APPLIED MATHEMATICS AND COMPUTATION

Feb. 11, 1998

Re: On the Fourier Transform of the Diamond Kernel of Marcel Riesz

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On The Fourier Transform of the Diamond Kernel of Marcel Riesz

AMNUAY KANANTHAI

ABSTRACT. **In this paper, the operator $\Diamond^k$ is introduced and named as the Diamond operator iterated k-times and is definded by**

$$\Diamond^k = \left(\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \dots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 \right)^k$$

where n is the dimension of the Euclidean space R^n, k is a nonnegative integer and $p+q=n$. The elementary solution of the operator $\Diamond^k$ is called the Diamond Kernel of Marcel Riesz. In this work we study the Fourier transform of the elementary solution and also the Fourier transform of their convolutions.

Keywords : Diamond operator, Fourier transform, Kernel of Marcel Riesz, Dirac delta distributions, Tempered distribution

(1991) AMS Mathematics Subject Classification: 46F10

1. INTRODUCTION

Consider the equation

$$\Diamond^k u(x) = \delta \qquad (1.1)$$

where $\Diamond^k$ is the Diamond operator iterated k-times ($k = 0, 1, 2, ...$) with $\Diamond^0 u(x) = u(x)$ and is defined by

$$\Diamond^k = \left(\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + ... + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right)^k \qquad (1.2)$$

where $p + q = n$, the dimension of the Euclidean space R^n and $u(x)$ is the generalized function, $x = (x_1, x_2, ...x_n) \in R^n$ and δ is the Dirac-delta distribution.

A. Kananthai([1], Theorem 1.3) has shown that the solution of convolutions form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is an unique elementary solution of (1.1) where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (2.2) and (2.4) respectively with $\alpha = 2k$. Now $(-1)^k S_{2k}(x) * R_{2k}(x)$ is a generalized function, see [1], and is called the Diamond Kernel of Marcel Riesz. In this paper we study the Fourier transform of $(-1)^k S_{2k}(x) * R_{2k}(x)$ and the Fourier transform of $[(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]$ where k and m are nonnegative integers.

2. Preliminaries

Definition 2.1 Let $E(x)$ be a function defined by

$$E(x) = \frac{|x|^{2-n}}{(2-n)\omega_n} \qquad (2.1)$$

where $x = (x_1, ..., x_n) \in R^n$, $|x| = (x_1^2 + ... + x_n^2)^{1/2}$ and $\omega_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ is a surface area of the unit sphere.

It is well known that $E(x)$ is an elementary solution of the Laplace operator Δ, that is $\Delta E(x) = \delta$ where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and δ is the Dirac-delta distribution.

Definition 2.2 Let $S_\alpha(x)$ be a function definded by

$$S_\alpha(x) = 2^{-\alpha}\pi^{-n/2}\Gamma\left(\frac{n-\alpha}{2}\right)\frac{|x|^{\alpha-n}}{\Gamma(\frac{\alpha}{2})} \qquad (2.2)$$

where α is a complex parameter, $|x| = (x_1^2 + ... + x_n^2)^{1/2}$, $x = (x_1, ..., x_n) \in R^n$.

$S_\alpha(x)$ is called the Elliptic Kernel of Marcel Riesz. Now $S_\alpha(x)$ is an ordinarly function for $Re(\alpha) \geq n$ and is a distribution of α for $Re(\alpha) < n$.

From (2.1) and (2.2) we obtain

$$E(x) = -S_2(x) \tag{2.3}$$

Definition 2.3 Let $x = (x_1, ..., x_n)$ be a point in R^n and write

$$V = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2$$

where $p + q = n$. Define $\Gamma_+ = \{x \in R^n : x_1 > 0 \text{ and} V > 0\}$ designates the interior of the forward cone and denote $\overline{\Gamma}_+$ by its closure and the following function introduced by Y. Nozaki ([4], p. 72) that

$$R_\alpha(x) = \begin{cases} \dfrac{V^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases}$$

Here $R_\alpha(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz and α is a complex parameter and n is the dimension of the space R^n.

The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}$$

Here $R_\alpha(x)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$.

Let $supp R_\alpha(x) \subset \overline{\Gamma}_+$ where $supp R_\alpha(x)$ denote the support of $R_\alpha(x)$.

Definition 2.4 Let f be continuous function, the Fourier transform of f denoted by

$$\mathcal{F}f = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-i\xi.x}\, dx \tag{2.5}$$

where $x = (x_1, \ldots, x_n) \in R^n$, $\xi = (\xi_1, \ldots, \xi_n) \in R^n$ and $\xi.x = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$. Sometimes we write $\mathcal{F}f(x) \equiv \hat{f}(\xi)$. By (2.5), we can define the inverse of Fourier transform of $\hat{f}(\xi)$ by

$$f(x) = \mathcal{F}^{-}1\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{R} e^{i\xi.x} \hat{f}(\xi)\, d\xi \tag{2.6}$$

If f is a distribution with compact supports by A.H. Zemanian ([5], Theorem 7.4-3, p. 187), (2.5) can be written as

$$\mathcal{F}f = \frac{1}{(2\pi)^{n/2}} < f(x), e^{-i\xi x} > \tag{2.7}$$

Lemma 2.1 The functions $S_\alpha(x)$ and $R_\alpha(x)$ defined by (2.2) and (2.4) respectively, for $Re(\alpha) < n$ are homogeneous distributions of order $\alpha - n$.

Proof. Since $R_\alpha(x)$ and $S_\alpha(x)$ satisfy the Euler equation, that is $(\alpha - n)R_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R_\alpha(x)$ and $(\alpha - n)S_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} S_\alpha(x)$, we have $R_\alpha(x)$ and $S_\alpha(x)$ are homogeneous distribution of order $\alpha - n$.

W.F.Donoghue ([3], p. 154 - 155) has proved that every homogeneous distribution is a tempered distribution.

That completes the proof.

Lemma 2.2 (The convolution of tempered distributions) *The convolution $S_\alpha(x) * R_\alpha(x)$ exists and is a tempered distribution.*

Proof. Choose $suppR_\alpha(x) = K \subset \overline{\Gamma}_+$ where K is a compact set. Then $R_\alpha(x)$ is a tempered distribution with compact support and by W.F. Donoghue ([3], p. 156 - 159) $S_\alpha(x) * R_\alpha(x)$ exists and is a tempered distribution.

Lemma 2.3. *Given the equation $\diamondsuit^k u(x) = \delta$ where the operator $\diamondsuit^k$ is defined by (1.2), $x = (x_1, \ldots, x_n) \in R^n$, k is nonnegative integer and δ is the Dirac-delta distribution. Then $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the equation where $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (2.2) and (2.4) respectively with $\alpha = 2k$.*

Proof. By Lemma 2.2, for $\alpha = 2k$, the distribution $(-1)^k S_{2k}(x) * R_{2k}(x)$ exists and is a tempered distribution.

Now the distribution $(-1)^k S_{2k}(x)$ is obtained by the convolution

$$\underbrace{E(x) * E(x) * \ldots * E(x)}_{k-times} = \underbrace{(-S_2(x)) * (-S_2(x)) * \ldots * (-S_2(x))}_{k-times}$$

where $E(x)$ is defined by (2.1) and by (2.3)

A.Kananthai ([2], Lemma 2.5) has shown that $\underbrace{-S_2(x) * (-S_2(x)) * \dots * (-S_2(x))}_{k-times} = (-1)^k S_{2k}(x)$ is an elementary solution of the Laplace operator Δ^k iterated k-times. By (1.2), $\diamondsuit^k$ can be written as

$$\diamondsuit^k = \Box^k \Delta^k \tag{2.8}$$

where $\Box^k = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \dots - \frac{\partial^2}{\partial^2_{p+q}})^k$

and $\Delta^k = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2})^k, p+q=n$

By A. Kananthai([1], Theorem 3.1) $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is unique elementary solution of the opertor $\diamondsuit^k$ as requied.

Lemma 2.4 (The Fourier transform of $\diamondsuit^k \delta$)
$\mathcal{F}\diamondsuit^k\delta = \frac{1}{(2\pi)^{n/2}}\left((\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2\right)^k$,
where $\mathcal{F}$ is the Fourier transform defined by (2.5) and if the norm of ξ is given by $\|\xi\| = (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2)^{1/2}$ then

$$|\mathcal{F}\diamondsuit^k\delta| \leq \frac{1}{(2\pi)^{n/2}}\|\xi\|^{4k} \tag{2.9}$$

that is $\mathcal{F}\diamondsuit^k\delta$ is bounded and continuous on the space S' of tempered distribution. Moreover, by (2.6)

$\diamondsuit^k\delta = \mathcal{F}^{-}1\frac{1}{(2\pi)^{n/2}}\left((\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^2\right)^k$

Proof. By (2.7)

$$\mathcal{F}\diamondsuit^k\delta = \frac{1}{(2\pi)^{n/2}} < \diamondsuit^k\delta, e^{-i\xi.x} >$$
$$= \frac{1}{(2\pi)^{n/2}} < \delta, \diamondsuit^k e^{-i\xi.x} >$$
$$= \frac{1}{(2\pi)^{n/2}} < \delta, \Box^k \Delta^k e^{-i\xi.x} > \text{ by (2.8)}$$
$$= \frac{1}{(2\pi)^{n/2}} < \delta, (-1)^k(\xi_1^2 + \xi_2^2 + ... + \xi_n^2)^k \Box^k e^{-i\xi.x}>$$
$$= \frac{1}{(2\pi)^{n/2}} < \delta, (-1)^k(\xi_1^2 + \xi_2^2 + ... + \xi_n^2)^k (-1)^k$$
$$\times (\xi_1^2 + \xi_2^2 + ... + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - ... - \xi_{p+q}^2)^k e^{-i\xi.x} >$$
$$= \frac{1}{(2\pi)^{n/2}}(-1)^{2k}((\xi_1^2 + ... + \xi_n^2)^k \times (\xi_1^2 + ... + \xi_p^2 - \xi_{p+1}^2 - ... - \xi_{p+q}^2)^k$$
$$= \frac{1}{(2\pi)^{n/2}}\left((\xi_1^2 + ... + \xi_p^2)^2 - (\xi_{p+1}^2 + ... + \xi_{p+q}^2)^2\right)^k.$$

Now

$$|\mathcal{F}\diamondsuit^k\delta| = \frac{1}{(2\pi)^{n/2}}\left(|\xi_1^2 + ... + \xi_n^2||\xi_1^2 + ... + \xi_p^2 - \xi_{p+1}^2 - ... - \xi_{p+q}^2|\right)^k$$
$$\leq \frac{1}{(2\pi)^{n/2}}\left(|\xi_1^2 + ... + \xi_n^2|\right)^k$$
$$= \frac{1}{(2\pi)^{n/2}}\|\xi\|^{4k}.$$

where $\|\xi\| = (\xi_1^2 + ... + \xi_n^2)^{1/2}$, $\xi_i (i = 1, 2, ..., n) \in R$. Hence we obtain (2.9) and $\mathcal{F}\diamondsuit^k\delta$ is bounded and continuous on the space S' of tempered distribution.

Since $\mathcal{F}$ is 1-1 transformation from the space S' of tempered distribution to the real space R, then by (2.6)

$$\diamondsuit^k\delta = \frac{1}{(2\pi)^{n/2}}\mathcal{F}^{-1}\left((\xi_1^2 + ... + \xi_p^2)^2 - (\xi_{p+1}^2 + ... + \xi_{p+q}^2)^2\right)^k$$

That completes the proof.

3. Main Results

Theorem 3.1 $\mathcal{F}((-1)^k S_{2k}(x)*R_{2k}(x)) = \dfrac{1}{(2\pi)^{n/2}[(\xi_1^2+\ldots+\xi_p^2)-(\xi_{p+1}^2+\ldots+\xi_{p+q}^2)^2]^k}$

and

$$|\mathcal{F}((-1)^k S_{2k}(x)*R_{2k}(x))| \leq \frac{1}{(2\pi)^{n/2}}.M \text{ for a large } \xi_i \in R \tag{3.1}$$

where M is a constant. That is $\mathcal{F}$ is a bounded and continuous on the space S' of tempered distributions.

Proof. By Lemma 2.3 $\Diamond^k((-1)^k S_{2k}(x)*R_{2k}(x)) = \delta$ or $(\Diamond^k\delta)*[(-1)^k S_{2k}(x)*R_{2k}(x)] = \delta$.

Taking the Fourier transform both sides, we obtain

$$\mathcal{F}\big((\diamond^k\delta)*[(-1)^k S_{2k}(x)*R_{2k}(x)]\big) = \mathcal{F}\delta = \frac{1}{(2\pi)^{n/2}} \ .$$

By (2.7) $\dfrac{1}{(2\pi)^{n/2}} < (\Diamond^k\delta)*[(-1)^k S_{2k}(x)*R_{2k}(x)], e^{-i\xi.x} >= \dfrac{1}{(2\pi)^{n/2}}$.

By the definition of convolution

$$\frac{1}{(2\pi)^{n/2}} < (\Diamond^k\delta), < [(-1)^k S_{2k}(r)*R_{2k}(r)], e^{-i\xi.(x+r)} >> = \frac{1}{(2\pi)^{n/2}}$$

$$\frac{1}{(2\pi)^{n/2}} < [(-1)^k S_{2k}(r)*R_{2k}(r)], e^{-i\xi.r} >< (\Diamond^k\delta), e^{-i\xi.x} > = \frac{1}{(2\pi)^{n/2}}$$

$$\mathcal{F}\big([(-1)^k S_{2k}(r)*R_{2k}(r)]\big)(2\pi)^{n/2}\mathcal{F}(\Diamond^k\delta) = \frac{1}{(2\pi)^{n/2}} \ .$$

By Lemma 2.4,

$\mathcal{F}\big([(-1)^k S_{2k}(x)*R_{2k}(x)]\big)\big((\xi_1^2+\ldots+\xi_p^2)^2-(\xi_{p+1}^2+\ldots+\xi_{p+q}^2)^2\big)^k = \dfrac{1}{(2\pi)^{n/2}}$. It follow that

$$\mathcal{F}\big([(-1)^k S_{2k}(x)*R_{2k}(x)]\big) = \frac{1}{(2\pi)^{n/2}[(\xi_1^2+\ldots+\xi_p^2)^2-(\xi_{p+1}^2+\ldots+\xi_{p+q}^2)^2]^k}$$

Now

$$\frac{1}{[(\xi_1^2+\ldots+\xi_p^2)^2-(\xi_{p+1}^2+\ldots+\xi_{p+q}^2)^2]^k} = \frac{1}{(\xi_1^2+\ldots+\xi_n^2)^k} \times \frac{1}{(\xi_1^2+\ldots+\xi_p^2-\xi_{p+1}^2-\ldots-\xi_{p+q}^2)^k} \tag{3.2}$$

Let $\xi = (\xi_1, \cdots, \xi_n) \in \Gamma_+$ where Γ_+ defined by definition 2.3. Then $(\xi_1^2 + ... + \xi_p^2 - \xi_{p+1}^2 - ... - \xi_{p+q}^2) > 0$ and for a large ξ_i and a large k, the right hand side of (3.2) tend to zero. It follows that it is bounded by a positive constant M say, that is we obtain (3.1) as required and also by (3.1) $\mathcal{F}$ is continuous on the space S' of tempered distribution.

Theorem 3.2

$$\begin{aligned}
&\mathcal{F}\big([(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]\big) \\
&= (2\pi)^{n/2}\mathcal{F}([(-1)^k S_{2k}(x) * R_{2k}(x)].\mathcal{F}[(-1)^m S_{2m}(x) * R_{2m}(x)] \\
&= \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{[(\xi_1^2 + ... + \xi_p^2)^2 - (\xi_{p+1}^2 + ... + \xi_{p+q}^2)^2]^{k+m}}.
\end{aligned}$$

where k and m are nonnegative integers and $\mathcal{F}$ is bounded and continuous on the space S' of tempered distribution.

Proof. Since $R_{2k}(x)$ and $S_{2k}(x)$ are tempered distribution with compact supports, we have

$$\begin{aligned}
&[(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)] \\
&= (-1)^{k+m}\big(S_{2k}(x) * S_{2m}(x)\big) * \big(R_{2k}(x) * R_{2m}(x)\big) \\
&= (-1)^{k+m}\big(S_{2(k+m)}(x) * R_{2(k+m)}(x)\big)
\end{aligned}$$

by W.F.Donoghue ([3], p 156-159) and A.Kananthai ([2], lemma 2.5). Taking Fourier transform both sides and use Theorem 3.1 we obtain

$$\begin{aligned}
&\mathcal{F}\big([(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]\big) \\
&= \frac{1}{(2\pi)^{n/2}\big((\xi_1^2 + ... + \xi_p^2)^2 - (\xi_{p+1}^2 + ... + \xi_{p+q}^2)^2\big)^{k+m}} \\
&= \frac{1}{(2\pi)^{n/2}\big((\xi_1^2 + ... + \xi_p^2)^2 - (\xi_{p+1}^2 + ... + \xi_{p+q}^2)^2\big)^{k}} \times \\
&\frac{(2\pi)^{n/2}}{(2\pi)^{n/2}\big((\xi_1^2 + ... + \xi_p^2)^2 - (\xi_{p+1}^2 + ... + \xi_{p+q}^2)^2\big)^{m}} \\
&= (2\pi)^{n/2}\mathcal{F}\big[(-1)^k S_{2k}(x) * R_{2k}(x)\big].\mathcal{F}\big[(-1)^m S_{2m}(x) * R_{2m}(x)\big] \quad .
\end{aligned}$$

Since $(-1)^k S_{2(k+m)}(x) * R_{2(k+m)}(x) \in S'$ the space of tempered distribution and by Theorem 3.1 we obtain $\mathcal{F}$ is bounded and continuous on S'.

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APPENDIX 4

Papers Submitted for International Publications

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July 30, 1998

Professor G.P. Felcher
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IL 60439-8296, U.S.A

Dear Prof. G.P. Felcher,

Three copies of the revised manuscript, your file copy #JR 2464, a list of the changes, two copies of the original improved figures, and a "Transfer of Copyright Agreement" form are enclosed.

The manuscript has been revised to comply with the reviewer's comments and suggestions. Grammar, sentence structure and spelling errors have been corrected by Edtext, the Academics' Editing Service.

Thank you in advance for your consideration.

Yours sincerely,

Mayuree Natenapit

Dr. Mayuree Natenapit
Assoc. Prof. of Physics

Capture radius of magnetic particles in random cylindrical matrices in high gradient magnetic separation

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An effective medium treatment (EMT) was used to model the magnetic field around randomly distributed magnetic cylindrical fine wires and applied to calculate the capture radius of paramagnetic particles in a filter operating either in the longitudinal or transverse design mode. This paper reports capture radius as a function of the ratio of magnetic velocity to fluid entrance velocity with a magnetic parameter which determines the strength of the magnetic short-range force, as a parameter. Finally, comparisons of the results based on the EMT approach, with those obtained by using the single-wire model, are given along with discussion on the criteria for validity of the simple single-wire model.

