e^{2\pi ji/n})^k \\ &= \sum_{k\in G_n(d_1)} a_{0,k}(e^{2\pi ji/n})^k + \cdots + \sum_{k\in G_n(d_l)} a_{0,k}(e^{2\pi ji/n})^k \\ &= c_{d_1} \sum_{k\in G_n(d_1)} (e^{2\pi ji/n})^k + \cdots + c_{d_l} \sum_{k\in G_n(d_l)} (e^{2\pi ji/n})^k \in \mathbb{Z}. \end{aligned}$$

Conversely, suppose that H is integral. Then $\lambda_j \in \mathbb{Z}$ for any $0 \le j \le n-1$. We consider the vector $v \in \mathbb{Q}^{n-1}$ with $a_{0,k}$ for the kth entry for any $k \in C$ and 0 elsewhere. Then

$$Fv = \begin{bmatrix} \omega^{1\cdot 1} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{(n-1)\cdot 1} & \omega^{(n-1)\cdot 2} & \cdots & \omega^{(n-1)\cdot(n-1)} \end{bmatrix} \begin{bmatrix} a_{0,1} \\ a_{0,2} \\ \vdots \\ a_{0,n-1} \end{bmatrix} = \begin{bmatrix} \sum_{k\in C} a_{0,k}\omega^{1\cdot k} \\ \sum_{k\in C} a_{0,k}\omega^{2\cdot k} \\ \vdots \\ \sum_{k\in C} a_{0,k}\omega^{(n-1)\cdot k} \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} \in \mathbb{Z}^{n-1}.$$

It follows that $v \in \mathcal{A}$ in Proposition 2.1 (2), and hence $v = \sum_{d|n, d<n} c_d v_d$ for some rational coefficients c_d's. The definition of v implies that the coefficient $c_d \in \left\{0, 1, \ldots, \binom{n}{t-2}\right\}$. Therefore, C is a union of some $G_n(d)$'s where for each such d, we have $a_{0,k} = c_d$ for all $k \in G_n(d)$. □

Remark. In particular, for $t = 2$, we have $S = C$. Theorem 2.2 implies that $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$ is integral if and only if S is a union of some $G_n(d)$'s and for which d, $a_{0,k} = 1$ for all $k \in G_n(d)$. This coincides So's result recalled in Theorem 1.4.

Let $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ be a gcd-hypergraph. We shall use Theorem 2.2 to show that H is integral. By Example 1.1, $S = \bigcup_{e\in D} G_n(e)$ for some set D of proper divisors of n. Since $lG_n(e) = G_n(\gcd(l, n)e)$ for any $l \in \{1, 2, \ldots, n-1\}$, we have $C = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$ equals $\bigcup_{d\in D'} G_n(d)$ for some set D' of proper divisors of n and $D \subseteq D'$. Let $d \in D'$ and $k \in G_n(d)$. We aim to show that $a_{0,k}$'s are identical for all $k \in G_n(d)$. Let $d \in D'$ and $k, k' \in G_n(d)$. There is $u \in G_n(1)$ such that $k' = uk$. Since hyperedges containing 0 are $\{(i-j)x : 0 \le i \le t-1\}$ where $x \in S$ and $0 \le j \le t-1$, we count such hyperedges containing k. For each $e \in D$, let $N_{d,k}(e)$ be the number of hyperedges containing 0 and k of the form $\{(i-j)x : 0 \le i \le t-1\}$ with $x \in G_n(e)$. For any $e, f \in D$ with $e \ne f$, such hyperedges with $x \in G_n(e)$ and $x \in G_n(f)$ are distinct, so

$$a_{0,k} = \sum_{e\in D} N_{d,k}(e).$$

Let $S_k = \{l : 1 \le l \le t-1 \text{ and } k \in lG_n(e)\}$. Since $G_n(d) = lG_n(e)$ for all $l \in S_k$ and $k' = uk$, it follows that $N_{d,k}(e) = N_{d,k'}(e)$. Hence,

$$a_{0,k} = \sum_{e\in D} N_{d,k}(e) = \sum_{e\in D} N_{d,k'}(e) = a_{0,k'}.$$

Therefore, we can conclude that H is integral by Theorem 2.2. We record this result in the following theorem.

