รายงานวิจัยฉบับสมบูรณ์

การวิเคราะห์สูตรสำเร็จของความยาววันเฉลี่ยของแผนภูมิควบคุมค่าเฉลี่ยเคลื่อนที่ถ่วงน้ำหนักด้วยเลขชี้กำลังแบบปรับปรุงสำหรับการตรวจจับค่าเฉลี่ยของกระบวนการ

An Explicit Analytical Expression of Average Run Length of Modified EWMA Control Chart for Monitoring Process Mean

โดย ศาสตราจารย์ ดร. ยุพาภรณ์ อารีพงษ์
มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าพระนครเหนือ

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สัญญาเลขที่ RSA6280086

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และมหาวิทยาลัยเทคโนโลยีพระจอมเกล้าพระนครเหนือ

Abstract

Project Code: RSA6280086

Project Title: An Explicit Analytical Expression of Average Run Length of Modified EWMA Control Chart for Monitoring Process Mean

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The Modified Exponentially Weighted Moving Average (Modified EWMA) control chart is widely used in a great variety of practical applications such as finance, medicine, engineering, psychology and in other areas. There are many situations in which the process is serially correlation such as in the manufacturing industry, for example, the dynamics of the process will induce correlations in observations that are closely spaced in time. The Average Run Length (ARL) is a traditional measurement of the performance of control chart. The main goal of this research is to derive analytical solutions for the ARL of Modified EWMA control chart when observations are autoregressive and moving average processes with exponential white noise. In this research, we establish that the ARL is unique solution to the integral equation under some weak regularity conditions. Checking the accuracy of results, we compared the results obtained from explicit formulas with numerical integral equation based on Gauss-Legendre rule. In addition, the performance of Modified EWMA and the Exponentially Weighted Moving Average (EWMA) control charts are studied. The explicit formulas for the ARL of Modified EWMA chart can be applied to real data, empirical data, and real-world situations applications for a variety of data processes.

Keywords: Autoregressive process, Moving Average process, Modified Exponentially Moving Average control chart, Average Run Length

บทคัดย่อ

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แผนภูมิควบคุมค่าเฉลี่ยเคลื่อนที่ถ่วงน้ำหนักแบบเลขชี้กำลังปรับปรุง (Modified EWMA) มีการใช้กันอย่างแพร่หลาย โดยนำไปใช้งานจริงในหลายๆด้าน เช่น การเงิน การแพทย์ วิศวกรรมศาสตร์ จิตวิทยา และในด้านอื่นๆ ในปัจจุบันจะพบว่ามีหลายสถานการณ์ที่กระบวนการมีความสัมพันธ์กัน เช่น ในอุตสาหกรรมการผลิต ตัวอย่างเช่น พลวัตของกระบวนการจะทำให้เกิดความสัมพันธ์ในค่าสังเกตที่เมื่อเวลาใกล้กัน ในงานวิจัยนี้ได้ศึกษาการหาค่าความยาวรันเฉลี่ย (ARL) ซึ่งคือเกณฑ์ที่ใช้ในการเปรียบเทียบประสิทธิภาพของแผนภูมิควบคุมโดยวิธีสูตรสำเร็จของแผนภูมิควบคุม Modified EWMA สำหรับตรวจจับค่าเฉลี่ยของกระบวนการเมื่อข้อมูลมีกระบวนการถดถอยในตัวและค่าเฉลี่ยเคลื่อนที่ และนำผลลัพธ์ที่ได้จากสูตรสำเร็จเปรียบเทียบกับค่า ARL ที่ได้จากวิธีสมการปริพันธ์เชิงตัวเลขตามกฎ Gauss-Legendre ผลการวิจัยพบว่าค่าความยาวรันเฉลี่ยที่คำนวณได้จากสูตรสำเร็จมีความถูกต้องแม่นยำเมื่อเปรียบเทียบกับวิธีสมการปริพันธ์เชิงตัวเลข จากนั้นได้มีการเปรียบเทียบประสิทธิภาพของแผนภูมิควบคุม Modified EWMA และแผนภูมิควบคุมค่าเฉลี่ยเคลื่อนที่ถ่วงน้ำหนักแบบเลขชี้กำลัง (EWMA) นอกจากนี้ในงานวิจัยนี้ได้นำสูตรสำเร็จค่า ARL ของแผนภูมิ Modified EWMA ไปประยุกต์ใช้กับข้อมูลจริงสำหรับกระบวนการที่หลากหลาย

คำหลัก: กระบวนการถดถอยในตัว กระบวนการค่าเฉลี่ยเคลื่อนที่ แผนภูมิควบคุมค่าเฉลี่ยเคลื่อที่ถ่วงน้ำหนักแบบเลขชี้กำลังปรับปรุง ค่าความยาวรันเฉลี่ย

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Yupaporn Areepong

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Introduction

1.1 Background and Importance

Statistical Process Control (SPC) plays a vital role in monitoring and detecting changes in a process, and uses for measuring, controlling and improving quality in many areas such as industrial and manufacturing, finance and economics, computer sciences and telecommunications networks, epidemiology, and other areas of applications (see Frisen (1992); Knoth (2007); Mazalov and Zhuralav (2002)). Control charts are one of the efficient tools of SPC for detecting changes in mean or variations of a process. Examples of SPC charts are the Shewhart control chart proposed by Shewhart (1931), Cumulative Sum (CUSUM) control chart introduced by Page (1954), and Exponentially Weighted Moving Average (EWMA) control chart proposed by Roberts (1959). The Shewhart control chart is used for detecting larger disturbances in process parameters, whereas CUSUM and EWMA charts are used for smaller and moderate changes. The CUSUM and EWMA control charts are greatly well known for using not only in several industry and manufacturing processes but also surveillance problem. The literatures concerning to detect a small shift process are recommended such as Yashchin (1993), Zhang (1998), Psarakis and Papaleonida (2007), Prajapati (2015) and Mawonike and Nkomo (2015) shown that the CUSUM and EWMA control charts are much more effective than the Shewhart control chart in monitoring of small changes with autocorrelated process.

The modified EWMA control chart developed by Patel and Divecha (2011) is a simplified EWMA control chart for detecting shifts in the process mean regardless of size. It is used in various fields, especially in a chemical industry which the processes are frequently autocorrelated. Past observations are considered (similar to the EWMA scheme) along with the past changes as well as the latest change in the process mean. Khan, Aslam, and Jun (2016) developed a new EWMA control chart based upon a modified EWMA statistic that considers the past and current behavior of a process; they compared it with the existing one by Patel and Divecha (2011) and found that the proposed control chart gains the ability to detect shifts more quickly.

A basic assumption of control charts is that observations from a process at different times are independent and identically distributed (i.i.d.) random variables. However, in many situations, a process does not yield sufficient observations for traditional SPC tools to be used effectively. However, production process observations often show some autocorrelation. Continuous product manufacturing operations such as the manufacture of food, chemicals, paper, and other wood products often exhibit serial correlation. Moreover, autocorrelation data can also be shown in data arising from computer intrusion detection, wind speeds, and the daily flow of a river. Several researchers in different fields of study have considered the problem of data correlation and how it relates to SPC.

Nowadays, there are a lot of techniques to analyze and forecast time series data. One of the widely used techniques is autoregressive integrated moving average (ARIMA) model. This technique uses univariate time series data to analyze its own trend, seasonal, and forecast future cycle. The general function model employed by the ARIMA model was discussed by Box and Tiao [3]. When an ARIMA model includes other time series as input variables, the model is sometimes referred to as an ARIMAX model. Pankratz (1991) refers to the ARIMAX model as a dynamic regression. Exponential white noise coordinated with time series has also been investigated. Jacob and Lewis (1971) studied autoregressive moving average process order (1,1) denoted by ARMA (1,1) when observations were exponentially distributed with exponential white noise. The exponential white noise was also used to analyze the autoregressive model, proposed by Mohamed and Hocine (2010).

The evaluation of the performance of a control chart is performed by Average Run Length (*ARL)*. Generally, the *ARL* is evaluated for a zero shift in the process level by its in-control *ARL,* which is denoted by ARL_0. The ARL_0 is the average number of observations taken before the signals. The *ARL* should be large when the process is in control and has no change. And for each level of shift values; on the other hand, it should be detected quickly by its out-of-control *ARL,* which is denoted by ARL_1, whereas the ARL_1 should be small when the process is out of control and undergoes a change.

Three standard methods that are often used to evaluate Average Run Length for the in control process (ARL_0) and the Average Run Length for the out of control process (ARL_1) are the Markov Chain Approach (MCA), the Numerical Integral Equation (NIE), and the Monte Carlo simulation (MC) methods. However, these methods have the following characteristics. The NIE method is the most advanced method but it requires a lot of programming and computation. MCA requires a discretization of the continuity of the process into many steps and a large number of calculations of matrix inverse. MC is a simple to program and good for checking accuracy but it requires a large number of sample trajectories. Therefore, this method is usually very time consuming; in addition, it is also difficult and laborious to find the optimal designs. All of the previous methods have limitations when computing ARL with the exception of the NIE method. This can also be extended to explicit formulas.

In this research, we derive explicit formulas for detecting changes in mean of the time series processes of Modified EWMA control chart with exponential white noise. The performance of the explicit formulas and the numerical integral equation method of ARL are compared. In addition, the performance of Modified EWMA and EWMA control charts are studied, and also apply the explicit formulas for the ARL of Modified EWMA chart to practical and empirical data.

1.2 Objectives

The objectives of the research project are as follows:

1.2.1 To derive explicit formulas and approximate the numerical integral equation method of average run length on Modified EWMA control chart for moving average process in the case of exponential white noise.

1.2.2 To derive explicit formulas and approximate the numerical integral equation method of average run length on Modified EWMA control chart for autoregressive process with exogenous variable (*ARX(p,r)*) in the case of exponential white noise.

1.2.3 To compare the efficiency of the explicit formulas and the numerical integral equation method of ARL.

1.2.4 To compare the performance of Modified EWMA and EWMA control charts.

1.2.5 To apply the explicit formulas for the ARL of Modified EWMA chart to real data or other control charts for a variety of data processes

1.3 Scope of the Study

In this study, we focus on analytical explicit formulas and numerical integral equation approximation of ARL for Modified EWMA and EWMA control charts. These include:

1.3.1 We define the parameters of the exponential distribution on Modified EWMA and EWMA control chart as follows:

Parameter $\alpha_0 = 1$, where the process is in-control.

Parameter $\alpha_1 = (1+\delta)\alpha_0,\ \alpha_1 > 1$, where the process is out-of-control and $\delta =$0.005, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.20, 0.50, 1.00, 2.00, 5.00

1.3.2 The recursive equation of the Modified EWMA control chart is as follows:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t + (Y_t - Y_{t-1}) \text{ , } t = 1, 2, ...$$

where λ is a smoothing parameter, $0 < \lambda \le 1$. The corresponding stopping time shows out-of-control as equation $\tau = \inf\{t > 0; Z_t > b\}$, $b > u$, where b is a control limit and $\lambda = 0.05, 0.1$ with initial value $Y_0 = u = 1$.

1.3.3 The autoregressive process with exogenous variable (ARX(p,r)) is defined as

$$Y_t = \mu + \varepsilon_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - ... - \phi_p Y_{t-p} + \sum_{i=1}^{r} \beta_i X_{it}$$

where $|\phi_i| < 1$ for $\forall_i = 1, 2, ...$,and $\varepsilon_t \sim Exp(\alpha)$.

1.3.4 The moving average process with exogenous variable (*MAX(q,r)*) is defined as:

$$Y_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - ... - \theta_q \varepsilon_{t-q} + \sum_{i=1}^{r} \beta_i X_{it}$$

where $|\theta_i| < 1$ for $\forall_i = 1, 2, ...$ and $\varepsilon_t \sim Exp(\alpha)$.

1.3.5 Fredholm Integral Equation of Second Type is used to analyze explicit formulas of ARL_0 and ARL_1 .

1.3.6 Gauss-Legendre quadrature rule is used to approximate of ARL_0 and ARL_1 in the numerical integral equation method.

1.3.7 Set in-control ARL : $ARL_0 = 370$ and 500.

1.3.8 The ARL result obtained from the NIE method uses the division point $m = 1,000$ nodes.

1.3.9 The comparisons of numerical result between the explicit formulas and NIE method are determined with the absolute percentage relative error (*Diff*) to verify the accuracy of the ARL results.

1.4 Utilization of the Study

1.4.1 The explicit formula of ARL for $MA(1)$, $ARX(p,r)$ processes with exponential white noise for modified EWMA control chart could be applied in real applications, for instance economics and financial, environment and ecologic, manufacturing process, and etc.

1.4.2 The explicit formulas of ARL_0 and ARL_1 by using Fredholm Integral Equation of second type technique can be extended to another process, for example $ARIMA(p,d,q)$, and etc. and can be applied in a new control chart.

Chapter 2
Literature Review

This chapter comprises four sections. Section 2.1 contains the characteristic of control charts and their properties. Section 2.2 is presented the process observations used in this study. An evaluation of the ARL is discussed in Section 2.3, while the Gaussian rules are described in Section 2.4, and finally, numerical methods for solving the NIE are detailed in Section 2.5.

2.1 Characteristics of Control Charts and Their Properties

2.1.1 Exponentially Weighted Moving Average (EWMA) control chart

The EWMA control chart was first introduced by Roberts (1959). The EWMA control chart can be expressed by the recursive equation below,

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t \text{ , } t = 1,2,3,..., \tag{2.1}$$

where λ is an exponential smoothing parameter, which is $0 < \lambda \leq 1$.The starting value is $Z_0 = X_0$ and the target value μ_0 and X_t are the process with mean μ and variance σ^2. Then, the variance of Z_t is $\sigma^2_{Z_i} = \sigma^2 \left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2i}\right)$. If i gets large, the term $(1-\lambda)^{2i}$ converges to 0. Therefore, the general upper control limit (UCL) and lower control limit (LCL) to detect the sequence Z_t is given by,

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \tag{2.2}$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \tag{2.3}$$

where μ_0 is the target mean, σ is the process standard deviation, and L is the appropriate control width limit.

The stopping time of EWMA chart is given by

$$\tau_h = \inf\{t > 0;\ Z_t > h\}, h > u \text{ ,} \tag{2.4}$$

here τ_h is the stopping time and h is the upper control limit (UCL).

2.1.2 Modified Exponentially Weighted Moving Average (modified EWMA) control chart

A modified EWMA control chart statistic that is a correction of the original EWMA control statistic and is also free from the inertia problem was developed and presented by Patel and Divecha (2011). It is very effective at detecting small and abrupt shifts in the process mean for observations that are independent and normally distributed or autocorrelated. Subsequently, the modified EWMA control statistic was redesigned by Khan et al. (2016) to be more efficient than either the traditional or modified EWMA control charts.

The modified EWMA control chart can be defined by the recursive equation below,

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t + c(X_t - X_{t-1})\ ,\ t = 1,2,3,..., \tag{2.5}$$

where c is a constant.

The mean of the modified EWMA control statistic is $E(Z_t) = \mu_0$. It may be shown that

$$\begin{aligned}
Z_t &= (1-\lambda)Z_{t-1} + \lambda X_t + c(X_t - X_{t-1}) \\
&= (1-\lambda)\left[(1-\lambda)Z_{t-2} + \lambda X_{t-1} + c(X_{t-1} - X_{t-2})\right] + \lambda X_t + c(X_t - X_{t-1}) \\
&= (1-\lambda)^3 Z_{t-3} + \lambda(1-\lambda)^2 X_{t-2} + (1-\lambda)^2 c(X_{t-2} - X_{t-3}) + \lambda(1-\lambda)X_{t-1} \\
&\quad + (1-\lambda)c(X_{t-1} - X_{t-2}) + \lambda X_t + (1-\lambda)^0 c(X_t - X_{t-1}).
\end{aligned}$$

And continuing like this recursively for $X_{t-j}\,;\, j = 1,2,3,...,t$, we obtain

$$Z_t = \lambda \sum_{j=0}^{t-1} (1-\lambda)^j X_{t-j} + (1-\lambda)^t Z_0 + c\sum_{j=0}^{t-1}(1-\lambda)^j (X_{t-j} - X_{t-j-1}).$$

Hence, $c\sum_{j=0}^{t-1}(1-\lambda)^j (X_{t-j} - X_{t-j-1})$ accounts for sum of the past and latest change in the process. The unaccounted current fluctuations accumulated to time t in EWMA statistic.

Let $$Z_t = (1-\lambda)^t Z_0 + \sum_{j=0}^{t-1}(1-\lambda)^j \left[(\lambda + c)(X_{t-j} - cX_{t-j-1})\right].$$

Take the expectation on both side, we have

$$E(Z_t) = (1-\lambda)^t E(Z_0) + \sum_{j=0}^{t-1}(1-\lambda)^j E\left[(\lambda + c)X_{t-j} - cX_{t-j-1}\right].$$

Since

$$\sum_{j=0}^{t}(1-\lambda)^j = \frac{\left[1-(1-\lambda)^t\right]}{\left[1-(1-\lambda)\right]} = \frac{\left[1-(1-\lambda)^t\right]}{\lambda}.$$

Then,

$$\begin{aligned}
E(Z_t) &= (1-\lambda)^t \mu_0 + \frac{\left[1-(1-\lambda)^t\right]}{\lambda}\left[(\lambda + c)\mu_0 - c\mu_0\right] \\
&= (1-\lambda)^t \mu_0 - (1-\lambda)^t \mu_0 + \mu_0 \\
E(Z_t) &= \mu_0
\end{aligned}$$

And variance is $V(Z_t) = \left[\dfrac{\left(\lambda + 2\lambda c + 2c^2\right)}{2-\lambda}\right]\sigma^2$. The derive of the variance of Z_t is

$$V\left(Z_t\right)=(1-\lambda)^{2t}V\left(Z_0\right)+\sum_{j=0}^{t-1}(1-\lambda)^{2j}\left[(\lambda+c)^2V(X_{t-j})+cV(X_{t-j-1})\right]$$

$$-2(\lambda+c)c\sum_{j=0}^{2t+1}(1-\lambda)^{2j+1}Cov(X_{t-j},X_{t-j-1})$$

$$=0+\frac{\left[1-(1-\lambda)^{2t}\right]}{1-(1-\lambda)^2}\left[(\lambda+c)^2+c^2\right]\left[\sigma^2\right]$$

$$+\frac{\left[1-(1-\lambda)^{2t}\right]}{1-(1-\lambda)^2}\left[-2(\lambda+c)c(1-\lambda)\right]\left[\rho\sigma^2\right].$$

For simply, we consider case $\rho\to 0$,

$$V\left(Z_t\right)=\frac{\left[1-(1-\lambda)^{2t}\right]}{\lambda(2-\lambda)}\left[\lambda^2+2\lambda c+c^2+c^2-2\lambda c-2c^2+2\lambda^2c+2\lambda c^2\right]\left[\sigma^2\right]$$

$$=\frac{\lambda}{\lambda(2-\lambda)}(\lambda+2\lambda c+2c^2)\sigma^2$$

$$V\left(Z_t\right)=\frac{\lambda+2\lambda c+2c^2}{2-\lambda}\sigma^2 .$$

Therefore, the upper control limit (UCL) and lower control limit (LCL) of the modified EWMA control chart are as follows,

$$UCL=\mu_0+Q\sigma\sqrt{\frac{\left(\lambda+2\lambda c+2c^2\right)}{2-\lambda}}, \tag{2.6}$$

$$UCL=\mu_0-Q\sigma\sqrt{\frac{\left(\lambda+2\lambda c+2c^2\right)}{2-\lambda}}, \tag{2.7}$$

where μ_0 is the target mean, σ is the process standard deviation, Q is the appropriate control width limit, and X_t is a sequence of observations. $Z_0=u$ and $X_0=v$ are the initial values and λ is an exponential smoothing parameter that is $0<\lambda\le 1$.

The stopping time of the EWMA chart is given by

$$\tau_b=\inf\{t>0;Z_t>b\},b>u \tag{2.8}$$

where τ_b is the stopping time and b is the upper control limit (UCL).

2.2 Process Observations

Autocorrelation is a characteristic of data that shows the degree of similarity among values of the same variables over successive time intervals where the basic assumption of instance independence, which underlies most conventional models, is violated. It frequently exists in datasets in which the data are from the same source, and since it occurs mostly due to dependencies within the data, the observations collected periodically need to be checked by applying a statistical analysis for time series. Various processes of time series modeling have been used in many applications.

2.2.1 Moving Average Process

Let $MA(q)$ process be observations of the q-order of moving average process. This can be described by the following as a recursion,

$$X_t = \mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q}, \qquad (2\text{-}9)$$

where ε_t is a white noise process assumed with the exponential distribution, θ is a moving average coefficient and $|\theta_i| < 1;\ i = 1,2,3,\ldots q$. In other word, the first order moving average $MA(1)$ can be written in the form as following

$$X_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1}. \qquad (2\text{-}10)$$

2.2.2 The Moving Average Process with explanatory variables

Let $\{Z_t, t = 1,2,\ldots.,\}$ be a sequence of MAX(q,r) process given as

$$Z_t = \mu + (1 - \theta_1 \mathrm{B} - \theta_2 \mathrm{B}^2 - \ldots - \theta_q \mathrm{B}^q)\varepsilon_t + \sum_{i=1}^{r}\beta_i X_{it}, \qquad (2\text{-}11)$$

and $\{\zeta_t, t = 1,2,\ldots.,\}$ be a sequence of SMAX(Q,r)$_L$ process given by the expression

$$\zeta_t = \mu + (1 - \theta_L \mathrm{B}^L - \theta_{2L} \mathrm{B}^{2L} - \ldots - \theta_{QL} \mathrm{B}^{QL})\varepsilon_t + \sum_{i=1}^{r}\beta_i X_{it}, \qquad (2\text{-}12)$$

where ε_t is a exponential white noise process,

μ is a process mean,

$(1 - \theta_1 \mathrm{B} - \theta_2 \mathrm{B}^2 - \ldots - \theta_q \mathrm{B}^q)$ is the moving average polynomials in B of order q.

$(1 - \theta_L \mathrm{B}^L - \theta_{2L} \mathrm{B}^{2L} - \ldots - \theta_{QL} \mathrm{B}^{QL})$ is the seasonal moving average polynomials in B of order Q; L is a natural number,

B is the backward shift operator, i.e., $B^q\varepsilon_t = \varepsilon_{t-q}$,

X_{it} is exogenous variable and β_i is a coefficient of X_{it}.

2.2.3 The Autoregressive Processes with explanatory variables (ARX(p,r))

The ARX(p,r) process is defined as

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \sum_{j=1}^{r}\beta_j X_j + \varepsilon_t \qquad ;t = 1,2,3,\ldots, \qquad (2.13)$$

where δ is a constant $(\delta \geq 0)$, ϕ_i is an autoregressive coefficient for $i = 1,2,...,p$ $(-1 < \phi_i < 1)$; ε_t is an independent and identically distributed (iid) sequence; $\varepsilon_t \sim Exp(\alpha)$; X_j are explanatory variables of Y_t; and β_j are coefficients of X_j; $j = 1,2,...,r$. The initial value for the ARX(p,r) process mean is $Y_{t-1}, Y_{t-2}, ..., Y_{t-p} = 1$ and the initial value for the explanatory variables $X_1, X_2, ..., X_r = 1$.

2.2.4 Exponential Distribution

The sequential observations $Y_1, Y_2, Y_2, ...$ are sequence of independent and identically distribution random variables with a probability density function $f(y, \beta)$. The change-point time (α) for exponential distribution can be written as

$$y_t \sim \begin{cases} Exp(\beta_0); t = 1,2,3,4,...,\alpha - 1 \\ Exp(\beta_1)\ ; t = \alpha, \alpha+1, \alpha+2, ... \end{cases}$$

The probability density function of exponential distribution is

$$f(y;\beta) = \begin{cases} \dfrac{1}{\beta} e^{\frac{-y}{\beta}}; y > 0, \beta > 0 \\ 0 \qquad ; \quad otherwise, \end{cases}$$

where β is exponential parameter.

The mean and variance of the exponential distribution are as following

$$E(Y) = \beta \text{ and } V(Y) = \beta^2.$$

2.2.5 White Noise

An example of stationary process with constant mean and covariance or no correlation between time series in different lag time or constant variance is called white noise. A time series X_t is a white noise process if for all $t = 1,2,3,...$ satisfies

(1) $E(X_t) = \mu$,

(2) $V(X_t) = E(X_t - \mu)^2 = E(X_{t-m} - \mu)^2 = \sigma^2 < \infty$, $\forall m \in N$ and

(3) $Cov(X_t, X_{t-m}) = E[(X_t - \mu)(X_{t-m} - \mu)] = 0$, $\forall m \in N$.

These properties indicated that the white noise is the process with a constant mean and variance of independent identically random variables. In this study the error ε_t term is assumed to be the exponential distribution. It can be written in the form of probability density function of ε_t in the form as

$$f(\varepsilon_t) = \frac{1}{\beta} e^{\frac{-\varepsilon_t}{\beta}}; \varepsilon_t > 0, \beta > 0, \qquad (2\text{-}14)$$

where β is shift parameter in the quality control or rate parameter of exponential distribution with the man and variance as mentioned above.

2.3 Evaluation the Average Run Length

The evaluation of ARL was proposed by Crowder in 1987. He carried out in the case of Gaussian distribution and evaluated the *ARL* by using Fredholm integral equation of the second kind. Afterward, Srivastava and Wu proposed the *ARL* for a continuous-time model and discrete process by using Fredholm integral equation of the second kind method. In this section, the main concept of method for evaluation of *ARL* on the modified EWMA control chart will be given.

Let $L(u)$ be the average run length on the modified EWMA control chart and suppose that Z_t is the modified EWMA statistics by given in Equation (2-5). Assume that process is in-control at time t if the modified EWMA statistics Z_t is in range $LCL < Z_t < UCL$ and the process is out-of-control if $Z_t < LCL$ or $Z_t > UCL$. To define function $L(u)$ following

$$ARL = L(u) = \mathbf{E}_\infty(\tau) \tag{2-15}$$

where $\mathbf{E}_\infty$ is the expectation and τ is stopping time.

According to Equation (2-5), the modified EWMA statistics Z_t can be written is this form

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t + c(X_t - X_{t-1}),$$

where t = 1,2,3,... , $0 < \lambda \le 1$. So,

$$Z_1 = (1-\lambda)Z_0 + (\lambda + c)X_1 - cX_0.$$

If Z_1 give an in-control process for Z_1, then $LCL < Z_1 < UCL$. So,

$$LCL < (1-\lambda)Z_0 + (\lambda + c)X_1 - cX_0 < UCL. \tag{2-16}$$

The observations will be produce before out-of-control signal occurs. The inequality in Equation (2-16) can be rewritten as following

$$\frac{LCL - (1-\lambda)Z_1 + cX_0}{\lambda + c} < X_1 \le \frac{UCL - (1-\lambda)Z + cX_0}{\lambda + c}. \tag{2-17}$$

The bounds of Equation (2-17) are satisfied probability X_1 with the probability distribution function $f(x_1)$ as following

$$\int_{\frac{UCL-(1-\lambda)Z_0+cX_0}{\lambda+c}}^{\frac{LCL-(1-\lambda)Z_0+cX_0}{\lambda+c}} f(y)dy \ , \tag{2-18}$$

where $f(y)$ is the probability density function.

If the initial values in the process mean are set as $Z_0 = u$, $X_0 = v$, then following Champ and Ridgon's (1991) method, the formula for $L(u)$ can be expressed in the form

$$L(u) = \left(1 - P\left[\frac{LCL-(1-\lambda)u+cv}{\lambda+c} < X_1 \le \frac{UCL-(1-\lambda)u+cv}{\lambda+c}\right]\right)$$

$$+ \int_{\frac{LCL-(1-\lambda)u+cv}{\lambda+c}}^{\frac{UCL-(1-\lambda)u+cv}{\lambda+c}} \left(1+L\left((1-\lambda)u-cv+(\lambda+c)y\right)\right) f(y)dy$$

$$= 1 + \int_{\frac{LCL-(1-\lambda)u+cv}{\lambda+c}}^{\frac{UCL-(1-\lambda)u+cv}{\lambda+c}} \left(1+L\left((1-\lambda)u-cv+(\lambda+c)y\right)\right) f(y)dy,$$

The integration variable is changed, we have

$$L(u) = 1 + \frac{1}{\lambda+c}\int_{LCL}^{UCL} L(y) f\left(\frac{y-(1-\lambda)u+cv}{\lambda+c}\right)dy, \qquad (2\text{-}19)$$

where $f(\cdot)$ is the probability density of X_t. In this study, the upper control limit is on focused for the nonnegative random variable X_t. Therefore, the case of nonnegative X_t due to $Z_t \ge 0$, we can assume $LCL = 0$ and $UCL = b$. The Equation (2-19) can be written in the form as

$$L(u) = 1 + \frac{1}{\lambda+c}\int_{0}^{b} L(y) f\left(\frac{y-(1-\lambda)u+cv}{\lambda+c}\right)dy. \qquad (2\text{-}20)$$

In the other hand, the ARL_1 is also evaluate by the Equation (2-20) where $f(\cdot)$ is probability density function with an out-of-control parameter β.

2.4 Numerical Methods for Solving the Integral Equation

The quadrature rule is used to approximate the integral equation by a finite sum. Since quadrature rule is defined by a set of points $\{a_j, j = 1, 2, 3, ..., n\}$ on the interval $[LCL, UCL]$ and set of constant weights $\{w_j, j = 1, 2, 3, ..., n\}$. The approximation of the integral equation is given as following

$$\int_{LCL}^{UCL} w(y) f(y)dy = \sum_{j=1}^{n} w_j f(a_j), \qquad (2\text{-}21)$$

where $w(y)$ and $f(y)$ are given function. Therefore, the Equation (2-19) is approximated of Equation (2-21). It can be approximated by $\tilde{L}(u)$ as the system of algebraic linear equations:

$$\tilde{L}(a_i) = 1 + \frac{1}{\lambda+c}\sum_{j=1}^{m} w_j \tilde{L}(a_j) f\left(\frac{a_j-(1-\lambda)a_i+cv}{\lambda+c}\right) \; ; i = 1, 2, 3, ..., n. \qquad (2\text{-}22)$$

The system of m linear equations is showed as:

$$\tilde{L}(a_1)=1+\frac{1}{\lambda+c}\sum_{j=1}^{m}w_j\tilde{L}(a_j)\,f\left(\frac{a_j-(1-\lambda)a_1+cv}{\lambda+c}\right),$$

$$\tilde{L}(a_2)=1+\frac{1}{\lambda+c}\sum_{j=1}^{m}w_j\tilde{L}(a_j)\,f\left(\frac{a_j-(1-\lambda)a_2+cv}{\lambda+c}\right),$$

$$\tilde{L}(a_3)=1+\frac{1}{\lambda+c}\sum_{j=1}^{m}w_j\tilde{L}(a_j)\,f\left(\frac{a_j-(1-\lambda)a_3+cv}{\lambda+c}\right)$$

$$\vdots$$

$$\tilde{L}(a_n)=1+\frac{1}{\lambda+c}\sum_{j=1}^{m}w_j\tilde{L}(a_j)\,f\left(\frac{a_j-(1-\lambda)a_n+cv}{\lambda+c}\right).$$

The n equations as mentioned above can be rewritten in a matrix form as following

$$L_{m\times 1}=1_{m\times 1}+R_{m\times m}L_{m\times 1}\ or\ \left(I_m-R_{m\times m}\right)L_{m\times 1}=1_{m\times 1}\ or\ L_{m\times 1}=\left(I_m-R_{m\times m}\right)^{-1}1_{m\times 1} \quad \text{(2-30)}$$

where $L_{m\times 1}=\begin{bmatrix} L(a_1) \\ L(a_2) \\ \vdots \\ L(a_m) \end{bmatrix}$, $I_m = diag(1,1,...,1)$, $1_{m\times 1}=\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

and

$$R_{n\times n}=\begin{bmatrix} \frac{1}{\lambda+c}w_1 f\left(\frac{a_1-(1-\lambda)a_1+cv}{\lambda+c}\right) & \cdots & \frac{1}{\lambda+c}w_n f\left(\frac{a_n-(1-\lambda)a_1+cv}{\lambda+c}\right) \\ \frac{1}{\lambda+c}w_1 f\left(\frac{a_1-(1-\lambda)a_2+cv}{\lambda+c}\right) & \cdots & \frac{1}{\lambda+c}w_n f\left(\frac{a_n-(1-\lambda)a_2+cv}{\lambda+c}\right) \\ \vdots & & \\ \frac{1}{\lambda+c}w_1 f\left(\frac{a_1-(1-\lambda)a_n+cv}{\lambda+c}\right) & \cdots & \frac{1}{\lambda+c}w_n f\left(\frac{a_n-(1-\lambda)a_n+cv}{\lambda+c}\right) \end{bmatrix}$$

Therefore, an approximation the NIE method for function $L(u)$ is

$$\tilde{L}(u)=1+\frac{1}{\lambda+c}\sum_{j=1}^{n}w_jL(a_j)\,f\left(\frac{a_j-(1-\lambda)u+cv}{\lambda+c}\right). \quad \text{(2-23)}$$

Chapter 3
Methodology

In this research, the explicit formulas and NIE for the evaluation of the *ARL* on a modified EWMA control chart are proposed when the observations follow $MA(1),$ and $ARX(p,r)$ processes with exponential white noise. In addition, the explicit formulas for the ARL of Modified EWMA chart can be applied to real data or other control charts. The solution for the integral equation is implemented to derive explicit formulas based on the Fredholm integral equation of the second kind. The NIE method produces an approximated *ARL* by applying the Gauss-Legendre quadrature rule. Moreover, the performances of the modified EWMA and EWMA control charts are compared by applying the *ARL* using explicit formulas and NIE.