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I. INTRODUCTION

The theory of magnetic filtration has long been investigated; however most of the theories published are based on the simplest single collector model. Among those, the formerly developed theory by Watson[1] has been referred to by many authors. This theory explains capture of the weakly magnetic particles carried by fluid of potential flow type, defined by Renold's number $Re = \rho V_0 a/\eta >> 1$, where ρ, V_0, η and a are the fluid density, entrance velocity, viscosity and collector radius, respectively. The theoretical model used consists of an isolate fine ferromagnetic cylindrical wire in the background of a uniform applied magnetic field. Later, Watson[2] calculated capture radius of paramagnetic particles in a filter consisting of fine ferromagnetic wires using the same approach as reported in the previous publication[1]. This paper includes analysis of the relation between the capture radius and the external uniform magnetic field with a magnetic parameter K_S which measures the short-range force as a parameter.

Particle capture in the random matrix at low field intensity limit has been treated by Sheerer et al.[3]. The capture radius of a single wire with arbitrary orientation with respect to the applied magnetic field direction was evaluated and used to determine the mean capture radius in describing an overall filter efficiency. In this research, the single-wire theory is generalized and the results of Watson[2] are extended by using the effective medium treatment (EMT) to predict the magnetic field around the filter matrices consisting of parallel wires distributed randomly. The same treatment was applied to a similar system of random sphere assemblage presented by Moyer et al.[4,5] and Natenapit[6]. The capture radius results of this study were reported and compared with those of Watson[2] based on the single-wire model. Finally, the criteria for validity of the single-wire model used to determine magnetic field around the filter matrices are discussed.

II. THE MAGNETIC FIELD AND FORCES

To determine the magnetic field around parallel cylindrical wires of high permeability, which are randomly distributed in a formerly uniform external magnetic field applied perpendicular to the wires' axes, the effective medium treatment originally conceived by Hashin[7] to describe the effective conductivity of spherical particulate composites was employed. In this approach, the system of magnetic cylinders and surrounding fluid medium is considered to be composed of cylindrical cells, each containing one of the cylinders. The ratio of the cylinder to cell volume (a^2/b^2) is set equal to the packing fraction of cylinders in the medium (F). Adjacent to each cylinder (permeability μ_s) is the surrounding fluid medium (permeability μ_f). In this model, only a representative cell is considered while the neighbor cells are replaced by a homogeneous medium with effective permeability μ^* to be determined. Self-consistency is achieved by requiring the magnetic induction averaged over the composite cylinder (cylindrical wire plus fluid shell) to be the volume average of the magnetic induction over the effective medium[7] (see (A13)). Taking $\mathbf{H}_0$ along x axis and the wire crossection on the xy plane, the following equations were obtained (see Appendix for details)

$$\mathbf{H} = A\,H_0\left[\left(1+\frac{K_c}{r_a^2}\right)\cos\theta\,\hat{r} - \left(1-\frac{K_c}{r_a^2}\right)\sin\theta\,\hat{\theta}\right],\quad 1 < r_a < \frac{b}{a} \qquad (1.1)$$

$$= H_0\,,\qquad \frac{b}{a} < r_a < \infty \qquad (1.2)$$

where, $A = \dfrac{1}{1-FK_c}$, $K_c = \dfrac{\nu-1}{\nu+1}$, $\nu = \mu_s/\mu_f$ and $r_a = r/a$.

Alternatively and equivalently, the effective permeability has been defined interms of the magnetic energy integral and determined by using variational theorems[8]. The consistent expression for the effective permeability μ^* has been obtained. From Eq. (1.1), one can see that the magnetization of the matrix increases the local field in the fluid shell, depending on the overall shape of the matrix volume and the magnetic parameter K_c. Eq. (1.2) is true for the EMT model used here, resulting from the boundary condition (i) in the appendix and the obtained μ^* without further assumption on the magnetic field.

The magnetic force acting on a small particle of radius r_p and magnetic susceptibility χ_p located in the fluid of susceptibility χ_f is[9]

$$f_m = \frac{2\pi}{3} r_p^3 \mu_o \chi \vec{\nabla} H^2 , \qquad \chi = \chi_p - \chi_f . \tag{2}$$

The particle is said to be paramagnetic if $\chi_p > \chi_f$ and diamagnetic if $\chi_p < \chi_f$.

Substituting **H** from Eq. (1) into Eq. (2), the magnetic force which depends on the particle radius, external field H_o, magnetic parameters K_c and A^2 is obtained. Figure 1 shows the variation of A^2 as a function of the packing fraction for $K_c = 0.2$ and 2. The other major force to be considered is the viscous drag force which is generally assumed to obey Stokes' law

$$f_d = -6\pi\eta\, r_p (\mathbf{v} - \mathbf{v}_f) . \tag{3}$$

Here, $\mathbf{v}_f$ is the fluid velocity, $\mathbf{v} = dr/dt$ the particle velocity, and η the viscosity.

III. EQUATIONS OF MOTION

By using the magnetic field developed here and the single-wire potential flow field, the equations of motion similar to those reported by Watson[2] were obtained as follows :

$$\frac{dr_a}{dt} = \left(\frac{V_o}{a}\right)\left(1-\frac{1}{r_a^{\,2}}\right)\cos(\theta-\alpha)-\left(\frac{V_m}{a}\right)A^2\left[\frac{K_c}{r_a^{\,5}}+\frac{\cos 2\theta}{r_a^{\,3}}\right], \qquad (4)$$

$$r_a\frac{d\theta}{dt} = -\left(\frac{V_o}{a}\right)\left(1+\frac{1}{r_a^{\,2}}\right)\sin(\theta-\alpha)-\left(\frac{V_m}{a}\right)A^2\frac{\sin 2\theta}{r_a^{\,3}}, \qquad (5)$$

$$\frac{dz_a}{dt} = 0, \qquad (6)$$

where, $\alpha = 0$ or $\frac{\pi}{2}$ for a filter in longitudinal ($\mathbf{H_0} \parallel \mathbf{V_0}$) or transverse ($\mathbf{H_0} \perp \mathbf{V_0}$) design, respectively. Here, the magnetic velocity $V_m = \frac{4}{9}\frac{\chi\mu_o H_o^{\,2} r_p^{\,2} K_c}{\eta a}$ is multiplied by the factor $A^2 = \frac{1}{(1-FK_c)^2}$ to account for the influence of the neighboring wires on the pattern of magnetic field around the filter matrices . It is noted that the magnetic parameter K_c is equivalent to the familiar magnetic parameter $K_S = \frac{M_s}{2\mu_o H_o}$ for a single-wire model. For a normal matrix with a very low coercive force the maximum value of $K_S = 1$. $K_s > 1$ can only occur for an hysteretic matrix[10].

The equations of motion are solved numerically for particle trajectories at varying initial positions on the xy plane. The inspection of the particle trajectories yields the critical capture radius (R_C) which depends on the following parameters V_m/V_o, F and K_C.

IV. RESULTS AND DISCUSSION

In this research, capture radius for paramagnetic particles as a function of the ratio of magnetic velocity to fluid entrance velocity (V_m/V_o) in both longitudinal ($H_o \| V_o$) and transverse ($H_o \perp V_o$) magnetic filters with parameters K_c = 0.2 and 2 were determined. Three cases of the magnetic filters with different values of packing fraction are considered. First, for a very dilute limit of filter packing fraction ($F = 0.0001$) R_c as a function of V_m/V_o is shown in Figs. 2 and 3. In this limit of packing fraction, $A^2 \cong 1$ for all values of K_c as can be observed from Fig. 1. This indicates that the EMT results of R_c obtained are consistent with the corresponding single-wire model results reported by Watson[2]. These are confirmed for the case of magnetic filters operating in longitudinal and transverse modes as shown in Figs. 2 and 3, respectively. Furthermore, Fig. 1 also indicates that the single-wire model is a good approximation for all values of K_c up to filter packing fraction $F \sim 0.05$.

Secondly, for a filter packing fraction $F = 0.1$, the relation between R_c and V_m/V_o based on the EMT magnetic field developed here, are compared with those based on the single-wire model as shown in Figs. 4 and 5 for longitudinal and transverse modes, respectively. Again two values of $K_c = 0.2$ and 2 were used. For all cases, the EMT results for R_c are higher than the corresponding single-wire model results; however, the difference is insignificant for $K_c = 0.2$, especially for the longitudinal mode. Thirdly, for a higher value of packing fraction ($F = 0.2$), the similar dependence of R_c on V_m/V_o are illustrated in Figs. 5 and 6 for the longitudinal and transverse modes, respectively. Here the difference between the EMT and single-wire results for $K_c = 2$ is more pronounced than the former case of a lower packing fraction $F = 0.1$. However, the difference is still very little for the smaller magnetic parameter $K_c = 0.2$.

V. CONCLUSION

The two dimensional description of the flow field around cylindrical matrix wires is applied to a three dimensional problem and the geometry used is similar to that used by Kuwabara[11] to discuss the flow around cylindrical matrix wires. It should be noted that the changes in flow produced by the presence of the wire falls as r_a^{-2} and so these are considerably more important at low operating parameter V_m/V_0 , than the changes in magnetic field produced by the matrix where there is a r_a^{-3} and a r_a^{-5} dependence for cylinders. In the case of spheres the fall off is even more rapid[12] and the capture for a system of randomly packed spheres is dominated by the short range term r_a^{-7}.

The studies also indicate that the effects of neigboring wires on the magnetic field pattern around the filter matrices may be neglected for a small packing fraction, e.g. $F \cong 0.05$, for all possible values of the magnetic parameter K_c which determines the strength of the magnetic short-range force (r_a^{-5} term in Eq. (4)). Here K_c depends on the ratio of the wire permeability to fluid permeability as $K_c = \dfrac{\mu_s/\mu_f - 1}{\mu_s/\mu_f + 1}$. However, for a higher packing fraction the single-wire approximation is still good only for the lower values of K_c (say < 0.2).

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APPENDIX

To determine the magnetic field in the cell, the boundary value problem of coaxial magnetic cylinders subject to the boundary condition of uniform magnetic field at far away from the cylindrical cell is solved. Taking z axis of cylindrical coordinate along the cylinder axis and let φ be the magnetic potential satisfying Laplace's equation for each region

$$\nabla^2\varphi_o = 0, \quad b < r < \infty \tag{A1}$$

$$\nabla^2\varphi_1 = 0, \quad a < r < b \tag{A2}$$

$$\nabla^2\varphi_2 = 0, \quad 0 < r < a \tag{A3}$$

with the boundary conditions

(i) $\varphi_o(r, \theta) = -H_o r\cos\theta \quad \text{at} \quad r \to \infty,$

(ii) $\dfrac{\partial\varphi_o(b,\theta)}{\partial\theta} = \dfrac{\partial\varphi_1(b,\theta)}{\partial\theta},$

(iii) $\dfrac{\partial\varphi_1(a,\theta)}{\partial\theta} = \dfrac{\partial\varphi_2(a,\theta)}{\partial\theta},$

(iv) $\mu^* \dfrac{\partial\varphi_o(r,\theta)}{\partial r}\Big|_{r=b} = \mu_f \dfrac{\partial\varphi_1(r,\theta)}{\partial r}\Big|_{r=b},$

and

(v) $\mu_f \dfrac{\partial\varphi_1(r,\theta)}{\partial r}\Big|_{r=a} = \mu_s \dfrac{\partial\varphi_2(r,\theta)}{\partial r}\Big|_{r=a}.$

The general solutions of Laplace's Eqs. (A1) - (A3) are

$$\varphi_o(r, \theta) = -H_o r\cos\theta + \sum_{n=1}^{\infty} A_n r^{-n} \cos n\theta, \tag{A4}$$

$$\varphi_1(r,\theta) = \sum_{n=1}^{\infty} [B_n r^n + C_n r^{-n}] \cos n\theta , \qquad (A5)$$

and

$$\varphi_2(r,\theta) = \sum_{n=1}^{\infty} D_n r^n \cos n\theta , \qquad (A6)$$

where the boundary condition (i) was imposed.

Applying the boundary conditions (ii)-(v), the constant coefficients were obtained as follows :

$$A_n = B_n = C_n = D_n = 0, \qquad \text{for} \quad n \neq 1 \qquad (A7)$$

and
$$A_1 = \frac{H_o a^2}{IF} [F(\nu^* + 1)(\nu - 1) - (\nu^* - 1)(\nu + 1)] \qquad (A8)$$

$$B_1 = -\frac{2H_o \nu^*}{I}(\nu + 1) \qquad (A9)$$

$$C_1 = \frac{2H_o a^2 \nu^*}{I}(\nu - 1) \qquad (A10)$$

$$D_1 = -\frac{4H_o \nu^*}{I} \qquad (A11)$$

where $\nu^* = \mu^*/\mu_f$, $\nu = \mu_s/\mu_f$ and $I = [(\nu^* + 1)(\nu + 1) - F(\nu - 1)(\nu^* - 1)]$.

The magnetic field related to φ by

$$H = -\bar{\nabla}\varphi \qquad (A12)$$

is now obtained everywhere by inserting Eqs. (A4) - (A11) into the above equation. However, the results are given interms of the unknown effective permeability μ^*. μ^* is determined self-consistency by requiring the magnetic induction averaged over the composite cell (cylinder plus cell medium) to be the volume average of the magnetic induction over the effective medium. That is

$$F\mu_s < H_2 >_i + (1 - F)\mu_f < H_1 >_i = \mu^* < H_{Eff} >_i , \; (H_{Eff} = -\bar{\nabla}\varphi_o) \qquad (A13)$$

where i referred to x, y or z. Substituting the magnetic field into Eq. (A13) and taking the x component, we obtain the relative effective permeability

$$\nu^* = \frac{\nu(1+F)+(1-F)}{\nu(1-F)+(1+F)}, \qquad (\nu^* = \frac{\mu^*}{\mu_f}, \nu = \frac{\mu_s}{\mu_f}). \qquad \text{(A14)}$$

Then, the magnetic fields in the cell and the effective medium are obtained as

$$\mathbf{H} = A\,H_0\,[(1+\frac{K_c}{r_a^2})\cos\theta\,\hat{r} - (1-\frac{K_c}{r_a^2})\sin\theta\hat{\theta}],\ 1 < r_a < \frac{b}{a} \qquad \text{(A15.1)}$$

$$= \mathbf{H}_0, \qquad \frac{b}{a} < r_a < \infty \qquad \text{(A15.2)}$$

where $A = \frac{1}{1-FK_c}$, $K_c = \frac{\nu-1}{\nu+1}$ and $r_a = r/a$.

We note that in the limit of $F\ (=\frac{a^2}{b^2}) \rightarrow 0$, $\nu^* = 1$ (or $\mu^* = \mu_f$) and Eq. (A15.1) is reduced to the single cylinder solution as expected. For $\mu_s = \mu_f$ (i.e. $K_c = 0$, $A = 1$), the homogeneous magnetic field $\mathbf{H} = \mathbf{H}_0$ is obtained.