Theorem 2.3. *Let $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ be a gcd-hypergraph of $\mathbb{Z}_n$ where $S = \bigcup_{e\in D} G_n(e)$ for some set D of proper divisors of n and $C = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\} = \bigcup_{d\in D'} G_n(d)$ for some set D' of proper divisors of n and $D \subseteq D'$. Let $d \in D'$ and*

$k \in G_n(d)$. For each $e \in D$, let $N_{d,k}(e)$ be the number of hyperedges containing 0 and k of the form $\{(i-j)x : 0 \leq i \leq t-1\}$ with $x \in G_n(e)$. Then

$$a_{0,k} = \sum_{e \in D} N_{d,k}(e).$$

Moreover, $a_{0,k}$'s are identical for all $k \in G_n(d)$ and H is an integral hypergraph.

Remark. Let $d \in D'$, $k \in G_n(d)$ and $e \in D$. If $k \notin lG_n(e)$ for all $l \in \{1, 2, \dots, t-1\}$, then hyperedges containing 0 of the form $\{(i-j)x : 0 \leq i \leq t-1\}$ where $x \in G_n(e)$ and $0 \leq j \leq t-1$ do not contain k, so $N_{d,k}(e) = 0$. Next, assume that $S_k = \{l : 1 \leq l \leq t-1 \text{ and } k \in lG_n(e)\} \neq \emptyset$. Note that $o(x) = \frac{n}{e}$ for all $x \in G_n(e)$. If $\frac{n}{e} \leq t$, then $\{(i-j)x : 0 \leq i \leq t-1\} = \langle x \rangle = e\mathbb{Z}_n$ for all $x \in G_n(e)$ and $0 \leq j \leq t-1$, so we have only one hyperedge containing 0 and k and $N_{d,k}(e) = 1$. Suppose that $\frac{n}{e} > t$. Let $l \in S_k$. Since $k \in lG_n(e)$, there is $x \in G_n(e)$ such that $k = lx$. We wish to find the number of elements y in $G_n(e)$ such that $k = ly$. Since $k \in lG_n(e)$, we have $d = \gcd(l, n)e$, so

$$G_n(d) = G_n(\gcd(l, n)e) = lG_n(e) = leG_{\frac{n}{e}}(1).$$

Suppose that $k = lx = leu$ for some $u \in G_{\frac{n}{e}}(1)$. To find the number of such y's in $G_n(e)$, it is equivalent to find the number of elements v in $G_{\frac{n}{e}}(1)$ such that $k = lev$. Now, we count such v's. For any $v \in G_{\frac{n}{e}}(1)$ with $k = lev$, we have $lev \equiv leu \mod n$, so $l(v-u) \equiv 0 \mod \frac{n}{e}$. If $v - u \equiv 0 \mod \frac{n}{e}$, then $l \cdot 0 \equiv 0 \mod \frac{n}{e}$, and if $v - u \not\equiv 0 \mod \frac{n}{e}$, then there are $q \in \mathbb{Z}$ and $r \in \{1, 2, \dots, \frac{n}{e} - 1\}$ such that $v = u + \frac{n}{e}q + r$. Consequently, $l(v-u) \equiv 0 \mod \frac{n}{e}$ if and only if $lr \equiv 0 \mod \frac{n}{e}$. Thus, the number of v in $G_{\frac{n}{e}}(1)$ such that $k \equiv lev \mod n$ equals the number of r in $\{0, 1, \dots, \frac{n}{e} - 1\}$ such that $lr \equiv 0 \mod \frac{n}{e}$. Note that $|G_n(e)| = \phi\left(\frac{n}{e}\right)$ if e is a divisor of n. Since this number is independent of k, there are exactly $\frac{\phi(n/e)}{\phi(n/d)}$ elements, say $v_1, v_2, \dots, v_{\frac{\phi(n/e)}{\phi(n/d)}}$, in $G_{\frac{n}{e}}(1)$ such that $k = lev_i$ for all $i \in \left\{1, 2, \dots, \frac{\phi(n/e)}{\phi(n/d)}\right\}$. Let $y_i = ev_i$ for all $i \in \left\{1, 2, \dots, \frac{\phi(n/e)}{\phi(n/d)}\right\}$. Since $o(y_i) = \frac{n}{e} > t$, the sets