3.1 Planning of the Study

In this research, the evaluation *ARL* on modified EWMA control chart for autocorrelated observation is planned as following

3.1.1 To derived explicit formulas and approximation of numerical integral equation for *ARL* on modified EWMA control chart for the first-order moving average (*MA*(*1*)) process with exponential white noise.

3.1.2 To derived explicit formulas and approximation of numerical integral equation for *ARL* on modified EWMA control chart for *ARX(p,r)* process with exponential white noise.

3.1.3 To derived explicit formulas and approximation of numerical integral equation for *ARL* on CUSUM control chart for with *MAX(q,r)* and $SMAX(Q,r)_L$ processes

3.2 Step of the Study

In this research, the steps of the study of evaluation *ARL* on modified EWMA control chart are proceeded as following

3.2.1 The *ARL* of *MA*(*1*) process with exponential white noise for modified EWMA control chart.

3.2.1.1 Derive the explicit formulas for *MA*(*1*) process with exponential distribution white noise by using the Fredholm integral equation of second type technique on modified EWMA control chart.

3.2.1.2 Compute the ARL_0 by setting in-control $ARL_0 = 370$ and 500.

3.2.1.3 Compute the ARL_1 of the out-of-control process

3.2.1.4 Approximate the ARL on modified EWMA control chart by using NIE method.

3.2.1.5 Compare the *ARL* results between explicit formulas and NIE method.

3.2.1.6 Compare the performance of *ARL* result between modified EWMA and EWMA control charts.

3.2.2 The explicit formulas and approximation of numerical integral equation for *ARL* on modified EWMA control chart for ARX(p,r) process with exponential white noise.

3.2.2.1 Derive the explicit formulas for ARX(p,r) process with Exponential white noise by using the Fredholm integral equation of second type technique on modified EWMA control chart.

3.2.2.2 Compute the ARL_0 by setting in-control $ARL_0 = 370$ and 500.

3.2.2.3 Compute the ARL_1 of the out-of-control process

3.2.2.4 Approximate the ARL on modified EWMA control chart by using NIE method.

3.2.2.5 Compare the *ARL* results between explicit formulas and NIE method.

3.2.2.6 Compare the performance of *ARL* result between modified EWMA and EWMA control charts.

3.2.3 The explicit formulas and approximation of numerical integral equation for *ARL* on CUSUM control chart for *MAX(q,r)* and $SMAX(Q,r)_L$ processes

3.2.3.1 Derive the explicit formulas for *MAX(q,r)* and $SMAX(Q,r)_L$ processes with exponential distribution white noise by using the Fredholm integral equation of second type technique on CUSUM control chart.

3.2.3.2 Compute the ARL_0 by setting in-control $ARL_0 = 370$ and 500.

3.2.3.3 Compute the ARL_1 of the out-of-control process.

3.2.3.4 Approximate the ARL on CUSUM control chart by using NIE method.

3.2.3.5 Compare the *ARL* results between explicit formulas and NIE method.

3.2.3.6 Compare the performance of *ARL* result between CUSUM and EWMA control charts.

Chapter 4
Experimental Results

The evaluation of the *ARL* by using explicit formulas and NIE are presented. The performances of the EWMA and modified EWMA control charts were revealed by applying the *ARL* by using explicit formulas and NIE method. The results of the study are organized into three sections: the first three contains evaluations of the *ARL* for *MA(1)* process on modified EWMA and EWMA control charts. Next, the explicit Formulas for the ARL of the Modified EWMA control chart for an *ARX(p,r)* process with exponential white noise are presented while the third section the explicit formula of ARL for modified EWMA is applied to other control chart.

4.1 Evaluation the *ARL* for *MA*(*1*) Process

4.1.1 Explicit formulas of *ARL* for *MA*(*1*) process

The first order moving average process denoted by *MA*(*1*) is described by the follow as recursion

$$X_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1}, \tag{4-1}$$

where ε_t is a white noise process assumed with exponential distribution, θ is a moving average coefficient which $|\theta| \le 1$ or $-1 \le \theta \le 1$ and given the initial value of $\varepsilon_0 = s$.
Therefore, the modified EWMA statistics Z_t can be written is this form

$$\begin{aligned} Z_t &= (1-\lambda)Z_{t-1} + \lambda X_t + c(X_t - X_{t-1}) \\ &= (1-\lambda)Z_{t-1} + \lambda\left(\mu + \varepsilon_t - \theta\varepsilon_{t-1}\right) + c\left(\mu + \varepsilon_t - \theta\varepsilon_{t-1} - X_{t-1}\right) \\ &= (1-\lambda)Z_{t-1} - cX_{t-1} + \left(\lambda + c\right)\varepsilon_t - \left(\lambda\theta + c\theta\right)\varepsilon_{t-1} + \left(\lambda + c\right)\mu \end{aligned}$$

where t = 1,2,3,... , $0 < \lambda \le 1$ and the initial value in the process mean $Z_0 = u$, $X_0 = v$, $\varepsilon_0 = s$, $LCL = 0$ and $UCL = b$. So,

$$\begin{aligned} Z_1 &= (1-\lambda)Z_0 - cX_0 + \left(\lambda + c\right)\varepsilon_1 + \left(\lambda + c\right)\mu - \left(\lambda\theta + c\theta\right)s \\ &= (1-\lambda)u - cv + \left(\lambda + c\right)\varepsilon_1 + \left(\lambda + c\right)\mu - \left(\lambda\theta + c\theta\right)s. \end{aligned}$$

The stopping time of modified EWMA chart is given by

$$\tau_b = \inf\{t > 0; Z_t > b\}, b > u \tag{4-2}$$

where τ_b is the stopping time, b is the upper control limit (*UCL*). Assume that process is in-control at time t if the modified EWMA statistics Z_t is in range $0 < Z_t < b$ and the process is out-of-control if $Z_t > b$. Let $L(u)$ denote the *ARL* for $MA(1)$ process. To define function $L(u)$ following

$$ARL = L(u) = \mathbf{E}_{\infty}(\tau_b) \tag{4-3}$$

where $\mathbf{E}_{\infty}$ is the expectation.

If ε_1 give an in-control process for Z_1, then $0 < Z_1 \le b$. So,

$$0 \le (1-\lambda)u - cv + (\lambda + c)\varepsilon_1 - (\lambda\theta + c\theta)s + (\lambda + c)\mu \le b \tag{4-4}$$

The inequality in Equation (10) can be rewritten as following

$$\frac{-(1-\lambda)u + cv + (\lambda\theta + c\theta)s - (\lambda + c)\mu}{\lambda + c} \le \varepsilon_1 \le \frac{b-(1-\lambda)u + cv + (\lambda\theta + c\theta)s - (\lambda + c)\mu}{\lambda + c}$$

According to method similar to Crowder (1987) and VanBrackle and Reynold (1997), we can write the formula for $L(u)$ by the integral equation

$$L(u) = 1 + \int_{\frac{-(1-\lambda)u+cv+(\lambda\theta+c\theta)s-(\lambda+c)\mu}{\lambda+1}}^{\frac{b-(1-\lambda)u+cv+(\lambda\theta+c\theta)s-(\lambda+c)\mu}{\lambda+1}} L\left[(1-\lambda)u - cv + (\lambda + c)y - (\lambda\theta + c\theta)s + (\lambda + c)\mu\right] f(y)dy$$ Change the variable, we obtain

$$L(u) = 1 + \frac{1}{\lambda + c}\int_0^b L(k) f\left(\frac{k-(1-\lambda)u + cv + (\lambda\theta + c\theta)s}{\lambda + c} - \mu\right)dk. \tag{4-5}$$

The $L(u)$ is a Fredholm integral equation of the second kind. If $Y \sim Exp(\beta)$, then we have $f(y) = \frac{1}{\beta}e^{-y/\beta}$; $y \ge 0$. So,

$$L(u) = 1 + \frac{1}{\lambda + c}\int_0^b L(k)\cdot\frac{1}{\beta}\cdot e^{-\frac{k}{\beta(\lambda+c)}}\cdot e^{\frac{(1-\lambda)u}{\beta(\lambda+c)}}\cdot e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}dk \tag{4-6}$$

Suppose that

$$C(u) = e^{\frac{(1-\lambda)u}{\beta(\lambda+c)}}\cdot e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}} \; ; \; 0 \le u \le b$$

and

$$D = \int_0^b L(k)\cdot e^{-\frac{k}{\beta(\lambda+c)}}dk$$, where D is a constant.

So, we have

$$L(u) = 1 + \frac{C(u)}{\beta(\lambda + c)}D \tag{4-7}$$

Consider

$$\begin{aligned} D &= \int_0^b L(k)\cdot e^{-\frac{k}{\beta(\lambda+c)}}dk \\ &= \int_0^b \left[1 + \frac{C(k)}{\beta(\lambda + c)}D\right]\cdot e^{-\frac{k}{\beta(\lambda+c)}}dk \end{aligned}$$

$$=\int_0^b e^{-\frac{k}{\beta(\lambda+c)}}dk+\int_0^b \frac{e^{\frac{(1-\lambda)k}{\beta(\lambda+c)}}\cdot e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}}{\beta(\lambda+c)}\cdot e^{-\frac{k}{\beta(\lambda+c)}}Ddk$$

$$=-\beta(\lambda+c)\left[e^{-\frac{k}{\beta(\lambda+c)}}\Big|_0^b\right]+\frac{D}{\beta(\lambda+c)}e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}\int_0^b e^{-\frac{\lambda k}{\beta(\lambda+c)}}dk$$

$$=-\beta(\lambda+c)\left[e^{-\frac{b}{\beta(\lambda+c)}}-1\right]-\frac{D}{\lambda}e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}\left[e^{-\frac{\lambda b}{\beta(\lambda+c)}}-1\right]$$

$$\therefore \qquad D=\frac{-\beta(\lambda+c)\left[e^{-\frac{b}{\beta(\lambda+c)}}-1\right]}{1+\frac{1}{\lambda}e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}\left[e^{-\frac{\lambda b}{\beta(\lambda+c)}}-1\right]} \tag{4-8}$$

Finally, the equation (4-7) is substituted by equation (4-8). The solution for the integral equation in equation (4-5) as following

$$L(u)=1+\frac{C(u)}{\beta(\lambda+c)}D$$

$$=1+\left[\frac{e^{\frac{(1-\lambda)u}{\beta(\lambda+c)}}\cdot e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}}{\beta(\lambda+c)}\right]\cdot\left[\frac{-\beta(\lambda+c)\left[e^{-\frac{b}{\beta(\lambda+c)}}-1\right]}{1+\frac{1}{\lambda}e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}\left[e^{-\frac{\lambda b}{\beta(\lambda+c)}}-1\right]}\right]$$

$$=1-\left[e^{\frac{(1-\lambda)u}{\beta(\lambda+c)}}\cdot e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}\right]\cdot\left[\frac{\lambda\left[e^{-\frac{b}{\beta(\lambda+c)}}-1\right]}{\left[\lambda+e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\cdot e^{\frac{\mu}{\beta}}\left[e^{-\frac{\lambda b}{\beta(\lambda+c)}}-1\right]\right]}\right]$$

$$=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\beta(\lambda+c)}}\cdot e^{-\frac{cv}{\beta(\lambda+c)}}\cdot e^{-\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}\left[e^{-\frac{b}{\beta(\lambda+c)}}-1\right]}{e^{-\frac{(\lambda\theta+\theta)s}{\beta(\lambda+c)}}\left[\lambda e^{\frac{-\mu}{\beta}}\cdot e^{\frac{(\lambda\theta+c\theta)s}{\beta(\lambda+c)}}+e^{-\frac{cv}{\beta(\lambda+c)}}\left[e^{-\frac{\lambda b}{\beta(\lambda+c)}}-1\right]\right]}$$

$$L(u)=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\beta_0(\lambda+c)}}\left[e^{-\frac{b}{\beta_0(\lambda+c)}}-1\right]}{\lambda e^{\frac{-\mu}{\beta_0}}\cdot e^{\frac{cv}{\beta_0(\lambda+c)}}\cdot e^{\frac{(\lambda\theta+c\theta)s}{\beta_0(\lambda+c)}}+e^{-\frac{\lambda b}{\beta_0(\lambda+c)}}-1} \tag{4-9}$$

The expectation of stopping time in equation (4-9) corresponds to the $L(u)=ARL_0$ when $\beta_0=1$. An out-of-control process when $\beta_0>1$ we obtain $L(u)=ARL_1$.

4.1.2 Numerical Integral Equation of ARL for MA (1) process

From the integral equation in equation (4-5) follow as

$$L(u)=1+\frac{1}{\lambda+c}\int_0^b L(k)\,f\left(\frac{k-(1-\lambda)u+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right)dk\,.$$

The numerical method for solving the integral equation by quadrature rule approach is approximated the integral by finite sum of areas of rectangles with base b/m with heights chosen as the values of f at midpoints of the one-sided interval which divide $[0,b]$ into a partition $0\le a_1\le a_2\le\ldots\le a_m\le b$ and a set of constant weights $w_j=b/m\ge 0$. The approximate for an integral can be expression by

$$\int_0^b L(\mathrm{k})\,\mathrm{f}(\mathrm{k})\,\mathrm{dk}\approx\sum_{j=1}^{m} w_j f(a_j)\ , \tag{4-10}$$

where $a_j=\frac{b}{m}\left(j-\frac{1}{2}\right)$ and $w_j=\frac{b}{m}$; $j=1,2,3,\ldots,m$.

The integral equation (4-5) can be approximated by using the Gauss-Legendre quadrature rule as follows

$$L(a_i)=1+\frac{1}{\lambda+c}\sum_{j=1}^{m} w_j L(a_j)\,f\left(\frac{a_j-(1-\lambda)a_i+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right), \tag{4-11}$$

where ; $i=1,2,3,\ldots,m$, $a_j=\frac{b}{m}\left(j-\frac{1}{2}\right)$ and $w_j=\frac{b}{m}$; $j=1,2,3,\ldots,m$.

The system of m linear equations is showed as:

$$L(a_1)\approx 1+\frac{1}{\lambda+c}\sum_{j=1}^{m} w_j L(a_j)\,f\left(\frac{a_j-(1-\lambda)a_1+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right)$$

$$L(a_2)\approx 1+\frac{1}{\lambda+c}\sum_{j=1}^{m} w_j L(a_j)\,f\left(\frac{a_j-(1-\lambda)a_2+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right)$$

$$L(a_3)\approx 1+\frac{1}{\lambda+c}\sum_{j=1}^{m} w_j L(a_j)\,f\left(\frac{a_j-(1-\lambda)a_3+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right)$$

$$\vdots$$

$$L(a_m)\approx 1+\frac{1}{\lambda+c}\sum_{j=1}^{m} w_j L(a_j)\,f\left(\frac{a_j-(1-\lambda)a_m+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right)$$

It can be written by matrix form as

$$L_{m\times 1}=1_{m\times 1}+R_{m\times m}L_{m\times 1}\quad or\quad \left(I_m-R_{m\times m}\right)L_{m\times 1}=1_{m\times 1}\quad or\quad L_{m\times 1}=\left(I_m-R_{m\times m}\right)^{-1}1_{m\times 1} \tag{4-12}$$

where $L_{m\times 1} = \begin{bmatrix} L(a_1) \\ L(a_2) \\ \vdots \\ L(a_m) \end{bmatrix}$, $I_m = diag(1,1,...,1)$, $1_{m\times 1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

and

$$R_{m\times m} = \begin{bmatrix} \frac{1}{\lambda+c} w_1 f\left(\frac{a_1-(1-\lambda)a_1+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right) & \cdots & \frac{1}{\lambda+c} w_m f\left(\frac{a_m-(1-\lambda)a_1+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right) \\ \frac{1}{\lambda+c} w_1 f\left(\frac{a_1-(1-\lambda)a_2+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right) & \cdots & \frac{1}{\lambda+c} w_m f\left(\frac{a_m-(1-\lambda)a_2+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right) \\ \vdots & & \\ \frac{1}{\lambda+c} w_1 f\left(\frac{a_1-(1-\lambda)a_m+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right) & \cdots & \frac{1}{\lambda+c} w_m f\left(\frac{a_m-(1-\lambda)a_m+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right) \end{bmatrix}$$

An approximation the NIE method for function $L(u)$ is

$$\tilde{L}(u) = 1 + \frac{1}{\lambda+c}\sum_{j=1}^{m} w_j L(a_j)\, f\left(\frac{a_j-(1-\lambda)u+cv+(\lambda\theta+c\theta)s}{\lambda+c}-\mu\right), \tag{4-13}$$

where $a_j = \frac{b}{m}\left(j-\frac{1}{2}\right)$ and $w_j = \frac{b}{m}$; $j = 1,2,3,...,m$.

Table 4-1 Comparison of ARL for $MA(1)$ on modified EWMA control chart by using explicit formulas for $ARL_0 = 370$, $\lambda = 0.1$, $\mu = 2$ and $\theta = 0.1$ with $c = 5\lambda,\ 10\lambda,\ 15\lambda, 20\lambda$.

Shift size (δ)	$c = 5\lambda$ $(b = 0.49058401)$	$c = 10\lambda$ $(b = 0.413935708)$	$c = 15\lambda$ $(b = 0.45153011)$	$c = 20\lambda$ $(b = 0.5099272685)$
0.00	370.0000691020	370.0000309970	370.0001954173	370.0000435986
0.005	217.2408039291	127.6995634708	100.3531155158	87.8016095772
0.01	153.4066744579	77.2119607183	58.2153981235	50.0311299754
0.02	96.2192991774	43.1673895860	31.7935230083	27.0856419508
0.03	69.8186997007	29.9952343685	21.9659955933	18.6908127359
0.04	54.6283560236	23.0075901833	16.8377631296	14.3395005117
0.05	44.7661512893	18.6791852178	13.6898843745	11.6782708920
0.06	37.8522379425	15.7361573040	11.5617663552	9.8832524829
0.07	32.7401757436	13.6061885354	10.0275786767	8.5911905151
0.08	28.8095491949	11.9940081173	8.8695901874	7.6170170786
0.09	25.6953252186	10.7318510554	7.9648895720	6.8565294360
0.10	23.1687163301	9.7173402740	7.2388375370	6.2465726992
0.20	11.4755918453	5.1205235779	3.9593181049	3.4943690487
0.50	4.5099214488	2.4242569533	2.0352387883	1.8782006479
1.00	2.4518399866	1.6105506917	1.4487568794	1.3828124251
2.00	1.5992992378	1.2606655170	1.1933351586	1.1656302034
5.00	1.1979163923	1.0887275633	1.0662536122	1.0569228987

Table 4-1 reports the ARL by using explicit formulas versus NIE for the $MA(1)$ process on modified EWMA control chart for $\mu = 2, \theta = 0.1$ and $\lambda = 0.1$. For $ARL_0 = 370$ with $c = 5\lambda,\ 10\lambda, 15\lambda,$ and 20λ, the results indicated that when *c* was increased, the *ARL* gradually decreased by small amounts. Subsequently, $c = 1$ was employed to evaluate the *ARL* in the rest of the experiments.

Next, the evaluation of the *ARL* obtained by applying the explicit formulas and the NIE method with 1,000 nodes on the modified EWMA control chart for the *MA*(*1*) process with exponential white noise is presented.

Table 4-2 Comparison of ARL for $MA(1)$ on modified EWMA control chart by using explicit formulas versus numerical integral equation for $ARL_0 = 370$, $\mu = 2$ and $\theta = 0.1$ with $\lambda = 0.05$, 0.1.

Shift size (δ)	$\lambda = 0.05$ $(b = 0.408730497)$			$\lambda = 0.1$ $(b = 0.413935708)$		
	Explicit	NIE	Diff	Explicit	NIE	Diff
0.00	370.000048935 (0.000)	370.000045565 (19.188)	9.107 x10^{-7}	370.000030997 (0.000)	370.000024645 (19.438)	1.717 x10^{-7}
0.005	135.115656100 (0.000)	135.115655125 (19.188)	7.215 x10^{-7}	127.699563470 (0.000)	127.699562241 (19.344)	9.624 x10^{-7}
0.01	82.6505751194 (0.000)	82.6505745638 (19.157)	6.722 x10^{-7}	77.2119607183 (0.000)	77.2119601018 (19.360)	7.985 x10^{-7}
0.02	46.5318371778 (0.000)	46.5318368860 (19.250)	6.271 x10^{-7}	43.1673895860 (0.000)	43.1673892938 (19.329)	6.769 x10^{-7}
0.03	32.3937161010 (0.000)	32.3937159070 (19.157)	5.989 x10^{-7}	29.9952343685 (0.000)	29.9952341825 (19.266)	6.201 x10^{-7}
0.04	24.8564642024 (0.000)	24.8564640590 (19.313)	5.769 x10^{-7}	23.0075901833 (0.000)	23.0075900491 (19.484)	5.833 x10^{-7}
0.05	20.1748191795 (0.000)	20.1748190671 (19.984)	5.571 x10^{-7}	18.6791852178 (0.000)	18.6791851142 (19.594)	5.546 x10^{-7}
0.06	16.9861431966 (0.000)	16.9861431049 (19.142)	5.399 x10^{-7}	15.7361573040 (0.000)	15.7361572205 (19.437)	5.306 x10^{-7}
0.07	14.6757215370 (0.000)	14.6757214603 (19.391)	5.226 x10^{-7}	13.6061885354 (0.000)	13.6061884660 (19.344)	5.101 x10^{-7}
0.08	12.9255345606 (0.000)	12.9255344951 (19.126)	5.067 x10^{-7}	11.9940081173 (0.000)	11.9940080584 (19.391)	4.911 x10^{-7}
0.09	11.5545317625 (0.000)	11.5545317057 (19.250)	4.916 x10^{-7}	10.7318510554 (0.000)	10.7318510044 (19.344)	4.752 x10^{-7}
0.10	10.4520618174 (0.000)	10.4520617675 (19.251)	4.774 x10^{-7}	9.7173402740 (0.000)	9.7173402295 (19.281)	4.579 x10^{-7}
0.20	5.4533669974 (0.000)	5.4533669778 (19.390)	3.594 x10^{-7}	5.1205235779 (0.000)	5.1205235607 (19.391)	3.359 x10^{-7}
0.50	2.5258096911 (0.000)	2.5258096868 (19.142)	1.702 x10^{-7}	2.4242569533 (0.000)	2.4242569495 (19.609)	1.567 x10^{-7}
1.00	1.6482054592 (0.000)	1.6482054582 (20.062)	6.067 x10^{-8}	1.6105506917 (0.000)	1.6105506908 (19.515)	5.588 x10^{-8}
2.00	1.2742928186 (0.000)	1.2742928184 (20.608)	1.569 x10^{-8}	1.2606655170 (0.000)	1.2606655169 (20.108)	7.932 x10^{-9}
5.00	1.0925601855 (0.000)	1.0925601855 (19.968)	0.000 x10^{-7}	1.0887275633 (0.000)	1.0887275633 (19.999)	0.000 x10^{-7}

Table 4-3 Comparison of ARL for $MA(1)$ on modified EWMA control chart by using explicit formulas versus numerical integral equation for $ARL_0 = 370$, $\mu = 2$ and $\theta = -0.1$ with $\lambda = 0.05$, 0.1.

Shift size (δ)	$\lambda = 0.05$ $(b = 0.333987011)$			$\lambda = 0.1$ $(b = 0.337683969)$		
	Explicit	NIE	Diff	Explicit	NIE	Diff
0.00	370.000088128 (0.000)	370.000085899 (19.203)	6.023×10^{-7}	370.000045250 (0.000)	370.000041103 (19.234)	1.121×10^{-6}
0.005	129.368446676 (0.000)	129.368446059 (19.329)	4.767×10^{-7}	121.977760605 (0.000)	121.977759847 (19.828)	6.218×10^{-7}
0.01	78.3785837061 (0.000)	78.3785833571 (19.281)	4.453×10^{-7}	73.0616312145 (0.000)	73.0616308355 (19.157)	5.187×10^{-7}
0.02	43.8283778836 (0.000)	43.8283777012 (19.640)	4.162×10^{-7}	40.5837742809 (0.000)	40.5837741013 (19.001)	4.425×10^{-7}
0.03	30.4255790759 (0.000)	30.4255789548 (19.439)	3.980×10^{-7}	28.1261295453 (0.000)	28.1261294309 (19.142)	4.067×10^{-7}
0.04	23.3084775806 (0.000)	23.3084774913 (19.532)	3.831×10^{-7}	21.5422828411 (0.000)	21.5422827586 (19.219)	3.830×10^{-7}
0.05	18.8979332134 (0.000)	18.8979331435 (19.454)	3.699×10^{-7}	17.4728622911 (0.000)	17.4728622274 (19.235)	3.646×10^{-7}
0.06	15.8985529165 (0.000)	15.8985528596 (19.421)	3.579×10^{-7}	14.7099776779 (0.000)	14.7099776266 (19.750)	3.487×10^{-7}
0.07	13.7277906602 (0.000)	13.7277906126 (19.266)	3.467×10^{-7}	12.7125612735 (0.000)	12.7125612309 (19.281)	3.351×10^{-7}
0.08	12.0849068085 (0.000)	12.0849067679 (19.500)	3.360×10^{-7}	11.2020224738 (0.000)	11.2020224376 (19.313)	3.232×10^{-7}
0.09	10.7989508728 (0.000)	10.7989508376 (19.797)	3.260×10^{-7}	10.0203022006 (0.000)	10.0203021694 (19.250)	3.114×10^{-7}
0.10	9.7655660837 (0.000)	9.7655660529 (19.437)	3.154×10^{-7}	9.0710522097 (0.000)	9.0710521824 (19.172)	3.010×10^{-7}
0.20	5.0909701574 (0.000)	5.0909701454 (19.329)	2.357×10^{-7}	4.7794459115 (0.000)	4.7794459010 (19.313)	2.197×10^{-7}
0.50	2.3731513934 (0.000)	2.3731513908 (19.329)	1.096×10^{-7}	2.2799397980 (0.000)	2.2799397957 (19.204)	1.009×10^{-7}
1.00	1.5707976721 (0.000)	1.5707976714 (19.703)	4.456×10^{-8}	1.5368909759 (0.000)	1.5368909753 (19.547)	3.904×10^{-8}
2.00	1.2359759882 (0.000)	1.2359759881 (20.187)	8.091×10^{-9}	1.2239327502 (0.000)	1.2239327501 (19.891)	8.170×10^{-9}
5.00	1.0776705703 (0.000)	1.0776705703 (20.109)	0.000×10^{-7}	1.0743419594 (0.000)	1.0743419594 (19.797)	0.000×10^{-7}

The values in parentheses are CPU times in the explicit formulas and NIE method (seconds)

Table 4-4 Comparison of ARL for $MA(1)$ on modified EWMA control chart by using explicit formulas versus numerical integral equation for $ARL_0 = 370$, $\mu = 2$ and $\theta = 0.3$ with $\lambda = 0.05$, 0.1.

Shift size (δ)	$\lambda = 0.05$ $(b = 0.500416482)$			$\lambda = 0.1$ $(b = 0.5078210852)$		
	Explicit	NIE	Diff	Explicit	NIE	Diff
0.00	370.000037555 (0.000)	370.000032443 (19.188)	1.382 x10^{-6}	370.000021198 (0.000)	370.000011412 (17.534)	2.645 x10^{-6}
0.005	141.365833037 (0.000)	141.365831489 (19.609)	1.095 x10^{-6}	133.968118708 (0.000)	133.968116699 (19.189)	1.499 x10^{-6}
0.01	87.3893405734 (0.000)	87.3893396843 (19.172)	1.017 x10^{-6}	81.8485965093 (0.000)	81.8485954972 (19.640)	1.237 x10^{-6}
0.02	45.5721131988 (0.000)	45.5721127297 (19.859)	1.029 x10^{-6}	46.0922055086 (0.000)	46.0922050291 (19.438)	1.040 x10^{-6}
0.03	34.6191639509 (0.000)	34.6191636381 (19.313)	9.035 x10^{-7}	32.1221812787 (0.000)	32.1221809738 (19.562)	9.492 x10^{-7}
0.04	26.6120557010 (0.000)	26.6120554695 (19.266)	8.699 x10^{-7}	24.6797238073 (0.000)	24.6797235875 (19.422)	8.906 x10^{-7}
0.05	21.6257283472 (0.000)	21.6257281654 (19.921)	8.407 x10^{-7}	20.0582469267 (0.000)	20.0582467570 (19.251)	8.460 x10^{-7}
0.06	18.2236441802 (0.000)	18.2236440319 (19.173)	8.138 x10^{-7}	16.9107669375 (0.000)	16.9107668006 (19.235)	8.095 x10^{-7}
0.07	15.7554375488 (0.000)	15.7554374244 (19.219)	7.896 x10^{-7}	14.6300579943 (0.000)	14.6300578805 (19.266)	7.779 x10^{-7}
0.08	13.8838344586 (0.000)	13.8838343523 (19.220)	7.656 x10^{-7}	12.9021228096 (0.000)	12.9021227129 (19.172)	7.495 x10^{-7}
0.09	12.4164876510 (0.000)	12.4164875587 (19.250)	7.434 x10^{-7}	11.5482611883 (0.000)	11.5482611047 (19.360)	7.239 x10^{-7}
0.10	11.2356855651 (0.000)	11.2356854839 (19.345)	7.227 x10^{-7}	10.4592857838 (0.000)	10.4592857107 (19.329)	6.989 x10^{-7}
0.20	5.8687516934 (0.000)	5.8687516612 (19.142)	5.487 x10^{-7}	5.5135573322 (0.000)	5.5135573037 (19.329)	5.169 x10^{-7}
0.50	2.7023142781 (0.000)	2.7023142709 (19.157)	2.664 x10^{-7}	2.5919056831 (0.000)	2.5919056768 (19.297)	2.431 x10^{-7}
1.00	1.7386942496 (0.000)	1.7386942479 (19.422)	9.777 x10^{-8}	1.6970258272 (0.000)	1.6970258257 (19.500)	8.839 x10^{-7}
2.00	1.3196959577 (0.000)	1.3196959573 (19.797)	3.031 x10^{-8}	1.3043639109 (0.000)	1.3043639106 (19.999)	2.300x10^{-8}
5.00	1.1104892940 (0.000)	1.1104892940 (19.718)	0.000 x10^{-7}	1.1061152594 (0.000)	1.1061152594 (20.031)	0.000 x10^{-7}

Table 4-5 Comparison of ARL for $MA(1)$ on modified EWMA control chart by using explicit formulas versus numerical integral equation for $ARL_0 = 370$, $\mu = 2$ and $\theta = -0.3$ with $\lambda = 0.05, 0.1$.