List of Figures

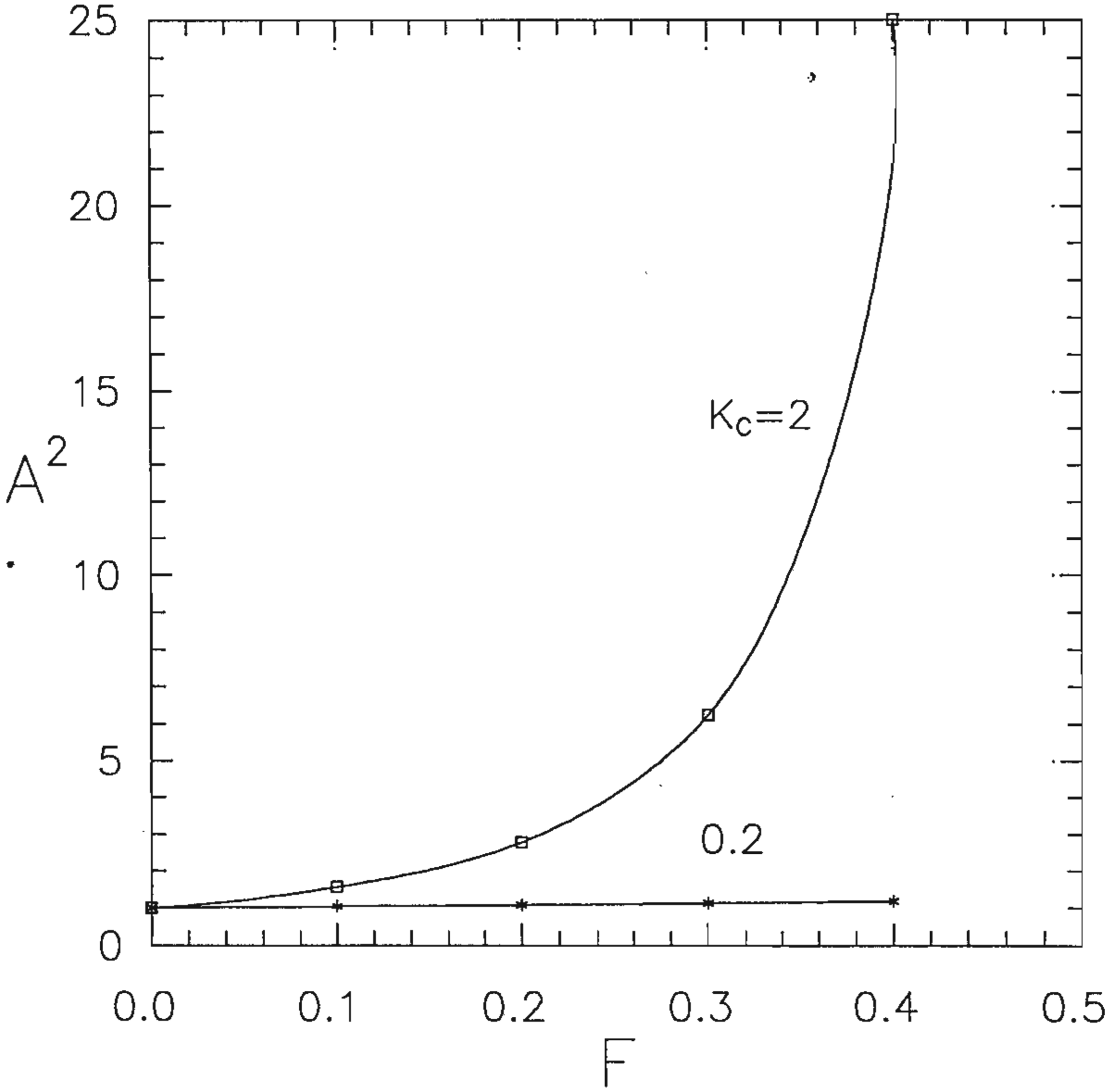

Fig. 1
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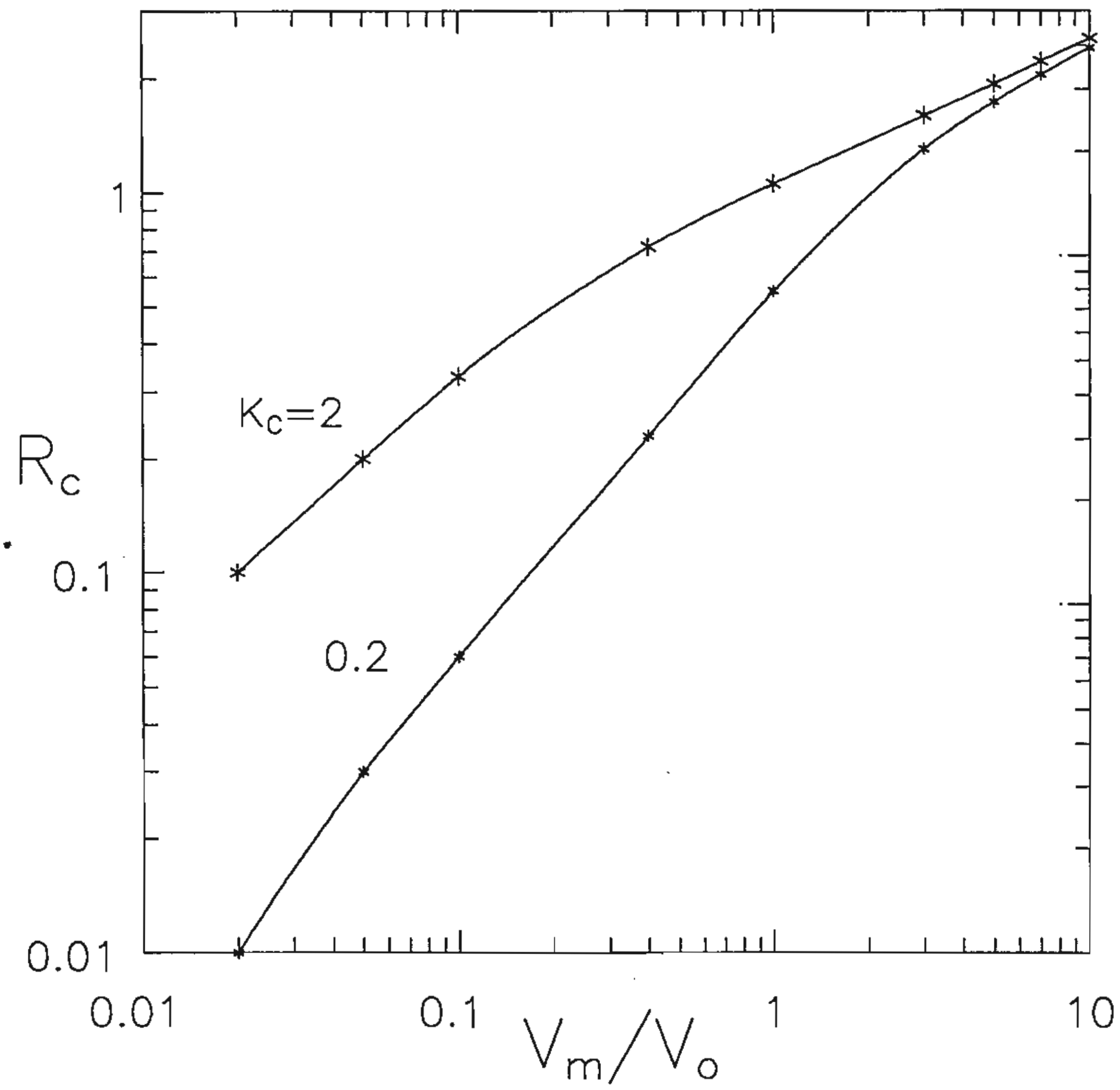

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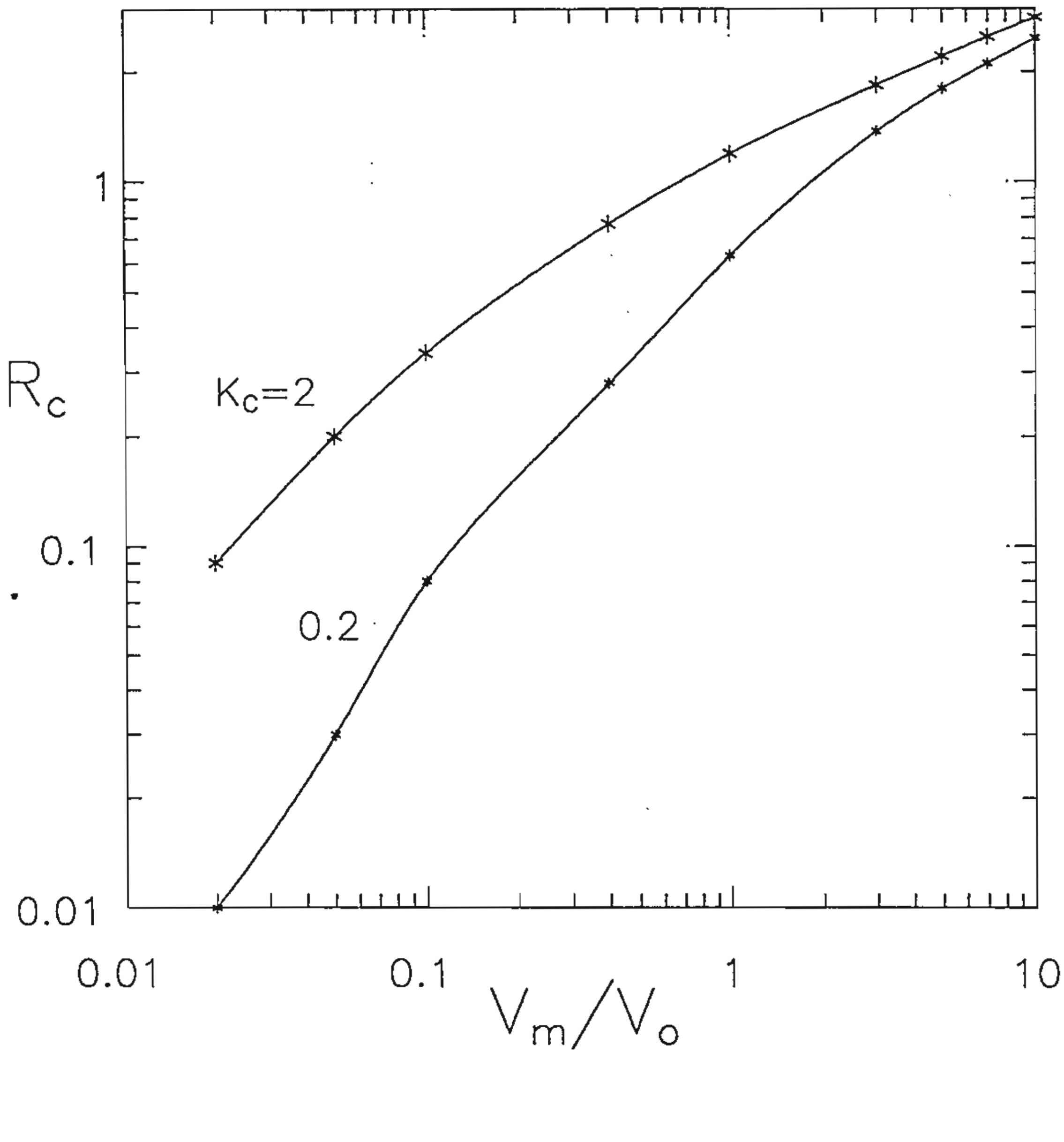

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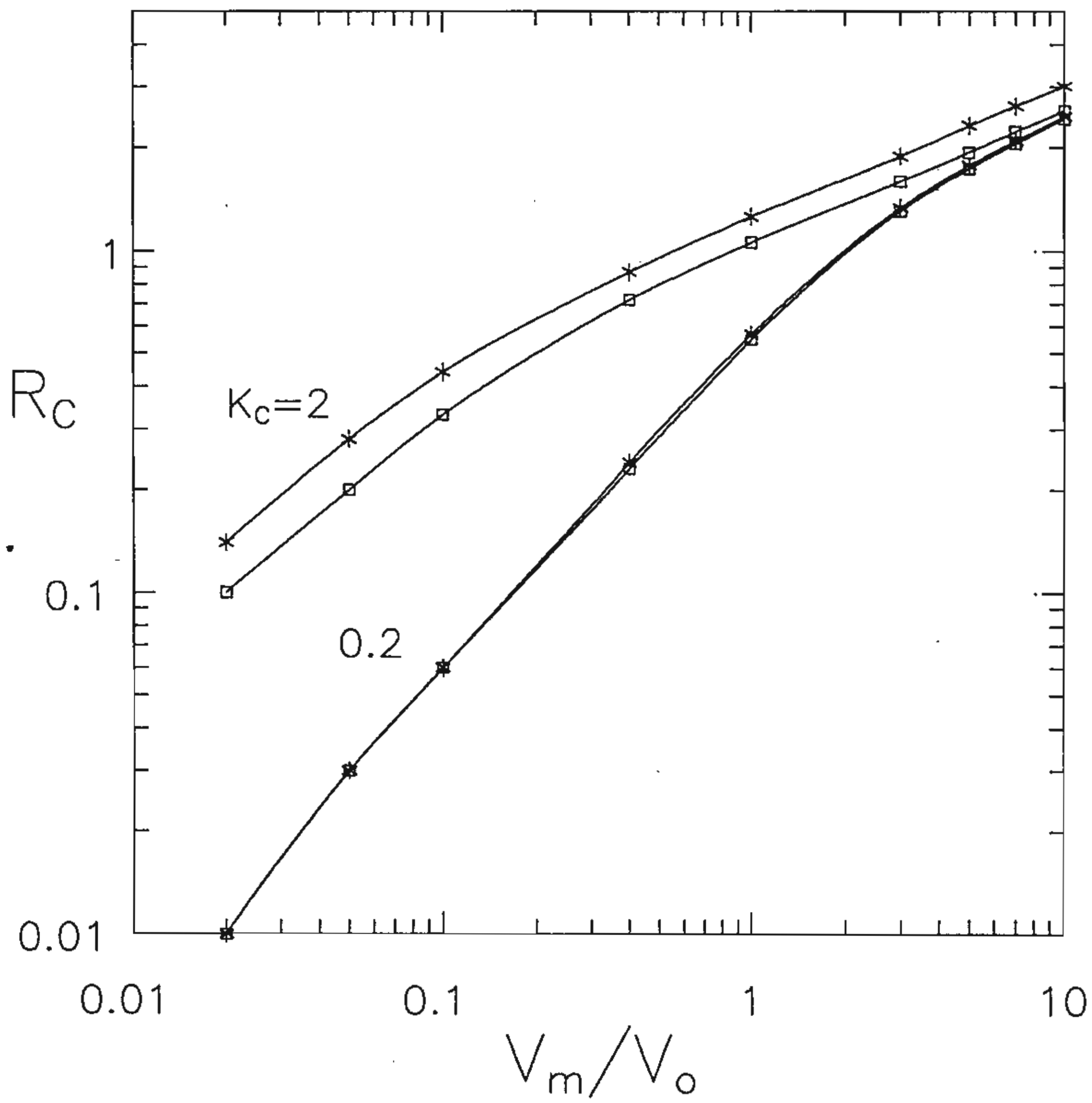

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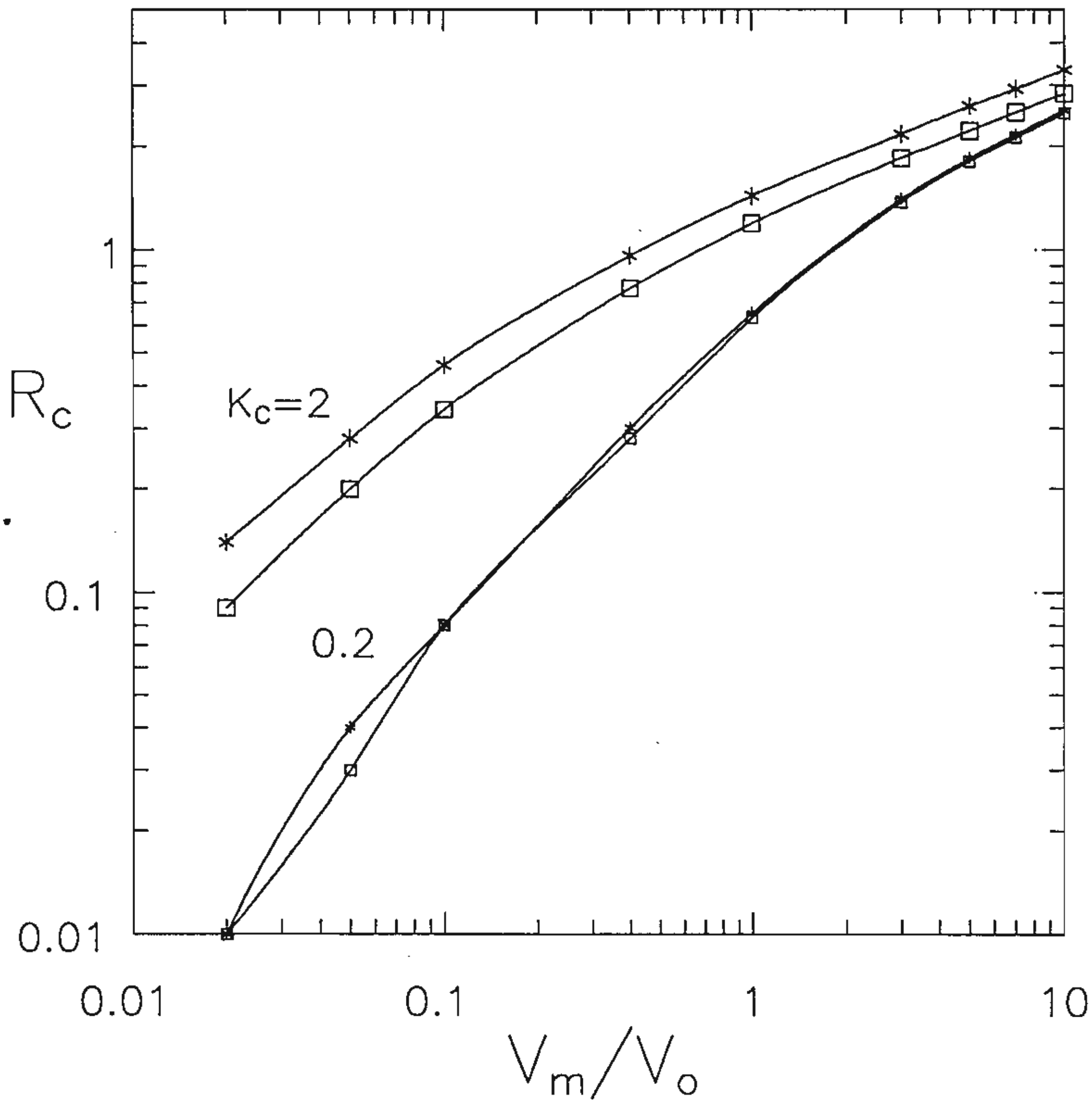

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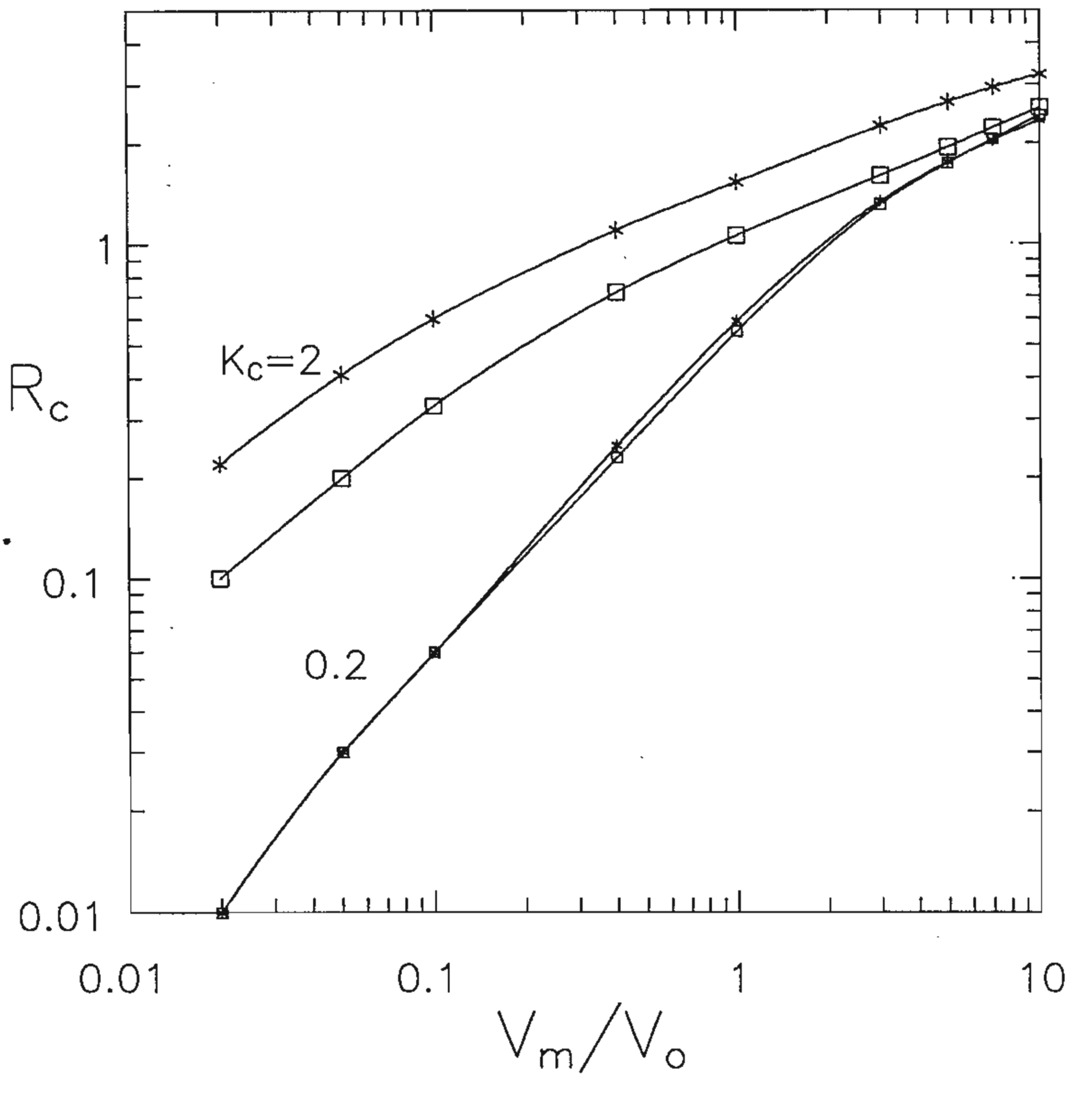