$$\{(l-t+1)y_i, (l-t)y_i, \dots, 0, \dots, ly_i\}, \{(l-t)y_i, (l-t-1)y_i, \dots, 0, \dots, ly_i, (l+1)y_i\}, \\ \dots, \{0, \dots, ly_i, (l+1)y_i, \dots, (t-1)y_i\}$$

are hyperedges of H containing 0 and k for all $i \in \left\{1, 2, \dots, \frac{\phi(n/e)}{\phi(n/d)}\right\}$. Thus,

$$N_{d,k}(e) = \left| \bigcup_{l \in S_k} \left\{ \{(i-j)y_m : 0 \leq i \leq t-1\} : 0 \leq j \leq t-1-l, 1 \leq m \leq \frac{\phi(n/e)}{\phi(n/d)} \right\} \right|.$$

However, these hyperedges may not be distinct, so $N_{d,k} \leq \sum_{l \in S_k} (t-l) \cdot \frac{\phi(n/e)}{\phi(n/d)}$.

Example 2.4. By Theorem 1.4, an integral 2-Cay$(\mathbb{Z}_n, S)$ is a gcd-graph. However, an integral t-Cay$(\mathbb{Z}_n, S)$ may not be a gcd-hypergraph when $t \geq 3$. For example, if $\mathrm{H} = 5\text{-Cay}(\mathbb{Z}_5, \{\pm 1\})$ which is not a gcd-hypergraph of $\mathbb{Z}_5$, then $E(\mathrm{H}) = \{\{0, 1, 2, 3, 4\}\}$. Hence, $C = \mathbb{Z}_5 \setminus \{0\} = G_5(1)$ and $a_{0,k} = 1$ for any $k \in C$, but H is integral by Theorem 2.2.

Finally, we study L-integral and D-integral t-Cayley hypergraphs. We start with a simple result on L-integral t-Cayley hypergraphs obtained by Proposition 1.3, Theorems 2.2 and 2.3. Let $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$. By Proposition 1.3, H is regular, so there exists $d \in \mathbb{N}$ such that $\deg k = d$ for any $0 \leq k \leq n-1$. It follows that

$$L(\mathrm{H}) = \mathcal{D}(\mathrm{H}) - A(\mathrm{H}) = dI_n - A(\mathrm{H}).$$

Hence,

$$\mathrm{Lspec}(\mathrm{H}) = \{d - \lambda : \lambda \in \mathrm{Spec}(\mathrm{H})\}.$$

By Theorems 2.2 and 2.3, we easily get

Corollary 2.5. *Let $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$. Then H is L-integral if and only if H is integral. In particular, a gcd-hypergraph of $\mathbb{Z}_n$ is L-integral.*

Now, we consider D-integral t-Cayley hypergraphs. For $t = 2$, Ilić [10] showed that a gcd-graph of $\mathbb{Z}_n$ is D-integral. Assume that $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ is connected. That is, $\langle S \rangle = G$ by Proposition 1.2 (1). By the natural labeling in $D(\mathrm{H})$, it is clear that $D(\mathrm{H})$ is circulant. Thus, it suffices to consider the first row of $D(\mathrm{H})$. Since H is connected, the set $\{k : d(0, k) \neq 0\} = \{1, 2, \dots, n-1\}$. Hence, we get a characterization of D-integral t-Cayley hypergraphs similar to Theorem 2.2.

Theorem 2.6. *Assume that $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ is connected. Then H is D-integral if and only if for each $d \mid n$, there is $c_d \in \{1, 2, \dots, \mathrm{diam}(\mathrm{H})\}$ such that $d(0, k) = c_d$ for all $k \in G_n(d)$.*

Let H $= t$-Cay$(\mathbb{Z}_n, S)$. We observe that $d(0, k)$ is the distance between 0 and k in 2-Cay$(\mathbb{Z}_n, C)$ where $C = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$. Hence, the distance matrix $D(\mathrm{H}) = D(2\text{-Cay}(\mathbb{Z}_n, C))$. If H is a gcd-hypergraph, then 2-Cay$(\mathbb{Z}_n, C)$ is also a gcd-graph. This implies that 2-Cay$(\mathbb{Z}_n, C)$ is D-integral [10]. Consequently, H is D-integral and we obtain the following theorem.