Shift size (δ)	$\lambda = 0.05$ $(b = 0.2730080154)$			$\lambda = 0.1$ $(b = 0.275661646)$		
	Explicit	NIE	Diff	Explicit	NIE	Diff
0.00	370.000059817 (0.000)	370.000058340 (18.985)	3.993×10^{-7}	370.000096869 (0.000)	370.000094148 (18.876)	7.353×10^{-7}
0.005	124.031896772 (0.000)	124.031896380 (19.079)	3.156×10^{-7}	116.701897449 (0.000)	116.701896978 (19.110)	4.038×10^{-7}
0.01	74.4835209978 (0.000)	74.4835207777 (19.298)	2.955×10^{-7}	69.3024972666 (0.000)	69.3024970321 (19.189)	3.384×10^{-7}
0.02	41.3942561115 (0.000)	41.3942559970 (19.328)	2.766×10^{-7}	38.2716959977 (0.000)	38.2716958865 (19.204)	2.906×10^{-7}
0.03	28.6625016773 (0.000)	28.6625016014 (19.297)	2.648×10^{-7}	26.4614633783 (0.000)	26.4614633074 (19.126)	2.679×10^{-7}
0.04	21.9257283323 (0.000)	21.9257282764 (19.376)	2.550×10^{-7}	20.2407247026 (0.000)	20.2407246515 (19.360)	2.525×10^{-7}
0.05	17.7595096535 (0.000)	17.7595096099 (19.547)	2.455×10^{-7}	16.4032325840 (0.000)	16.4032325445 (19.204)	2.408×10^{-7}
0.06	14.9302560132 (0.000)	14.9302559777 (19.407)	2.378×10^{-7}	13.8012538432 (0.000)	13.8012538114 (19.141)	2.304×10^{-7}
0.07	12.8847709963 (0.000)	12.8847709667 (19.188)	2.297×10^{-7}	11.9220242644 (0.000)	11.9220242380 (19.219)	2.214×10^{-7}
0.08	11.3380063142 (0.000)	11.3380062889 (19.484)	2.231×10^{-7}	10.5019946579 (0.000)	10.5019946355 (19.094)	2.133×10^{-7}
0.09	10.1281525340 (0.000)	10.1281525122 (19.360)	2.152×10^{-7}	9.3918313761 (0.000)	9.3918313568 (19.032)	2.055×10^{-7}
0.10	9.1565357524 (0.000)	9.1565357332 (19.329)	2.097×10^{-7}	8.5005935705 (0.000)	8.5005935536 (19.126)	1.988×10^{-7}
0.20	4.7711256680 (0.000)	4.7711256606 (19.313)	1.551×10^{-7}	4.4798048938 (0.000)	4.4798048873 (19.125)	1.451×10^{-7}
0.50	2.2398739905 (0.000)	2.2398739889 (19.220)	7.143×10^{-8}	2.1544387573 (0.000)	2.1544387560 (18.923)	6.034×10^{-8}
1.00	1.5040903607 (0.000)	1.5040903604 (19.282)	1.995×10^{-8}	1.4736329236 (0.000)	1.4736329233 (19.312)	2.036×10^{-8}
2.00	1.2034664108 (0.000)	1.2034664107 (19.890)	8.309×10^{-9}	1.1928656596 (0.000)	1.1928656595 (19.765)	8.383×10^{-9}
5.00	1.0652660139 (0.000)	1.0652660139 (19.921)	0.000×10^{-7}	1.0623934678 (0.000)	1.0623934678 (19.719)	0.000×10^{-7}

The values in parentheses are CPU times in the explicit formulas and NIE method (seconds)

Tables 4-2 to 4-5 show the ARL by using explicit formulas versus NIE for the $MA(1)$ process on modified EWMA control chart for $\mu = 2$, $\theta = 0.1, -0.1$, $0.3, -0.3$ with $\lambda = 0.05, 0.1.$ For $ARL_0 = 370, 500$ the results from both methods are excellent agreement. However, the CPU time when using explicit formulas was 0 second whereas it was around 20 seconds for NIE.

Table 4-6 Comparison ARL for $MA(1)$ on EWMA and Modified EWMA control charts using by Explicit formulas for $ARL_0 = 370$, $\mu = 2$ and $\theta = 0.1$ with $\lambda = 0.05, 0.1$.

Shift size (δ)	$\lambda = 0.05$		$\lambda = 0.10$	
	EWMA (h=0.000000691135)	Modified (b=0.408730497)	EWMA (h=0.034133896)	Modified (b=0.413935708)
0.00	370.0000	370.0000	370.0000	370.0000
0.005	338.2197	135.1157	355.7906	127.6996
0.01	309.4438	82.6506	342.2527	77.2120
0.02	259.7080	46.5318	317.0453	43.1674
0.03	218.7135	32.3937	294.1084	29.9952
0.04	184.8066	24.8565	273.2048	23.0076
0.05	156.6673	20.1748	254.1253	18.6792
0.06	133.2381	16.9861	236.6847	15.7362
0.07	113.6682	14.6757	220.7193	13.6062
0.08	97.2712	12.9255	206.0837	11.9940
0.09	83.4913	11.5545	192.6487	10.7319
0.10	71.8767	10.4521	180.2995	9.7173
0.20	18.7872	5.4534	98.5052	5.1205
0.50	1.8231	2.5258	25.7014	2.4243
1.00	1.0357	1.6482	6.8447	1.6106
2.00	1.0014	1.2743	2.2199	1.2607
5.00	1.0000	1.0926	1.1884	1.0887

Table 4-7 Comparison ARL for $MA(1)$ on EWMA and Modified EWMA control charts using by Explicit formulas for $ARL_0 = 500$, $\mu = 2$ and $\theta = -0.3$ with $\lambda = 0.05$, 0.1.

Shift size (δ)	$\lambda = 0.05$		$\lambda = 0.10$	
	EWMA (h=1.3943043x10^{-6})	Modified (b=0.2734313281)	EWMA (h=0.08324898)	Modified (b=0.2760602695)
0.00	500.0000	500.0000	500.0000	500.0000
0.005	457.9318	135.9195	482.3789	127.1542
0.01	419.7648	78.6269	465.5354	72.8675
0.02	353.6066	42.6511	434.0220	39.3388
0.03	298.8649	29.2629	405.1639	26.9690
0.04	253.4160	22.2771	378.6987	20.5372
0.05	215.5577	17.9904	354.3935	16.5979
0.06	183.9212	15.0937	332.0411	13.9390
0.07	157.4013	13.0068	311.4571	12.0248
0.08	135.1032	11.4326	292.4766	10.5818
0.09	116.2992	10.2038	274.9525	9.4556
0.10	100.3964	9.2184	258.7528	8.5528
0.20	26.7119	4.7878	148.6098	4.4940
0.50	2.2719	2.2430	42.9740	2.1572
1.00	1.0590	1.5051	12.1432	1.4745
2.00	1.0024	1.2038	3.5904	1.1932
5.00	1.0001	1.0654	1.4368	1.0625

Tables 4-6 and 4-7 summarize the *ARLs* using the explicit formulas on both standard and modified EWMA control charts for the *MA*(1) process given ARL_0 = 370, 500; $\mu = 2$, $\lambda = 0.05$ and 0.1; and $\theta = 0.1, -0.3$. The results indicated that the modified EWMA control chart immediately reduced ARL_1 by more than the standard one when detecting a small shift in the process mean, except for shift sizes of 0.5, 1, 2, or 5 for θ = 0.1 and shift sizes of 1, 2, or 5 for θ = -0.3 in the case of λ = 0.05.

4.2 Evaluation the *ARL* for *ARX(p,r)* Process

4.2.1 The Explicit formulas of ARL for *ARX(p,r)* process

Explicit formulas for the ARL of the modified EWMA control chart for an *ARX(p,r)* process are derived as follows:

The modified EWMA control chart proposed by Khan et al. (2016) is defined as

$$M_t = (1-\lambda)M_{t-1} + \lambda Y_t + k(Y_t - Y_{t-1}) \qquad ;t = 1,2,3,\ldots,$$

where M_t is the modified EWMA statistic, Y_t is the sequence of the *ARX(p,r)* process with exponential white noise, λ is an exponential smoothing parameter $(0 < \lambda \le 1)$, and k is a constant $(k > 0)$.

Substituting Y_t into M_t, then

$$M_t = (1-\lambda)M_{t-1} + (\lambda+k)\delta + (\lambda+k)\phi_1 Y_{t-1} + \ldots + (\lambda+k)\phi_p Y_{t-p}$$
$$+(\lambda+k)\sum_{j=1}^{r}\beta_j X_j + (\lambda+k)\varepsilon_t - kY_{t-1}\ .$$

If Y_1 gives the out-of-control state for M_1, $M_0 = u$ and $Y_0 = v$, then

$M_1 = (1-\lambda)u + (\lambda+k)\delta + (\lambda+k)\phi_1 v + \ldots + (\lambda+k)\phi_p v + (\lambda+k)\sum_{j=1}^{r}\beta_j X_j + (\lambda+k)\varepsilon_1 - kv$ If ε_1 is the in-control limit for M_1, then $0 \le M_1 \le h$.

The function $G(u)$ can be derived by the Fredholm integral equation of the second type (Mititelu et al., 2010), and thus $G(u)$ can be written as

$$G(u) = 1 + \int G(M_1) f(\varepsilon_1) d(\varepsilon_1)\ , \tag{4-14}$$

by substituting ε_1 with y. Therefore, the function $G(u)$ is obtained as

$$G(u) = 1 + \int_0^h L\left\{\begin{array}{l}(1-\lambda)u + (\lambda+k)\delta + (\lambda+k)\phi_1 v + \ldots + (\lambda+k)\phi_p v \\ +(\lambda+k)\sum_{j=1}^{r}\beta_j X_j - kv + (\lambda+k)y\end{array}\right\} f(y)dy\ .$$

Let $w = (1-\lambda)u + (\lambda+k)\delta + (\lambda+k)\phi_1 v + \ldots + (\lambda+k)\phi_p v + (\lambda+k)\sum_{j=1}^{r}\beta_j X_j - kv + (\lambda+k)y.$

By changing the integral variable, we obtain the integral equation as follows:

$$G(u) = 1 + \frac{1}{\lambda+k}\int_0^h G(w)\, f\left\{\frac{w-(1-\lambda)u}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\}dw\ . \tag{4-15}$$

If $Y_t \sim Exp(\alpha)$ the $f(y) = \frac{1}{\alpha}e^{\frac{-y}{\alpha}}$; $y \ge 0$, so

$$G(u) = 1 + \frac{1}{\lambda+k}\int_0^h G(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\}}dw\ .$$

Let the function $D(u)=e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}-\frac{kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}$, then we have

$$G(u)=1+\frac{D(u)}{\alpha(\lambda+k)}\int_0^h L(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw \qquad ; 0\le u\le h .$$

Let $g=\int_0^h G(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw$, then $G(u)=1+\frac{D(u)}{\alpha(\lambda+k)}\cdot g$. Consequently, we obtain

$$G(u)=1+\frac{1}{\alpha(\lambda+k)}e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}-\frac{kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}\cdot g .$$

(4-16)

Solving a constant g

$$\begin{aligned}
g&=\int_0^h G(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw\\
&=\int_0^h e^{\frac{-w}{\alpha(\lambda+k)}}dw+\int_0^h \frac{g}{\alpha(\lambda+k)}D(w)\cdot e^{\frac{-w}{\alpha(\lambda+k)}}dw\\
&=-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)+\frac{ge^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\alpha(\lambda+k)}\int_0^h e^{\frac{-\lambda w}{\alpha(\lambda+k)}}dw\\
&=-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)-\frac{de^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\lambda}\left(e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right)
\end{aligned}$$

$$g=\frac{-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)}{1+\frac{e^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\lambda}\left(e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right)} . \tag{4-17}$$

Substituting g from Equation (4-17) into Equation (4-16), then

$$G(u)=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}}\left[e^{\frac{-h}{\alpha(\lambda+k)}}-1\right]}{\lambda e^{\frac{kv}{\alpha(\lambda+k)}-\frac{\delta}{\alpha}-\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}-\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}+\left[e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right]}. \tag{4-18}$$

When the process is in the in-control state with exponential parameter $\alpha=\alpha_0$, we obtain the explicit solution for ARL_0 as follows:

$$ARL_0=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_0(\lambda+k)}}\left[e^{\frac{-h}{\alpha_0(\lambda+k)}}-1\right]}{\lambda e^{\frac{kv}{\alpha_0(\lambda+k)}-\frac{\delta}{\alpha_0}-\frac{v\sum_{i=1}^{p}\phi_i}{\alpha_0}-\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha_0}}+\left[e^{\frac{-\lambda h}{\alpha_0(\lambda+k)}}-1\right]}. \tag{4-19}$$

Similarly, when the process is in the out-of-control state with exponential parameter $\alpha=\alpha_1$, the explicit solution for ARL_1 can be written as

$$ARL_1=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_1(\lambda+k)}}\left[e^{\frac{-h}{\alpha_1(\lambda+k)}}-1\right]}{\lambda e^{\frac{kv}{\alpha_1(\lambda+k)}-\frac{\delta}{\alpha_1}-\frac{v\sum_{i=1}^{p}\phi_i}{\alpha_1}-\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha_1}}+\left[e^{\frac{-\lambda h}{\alpha_1(\lambda+k)}}-1\right]}. \tag{4-20}$$

4.2.2 Numerical Integral Equation of ARL for *ARX(p,r)* process

An integral equation of the second type for the *ARL* on the modified EWMA control chart for the *ARX(p,r)* process can be approximated by using the quadrature formula. In this study, the Gauss-Legendre quadrature rule is applied as follows:

Given $f\left(a_j\right)=f\left\{\frac{a_j-(1-\lambda)a_i}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\}$ (4-21)

The approximation for the integral is in the form

$$\int_0^h G(w)f(w)dw\approx\sum_{j=1}^{m}w_j f\left(a_j\right),$$

where $a_j=\frac{h}{m}\left(j-\frac{1}{2}\right)$ and $w_j=\frac{h}{m}; j=1,2,...,m$.

Using the quadrature formula, the numerical approximation $\tilde{G}(u)$ for the integral equation can be found as a solution of the linear equations as follows:

$$\tilde{G}(a_i)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_i}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\};i=1,2,...,m.$$

Thus,

$$\tilde{G}(a_1)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_1}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\},$$

$$\tilde{G}(a_2)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_2}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\},$$

$$\tilde{G}(a_3)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_3}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\},$$

$$\vdots$$

$$\tilde{G}(a_m)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_m}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\}.$$

The set of m equations with m unknowns can be rewritten in matrix form. The column vector of $\tilde{G}(a_i)$ is $\mathbf{G}_{m\times1}=(\tilde{G}(a_1),\tilde{G}(a_2),...,\tilde{G}(a_m))'$. Since $\mathbf{1}_{m\times1}=(1,1,...,1)'$ is a column vector of ones and $\mathbf{R}_{m\times m}$ is a matrix, we can define m to m^{th} as elements of matrix $\mathbf{R}$ as follows:

$$\left[R_{ij}\right]\approx\frac{1}{\lambda+k}w_j\,f\left\{\frac{a_j-(1-\lambda)a_i}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\},$$

and $\mathbf{I}_m=diag(1,1,...,1)$ is a unit matrix of order m. If $(\mathbf{I}-\mathbf{R})^{-1}$ exists, the numerical approximation for the integral equation in terms of the matrix can be written as

$$\mathbf{G}_{m\times1}=(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times1}.$$

Finally, by substituting a_i by u in $\tilde{G}(a_i)$, the numerical integration equation for function $\tilde{G}(u)$ can be derived as

$$\tilde{G}(u)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)u}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\}\qquad(4.22)$$

A comparison of the efficacies of the NIE method ($\tilde{G}(u)$) and the explicit formulas ($G(u)$) for the *ARL* of an *ARX(p,r)* process on the modified EWMA control chart is presented. The parameter values were set as $ARL_0=370$ and 500; $\lambda=0.01$, 0.05, 0.1, and 0.2; the in-control parameter $\alpha_0 = 1$; and the shift size was varied as 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.3, and 0.5. In general, the popular setting the initial value is equal to the expected value of the distribution. For setting the in-control parameter, $\alpha_0 = 1$ which is the initial value as 1. The coefficient has a value from -1 to 1, which can be specified as any value. The configuration does not affect the accuracy of the explicit formulas and the NIE methods. The absolute percentage difference to measure the accuracy of the *ARL* is defined as

$$Diff\,(\%) = \frac{\left|G(u) - \tilde{G}(u)\right|}{G(u)} \times 100\,.$$

Table 4-8 Comparison of the *ARL* on a modified EWMA control chart using explicit formulas with the NIE method for ARX(1,1) with $u=1,$ $v=1,$ $k=1,$ $\beta_1=0.2,$ and $ARL_0=370$.

λ	δ	ϕ_1	h	shift	Explicit	NIE	Time[a]	Diff%
0.05	0	0.1	2.11284	0.00	370.514622	370.51416	13.468	0.000123
				0.01	185.632808	185.63263	13.555	0.000093
				0.02	123.215119	123.21501	13.494	0.000082
				0.03	91.885107	91.885037	13.428	0.000076
				0.04	73.065410	73.065357	13.551	0.000072
				0.05	60.520993	60.520951	13.552	0.000068
				0.10	32.116753	32.116734	13.504	0.000058
				0.30	10.731776	10.731772	13.546	0.000037
				0.50	6.457709	6.457707	13.440	0.000026
		-0.1	2.61195	0.00	370.424900	370.42415	13.511	0.000201
				0.01	274.686377	274.68590	13.563	0.000172
				0.02	214.349128	214.34880	13.478	0.000153
				0.03	173.294237	173.29399	13.511	0.000140
				0.04	143.823485	143.82329	13.446	0.000130
				0.05	121.811602	121.81145	13.445	0.000122
				0.10	64.531777	64.531713	13.421	0.000098
				0.30	17.824627	17.824616	13.541	0.000060
				0.50	9.519466	9.519462	13.452	0.000042
0.10	1	0.2	0.69141	0.00	370.555941	370.55586	13.502	0.000021
				0.01	90.098635	90.098626	13.498	0.000010
				0.02	51.385975	51.385971	13.460	0.000008
				0.03	35.997312	35.997309	13.434	0.000007
				0.04	27.736604	27.736602	13.467	0.000007
				0.05	22.584474	22.584472	13.485	0.000006
				0.10	11.823565	11.823565	13.518	0.000005
				0.30	4.369675	4.369675	13.506	0.000003
				0.50	2.902154	2.902154	13.467	0.000002
		-0.2	1.04870	0.00	370.273926	370.27373	13.537	0.000052
				0.01	105.208634	105.20860	13.519	0.000025
				0.02	61.437849	61.437836	13.532	0.000020
				0.03	43.452288	43.452281	13.542	0.000018
				0.04	33.653645	33.653639	13.449	0.000017
				0.05	27.489870	27.489866	13.471	0.000016
				0.10	14.478770	14.478768	13.508	0.000013
				0.30	5.327500	5.327500	13.488	0.000008
				0.50	3.494115	3.494115	13.434	0.000005

Table 4-9 Comparison of the *ARL* on a modified EWMA control chart using explicit formulas with the NIE method for ARX(2,1) with $u=1,\ \ v=1,\ \ \delta=0,\ \ k=1,\ \ \beta_1=0.2$ and $ARL_0=370$.

λ	ϕ_1	ϕ_2	h	shift	Explicit	NIE	Time	Diff%
0.05	0.1	0.1	1.90196	0.00	370.104536	370.104178	13.492	0.000097
				0.01	164.156587	164.156470	13.512	0.000072
				0.02	105.192627	105.192560	13.467	0.000063
				0.03	77.258066	77.258020	13.551	0.000059
				0.04	60.970239	60.970204	13.466	0.000056
				0.05	50.307351	50.307324	13.532	0.000054
				0.10	26.685128	26.685116	13.545	0.000046
				0.30	9.216663	9.216660	13.436	0.000029
				0.50	5.688211	5.688210	13.529	0.000020
		-0.1	2.34842	0.00	370.111274	370.110694	13.520	0.000157
				0.01	218.233870	218.233600	13.487	0.000124
				0.02	153.301535	153.301368	13.495	0.000109
				0.03	117.350069	117.349952	13.416	0.000100
				0.04	94.565641	94.565553	13.479	0.000094
				0.05	78.865508	78.865438	13.492	0.000089
				0.10	41.847694	41.847663	13.492	0.000074
				0.30	13.172037	13.172030	13.492	0.000047
				0.50	7.603031	7.603028	13.517	0.000033
0.10	0.1	0.2	1.78838	0.00	370.299172	370.298515	13.519	0.000178
				0.01	144.464904	144.464768	13.518	0.000094
				0.02	89.685034	89.684969	13.459	0.000073
				0.03	65.007924	65.007883	13.455	0.000063
				0.04	50.973254	50.973225	13.535	0.000057
				0.05	41.920612	41.920590	13.490	0.000052
				0.10	22.222112	22.222103	13.425	0.000041
				0.30	7.893825	7.893823	13.482	0.000024
				0.50	4.987173	4.987172	13.499	0.000016
		-0.2	2.78993	0.00	370.115966	370.114002	13.501	0.000531
				0.01	286.090889	286.089671	13.542	0.000426
				0.02	228.394081	228.393273	13.492	0.000354
				0.03	187.010941	187.010376	13.531	0.000302
				0.04	156.285272	156.284860	13.569	0.000264
				0.05	132.822686	132.822375	13.438	0.000234
				0.10	70.157635	70.157527	13.440	0.000153
				0.30	18.713286	18.713273	13.557	0.000072
				0.50	9.832924	9.832919	13.520	0.000047
0.20	0.2	0.1	1.95666	0.00	370.295141	370.292831	13.437	0.000624
				0.01	137.992552	137.992202	13.490	0.000253
				0.02	84.820251	84.820109	13.462	0.000168
				0.03	61.246548	61.246469	13.427	0.000130
				0.04	47.941844	47.941792	13.486	0.000108
				0.05	39.398626	39.398589	13.534	0.000093
				0.10	20.908407	20.908394	13.466	0.000060
				0.30	7.523560	7.523557	13.512	0.000028
				0.50	4.802271	4.802270	13.539	0.000018
		-0.1	2.49307	0.00	370.002909	369.998734	13.478	0.001128
				0.01	183.558057	183.556989	13.456	0.000582
				0.02	121.435201	121.434716	13.457	0.000399
				0.03	90.409051	90.408773	13.539	0.000307
				0.04	71.822583	71.822402	13.520	0.000252
				0.05	59.455236	59.455108	13.515	0.000215
				0.10	31.519801	31.519760	13.498	0.000129
				0.30	10.562568	10.562562	13.519	0.000054
				0.50	6.382784	6.382781	13.433	0.000033

The results in Tables 4-7 and 4-8 report the numerical values of the *ARL* derived from the explicit formulas and NIE method, and the absolute percentage difference between them. From the results, we can see that the *ARL* values derived from the explicit formulas give the same results as the NIE method. The numerical approximations had an absolute percentage difference of less than 0.003%. However, the computational time of the NIE method was 13.42–13.58 s whereas that of the explicit formulas was < 1 s.

We use the relative mean index (RMI) to compare the performance of the *ARL* of an *ARX(p,r)* process on the EWMA and modified EWMA control charts. The RMI is defined as

$$RMI = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_{shift,i} - Min\left[ARL_{shift,i}\right]}{Min\left[ARL_{shift,i}\right]}\right).$$

where $ARL_{shift,i}$ is the *ARL* of the control chart when the position process shift, $Shift,i$ is the shift size for $i = 1,2,...,n$, $Min\left[ARL_{shift,i}\right]$ denotes the smallest *ARL* of two control charts in comparison when the position process shift. The control chart with a smallest RMI performs the best in detecting mean changes on the whole.

For the comparison of the *ARL*s on the EWMA and modified EWMA control charts for an *ARX(1,1)* process, the parameter values were set as $ARL_0 = 370$; $\lambda =$ 0.05, 0.1, and 0.2; the in-control parameter $\alpha_0 = 1$; the shift size was varied as 0.001, 0.003, 0.005, 0.007, 0.009, 0.01, 0.03, 0.05, 0.07 and 0.09. The results are reported in Table 4-9.

Table 4-1 0 Comparison of the *ARL* of EWMA and modified EWMA control charts using explicit formulas for an *ARX(1,1)* with $u=1,$ $v=1,$ $\delta=1,$ $k=40\lambda,$ $\beta_1=0.2$ and $ARL_0=370$.

shift	$\lambda=0.05,\ \phi_1=0.2$		$\lambda=0.1,\ \phi_1=0.2$		$\lambda=0.2,\ \phi_1=0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b=2.5496\times10^{-8}$	$h=1.3590441$	$b=0.00107964$	$h=2.7665764$	$b=0.04441$	$h=5.734902$
0.00	370.071291	370.076891	370.004307	370.003373	370.722568	370.715572
0.00	362.264617	257.030787	365.787507	239.166659	362.075941	233.363671
0.00	347.186399	159.726241	357.521610	140.347811	345.767724	134.327916
0.00	332.792484	115.984299	349.473732	99.484050*	330.655613	94.494822*
0.00	319.049437	91.124031*	341.637371	77.151829*	316.615806	72.994088*
0.00	305.925567	75.090081*	334.006238	63.075235*	303.540954	59.537517*
0.01	299.586374	69.033841*	330.265724	57.823254*	297.335678	54.535706*
0.03	198.799944	26.743266*	264.852791	22.105174*	206.492192	20.787108*
0.05	134.056104	16.821242*	214.163689	13.957965*	153.166554	13.153305*
0.07	91.808369	12.393346*	174.541323	10.348427*	118.601180	9.778282*
0.09	63.828457	9.886695*	143.313881	8.311410*	94.690406	7.875239*
RMI	3.763440	0	7.445925	0	5.824404	0

shift	$\lambda=0.05,\ \phi_1=-0.2$		$\lambda=0.1,\ \phi_1=-0.2$		$\lambda=0.2,\ \phi_1=-0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b=3.8036\times10^{-8}$	$h=2.0452044$	$b=0.0016149$	$h=4.200333$	$b=0.067334$	$h=8.88252$
0.00	370.075490	370.074437	370.035318	370.040056	370.017072	370.021854
0.00	362.413120	272.589846	365.971526	257.425111	362.407567	254.808465
0.00	347.604772	178.698291	358.000435	160.267524	347.947355	157.288491
0.00	333.457209	133.038467	350.232942	116.523643	334.419802	113.936839
0.00	319.938646	106.040681	342.663148	91.643263*	321.740040	89.431083*
0.00	307.018935	88.205073*	335.285344	75.588906*	309.833149	73.676321*
0.01	300.774653	81.379575*	331.666704	69.523408*	304.148484	67.735772*
0.03	201.120167	32.306069*	268.119397	27.141638*	218.327779	26.405507*
0.05	136.616346	20.420367*	218.476210	17.187944*	165.516539	16.743211*
0.07	94.217093	15.067503*	179.369496	12.742094*	130.215179	12.432517*
0.09	65.938916	12.022993*	148.318390	10.222777*	105.253345	9.990907*
RMI	3.062808	0	6.078151	0	4.802528	0

*The smallest *ARL* on each shift size according to the case.

From the results in Tables 4-9, it is evident that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every value of λ. For example, when $\phi_1=0.2$, $\lambda=0.05$ and shift = 0.05, the *ARL* is 332.792484 from the EWMA control chart while the *ARL* is 115.984299 from the modified EWMA control chart, which corresponds to the RMI values for the modified EWMA control chart being less than those for the EWMA control chart.

We applied the explicit formulas for the *ARL*s on the EWMA and modified EWMA control charts for an *ARX(1,1)* process using 55 real-world data observations on the value of exports and imports of agricultural products to and from Thailand (Unit: Ten billion baht) from January 2016 to July 2019, where the value of the imports is the explanatory variable (data from the Office of Agricultural Economics of Thailand). The parameters were set as $\lambda = 0.05$, 0.1, and 0.2; $\alpha_0 = u = 0.589259$; shift size values of 0.001, 0.003, 0.005, 0.007, 0.009, 0.01, 0.03, 0.05, 0.07, and 0.09; and autoregressive coefficients $\phi_1 = 0.326152,$ $\delta = 6.652233,$ $v = 10.4918,$ $\beta_1 = 0.933313$, and $X_1 = 4.1439$. The results are given in Table 4-10.

Table 4-11 Comparison of the *ARL* of ARX(1,1) of EWMA and modified EWMA control charts for the value of exports and imports of agricultural products for $ARL_0 = 370$.

shift	$\lambda = 0.05$ $(k = 100\lambda)^{a}$		$\lambda = 0.1$ $(k = 40\lambda)$		$\lambda = 0.2$ $(k = 60\lambda)$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 3.27104 \times 10^{-18}$	$h = 0.007128814$	$b = 1.3617 \times 10^{-13}$	$h = 0.0044711692$	$b = 5.446519 \times 10^{-12}$	$h = 0.0153495067$
0.00	374.505189	374.51027	370.015353	370.013368	370.026479	370.02620
0.00	348.463281	80.027591	348.920418	79.529679*	276.359424	71.606027*
0.00	301.920274	31.405097	310.520857	31.227647*	178.925791	27.747820*
0.00	261.864143	19.690008	276.644980	19.575453*	128.916300	17.383968*
0.00	227.356776	14.422145	246.734101	14.334171*	98.643720	12.748838*
0.00	197.600498	11.428518	220.280627	11.355250*	78.453653	10.121645*
0.01	184.287726	10.369330	208.221352	10.301213*	70.713067	9.193284*
0.03	48.289170	3.899445*	70.946841	3.863411*	17.089659	3.535219*
0.05	14.280238	2.596183*	26.449591	2.568076*	6.766353	2.397466*
0.07	5.028239	2.046506*	10.869219	2.022750*	3.406272	1.917426*
0.09	2.310792	1.748461*	5.049280	1.727746*	2.092499	1.656939*
RMI	8.975697	0	11.224773	0	4.159967	0

k for the modified EWMA control chart. [a]a constant value. *The smallest *ARL* on each shift size according to the case.

From the results in Tables 4-10, it is evident that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every value of λ. For example, when λ = 0.05 and shift = 0.009, the *ARL* is 197.600498 from the EWMA control chart while the *ARL* is 11.428518 from the modified EWMA control chart. This corresponds to an RMI value of 0 for the modified EWMA control chart, which is less than that for the EWMA control chart. The results from Tables 4-10 is plotted on the charts in Figure 4-1.

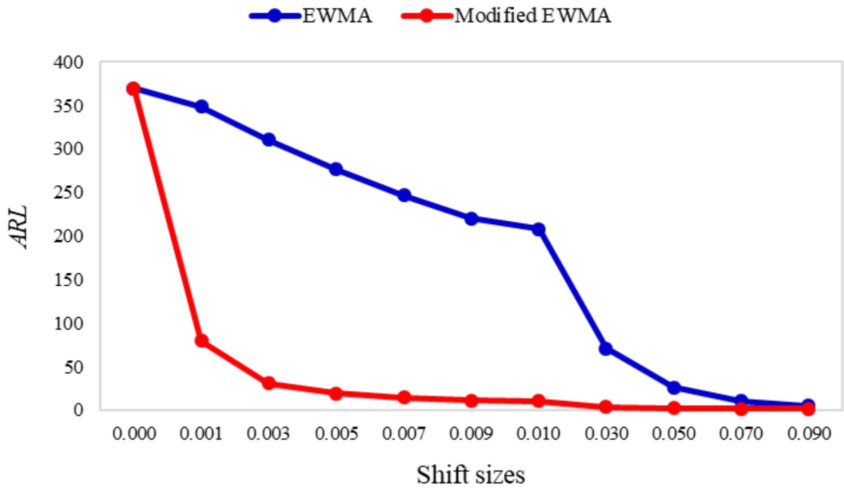

Figure 4-1 Comparison of the *ARL* for an ARX(1,1) on EWMA and modified EWMA control charts for real data in table 6, where $\lambda = 0.10$.