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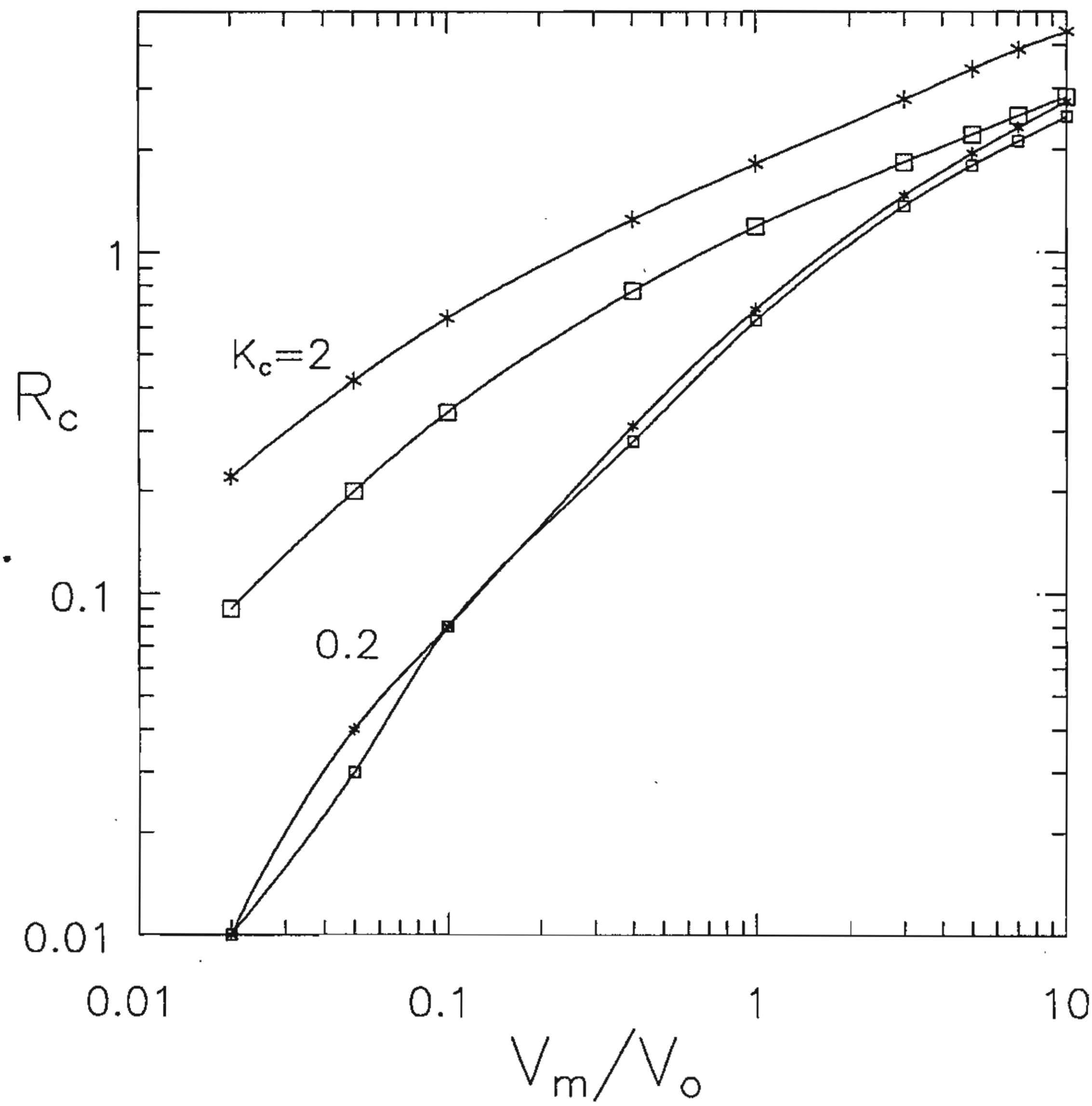

Fig. 7
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APPENDIX 5

Papers Published in International Proceedings

Proceedings of

DUBNA JOINT MEETING

of

INTERNATIONAL SEMINAR

"PATH INTEGRALS: THEORY & APPLICATIONS"

and

5th INTERNATIONAL CONFERENCE

"PATH INTEGRALS FROM meV TO MeV"

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A Model of Electron Transport in Solids: Path Integral Approach

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Abstract

The Feynman path integral method developed by Feynman-Thornber for treating the non-linear electron transport in the polaron problem is applied to the model of an electron moving in a two-dimensional random system in the present of strong electric and magnetic fields. The random system is assumed to be a Gaussian random potential with Gaussian autocorrelation function. With the implementation of the self-consistent condition, the analytical expression for the average velocity for the model system is derived. Furthermore it is shown that in the lowest order approximation, as well as in the linear limit, the relaxation time expression is obtained. The implication of these results to the Quantum Hall mobility is discussed.

1 Introduction

At the present there are several theoretical approaches to the quantum transport in condensed matter physics such as Wigner function, the density matrices, the Green's functions and the Feynman path integrals. All of these approaches are equivalent to the representation of the quantum nature of transport. Unfortunately there is no a single theory that can unified and described the transport phenomena correctly. All of these theories have their application and computational strength and weakness.

In the simplest Boltzman approach to the transport theory, in solid, one deals with well-defined probability distribution which change in space and time and are governed by the integro-differntial equation involving complicated scattering rates. In the quantum transport theory the problem completely difference, namely, carriers can scatter so rapidly that the scattering process is no longer represented in terms of scattering rate alone. Instead more details of scattering amplitude must be considered and include in the description of the transport phenomena. Furthermore the case of high electric and magnetic fields, the situation is more complicate.

The quantum transport can be handle by the Wigner function, density matrices and the Green's function approaches. In the Wigner function approach one attempt to retain as much as the classical formalism in order to be able to express the result in terms of the momentum or velocity which is of greatest experimental interest. The Wigner function has a maximum flexibility. By contrast the density matrices and the Green's function approach adhere closely to the actual quantum states. These approaches can be obtained greatest sensitivity but are relatively inflexible in studying the nonlinear properties in the present of full scattering.

The Feynman path integral method relying on the influence functional technique in which the sources of dissipation such as phonons, plasmons, fluctuations, has been integrated out. The elimination of fluctuation can then lead to the model influence functional in which the dissipation due to fluctuation can be represented by the interaction with the collection of harmonic oscillator modes in which the translation invariance of the system is preserved. In this paper we will apply the Feynman path integral [1] method to the electron transport in two dimensions.

The starting point for the transport phenomena is to calculate the expectation value of the velocity

$$\vec{v} = \langle \hat{\vec{v}} \rangle = \lim_{t\to\infty} Tr(\dot{\hat{\vec{r}}}\hat{\rho}_t), \tag{1}$$

where $\hat{\rho}_t$ is the density matrix of the system. In the expression of Eq. (1) , $\hat{\rho}_t$ must be known very accurately in order to get a sensible result. At zero temperature, as in the case of a random potential, the system is in a well defined state at all times,

$$\vec{v} = \langle \hat{\vec{v}} \rangle = \lim_{t\to\infty} \int \psi^*(\vec{r},t)\dot{\vec{r}}\psi(\vec{r},t)d\vec{r}, \tag{2}$$

where $\psi(\vec{r},t)$ is the wave function of the system of electrons. In both cases, either the density matrix or the state wave functions must be known very accurately. These problems can be overcome by transforming the expression in such a way as to calculate the gradient of the potential. The Feynman path integral method introduced by Feynman and Thornber [2] for handling the polaron transport theory is used to achieve the general expression. In Section II, we present a model system for electron in two dimensions in the present of strong electric and magnetic fields which is appropriate to the quantum Hall problem [3]. However, the present approach could be applied to any dimensions. In Section III, we derive the equation of motion. In section IV, the self-consistent condition is implemented. Section V is devoted to numerical result and the final Section is devoted to the discussion.

2 Model System

We consider a model of an electron moving in a two-dimensional random potential $V(\vec{r})$ in the presence of an electric field $\vec{E}$ and a strong magnetic field $\vec{B}$. The lagrangian of this system is given by

$$L(\dot{\vec{r}},\vec{r};\vec{E}) = \frac{m}{2}\dot{\vec{r}}^2 + \frac{e}{c}\vec{A}\cdot\dot{\vec{r}} + e\vec{E}\cdot\vec{r} - V(\vec{r}) \tag{3}$$

where m = the mass of the electron, $\vec{r} = (x,y)$, $\vec{E} = (E_x, E_y)$ and $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ where $\vec{A}$ is the vector potential of the constant magnetic field. The random potential can be assumed to be the sum of the impurity potentials $v(\vec{r}-\vec{R}_i)$ located at $\vec{R}_i$

$$V(\vec{r}) = \sum_i v(\vec{r}-\vec{R}_i). \tag{4}$$

The density matrix of this system can be written in the Feynman path integral representation as

$$\begin{aligned}\rho(\vec{r}_2,\vec{r}_2';t_2;\{\vec{R}_i\}) &= \int d\vec{r}_1 \int d\vec{r}_1' \rho(\vec{r}_1,\vec{r}_1') \\ &\times \int_{\vec{r}_1}^{\vec{r}_2} D(\vec{r}(\tau)) \int_{\vec{r}_1'}^{\vec{r}_2'} D(\vec{r}'(\tau)) e^{\frac{i}{\hbar}S(\vec{r},\vec{E};\{\vec{R}_i\})} \\ &\times e^{-\frac{i}{\hbar}S(\vec{r}',\vec{E};\{\vec{R}_i\})}\end{aligned} \tag{5}$$

where

$$S(\vec{r},\vec{E};\{\vec{R}_i\}) = \int_{t_1}^{t_2} d\tau L(\dot{\vec{r}},\vec{r},\vec{E};\{\vec{R}_i\}) \tag{6}$$

with

$$L(\dot{\vec{r}},\vec{r},\vec{E};\{\vec{R}_i\}) = \frac{m}{2}\dot{\vec{r}}^2 + \frac{e}{c}\vec{A}\cdot\dot{\vec{r}} + e\vec{E}\cdot\vec{r} - \sum_i v(\vec{r}-\vec{R}_i). \tag{7}$$

In writing Eq. (5), we assumed that the initial density matrix of the electron has been decoupled from the impurities.

We now introduce randomness into the system by assuming that the impurities are located completely at random within the volume V. We further assume that the potential $v(\vec{r}-\vec{R}_i)$ is weak and the density $\frac{N}{V} = \rho$ is so high that the density ρv^2 is finite. Defining the random average as

$$\langle O\rangle_{\{\vec{R}_i\}} = \int \frac{d\vec{R}_1}{V}\int\frac{d\vec{R}_2}{V}\cdots\int\frac{d\vec{R}_N}{V}O \tag{8}$$

and substituting the random average in Eq. (5), we obtain

$$\begin{aligned}\rho(\vec{r}_2,\vec{r}_2';t_2) &= \int d\vec{r}_1 \int d\vec{r}_1' \rho(\vec{r}_1,\vec{r}_1';t_1) \int_{\vec{r}_1}^{\vec{r}_2} D(\vec{r}(\tau)) \int_{\vec{r}_1'}^{\vec{r}_2'} D(\vec{r}'(\tau)) \\ &\times \exp\left[\frac{i}{\hbar}\{S(\vec{r}.\vec{E}) - S(\vec{r}';\vec{E})\} + \frac{i}{\hbar}S(\vec{r},\vec{r}')\right].\end{aligned} \tag{9}$$

Here

$$S(\vec{r};\vec{E}) = \int_{t_1}^{t_2} \left(\frac{1}{2m}\dot{\vec{r}}^2 + \frac{e}{c}\vec{A}\cdot\dot{\vec{r}} + e\vec{E}\cdot\vec{r}\right) d\tau \tag{10}$$

and

$$\begin{aligned} S(\vec{r},\vec{r}') = & \frac{i}{2\hbar}\rho\int_{t_1}^{t_2}\int_{t_1}^{t_2} d\tau d\sigma [W(\vec{r}(\tau)-\vec{r}(\sigma)) \\ & -2W(\vec{r}(\tau)-\vec{r}'(\sigma)) - W(\vec{r}'(\tau)-\vec{r}'(\sigma))] \end{aligned} \tag{11}$$

where

$$W(\vec{r}(\tau)-\vec{r}(\sigma)) = \sum_{\vec{k}} | v(\vec{k}) |^2 e^{i\vec{k}\cdot(\vec{r}(\tau)-\vec{r}(\sigma))}. \tag{12}$$

Here $v(\vec{k})$ is the Fourier component of the potential $v(\vec{r}-\vec{R}_i)$.

3 Equation of Motion

The equation of motion of the system can be obtained by making a variation $\delta\vec{u}(t)$ on the path $\vec{u}(t)$ or on the action function of Eq. (9), where $\vec{u} = \vec{r} - \vec{R}$,

$$\langle\ddot{\vec{u}}\rangle + \frac{e}{c}\langle\dot{\vec{u}}\rangle \times \vec{B} + (\vec{F} - m\ddot{\vec{R}} + \frac{e}{c}\dot{\vec{R}}\times\vec{B}) + \left\langle\frac{\delta S(\vec{u},\vec{u}';\vec{R})}{\delta\vec{u}}\right\rangle = 0. \tag{13}$$

The variation of the action $S(\vec{u},\vec{u}';\vec{R})$ with respect to $\delta\vec{u}(t)$ gives

$$\begin{aligned} \frac{\delta S(\vec{u},\vec{u}';\vec{R})}{\delta\vec{u}(t)} = & \frac{i\rho}{2\hbar}\int_{t_1}^{t_2} d\tau \int_{t_1}^{t_2} d\sigma \nabla_{\vec{u}(t)}\{e^{(\vec{u}(\tau)-\vec{u}(\sigma))\cdot\nabla_x} \\ & -2e^{(\vec{u}(\tau)-\vec{u}'(\sigma))\cdot\nabla_x} + e^{(\vec{u}'(\tau)-\vec{u}'(\sigma))\cdot\nabla_x}\}W(\vec{x}). \end{aligned} \tag{14}$$

Here we have written the correlation function in terms of the operator $\nabla_{\vec{x}}$.

$$W(\vec{u}(\tau)-\vec{u}(\sigma)+\vec{R}(\tau)-\vec{R}(\sigma)) = e^{(\vec{u}(\tau)-\vec{u}(\sigma))\cdot\nabla_x}W(\vec{x}) \tag{15}$$

where $\vec{x}$ stands for $\vec{R}(\tau)-\vec{R}(\sigma)$. Proceeding the differentiation inside of Eq. (14) with respect to $\delta\vec{u}(t_2)$ we have finally

$$\frac{\delta S(\vec{u},\vec{u}';\vec{R})}{\delta\vec{u}(t_2)} = \frac{i\rho}{\hbar}\int_{t_1}^{t_2} d\sigma \left[e^{(\vec{u}(\tau)-\vec{u}(\sigma))\cdot\nabla_x} - e^{(\vec{u}(\tau)-\vec{u}'(\sigma))\cdot\nabla_x}\right]\nabla_{\vec{x}}W(\vec{x}). \tag{16}$$

In the symbol $\langle ... \rangle$ denotes the average with respect to the density matrix of Eq. (9). Thus, in principle, given an explicit form of $W(\vec{x})$ coupled with the aid of generating functional, we can carry out all the analytical calculations. We shall consider a special case in the following sections.

4 Self-Consistent Solution

The equation of motion of Eq. (13) contains fluctuations around the chosen path. Without the external driving force we can neglect the fluctuation $\langle\dot{\vec{u}}\rangle$ and $\langle\ddot{\vec{u}}\rangle$. The equation of motion for the steady state condition then becomes

$$e\vec{E} + \frac{e}{c}\vec{v}_d \times \vec{B} + \left\langle \frac{\delta S(\vec{u}, \vec{u}'; \vec{v}_d)}{\delta \vec{u}(t_2)} \right\rangle = 0. \tag{17}$$

This equation is still quite complicated to handle especially the driving force terms. In this section we shall implement the self-consistent condition which have been proved to be a very successful in the polaron mobility [2, 4, 5].

The self consistency condition corresponds to choosing $\vec{v}_d$ such that

$$e\vec{E} + \frac{e}{c}\vec{v}_d \times \vec{B} = -\langle \nabla_{\vec{u}(\tau)} S(\vec{u}, \vec{u}'; \vec{v}_d) \rangle \tag{18}$$

For the linear electron transport, the equation of motion now becomes

$$e\vec{E} + \frac{e}{c}\vec{v}_d \times \vec{B} = m\vec{v}\frac{1}{\tau_{eff}} \tag{19}$$

where we have obtained the effective relaxation time τ_{eff} as

$$\tau_{eff} = \frac{h E_L (\frac{E_e}{E_L})^{3/2}}{2\sqrt{\pi}\xi_L} \frac{(1 + (\frac{E_\omega}{E_L})^2)^2}{(1 + \frac{E_\omega}{E_L})^{3/2}} e^{-\frac{E_\omega^2 E_L}{64 E_v}} \tag{20}$$

This expression has the following physical interpretation. In the case of strong magnetic field, the effective relaxation time is directly proportional to the energy of the electron but inversely proportional to fluctuation parameter. From (19) we can find the mobility which is defined as

$$\begin{aligned} \vec{\mu} &= \frac{\vec{v}_d}{E} \\ &= \frac{e}{m}\begin{pmatrix} \frac{1}{\tau_{eff}} & -\Omega \\ \Omega & \frac{1}{\tau_{eff}} \end{pmatrix}. \end{aligned} \tag{21}$$

Then the two components of the mobilities are

$$\mu_{xx} = \frac{e}{m}\frac{\tau_{eff}}{[1 + (\Omega\tau_{eff})^2]} \tag{22}$$

$$\mu_{xy} = -\frac{e}{m}\frac{\Omega\tau_{eff}^2}{[1 + (\Omega\tau_{eff})^2]} \tag{23}$$

The conductivities are now defined

$$\sigma_{xx} = \int_{-\infty}^{\infty} (-\frac{df(E)}{dE}) n(E) \mu_{xx} dE \tag{24}$$

and

$$\sigma_{xy} = \int_{-\infty}^{\infty} f(E) \mu_{xy} dE \tag{25}$$

5 Numerical Results

In the previous section we derived the conductivity expressions in the finite temperature expressed in term of the electron mobility $\bar{\mu}$ and the density of states (DOS) of the Quantum Hall system. The mobility components are complicated functions depending on five physical parameters, the Fermi energy, the energy fluctuation parameter ξ_L, the energy E_L associated with the correlation length of the random system, the magnetic energy $E_\Omega = \hbar\Omega$ and finally the energy of the moving electron $E_{\bar{v}}$.

In proceeding to the detailed calculation, it is convenient to express all physical quantities and parameters in term of dimensionless quantities measured with respect to E_L. This scale of units had been proved to be very useful in many previous problems such as heavily doped semiconductor, Urbach Tail, Quantum Hall etc..