Theorem 2.7. *A* gcd-*hypergraph of* $\mathbb{Z}_n$ *is* D-*integral.*

Remark. Let $S = S_1 \times S_2$ be a subset of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \setminus \{(0, 0)\}$ such that $S = -S$ and H $= t$-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S)$. Suppose that $S_1 \neq \{0\}$ and $S_2 \neq \{0\}$. We observe that t-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S_1 \times S_2)$ is a subgraph of t-Cay$(\mathbb{Z}_{n_1}, S_1) \otimes t$-Cay$(\mathbb{Z}_{n_2}, S_2)$. Fix two vertices $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Let $\{x + ix' : 0 \leq i \leq t-1\}$ be a hyperedge in t-Cay$(\mathbb{Z}_{n_1}, S_1)$ containing both of x_1 and x_2 and let $\{y + iy' : 0 \leq i \leq t-1\}$ a hyperedge in t-Cay$(\mathbb{Z}_{n_2}, S_2)$ containing both of y_1 and y_2. Then $\{(x, y) + i(x', y') : 0 \leq i \leq t-1\}$ is a hyperedge in t-Cay$(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S_1 \times S_2)$. But when $t \geq 3$, the problem is that it may not contain (x_1, y_1) and (x_2, y_2). This means that $A(t\text{-Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, S_1 \times S_2))$ may not equal to $A(t\text{-Cay}(\mathbb{Z}_{n_1}, S_1) \otimes t\text{-Cay}(\mathbb{Z}_{n_2}, S_2))$ when $t \geq 3$. Hence, a characterization of integral t-Cayley hypergraphs of finite abelian groups is still an open problem when $t \geq 3$.

3. Algebraic degree of spectra of t-Cayley hypergraphs of $\mathbb{Z}_n$

Let H be a hypergraph on m vertices and $f(x) = \det(xI_m - A(\mathrm{H}))$ the characteristic polynomial of H. Let E_f be the splitting field of $f(x)$ over $\mathbb{Q}$. The *algebraic degree* of H is $[E_f : \mathbb{Q}]$ and denoted by deg H. By Theorem 2.2, we have a characterization of integral t-Cayley hypergraphs of $\mathbb{Z}_n$. They are hypergraphs of $\mathbb{Z}_n$ of algebraic degree one. We study the algebraic degree of t-Cayley hypergraphs of $\mathbb{Z}_n$ in this section.

Let $n \geq 3$ and H $= t$-Cay$(\mathbb{Z}_n, S)$. Recall from the beginning of Section 2 that the eigenvalues of H are

$$\lambda_j = \sum_{k \in C} a_{0,k} (e^{2\pi j i/n})^k$$

where $C = \{k : a_{0,k} \neq 0\} = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$ and $j \in \{0, 1, \ldots, n-1\}$. Let $\omega = e^{2\pi i/n}$ be a primitive nth root of unity. By the fundamental theorem of Galois theory,

$$\deg \mathrm{H} = [\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) : \mathbb{Q}] = \frac{\phi(n)}{|\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}))|}, \tag{C}$$

where $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})) = \{\sigma \in \mathrm{Aut}(\mathbb{Q}(\omega)) : \sigma$ is a $\mathbb{Q}$-automorphism and $\sigma(\lambda_j) = \lambda_j$ for all $j \in \{0, 1, \ldots, n-1\}\}$. We shall determine the size of this group and obtain the algebraic degree of H.

Lemma 3.1. *Let* $y \in \{0, 1, \ldots, n-1\}$ *be such that* $\gcd(y, n) = 1$ *and* $\sigma_y \in \mathrm{Aut}(\mathbb{Q}(\omega))$ *be the* $\mathbb{Q}$-*automorphism defined by* $\omega \mapsto \omega^y$. *Then* $\sigma_y(\lambda_j) = \lambda_j$ *for all* $j \in \{0, 1, \ldots, n-1\}$ *if and only if there is* $n_y \in \mathbb{N}$ *with* $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ *and* $a_{0,k} = a_{0,yk}$ *for all* $k \in C_l$ *and* $l \in \{1, 2, \ldots, n_y\}$.