From Figure 4-1, it can be seen that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every case. For example, when shift = 0.009, the *ARL* from the modified EWMA control chart is less than that of the EWMA control chart.

4.3 Evaluation the ARL for *MAX(q,r)* and *SMAX(Q,r)$_L$* Processes

4.3.1 Explicit formulas of *ARL* for MAX(q,r) and SMAX(Q,r)$_L$ processes

The derivations of the explicit fomulas for the ARL of CUSUM chart, when observations are *MAX(q,r)* and *SMAX(Q,r)$_L$* processes with exponential white noise are presented as follows:

1.The explicit formula $C(s)$ for the ARL of MAX(q,r) process with an exponential white noise is

$$C(s) = 1 + \int_0^h C(y)\lambda e^{\lambda(s-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it})} e^{-\lambda y} dy$$

$$+\left(1-e^{-\lambda\left(a-s-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)C(0) \quad ; s \in [0,a) \qquad (4.23)$$

Let g be constant as $g = \int_0^h C(y)e^{-\lambda y}dy.$

$C(s)$ can be written as

$$C(s)=1+\lambda g e^{\lambda(s-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it})}+\left(1-e^{-\lambda\left(a-s-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)C(0)$$

For $s=0$, then

$$C(0)=1+\lambda g e^{\lambda\left(-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it}\right)}+\left(1-e^{-\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)C(0)$$

$$=e^{\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}+\lambda g\ .$$

Then

$$C(s)=1+\lambda g e^{\lambda(s-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it})}$$
$$+\left(1-e^{-\lambda\left(a-s-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)e^{\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}+\lambda g$$
$$=1+e^{\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}+\lambda g-e^{\lambda s}$$
$$=1+\lambda g+e^{\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}-e^{\lambda s}. \tag{4-24}$$

Now, constant g can be found as the following

$$g=\int_0^h C(y)e^{-\lambda y}dy$$
$$=\int_0^h (1+\lambda g+e^{\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}-e^{\lambda y})e^{-\lambda y}dy$$
$$=\left((1+\lambda g+e^{\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)\int_0^h e^{-\lambda y}dy-\int_0^h e^{\lambda y-\lambda y}dy$$
$$=\frac{e^{\lambda h}}{\lambda}\left(1-e^{-\lambda h}\right)\left(1+e^{\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)-he^{\lambda h}.$$

Substituting the constant g into Equation (7), then

$$C(s)=e^{\lambda h}\left(1+e^{\lambda\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}-\lambda h\right)-e^{\lambda s},\quad s\ge 0.$$

As previously mentioned, the value of exponential parameter $\lambda=\lambda_0,$; this implies that the process is an in-control state. Hereby, the explicit analytical solution for *ARL₀* can be written as

$$ARL_0=\ e^{\lambda_0 h}\left(1+e^{\lambda_0\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}-\lambda_0 h\right)-e^{\lambda_0 s},\quad s\ge 0. \tag{4-25}$$

On the contrary, if the process is in an out-of-control state, the value of exponential parameter $\lambda_1 = \lambda_0(1+\delta)$, where $\lambda_1 > \lambda_0$, and δ is the shift size. The explicit analytical solution for ARL_1 can be written as

$$ARL_1 = e^{\lambda_1 h}\left(1+e^{\lambda_1\left(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}-\lambda_1 h\right)-e^{\lambda_1 s}, \quad s\geq 0. \qquad (4\text{-}26)$$

2. The explicit formula $D(s)$ for the ARL of *SMAX(Q,r)_L* process is

$$D(s)=1+\int_0^h D(y)\lambda e^{\lambda(s-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it})}e^{-\lambda y}dy$$
$$+\left(1-e^{-\lambda\left(a-s-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)D(0) \qquad ;s\in[0,a) \qquad (4\text{-}27)$$

Let k be constant as $k=\int_0^h D(y)e^{-\lambda y}dy.$

$D(s)$ can be written as

$$D(s)=1+\lambda k e^{\lambda(s-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it})}$$
$$+\left(1-e^{-\lambda\left(a-s-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)D(0).$$

For $s=0$, then

$$D(0)=1+\lambda k e^{\lambda\left(-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it}\right)}+\left(1-e^{-\lambda\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)D(0)$$
$$=e^{\lambda\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}+\lambda k\,.$$

Then,

$$D(s)=1+\lambda k e^{\lambda(s-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it})}$$
$$+\left(1-e^{-\lambda\left(a-s-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)e^{\lambda\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}+\lambda k$$
$$=1+\lambda k+e^{\lambda\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}-e^{\lambda s} \qquad (4\text{-}28)$$

Now, constant k can be found as the following

$$k=\int_0^h D(y)e^{-\lambda y}dy$$

$$= \int_0^h (1+\lambda k + e^{\lambda\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+...+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)} - e^{\lambda y})e^{-\lambda y}dy$$

$$= \left((1+\lambda k + e^{\lambda\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+...+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right)\int_0^h e^{-\lambda y}dy - \int_0^h e^{\lambda y-\lambda y}dy$$

$$= \frac{e^{\lambda h}}{\lambda}\left(1-e^{-\lambda h}\right)\left(1+e^{\lambda\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+...+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)}\right) - he^{\lambda h}.$$

Substituting the constant k into Equation 4-28, then

$$D(s) = e^{\lambda h}\left(1+e^{\lambda\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+...+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)} - \lambda h\right) - e^{\lambda s}, \quad s \ge 0.$$

As previously mentioned, the value of exponential parameter $\lambda = \lambda_0$,; this implies that the process is an in-control state. Hereby, the explicit analytical solution for ARL_0 can be written as

$$ARL_0 = e^{\lambda_0 h}\left(1+e^{\lambda_0\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+...+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)} - \lambda_0 h\right) - e^{\lambda_0 s}, \quad s \ge 0 \qquad (4\text{-}29)$$

On the contrary, if the process is in an out-of-control state, the value of exponential parameter $\lambda_1 = \lambda_0(1+\delta)$, where $\lambda_1 > \lambda_0$, and δ is the shift size. The explicit analytical solution for ARL_1 can be written as

$$ARL_1 = e^{\lambda_1 h}\left(1+e^{\lambda_1\left(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+...+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it}\right)} - \lambda_1 h\right) - e^{\lambda_1 s}, \quad s \ge 0 \qquad (4\text{-}30)$$

4.3.2. Numerical Integral Equation of ARL for *MAX(q,r) and SMAX(Q,r)$_L$* processes

The numerical integral equation method can evaluate the solution by the Gauss-Legendre quadrature rule as

$$ARL(a_i) = 1 + \frac{1}{\lambda+1}\sum_{j=1}^{m} w_j ARL(a_j) f\left(\frac{a_j - (1-\lambda)a_i + v + (\lambda\theta+\theta)s}{\lambda+1} - \mu\right)$$

where $i = 1,2,3,...,m$, $a_j = \frac{h}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{h}{m}$; $j = 1,2,3,...,m$.

It can be rewritten in matrix form as

$$ARL_{m\times1} = 1_{m\times1} + R_{m\times m}ARL_{m\times1} \text{ or } ARL_{m\times1} = \left(I_m - R_{m\times m}\right)^{-1}1_{m\times1}$$

where $ARL_{m\times1} = \begin{bmatrix} ARL(a_1) \\ ARL(a_2) \\ \vdots \\ ARL(a_m) \end{bmatrix}$, $I_m = diag(1,1,...,1)$, $1_{m\times1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

and

$$R_{m\times m}=\begin{bmatrix} \frac{1}{\lambda+1}w_1 f\left(\frac{a_1-(1-\lambda)a_1+v+(\lambda\theta+\theta)s}{\lambda+1}-\mu\right) & \cdots & \frac{1}{\lambda+1}w_m f\left(\frac{a_m-(1-\lambda)a_1+v+(\lambda\theta+\theta)s}{\lambda+1}-\mu\right) \\ \frac{1}{\lambda+1}w_1 f\left(\frac{a_1-(1-\lambda)a_2+v+(\lambda\theta+\theta)s}{\lambda+1}-\mu\right) & \cdots & \frac{1}{\lambda+1}w_m f\left(\frac{a_m-(1-\lambda)a_2+v+(\lambda\theta+\theta)s}{\lambda+1}-\mu\right) \\ \vdots & & \\ \frac{1}{\lambda+1}w_1 f\left(\frac{a_1-(1-\lambda)a_m+v+(\lambda\theta+\theta)s}{\lambda+1}-\mu\right) & \cdots & \frac{1}{\lambda+1}w_m f\left(\frac{a_m-(1-\lambda)a_m+v+(\lambda\theta+\theta)s}{\lambda+1}-\mu\right) \end{bmatrix}$$

Therefore, the approximation of average run length is evaluated by the numerical integral equation method for $ARL(u)$ is

$$ARL(u)=1+\frac{1}{\lambda+1}\sum_{j=1}^{m} w_j ARL(a_j)\, f\left(\frac{a_j-(1-\lambda)u+v+(\lambda\theta+\theta)s}{\lambda+1}-\mu\right)$$

where $a_j=\frac{h}{m}\left(j-\frac{1}{2}\right)$ and $w_j=\frac{h}{m}$, $j=1,2,3,...,m.$

An absolute percentage relative error (APRE) criterion is used as a performance criterion, and the APRE can be expressed as

$$APRE(\%)=\left|\frac{ARL(s)_{\exp}-ARL(s)_{app}}{ARL(s)_{\exp}}\right|\times 100\%$$

where $ARL(s)_{\exp}$ is the ARL results from the explicit formula and $ARL(s)_{app}$ is an approximation of ARL from NIE method.

The numerical procedure for obtaining optimal parameters for MAX designs

1. Select an acceptable in-control value of ARL_0 and decide on the change parameter value (λ_1) for an out-of-control state.;

2. For the given values λ_0 and T, find the optimal values of a and h to minimize the ARL_1 values given by Equation 4-26, subject to the constraint that ARL_0 = T in Equation 4-25, i.e., a and h are solutions of the optimality problem.

In addition, the numerical procedure for obtaining optimal parameters for SMAX designs is the same as the MAX procedure, by using Equations 4-29 and 4-30 for ARL_0 and ARL_1, respectively. The optimal (a,h) values for T = 370 and magnitudes of change are shown in Table 4-15.

Table 4-12 ARL values for in control process for *MAX (2,1)* using explicit formula against NIE given $\beta_1 = 0.5$ for $ARL_0 = 370$

a	θ_1	θ_2	h	Explicit	NIE	APRE (%)
1.5	0.1	0.2	4.889	370.485	369.123 (12.821)	0.365197
	0.1	-0.2	5.442	370.211	368.657 (12.885)	0.317387
2.5	0.1	0.2	3.971	370.432	369.123 (12.897)	0.353371
	0.1	-0.2	4.887	370.008	368.657 (12.777)	0.365127
3	0.1	0.2	3.265	370.225	369.087 (12.818)	0.307381
	0.1	-0.2	3.811	370.154	368.879 (12.843)	0.344451

Table 4-13 ARL values for in control process for *SMAX* $(3,1)_{12}$ using explicit formula against NIE given $\beta_1 = 0.5$ for $ARL_0 = 370$

a	Θ_1	Θ_2	Θ_3	h	Explicit	NIE	APRE (%)
1.5	0.1	0.2	0.3	3.525	370.411	369.202 (13.159)	0.326394
	0.1	-0.2	0.3	4.145	370.071	369.652 (13.258)	0.396805
2.5	0.1	0.2	0.3	3.525	370.411	369.202 (13.810)	0.326394
	0.1	-0.2	0.3	4.145	370.132	368.797 (13.574)	0.360682
3	0.1	0.2	0.3	2.906	370.008	368.979 (13.763)	0.278102
	0.1	-0.2	0.3	3.392	370.202	369.029 (13.564)	0.316854

Table 4-14 Comparison of ARL values for *MAX (2,1)* process using explicit formula against NIE given $a = 3, \theta_1 = 0.1, \theta_2 = 0.2,\ \beta_1 = 0.5,\ h = 3.265$ for $ARL_0 = 370$ and $h = 3.588$ for $ARL_0 = 500$

δ (shift size)	$ARL_0 = 370$		APRE (%)	$ARL_0 = 500$		APRE (%)
	Explicit	NIE		Explicit	NIE	
0.00	370.225	369.087 (12.235)	0.307381	500.080	498.380 (11.088)	0.339946
0.01	347.839	346.783 12.684)	0.303589	468.139	466.570 (11.137)	0.335157
0.03	308.154	307.243 (12.486)	0.295631	411.811	410.466 (11.199)	0.326606
0.05	274.253	273.462 (12.264)	0.28842	364.017	362.859 (10.497)	0.318117
0.07	245.143	244.453 (12.994)	0.281468	323.248	322.245 (10.706)	0.310288
0.1	208.758	208.191 (11.092)	0.271606	272.684	271.869 (11.218)	0.298881
0.3	86.578	86.391 (11.165)	0.21599	107.354	107.100 (11.242)	0.23660
0.5	45.641	45.561 (11.441)	0.175281	54.493	54.389 (11.282)	0.19085
1.0	16.512	16.494 (11.559)	0.109012	18.611	18.588 (11.670)	0.123583
1.5	9.183	9.176 (11.412)	0.076228	10.045	10.037 (11.363)	0.079642
2.0	6.288	6.256 (11.012)	0.38610	6.761	6.758 (11.360)	0.044372

Table 4-15 Comparison of ARL values using explicit formula against NIE for *SMAX(3,1)*$_{12}$ given $a=3$, $\Theta_1=0.1, \Theta_2=0.2, \Theta_3=0.3$, $\beta_1=0.5$, $h=2.906$ for $ARL_0=370$ and $h=3.223$ for $ARL_0=500$

δ (shift size)	$ARL_0=370$		APRE (%)	$ARL_0=500$		APRE (%)
	Explicit	NIE		Explicit	NIE	
0.00	370.008	368.979 (13.753)	0.278102	500.438	498.887 (13.859)	0.309929
0.01	348.077	347.120 (13.961)	0.274939	469.150	467.714 (14.024)	0.306085
0.03	309.124	308.294 (13.439)	0.268501	413.854	412.618 (14.150)	0.298656
0.05	275.763	275.040 (11.606)	0.262182	366.803	365.733 (14.016)	0.29171
0.07	247.047	246.414 (11.660)	0.256227	326.556	325.625 (14.204)	0.285097
0.1	211.048	210.525 (13.293)	0.247811	276.476	275.714 (13.411)	0.275612
0.3	88.943	88.764 (13.893)	0.201252	110.871	110.625 (11.886)	0.221879
0.5	47.311	47.233 (13.971)	0.164867	56.868	56.764 (11.878)	0.18288
1.0	17.208	17.190 (14.135)	0.104603	19.542	19.518 (12.244)	0.122812
1.5	9.530	9.523 (13.920)	0.073452	10.493	10.484 (11.859)	0.085771
2.0	6.486	6.482 (14.323)	0.061671	7.011	7.006 (12.324)	0.071317

Table 4-16 Optimal design parameters and ARL_1^* of *MAX (1,2)* and *SMAX (2,2)*$_{12}$ processes for $ARL_0=370$

δ (shift size)	*MAX (1,2)*		ARL_1^*	*SMAX (2,2)*$_{12}$		ARL_1^*
	a	h		a	h	
0.01	1.66999118602	5.973714487571	315.357	2.354154802	5.4156807185	165.571
005	1.66999118587	5.973714491718	207.994	2.360429504	5.2839458794	126.065
0.1	1.66999118585	5.973714492065	128.53	2.361119789	5.2874245951	87.092
0.3	1.66999118582	5.973714493059	29.355	2.361410035	5.2778673182	29.659
0.5	1.66999118582	5.973714493059	13.552	2.361410035	5.2778673147	16.122
1.0	1.66999118585	5.973714492065	7.619	2.361410035	5.2778673174	8.158
2.0	1.66999118611	5.973714485385	4.936	2.354154797	5.415680824	4.641

The numerical results were obtained from the simulation. The chart was set up with reference value (a) that is greater than s, where s is an initial value (s =0). The parameter a could be any number but it combines with the control limit (s) that corresponds to the in-control $ARL_0=370$ and 500. In this paper, the parameter a is varied between 1.5 to 3.0. A comparison of the solution of the explicit formula (Explicit) with the NIE method for the CUSUM chart, when $\lambda_0=1, \beta_1=0.5$ for *MAX(2,1)* and *SMAX (3,1)*$_{12}$ are reported in Tables 4-11 and 4-12 for $ARL_0 = 370,500$ with which they are in good agreement. Notice that the absolute percentage relative error is small. In Tables 4-13 and 4-14, we use Equations 4-25 and 4-26 to show ARL_0 and ARL_1 for the *MAX(2,1)* process with parameter $a = 3$ and the coefficient parameters of the process $\theta_1 = 0.1, \theta_2 = 0.2,$ and $\beta_1 = 0.5.$ For Equations 4-29 and 4-30, the parameters $a = 3, \Theta_1 = 0.1,\ \Theta_2 = 0.2,\ \Theta_3 = 0.3$, and $\beta_1 = 0.5.$ were used for the SMAX $(3,1)_{12}$ process. The parameter value $\lambda_0 = 1$ was applied to the in-control process. Meanwhile, for the out-of-control process ($\lambda_1 > \lambda_0$), parameter values $\lambda_1 = \lambda_0(1+\delta)$ were used, with shift sizes of 0.01, 0.03, 0.05, 0.10, 0.30, 0.50, 1.0, 1.5, and 2.0. The first row in both tables shows that the results of ARL_0 with the explicit formula were close to the NIE method, when ARL_0 approached 370 and 500. The values in parentheses are the CPU times of the ARL from both methods. The ARL values of the explicit formula and the NIE method were similar and tended to decrease when the level of the shift size increased. Note that the absolute percentage relative error was very small and the CPU time with the explicit formula was just a fraction of a second, while the NIE method took around 11–13 minutes.

In Table 4-15, the results in terms of the optimal reference value (a) and optimal width of the control limit (h) and the minimum ARL_1 (ARL_1^*) of *MAX (1,2)* and *SMAX* $(2,2)_{12}$ processes for $ARL_0 = 370$ are shown. For example, if we want to detect a parameter change from $\lambda_0 = 1$ to $\lambda_1 > \lambda_0$ and the ARL_0 is 370, then the optimality procedure given above will give the optimal parameter values a=1.66999118582 and h = 5.9737144930593 and ARL_1^* value = 13.552. The suggested explicit formulas are useful to practitioners, especially when finding the optimal parameters of the MAX and SMAX processes for the CUSUM chart.

Application to real-world data was conducted to evaluate the ARL by the explicit formula and NIE method, as reported in Table 4-17. The AEONTS share prices in the SET with two exogenous variables, the US/THB exchange rate and the interest rate, were collected monthly from January 2012 to December 2016 as the dataset of real observations. The first-order MA model is suitable for fitting the AEONTS share price with two exogenous variables, because the error of estimation is smallest compared to other models. Therefore, the first-order MA model was constructed with the process coefficient $\mu = 650.0555, \theta_1 = 0.746083, \beta_1 = -11.773, \beta_2 = -61.94972,$ and the error as exponential white noise with $\lambda_0 = 6.9094.$ For the ARL performance comparison, the boundary values $h = 22.48,\ 24.71$ for the CUSUM control chart and $\kappa = 0.5333,\ 0.8791$ for the EWMA control chart were used with conditions of $ARL_0 = 370$ and $ARL_0 = 500$, respectively. The smoothing parameter of the EWMA control chart was set to 0.1. The results in Table 4-16 are similar to the results in Tables 4-13–4-14, in that the NIE results approached the explicit formula results. In Table 4-17, the performance of CUSUM control chart with the explicit formula is compared with EWMA control chart by using the NIE method. The results of performance comparison show that the CUSUM control chart provided a smaller ARL_1 than the EWMA control chart when the shift size was small, but when the shift size was $\delta \geq 0.015$, the EWMA control chart performed better than the CUSUM control chart.

Table 4-17 Comparison of ARL using explicit formula against NIE for *MAX (1,2)* when given $a = 550, \theta_1 = 0.746083, \beta_1 = -11.773, \beta_2 = -61.94972, h = 22.48$ for $ARL_0 = 370$ and $h = 24.71$ for $ARL_0 = 500$ $\lambda_0 = 6.9094$

δ (shift size)	$ARL_0 = 370$ Explicit	NIE	APRE (%)	$ARL_0 = 500$ Explicit	NIE	APRE (%)
0.00	370.437	369.302 (10.642)	0.306394	500.389	498.693 (10.630)	0.338936
0.01	367.082	365.959 (11.017)	0.305926	495.596	493.919 (11.0590	0.33838
0.05	354.056	352.981 (11.106)	0.303624	477.007	475.406 (11.122)	0.335634
0.1	338.617	337.598 (11.103)	0.300929	455.083	453.511 (11.586)	0.345431
0.3	285.049	284.222 (10.352)	0.290125	379.214	378.000 (11.111)	0.320136
0.5	242.186	241.508 (10.965)	0.279950	319.135	318.150 (10.628)	0.308647
0.7	207.537	206.976 (11.193)	0.270313	271.011	270.204 (10.755)	0.297774
1.0	167.067	166.638 (11.902)	0.256783	215.395	214.787 (11.016)	0.282272
2.0	90.192	89.995 (11.758)	0.218422	112.120	111.851 (11.116)	0.239922
3.0	55.293	55.189 (11.212)	0.188088	66.776	66.639 (11.467)	0.205164

Table 4-18 Performance comparison of CUSUM and EWMA control charts using explicit formula for *MAX (1,2)* when $\lambda_0 = 6.9094$ for $ARL_0 = 370$ and $ARL_0 = 500$

δ (shift size)	$ARL_0 = 370$ CUSUM	EWMA $h = 0.5333$	$ARL_0 = 500$ CUSUM	EWMA $h = 0.8791$
0.000	370.437	370.423 (16.215)	500.389	500.433 (16.036)
0.005	**368.755**	369.083 (16.202)	**497.985**	498.303 (16.265)
0.010	**367.082**	368.191 (16.112)	**495.596**	495.817 (16.085)
0.015	365.420	**365.311** (15.489)	493.221	**490.892** (16.878)
0.050	354.056	**348.928** (16.156)	477.007	**471.804** (16.966)
0.1	338.617	**335.141** (15.005)	455.083	**453.656** (16.255)
0.3	285.049	**275.716** (16.485)	379.214	**375.205** (16.255)
0.5	242.186	**229.117** (15.152)	319.135	**313.385** (16.008)
1.0	167.067	**149.136** (15.156)	215.395	**207.585** (16.026)

CHAPTER 5

CONCLUSIONS

The focus of this study was on the derivation of explicit formulas for the *ARL* on a modified EWMA control chart when the observations are autocorrelated with exponential white noise. In this chapter, the findings from the study are summarized. Then, suggestions for further study are presented at the end.

5.1 Conclusions

Explicit formulas for the *ARL* on a modified EWMA control chart for *MA(1)* and *ARX(p,r)* processes with exponential white noise are proposed. In addition, the explicit formulas for the ARL of modified EWMA chart can be applied to real data or other control charts. Furthermore, an NIE method for approximating the *ARL* on the modified EWMA and CUSUM control charts was obtained by using the Gauss-Legendre quadrature rule. The accuracy of the explicit formulas for the *ARL* was compared with the NIE method using the relative of absolute percentage error. In addition, the performance of the *ARL* by using explicit formulas was determined by a comparison on the modified and EWMA control charts.

The results indicate that the *ARL* by using the explicit formulas and NIE method were in excellent agreement in that the values obtained with the NIE method were smaller than those with the explicit formulas by an absolute percentage difference of less than 0.001%. Therefore, the explicit formulas are highly appropriate for evaluating the exact *ARL*. In addition, the CPU time for the *ARL* obtained by the NIE method were around 19–23 seconds while explicit formulas were 0 second. Thus, the explicit formulas for the *ARL* on the modified EWMA control were easy to derive and they substantially saved on the computation time.

The findings reveal that the modified EWMA control chart performed excellently in monitoring and detecting the small shift size in the process mean. The experimental results indicated that the performance of the modified EWMA control chart was apparently worse than the standard chart for detecting a large shift in the process mean when the smoothing parameter was 0.05. Subsequently, a smoothing parameter value of 0.10 is considered to be appropriate for efficiently applying the modified EWMA control chart. In the application of the proposed method to real observations, the modified EWMA control chart performed more efficient than the standard one for all smoothing parameter values.

5.2 Further Study

The derivation of the explicit formulas of the ARL on a modified EWMA control chart for *MA(1)* and *ARX(p,r)* processes with exponentially distributed white noise were proposed. This approach could be extended to areas involving serially correlated observations in control schemes for detecting shifts in the process mean as follows.

5.2.1 The explicit formulas approach could be extended to obtain the two-sided of *ARL* on the modified EWMA control chart when the observations are *MA* or *ARMA* processes with exponential white noise.

5.2.2 Other types of autocorrelated observations in processes such as *ARIMA*(*p*,*d*,*q*) and *ARFIMA*(*p*,*d*,*q*) processes could be considered when developing the *ARL* using explicit formulas on the modified EWMA control scheme.

5.2.3 The explicit formulas of ARL_0 and ARL_1 by using Fredholm Integral Equation of second type technique can be can be applied in a new control chart.

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APPENDIX

Publications

Publications

1. Rapin Sunthornwat and Yupaporn Areepong* "Average Run Length on CUSUM Control Chart for Seasonal and Non-Seasonal Moving Average Processes with Exogenous Variables" Symmetry, 2020, 12(1).

2. Supharakonsakun Y, Areepong Y* and Sukparungsee, S. "The exact solution of the average run length on a modified EWMA control chart for the first-order moving-average process" SCIENCEASIA, 2020, 46(1), 109-118.

3. Korakoch Silpakob, Yupaporn Areepong*, Saowanit Sukparungsee, and Rapin Sunthornwat "Explicit analytical solutions for the average run length of modified EWMA control chart for ARX(p,r) processes" Songklanakarin Journal of Science and Technology, 2021, 43 (5), 1414-1427.

symmetry

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Article

Average Run Length on CUSUM Control Chart for Seasonal and Non-Seasonal Moving Average Processes with Exogenous Variables

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Abstract: The aim of this study was to derive explicit formulas of the average run length (ARL) of a cumulative sum (CUSUM) control chart for seasonal and non-seasonal moving average processes with exogenous variables, and then evaluate it against the numerical integral equation (NIE) method. Both methods had similarly excellent agreement, with an absolute percentage error of less than 0.50%. When compared to other methods, the explicit formula method is extremely useful for finding optimal parameters when other methods cannot. In this work, the procedure for obtaining optimal parameters—which are the reference value (a) and control limit (h)—for designing a CUSUM chart with a minimum out-of-control ARL is presented. In addition, the explicit formulas for the CUSUM control chart were applied with the practical data of a stock price from the stock exchange of Thailand, and the resulting performance efficiency is compared with an exponentially weighted moving average (EWMA) control chart. This comparison showed that the CUSUM control chart efficiently detected a small shift size in the process, whereas the EWMA control chart was more efficient for moderate to large shift sizes.

Keywords: CUSUM control chart; moving average process with exogenous variable; average run length; explicit formula; numerical integral equation

1. Introduction

Statistical process control (SPC) has been widely used to monitor processes and services, so as to avoid any instabilities and inconsistencies. The control chart is the main tool for SPC. Page first introduced the cumulative sum (CUSUM) control chart [1] and Roberts initially devised the exponentially weighted moving average (EWMA) control chart [2]. The CUSUM and EWMA control charts were developed from the Shewhart control chart [3], which is suitable for processes with a large shift size in the parameters of interest (the mean or variance) when the observations follow a normal distribution. Meanwhile, the CUSUM and EWMA control chart can detect small shift sizes in the parameters of interest and they are suitable for observations following more complex patterns, such as auto-correlated observations, trending and seasonal observations, and changing point observations [4–6].

SPC has been adopted for monitoring production and service processes in several fields, such as medical sciences, industrial manufacturing, network analysis, mechanical trading on securities, and healthcare. A systematic review of the researches on the limitations and benefits of SPC for the quality improvement of healthcare systems can be found in [7,8], and a comparison of SPC to several control charts that were implemented in the manufacturing industry was given by Saravanan and Nagaragan [9]. Moreover, range and X-bar control charts to monitor production at Swat Pharmaceutical Company were researched by Muhammad [10], while SPC in a computer-integrated manufacturing environment was investigated by Montgomery and Friedman [11] and traffic observations for IP networks that were monitored by CUSUM, Shewhart and EWMA control charts were studied and analyzed by Matias et al. [12].

An advantage of the CUSUM control chart is that it is suitable for detecting unstable processes for a subgroup of data and single observations. The following researches focus on the application of the CUSUM control chart for real observations. Sunthornwat et al. [4] designed optimal differencing and smoothing parameters for EWMA and CUSUM control charts to evaluate average run length (ARL) with an autoregressive fractionally integrated moving average (ARFIMA) model. The practical observations were obtained from the time intervals in days between explosions in mines in Great Britain from 1875 to 1951 [13]. Meanwhile, Sheng-Shu and Fong-Jung [14] reported that the CUSUM control chart performed better than the EWMA control chart in monitoring the failure mechanism of wafer production quality control. Benoit and Pierre detected the persistent changes in the mean and variance in the state of marine ecosystems, while using indicators of North Sea cod from the International Bottom Trawl Survey [15]. Their results showed that the CUSUM control chart is suitable for detecting small, persistent changes.

The observations or data in practical situations are often collected from stochastic processes that are dependent on time-space or time series. In other words, the models established under the econometric models are sometimes specified in time-series models. The observations in econometrics as a time-series model comprise of autoregressive (AR) and moving average (MA) elements. Moreover, it is very simple to identify the movement patterns of time-series observations in MA models and, often, the seasonal factor can be embedded in the observations, being modeled as a seasonal moving average (SMA) model. Moreover, the error, which is the difference between the exact value and approximated value, should be considered in the modeling. A smaller error signifies better accuracy. The error of a time-series model, which is called white noise, usually follows a normal distribution, but another form of time-series model with auto-correlated observations is exponential white noise [16–20].

Econometric models are related to the economic indicators or variables that affect economic forecasting. An exogenous variable is a variable that is not affected by other variables in the system. For example, the exogenous variable may depend on the government's investment policies. Exogenous variables that are popularly used in econometric models are the exchange rate, interest rate, and inflation rate, among others. These variables affect the econometric model when forecasting economic situations in the future. When forecasting in economics and other fields, if the forecasting model includes an exogenous variable, the model is usually more accurate than one without it. X denotes the exogenous variable in economic models, and models that are based on an MA or SMA time series with an exogenous variable are denoted as MAX and SMAX, respectively. For the quality control of processes and services, the efficiency of a control chart can be measured in terms of the proposed ARL [5,21]. The ARLs for the in-control state (ARL_0) and the out-of-control state (ARL_1) are evaluated as the efficiency criteria; a large ARL_0 means that the control chart can be applied to efficiently control the process, whereas a small ARL_1 demonstrates that the control chart can detect a change in the process quickly. The Numerical Integral Equation (NIE) method is widely used for the continuous distribution of observations in real-world applications. Furthermore, Banach's fixed-point theory has been adopted to prove the existence and uniqueness of the ARL in the following researches. Sunthornwat et al. investigated an explicit formula for the ARL on an EWMA control chart for the Autoregressive Fractionally Integrated Moving Average (ARFIMA) model with exponential white noise [4], while Mititelu et al. [22] solved one representing the ARL on a CUSUM control chart based on observations in a hyper-exponential distribution. Their findings show that the explicit formula ARL was more quickly evaluated than the NIE ARL. Petcharat et al. derived an explicit formula ARL, while using an integral equation method on a CUSUM control chart for an MA model with exponential white noise [23], with its existence and uniqueness being proved via Banach's fixed-point theory. However, the optimal parameters could not be obtained in an MA model.