The first dimensionless parameter is $E'_\Omega = E_\Omega/E_L$. This parameter measure the strength of a magnetic field. In the Quantum Hall problem E'_Ω usually larger than 1. Then DOS can be expressed as

$$n(E'_{\bar{v}}) \;=\; \frac{n_0 E'_\Omega}{(2\pi\Gamma'^2)^{1/2}} \sum_{n=0}^{\infty} \exp\left\{-\frac{1}{2}\frac{(E'_{\bar{v}}-(n+\frac{1}{2})E'_\Omega)^2}{\Gamma^2}\right\}. \tag{26}$$

Here we have introduced the dimensionless energy $E'_{\bar{v}} = E_{\bar{v}}/E_L$ and dimensionless Landau Level width Γ' defined by

$$\Gamma'^2 = \frac{\Gamma^2}{E_L^2} = \frac{\xi'}{\left(1+\frac{4}{E'_\Omega}\right)} \tag{27}$$

with $\xi' = \xi_L/E_L$. Using these dimensionless parameters, we can express the conductivities in terms of E'_Ω, $E'_{\bar{v}}$ and Γ',

$$\begin{aligned}\sigma_{xx} \;=\; & \frac{e^2}{\pi\hbar}\int_{-\infty}^{\infty}\left(-\frac{df(E'_{\bar{v}})}{dE'_{\bar{v}}}\right)\frac{\omega\tau_{eff}}{[1+(\omega\tau_{eff})^2]} \\ & \times\sum_{n=0}^{\infty}\frac{1}{(2\pi\Gamma'^2)^{1/2}}\exp\left[-\frac{1}{2}\frac{\{E'_{\bar{v}}-(n+\frac{1}{2})E'_\Omega\}^2}{\Gamma'^2}\right]dE'_{\bar{v}}\end{aligned} \tag{28}$$

and similarly for

$$\begin{aligned}\sigma_{xy} \;=\; & \frac{e^2}{\pi\hbar}\int_{-\infty}^{\infty} f(E'_{\bar{v}})\frac{[\omega\tau_{eff}]^2}{[1+(\omega\tau_{eff})^2]} \\ & \times\sum_{n=0}^{\infty}\frac{1}{(2\pi\Gamma'^2)^{1/2}}\exp\left[-\frac{1}{2}\frac{\{E'_v-(n+\frac{1}{2})E'_\Omega\}^2}{\Gamma'^2}\right]dE'_{\bar{v}}\end{aligned} \tag{29}$$

where E'_F is the dimensionless Fermi energy defined through $\frac{E_F}{E_L}$.

We are now ready to perform the numerical evaluation of DOS $n(E'_{\bar{v}})$, $\sigma_{xx}(E'_F)$ and $\sigma_{xy}(E'_F)$. We first choose the strength of the magnetic field which for the most appropriate value is $E'_\Omega = 4$. This value implies that the energy associated with of the magnetic field is four times larger than the energy associated with the random potential.This value is quite close to the value we used in our previous paper to explain the experimental result [8]. We note that for $E_L = 1meV$ and $1 \leq B \leq 10T$, E'_Ω takes the value $1 \leq E'_\Omega \leq 15$ whereas Γ' and ξ'_L are of the order 1-10. As shown in our previous paper, $\xi'_L = 4$ and $E'_\Omega = 5$ give the DOS in agreement with that extract from the experimental work of Einstein et al. [9].

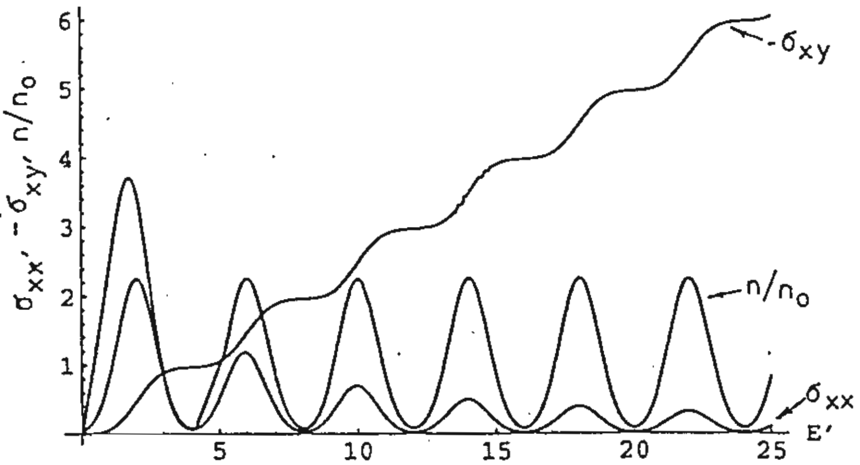

Figure 1: *Comparative plot of the normalized DOS n/n_0, Σxx, and $-\Sigma_{xy}$ in unit of $e^2/\pi\hbar$ as a function of the dimensionless Fermi enrgies E'_F for the dimensionless fluctuation parameter $\xi'_L = 1, kT = 0.01$.*

Having the DOS available we can now proceed to the evaluation of the conductivities. Again let us first consider $E'_\Omega = 4$ and $\Gamma' = 1$. It is convenient to express σ_{xx} and σ_{xy} in unit of $(\frac{e^2}{\pi\hbar})$. Since $\sigma_{xy} \gg \sigma_{xx}$, a difference of two orders of magnitude for comparison with the σ_{xy} and DOS, we multiply σ_{xx} by a factor of 50. The results for σ_{xy}, $\sigma_{xx}$and DOS are given in the figure. All of these quantities are calculated up to $E'_F = 25$. From the figure it is evident that for small DOS, σ_{xy} approaches a constant integer value. This result demonstrates that indeed plateaus occur at exactly the integer quantum numbers. This clearly confirms the Integer Quantum Hall Effect from the fundamental theory. The plateau is even more evident when gets larger E'_Ω .

6 Discussion

In this paper we have presented a general theory of electron transport in two dimensions using the Feynman path integral method. The technique used to derive the result is that taken from the Feynman-Thornber approach to the polaron problem. We have succeeded previously in applying the same technique to other problems such as electrons in random potentials [10], heavily doped semiconductors [11], Urbach Tails [10] and Quantum Hall [8, 7, 12]. However, in obtaining the results we have made several assumptions, which we will justify now. The first assumption was that the electron coordinate and the impurity coordinates are decoupled. The justification of this assumption is based on the fact that disordered system contain no dynamical variables in contrast to the polaron problem where the phonon plays an important role in the dynamic electron-phonon interaction. In the Feynman-Thornber theory of polaron mobility, this assumption was employed.

The second assumption was that only the ground state contributed to the initial density matrix. This assumption is only valid at zero temperature. Since the electron in a two-dimensional system behaves as a degenerate electron gas, therefore the assumption is justified. Furthermore we have taken the initial ground state to be that of a free particle in the presence of a magnetic field. In principle we should have taken a ground state containing both contributions from the electron as well as from random impurities. This question will bring us to the complicated problem of decoupling between different degrees of freedom. This complicated decoupling of the initial density matrix is still one of the complicated problems to be solved. This problem has been addressed in detail in Ref. [7].

In conclusion we have shown that the powerful method of the Feynman-Thornber theory of polarons could be applied successfully to the transport phenomena of Quantum Hall problems. In contrast to other transport theories, such as linear response theory, the present theory starts from the general equation of motion which contains non-linear effects. The linear approximation leads to the usual theory of linear transport. The theory presented here could be applied to other non-linear problems in Semiconductors such as hot electron transport.

Acknowledgements

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EFFICIENCY OF HIGH GRADIENT MAGNETIC SEPARATION

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Thailand

INTRODUCTION

Magnetic separation has received increasing attention from industries as a means for reducing the levels of undesirable particulate products from fluid systems. Commercial applications of magnetic separation, such as in removing iron oxides and other magnetic debris from fuel lines in power plants, and cleaning of municipal and industrial waste water by adding colloidal paramagnetic particles, have been achieved[1). Magnetic separation has an advantage over conventional fluid filtration in that it can remove particulates at a much higher rate (~ 10 times) and process greater quantities of material (operating at a rate of 5 $gal/ft^2\ sec^{-1}$ for 12 hours/day a filter plant can process sewage at a rate of 2.5×10^7 gal/day, which is sufficient for a city of 0.5 million people)[2).

1. CONCEPT AND APPLICATIONS

In high gradient magnetic separation (HGMS), a small magnetic particle is attracted by a ferromagnetic or paramagnetic collector with high susceptibility placed in a uniform external magnetic field. Figure 1 illustrates the phenomena that the external field magnetizes the collector sphere and induces a magnetic dipole in the particle. The convergence of the magnetic field near the sphere produces regions of high gradient magnetic fields both sides of the sphere that attracts micron-sized weakly magnetic particles from suspensions.

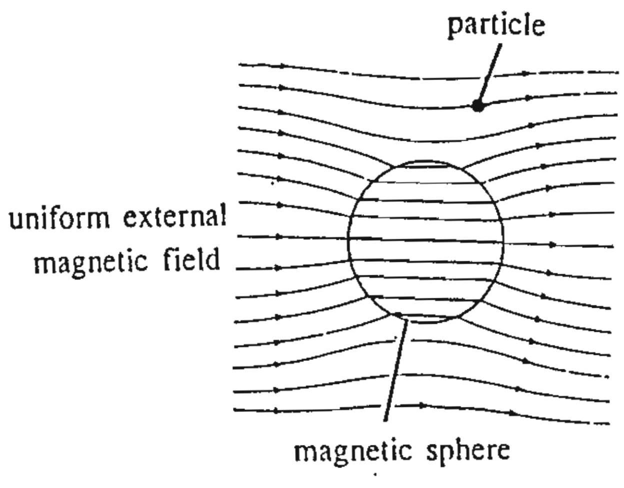

Fig. 1. Conceptual illustration of HGMS.

Figure 2 shows a HGMS cyclic system which consists of a non-magnetic canister filled with magnetic filter matrix elements typically 50 - 200 μm in diameter. The canister is placed in a uniform external magnetic field (H_o ~2 Tesla) usually generated by a solenoid. As the fluid passes through the canister, magnetic particles in suspensions are captured on the filter elements and the purified product leaves the system. The filter is cleaned by switching off the magnetic field, and the magnetic particles are flushed out.

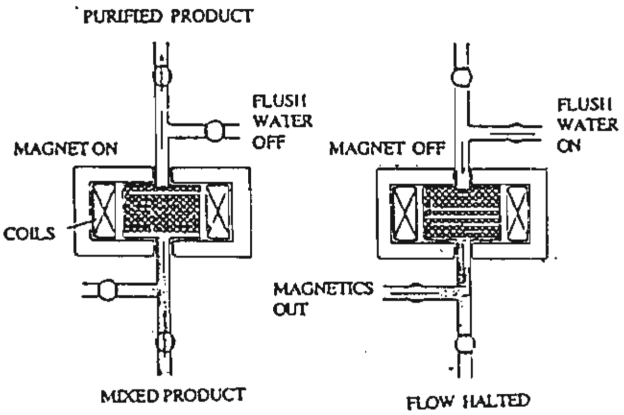

Fig. 2. A Cyclic High Gradient Magnetic Separator (after Ref. 1).

Paralleling the development of filamentary matrix, designs also have appeared using a collection of magnetic spheres as filter elements[3]. The diffusion bonded matrix designed for steel mill process waste water treatment has shown an excellent improvement of filtering efficiency[4]. The uses of HGMS in nuclear power plants[5] have been acheived for clean-up of fuel and stream lines. Extensive applications in the treatment of effuent water from different municipal and industrial sources have been reported and discussed in reference 1.

2. THEORY

In this study, we emphasize on the case of randomly distributed magnetic spheres as filter elements and two cases of design, the longitudinal and transverse modes which the external magnetic field ($\vec{H}_o$) and the fluid entrance velocity ($\vec{V}_o$) are parallel and perpendicular, respectively. In describing the particle capture process, the forces acting on the individual particles are required. For particles less than 200 μm in diameter, the inertial and gravitational forces are insignificant. The electric forces - the London dispersion force and the ionic double-layer force - are of extremely short range, and therefore play no major role in the capture of microscopic particles. The dominant forces acting upon the individual particles are the viscous drag force ($\vec{f}_d$) and the magnetic force. The viscous drag force is assumed to obey Stokes's law,

$$\vec{f}_d = -6\pi\eta r_p(\vec{V} - \vec{V}_f) \qquad (1)$$

where $\vec{V}$ is the particle velocity, $\vec{V}_f$ the fluid velocity, η the viscosity and r_p the particle radius. The fluid flow can be described by laminar or creeping flow with the condition that the Reynolds number $Re = \rho V_o a/\eta < 1$, where ρ, V_o, η and a are the fluid density, entrance velocity, viscousity and sphere radius, respectively.

Let $\vec{m}$ be the particle magnetic dipole moment and $\vec{B}$ magnetic induction at the center of the particle treated as a point dipole. The magnetic potential energy is

$$u = -\vec{m}.\vec{B} \qquad (2)$$

and the magnetic force acting on the particle is

$$\vec{f}_m = -\vec{\nabla} u . \qquad (3)$$

Assuming that the magnetic dipole is constant and $\vec{B} = \mu\vec{H} = \mu_o(1+\chi_p)\vec{H} \approx \mu_o\vec{H}$ for weak paramagentic or diamagnetic particles, we obtain

$$\vec{f}_m = \tfrac{1}{2}\mu_o\chi_p V_p \vec{\nabla} H^2 \qquad (4)$$

where χ_p is the particle susceptibility and V_p is the particle volume. For the particle carried by a fluid of susceptibility χ_f, the magnetic force is

$$\vec{f}_m = (2\pi/3)r_p^3\mu_o(\chi_p - \chi_f)\vec{\nabla} H^2. \qquad (5)$$

The magnetic field around the magnetized collector spheres in a uniform magnetic field taking into account the effects of neighboring spheres was discused by Moyer et al.[6], is applied to determine the magnetic force acting on the particle. By taking the Happel flow field[7], the drag force is calculated and the equations of motion[8] for particles traversing a magnetic filter operating in the longitudinal and transvere modes are obtained. Then the particle trajectories are determined by numerical integration as a function of V^*_{ma}/V_{oa} and γ. The normalized magnetic velocity $V^*_{ma} = V^*_m/a = 2(\chi_p - \chi_f)\mu_o K_s H_o^2 r_p^2/3\eta a^2$, the normalized fluid entrance velocity $V_{oa} = V_o/a$, and the collector packing function (γ^3) are operational parameters. While $K_s = (\nu - 1)/(\nu + 2)$, $\nu = \mu_s/\mu_f =$ the relative permeability of the collecter sphere, is the magnetic constant. By inspection of the particle trajectories, the critical capture trajectory and the corresponding capture distance r_c is obtained, as shown in Fig. 3. We observe that paramagnetic particles are captured on both sides of the collector where the magnetic field gradient is highest, consistent with the experimental observation[9] as shown in Fig. 4.

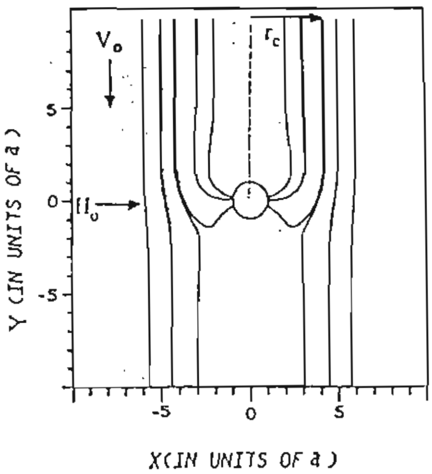

Fig. 3. Trajectories of paramagnetic particles for a transverse mode ($\vec{H}_o \perp \vec{V}_o$) and capture radius ($r_c$).

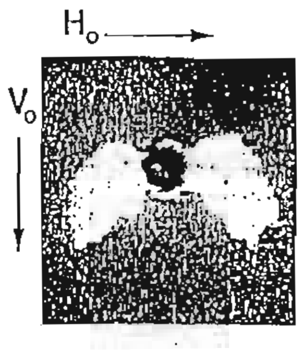

Fig. 4. Buildup of $Mn_2P_2O_7$ particles on a HGMS collectors (after Ref. 9).

3. FILTER EFFICIENCY

Filter efficiency can be predicted from capture efficiency of individual elements by considering a thin section of filter having cross section A and thickness dx oriented perpendicular to the fluid entrance velocity. The number of collector spheres in the layer is $\gamma^3 A dx/(4/3\ \pi a^3)$. Let A_C be the cross section for capture by a single collector, and n the particle concentration at x. Then

$$-\frac{dn}{n} = \frac{A_C \gamma^3 A dx/(4/3 \pi a^3)}{A} \tag{6}$$

and the integration from x = 0 to L (L = the filter depth) gives

$$n_L = n_o \exp(-3\ A_C\ \gamma^3 L/4\pi a^3). \tag{7}$$

The filter performance, defined as

$$\varepsilon = \left(\frac{n_o - n_L}{n_o}\right) \tag{8}$$

is given by the formula

$$\varepsilon = 1 - \exp(-3\ A_C\ \gamma^3 L/4\pi a^3). \tag{9}$$

We note that the expression for ε neglects nonuniformities in particle density caused by the capture of particles from the stream, i.e., shadows cast by upstream spheres are not recognized. For longitudinal design $A_C = \pi r_c^2$; in the transverse design r_c varies with the orientation of the incident plane and A_C can be found numerically.

4. RESULTS

Figure 5 shows the dependence of the efficiency as a function of external field and packing fraction, respectively, for several different filter lengths of a magnetic separator operating in the longitudinal mode. We observe that the efficiency increases with increasing external magnetic field (H_o) and, then, the saturation effects starting to appear at higher fields which is in agreement with the experiment of Anand et al.[10] The similar saturation behavior of the efficiency at higher range of packing fraction is shown in Fig. 6. The general behaviors of the filter efficiency, as mentioned, are also observed in the transverse mode design [11]. Furthermore, our preliminary investigation suggests that the efficiency for the transverse design is better than that of the longitudinal design under the same operational parameters V^*_{ma} and V_{oa}, especially for higher collector packing fraction.

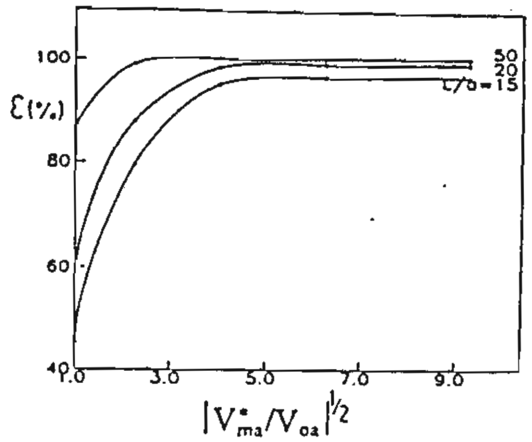

Fig. 5. Filter efficiency as a function of operating magnetic field strength. The magnetic velocity V^*_{ma} is proportional to the square of the external field.

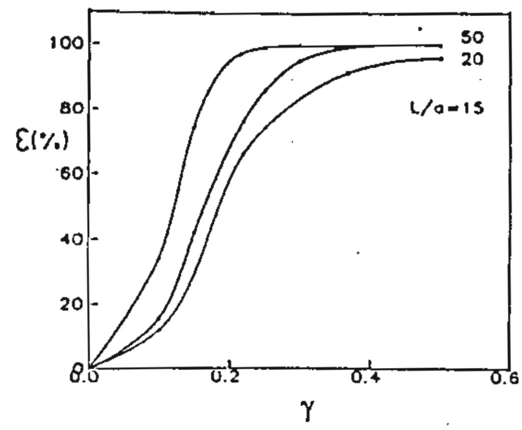

Fig. 6. Filter efficiency as a function of γ (γ^3 is the packing fraction of the collector spheres comprising the filter).