Proof. If there is an $n_y \in \mathbb{N}$ with $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ and $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$, then

$$\begin{aligned}\sigma_y(\lambda_j) = \sigma_y\left(\sum_{k \in C} a_{0,k}\omega^{jk}\right) &= \sum_{l=1}^{n_y}\sum_{k \in C_l} a_{0,k}\sigma_y\left(\omega^{jk}\right) = \sum_{l=1}^{n_y}\sum_{k \in C_l} a_{0,k}\omega^{jky} \\ &= \sum_{l=1}^{n_y}\sum_{k \in C_l} a_{0,yk}\omega^{jky} = \sum_{k \in C} a_{0,yk}\omega^{jyk} = \sum_{yk \in C} a_{0,yk}\omega^{jyk} = \lambda_j\end{aligned}$$

for all $j \in \{0, 1, \ldots, n-1\}$. On the other hand, suppose that $\sigma_y(\lambda_j) = \lambda_j$ for all $j \in \{0, 1, \ldots, n-1\}$. Then $\sum_{k \in C} a_{0,k}\left(\omega^j\right)^{yk} = \sum_{k \in C} a_{0,k}\left(\omega^j\right)^k$ for all $j \in \{0, 1, \ldots, n-1\}$. Let $p(x) = \sum_{k \in C} a_{0,k}x^{yk} - \sum_{k \in C} a_{0,k}x^k$. It is a polynomial of degree at most $n-1$. Since $1, \omega, \ldots, \omega^{n-1}$ are distinct roots of $p(x)$, we have $p(x) = 0$. Define an equivalence relation on C by $k \sim k'$ whenever $a_{0,k} = a_{0,k'}$. Let $C_1, \ldots, C_{n_y}$ be all equivalence classes of $\sim$. Then $C = C_1 \cup \cdots \cup C_{n_y}$. Since $p(x) = 0$, we have $yC_l \equiv C_l \mod n$ and so $a_{0,k} = a_{0,yk}$ for all $k \in C_l$ and $l \in \{1, 2, \ldots, n_y\}$. □

Theorem 3.2. *Let* H $= t$-Cay$(\mathbb{Z}_n, S)$ *and* $C = S \cup 2S \cup \cdots \cup (t-1)S \setminus \{0\}$. *Let* m *be the number of* y *in* $\{0, 1, \ldots, n-1\}$ *such that* $\gcd(y, n) = 1$ *and there is an* $n_y \in \mathbb{N}$ *with* $C = C_1 \cup \cdots \cup C_{n_y}$, $yC_l \equiv C_l \mod n$ *and* $a_{0,k} = a_{0,yk}$ *for all* $k \in C_l$ *and* $l \in \{1, 2, \ldots, n_y\}$. *Then*

$$\deg \mathrm{H} = \frac{\phi(n)}{m}.$$

Moreover, $\deg \mathrm{H} \leq \frac{\phi(n)}{2}$.

Proof. By Lemma 3.1, m is the size of $\mathrm{Gal}\,(\mathbb{Q}(\omega)/\mathbb{Q}\,(\lambda_0, \lambda_1, \dots, \lambda_{n-1}))$. It follows from Eq. (C) that $\deg \mathrm{H} = \frac{\phi(n)}{m}$. From $S \equiv -S \mod n$, we have $C = -C \mod n$. Since $\{\pm k\} = -\{\pm k\}$ and $a_{0,k} = a_{0,-k}$ for any $k \in C$, 1 and -1 are such y. Hence, $m \geq 2$, so $\frac{\phi(n)}{m} \leq \frac{\phi(n)}{2}$. $\square$

Consider $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$. Then $C = S$ and $a_{0,k} = 1$ for any $k \in S$ and $a_{0,k} = 0$ otherwise. The assumption of Theorem 3.2 can be reduced to $yS \equiv S \mod n$. In addition, if $n = p$ is a prime number, Mönius showed in the proof of Theorem 2.5 of [13] that m in Theorem 3.2 is the maximum number of $M \in \{1, 2, \dots, |S|\}$ such that M divides $\gcd(|S|, p-1)$ and

$$S = \bigcup_{l=1}^{|S|/M} S_l$$

where $|S_l| = M$ and for each $l \in \{1, 2, \dots, |S|/M, k^M = (k')^M \mod p\}$ for all $k, k' \in S_l$. The next corollary gives the algebraic degree of Cayley graph of $\mathbb{Z}_n$ over S which generalizes Theorem 2.5 of [13].