As mentioned in this literature review, the ARL, for measuring the efficiency of a CUSUM control chart, is very important for comparing control chart performance. Moreover, the application of control charts to detect shifts in the processes from autocorrelated data, which needs a high accuracy model with the study of exogenous variables, is very interesting. For example, the MAX and SMAX models are widely used in the field of economics. In addition, finding the optimal values for the parameters in a CUSUM chart is important for these models to detect changes in a process as fast as possible. Therefore, the aim of this research was to evaluate the ARL on a CUSUM control chart of MAX and SMAX processes, and to apply Banach's fixed-point theory to prove the existence and uniqueness of the ARL. The stock price for Aeon Thana Sinsap (Thailand) PCL (AEONTS) with exogenous variables, which are the USD/THB exchange rate and the interest rate, were applied to analyze the ARL of a CUSUM control chart. The rest of this paper is organized, as follows. In the next section, preliminaries regarding the definitions and theory used in the study are given. Section 3 illustrates the explicit formula. Section 4 presents the existence and uniqueness of the explicit formula, the NIE method and the numerical procedure for obtaining optimal parameters for ARL. The computational results, a comparison, and an application to real data are reported in Section 5. Section 6 consists of the conclusion and a discussion of this research.

2. Preliminaries

In this section, the definitions and theories related to the fixed point theorem, MAX and SMAX processes, CUSUM control chart, and characteristic are proposed, as follows:

2.1. The MAX(q,r) and SMAX(Q,r)$_L$ Processes

Definition 1. *Let $\{Z_t,\ t = 1, 2, \ldots,\}$ be a sequence of MAX(q,r) process given as*

$$Z_t = \mu + (1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q)\varepsilon_t + \sum_{i=1}^{r} \beta_i X_{it}, \tag{1}$$

and $\{\zeta_t,\ t = 1, 2, \ldots,\}$ be a sequence of SMAX(Q,r)$_L$ process given by the expression

$$\zeta_t = \mu + (1 - \theta_L B^L - \theta_{2L} B^{2L} - \ldots - \theta_{QL} B^{QL})\varepsilon_t + \sum_{i=1}^{r} \beta_i X_{it}, \tag{2}$$

where

ε_t is a exponential white noise process,
μ is a process mean,
$(1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q)$ is the moving average polynomials in B of order q.
$(1 - \theta_L B^L - \theta_{2L} B^{2L} - \ldots - \theta_{QL} B^{QL})$ is the seasonal moving average polynomials in B of order Q; L is a natural number,
B is the backward shift operator, i.e., $B^q \varepsilon_t = \varepsilon_{t-q}$, and
X_{it} is exogenous variable and β_i is a coefficient of X_{it}.

2.2. Fixed Point and Metric Space

Definition 2. *The point s^* is called a fixed point in the domain of the function $ARL(s)$ if $ARL(s^*) = s^*$.*

Definition 3. *Let M be a nonempty set and R denote the set of real numbers. A distance function from x to y, $\rho : M \times M \to R$ is said to be a metric on M denoted by (M, ρ) if it satisfies the following conditions. For all, $x, y, z \in M$,*

(1) $\rho(x, y) \geq 0$ *i.e., ρ is finite and non-negative real valued function.*
(2) $\rho(x, y) = 0$ *if, and only if, $x = y$.*
(3) $\rho(x, y) = \rho(y, x)$, *(Symmetric property)*
(4) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, *(Triangular inequality).*

Definition 4. *A sequence (x_n) of points of (M, ρ) is a Cauchy's sequence if $\rho(x_m, x_n) \to 0$ for $m, n \to \infty$.*

Definition 5. *A metric space is a complete metric space if every Cauchy's sequence converges to $x \in M$. That is to say, if $\rho(x_m, x_n) \to 0$ as $m, n \to \infty$, then there exists $x \in M$, such that $\rho(x_m, x) \to 0$ as $m \to \infty$.*

Definition 6. *An operator $C^* : M \to M$ is a contraction mapping, or contraction, if there exists $\omega \in [0, 1)$ such that $\rho(C^*x, C^*y) \leq \omega\rho(x, y)$, for all $x, y \in M$.*

Definition 7. *An element $x \in M$ is a fixed point of an operator C^*, if $C^* x = x$.*

Definition 8. *A supremum norm $\|.\|_\infty$ in the domain of continuous function $ARL(s) : Dom(ARL(s)) \to R$ is defined as $\|.\|_\infty = \sup\limits_{0 \leq s \leq h} |ARL(s)| = sup\{|ARL(s)| : s \in Dom(ARL(s))\}$.*

Definition 9. *$ARL : [0, h] \to R$ is a twice differentiable function, being denoted by $ARL \in C^2[0, h]$.*

Theorem 1. *(Banach's Fixed Point Theorem (see Richard [24]))*
If $C^* : M \to M$ is a contraction mapping on a complete metric space (M, ρ), then there exists an unique solution $s \in M$ of $C^*s = s$.

2.3. CUSUM and EWMA Control Charts for the MAX(q,r) and SMAX(Q,r)$_L$ Processes

The CUSUM statistic under the assumption $\{Y_t;\ t = 1, 2, 3, \ldots\}$, as a sequence of i.i.d continuous random variables with common probability density function, is considered. The CUSUM statistic (Y_t) is referred to as an upper CUSUM statistic, being based on MAX(q,r) and SMAX(Q,r)$_L$ processes. Y_t can be expressed by the recursive formula, as

$$Y_t = \max\{0,\ Y_{t-1} + Z_t - a\}, \text{ for } t = 1,\ 2, \ldots\ , \tag{3}$$

where Z_t is a sequence of the MAX(q,r) and SMAX(Q,r)$_L$ processes with exponential white noise, the starting value $Y_0 = s$ is an initial value; $s \in [0, h]$, where h is a control limit and a is usually called the reference value of CUSUM chart. The CUSUM stopping time (τ_h) with predetermined threshold h is defined as

$$\tau_h = \inf\{t > 0;\ Y_t \geq h\}, \text{ for } h > u \tag{4}$$

Meanwhile, the EWMA statistic for constructing EWMA control chart with smoothing parameter $\eta \in (0, 1]$, mean μ, variance σ^2, initial value of the process mean: $M_0 = \mu$ is defined as

$$M_t = (1 - \eta)M_{t-1} + \eta Z_t\ ;\ t = 1, 2, 3, \ldots, n \tag{5}$$

where Z_t is generated from the MAX(q,r) and SMAX(Q,r)$_L$ processes with exponential white noise.

The control limits of EWMA control chart consist of

Upper control limit: $UCL_t = \mu + \kappa\sigma\sqrt{\frac{\eta}{(2-\eta)}\left[1 - (1-\eta)^{2t}\right]}$

Center Line: $CL = \mu$

Lower control limit: $LCL_t = \mu - \kappa\sigma\sqrt{\frac{\eta}{(2-\eta)}\left[1-(1-\eta)^{2t}\right]}$

where κ is the width of the control limits.

2.4. Characteristics of Average Run Length

Let $\{\varepsilon_t,\ t = 1,2,3,\ldots\}$ be a sequence of independent and identically distributed random variables with a probability density function $f(x)$ with the parameter $\lambda = \lambda_0$, which is before a change-point time $\theta \le \infty$; the parameters $\lambda_1 > \lambda_0$ are after the change-point time. Generally, the change-point times are considered. The expectation $E_\theta(.)$ for fixed θ under probability density function $f(x)$ with parameter λ_1 is that the change-point occurs at point θ. The appropriate chart provides a large ARL for $\theta = \infty$. There is the behavior of in-control state of ARL, being denoted by $\mathrm{ARL}_{0,}$ or the state of no change $(\lambda = \lambda_0)$. The expectation of the run length τ_h in the -control state can be defined as

$$ARL_0 = E_\theta(\tau_h)$$

Meanwhile, if $\theta = 1$, in the case of the change-point time from λ_0 to λ_1, then the *ARL* is evaluated as the out-of -control state of *ARL*, being denoted by ARL_1, which can be defined as

$$ARL_1 = E_\theta(\tau_h \big| \tau_h \ge 1)$$

3. The Explicit Formulas for Average Run Lengths with MAX(*q*,*r*) and SMAX(*Q*,*r*)$_L$ Processes

In this section, the derivations of the explicit formulas for the *ARL* of CUSUM chart, when observations are MAX(*q*,*r*) and SMAX(*Q*,*r*)$_L$ processes with exponential white noise from the integral equations, after checking the existence and uniqueness of the solutions for the *ARL*, are presented, as follows:

Theorem 2. *The explicit formula $C(s)$ for the ARL of* MAX*(q,r) process with an exponential white noise is*

$$C(s) = e^{\lambda h}(1 + e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum\limits_{i=1}^{r}\beta_i X_{it})} - \lambda h) - e^{\lambda s}; s \ge 0$$

Proof. Let $C(s)$ be the explicit *ARL* of MAX(*q*,*r*) process with an exponential white noise.

$$C(s) = 1 + \int\limits_0^h C(y)\lambda e^{\lambda(s-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\ldots-\theta_q\varepsilon_{t-q}+\sum\limits_{i=1}^{r}\beta_i X_{it})} e^{-\lambda y} dy$$
$$+\left(1 - e^{-\lambda(a-s-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum\limits_{i=1}^{r}\beta_i X_{it})}\right)C(0); s \in [0,a) \quad (6)$$

Let g be constant as

$$g = \int\limits_0^h C(y)e^{-\lambda y}dy.$$

$C(s)$ can be written as

$$C(s) = 1 + \lambda g e^{\lambda(s-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\ldots-\theta_q\varepsilon_{t-q}+\sum\limits_{i=1}^{r}\beta_i X_{it})} + \left(1 - e^{-\lambda(a-s-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum\limits_{i=1}^{r}\beta_i X_{it})}\right)C(0)$$

For $s = 0$, then

$$C(0) = 1 + \lambda g e^{\lambda(-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it})} + \left(1 - e^{-\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})}\right)C(0)$$

$$= e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} + \lambda g$$

Subsequently,

$$C(s) = 1 + \lambda g e^{\lambda(s-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it})}$$

$$+ \left(1 - e^{-\lambda(a-s-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})}\right) e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} + \lambda g$$

$$= 1 + e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} + \lambda g - e^{\lambda s}$$

$$= 1 + e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} + \lambda g - e^{\lambda s}$$

$$= 1 + \lambda g + e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} - e^{\lambda s}. \tag{7}$$

Now, constant g can be found as the following

$$g = \int_0^h C(y) e^{-\lambda y} dy$$

$$= \int_0^h (1 + \lambda g + e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} - e^{\lambda y}) e^{-\lambda y} dy$$

$$= \left((1 + \lambda g + e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})}\right) \int_0^h e^{-\lambda y} dy - \int_0^h e^{\lambda y - \lambda y} dy$$

$$= \frac{e^{\lambda h}}{\lambda}\left(1 - e^{-\lambda h}\right)\left(1 + e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})}\right) - h e^{\lambda h}$$

Substituting the constant g into Equation (7), then

$$C(s) = e^{\lambda h}\left(1 + e^{\lambda(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} - \lambda h\right) - e^{\lambda s}, \quad s \geq 0.$$

□

As previously mentioned, the value of exponential parameter $\lambda = \lambda_0$,; this implies that the process is an in-control state. Hereby, the explicit analytical solution for ARL_0 can be written as

$$ARL_0 = e^{\lambda_0 h}\left(1 + e^{\lambda_0(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} - \lambda_0 h\right) - e^{\lambda_0 s}, \quad s \geq 0. \tag{8}$$

On the contrary, if the process is in an out-of-control state, the value of exponential parameter $\lambda_1 = \lambda_0(1+\delta)$, where $\lambda_1 > \lambda_0$, and δ is the shift size. The explicit analytical solution for ARL_1 can be written as

$$ARL_1 = e^{\lambda_1 h}\left(1 + e^{\lambda_1(a-\mu+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it})} - \lambda_1 h\right) - e^{\lambda_1 s}, \quad s \geq 0. \tag{9}$$

Theorem 3. *The explicit formula $D(s)$ for the ARL of* SMAX$(Q,r)_L$ *process is*

$$D(s) = e^{\lambda h}(1 + e^{\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})} - \lambda h) - e^{\lambda s}; s \geq 0$$

Proof. Let $D(s)$ be the explicit ARL of SMAX(q,r) process with an exponential white noise.

$$D(s) = 1 + \int_0^h D(y)\lambda e^{\lambda(s-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it})} e^{-\lambda y} dy$$

$$+\left(1 - e^{-\lambda(a-s-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})}\right) D(0); s \in [0,a) \tag{10}$$

Let k be constant as

$$k = \int_0^h D(y) e^{-\lambda y} dy.$$

$D(s)$ can be written as

$$D(s) = 1 + \lambda k e^{\lambda(s-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it})}$$

$$+\left(1 - e^{-\lambda(a-s-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})}\right) D(0).$$

For $s = 0$, then

$$D(0) = 1 + \lambda k e^{\lambda(-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it})} + \left(1 - e^{-\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})}\right) D(0)$$

$$= e^{\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})} + \lambda k$$

Afterwards,

$$D(s) = 1 + \lambda k e^{\lambda(s-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it})}$$

$$+\left(1 - e^{-\lambda(a-s-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})}\right) e^{\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})} + \lambda k$$

$$= 1 + \lambda k + e^{\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum_{i=1}^{r}\beta_i X_{it})} - e^{\lambda s} \tag{11}$$

Now, constant k can be found as the following

$$k = \int_0^h D(y)e^{-\lambda y}dy$$

$$= \int_0^h (1 + \lambda k + e^{\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum\limits_{i=1}^{r}\beta_iX_{it})} - e^{\lambda y})e^{-\lambda y}dy$$

$$= \left((1 + \lambda k + e^{\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum\limits_{i=1}^{r}\beta_iX_{it})}\right)\int_0^h e^{-\lambda y}dy - \int_0^h e^{\lambda y-\lambda y}dy$$

$$= \frac{e^{\lambda h}}{\lambda}\left(1 - e^{-\lambda h}\right)\left(1 + e^{\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum\limits_{i=1}^{r}\beta_iX_{it})}\right) - he^{\lambda h}$$

Substituting the constant k into Equation (11), then

$$D(s) = e^{\lambda h}\left(1 + e^{\lambda(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum\limits_{i=1}^{r}\beta_iX_{it})} - \lambda h\right) - e^{\lambda s}, \quad s \geq 0.$$

□

As previously mentioned, the value of exponential parameter $\lambda = \lambda_0$,; this implies that the process is an in-control state. Hereby, the explicit analytical solution for ARL_0 can be written as

$$ARL_0 = e^{\lambda_0 h}\left(1 + e^{\lambda_0(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum\limits_{i=1}^{r}\beta_iX_{it})} - \lambda_0 h\right) - e^{\lambda_0 s}, \quad s \geq 0. \tag{12}$$

On the contrary, if the process is in an out-of-control state, the value of exponential parameter $\lambda_1 = \lambda_0(1 + \delta)$, where $\lambda_1 > \lambda_0$, and δ is the shift size. The explicit analytical solution for ARL_1 can be written as

$$ARL_1 = e^{\lambda_1 h}\left(1 + e^{\lambda_1(a-\mu+\theta_L\varepsilon_{t-L}+\theta_{2L}\varepsilon_{t-2L}+\ldots+\theta_{QL}\varepsilon_{t-QL}-\sum\limits_{i=1}^{r}\beta_iX_{it})} - \lambda_1 h\right) - e^{\lambda_1 s}, \quad s \geq 0. \tag{13}$$

4. Explicit Formulas and Numerical Integral Equation Method for Average Run Length

In this section, the existence and uniqueness of the explicit formulas for the analytical *ARLs* and numerical integral equation method for numerical *ARLs* are shown as the following.

4.1. Existence and Uniqueness of the Explicit Formulas for Average Run Lengths

Theorem 4. *Explicit formulas for $ARL(s)$—$C(s)$ and $D(s)$—derived from the integral equations on the CUSUM control chart of MAX and SMAX processes with exponential white noise, respectively, have existence and uniqueness.*

Proof (Existence). Let $Con\,[0, h]$ be a set of all continuous functions of ARL on $[0, h]$. Let $ARL_0 \in Con([0, h]$ and $(ARL_n)_{n\geq 0}$ be a Cauchy's sequence of explicit formula ARL that satisfies $ARL_{n+1} = C^*ARL_n$.

$$\rho(ARL_{n+1}, ARL_n) = \rho(C^*ARL_n, C^*ARL_{n-1})$$
$$\leq \omega\rho(C^*ARL_n, C^*ARL_{n-1});\ \omega \in [0, 1).$$

Repeatedly, $\rho(ARL_{n+1}, ARL_n) \le \omega^n \rho(ARL_1, ARL_0)$, for $n \ge 0$.

Thus, for $n \ge m$

$$\begin{aligned}\rho(ARL_n, ARL_m) &\le \rho(ARL_n, ARL_{n-1}) + \ldots + \rho(ARL_{m+1}, ARL_m)\\ &\le (\omega^{n-1} + \ldots + \omega^m)\,\rho(ARL_1, ARL_0)\\ &\le \frac{\omega^m}{1-\omega}\rho(ARL_1, ARL_0)\end{aligned}$$

$(ARL_n)_{n\ge 0}$ is a Cauchy's sequence and $\lim_{n\to\infty} C^{*n} ARL_n \to ARL$. That is, $C^* ARL = ARL$. □

Proof (Uniqueness). Let

$$k(s,y) = \lambda e^{\lambda(s-a+\mu-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-\ldots-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it})} e^{-\lambda y}$$

$$k(s,y) = \lambda e^{\lambda(s-a+\mu-\theta_L\varepsilon_{t-L}-\theta_{2L}\varepsilon_{t-2L}-\ldots-\theta_{QL}\varepsilon_{t-QL}+\sum_{i=1}^{r}\beta_i X_{it})} e^{-\lambda y}$$

be a kernel function of integral equation for $C(s)$ and be a kernel function of integral equation for $D(s)$ representing $ARLs$, where $k : [0,h] \times [0,h] \to R$ and Z_t is MAX or SMAX process. The inequality $\sup_{0\le y\le h}\left\{\int_0^h \left|k(s,y)\right| dy\right\} < 1$ will be shown to prove that C^* is a contraction mapping on the complete metric space $(Con([0,h]), \|.\|_\infty)$.

For any ARL_i, $ARL_j \in Con([0,h])$,

$$\| C^* ARL_i - C^* ARL_j \|_\infty = \sup_{0\le y\le h}\left|\int_0^h k(s,y)[ARL_i(y) - ARL_j(y)]dy\right|$$

$$\le \sup_{0\le y\le h}\int_0^h \left|k(s,y)\right|\left|[ARL_i(y) - ARL_j(y)]\right|dy$$

$\le \omega\|ARL_i - ARL_j\|_\infty$, where $\omega = \sup_{0\le y\le h}\left\{\int_0^h \left|k(s,y)\right| dy\right\} < 1$. □

Therefore, the explicit formulas ARL on the CUSUM control chart of MAX or SMAX processes with exponential white noise have existence and uniqueness.

4.2. The Numerical Integral Equation Method

According to the integral equation in Equations (6) and (10), the numerical integral equation method can evaluate the solution by the Gauss–Legendre quadrature rule as

$$ARL(a_i) = 1 + \frac{1}{\lambda+1}\sum_{j=1}^{m} w_j ARL(a_j) f\left(\frac{a_j - (1-\lambda)a_i + v + (\lambda\theta + \theta)s}{\lambda+1} - \mu\right)$$

where

$$i = 1,2,3,\ldots,m,\ a_j = \frac{h}{m}\left(j - \frac{1}{2}\right) \text{ and } w_j = \frac{h}{m}\,;\ j = 1,2,3,\ldots,m$$

It can be rewritten in matrix form as

$$ARL_{m\times 1} = 1_{m\times 1} + R_{m\times m} ARL_{m\times 1} \text{ or } ARL_{m\times 1} = (I_m - R_{m\times m})^{-1} 1_{m\times 1}$$

where

$$ARL_{m\times 1} = \begin{bmatrix} ARL(a_1) \\ ARL(a_2) \\ \vdots \\ ARL(a_m) \end{bmatrix}, \; I_m = diag(1,1,\ldots,1), \; 1_{m\times 1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and

$$R_{m\times m} = \begin{bmatrix} \frac{1}{\lambda+1} w_1 f\left(\frac{a_1-(1-\lambda)a_1+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right) & \cdots & \frac{1}{\lambda+1} w_m f\left(\frac{a_m-(1-\lambda)a_1+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right) \\ \frac{1}{\lambda+1} w_1 f\left(\frac{a_1-(1-\lambda)a_2+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right) & \cdots & \frac{1}{\lambda+1} w_m f\left(\frac{a_m-(1-\lambda)a_2+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right) \\ \vdots & & \\ \frac{1}{\lambda+1} w_1 f\left(\frac{a_1-(1-\lambda)a_m+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right) & \cdots & \frac{1}{\lambda+1} w_m f\left(\frac{a_m-(1-\lambda)a_m+v+(\lambda\theta+\theta)s}{\lambda+1} - \mu\right) \end{bmatrix}$$

Therefore, the approximation of average run length is evaluated by the numerical integral equation method for $ARL(u)$ is

$$ARL(u) = 1 + \frac{1}{\lambda+1}\sum_{j=1}^{m} w_j ARL(a_j) f\left(\frac{a_j - (1-\lambda)u + v + (\lambda\theta+\theta)s}{\lambda+1} - \mu\right)$$

where $a_j = \frac{h}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{h}{m}$, $j = 1,2,3,\ldots,m$.

An absolute percentage relative error (*APRE*) criterion is used as a performance criterion and the *APRE* can be expressed as

$$APRE(\%) = \left|\frac{ARL(s)_{\exp} - ARL(s)_{app}}{ARL(s)_{\exp}}\right| \times 100\% \tag{14}$$

where $ARL(s)_{\exp}$ is the *ARL* results from the explicit formula and $ARL(s)_{app}$ is an approximation of *ARL* from the NIE method.

4.3. The Numerical Procedure for Obtaining Optimal Parameters for MAX Designs

Step 1. Select an acceptable in-control value of ARL_0 and decide on the change parameter value (λ_1) for an out-of-control state;

Step 2. For the given values λ_0 and T, find the optimal values of a and h to minimize the ARL_1 values that are given by Equation (9), subject to the constraint that ARL_0 = T in Equation (8), i.e., a and h are solutions of the optimality problem.

In addition, the numerical procedure for obtaining optimal parameters for SMAX designs is the same as the MAX procedure, by using Equations (12) and (13) for ARL_0 and ARL_1, respectively. Table shows the optimal (a, h) values for T = 370 and magnitudes of change.

5. Computational Results and Real Application

In this section, the results of the *ARL* with the explicit formula and the NIE method are provided and compared.

The *ARL* results of the proposed explicit formula (explicit) were compared with the NIE method while using the Gauss–Legendre quadrature rule with 1000 nodes for the CUSUM chart, based on the *APRE* criterion in Equation (14). The numbers in parentheses are CPU time (minutes) with the NIE method.

5.1. Numeric Results

The numerical results were obtained from the simulation. The chart was set up with reference value (a) that is greater than s, where s is an initial value ($s = 0$). The parameter a could be any number, but it combines with the control limit (s) that corresponds to the in-control $ARL_0 = 370$ and 500. In this paper, the parameter a is varied between 1.5 to 3.0. A comparison of the solution of the explicit formula (Explicit) with the NIE method for the CUSUM chart, when $\lambda_0 = 1, \beta_1 = 0.5$ for MAX (2,1) and SMAX $(3,1)_{12,}$ are reported in Tables 1 and 2 for $ARL_0 = 370, 500$, with which they are in good agreement. Notice that the absolute percentage relative error is small. In Tables 3 and 4, we use Equations (8) and (9) to show ARL_0 and ARL_1 for the MAX (2,1) process with parameter $a = 3$ and the coefficient parameters of the process $\theta_1 = 0.1, \theta_2 = 0.2$, and $\beta_1 = 0.5$. For Equations (12) and (13), the parameters $a = 3$, $\Theta_1 = 0.1$, $\Theta_2 = 0.2$, $\Theta_3 = 0.3$, and $\beta_1 = 0.5$. were used for the SMAX $(3,1)_{12}$ process. The parameter value $\lambda_0 = 1$ was applied to the in-control process. Meanwhile, for the out-of-control process ($\lambda_1 > \lambda_0$), parameter values $\lambda_1 = \lambda_0(1 + \delta)$ were used, with shift sizes of 0.01, 0.03, 0.05, 0.10, 0.30, 0.50, 1.0, 1.5, and 2.0. The first row in both tables shows that the results of ARL_0 with the explicit formula were close to the NIE method, when ARL_0 approached 370 and 500. The values in parentheses are the CPU times of the *ARL* from both of the methods. The *ARL* values of the explicit formula and the NIE method were similar and tended to decrease when the level of the shift size increased. Note that the absolute percentage relative error was very small and the CPU time with the explicit formula was just a fraction of a second, while the NIE method took around 11–13 min.

Table 1. *ARL* values for in control process for MAX (2,1) using explicit formula against numerical integral equation given $\beta_1 = 0.5$ for $ARL_0 = 370$.

a	θ_1	θ_2	h	Explicit	NIE	*APRE* (%)
1.5	0.1	0.2	4.889	370.485	369.123 (12.821)	0.365197
	0.1	−0.2	5.442	370.211	368.657 (12.885)	0.317387
2.5	0.1	0.2	3.971	370.432	369.123 (12.897)	0.353371
	0.1	−0.2	4.887	370.008	368.657 (12.777)	0.365127
3	0.1	0.2	3.265	370.225	369.087 (12.818)	0.307381
	0.1	−0.2	3.811	370.154	368.879 (12.843)	0.344451

Table 2. ARL values for in control process for SMAX $(3,1)_{12}$ using explicit formula against numerical integral equation given $\beta_1 = 0.5$ for $ARL_0 = 370$.

a	Θ_1	Θ_2	Θ_3	h	Explicit	NIE	*APRE* (%)
1.5	0.1	0.2	0.3	3.525	370.411	369.202 (13.159)	0.326394
	0.1	−0.2	0.3	4.145	370.071	369.652 (13.258)	0.396805
2.5	0.1	0.2	0.3	3.525	370.411	369.202 (13.810)	0.326394
	0.1	−0.2	0.3	4.145	370.132	368.797 (13.574)	0.360682
3	0.1	0.2	0.3	2.906	370.008	368.979 (13.763)	0.278102
	0.1	−0.2	0.3	3.392	370.202	369.029 (13.564)	0.316854

Table 3. *ARL* values for MAX (2,1) process using explicit formula against numerical integral equation given $a = 3$, $\theta_1 = 0.1, \theta_2 = 0.2, \beta_1 = 0.5, h = 3.265$ for $ARL_0 = 370$ and $h = 3.588$ for $ARL_0 = 500$.

δ	ARL_0=370		APRE (%)	ARL_0=500		APRE (%)
(Shift Size)	Explicit	NIE		Explicit	NIE	
0.00	370.225	369.087 (12.235)	0.307381	500.080	498.380 (11.088)	0.339946
0.01	347.839	346.783 12.684)	0.303589	468.139	466.570 (11.137)	0.335157
0.03	308.154	307.243 (12.486)	0.295631	411.811	410.466 (11.199)	0.326606
0.05	274.253	273.462 (12.264)	0.28842	364.017	362.859 (10.497)	0.318117
0.07	245.143	244.453 (12.994)	0.281468	323.248	322.245 (10.706)	0.310288
0.1	208.758	208.191 (11.092)	0.271606	272.684	271.869 (11.218)	0.298881
0.3	86.578	86.391 (11.165)	0.21599	107.354	107.100 (11.242)	0.23660
0.5	45.641	45.561 (11.441)	0.175281	54.493	54.389 (11.282)	0.19085
1.0	16.512	16.494 (11.559)	0.109012	18.611	18.588 (11.670)	0.123583
1.5	9.183	9.176 (11.412)	0.076228	10.045	10.037 (11.363)	0.079642
2.0	6.288	6.256 (11.012)	0.38610	6.761	6.758 (11.360)	0.044372

Table 4. *ARL* values using explicit formula against numerical integral equation for SMAX $(3,1)_{12}$ given $a = 3, \Theta_1 = 0.1, \Theta_2 = 0.2, \Theta_3 = 0.3, \beta_1 = 0.5, h = 2.906$ for $ARL_0 = 370$ and $h = 3.223$ for $ARL_0 = 500$.

δ	ARL_0=370		APRE (%)	ARL_0=500		APRE (%)
(Shift Size)	Explicit	NIE		Explicit	NIE	
0.00	370.008	368.979 (13.753)	0.278102	500.438	498.887 (13.859)	0.309929
0.01	348.077	347.120 (13.961)	0.274939	469.150	467.714 (14.024)	0.306085
0.03	309.124	308.294 (13.439)	0.268501	413.854	412.618 (14.150)	0.298656
0.05	275.763	275.040 (11.606)	0.262182	366.803	365.733 (14.016)	0.29171
0.07	247.047	246.414 (11.660)	0.256227	326.556	325.625 (14.204)	0.285097
0.1	211.048	210.525 (13.293)	0.247811	276.476	275.714 (13.411)	0.275612
0.3	88.943	88.764 (13.893)	0.201252	110.871	110.625 (11.886)	0.221879
0.5	47.311	47.233 (13.971)	0.164867	56.868	56.764 (11.878)	0.18288
1.0	17.208	17.190 (14.135)	0.104603	19.542	19.518 (12.244)	0.122812
1.5	9.530	9.523 (13.920)	0.073452	10.493	10.484 (11.859)	0.085771
2.0	6.486	6.482 (14.323)	0.061671	7.011	7.006 (12.324)	0.071317

In Table 5, the results in terms of the optimal reference value (a) and optimal width of the control limit (h) and the minimum ARL_1 (ARL_1^*) of MAX (1,2) and SMAX $(2,2)_{12}$ processes for $ARL_0 = 370$ are shown. For example, if we want to detect a parameter change from $\lambda_0 = 1$ to $\lambda_1 > \lambda_0$ and the ARL_0 is 370, then the optimality procedure given above will give the optimal parameter values a = 1.66999118582 and h = 5.9737144930593 and $ARL_1{}^*$ value = 13.552. The suggested explicit formulas are useful to practitioners, especially when finding the optimal parameters of the MAX and SMAX processes for the CUSUM chart.

Table 5. Optimal design parameters and ARL_1^* of MAX (1,2) and SMAX $(2,2)_{12}$ processes for $ARL_0 = 370$.

δ	MAX (1,2)		ARL_1^*	SMAX $(2,2)_{12}$		ARL_1^*
(Shift Size)	*a*	*h*		*a*	*h*	
0.01	1.66999118602	5.9737144875716	315.357	2.354154802	5.4156807185	165.571
005	1.66999118587	5.973714491718	207.994	2.360429504	5.28394587940	126.065
0.1	1.66999118585	5.9737144920653	128.53	2.361119789	5.28742459511	87.092
0.3	1.66999118582	5.9737144930593	29.355	2.361410035	5.2778673182	29.659
0.5	1.66999118582	5.9737144930593	13.552	2.361410035	5.2778673147	16.122
1.0	1.66999118585	5.9737144920654	7.619	2.361410035	5.2778673174	8.158
2.0	1.66999118611	5.973714485385	4.936	2.354154797	5.415680824	4.641

5.2. Real-World Application

Application to real-world data was conducted to evaluate the ARL by the explicit formula and NIE method, as reported in Table 6. The AEONTS share prices in the SET with two exogenous variables, the US/THB exchange rate, and the interest rate, were collected monthly from January 2012 to December 2016 as the dataset of real observations. The first-order MA model is suitable for fitting the AEONTS share price with two exogenous variables, because the error of estimation is the smallest as compared to other models. Therefore, the first-order MA model was constructed with the process coefficients $\mu = 650.0555$, $\theta_1 = 0.746083$, $\beta_1 = -11.773$, $\beta_2 = -61.94972$, and the error as exponential white noise with $\lambda_0 = 6.9094$. For the ARL performance comparison, the boundary values $h = 22.48,\ 24.71$ for the CUSUM control chart and $\kappa = 0.5333,\ 0.8791$ for the EWMA control chart were used with conditions of $ARL_0 = 370$ and $ARL_0 = 500$, respectively. The smoothing parameter of the EWMA control chart was set to 0.1. The results in Table 6 are similar to the results in Tables 3 and 4, in that the NIE results approached the explicit formula results. In Table 7, the performance of CUSUM control chart with the explicit formula is compared with EWMA control chart by using the NIE method. The results of the performance comparison show that the CUSUM control chart provided a smaller ARL_1 than the EWMA control chart when the shift size was small, but the EWMA control chart performed better than the CUSUM control chart when the shift size was $\delta \geq 0.015$.