5. ACKNOWLEDGEMENTS

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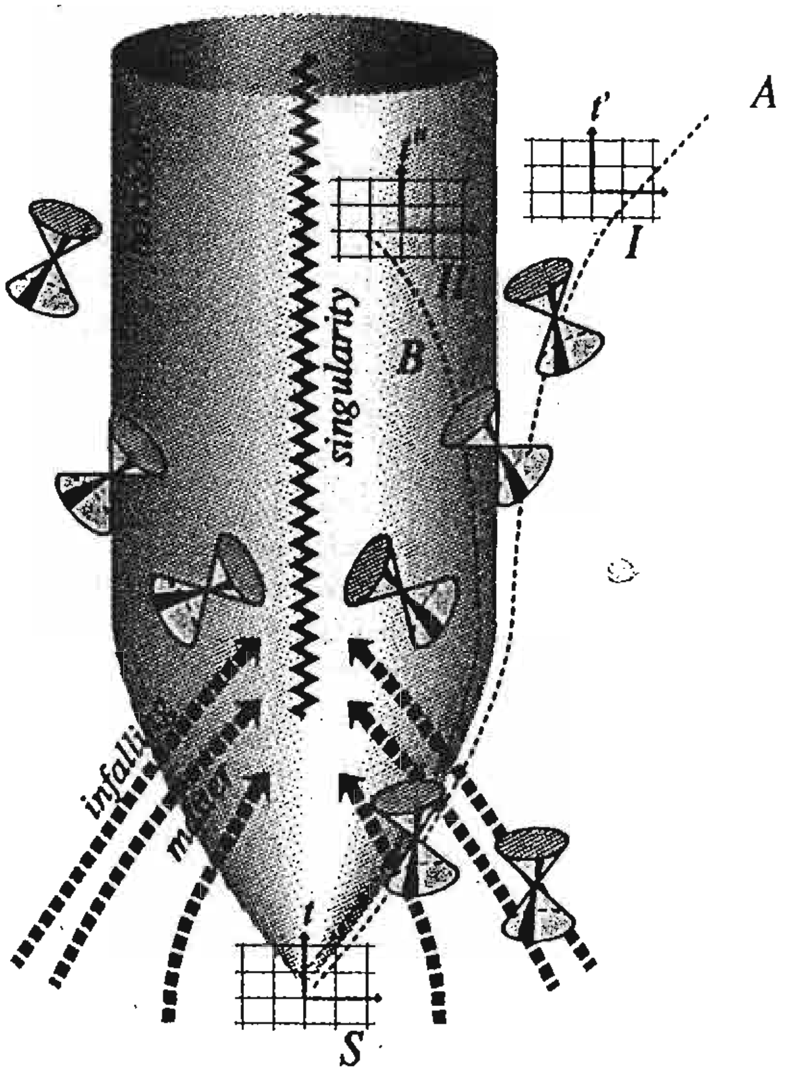

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Static and Dynamic Effective Mass of the Polaron

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Abstract. The approximated density matrix of the polaron system obtained in our previous paper for deriving Feynman effective mass m_F is used to calculate the static and dynamic mass of the polaron. It is shown that to obtain a consistent definition of the polaron effective mass one must impose the condition $m_F = m_{KP}$. This condition leads to a new definition of effective mass and a new variation principle for treating excited states. Numerical results are presented for various coupling constants. The dynamic mass of the polaron is obtained from the quartic term of the off-diagonal parts of the density matrix and compared with the recent work of Wang et. al.

The path integral approach to the polaron effective mass was first discussed by Feynman. He showed that given a partition function, it is possible to calculate the ground state energy as well as the effective mass of the polaron m_F. The ground state energy was obtained by variation calculation based on the Feynman-Jensen inequality. The variational parameters obtained are used to calculate the ground state energy as well as the effective mass of the polaron. As pointed out by Feynman in [1] the calculation of the effective mass is not rigorous since there is no variational principle for calculating the excited state. Nevertheless, the variational parameters can be used to calculate the effective mass of the polaron. The partition function was also used by Krivoglaz and Pekar [2] in obtaining the different definition of the effective mass of the polaron m_F.

In this paper we show that starting from the density matrix instead of the partition function, we obtain a consistent definition of the polaron effective mass. As pointed out in Sa-yakanit [3], the Feynman mass and the Krivoglaz- Pekar mass are of the same density matrix which is obtained from the zero temperature limit expressed in the free particle form, this limit is:

$$\rho\left(\vec{R}_2 - \vec{R}_1; \beta \rightarrow \infty\right) = \left(\frac{m_{KP}}{2\pi\beta}\right)^{\frac{3}{2}} \exp\left[-E_\circ \beta - \frac{m_F\left|\vec{R}_2 - \vec{R}_1\right|^2}{2\beta}\right], \quad (1)$$

where E_0 is the ground state energy of the polaron, β denote the imaginary time. This expression suggests that a consistent definition of the effective mass of the polaron should be such that $m_F = m_{KP}$. We shall show that this condition is necessary in order for the wave function obtained from the density matrix to be normalised. The effective density matrix can also be used for obtaining the excited state wave function . The orthogonal requirement applied to each excited state leads to a new variation principle.

The starting point of our discussion is the density matrix for the polaron system

$$\rho(\vec{x}_2,\vec{x}_1;\beta)=\int D[\vec{x}(\tau)]\exp(S) \tag{3}$$

where $D[\vec{x}(\tau)]$ is the path integral from $\vec{x}(0)=\vec{x}_1$ and $\vec{x}(\beta)=\vec{x}_2$.

$$S=\frac{m}{2}\int_0^\beta \dot{\vec{x}}(t)^2dt-\frac{\alpha}{2^{3/2}m^{1/2}}\int_0^\beta\int_0^\beta dtds\frac{\cosh(\beta/2-|t-s|)/\sinh(\beta/2)}{|\vec{x}(t)-\vec{x}(s)|}. \tag{4}$$

where α is a coupling constant between electron and phonon and m is the electron band mass. By following Feynman, a trial action was introduced

$$S_{\circ}=\frac{m}{2}\int_0^\beta \dot{\vec{x}}(t)^2dt-\frac{\kappa w}{8}\int_0^\beta\int_0^\beta dtds\,(\vec{x}(t)-\vec{x}(s))^2\frac{\cosh w(\beta/2-|t-s|)}{\sinh(w\beta/2)}. \tag{5}$$

This action corresponding to an electron coupled to a fictitious particle with κ and w are two parameters corresponding to the spring constant and the frequency of a harmonic oscillator respectively. Within the first cumulant expansion, we have

$$\rho(\vec{x}_2-\vec{x}_1;\beta)=\rho_{\circ}\exp\langle S-S_{\circ}\rangle_{S_{\circ}} \tag{6}$$

where $\langle O\rangle_{S_{\circ}}=\int D[\vec{x}(t)]Oe^{S_{\circ}}/\int D[\vec{x}(t)]e^{S_{\circ}}$

Carrying out the path integral we have the density matrix

$$\rho_1(\vec{x}_2-\vec{x}_1;\beta)=\left(\frac{m}{2\pi\beta}\right)^{3/2}\left(\frac{v\sinh(w\beta/2)}{w\sinh(v\beta/2)}\right)^3\exp\Bigg(-\left[\frac{v\mu}{4}\coth(v\beta/2)+\frac{\mu}{2M\beta}\right]|\vec{x}_2-\vec{x}_1|^2$$
$$+\Bigg\{\frac{m^{1/2}\alpha}{2^{3/2}}\int_0^\beta\int_0^\beta d\sigma d\tau\int_{-\infty}^{\infty}\frac{d^3k}{2\pi^2k^2}\exp\Bigg[i\vec{k}\cdot(\vec{x}_2-\vec{x}_1)\mu\left(\frac{\sinh(v(\tau-\sigma)/2)\cosh(v(\beta-\tau-\sigma)/2)}{m\sinh(v\beta/2)}\right.$$
$$\left.+\frac{\tau-\sigma}{M\beta}\right)-\frac{\vec{k}^2}{2mv^2}F(|\tau-\sigma|,\beta)\Bigg]\frac{\cosh(\beta/2-|\tau-\sigma|)}{\sinh(\beta/2)}-\frac{3}{2}\left(1-\frac{w^2}{v^2}\right)\left[\frac{v\beta}{2}\coth\left(\frac{v\beta}{2}\right)-1\right]$$
$$-\frac{C}{2}\int_0^\beta\int_0^\beta d\tau d\sigma\mu^2\left(\frac{\sinh(v(\tau-\sigma)/2)\cosh(v[\beta-\tau-\sigma]/2)}{\sinh(v\beta/2)}+\frac{\tau-\sigma}{\beta M}\right)^2$$
$$\times\frac{\cosh w(\beta/2-|\tau-\sigma|)}{\sinh(w\beta/2)}|\vec{x}_2-\vec{x}_1|^2\Bigg\}\Bigg),$$

where

$$F(|\tau-\sigma|,\beta)=\mu\left(\frac{2v\sinh(v(\tau-\sigma)/2)\sinh(v(\beta-\tau+\sigma)/2)}{m\sinh(w\beta/2)}+\frac{v^2[\beta-\tau+\sigma](\tau-\sigma)}{M\beta}\right). \tag{8}$$

Here v and w are alternative set of the variational parameters related to the previous set by the relation $v^2=\kappa/\mu$, M is the mass of a fictitious particle and $\mu=mM/(m+M)$ is a reduced mass of the system. As pointed out earlier [3], to obtain the effective mass from the above expressions it is necessary to go over to the center of mass co-ordinates $\vec{R}=\frac{m\vec{x}+M\vec{y}}{m+M}$, where $\vec{x}$ is the electron co-ordinate and $\vec{y}$ is the fictitious particle co-ordinate and M is the fictitious particle mass. In averaging out the fictitious co-ordinates, the end points of $\vec{y}$ were set to be equal so we have the transformation $\vec{R}_2 - \vec{R}_1 = \frac{m}{m_*}(\vec{x}_2 - \vec{x}_1)$, where $m_* = m + M$, is the total mass of the system. For $(\vec{x}_2 - \vec{x}_1) \to 0$ and $\beta \to \infty$, we can expand the exponent depending on the co-ordinates in equation (8) and keep terms up to the 2nd order of $(\vec{R}_2 - \vec{R}_1)$. We then arrive at the density matrix of the polaron at zero temperature limit as in the equation (1) with the ground state energy and the Feynman and Krivoglaz-Pekar effective mass defined respectively as

$$E_* = \frac{3}{4}\frac{(v-w)^2}{v} - \frac{\alpha v}{\sqrt{\pi}}\int_0^\infty dx\, e^{-x} F(x)^{-\frac{1}{2}} \tag{9}$$

$$m_F = 1 + \frac{\alpha}{3\sqrt{\pi}}\int_0^\infty dx v^3 x^2 e^{-x} F(x)^{-\frac{3}{2}} \tag{10}$$

$$m_{KP} = \left(\frac{v}{w}\right)^2 \exp\left(\frac{w^2}{v^2} - 1 + \frac{w^2}{v^2}\frac{\alpha v^3}{\sqrt{\pi}}\int_0^\infty dx\, x^2 e^{-x} F(x)^{-\frac{3}{2}}\right) \tag{11}$$

The wave function of the polaron can be obtained from this density matrix by noticing that

$$\int_{-\infty}^{\infty} d^3k \exp\left[-\frac{2\pi^2\beta}{m^*}\vec{k}^2 + 2\pi i\vec{k}\cdot\left|\vec{R}_2 - \vec{R}_1\right|\sqrt{\frac{m_F}{m^*}}\right] = \left(\frac{m^*}{2\pi\beta}\right)^{\frac{3}{2}} \exp\left(-\frac{m_F\left|\vec{R}_2 - \vec{R}_1\right|^2}{2\beta}\right) \tag{12}$$

Then
$$\rho = \int \frac{Vd^3p}{(2\pi)^3}\left(\frac{m^*}{m_F}\right)^{\frac{3}{2}} \frac{1}{V}\exp\left[i\vec{p}\cdot\left|\vec{R}_2 - \vec{R}_1\right| - \left(E_* + \frac{\vec{p}^2}{2m_F}\right)\beta\right] \tag{13}$$

From this expression, we can obtain the unnormalized wave functions and the energies of the polaron as

$$\Psi_p(\vec{R}) = \frac{1}{\sqrt{V}}\sqrt{\frac{m^*}{m_F}}\exp[i\vec{p}\cdot\vec{R}], \quad E_p = E_* + \frac{\vec{p}^2}{2m_F}. \tag{14}$$

Because of the translational invariant, the wave functions behave like plane waves. Therefore all wave functions become orthonormal because

$$\int d^3R\, \Psi_p^*(\vec{R})\Psi_{p'}(\vec{R}) = \left(\frac{m_{KP}}{m_F}\right)^{\frac{3}{2}} \delta(\vec{p} - \vec{p}') \tag{15}$$

Since all wave function must be orthonormal, we must have $m_F = m_{PK}$ as in our conjecture. Thus we have shown that the Feynman mass and the Krivoglaz-Pekar mass should be equal. In order to obtain the variational parameters for the effective mass or for excited states, we minimize the energy with respect to the two parameter

v and w with constraint $m_F = m_{KP}$. Note that any constraints on the ground state energy will lead to a higher energy. Thus we consider this method should be more appropriate variational principle for the effective mass. The calculated energies are presented in table 1. It is evident that our E_{new} are slightly higher than the ground state energy of Feynman. These are not unexpected results since any constraint variational calculation will give a energy higher than the ground state indicating the low lying excited state. The difference in energy is quite small. In figure 1 we present the new effective mass normalized by m_F and m_{KP}.

Table 1 Our results for variational parameters, the effective masses, and the ground state energies at various coupling constants.

α	v	w	m_{eff}	E_o
1	2.716	2.48	1.19819	-1.01293
2	2.841	2.33	1.48568	-2.05494
3	3.028	2.18	1.93148	-3.13234
4	3.264	1.99	2.68936	-4.25447
5	3.663	1.80	4.13918	-5.43697
6	4.396	1.61	7.45220	-6.70704
7	5.575	1.43	15.2030	-8.10945
8	7.414	1.30	35.5344	-9.69335
9	9.877	1.22	65.5548	-11.4846
10	12.72	1.17	118.221	-13.4888
11	15.34	1.13	184.296	-15.7094
12	19.00	1.11	293.163	-18.1419
13	22.40	1.09	422.105	-20.7901
14	25.51	1.07	568.366	-23.6497
15	29.62	1.06	781.014	-26.7234

Figure 1 Our new results of the effective mass normalized to the Feynman's masses m_F as a function of the coupling constant α.

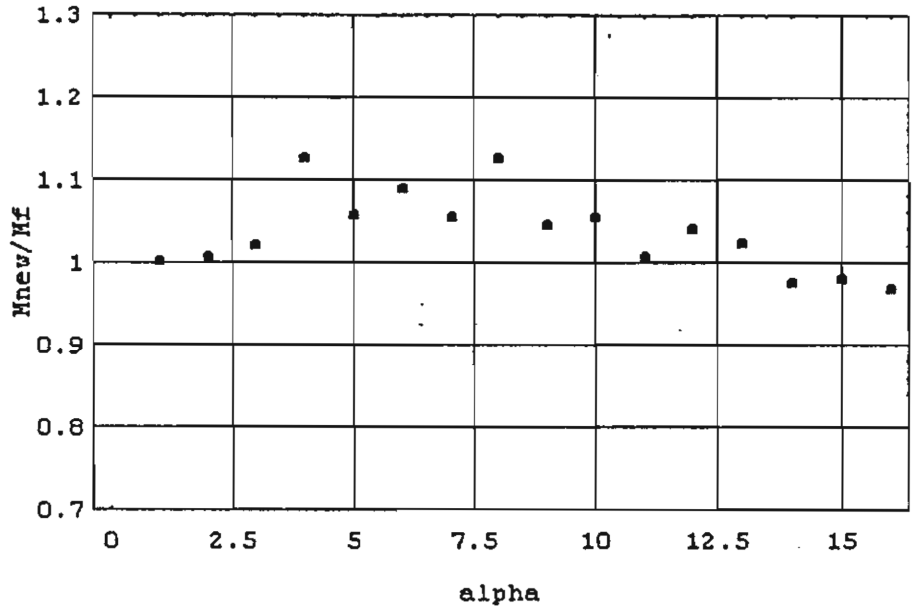
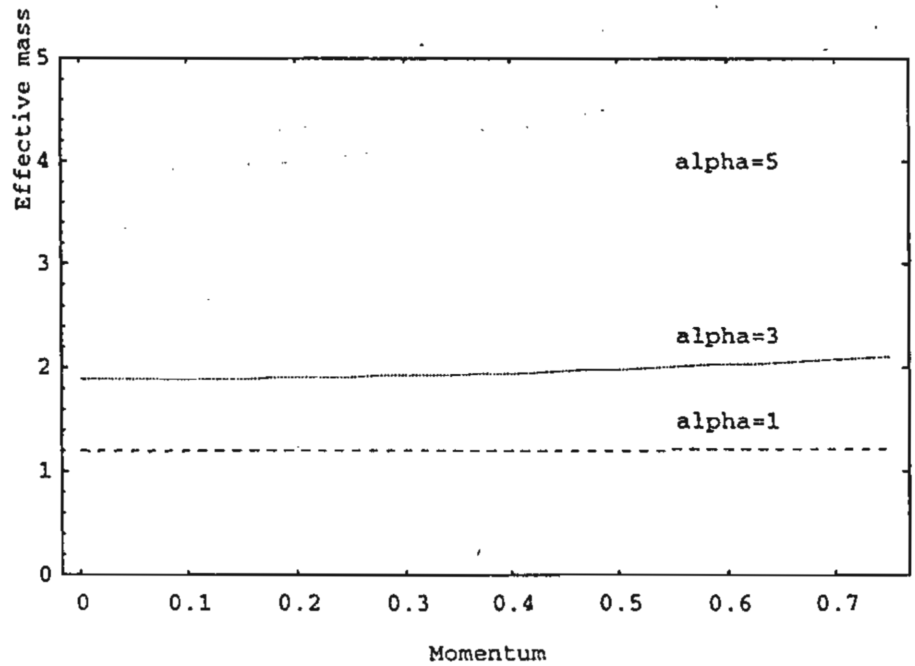


Figure 2 The dynamic mass of the poaron as a function of the momentum at various coupling constants.



In order to see the dynamic mass of the polaron, we expand the exponential as a power series depending on the co-ordinates up to the 4th order in the co-ordinates $\left(\vec{R}_2 - \vec{R}_1\right)$. We can separate the quartic term by neglecting fluctuation in momentum and inserting the quadratic term in the mass and then define the dynamic mass as

$$m_F(\bar{P}) = 1 + \frac{\alpha}{3\sqrt{\pi}} \int_0^{\infty} dx \frac{v^3 x^2 e^x}{F^{\frac{3}{2}}(x)} + \frac{\alpha}{18\sqrt{\pi}} \int_0^{\infty} dx \frac{v^5 x^4 e^{-x}}{F^{\frac{5}{2}}(x)} \bar{P}^2 \qquad (20)$$

The numerical results of this dynamic mass are shown in .figure 2. Note that this results corresponding to the work be Wang et.al. [4] which is valid only for small momentum and weak coupling.