Corollary 3.3. *Let $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_n, S)$. If m is the number of y in $\{0, 1, \dots, n-1\}$ such that $yS \equiv S \mod n$, then*

$$\deg \mathrm{H} = \frac{\phi(n)}{m}.$$

Example 3.4. Consider $\mathrm{H} = 2\text{-Cay}(\mathbb{Z}_{31}, S)$ where $S = \{\pm 2, \pm 3, \pm 10, \pm 12, \pm 13, \pm 15\} = C$. Since $\pm 1, \pm 5, \pm 6$ are all elements of y such that $\gcd(y, 31) = 1$ and $yC \equiv C \mod 31$, by Corollary 3.3, $\deg \mathrm{H} = \frac{\phi(31)}{6} = 5$. This coincides Example 2.10 of [13].

In the proof of Theorem 3.2, we have known that 1 and -1 are always such y satisfying $yC \equiv C \mod n$. If only they satisfy this congruence, we have a special case of Theorem 3.2 as follows.

Corollary 3.5. *Let $\mathrm{H} = t\text{-Cay}(\mathbb{Z}_n, S)$ and $C = S \cup 2S \cup \dots \cup (t-1)S \setminus \{0\}$. If $y = 1$ and $y = -1$ are the only elements in $\mathbb{Z}_n$ such that $\gcd(y, n) = 1$ and $yC \equiv C \mod n$, then*

$$\deg \mathrm{H} = \frac{\phi(n)}{2}.$$

We provide some numerical examples using Theorem 3.2 and Corollary 3.5 as follows.

Example 3.6. Consider $\mathrm{H} = 3\text{-Cay}(\mathbb{Z}_{12}, \{\pm 1\})$. We have $C = \{\pm 1, \pm 2\}$. In addition, $a_{0,\pm 1} = 2$ and $a_{0,\pm 2} = 1$. The characteristic polynomial of $A(\mathrm{H})$ is

$$(x-1)^2(x+2)^3(x+3)^2(x-6)(x^2-2x-11)^2$$

and hence $\deg \mathrm{H} = 2$. Since 1 and -1 are the only elements y in $\mathbb{Z}_{12}$ such that $\gcd(y, 12) = 1$ and $yC \equiv C \mod 12$, by Corollary 3.5, $\deg \mathrm{H} = \frac{\phi(12)}{2} = 2$.

Example 3.7. Let $S = \{\pm 1\}$ be a subset of $(\mathbb{Z}_9, +)$. Them $\max\{o(x) : x \in S\} = 9$, so $2 \leq t \leq 9$. The algebraic degree of t-Cayley hypergraph of $\mathbb{Z}_9$ over S for all t are presented in the following table. The cases $t \in \{2, 3, 4\}$ are computed by Corollary 3.5 and the others are obtained from Theorem 3.2.

t	$a_{0,\pm 1}$	$a_{0,\pm 2}$	$a_{0,\pm 3}$	$a_{0,\pm 4}$	y with $yC \equiv C \mod 9$	$\deg t\text{-Cay}(\mathbb{Z}_9, S)$
2	1				± 1	3
3	2	1			± 1	3
4	3	2	1		± 1	3
5	4	3	2	1	$\pm 1, \pm 2, \pm 4$	3
6	5	4	3	3	$\pm 1, \pm 2, \pm 4$	3
7	6	5	5	5	$\pm 1, \pm 2, \pm 4$	3
8	7	7	7	7	$\pm 1, \pm 2, \pm 4$	1
9	1	1	1	1	$\pm 1, \pm 2, \pm 4$	1

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