Table 6. Comparison of ARL values using explicit formula against numerical integral equation for MAX (1,2) when given a = 550, $\theta_1 = 0.746083$, $\beta_1 = -11.773$, $\beta_2 = -61.94972$, $h = 22.48$ for $ARL_0 = 370$ and $h = 24.71$ for $ARL_0 = 500$ $\lambda_0 = 6.9094$.

δ	***ARL*₀=370**		***APRE* (%)**	***ARL*₀=500**		***APRE* (%)**
(Shift Size)	**Explicit**	**NIE**		**Explicit**	**NIE**	
0.00	370.437	369.302 (10.642)	0.306394	500.389	498.693 (10.630)	0.338936
0.01	367.082	365.959 (11.017)	0.305926	495.596	493.919 (11.0590	0.33838
0.05	354.056	352.981 (11.106)	0.303624	477.007	475.406 (11.122)	0.335634
0.1	338.617	337.598 (11.103)	0.300929	455.083	453.511 (11.586)	0.345431
0.3	285.049	284.222 (10.352)	0.290125	379.214	378.000 (11.111)	0.320136
0.5	242.186	241.508 (10.965)	0.279950	319.135	318.150 (10.628)	0.308647
0.7	207.537	206.976 (11.193)	0.270313	271.011	270.204 (10.755)	0.297774
1.0	167.067	166.638 (11.902)	0.256783	215.395	214.787 (11.016)	0.282272
2.0	90.192	89.995 (11.758)	0.218422	112.120	111.851 (11.116)	0.239922
3.0	55.293	55.189 (11.212)	0.188088	66.776	66.639 (11.467)	0.205164

Table 7. Performance comparison of cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts using explicit formula (Explicit) for MAX (1,2) when $\lambda_0 = 6.9094$ for $ARL_0 = 370$ and $ARL_0 = 500$.

δ	***ARL*₀=370**		***ARL*₀=500**	
(Shift Size)	**CUSUM**	**EWMA** *h*=0.5333	**CUSUM**	**EWMA** *h*=0.8791
0.000	370.437	370.423 (16.215)	500.389	500.433 (16.036)
0.005	**368.755**	369.083 (16.202)	**497.985**	498.303 (16.265)
0.010	**367.082**	368.191 (16.112)	**495.596**	495.817 (16.085)
0.015	365.420	**365.311** (15.489)	493.221	**490.892** (16.878)
0.050	354.056	**348.928** (16.156)	477.007	**471.804** (16.966)
0.1	338.617	**335.141** (15.005)	455.083	**453.656** (16.255)
0.3	285.049	**275.716** (16.485)	379.214	**375.205** (16.255)
0.5	242.186	**229.117** (15.152)	319.135	**313.385** (16.008)
1.0	167.067	**149.136** (15.156)	215.395	**207.585** (16.026)

6. Conclusions and Discussion

In the theoretical computation, the *ARL* that was calculated from the explicit formula was in excellent agreement with the *ARL* obtained from the NIE method with the percentage error at less than 0.50%. However, the CPU time of the NIE method took between 10 and 16 min., whereas that of the explicit formula was less than one second. Moreover, the explicit formula for evaluating the *ARL* of the CUSUM control chart could not only significantly reduce the computational time, but also obtain the optimal parameters. In addition, the results from the experiment using a real-world dataset were similar to those of the theoretical computation. This shows that the CUSUM control chart is good for detecting processes with a small shift size, while the EWMA control chart can efficiently detect processes with a moderate to large shift size. Thus, it is suggested that the explicit formulas for the *ARL* of the CUSUM chart have real-world applications for a variety of data processes, including finance, agriculture, hydrology, and environmental. These issues should be addressed in future research. Future research could compare the results of the *ARL* for the CUSUM control chart with nonparametric control charts, such as the Tukey control chart. Moreover, a variety of data processes could be extended to other models or the explicit formula of *ARL* could be developed for other observations that correspond to the exponential family.

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The exact solution of the average run length on a modified EWMA control chart for the first-order moving-average process

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ABSTRACT: Statistical process control methods are widely used in several fields for monitoring and detecting process problems. One of them is the control chart which is powerful and effective for monitoring many types of processes, and its capability is usually measured using the average run length (ARL). In this article, we investigate explicit formulas for both the one- and the two-sided ARL on a modified exponentially weighted moving-average (EWMA) control chart for a first-order moving-average process with exponential white noise. The accuracy of the solution obtained with the modified EWMA control chart was compared to the numerical integral equation method and extended to a comparative performance with the standard EWMA control chart. The results show that the ARL obtained by the explicit formulas and the numerical integral equation method are in close agreement. The performance comparison shows that the modified EWMA control chart is dramatically more sensitive than the standard EWMA control chart for almost all of the studied exponential smoothing parameters and magnitudes of shift size. To demonstrate its capability, the proposed approach was applied to Thailand/US monthly foreign exchange rates data, yielding in good performance.

KEYWORDS: autocorrelated data, explicit formulas, monitoring process, statistical process control

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INTRODUCTION

Currently, the quality control of products or services, which plays an important role in business, industry, and manufacturing, can help in determining the characteristics of a product corresponding to customer demand. Moreover, products or services should have consistently maintained quality and standards. One of the most powerful sets of problem-solving tools for achieving process stability and improving capability through the reduction of variability is collectively called statistical process control (SPC) [1]. There are quite a few tools available for monitoring processes and detecting process deviation, and a most popular one is the control chart.

The first control chart widely used for monitoring processes and detecting out-of-control occurrences is the Shewhart control chart. It is effective for detecting large changes in the process mean and variance, but its performance will be degraded when small changes are detected, or the distribution of the assumption is not freeform. Many researchers have later proposed various control charts that are effective in detecting small changes, such as the cumulative sum (CUSUM) [2] and the exponentially weighted moving-average (EWMA) [3]. These control charts are well known for detecting SPC problems. More detection methods for small process shifts have been suggested by Yashchin [4], Zhang [5], Psarakis and Papaleonida [6], Prajapati [7], and Mawonike and Nkomo [8]. They have shown that the CUSUM and EWMA control charts are more effective than the Shewhart control chart for monitoring small changes and for autocorrelated processes. Recently, Patel and Divecha [9] proposed the modified EWMA control chart, developed from the original one, which is effective in detecting small and abrupt shifts in the mean of the monitored process. One of the benefits of using this control chart is that it performs well with observations that are serially correlated. Afterwards, Khan et al [10] proposed a new generalized structure for the modified EWMA statistic. The performance of the proposed control chart was compared with the original EWMA and modified EWMA control charts, as determined by their ARLs. The results indicate that the proposed control chart performed better than the two existing ones with each exponential smoothing parameter value.

One of the comparative performance methods for control charts is the average run length (ARL), which can be classified into two categories: ARL_0 and ARL_1, indicating the average number of points or observations plotted within the control limit until there exists a point or observation falsely identified as outside of the control limit. ARL_0 is accepted if it is large enough to maintain false alarms at an acceptable level and thus it should be large. Meanwhile, ARL_1 is the average number of points or observations plotted within the control limit from the change point time until there is a point or observation falsely identified outside of the control limit. ARL_0 should be large whereas ARL_1 should be as small as possible.

Many methods have been used to solve the exact solution of the ARL of a control chart, such as explicit formulas, along with several approaches to estimate the ARL, such as the Markov chain, Monte Carlo simulation, Martingale, and numerical integral equation (NIE).

Previous literature has focused on approximation of the ARL to represent an efficient control chart using many methods. Mastrangelo and Montgomery [11] evaluated the ARL of the traditional EWMA chart for serially correlated processes by using the Monte Carlo simulation method. Vanbrackle and Reynold [12] investigated EWMA and CUSUM control charts using the NIE and Markov chain approaches to find the ARL for a first-order autoregressive (AR(1)) process with additional random error. Fu et al [13] determined the run length and the ARL on CUSUM, EWMA and Shewhart control charts based on the Markov chain approach. Zhang et al [14] studied the efficiency of the EWMA control chart to further enhance the monitoring of the coefficient of variation when the process mean and standard deviation were not constant to determine the ARL, which was evaluated using Monte Carlo simulation approach.

In addition, there have been several researchers who computed the ARL by deriving explicit formulas and then comparing the ARL results with other methods to check the accuracy. Sukparungsee and Areepong [15] implemented an explicit formula for the ARL on a EWMA control chart, and compared the accuracy of the numerical results via Monte Carlo simulations. Busaba et al [16] analyzed explicit formulas for ARL on a CUSUM control chart for the case of a stationary AR(1) process and compared them with the NIE method. Meanwhile, Suriyakat et al [17] used the NIE method to solve the ARL and derived an explicit expression for AR(1) procedure on the EWMA control chart process. Areepong [18] proposed the explicit formulas on the moving-average (MA) control chart when observations are binomially distributed. Additionally, the new formulas were compared with the numerical simulation results of the Shewhart and EWMA charts. After that, Petcharat et al [19] derived explicit formulas for the ARL on an EWMA control chart for an MA process. Petcharat et al [20] analyzed explicit formulas and used the NIE method for the ARL on a CUSUM control chart when the observations comprised an MA of order q (MA(q)). Phantu et al [21] proposed explicit formulas to evaluate the ARL of an MA control chart for an AR(1) serially dependent Poisson process. Peerajit et al [22] compared the efficiency of explicit solutions to the NIE method of ARL on a CUSUM control chart for a long memory process with a seasonally adjusted autoregressive fractionally integrated moving-average (ARFIMA) model. Later, Suntornwat et al [23] proposed an analytical solution for the ARL on EWMA control chart for an ARFIMA process by comparing it with the integral equation technique. Their findings show that the proposed methods of evaluation were in good agreement, but the explicit formulas had a faster computational time than the numerical ARL.

The performance of the EWMA and CUSUM control charts has been compared by a number of researchers. Vargas et al [24] , Areepong[25], Suriyakat et al [26] and Phanyaem et al [27] which were determined by the ARL of the control charts. The results of these studies indicate that the EWMA control chart is more powerful than the others, and as previously mentioned, the modified EWMA control chart is more effective than the standard one for detecting the small shifts in the process and for autocorrelated data. The findings also indicate that the ARL is useful for comparing the efficiency of the control charts and that explicit formulas take much less computational time to evaluate the ARL than the other methods. Therefore, the derivation of explicit formulas on a modified EWMA chart is of interest in this study. The focus of this paper is on a comparison of explicit formulas with the NIE method by comparing their efficiencies on a modified EWMA control chart for a first-order moving-average (MA(1)) process with exponential white noise.

MODIFIED EWMA CONTROL CHART

The EWMA control chart continually monitors and detects small changes in the process mean. This was first proposed by Robert [3] in 1959. The EWMA control chart can be expressed by the recursive equation below.

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t, \quad t = 1, 2, 3, \ldots, \qquad (1)$$

where $0 < \lambda < 1$ is an exponential smoothing parameter. The starting value is $Z_0 = X_0$, the target value μ_0, and X_t is a processes with mean μ and variance σ^2. Then, the variance of Z_t is $\sigma^2_{Z_t} = \sigma^2(\frac{\lambda}{2-\lambda})(1-(1-\lambda)^{2i})$. When i gets large, the term $(1-\lambda)^{2i})$ converges to 0. Therefore, the general upper control limit (UCL) and lower control limit (LCL) to detect the sequence is given by,

$$\mathrm{UCL} = \mu_0 + L\sigma\sqrt{\tfrac{\lambda}{2-\lambda}}, \qquad (2)$$

$$\mathrm{LCL} = \mu_0 - L\sigma\sqrt{\tfrac{\lambda}{2-\lambda}}, \qquad (3)$$

where μ_0 is the target mean, σ is the process standard deviation, and L is an appropriate control width limit.

A modified EWMA was developed and presented by Patel and Divecha [3] in 2011. It is a correction of the EWMA control statistic, and it is also free from the inertia problem. This is effective in detecting small and abrupt shifts when monitoring the process mean for observations that are independent and normally distributed or autocorrelated. Subsequently, the modified EWMA control statistic was redesigned by Khan et al [3] in 2017. They proposed the structure of control statistic which was developed of this chart to the more efficient with the traditional of EWMA and modified EWMA control charts.

The newly modified EWMA chart can be defined by the recursive equation below.

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t + c(X_t - X_{t-1}), \qquad (4)$$

where c is a constant. The expected value and the variance of the modified EWMA control statistic are $E(Z_0) = \mu_0$ and $V(Z_t) = [\frac{(\lambda+2\lambda c+2c^2)}{2-\lambda}]\sigma^2$, respectively. Therefore, the upper control limit (UCL) and lower control limit (LCL) of the modified EWMA control chart are as follows,

$$\mathrm{UCL} = \mu_0 + Q\sigma\sqrt{\tfrac{\lambda+2\lambda c+2c^2}{2-\lambda}}, \qquad (5)$$

$$\mathrm{LCL} = \mu_0 - Q\sigma\sqrt{\tfrac{\lambda+2\lambda c+2c^2}{2-\lambda}}, \qquad (6)$$

where μ_0 is the target mean, σ is the process standard deviation, Q is an appropriate control width limit, and X_t is a sequence of observations. $Z_0 = u$ and $X_0 = v$ are the initial values and $0 < \lambda \leqslant 1$ is an exponential smoothing parameter.

The stopping time of the EWMA chart is given by

$$\tau_h = \inf\{t > 0 : Z_t < l \text{ or } Z_t > h\}, \qquad (7)$$

where τ_k is the stopping time, h is the upper control limit (UCL) and l is the lower control limit (LCL).

ARL FOR MODIFIED EWMA CHART OF MA(1)

Let MA(1) be observations of the first-order moving-average process. This can be described by the following recursion.

$$X_t = \mu + \epsilon_t - \theta\epsilon_{t-1}, \qquad (8)$$

where ϵ_t is a white noise process assumed with the exponential distribution and θ is a moving-average coefficient with an initial value of $\epsilon_0 = s$.

Therefore, the modified EWMA statistics Z_t can be written as,

$$\begin{aligned} Z_t = (1-\lambda)Z_{t-1} - X_{t-1} + (\lambda+c)\epsilon_t \\ - (\lambda\theta + \theta c)\epsilon_{t-1} + (\lambda+c)\mu, \end{aligned}$$

where $0 < \lambda \leqslant 1$ and the initial values $Z_0 = u$, $X_0 = v$, $\epsilon_0 = s$, LCL $= a$, and UCL $= b$. Thus,

$$Z_1 = (1-\lambda)u - v + (\lambda+c)\epsilon_1 - (\lambda\theta+\theta c)s + (\lambda+c)\mu.$$

The stopping time of the modified EWMA chart is given by

$$\tau_b = \inf\{t > 0 : Z_t < a \text{ or } Z_t > b\}, \qquad (9)$$

where τ_b is the stopping time, b is the upper control limit (UCL), and a is the lower control limit (LCL).

Let $L(u)$ denote the average run length on the modified EWMA control chart. The integral equation can be written in the form

$$L(u) = 1 + \int_{\frac{a-w}{\lambda+c}}^{\frac{b-w}{\lambda+c}} L((\lambda+c)y + w) f(y)\,\mathrm{d}y,$$

where $w := (1-\lambda)u - v - (\lambda\theta + \theta c)s + (\lambda+c)\mu$. Changing the variable, the integral equation becomes

$$L(u) = 1 + \frac{1}{\lambda+c}\int_a^b L(k) f\left(\frac{k-w}{\lambda+c}\right)\mathrm{d}k. \qquad (10)$$

Therefore,

$$L(u) = 1+\frac{1}{\lambda+c}\int_a^b L(k)\frac{1}{\beta}\,\mathrm{e}^{\frac{-k+(1-\lambda)u-v-(\lambda\theta+\theta c)s}{\beta(\lambda+c)}+\frac{\mu}{\beta}}\,\mathrm{d}k$$
$$= 1+\frac{C(u)}{\beta(\lambda+c)}D, \quad (11)$$

where $C(u) = \mathrm{e}^{\frac{(1-\lambda)u-v-(\lambda\theta+\theta c)s}{\beta(\lambda+c)}+\frac{\mu}{\beta}}$ and

$$\begin{aligned} D &= \int_a^b L(k)\,\mathrm{e}^{-\frac{k}{\beta(\lambda+c)}}\,\mathrm{d}k \\ &= \int_a^b \left[1+\frac{C(k)}{\beta(\lambda+c)}D\right]\mathrm{e}^{-\frac{k}{\beta(\lambda+c)}}\,\mathrm{d}k \\ &= \int_a^b \mathrm{e}^{-\frac{k}{\beta(\lambda+c)}}\,\mathrm{d}k + \int_a^b \frac{\mathrm{e}^{\frac{(1-\lambda)k-v-(\lambda\theta+\theta c)s}{\beta(\lambda+c)}+\frac{\mu}{\beta}}}{\beta(\lambda+c)}\,\mathrm{e}^{-\frac{k}{\beta(\lambda+c)}}D\,\mathrm{d}k \\ &= \frac{-\beta(\lambda+c)\left[\mathrm{e}^{-\frac{b}{\beta(\lambda+c)}}-\mathrm{e}^{-\frac{a}{\beta(\lambda+c)}}\right]}{1+\frac{1}{\lambda}\,\mathrm{e}^{-\frac{v}{\beta(\lambda+c)}-\frac{(\lambda\theta+\theta c)s}{\beta(\lambda+c)}+\frac{\mu}{\beta}}\left[\mathrm{e}^{-\frac{\lambda b}{\beta(\lambda+c)}}-\mathrm{e}^{-\frac{\lambda a}{\beta(\lambda+c)}}\right]}. \end{aligned}$$

Therefore,

$$L(u) = 1+\frac{\mathrm{e}^{\frac{(1-\lambda)u-v-(\lambda\theta+\theta c)s}{\beta(\lambda+c)}+\frac{\mu}{\beta}}}{\beta(\lambda+c)} \times \frac{-\beta(\lambda+c)\left[\mathrm{e}^{-\frac{b}{\beta(\lambda+c)}}-\mathrm{e}^{-\frac{a}{\beta(\lambda+c)}}\right]}{1+\frac{1}{\lambda}\,\mathrm{e}^{-\frac{v-(\lambda\theta+\theta c)s}{\beta(\lambda+c)}+\frac{\mu}{\beta}}\left[\mathrm{e}^{-\frac{\lambda b}{\beta(\lambda+c)}}-\mathrm{e}^{-\frac{\lambda a}{\beta(\lambda+c)}}\right]}.$$

Hence, the explicit two-sided formula of average run length for the first-order moving-average process on the modified EWMA control chart can be defined using the Fredholm integral equation of the second kind as follows,

$$\mathrm{ARL}_{2-\text{sided}} = 1-\frac{\lambda\,\mathrm{e}^{\frac{(1-\lambda)u}{\beta(\lambda+c)}}\left[\mathrm{e}^{-\frac{b}{\beta(\lambda+c)}}-\mathrm{e}^{-\frac{a}{\beta(\lambda+c)}}\right]}{\lambda\,\mathrm{e}^{\frac{-\mu}{\beta}+\frac{v+(\lambda\theta+\theta c)s}{\beta(\lambda+c)}}+\mathrm{e}^{-\frac{\lambda b}{\beta(\lambda+c)}}-\mathrm{e}^{-\frac{\lambda a}{\beta(\lambda+c)}}}. \quad (12)$$

When $a = 0$, the explicit one-sided formulas of average run length on the modified EWMA control chart can be defined as follows,

$$\mathrm{ARL}_{1-\text{sided}} = 1-\frac{\lambda\,\mathrm{e}^{\frac{(1-\lambda)u}{\beta(\lambda+c)}}\left[\mathrm{e}^{-\frac{b}{\beta(\lambda+c)}}-1\right]}{\lambda\,\mathrm{e}^{\frac{-\mu}{\beta}+\frac{v+(\lambda\theta+\theta c)s}{\beta(\lambda+c)}}+\mathrm{e}^{-\frac{\lambda b}{\beta(\lambda+c)}}-1}, \quad (13)$$

with in-control process parameter $\beta = \beta_0 = 1$ and out of control process parameter $\beta = \beta_1 > 1$.

EXISTENCE AND UNIQUENESS OF ARL

The solution of average run length shows that there uniquely exists the integral equation for explicit formulas by the Banach's Fixed-point Theorem. In this study, let T be an operation in the class of all continuous functions defined by

$$T(L(u)) = 1+\frac{1}{(\lambda+c)}\int_a^b L(k)\frac{1}{\beta}\,\mathrm{e}^{\frac{-k+(1-\lambda)u-v-(\lambda\theta+\theta c)s}{\beta(\lambda+c)}+\frac{\mu}{\beta}}\,\mathrm{d}k. \quad (14)$$

According to Banach's Fixed-point Theorem, if an operator T is a contraction, then the fixed-point equation $T(L(u)) = L(u)$ has a unique solution. To show that (14) exists and has a unique solution, the following theorem can be used.

Theorem 1 (Banach Fixed-point) *Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping with contraction constant $0 \leqslant r < 1$ such that $\|T(L_1)-T(L_2)\| \leqslant r\|L_1-L_2\|$ for all $L_1, L_2 \in X$. Then there exists a unique $L(\cdot) \in X$ such that $T(L(u)) = L(u)$, i.e., a unique fixed-point in X [28].*

Proof: To show that T defined in (14) is a contraction mapping for $L_1, L_2 \in C[a,b]$, we will show that $\|T(L_1)-T(L_2)\| \leqslant r\|L_1-L_2\|$ for all $L_1, L_2 \in C[a,b]$ with $0 \leqslant r < 1$ under the norm $\|L\|_\infty = \sup_{u\in(a,b)}|L(u)|$. From (11) and (14),

$$\begin{aligned} &\|T(L_1)-T(L_2)\|_\infty \\ &= \sup_{u\in(a,b)}\left|\frac{C(u)}{\beta(\lambda+c)}\int_a^b (L_1(k)-L_2(k))\,\mathrm{e}^{-\frac{k}{\beta(\lambda+c)}}\,\mathrm{d}k\right| \\ &\leqslant \sup_{u\in(a,b)}\left|\|L_1-L_2\|_\infty C(u)\left(\mathrm{e}^{\frac{-a}{\beta(\lambda+c)}}-\mathrm{e}^{\frac{-b}{\beta(\lambda+c)}}\right)\right| \\ &= \|L_1-L_2\|_\infty \sup_{u\in(a,b)}|C(u)|\left|\mathrm{e}^{\frac{-a}{\beta(\lambda+c)}}-\mathrm{e}^{\frac{-b}{\beta(\lambda+c)}}\right| \\ &\leqslant r\|L_1-L_2\|_\infty, \end{aligned}$$

where $r := \sup_{u\in(a,b)}|C(u)|\left|\mathrm{e}^{\frac{-a}{\beta(\lambda+c)}}-\mathrm{e}^{\frac{-b}{\beta(\lambda+c)}}\right|$ and $C(u) = \mathrm{e}^{\frac{(1-\lambda)u-v-(\lambda\theta+\theta c)s}{\beta(\lambda+c)}+\frac{\mu}{\beta}} = \mathrm{e}^{\frac{(1-\lambda)u}{\beta(\lambda+c)}-\frac{v}{\beta(\lambda+c)}-\frac{\theta s}{\beta}+\frac{\mu}{\beta}}$; $0 \leqslant r < 1$.

Therefore, the existence and the uniqueness of the solution are guaranteed by the Banach's Fixed-point Theorem. □

NUMERICAL INTEGRAL EQUATION METHOD

According to the integral equation in (10), the numerical integral equation (NIE) method can be used to evaluate the solution using the Gauss-Legendre quadrature rule as follows.

$$L(a_i) = 1+\frac{1}{\lambda+c}\sum_{j=1}^m w_j L(a_j) f\left(\frac{a_j-(1-\lambda)a_i+v}{\lambda+c}+\theta s-\mu\right), \quad (15)$$

where $i=1,2,3,\ldots,m$, $a_j=\frac{b-a}{m}(j-\frac{1}{2})+a$, and $w_j=\frac{b-a}{m}$, $j=1,2,3,\ldots,m$.

This can be rewritten in a matrix form as

$$L_{m\times1}=1_{m\times1}+R_{m\times m}L_{m\times1}$$
$$\text{or}\quad L_{m\times1}=(I_m-R_{m\times m})^{-1}1_{m\times1}, \tag{16}$$

where $L_{m\times1}=\begin{bmatrix}L(a_1)\\ \vdots\\ L(a_m)\end{bmatrix}$, $1_{m\times1}=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$, I_m is the identity, and $R_{m\times m}=\frac{1}{\lambda+c}\begin{bmatrix}w_1f_{11} & \cdots & w_mf_{1m}\\ w_1f_{21} & \cdots & w_mf_{2m}\\ \vdots & & \vdots\\ w_1f_{m1} & \cdots & w_mf_{mm}\end{bmatrix}$, when $f_{ij}:=f(\frac{a_j-(1-\lambda)a_i+v}{\lambda+c}+\theta s-\mu)$.

Therefore, the approximation of average run length is evaluated by NIE method for $L(u)$ is

$$\tilde{L}(u)=1+\frac{1}{\lambda+c}\sum_{j=1}^{m}w_jL(a_j)f\left(\frac{a_j-(1-\lambda)u+v}{\lambda+c}+\theta s-\mu\right). \tag{17}$$

NUMERICAL RESULTS

In this section, the ARL was approximated by NIE method using the Gauss-Legendre quadrature rule on a modified EWMA control chart with 1000 nodes. The accuracy of ARL was used to measure the absolute percentage relative error (APRE),

$$\text{APRE}\,(\%)=\frac{|L(u)-\tilde{L}(u)|}{L(u)}\times100, \tag{18}$$

where $\tilde{L}(u)$ is an approximation of ARL using NIE method. The numerical results are computed by MATHEMATICA. The evaluation of numerical results using (12) and (13) are shown in Table 1 and Table 2, respectively. The explicit one-sided formula of ARL and NIE method on modified EWMA control chart for the first-order moving-average when given $\text{ARL}_0=370$, $c=50\lambda$, $\lambda=0.05,0.1$, and $\theta=-0.1,0.1,-0.5,0.5$ were in dramatic agreement. But, the computational time of the explicit formulas is not much, while the numerical integral equation method took around 22 s for the one-sided of ARL and 28–30 s for the two-sided of ARL.

COMPARISON OF PERFORMANCE FOR EWMA AND MODIFIED EWMA CHARTS

In this section, the numerical comparative results of ARL on the modified EWMA and EWMA control charts are investigated. The proposed expression of modified EWMA chart was computed by (12) and (13). The explicit formulas of average run length in EWMA control chart for moving average order q process with exponential white noise was applied to the first-order moving-average process, as proposed by Petcharat et al.

Table 4 and Table 5 present the ARL of explicit formulas on EWMA and modified EWMA control charts for the first-order moving average process when $\text{ARL}_0=370,500$, $\mu=2$, $\lambda=0.05,0.1$, $\theta=-0.2,0.2,-0.4,0.4$, which are obtained by the upper control limit and lower control limit (Table 3). The results indicated that modified EWMA immediately reduced the ARL_1 more than the EWMA control chart when detecting the mean shift process. It can be said that the performance of modified EWMA control chart is more powerful than the performance of EWMA control chart for most scales of shift size.

APPLICATION

In this section, real data was applied to determine the average run length by the explicit formulas on modified EWMA and EWMA control charts for the Thailand/US foreign exchange rates observed from April 2014 to March 2019. This data is a stationary time series. By looking at the autocorrelation function (ACF) and partial autocorrelation (PACF). The data was analyzed and fitted with the first-order of moving-average process with the significant of mean and coefficient 33.567 and −0.869, respectively, and the white noise was significantly exponentially distributed with mean 0.8207. The average run lengths on modified EWMA and EWMA control charts with the results of real data are shown in Table 6 and Fig. 1.

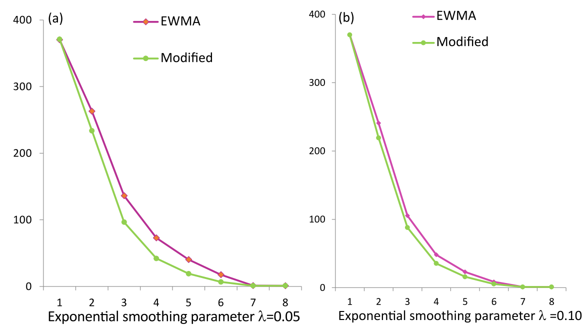

Fig. 1 The ARL of the EWMA and modified EWMA control charts with real data: (a) $\lambda=0.05$ and (b) $\lambda=0.10$.

Table 6 shows that the results are in similar agreement to Table 4 and Table 5. The modified EWMA control chart offers an ARL_1 that is more

Table 1 The one-sided of ARL for MA(1) on the modified EWMA chart using explicit formula and NIE when $\text{ARL}_0 = 370$, $\mu = 2$, and $c = 50\lambda$.

θ	Shift size	$\lambda = 0.05^{*}$			$\lambda = 0.1^{**}$		
	(δ)	Explicit	NIE	APRE	Explicit	NIE	APRE
−0.1	0.00	370.0000867370	370.0000858815 (21.918)	2.312×10^{-7}	370.0000606318	370.0000591485 (21.809)	4.009×10^{-7}
	0.01	44.0439249625	44.0439249000 (21.980)	1.419×10^{-7}	35.5693493235	35.5693492821 (22.059)	1.164×10^{-7}
	0.03	16.3214268274	16.3214268070 (21.949)	1.250×10^{-7}	13.1229621168	13.1229621050 (21.995)	8.992×10^{-8}
	0.05	10.2080506715	10.2080506598 (22.012)	1.146×10^{-7}	8.2715123064	8.2715122999 (22.028)	7.858×10^{-8}
	0.07	7.5274541620	7.5274541541 (22.012)	1.049×10^{-7}	6.1546487285	6.1546487241 (22.105)	7.149×10^{-8}
	0.10	5.4963051443	5.4963051391 (22.106)	9.461×10^{-8}	4.5543136310	4.5543136282 (22.090)	6.148×10^{-8}
	0.30	2.3371054262	2.3371054251 (22.152)	4.707×10^{-8}	2.0669730690	2.0669730685 (21.996)	2.419×10^{-8}
	0.50	1.7313103221	1.7313103217 (22.183)	4.707×10^{-8}	1.5875989978	1.5875989976 (22.074)	1.260×10^{-8}
0.1	0.00	370.0000191101	370.0000178192 (21.886)	3.489×10^{-7}	370.0000939896	370.0000917457 (21.784)	6.065×10^{-7}
	0.01	46.3721160788	46.3721159797 (21.924)	2.137×10^{-7}	37.4509233915	37.4509233251 (21.895)	1.773×10^{-7}
	0.03	17.2560948857	17.2560948533 (22.013)	1.878×10^{-7}	13.8543677226	13.8543677037 (21.893)	1.364×10^{-7}
	0.05	10.7988295175	10.7988294989 (22.028)	1.722×10^{-7}	8.7312602017	8.7312601912 (21.784)	1.203×10^{-7}
	0.07	7.9626558548	7.9626558422 (22.086)	1.582×10^{-7}	6.4928549093	6.4928549023 (22.014)	1.078×10^{-7}
	0.10	5.8110939946	7.9626558422 (22.098)	1.411×10^{-7}	4.7989996955	4.7989996909 (21.998)	9.585×10^{-8}
	0.30	2.4548556274	2.4548556256 (22.174)	7.332×10^{-8}	2.1594952549	2.1594952585 (22.087)	1.667×10^{-7}
	0.50	1.8060957035	1.8060957028 (22.192)	3.876×10^{-8}	1.6469191130	1.6469191126 (22.124)	2.429×10^{-7}

* $b = 0.4626313926$ for $\theta = -0.1$ and $b = 0.565666747$ for $\theta = 0.1$.
** $b = 0.763204934$ for $\theta = -0.1$ and $b = 0.933777249$ for $\theta = 0.1$.

promptly deducted and very small for all shift sizes and exponential smoothing parameter values. This means that the performance of the modified EWMA chart was more powerful than the EWMA chart for all cases when monitoring and detecting the change of mean.