In conclusion we have shown that giving the approximated density matrix proposed in Sa-yakanit [3], it is possible to obtain a new definition of effective mass from the fact that all wave functions derived from the density matrix should be orthonormal. The variational principle used to obtain the effective mass is appropriate for the excited state as can be demonstrated by calculating the constraint ground state energy which is slightly higher than the ground state energy of the polaron. Although we have no rigorous proof of this variational principle, we believe that the constraint variational principle should be the correct parameter for the excited state and therefore for the effective mass as well. We also shown that the dynamic mass can be obtained from our approximated density matrix by expanding the exponent of the co-ordinates difference up to quartic terms.

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APPENDIX 6

Papers Presented in International Conferences

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6th International Conference on
PATH-INTEGRALS FROM peV TO TeV
50 Years from Feynman's Paper
Florence, Italy, 25-29 August 1998

Web site: http://www.area.fi.cnr.it/pi98/
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Prof. Virulh Sa-yakanit
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Firenze, 7 May 1998

Dear Prof. Sa-yakanit,

on behalf of the Organizing Committee, we invite you to Florence in order to attend the *PI98* Conference and to present the plenary talk on

Path integral approach to the Landau level
mixing and levitation in 2-D random systems.

The Organizing Committee will waive your registration fee, as well as provide support for your accommodation expenses in Florence during the period of the conference (5 nights from 25 to 29 August 1998).

We believe your participation will contribute to the success of the conference and do hope you can come.

Looking forward to meeting you in Florence
Yours sincerely

Dr. Ruggero Vaia
Scientific secretary

Prof. Valerio Tognetti
Chairman of the Organizing Committee

PATH INTEGRAL APPROACH TO THE LANDAU LEVEL LEVITATION IN 2-D RANDOM SYSTEMS

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The Feynman path integral method developed in our previous paper for handling the Quantum Hall problem is applied to the study of the motion of electrons in the presence of a perpendicular magnetic field B with saddle-point potential $V_{sp}(x,y) = V_0 + (m/2)(\Omega_x^2 x^2 - \Omega_y^2 y^2)$. An exact propagator is obtained and is used to calculate the density of states, the energy of extended states, and Landau Level levitation. The ground state energy of extended state is given as $E_n = (n + 1/2)(\Omega_1 + \Omega_2)\hbar$ where $\Omega_1 = \sqrt{\omega^2/4 + \Omega_x^2}$ and $\Omega_2 = \sqrt{\omega^2/4 - \Omega_y^2}$. with $\omega = eB/mc$. In the strong field case, the energy of the extended states approaches the corresponding Landau Levels while for weak fields there is a floating of extended states. The floating is not increased indefinitely, but instead , it approaches a finite value proportional to Ω_x in agreement with the existing experimental results.

1 Introduction

Recently there has been substantial interest in the study of the behavior of the extended states of noninteracting electrons in a two-dimensional system under a strong magnetic field with random potential. This problem is important for the understanding of the integer Quantum Hall effect [1], in particular, the understanding of the Landau Level-mixing and levitation of extended states or the floating of extended states.

There are several approaches to the study of floating of the extended states [1,2,3,4,5,6,7]. The quadratic saddle point potential $V_{sp}(x,y) = V_0 + U_y y^2 - U_x x^2$ has been used by Ferring and Halperin (FH) [2] for calculating the transmission coefficient through the saddle point potential in a two-dimensional system with a strong magnetic field. This potential was also used by Haldane and Yang (HY) [3] for studying the effect of mixing of different Landau Levels (LL) for a two-dimensional system with a strong magnetic field and random potential. The problem of LL mixing is also discussed recently by Chang. et al. [4] for electrons in a random magnetic field.

In this paper, we show that by using the Feynman Path Integral method developed by us [8,9] in a previous paper for handling the Quantum Hall problem one can obtain the density of states as well as the energy of extended states. In order to discuss the localized and extended states transition, we employ the quadratic saddle point potential proposed by FH which explicitly breaks the translational symmetry. We calculate the classical action associated with this V_{sp} and obtain the exact propagator. Then the density of states and the energy of extended states for any value of the magnetic field B of the system can be obtained. From the energy of extended states we can show that for a strong magnetic field, the energy levels shift proportionally to $1/B$. We also calculate the higher order contributions and are found to be proportional to $(n+\frac{1}{2})/B^3$.

For weak magnetic fields the energy of extended states floats up to a finite value proportional to Ω_x of the saddle point potential contrary to the HY approach [3] where the floating diverses as $B \to 0$.

2 Exact Propagator

The starting point of our discussion is the following Lagrangian,

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m}{2}\omega(x\dot{y} - y\dot{x}) - \frac{m}{2}(\Omega_x^2 x^2 - \Omega_y^2 y^2) - V_0. \tag{1}$$

Note that we have defined $U_x = -(m\ \Omega_x^2)/2$ and $U_y = -(m\ \Omega_y^2)/2$. Here $\omega = eB/mc$ is the cyclotron frequency, Ω_x and Ω_y are parameters representing the saddle point potential, and V_0 is the top of the saddle point potential. Since the system is quadratic, the path integral can be carried out exactly. The propagator consists of the classical action S_{cl} and the prefactor $F(T)$ which can be calculated using the Van Vleck-Pauli formula. To find the classical action, we need to find the classical path which can be achieved by making a variation of Eq.(1). We obtain

$$\ddot{x} - \omega\dot{y} + \Omega_x^2 x = 0, \tag{2}$$

$$\ddot{y} + \omega\dot{x} - \Omega_y^2 y = 0. \tag{3}$$

To solve these two simultaneous equations, we use the method developed by Sa-yakanit, Choosiri and Robkob [10]. Then the classical action is

$$\begin{aligned} S_{cl}(\mathbf{r}_b, \mathbf{r}_a; T) = \\ \frac{m}{2}\Big[\frac{\Omega_1}{\sin\Omega_1 T}\big[\ (x_b^2 + x_a^2)\cos\Omega_1 T - 2x_b x_a \cos\frac{\omega}{2}T + (x_a y_b - x_b y_a)\sin\frac{\omega}{2}T\ \big] \\ + \frac{\Omega_2}{\sin\Omega_2 T}\big[\ (y_b^2 + y_a^2)\cos\Omega_2 T - 2y_b y_a \cos\frac{\omega}{2}T + (x_a y_b - x_b y_a)\sin\frac{\omega}{2}T\ \big]\Big] \end{aligned} \tag{4}$$

where $\Omega_1 = \sqrt{\omega^2/4 + \Omega_x^2}$ and $\Omega_2 = \sqrt{\omega^2/4 - \Omega_y^2}$. and we define V_0 to zero. The prefactor can be obtained, by using the Van Vleck-Pauli formula [10], to be

$$F(T) = (\frac{m}{2\pi i\hbar})[\frac{\Omega_1}{\sin\Omega_1 T}\frac{\Omega_2}{\sin\Omega_2 T} + \frac{1}{4}[\frac{\Omega_1}{\sin\Omega_1 T} - \frac{\Omega_2}{\sin\Omega_2 T}]^2 \sin^2\frac{\omega T}{2}]^{1/2}. \quad (5)$$

Thus the exact propagator for the two-dimensional random system with quadratic saddle point potential in a strong magnetic field is given by

$$K(\mathbf{r}_b, \mathbf{r}_a; T) = F(T)\exp\frac{i}{\hbar}S_{cl}(\mathbf{r}_b, \mathbf{r}_a; T). \quad (6)$$

It is noted that for $\Omega_x^2 = -\Omega_y^2$, then $\Omega_1 = \Omega_2$ corresponding to a particle under a strong magnetic field with a fixed harmonic potential [10] and for $\omega = 0$, then $\Omega_1 = \Omega_x$, $\Omega_2 = i\Omega_y$ corresponding to a particle with a saddle point potential. Furthermore, if $\Omega_x = \Omega_y = 0$, then $\Omega_1 = \Omega_2 = \frac{\omega}{2}$ is reduced to the free particle case in the presence of a magnetic field.

3 Density of States and Discussion

Starting from the exact propagator given in Eq.(6), it is possible to obtain the density of states by taking the trace of the propagator. The off-diagonal terms can be used to calculate the transmission coefficient. However, in this paper we will consider only the diagonal part which allows us to obtain the density of states, and energy of extended states.

The density of states $\rho(E)$ can be obtained from the following expression

$$\rho(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dT\ Tr\ K(\mathbf{r}_b, \mathbf{r}_a, T)\ \exp\frac{i}{\hbar}ET \quad (7)$$

To obtain the main contribution, we make a large T approximation so we can neglect oscillating term in the propagator. Then, we have

$$\rho(E) = \frac{1}{2\pi\hbar}\sum_n \int_{\infty}^{\infty} dT\ \exp\frac{i}{\hbar}[E - (n+\frac{1}{2})(\Omega_1 + \Omega_2)\hbar]T \quad (8)$$

where we can obtain the energy of the extended states

$$E_n = (n+\frac{1}{2})(\Omega_1 + \Omega_2)\hbar. \quad (9)$$

This result contains information about the LL levitation. To get in touch with the result of HY and FH, we assume that $\omega >> \Omega_x$ and $\omega >> \Omega_y$. Then we can expand Ω_1 and Ω_2 in power series of $1/\omega$ or $1/B$, and the result is

$$E_n = (n + \frac{1}{2})\hbar\omega \; [1 + \frac{1}{\omega^2}(\Omega_x^2 - \Omega_y^2) + O(\omega^{-4})] \tag{10}$$

The first term in the square bracket is the cyclotron frequency and the second term is the shift in the energy of the extended states which could be positive or negative depending on the magnitude of the anisotropic behavior of the saddle point potential and it is proportional to $1/B$ in agreement with HY. Note that for $\omega \gg 1$ the energy of the extended states approaches the corresponding LL energy $E_n = (n + \frac{1}{2})\hbar\omega$.

For a weak magnetic field, we can expand the energy dispersion E_n in powers of ω and consider the real part

$$E_n = (n + \frac{1}{2})\hbar\Omega_x(1 + \frac{\omega^2}{8\Omega_x^2} + O(\omega^4)) \tag{11}$$

This result indicates that as $\omega \rightarrow 0$, E_n approaches a finite value proportional to Ω_x. It should be noted that for $\omega/2 < \Omega_y$, Ω_2 becomes complex. Since we are interested in the energy of the extended states we simply take the real part of E_n. The complex part will contribute to the decay of the LL which will be important for the tunnelling problem.

We present now the results of the energy of the extended states as a function of the magnetic field or $\hbar\omega$. The results are shown in Fig.1 where the dashed lines are the corresponding LL to $E_n = (n + \frac{1}{2})\hbar\omega$ for the case of $n = 0$. We have also presented the calculation from Eq.(9) (full line) and the two dash-dot lines from the asymptotic expressions Eqs.(10) and (11). Our results show that there is a floating of the extended state which is due to the saddle point potential. The discontinuity in the full line is due to the energy becomes complex. The horizontal line indicates the Fermi level. Thus we can conclude that for weak fields the real part of E_n contributed to the floating of extended states while an imaginary part of E_n contributed to the decay of the extended states.

Acknowledgments

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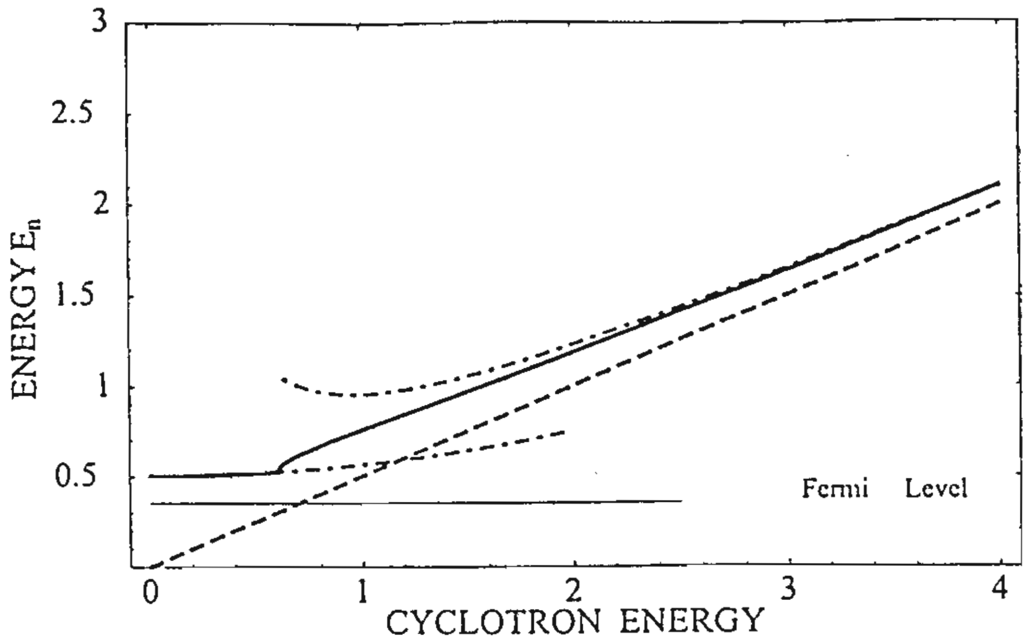

Figure 1. The energy of extended states E_n as a function of the cyclotron energy $\hbar\omega$. The dashed line correspond to LL $n = 0$. The full line is calculated from Eq. (9). The dash-dot lines are two asymptotic expressions calculated according to Eqs. (10) and (11), respectively. The horizontal line indicates the Fermi level and is set at 0.35. All calculations are performed for $\Omega_x = 1$ and $\Omega_y = 0.3$.

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05.05.1998 Poznań

Prof. Suantai Suthep,
Dep. of Mathematics,
Chiangmai, University,

Dear Prof. Suantai Suthep,
We are pleased to inform you that your talk "Matrix transformations of some vector-valued sequence spaces" is included to the program of the conference "Function spaces V".

On behalf of Organizing Committee sincerely,

dr Leszek Skrzypczak

Matrix Transformations on Some Vector-Valued Sequence Spaces

SUTHEP SUANTAI

ABSTRACT. **In this paper, we give the matrix characterizations from vector-valued sequence spaces of Maddox $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, and $\ell(X,p)$ into scalar-valued sequence spaces of Maddox $c_0(q)$,$c(q)$, and $\ell_\infty(q)$ where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers.**

(1991) AMS Mathematics Subject Classification: 46A45.

1. INTRODUCTION

Let $(X, \|.\|)$ be a Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. We write $x=(x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces of Maddox are defined as

$$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\},$$
$$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\},$$
$$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\},$$
$$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}$$

When $X = R$ or C, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\ell(p)$ respectively and each of them is called a *sequence space of Maddox.* These spaces were first introduced and studied by Simons [6], Maddox [3, 4], and Nakano [5]. In [1] the structure of the spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ have been investigated.

In this paper we consider the problem of characterizing those matrices that map an X-valued sequence spaces of Maddox into scalar-valued sequence spaces of Maddox. Grosse-Erdmann [2] has given characterizations of matrix transformations between the scalar-valued sequence spaces of Maddox. Wu and Liu [8] deal with some of this problem with some conditions on the sequences (p_k) and (q_k). Their characterizations are derived from functional analytic principles. Our approach here is different. We use a method of reduction introduced by Grosse-Erdmann [2]. In [1] it is pointed out that $c_0(p)$ is an echelon space of order 0 and that $\ell_\infty(p)$ is a co-echelon space of order ∞. In this paper we also show that $c_0(X,p)$ is an echelon space of order 0 and $\ell_\infty(X,p)$ is a co-echelon space of order ∞. Therefore these spaces are made up of simpler spaces. We will use certain auxiliary results(Section 3) to reduce our problem to the characterisations of matrix mapping between much simpler spaces.

Notation and Definitions

2.1 Let $(X, \|.\|)$ be a Banach space, the space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ is denoted for the space of all finite sequences in X. When $X = R$ or C, the corresponding spaces are written as W and Φ.

A sequence spaces in X is a linear subspace of $W(X)$. Let E be any X-valued sequence space. For $x \in E$ and $k \in N$, we write x_k stands for the k^{th} term of X. For $k \in N$ denote by e_k the sequence $(0,0,...,0,1,0,...)$ with 1 in the k^{th} position and by e the sequence $(1,1,1,...)$. For $x \in X$ and $k \in N$, let $e^k(x)$ be the sequence $(0,0,...,0,x,0,...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x,x,x,...)$. For a fixed scalar sequence $\mu = (\mu_k)$ the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} .$$

The sequence space E is called *normal* if $x \in E$ and $y \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $y \in E$.

2.2 Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = (A_n(x)) \in F$. Let (E,F) denote for the set of all infinite matrices mapping from E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E,F)\ \}$$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (\frac{1}{u^k})$. In this paper we are concerned with finding conditions on a matrix $A = (f_k^n)$ that characterise its membership to certain classes (E,F).

2.3 Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $n \in N$ the n^{th} coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E,τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space.* Now, suppose that E contains $\phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N\ \}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

The space $\ell(p)$ is an FK-space with AK under the paranorm $g(x) = \left(\sum_{k=1}^{\infty} |x_k|^{p_k}\right)^{1/M}$, where $M = max\{1, \sup_k p_k\}$. The space $c_0(p)$ is an FK-space with AK, $c(p)$ is an FK-space and $\ell_\infty(p)$ is a complete LBK-space with AB (see [1]). It is the same as above the space $\ell(X,p)$ is an FK-space with AK under the paranorm $g(x) = \left(\sum_{k=1}^{\infty} \|x_k\|_k^p\right)^{1/M}$, where $M = max\{1, \sup_k p_k\}$. In each of the space $\ell_\infty(X,p)$ and $c_0(X,p)$ we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = max\ \{1, \sup_k p_k\}$. It is known that $c_0(X,p)$ is an FK-space with AK under the paranorm g defined as above and $\ell_\infty(X,p)$ is a complete LBK-space with AB. In $c(X,p)$ we consider the function $g(x) = \sup_k \|x_k - a\|^{p_k/M} + \|a\|$ where a is the unique element in X with $x - e(a) \in c_0(X,p)$. Then g is a paranorm on $c(X,p)$ and $c(X,p)$ is an FK-space under this paranorm g.

Some Auxiliary Results

In this section we give various useful results that can be used to reduce our problems into some simpler forms.

Proposition 3.1 *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar sequence spaces, and let u and v be sequences of real numbers with $u_k \neq 0$, $v_k \neq 0$ for all $k \in N$. Then we have*

(i) $\left(\cup_{n=1}^{\infty} E_n, F\right) = \cap_{n=1}^{\infty}(E_n, F)$

(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty}(E, F_n)$

(iii) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$

(iv) $(E, F_1 \oplus F_2) = (E, F_1) \oplus (E, F_2)$ *if the following two conditions hold*

(1). E, F_1, and F_2 are FK-spaces and E has AK and

(2). If (x_n) is a sequence in X with $x_n \to 0$ as $n \to \infty$ implies $e^k(x_n) \to (0,0,0,...)$ as $n \to \infty$ in E for all $k \in N$.

(v) $(E, c(q)) = (E, c_0(q)) \oplus (E, < e >)$ *if E is normal containing $\Phi(X)$, where $q = (q_k)$ is a bounded sequence of positive real numbers.*

(vi) $(E, F_1) = (E, F_2) \cap (\phi(X), F_1)$ *if E is an FK-space with AD, F_2 is an FK-space and F_1 is a closed subspace of F_2.*

(vii) $(E_u, F_v) = {}_v(E, F)_{u^{-1}}$.