The exchange rate is an indicator that is measured in terms of national currency per US dollar. It is defined as the price of national currency in relation to the US dollar, and is expressed as the average rate for a period of time. In this case, there were 60 observations of monthly Thailand exchange rates of Thailand/US Dollar from April 2014 to March 2019 analyzed. The mean and standard deviation of the time series data were 33.5904 and 1.45699, respectively. The upper and lower control limits were established by (2) and (3) for the modified EWMA control chart and (5) and (6) for the EWMA control chart. The detection of the process with real data is presented in Fig. 2. These show that the modified EWMA control chart is superior to the EWMA control chart at the 5th observation when detecting changes, while the EWMA control chart can detect at the 25th observation for the out-of-control process.

Fig. 2 indicates that the modified EWMA chart was able to detect the exchange rate at the 17th–24th, the 26th, 27th, and 32th–34th observations. These observations were plotted above the upper control limit. One could suspect that the depreciation has occurred at or before that time. Thus, the depreciation of the currency is an important indicator in price determination. This situation directly affects the direct import business. On the other hand, if the observations are plotted below the lower control limit, it means appreciation will occur and will also impact the export business. Therefore, tracking changes in exchange rates is the key for high performance in both economic and financial areas.

Table 2 The two-sided of ARL for MA(1) on the modified EWMA chart using explicit formula and NIE for $ARL_0 = 370$, $\mu = 2$, and $c = 50\lambda$.

θ	Shift size (δ)	$\lambda = 0.05^*$ Explicit	NIE	APRE	$\lambda = 0.1^{**}$ Explicit	NIE	APRE
−0.5	0.00	370.0000694633	370.0000687600 (30.139)	1.901×10^{-7}	370.0000980167	370.0000965317 (28.533)	4.014×10^{-7}
	0.01	41.5661928805	41.5661928328 (28.735)	1.148×10^{-7}	34.9758342828	34.9758342426	1.149×10^{-7}
	0.03	15.3646262066	15.3646261911 (28.470)	1.009×10^{-7}	12.9043398195	12.9043398080 (30.311)	8.912×10^{-8}
	0.05	9.6209444263	9.6209444175 (28.548)	9.147×10^{-8}	8.1405113753	8.1405113690 (28.298)	7.739×10^{-8}
	0.07	7.1063784712	7.1063784652 (28.564)	8.443×10^{-8}	6.0625383282	6.0625383240 (29.032)	6.928×10^{-8}
	0.10	5.2026737276	5.2026737237 (28.985)	7.496×10^{-8}	4.4917979139	4.4917979111 (28.611)	6.148×10^{-8}
	0.30	2.2450744171	2.2450744162 (27.753)	4.009×10^{-8}	2.0501611023	2.0501611018	2.439×10^{-8}
	0.50	1.6789252310	1.6789252307 (29.469)	1.787×10^{-8}	1.5791522674	1.5791522672 (27.738)	1.267×10^{-8}
0.5	0.00	370.0000411179	370.0000394821 (28.954)	4.421×10^{-7}	370.0000889780	370.0000866536 (29.078)	6.065×10^{-7}
	0.01	46.2382379117	46.2382377884 (30.108)	2.667×10^{-7}	36.9785201623	36.9785200953 (28.705)	1.812×10^{-7}
	0.03	17.2318873875	17.2318873473 (29.515)	1.878×10^{-7}	13.6814661485	13.6814661295 (28.891)	1.389×10^{-7}
	0.05	10.8006895819	10.8006895588 (28.345)	2.139×10^{-7}	8.6289415891	8.6289415785 (29.235)	1.228×10^{-7}
	0.07	7.9756157868	7.9756157711 (29.731)	1.968×10^{-7}	6.4218897912	6.4218897842 (28.376)	1.090×10^{-7}
	0.10	5.8317979184	5.8317979081 (28.096)	1.766×10^{-7}	4.7518643793	4.7518643748 (30.701)	9.470×10^{-8}
	0.30	2.4815717824	2.4815717801 (28.283)	9.268×10^{-8}	2.1487289737	2.1487289727 (29.032)	4.654×10^{-8}
	0.50	1.8297312330	1.8297312320 (28.673)	5.465×10^{-8}	1.6424658227	1.6424658223 (29.297)	2.435×10^{-8}

* $a = 0.1$, $b = 0.5168237377$ for $\theta = -0.5$ and $a = 0.1$, $b = 0.7305875693$ for $\theta = 0.5$.
** $a = 0.1$, $b = 0.858734671$ for $\theta = -0.5$ and $a = 0.1$, $b = 1.043114241$ for $\theta = 0.5$.

Table 3 Upper control limit and lower control limit of EWMA and modified EWMA charts using explicit formulas.

ARL_0	θ	$\lambda = 0.05$ EWMA l	h	Mofified a	b	$\lambda = 0.1$ EWMA l	h	Modified a	b
370	−0.4	0.1	0.1000084203	0.1	0.534425014	0.1	0.5148186	0.1	0.875406997
	0.4	0.1	0.100003783309	0.1	0.7049886465	0.1	0.17579211	0.1	1.0228024732
500	−0.4	0.1	0.100011387083	0.1	0.5348217905	0.1	0.9169484	0.1	0.8760101704
	0.4	0.1	0.1000051162256	0.1	0.7055255095	0.1	0.210902	0.1	1.2035123745

CONCLUSION

Explicit formulas for the ARL on a modified EWMA control chart monitoring an MA(1) process with exponential white noise which provides its own expression that is easy to derive and saves on computation time was investigated in this research. The accuracy of the explicit formulas was checked by comparing its absolute percentage relative error with that of the NIE method. The results show that both methods were in dramatic agreement with absolute percentage errors of less than 9.212×10^{-7}. The exponential smoothing parameter in the range of 0.05–0.25 is usually recommended. A comparison of the ARL performance of the modified and standard EWMA control charts shows that the former was more sensitive when shift sizes were small and performed better when the exponential smoothing parameter value was 0.1. The exponential smoothing parameter 0.05 is unsuitable to the most effective of the modified EWMA chart employ-

Table 4 Comparison of one-sided of ARL for MA(1) on EWMA and modified EWMA charts using explicit formulas when $\mu = 2$ and $c = 50\lambda$.

λ	θ	Shift size (δ)	$\text{ARL}_0 = 370$ EWMA	$\text{ARL}_0 = 370$ Modified	$\text{ARL}_0 = 500$ EWMA	$\text{ARL}_0 = 500$ Modified
0.05	−0.2	0.00	370.0000756855	370.0000433201	500.0000550446	500.0000517554
		0.01	310.3613470534	42.9524104373	419.3504170920	44.2714306321
		0.03	220.6242217364	15.8864511633	297.9986166729	16.0587690464
		0.05	158.9071572943	9.9337191786	214.5384272538	9.9990002172
		0.07	115.9013063561	7.3256434174	156.3814690365	7.3601034492
		0.10	73.8363408663	5.3505613736	99.4968683696	5.3681504949
		0.30	6.8797381907	2.2829992334	8.9511975571	2.2853882186
		0.50	1.9096782195	1.6971565736	2.2301626398	1.6981625206
	0.2	0.00	370.0000465698	370.0000749812	500.0000461456	500.0000637535
		0.01	309.1385402792	47.6155843164	417.6968310349	49.2416289066
		0.03	218.0802984841	17.7591290169	294.5584663025	17.9747029006
		0.05	155.9278248873	11.1174992983	210.5094640643	11.1993602969
		0.07	112.9335052931	8.1977303757	152.3680959291	8.2409666592
		0.10	71.2352935291	5.9813923738	95.9794555925	6.0034609210
		0.30	6.3612841228	2.5190268777	8.2500877220	2.5220254650
		0.50	1.7961249424	1.8470923312	2.0766034871	1.8383611231
0.10	−0.2	0.00	370.0000618877	370.0000321922	500.0000893066	500.0000737559
		0.01	343.5592416319	34.6891029378	464.817836050466	35.5348763989
		0.03	297.4261061703	12.7828116715	403.3251666153	12.8906618547
		0.05	258.8319280712	8.0580837849	351.7620972277	8.0988617372
		0.07	226.3579081554	5.9978247468	308.2779840985	6.0193780131
		0.10	186.7358687239	4.4410088793	255.0797494461	4.4520489607
		0.30	64.3155063451	2.0244287200	89.2020341712	2.0259651623
		0.50	1.5604846313	2.0244287200	40.7569375637	1.5611422914
	0.2	0.00	370.0000576120	370.0000477061	500.0000655915	500.0000331334
		0.01	341.8375030358	38.4578607779	462.2452526816	39.5013005888
		0.03	293.0596802356	14.2482021597	396.7846634952	14.3827974754
		0.05	252.6450009210	8.9792734190	342.4723183628	9.0302521363
		0.07	218.9544992262	6.6755150638	297.1358039796	6.7024702255
		0.10	178.2991377127	4.9313310869	242.3400934144	4.9451380394
		0.30	57.2166373165	2.2098772863	78.2865567375	2.2118022389
		0.50	24.6972750753	1.6794079501	33.8062753712	1.6802371690

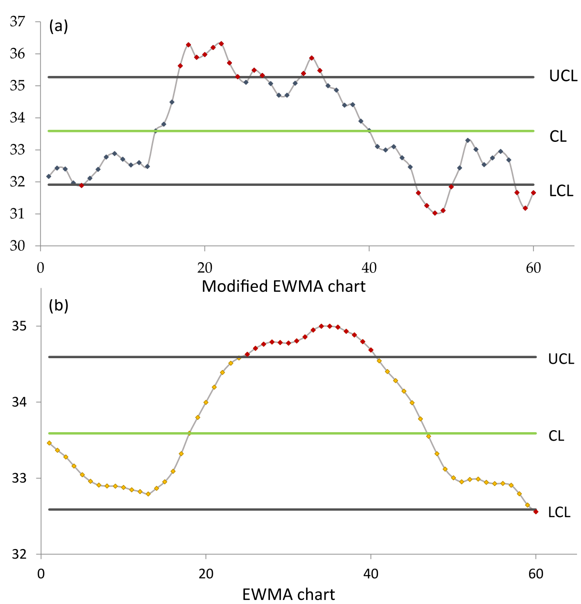

Fig. 2 The detection of the process with real data: (a) the modified EWMA and (b) the EWMA.

ing. The exponential smoothing parameter 0.1 is recommended. In addition, the performances of the modified and standard EWMA control charts were compared using real-world exchange rates data, the result indicating that the modified EWMA control chart was more efficient than the standard one.

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Table 5 Comparison of two-sided of ARL for MA(1) on EWMA and modified EWMA charts using explicit formulas when $\mu = 2$ and $c = 50\lambda$.

λ	θ	Shift size	$ARL_0 = 370$		$ARL_0 = 500$	
		(δ)	EWMA	Modified	EWMA	Modified
0.05	−0.4	0.00	370.0000555149	370.0000795892	500.0000440249	500.0000713397
		0.01	317.1741116242	41.9969256683	428.5634950788	43.25458952521
		0.03	235.1585236152	15.5352072476	317.6537185179	15.6990475039
		0.05	176.3481718462	9.7284228105	238.1243548315	9.7904827888
		0.07	133.6885773383	7.1854323906	180.4356556364	7.2182005835
		0.10	89.9632614742	5.2597771485	121.3057315405	5.2765129737
		0.30	10.7691191351	2.2829992334	14.2109276294	2.2686198289
		0.50	2.8940179432	1.69237742929	3.5613189677	1.6933433542
	0.4	0.00	370.0000314578	370.0000810411	500.0000652911	500.0000523870
		0.01	314.6794551262	45.7317530918	425.1899218557	47.2269890336
		0.03	229.7650783509	17.0276497843	310.3599859730	17.2247307716
		0.05	169.7934005538	10.6713120635	229.2601018536	10.7461007259
		0.07	126.9222352329	7.8801356635	171.2852755031	7.9196410929
		0.10	83.7224337918	5.7625654284	112.8660125465	5.7827413520
		0.30	9.1221667504	2.4553179300	11.9836764191	2.4580722709
		0.50	2.4506522991	1.8128724325	2.9617326913	1.8140418723
0.10	−0.4	0.00	370.0000518336	370.0000614837	500.0000776587	500.0000873733
		0.01	352.1496772110	35.1675420174	480.5671128914	36.0364577678
		0.03	319.9954347429	12.9784424848	445.5615061301	13.0894176155
		0.05	291.9356456494	8.1870306923	415.0336802266	8.2290068857
		0.07	267.3327564446	6.0967379490	388.3095198819	6.1189301589
		0.10	235.7921446525	4.5165264412	354.1704037426	4.5278976817
		0.30	119.5488019007	2.0594906242	234.0545686064	2.0610780140
		0.50	73.0760688952	1.5851215979	202.31849758374	1.5858037164
	0.4	0.00	370.0000434468	370.0000381773	500.0000679211	500.0000665742
		0.01	345.4281185687	36.7693566544	467.64025996536	37.7207921966
		0.03	302.2029432208	13.5999906834	410.5608960138	13.7220985121
		0.05	265.6509248565	8.5776745459	362.1207155824	9.0302521363
		0.07	234.5756240775	6.3841434405	320.7953553002	6.4085834870
		0.10	196.1945491439	4.7245233411	269.5427528719	4.7370462384
		0.30	72.6935285465	2.1383209265	102.2625833543	2.1400704197
		0.50	34.6069696123	1.6357556044	49.3435046695	1.6365094477

Table 6 Comparison of two-sided of ARL for MA(1) on EWMA and modified EWMA charts using explicit formulas when $ARL_0 = 370$, $\mu = 33.567$, $\theta = -0.869$, and $\beta_0 = 0.8207$.

Shift size	$\lambda = 0.05$		$\lambda = 0.10$	
(δ)	EWMA $h = 1.738 \times 10^{-11}$	Modified $b = 1.976 \times 10^{-14}$	EWMA $h = 2.3226 \times 10^{-14}$	Modified $b = 2.07 \times 10^{-15}$
0.00	370.1606511895	370.9704083281	370.1086602485	370.0950486051
0.01	262.9495082304	240.8173740827	240.8173740827	219.2229808369
0.03	136.0663459955	96.3929255627	105.3513596562	87.9774945611
0.05	72.7559093150	41.8842059927	48.1559932900	35.3549720373
0.07	40.1994636558	19.0826033618	23.0674462751	15.8581662799
0.10	17.6114490930	6.7114526651	8.5069999491	5.2864805809
0.30	1.1720315932	1.0123817372	1.0243050587	1.0065595855
0.50	1.0069206108	1.0001673887	1.0004345640	1.0000703639

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Original Article

Explicit analytical solutions for the average run length of modified EWMA control chart for ARX(p,r) processes

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Abstract

Statistical process control (SPC) plays a necessary role in manufacturing industry processes. An essential tool for SPC used for monitoring, measuring, controlling, and improving quality in various fields is the control chart. The modified exponentially weighted moving average (modified EWMA) control chart is widely used in various fields, and a measure commonly used to elucidate its efficiency is average run length (*ARL*). The main purpose of this study is to derive explicit formulas for the *ARL* to detect changes in the process mean of modified EWMA control chart for an autoregressive processes with explanatory variables (ARX(p,r)) with exponential white noise. In addition, the performances of the modified EWMA are compared with EWMA control charts based on the relative mean index (RMI). It was found that the explicit formulas for the *ARL* of the modified EWMA control chart performed better than on the EWMA control chart for monitoring process mean.

Keywords: average run length, modified EWMA control chart, ARX process, explanatory variables, Integral equation

1. Introduction

Statistical process control (SPC) plays a necessary role in manufacturing industry processes and is used for monitoring, measuring, controlling and improving quality in various fields (science, economics, engineering, finance, medicine, etc.). An important SPC tool is the control chart which is used for detecting changes in process means. The first control chart, introduced by Shewhart (1931), was used for detecting large shifts in process means ($>1.5\sigma$) (Montgomery, 2012). The cumulative sum (CUSUM) control chart proposed by Page (1954) is a good alternative to the Shewhart control chart for detecting small shifts in process means ($<1.5\sigma$) (Montgomery, 2012), as has been indicated by comparative studies on the two (Hawkins & Olwell, 1998; Lucas & Saccucci, 1990). In addition, another option for detecting small shifts is the exponentially weighted moving average (EWMA) control chart first presented by Roberts (1959), which has been used in various industries, However, these charts cannot be used directly for chemical and pharmaceutical processes due to the observations being frequently autocorrelated (Patel & Divecha, 2011). The EWMA technique is used in SPC to monitor the results of manufacturing processes by tracking the moving average of the efficiency throughout the lifetime of the process.

The modified EWMA control chart developed by Patel and Divecha (2011) is a simplified EWMA control chart for detecting shifts in the process mean regardless of size. It is used in various fields, especially in the chemical industry, in which the processes are frequently autocorrelated. Past observations are considered (similar to the EWMA scheme) along with past changes as well as the latest change in the process mean. Khan, Aslam and Jun (2016) developed a new EWMA control chart based upon a modified EWMA statistic that considers the past and current behavior of the process; they compared it with the existing one by Patel and Divecha (2011) and found that the proposed control chart had the ability to detect shifts more quickly.

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A commonly used measure for the efficiency of control charts is the average run length (*ARL*). Ryu, Wan and Kim (2010) applied in-control *ARL* (ARL_0), which refers to the average number of observations on the in-control process before a false out-of-control alarm is raised, as a measure of the false-alarm rate. The out-of-control *ARL* (ARL_1), is the average number of observations required to detect a specific process mean shift and represents the ability to detect shifts in the process mean.

Various methods that can be used to find the *ARL* of control charts have been proposed, such as Monte Carlo simulation, Markov chains, Martingales, numerical integration equations (NIEs), and explicit formulas. A NIE is a method for evaluating the *ARL* that has many rules, namely the midpoint rule, trapezoidal rule, Simson's rule, and the Gauss-Legendre rule. In this study, we used the Gauss-Legendre rule. An explicit formula is a method for evaluating the *ARL* that requires an integral equation for its derivation. In this study, we used the Fredholm integral equation of the second type (Mititelu, Areepong, Sukparungsee, & Novikov, 2010). In 1959, Robert (1959) proposed an EWMA control chart by using Monte Carlo simulations to estimate the *ARL*. Crowder (1987) used an NIE approach to find the *ARL* for a Gaussian distribution. Harris and Ross (1991) studied CUSUM with serially correlated observations via Monte Carlo simulations. Mititelu *et al.*, (2010) used a linear Fredholm-type integral equation approach to derive explicit formulas for the *ARL* in certain special cases. The *ARL* for a CUSUM control chart has been found when the random observations follow a hyperexponential distribution and the *ARL* for an EWMA control chart with observations following a Laplace distribution. Suriyakat, Areepong, Sukparungsee and Mititelu (2012) derived explicit formulas for the *ARL* of the EWMA statistic for first-order autoregressive (AR(1)) observations with errors following an exponential white noise process. Paichit (2016) used an NIE to find the exact expression for the *ARL* of an EWMA control chart for an AR process with exogenous input (ARX(p)). Paichit (2017) presented an exact expression for the *ARL* of the control chart for an ARX(p) procedure. Explicit formulas for the *ARL* of a modified EWMA control chart for an exponential AR(1) process were presented by Phanthuna, Areepong and Sukparungsee (2018).

The main purpose of this study is to derive explicit formulas for the *ARL* for detecting changes in the process mean of modified EWMA control chart based on Khan *et al.* (2016) for an autoregressive processes with explanatory variables (ARX(p,r)) with exponential white noise. In the present study, Fredholm-type integral equations are used to derive explicit formulas of ARL_0 and ARL_1. This paper is organized as follows. An introduction to the properties of control charts and the model for an ARX(p,r) process with exponential white noise is given in Section 2. The solutions for the *ARL*s of the EWMA and modified EWMA control charts for an ARX(p,r) process with exponential white noise are presented in Section 3. Next, the NIEs for the *ARL*s of the modified EWMA control charts are introduced in Section 4. Furthermore, numerical results for a comparison of the *ARL*s on the modified EWMA control charts for ARX(p,r) process with exponential white noise are offered in Section 5 and Section 6. The proposed explicit formulas are applied in Section 7. Finally, conclusions are given in Section 8.

2. The Properties of the Control Charts and ARX(p,r) Process with Exponential White Noise

2.1 The EWMA control chart

Roberts (1959) introduced the EWMA control chart for detecting small shifts in the process mean that is defined as

$$Z_t = (1-\lambda) Z_{t-1} + \lambda Y_t \qquad ; t = 1,2,3,... , \tag{2.1}$$

where Z_t is the EWMA statistic, Y_t is the sequence of the ARX(p,r) process with exponential white noise and λ is an exponential smoothing parameter $(0 < \lambda \le 1)$.

The stopping time will occur when an out-of-control observation is firstly detected, which is sufficient to decide that the process is out-of-control. The stopping time τ_b for the EWMA control chart can be written as

$$\tau_b = \inf\{t > 0;\ Z_t > b\}, \tag{2.2}$$

where b is a constant parameter known as the upper control limit $(b > 0)$. The upper side of the *ARL* for the ARX(p,r) process on the EWMA control chart with an initial value $(Z_0 = u)$ can be found. Now, the function $L(u)$ is defined as

$$L(u) = ARL = E_\infty(\tau_b) \ge T,\ Z_0 = u. \tag{2.3}$$

The mean and the variance of the EWMA control chart can be written as

$$E(Z_t) = \mu, \tag{2.4}$$

$$Var(Z_t) = \left(\frac{\lambda}{2-\lambda}\right)\sigma^2, \tag{2.5}$$

and the upper control limit (UCL) and the lower control limit (LCL) of the EWMA control chart is defined as follows:

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}, \tag{2.6a}$$

$$CL = \mu_0, \tag{2.6b}$$

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}, \tag{2.6c}$$

where μ_0 is the target mean, σ is the process standard deviation and L is an appropriate control width limit $(L > 0)$.

2.2 The modified EWMA control chart

Khan *et al.* (2016) developed a new EWMA control chart based upon the modified EWMA statistic that considers the past and current behavior of the process. They compared the proposed chart with the existing modified EWMA control chart developed by Patel and Divecha (2011) that is a simplified EWMA control chart for detecting shifts of all sizes in the process mean under the assumption that the observations follow a normal distribution. The modified EWMA control chart proposed by Khan *et al.* (2016) is defined as

$$M_t = (1-\lambda)M_{t-1} + \lambda Y_t + k(Y_t - Y_{t-1}) \qquad ;t = 1,2,3,..., \tag{2.7}$$

where M_t is the modified EWMA statistic, Y_t is the sequence of the ARX(p,r) process with exponential white noise, λ is an exponential smoothing parameter $(0 < \lambda \le 1)$, and k is a constant $(k > 0)$.

The modified EWMA control chart is based on two constants, λ and k, and comprises an extension to the existing EWMA control chart. The modified EWMA control chart by Khan *et al.* (2016) is reduced to the original EWMA control chart by Roberts (1959) if $k = 0$ and is reduced to the control chart based on the modified EWMA control chart by Patel and Divecha (2011) if $k = 1$.

The stopping time τ_h for the modified EWMA control chart can be written as

$$\tau_h = \inf\{t > 0;\ M_t > h\}, \tag{2.8}$$

where h is a constant parameter known as the upper control limit $(h > 0)$. The upper side of the *ARL* for the ARX(p,r) process on the modified EWMA control chart with an initial value ($M_0 = u$) can be found. Now, we define the function $G(u)$ as

$$ARL = G(u) = E_\infty(\tau_h) \ge T,\ M_0 = u, \tag{2.9}$$

where *T* is a fixed number (should be large) and $E_\infty(.)$ is the expectation under the assumption that observations ε_t have the distribution $F(y_t, \alpha)$.

The value of the mean and the variance of the modified EWMA control chart is defined as

$$E(M_t) = \mu, \tag{2.10}$$

$$Var(M_t) = \frac{(\lambda + 2\lambda k + 2k^2)\sigma^2}{(2-\lambda)}, \tag{2.11}$$

and the *UCL* and the *LCL* of the modified EWMA control chart can be written as

$$LCL = \mu_0 - L\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}}, \tag{2.12a}$$

$$CL = \mu_0, \tag{2.12b}$$

$$UCL = \mu_0 + L\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}}, \tag{2.12c}$$

where μ_0 is the target mean, σ^2 is the process variance and L is an appropriate control width limit $(L > 0)$.

2.3 The ARX(p,r) process with exponential white noise

The ARX(p,r) process is defined as

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \sum_{j=1}^{r} \beta_j X_j + \varepsilon_t \qquad ;t = 1,2,3,..., \tag{2.13}$$

where δ is a constant $(\delta \ge 0)$, ϕ_i is an autoregressive coefficient for $i = 1,2,...,p$ $(-1 < \phi_i < 1)$; ε_t is an independent and identically distributed (iid) sequence; $\varepsilon_t \sim Exp(\alpha)$; X_j are explanatory variables of Y_t; and β_j are coefficients of

$X_j\ ; j=1,2,...,r$. The initial value for the ARX(p,r) process mean is $Y_{t-1},Y_{t-2},...,Y_{t-p}=1$ and the initial value for the explanatory variables $X_1,X_2,...,X_r=1$.

3. Explicit Formulas for the *ARL*s of the Modified EWMA Control Chart for an ARX(p,r) Process with Exponential White Noise

Explicit formulas for the *ARL* of the modified EWMA control chart for an ARX(p,r) process are derived as follows:

Substituting Y_t from Equation (2.13) into Equation (2.7), then

$$M_t=(1-\lambda)M_{t-1}+(\lambda+k)\delta+(\lambda+k)\phi_1Y_{t-1}+...+(\lambda+k)\phi_pY_{t-p}$$
$$+(\lambda+k)\sum_{j=1}^{r}\beta_jX_j+(\lambda+k)\varepsilon_t-kY_{t-1}\ .$$

If Y_1 gives the out-of-control state for M_1, $M_0=u$ and $Y_0=v$, then

$$M_1=(1-\lambda)u+(\lambda+k)\delta+(\lambda+k)\phi_1v+...+(\lambda+k)\phi_pv+(\lambda+k)\sum_{j=1}^{r}\beta_jX_j+(\lambda+k)\varepsilon_1-kv$$

If ε_1 is the in-control limit for M_1, then $0\le M_1\le h$.

The function $G(u)$ can be derived by the Fredholm integral equation of the second type (Mititelu *et al.*, 2010), and thus $G(u)$ can be written as

$$G(u)=1+\int G(M_1)f(\varepsilon_1)d(\varepsilon_1)\,, \tag{3.1}$$

by substituting ε_1 with y. Therefore, the function $G(u)$ is obtained as

$$G(u)=1+\int_0^h L\left\{\begin{matrix}(1-\lambda)u+(\lambda+k)\delta+(\lambda+k)\phi_1v+...+(\lambda+k)\phi_pv\\+(\lambda+k)\sum_{j=1}^{r}\beta_jX_j-kv+(\lambda+k)y\end{matrix}\right\}f(y)dy\,\cdot$$

Let $w=(1-\lambda)u+(\lambda+k)\delta+(\lambda+k)\phi_1v+...+(\lambda+k)\phi_pv+(\lambda+k)\sum_{j=1}^{r}\beta_jX_j-kv+(\lambda+k)y.$

By changing the integral variable, we obtain the integral equation as follows:

$$G(u)=1+\frac{1}{\lambda+k}\int_0^h G(w)f\left\{\frac{w-(1-\lambda)u}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\}dw\,\cdot \tag{3.2}$$

If $Y_t \sim Exp(\alpha)$ the $f(y)=\frac{1}{\alpha}e^{\frac{-y}{\alpha}}\ ;\ y\ge 0$, so

$$G(u)=1+\frac{1}{\lambda+k}\int_0^h G(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_jX_j\right\}}dw\,\cdot$$

Let the function $D(u)=e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}-\frac{kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_jX_j}{\alpha}}$, then we have

$$G(u)=1+\frac{D(u)}{\alpha(\lambda+k)}\int_0^h L(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw \qquad ;0\le u\le h\cdot$$

Let $g=\int_0^h G(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw$, then $G(u)=1+\frac{D(u)}{\alpha(\lambda+k)}\cdot g$. Consequently, we obtain

$$G(u)=1+\frac{1}{\alpha(\lambda+k)}e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}-\frac{kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}\cdot g\cdot \tag{3.3}$$

Solving a constant g

$$g=\int_0^h G(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw \;=\int_0^h\left[1+\frac{g}{\alpha(\lambda+k)}D(w)\right]e^{\frac{-w}{\alpha(\lambda+k)}}dw$$

$$=\int_0^h e^{\frac{-w}{\alpha(\lambda+k)}}dw+\int_0^h\frac{g}{\alpha(\lambda+k)}D(w)\cdot e^{\frac{-w}{\alpha(\lambda+k)}}dw$$

$$=-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)+\frac{ge^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\alpha(\lambda+k)}\int_0^h e^{\frac{-\lambda w}{\alpha(\lambda+k)}}dw$$

$$=-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)-\frac{de^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\lambda}\left(e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right)$$

$$g=\frac{-\alpha(\lambda+k)\left(e^{\frac{-h}{\alpha(\lambda+k)}}-1\right)}{1+\frac{e^{\frac{-kv}{\alpha(\lambda+k)}+\frac{\delta}{\alpha}+\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}+\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}}{\lambda}\left(e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right)}. \tag{3.4}$$

Substituting g from Equation (3.4) into Equation (3.3), then

$$G(u)=1-\frac{\lambda e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}}\left[e^{\frac{-h}{\alpha(\lambda+k)}}-1\right]}{\lambda e^{\frac{kv}{\alpha(\lambda+k)}-\frac{\delta}{\alpha}-\frac{v\sum_{i=1}^{p}\phi_i}{\alpha}-\frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha}}+\left[e^{\frac{-\lambda h}{\alpha(\lambda+k)}}-1\right]}. \tag{3.5}$$

When the process is in the in-control state with exponential parameter $\alpha = \alpha_0$, we obtain the explicit solution for ARL_0 as follows:

$$ARL_0 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_0(\lambda+k)}}\left[e^{\frac{-h}{\alpha_0(\lambda+k)}} - 1\right]}{\lambda e^{\frac{kv}{\alpha_0(\lambda+k)} - \frac{\delta}{\alpha_0} - \frac{v\sum_{i=1}^{p}\phi_i}{\alpha_0} - \frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha_0}} + \left[e^{\frac{-\lambda h}{\alpha_0(\lambda+k)}} - 1\right]}. \tag{3.6}$$

Similarly, when the process is in the out-of-control state with exponential parameter $\alpha = \alpha_1$, the explicit solution for ARL_1 can be written as

$$ARL_1 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_1(\lambda+k)}}\left[e^{\frac{-h}{\alpha_1(\lambda+k)}} - 1\right]}{\lambda e^{\frac{kv}{\alpha_1(\lambda+k)} - \frac{\delta}{\alpha_1} - \frac{v\sum_{i=1}^{p}\phi_i}{\alpha_1} - \frac{\sum_{j=1}^{r}\beta_j X_j}{\alpha_1}} + \left[e^{\frac{-\lambda h}{\alpha_1(\lambda+k)}} - 1\right]}. \tag{3.7}$$

4. The NIE for the *ARL* on the Modified EWMA Control Chart

An integral equation of the second type for the *ARL* on the modified EWMA control chart for the ARX(p,r) process in Equation (3.5) can be approximated by using the quadrature formula. In this study, the Gauss-Legendre quadrature rule is applied as follows:

Given $$f(a_j) = f\left\{\frac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\}. \tag{4.1}$$

The approximation for the integral is in the form

$\int_0^h G(w) f(w) dw \approx \sum_{j=1}^{m} w_j f(a_j)$, where $a_j = \frac{h}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{h}{m}; j = 1,2,...,m$.

Using the quadrature formula, the numerical approximation $\tilde{G}(u)$ for the integral equation can be found as a solution of the linear equations as follows:

$$\tilde{G}(a_i) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{G}(a_j) f\left\{\frac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\}; i = 1,2,...,m.$$

Thus,

$$\tilde{G}(a_1) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{G}(a_j) f\left\{\frac{a_j - (1-\lambda)a_1}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\},$$

$$\tilde{G}(a_2) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{G}(a_j) f\left\{\frac{a_j - (1-\lambda)a_2}{(\lambda+k)} + \frac{kv}{(\lambda+k)} - \delta - v\sum_{i=1}^{p}\phi_i - \sum_{j=1}^{r}\beta_j X_j\right\},$$

$$\tilde{G}(a_3)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_3}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\},$$
$$\vdots$$
$$\tilde{G}(a_m)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)a_m}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\}.$$

The set of m equations with m unknowns can be rewritten in matrix form. The column vector of $\tilde{G}(a_i)$ is $\mathbf{G}_{m\times1}=(\tilde{G}(a_1),\tilde{G}(a_2),...,\tilde{G}(a_m))'$. Since $\mathbf{1}_{m\times1}=(1,1,...,1)'$ is a column vector of ones and $\mathbf{R}_{m\times m}$ is a matrix, we can define m to m^{th} as elements of matrix $\mathbf{R}$ as follows:

$$\left[R_{ij}\right]\approx\frac{1}{\lambda+k}w_j\,f\left\{\frac{a_j-(1-\lambda)a_i}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\},$$

and $\mathbf{I}_m=diag(1,1,...,1)$ is a unit matrix of order m. If $(\mathbf{I}-\mathbf{R})^{-1}$ exists, the numerical approximation for the integral equation in terms of the matrix can be written as

$$\mathbf{G}_{m\times1}=\left(\mathbf{I}_m-\mathbf{R}_{m\times m}\right)^{-1}\mathbf{1}_{m\times1}.$$

Finally, by substituting a_i by u in $\tilde{G}(a_i)$, the numerical integration equation for function $\tilde{G}(u)$ can be derived as

$$\tilde{G}(u)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\frac{a_j-(1-\lambda)u}{(\lambda+k)}+\frac{kv}{(\lambda+k)}-\delta-v\sum_{i=1}^{p}\phi_i-\sum_{j=1}^{r}\beta_j X_j\right\}. \qquad (4.2)$$

5. Comparison of the NIE Method and the Explicit Formulas

Here, a comparison of the efficacies of the NIE method ($\tilde{G}(u)$) and the explicit formulas ($G(u)$) for the *ARL* of an ARX(p,r) process on the modified EWMA control chart is presented. The parameter values were set as $ARL_0=370$ and 500; $\lambda=$ 0.01, 0.05, 0.1, and 0.2; the in-control parameter $\alpha_0=1$; and the shift size was varied as 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.3, and 0.5. In general, the popular setting of the initial value is equal to the expected value of the distribution. For setting the in-control parameter, $\alpha_0=1$, which is the initial value as 1. The coefficient has a value from -1 to 1, which can be specified as any value. The configuration does not affect the accuracy of the explicit formulas and the NIE methods. The absolute percentage difference to measure the accuracy of the *ARL* is defined as

$$Diff\,(\%)=\frac{\left|G(u)-\tilde{G}(u)\right|}{G(u)}\times100. \qquad (5.1)$$

Equations (3.5) and (4.2) are used to evaluate the *ARL* on the modified EWMA control chart for an ARX(p,r) process with exponential white noise. The number of nodes equal to 500 iterations was used to obtain the *ARL* results from the NIE method. The computations for the NIE method were carried out on a Windows 7 Professional 32-bit PC System with RAM of 2 GB and an AMD E1-1200 CPU.

The results in Tables 1–3 report the numerical values of the *ARL* derived from the explicit formulas and NIE method, and the absolute percentage difference between them. From the results, we can see that the *ARL* values derived from the explicit formulas give the same results as the NIE method. The numerical approximations had an absolute percentage difference of less than 0.003%. However, the computational time of the NIE method was 13.42–13.58 s whereas that of the explicit formulas was < 1 s.

Table 1. Comparison of the *ARL* on a modified EWMA control chart using explicit formulas with the NIE method for ARX(1,1) with $u=1$, $v=1$, $k=1$, $\beta_1=0.2$, and $ARL_0=370$.

λ	δ	ϕ_1	h	shift	Explicit	NIE	Time[a]	Diff%
0.05	0	0.1	2.11284	0.00	370.514622	370.514167	13.468	0.000123
				0.01	185.632808	185.632635	13.555	0.000093
				0.02	123.215119	123.215018	13.494	0.000082
				0.03	91.885107	91.885037	13.428	0.000076
				0.04	73.065410	73.065357	13.551	0.000072
				0.05	60.520993	60.520951	13.552	0.000068
				0.10	32.116753	32.116734	13.504	0.000058
				0.30	10.731776	10.731772	13.546	0.000037
				0.50	6.457709	6.457707	13.440	0.000026
		-0.1	2.61195	0.00	370.424900	370.424156	13.511	0.000201
				0.01	274.686377	274.685905	13.563	0.000172
				0.02	214.349128	214.348800	13.478	0.000153
				0.03	173.294237	173.293995	13.511	0.000140
				0.04	143.823485	143.823298	13.446	0.000130
				0.05	121.811602	121.811453	13.445	0.000122
				0.10	64.531777	64.531713	13.421	0.000098
				0.30	17.824627	17.824616	13.541	0.000060
				0.50	9.519466	9.519462	13.452	0.000042
0.10	1	0.2	0.69141	0.00	370.555941	370.555865	13.502	0.000021
				0.01	90.098635	90.098626	13.498	0.000010
				0.02	51.385975	51.385971	13.460	0.000008
				0.03	35.997312	35.997309	13.434	0.000007
				0.04	27.736604	27.736602	13.467	0.000007
				0.05	22.584474	22.584472	13.485	0.000006
				0.10	11.823565	11.823565	13.518	0.000005
				0.30	4.369675	4.369675	13.506	0.000003
				0.50	2.902154	2.902154	13.467	0.000002
		-0.2	1.04870	0.00	370.273926	370.273736	13.537	0.000052
				0.01	105.208634	105.208608	13.519	0.000025
				0.02	61.437849	61.437836	13.532	0.000020
				0.03	43.452288	43.452281	13.542	0.000018
				0.04	33.653645	33.653639	13.449	0.000017
				0.05	27.489870	27.489866	13.471	0.000016
				0.10	14.478770	14.478768	13.508	0.000013
				0.30	5.327500	5.327500	13.488	0.000008
				0.50	3.494115	3.494115	13.434	0.000005

[a]The computational times for the NIE methods in seconds (PC System: Windows 7 Professional 32-bit, RAM: 2 GB and CPU: AMD E1-1200)

Table 2. Comparison of the *ARL* on a modified EWMA control chart using explicit formulas with the NIE method for ARX(2,1) with $u=1$, $v=1$, $\delta=0$, $k=1$, $\beta_1=0.2$, and $ARL_0=370$.

λ	ϕ_1	ϕ_2	h	shift	Explicit	NIE	Time	Diff%
0.05	0.1	0.1	1.90196	0.00	370.104536	370.104178	13.492	0.000097
				0.01	164.156587	164.156470	13.512	0.000072
				0.02	105.192627	105.192560	13.467	0.000063
				0.03	77.258066	77.258020	13.551	0.000059
				0.04	60.970239	60.970204	13.466	0.000056
				0.05	50.307351	50.307324	13.532	0.000054
				0.10	26.685128	26.685116	13.545	0.000046
				0.30	9.216663	9.216660	13.436	0.000029
				0.50	5.688211	5.688210	13.529	0.000020
		-0.1	2.34842	0.00	370.111274	370.110694	13.520	0.000157
				0.01	218.233870	218.233600	13.487	0.000124
				0.02	153.301535	153.301368	13.495	0.000109
				0.03	117.350069	117.349952	13.416	0.000100
				0.04	94.565641	94.565553	13.479	0.000094
				0.05	78.865508	78.865438	13.492	0.000089
				0.10	41.847694	41.847663	13.492	0.000074
				0.30	13.172037	13.172030	13.492	0.000047
				0.50	7.603031	7.603028	13.517	0.000033

Table 2. Continued.

λ	ϕ_1	ϕ_2	h	shift	Explicit	NIE	Time	Diff%
0.10	0.1	0.2	1.78838	0.00	370.299172	370.298515	13.519	0.000178
				0.01	144.464904	144.464768	13.518	0.000094
				0.02	89.685034	89.684969	13.459	0.000073
				0.03	65.007924	65.007883	13.455	0.000063
				0.04	50.973254	50.973225	13.535	0.000057
				0.05	41.920612	41.920590	13.490	0.000052
				0.10	22.222112	22.222103	13.425	0.000041
				0.30	7.893825	7.893823	13.482	0.000024
				0.50	4.987173	4.987172	13.499	0.000016
		-0.2	2.78993	0.00	370.115966	370.114002	13.501	0.000531
				0.01	286.090889	286.089671	13.542	0.000426
				0.02	228.394081	228.393273	13.492	0.000354
				0.03	187.010941	187.010376	13.531	0.000302
				0.04	156.285272	156.284860	13.569	0.000264
				0.05	132.822686	132.822375	13.438	0.000234
				0.10	70.157635	70.157527	13.440	0.000153
				0.30	18.713286	18.713273	13.557	0.000072
				0.50	9.832924	9.832919	13.520	0.000047
0.20	0.2	0.1	1.95666	0.00	370.295141	370.292831	13.437	0.000624
				0.01	137.992552	137.992202	13.490	0.000253
				0.02	84.820251	84.820109	13.462	0.000168
				0.03	61.246548	61.246469	13.427	0.000130
				0.04	47.941844	47.941792	13.486	0.000108
				0.05	39.398626	39.398589	13.534	0.000093
				0.10	20.908407	20.908394	13.466	0.000060
				0.30	7.523560	7.523557	13.512	0.000028
				0.50	4.802271	4.802270	13.539	0.000018
		-0.1	2.49307	0.00	370.002909	369.998734	13.478	0.001128
				0.01	183.558057	183.556989	13.456	0.000582
				0.02	121.435201	121.434716	13.457	0.000399
				0.03	90.409051	90.408773	13.539	0.000307
				0.04	71.822583	71.822402	13.520	0.000252
				0.05	59.455236	59.455108	13.515	0.000215
				0.10	31.519801	31.519760	13.498	0.000129
				0.30	10.562568	10.562562	13.519	0.000054
				0.50	6.382784	6.382781	13.433	0.000033

Table 3. Comparison of the *ARL* on a modified EWMA control chart using explicit formulas with the NIE method for ARX(2,1) with $u=1,\ v=1,\ \delta=0,\ k=1,\ \phi_1=0.1,\ \phi_2=-0.2,\ \beta_1=0.2,$ and $ARL_0=500$.

shift	$\lambda=0.01,\ h=2.48512$				$\lambda=0.05,\ h=2.61385$			
	Explicit	NIE	Time	Diff%	Explicit	NIE	Time	Diff%
0.00	500.488509	500.487979	13.458	0.000106	500.640854	500.639672	13.476	0.000236
0.01	331.637724	331.637386	13.484	0.000102	340.395524	340.394878	13.463	0.000190
0.02	243.153820	243.153579	13.448	0.000099	252.391615	252.391203	13.531	0.000163
0.03	189.153991	189.153809	13.490	0.000096	197.352918	197.352629	13.536	0.000146
0.04	153.035120	153.034976	13.469	0.000094	160.016146	160.015932	13.469	0.000134
0.05	127.342375	127.342258	13.468	0.000092	133.230089	133.229922	13.533	0.000125
0.10	64.915512	64.915458	13.506	0.000083	67.592092	67.592025	13.497	0.000099
0.30	17.675818	17.675808	13.491	0.000056	18.040792	18.040781	13.502	0.000060
0.50	9.459167	9.459163	13.546	0.000040	9.576707	9.576703	13.510	0.000042
shift	$\lambda=0.1,\ h=2.79202$				$\lambda=0.2,\ h=3.22379$			
	Explicit	NIE	Time	Diff%	Explicit	NIE	Time	Diff%
0.00	500.386457	500.383045	13.496	0.000682	500.609662	500.595309	13.455	0.002867
0.01	358.298531	358.296709	13.537	0.000508	434.146693	434.136038	13.552	0.002454
0.02	272.247864	272.246767	13.458	0.000403	368.724286	368.716686	13.482	0.002061

Table 3. Continued.

shift	$\lambda = 0.1,\ h = 2.79202$				$\lambda = 0.2,\ h = 3.22379$			
	Explicit	NIE	Time	Diff%	Explicit	NIE	Time	Diff%
0.03	215.455768	215.455048	13.496	0.000334	310.585722	310.580379	13.495	0.001720
0.04	175.682830	175.682330	13.486	0.000285	261.603168	261.599405	13.447	0.001438
0.05	146.584892	146.584527	13.469	0.000249	221.388663	221.385984	13.530	0.001210
0.10	73.817022	73.816906	13.579	0.000157	108.236492	108.235857	13.506	0.000586
0.30	18.955138	18.955124	13.516	0.000072	23.381342	23.381309	13.507	0.000139
0.50	9.895165	9.895160	13.461	0.000047	11.381464	11.381456	13.513	0.000072

6. Comparison of the *ARL*s on the EWMA with modified EWMA control charts

After verifying the accuracy of the explicit formulas in the previous section, we used simulated data and the relative mean index (RMI) to compare the performances of the *ARL* of an ARX(p,r) process on EWMA and modified EWMA control charts. The RMI is defined as

$$RMI = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_{shift,i} - Min\left[ARL_{shift,i}\right]}{Min\left[ARL_{shift,i}\right]}\right). \tag{6.1}$$

where $ARL_{shift,i}$ is the *ARL* of the control chart when the position process shift, $shift,i$ is the shift size for $i = 1,2,...,n$, $Min\left[ARL_{shift,i}\right]$ denotes the smallest *ARL* of two control charts in comparison when the position process shift. The control chart with the smallest RMI performs the best in detecting mean changes on the whole.

For the comparison of the *ARL*s on the EWMA and modified EWMA control charts for an ARX(1,1) process, the parameter values were set as $ARL_0 = 370$; $\lambda =$ 0.05, 0.1, and 0.2; the in-control parameter $\alpha_0 = 1$; the shift size was varied as 0.001, 0.003, 0.005, 0.007, 0.009, 0.01, 0.03, 0.05, 0.07 and 0.09. The results are reported in Table 4.

Table 4. Comparison of the *ARL* of EWMA and modified EWMA control charts using explicit formulas for an ARX(1,1) with $u = 1,\ v = 1,\ \delta = 0,\ k = 40\lambda,\ \beta_1 = 0.2,$ and $ARL_0 = 370$.

shift	$\lambda = 0.05,\ \phi_1 = 0.2$		$\lambda = 0.1,\ \phi_1 = 0.2$		$\lambda = 0.2,\ \phi_1 = 0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 2.5496\times10^{-8}$	$h = 1.3590441$	$b = 0.00107964$	$h = 2.7665764$	$b = 0.04441$	$h = 5.734902$
0.000	370.071291	370.0768919	370.004307	370.003373	370.722568	370.715572
0.001	362.264617	257.030787*	365.787507	239.166659*	362.075941	233.363671*
0.003	347.186399	159.726241*	357.521610	140.347811*	345.767724	134.327916*
0.005	332.792484	115.984299*	349.473732	99.484050*	330.655613	94.494822*
0.007	319.049437	91.124031*	341.637371	77.151829*	316.615806	72.994088*
0.009	305.925567	75.090081*	334.006238	63.075235*	303.540954	59.537517*
0.010	299.586374	69.033841*	330.265724	57.823254*	297.335678	54.535706*
0.030	198.799944	26.743266*	264.852791	22.105174*	206.492192	20.787108*
0.050	134.056104	16.821242*	214.163689	13.957965*	153.166554	13.153305*
0.070	91.808369	12.393346*	174.541323	10.348427*	118.601180	9.778282*
0.090	63.828457	9.886695*	143.313881	8.311410*	94.690406	7.875239*
RMI	3.763440	0	7.445925	0	5.824404	0
shift	$\lambda = 0.05,\ \phi_1 = -0.2$		$\lambda = 0.1,\ \phi_1 = -0.2$		$\lambda = 0.2,\ \phi_1 = -0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 3.8036\times10^{-8}$	$h = 2.0452044$	$b = 0.0016149$	$h = 4.200333$	$b = 0.067334$	$h = 8.88252$
0.000	370.075490	370.074437	370.035318	370.040056	370.017072	370.021854
0.001	362.413120	272.589846*	365.971526	257.425111*	362.407567	254.808465*
0.003	347.604772	178.698291*	358.000435	160.267524*	347.947355	157.288491*
0.005	333.457209	133.038467*	350.232942	116.523643*	334.419802	113.936839*

Table 4. Continued.

shift	$\lambda = 0.05,\ \phi_1 = -0.2$		$\lambda = 0.1,\ \phi_1 = -0.2$		$\lambda = 0.2,\ \phi_1 = -0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 3.8036\times10^{-8}$	$h = 2.0452044$	$b = 0.0016149$	$h = 4.200333$	$b = 0.067334$	$h = 8.88252$
0.007	319.938646	106.040681*	342.663148	91.643263*	321.740040	89.431083*
0.009	307.018935	88.205073*	335.285344	75.588906*	309.833149	73.676321*
0.010	300.774653	81.379575*	331.666704	69.523408*	304.148484	67.735772*
0.030	201.120167	32.306069*	268.119397	27.141638*	218.327779	26.405507*
0.050	136.616346	20.420367*	218.476210	17.187944*	165.516539	16.743211*
0.070	94.217093	15.067503*	179.369496	12.742094*	130.215179	12.432517*
0.090	65.938916	12.022993*	148.318390	10.222777*	105.253345	9.990907*
RMI	3.062808	0	6.078151	0	4.802528	0

*The smallest *ARL* on each shift size according to the case.

For the *ARL* comparison for an ARX(2,1) process on the EWMA and modified EWMA control charts, the parameter values were set as $ARL_0 = 500$; $\lambda =$ 0.05, 0.1, and 0.2; the in-control parameter $\alpha_0 = 1$; shift sizes of 0.001, 0.003, 0.005, 0.007, 0.009, 0.01, 0.03, 0.05, 0.07 and 0.09. The results are reported in Table 5.

Table 5. Comparison of the *ARL* of EWMA and modified EWMA control charts using explicit formulas for an ARX(2,1) with $u = 1,\ v = 1,$ $\delta = 2,\ k = 50\lambda,\ \beta_1 = 0.3$ and $ARL_0 = 500$.

shift	$\lambda = 0.05,\ \phi_1 = 0.1,\ \phi_1 = -0.2$		$\lambda = 0.1,\ \phi_1 = 0.1,\ \phi_1 = -0.1$		$\lambda = 0.2,\ \phi_1 = 0.1,\ \phi_1 = 0.1$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 1.5491\times10^{-8}$	$h = 0.7568155$	$b = 0.00058356$	$h = 1.378502$	$b = 0.014952$	$h = 2.277413$
0.000	500.060978	500.076069	500.324113	500.360304	500.029224	500.022472
0.001	489.114475	267.090908*	494.127012	239.961773*	480.592452	221.762202
0.003	467.997069	138.496237*	481.998019	117.893325*	445.500726	105.295630
0.005	447.870153	93.629467*	470.213191	78.325628*	414.691287	69.234304
0.007	428.683824	70.801160*	458.761545	58.745762*	387.433378	51.680381
0.009	410.390862	56.975292*	447.632483	47.062407*	363.153404	41.295851
0.010	401.565333	51.923525*	442.185713	42.825869*	351.983744	37.546730
0.030	262.306360	19.020934*	347.882344	15.647366*	210.659066	13.700820
0.050	174.194272	11.857221*	276.201620	9.822354*	143.484292	8.635854
0.070	117.530522	8.724851*	221.192183	7.284761*	104.938276	6.434339
0.090	80.526574	6.969808*	178.593387	5.865033*	80.339327	5.203939
RMI	7.449113	0	14.096343	0	9.179088	0
shift	$\lambda = 0.05,\ \phi_1 = 0.2,\ \phi_1 = -0.1$		$\lambda = 0.1,\ \phi_1 = 0.2,\ \phi_1 = 0.1$		$\lambda = 0.2,\ \phi_1 = 0.2,\ \phi_1 = 0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 1.2685\times10^{-8}$	$h = 0.6187533$	$b = 0.00043188$	$h = 1.0175587$	$b = 0.0122187$	$h = 1.85691$
0.000	500.141339	500.150484	500.174945	500.189478	500.055307	500.024614
0.001	489.095541	259.298533*	493.822612	228.318970*	479.729360	214.054265*
0.003	467.792945	132.316088*	481.395919	109.704783*	443.211639	100.188030*
0.005	447.497611	88.958619*	469.329752	72.374872*	411.339382	65.592229*
0.007	428.158412	67.081769*	457.612366	54.098310*	383.287724	48.862371*
0.009	409.726985	53.891761*	446.232438	43.253693*	358.416006	38.999212*
0.010	400.837290	49.083498*	440.665594	39.332505*	347.010479	35.444488*
0.030	260.830440	17.912907*	344.589520	14.318623*	204.669047	12.911676*
0.050	172.580252	11.160506*	272.008910	8.990081*	138.296605	8.143100*
0.070	116.034271	8.212396*	216.634577	6.672959*	100.601078	6.072667*
0.090	79.236668	6.562091*	173.992869	5.378034*	76.705241	4.916320*
RMI	7.883248	0	15.215424	0	9.476407	0

*The smallest *ARL* on each shift size according to the case.

From the results in Tables 4 and 5, it is evident that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every value of λ. For example, in Table 4, when $\phi_1 = 0.2$, $\lambda = 0.05$ and shift = 0.05, the *ARL* is 332.792484 from the EWMA control chart while the *ARL* is 115.984299 from the modified EWMA control chart, which corresponds to the RMI values for the modified EWMA control chart being less than those for the EWMA control chart.

7. Application

In Section 6, we compared the performance of the *ARL* of an ARX(p,r) process on EWMA and modified EWMA control charts by using simulation data. The results show that the *ARL* values derived from the explicit formulas for the modified EWMA control chart were shorter than those for the EWMA control chart in every case. Hence, we applied the explicit formulas for the *ARL*s on the EWMA and modified EWMA control charts for an ARX(1,1) process using 55 real-world data observations on the value of exports and imports of agricultural products to and from Thailand (Unit: Ten billion baht) from January 2016 to July 2019, where the value of the imports is the explanatory variable (data from the Office of Agricultural Economics of Thailand (2019)) to confirm the above results. The parameters were set as $\lambda = 0.05$, 0.1, and 0.2; $\alpha_0 = u = 0.589259$; shift size values of 0.001, 0.003, 0.005, 0.007, 0.009, 0.01, 0.03, 0.05, 0.07, and 0.09; and autoregressive coefficients $\phi_1 = 0.326152$, $\delta = 6.652233$, $v = 10.4918$, $\beta_1 = 0.933313$, and $X_1 = 4.1439$. The results are given in Table 6.

Another *ARL* comparison for an ARX(2,1) process on the modified EWMA control charts was conducted using real-world data on the price of cassava (unit: Baht per kilogram, data from the Office of Agricultural Economics of Thailand (2019)) and diesel oil (unit: Baht per liter, data from Petroleum Authority of Thailand (2019)), with the latter being the explanatory variable. The parameters used were $\lambda = 0.1$, 0.15 and 0.2; $\alpha_0 = u = 0.136281$; shift size values of 0.0001, 0.0003, 0.0005, 0.0007, 0.0009, 0.001, 0.003, 0.005, 0.007, and 0.009; and autoregressive coefficients $\phi_1 = 0.623567$ and $\phi_2 = 0.292098$, $\delta = 0$, $v = 1.88$, $\beta_1 = 0.064905$, and $X_1 = 25.62$. The results are summarized in Table 7.

Table 6. Comparison of the *ARL* of ARX(1,1) of EWMA and modified EWMA control charts for the value of exports and imports of agricultural products for $ARL_0 = 370$.

shift	$\lambda = 0.05$ $(k = 100\lambda)^a$		$\lambda = 0.1$ $(k = 40\lambda)$		$\lambda = 0.2$ $(k = 60\lambda)$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 3.27104\times10^{-18}$	$h = 0.007128814$	$b = 1.3617\times10^{-13}$	$h = 0.0044711692$	$b = 5.446519\times10^{-12}$	$h = 0.0153495067$
0.000	374.505189	374.510271	370.015353	370.013368	370.026479	370.026204
0.001	348.463281	80.027591*	348.920418	79.529679*	276.359424	71.606027*
0.003	301.920274	31.405097*	310.520857	31.227647*	178.925791	27.747820*
0.005	261.864143	19.690008*	276.644980	19.575453*	128.916300	17.383968*
0.007	227.356776	14.422145*	246.734101	14.334171*	98.643720	12.748838*
0.009	197.600498	11.428518*	220.280627	11.355250*	78.453653	10.121645*
0.010	184.287726	10.369330*	208.221352	10.301213*	70.713067	9.193284*
0.030	48.289170	3.899445*	70.946841	3.863411*	17.089659	3.535219*
0.050	14.280238	2.596183*	26.449591	2.568076*	6.766353	2.397466*
0.070	5.028239	2.046506*	10.869219	2.022750*	3.406272	1.917426*
0.090	2.310792	1.748461*	5.049280	1.727746*	2.092499	1.656939*
RMI	8.975697	0	11.224773	0	4.159967	0

k for the modified EWMA control chart. [a]a constant value. *The smallest *ARL* on each shift size according to the case.

Table 7. Comparison of the *ARL* of ARX(2,1) of EWMA and modified EWMA control charts for the price of cassava and diesel oil for $k = 40\lambda$ and $ARL_0 = 500$.

shift	$\lambda = 0.1$		$\lambda = 0.15$		$\lambda = 0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 1.2971\times10^{-14}$	$b = 6.399136\times10^{-6}$	$b = 2.1239\times10^{-13}$	$b = 9.600579\times10^{-6}$	$b = 4.0308\times10^{-13}$	$b = 1.2801949\times10^{-5}$
0.0000	500.039396	500.039009	500.179242	500.174921	500.226696	500.227872
0.0001	486.931023	108.038894*	473.487004	101.389201*	417.839400	98.180795*
0.0003	461.702687	42.328712*	426.334701	39.364297*	311.758448	37.955938*
0.0005	437.968562	26.452894*	385.987063	24.571033*	246.391305	23.680218*

Table 7. Continued.

shift	$\lambda = 0.1$		$\lambda = 0.15$		$\lambda = 0.2$	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$b = 1.2971\times10^{-14}$	$b = 6.399136\times10^{-6}$	$b = 2.1239\times10^{-13}$	$b = 9.600579\times10^{-6}$	$b = 4.0308\times10^{-13}$	$b = 1.2801949\times10^{-5}$
0.0007	415.503914	19.307837*	351.191004	17.937223*	202.151348	17.289440*
0.0009	394.328865	15.245271*	320.856371	14.171951*	170.215918	13.665109*
0.0010	384.125979	13.807393*	307.136133	12.840435*	157.387407	12.383952*
0.0030	230.568928	5.011479*	146.767998	4.707794*	54.533946	4.564598*
0.0050	141.203211	3.229155*	81.398115	3.062145*	27.781114	2.983359*
0.0070	88.045417	2.471605*	48.796600	2.362680*	16.357609	2.311262*
0.0090	55.861517	2.057219*	30.752391	1.979945*	10.453770	1.943442*
RMI	24.968946	0	18.129071	0	8.344739	0

*The smallest *ARL* on each shift size according to the case.

From the results in Tables 6 and 7, it is evident that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every value of λ. For example, in Table 6 when $\lambda = 0.05$ and shift = 0.009, the *ARL* is 197.600498 from the EWMA control chart while the *ARL* is 11.428518 from the modified EWMA control chart. This corresponds to an RMI value of 0 for the modified EWMA control chart, which is less than that for the EWMA control chart. The results from Tables 6 and 7 are plotted on the charts in Figures. 1 and 2, respectively.

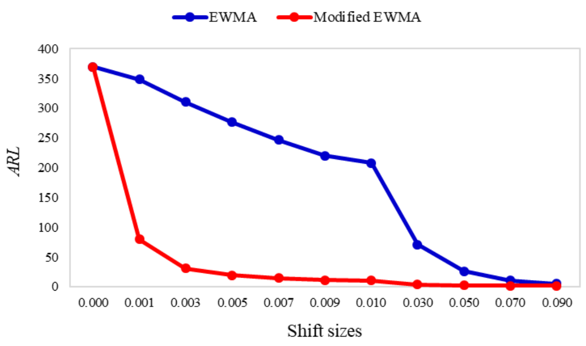

Figure 1. Comparison of the *ARL* for an ARX(1,1) on EWMA and modified EWMA control charts for real data in table 6, where $\lambda = 0.10$.

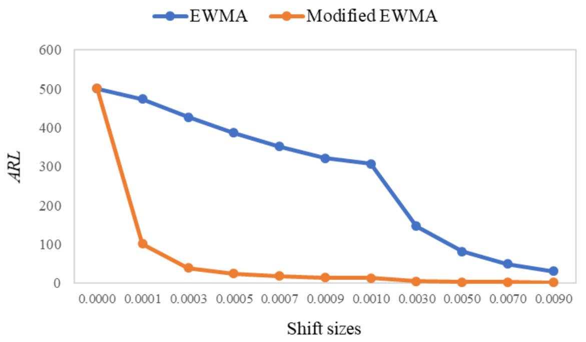

Figure 2. Comparison of the *ARL* for an ARX(2,1) on EWMA and modified EWMA control charts for real data in table 7, where $\lambda = 0.15$.

From Figures 1 and 2, it can be seen that the *ARL* values derived from the explicit formulas for the modified EWMA control chart are less than those for the EWMA control chart for every case. For example, in Figure 1, when shift = 0.009, the *ARL* from the modified EWMA control chart (*ARL* = 11.355250) is less than that of the EWMA control chart (*ARL* = 220.280627).

From Tables 6 and 7 and Figures 1 and 2, it is evident that the *ARL* for the modified EWMA control chart is smaller than that of the EWMA control chart for every case. Such that the *ARL* values derived from the explicit formulas for the modified EWMA control chart outperformed that for the EWMA control chart.

8. Conclusions

In this study, we derived explicit formulas for the *ARL*s on the EWMA and modified EWMA control charts for an ARX(p,r) process with exponential white noise using real-world data observations and compared the performance of the *ARL* of an ARX(p,r) process on both control charts using the RMI. The suggested formulas are easy to calculate and program. The explicit formulas clearly take much less computational time than the numerical Integral Equation method (NIE). Our results show that they performed better for an ARX(p,r) process on the modified EWMA control chart compared to the EWMA control chart for the case of a one-sided shift with constant *k*. However, the conclusions drawn in this study are only applicable to an ARX(p,r) process and may not be relevant for other processes. In future work, it would be of interest to derive explicit formulas for the *ARL* of other control charts and processes using the Fredholm integral equation of the second type technique. Based on the findings, the *ARL* explicit formula for an ARX(p,r) process on the modified EWMA control chart outperformed the EWMA control chart. Thus, the modified EWMA control chart could be applied to other processes.

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