Proof. Assertions (i), (ii), (iii), and (vii) are immediate.

To show (iv), suppose that the conditions (1) and (2) hold. It is clear that $(E, F_1) + (E, F_2) \subseteq (E, F_1 + F_2)$. Moreover, if $A \in (E, F_1) \cap (E, F_2)$, then $A \in (E, F_1 \cap F_2) = (E, 0)$, which implies that $A = 0$ because E contains ϕ. Hence $(E, F_1) + (E, F_2)$ is a direct sum. Now we will show that $(E, F_1 \oplus F_2) \subseteq (E, F_1) + (E, F_2)$. Let $A = (f_k^n) \in (E, F_1 \oplus F_2)$. For $x \in X$ and $k \in N$, we have $\left(f_k^n(x)\right)_{n=1}^{\infty} = Ae^k(x) \in F_1 \oplus F_2$, so that there are unique sequences $\left(b_k^n(x)\right)_{n=1}^{\infty} \in F_1$ and $\left(c_k^n(x)\right)_{n=1}^{\infty} \in F_2$ with

$$\left(f_k^n(x)\right)_{n=1}^{\infty} = \left(b_k^n(x)\right)_{n=1}^{\infty} + \left(c_k^n(x)\right)_{n=1}^{\infty} \tag{3.1}$$

For each n, $k \in N$, let g_k^n and h_k^n be functionals on X defined by

$$g_k^n(x) = b_k^n(x) \text{ and } h_k^n(x) = c_k^n(x) \text{ for all } x \in X$$

Clearly, g_k^n and h_k^n are linear and by (3.1)

$$f_k^n = g_k^n + h_k^n \quad \text{for all } n, k \in N . \tag{3.2}$$

Note that $F_1 \oplus F_2$ is an FK-space in its direct sum topology. By Zeller theory, $A : E \to F_1 \oplus F_2$ is continuous. For each $k \in N$, let $T_k : X \to E$ be defined by $T_k x = e^k(x)$. It follows from the condition (2) that T_k is continuous for all $k \in N$. Since the projection P_i of $F_1 \oplus F_2$ onto $F_i (i = 1, 2)$ are continuous and $g_k^n = P_1 \circ A \circ T$, and $h_k^n = P_2 \circ A \circ T$ for all $n, k \in N$, we have g_k^n and h_k^n are continuous, so $g_k^n, h_k^n \in X'$ for all $n, k \in N$. Let $B = (g_k^n)$ and $C = (h_k^n)$. By (3.2) we have $A = B + C$ and it is clear that $B \in (\Phi(X), F_1)$ and $C \in (\Phi(X), F_2)$. We will show that $B \in (E, F_1)$ and $C \in (E, F_2)$. To do this, let $x = (x_k) \in E$. By the continuity of the matrix $A : E \to F_1 \oplus F_2$ and the AK property for E we find that $A(\sum_{k=1}^{n} e^k(x_k)) \to Ax$ as $n \to \infty$. Since the projection P_i of $F_1 \oplus F_2$ onto $F_i (i = 1, 2)$ are continuous, we have

$$B\left(\sum_{k=1}^{n} e^k(x_k)\right) = P_1\left(A(\sum_{k=1}^{n} e^k(x_k))\right) \to P_1(Ax) \in F_1 \quad \text{and}$$

$$C\left(\sum_{k=1}^{n} e^k(x_k)\right) = P_2\left(A(\sum_{k=1}^{n} e^k(x_k))\right) \to P_2(Ax) \in F_2$$

Hence $B \in (E, F_1)$ and $C \in (E, F_2)$,Therefore, we have $A \in (E, F_1) \oplus (E, F_2)$, as desired.

To show (v), suppose E is normal containing $\Phi(X)$. Since $c(q) = c_0(q) \oplus < e >$, using the same proof as in (iv) we have $(E, c_0(q)) + (E, < e >) \subseteq (E,\ c_0(q) \oplus < e >) = (E, c(q))$ and $(E, c_0(q)) + (E, < e >)$ is a direct sum. If $A = (f_k^n) \in (E,\ c(q)) = (E,\ c_0(q) \oplus < e >)$, the same as in (iv) we can write $A = B + C$ with $B = (g_k^n) \in (\Phi(X), c_0(q))$ and $C = (h_k^n) \in (\Phi(X), < e >)$. Let $x \in E$. Then for $\alpha = (\alpha_k) \in \ell_\infty$, we have

$$\|\alpha_k x_k\| = |\alpha_k| \|x_k\| \leq \|M x_k\| \quad , \text{where } M = \sup_k |\alpha_k| \, .$$

By the normality of E implies that $(\alpha_k x_k) \in E$, it follows that $(f_k^n(x_k))_{n,k} \in (\ell_\infty,\ c_0(q) \oplus < e >)$; Since ℓ_∞ is normal, it follows form [2, Proposition 3.1 (vi)] that $(g_k^n(x_k))_{n,k} \in (\ell_\infty, c_0(q))$ and $(h_k^n(x_k))_{n,k} \in (\ell_\infty,\ < e >)$. This implies that $Bx \in c_0(q)$ and $Cx \in < e >$, so we have $B \in (E, c_0)$ and $C \in (E,\ < e >)$, hence $A \in (E, c_0) \oplus (E, < e >)$, so we obtain (v).

It remains to show (vi). Assume that E is an FK-space with AD, F_2 is an FK-space and F_1 is a closed subspace of F_2. Clearly, $(E, F_1) \subseteq (E, F_2) \cap (\phi(X), F_1)$ is always the case. Now, assume that $A = (f_k^n) \in (E, F_2) \cap (\phi(X), F_1)$ and $x \in E$. By Zeller theorem, $A : E \to F_2$ is continuous. Since E has AD, there is a sequence $(y^{(n)})$ with $y^{(n)} \in \phi(X)$ for all $n \in N$ such that $y^{(n)} \to x$ in E as $n \to \infty$. By the continuity of A, we have $Ay^{(n)} \to Ax$ in F_2 as $n \to \infty$. Since $Ay^{(n)} \in F_1$ for all $n \in N$ and F_1 is a closed subspace of F_2, we obtain that $Ax \in F_1$. Hence $A \in (E, F_1)$, so that $(E, F_2) \cap (\phi(X), F_1) \subseteq (E, F_1)$. This complete the proof. □

Proposition 3.2 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

(i) $c(X, p) = c_0(X, p) + \{e(x) : x \in X\}$.

(ii) $c_0(X, p) = \cap_{n=1}^{\infty} c_0(X)_{(n^{1/p_k})}$. *Hence $c_0(X, p)$ is an echelon space of order* 0.

(iii) $\ell_\infty(X, p) = \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$. *Hence $\ell_\infty(X, p)$ is a co-echelon space of order* ∞.

Proof. Assertion (i) is immediate. To show (ii), let $x \in c_0(X, p)$. Then $\|x_k\|^{p_k} \to 0$ as $k \to \infty$. For each $n \in N$, let $\delta_k = \|x_k\|^{p_k}.n$ for all $k \in N$. We have that $\delta_k \to 0$ as $k \to \infty$; hence $\|x_k\|.n^{1/p_k} = \delta_k^{1/p_k} \to 0$ as $k \to \infty$ (because $p \in \ell_\infty$), so we have $x \in c_0(X)_{(n^{1/p_k})}$. Conversely, assume that $x \in \cap_{n=1}^{\infty} c_0(X)_{(n^{1/p_k})}$. Then $\lim_{k\to\infty} \|x_k\|.n^{1/p_k} = 0$ for every $n \in N$. Then for $n \in N$ we have $|x_k|^{p_k} \leq \frac{1}{n}$ for large k, hence $x \in c_0(X,p)$.

It remains to show (iii). If $x \in \ell_\infty(X,p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \leq n$ for all $k \in N$. Hence $\|x_k\|.n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\|.n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \leq n.M^{p_k} \leq n.M^{\alpha}$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$ □.

Main Results

We now turn to our main objective, the characterisations of matrix transformations from the vector-valued sequence spaces of Maddox $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, and $\ell(X,p)$ into scalar sequence spaces $c_0(q)$, $c(q)$, and $\ell_\infty(q)$. Some results generalize some in [2, 6, 7, 8]. We begin with the following theorem which generalizes [8, Theorem 2.1].

Theorem 4.1 Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A \in (c_0(X,p), c_0(q))$ if and only if

(1) $m^{1/q_n} f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for every m, $k \in N$ and

(2) $\sum_{k=1}^{\infty} m^{1/q_n} \|f_k^n\| r^{-1/p_k} \to 0$ as n, $r \to \infty$ for every fixed $m \in N$.

Proof. By Proposition 3.2 (ii) we have $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/q_k})}$. It follows from Proposition 3.1 (ii) and (vii) that $A \in (c_0(X,p), c_0(q))$ if and only if $(m^{1/q_n} f_k^n)_{n,k} \in (c_0(X,p), c_0)$ for all $m \in N$. By [8, Theorem 2.4], we have $(m^{1/q_n} f_k^n)_{n,k} \in (c_0(X,p), c_0)$ if and only if (1) and (2) hold. □

The next theorem gives a characterization of infinite matrix A such that $A \in (c_0(X,p), c(q))$. To do this we need a lemma.

Lemma 4.2 *Let (f_k) be a sequence of continuous linear functional on X. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$ if and only if $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$.*

Proof. Suppose that $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$. Let $x = (x_k) \in c_0(X,p)$. Then there is a positive integer K such that $\|x_k\|^{p_k} < \frac{1}{M}$ for all $k \geq K$, hence $\|x_k\| < M^{-1/p_k}$ for all $k \geq K$. Then we have

$$\sum_{k=K}^{\infty} |f_k(x_k)| \leq \sum_{k=K}^{\infty} \|f_k\| \|x_k\| \leq \sum_{k=K}^{\infty} \|f_k\| M^{-1/p_k} < \infty .$$

It follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

On the other hand, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x \in c_0(X,p)$. For each $x = (x_k) \in c_0(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_k x_k) \in c_0(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k))$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \text{for all } x \in c_0(X,p) . \tag{4.1}$$

Now, suppose that $\sum_{k=1}^{\infty} \|f_k\| m^{-1/p_k} = \infty$ for all $m \in N$. Choose m_1, $k_1 \in N$ such that

$$\sum_{k \leq k_1} \|f_k\| m_1^{-1/p_k} > 1 ,$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1 < k \leq k_2} \|f_k\| m_2^{-1/p_k} > 2 .$$

Proceeding in this way, we can choose $m_1 < m_2 < \ldots$, and $0 = k_1 < k_2 < \ldots$ such that

$$\sum_{k_{i-1} < k \leq k_i} \|f_k\| m_i^{-1/p_k} > i .$$

Take x_k in X with $\|x_k\| = 1$ for all k, $k_{i-1} < k \leq k_i$ such that

$$\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \text{for all } \ i \in N$$

Put $y = (y_k)$, $y_k = m_i^{-1/p_k}.x_k$ for $k_{i-1} < k \leq k_i$, then $y \in c_0(X,p)$, and we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| \geq \sum_{k_{i-1}<k\leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \text{ for all } i \in N .$$

Hence we have $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$ which contradicts with (4.1). This complete the proof.□

Theorem 4.3 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (c_0(X,p), c(q))$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in N$,

(2) $m^{1/q_n}.(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every m, $k \in N$ and

(3) $\sum_{k=1}^{\infty} m^{1/q_n}.\|f_k^n - f_k\| r^{-1/p_k} \to 0$ as $n,r \to \infty$ for each fixed $m \in N$.

Proof. If $A \in (c_0(X,p), c(q))$, we have $A \in (c_0(X,p),\ c_0(q) \oplus < e >)$ since $c(q) = c_0(q) \oplus < e >$. It follows from Proposition 3.1(v) that $A = B + C$, where $B \in (c_0(X,p), c_0(q))$ and $C \in (c_0(X,p), < e >)$. Let $C = (g_k^n)$. Since $\phi(X) \subseteq c_0(X,p)$, we have $\left(g_k^n(x)\right)_{n=1}^{\infty} \in < e >$ for all $x \in X$ and $k \in N$, which implies that $g_k^n = g_k^{n+1}$ for all n, $k \in N$. For each $k \in N$, let $f_k = g_k^1$. Then we have $(f_k^n - f_k)_{n,k} \in (c_0(X,p), c_0(q))$. Hence (2) and (3) are obtained by Theorem 4.1. Since $C = (f_k)_{n,k} \in (c_0(X,p), < e >)$, we have $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$, hence (1) is obtained by Lemma 4.2

Conversely, assume that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that the conditions (1), (2), and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By the conditions (2) and (3), we obtain by Theorem 4.1 that $B \in (c_0(X,p), c_0(q))$. The condition (1) implies by Lemma 74.2 that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$. This implies $C \in (c_0(X,p), < e >)$. Hence we have by Proposition 4.1(v) that $A \in (c_0(X,p), c(q))$. This completes the proof. □

Theorem 4.4 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A \in (\ell_\infty(X,p), c_0(q))$ if and only if*

(1) $m^{1/q_n}.f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for every k and $m \in N$ and

(2) for each m, $M \in N$, $\sum_{j>k} \|f_j^n\| m^{1/q_n} M^{1/p_j} \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Proof. Since $c_0(q) = \cap_{m=1}^{\infty} c_{0(m^{1/q_k})}$, we have by Proposition 3.1(ii) and (vii) that

$$
\begin{aligned}
A \in (\ell_\infty(X,p), c_0(q)) &\iff A \in (\ell_\infty(X,p), c_{0(m^{1/q_k})}) \text{ for all } m \in N\ . \\
&\iff \left(m^{1/q_n} f_k^n\right)_{n,k} \in (\ell_\infty(X,p), c_0) \text{ for all } m \in N\ . \\
&\iff \text{the conditions (1) and (2) hold (by [8, Theorem 2.9].)}
\end{aligned}
$$

□

Note that Theorem 4.4 generalizes the result in [8, Theorem 2.8].

We now give a characterization of an infinite matrix A such that $A \in (\ell_\infty(X,p), c(q))$ by using the previous auxiliary results and Theorem 4.4. However, in order to get this we need the following lemma.

Lemma 4.5 *Let $p = (p_k)$ be bounded sequence of positive real numbers and (f_k) a sequence with $f_k \in X'$ for all $k \in N$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$ if and only if $\sum_{k=1}^{\infty} \|f_k\| n^{1/p_k} < \infty$ for all $n \in N$.*

Proof. If $\sum_{k=1}^{\infty} \|f_k\| n^{1/p_k} < \infty$ for all $n \in N$, then we have that for each $x = (x_k) \in \ell_\infty(X,p)$, there is $m \in N$ such that $\|x_k\| \le m^{1/p_k}$ for all $k \in N$, hence $\sum_{k=1}^{\infty} |f_k(x_k)| \le \sum_{k=1}^{\infty} \|f_k\| \|x_k\| \le \sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$, which implies $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

Conversely, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$. We first note that, by using the same proof as in Lemma 4.2, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \text{ for all } x = (x_k) \in \ell_\infty(X,p)\,. \tag{4.2}$$

Now, suppose that $\sum_{k=1}^{\infty} \|f_k\| n^{1/p_k} = \infty$ for some $n \in N$. Then we can choose a sequence (k_i) of positive integer with $0 = k_0 < k_1 < k_2 < \ldots$ such that

$$\sum_{k_{i-1}<k\le k_i} \|f_k\| n^{1/p_k} > i \text{ for all } i \in N\,.$$

Taking x_k in X with $\|x_k\| = 1$ such that for all $i \in N$,

$$\sum_{k_{i-1}<k\le k_i} |f_k(x_k)| n^{1/p_k} > i\,.$$

Put $y = (y_k) = \left(n^{1/p_k} x_k\right)_{k=1}^{\infty}$. Clearly, $y \in \ell_\infty(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \ge \sum_{k_{i-1}<k\le k_i} |f_k(x_k)| n^{1/p_k} > i \text{ for all } i \in N\,.$$

Hence $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts with (4.2). The proof is now complete. □

Theorem 4.6 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (\ell_\infty(X,p), c(q))$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k\| n^{1/p_k} < \infty$ for all $n \in N$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every k and $m \in N$ and

(3) for each m, $M \in N$, $\sum_{j>k} \|f_j^n - f_j\| m^{1/q_n} M^{1/p_j} \to 0$ as $k \to \infty$ uniformly on n.

Proof. If $A \in (\ell_\infty(X,p), c(q))$, it follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell_\infty(X,p)$, hence (1) holds by Lemma 4.5. Since $c(q) = c_0(q) \oplus < e >$, we have by

Proposition 3.1(v) that $A = B + C$ where $B \in (\ell_\infty(X,p), c_0(q))$ and $C \in (\ell_\infty(X,p), <e>)$. Since $\Phi(X) \subseteq \ell_\infty(X,p)$, it implies that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$, so we have $(f_k^n - f_k)_{n,k} = B \in (\ell_\infty(X,p), c_0(q))$. Hence we obtain (2) and (3) by Theorem 4.4.

Conversely, assume that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that the conditions (1), (2), and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. The condition (1) implies that $C \in (\ell_\infty(X,p), <e>)$ (by application of Lemma 4.5) and the condition (2) and (3), by Theorem 4.4, implies that $B \in (\ell_\infty(X,p), c_0(q))$. By Proposition 3.1(v), we obtain that $A \in (\ell_\infty(X,p), c(q))$. This complete the proof. □

Theorem 4.7 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (c(X,p), \ell_\infty(q))$ if and only if*

(1) $\sup_n \left(\sum_{k=1}^{\infty} \|f_k^n\| M^{-1/p_k}\right)^{q_n} < \infty$ for some $M \in N$ and

(2) $\sup_n \|T_n\|^{q_n} < \infty$ where $T_n \in X'$ is defined by $T_n x = \sum_{k=1}^{\infty} f_k^n(x)$ for all $x \in X$.

Proof. Assume that $A \in (c(X,p), \ell_\infty(q))$. Since $c(X,p) = c_0(X,p) + E$ where $E = \{e(x) : x \in X\}$, we have by Proposition 3.1(iii) that $A \in (c_0(X,p), \ell_\infty(q))$ and $A \in (E, \ell_\infty(q))$. It follows from [8, Theorem 2.10] that the condition (1) holds. Since $A \in (E, \ell_\infty(q))$, we have $\sum_{k=1}^{\infty} f_k^n(x)$ converges for every $x \in X$ and $\left(\sum_{k=1}^{\infty} f_k^n(x)\right)_{n=1}^{\infty} \in \ell_\infty(q)$. For each $n \in N$, let $T_n x = \sum_{k=1}^{\infty} f_k^n(x)$ for all $x \in X$. It follows by Banach Steinhaus Theorem that $T_n \in X'$. Since $\sup_n |T_n(x)|^{q_n} = \sup_n |\sum_{k=1}^{\infty} f_k^n(x)|^{q_n} < \infty$, by [8, Theorem 1.1] we have $\sup_n \|T_n\|^{q_n} < \infty$, so (2) is obtained.

Conversely, assume that the conditions (1) and (2) hold. It follows from [8, Theorem 2.10] that $A \in (c_0(X,p), \ell_\infty(q))$. We have by (2) that for each $x \in X$,

$$\sup_n |\sum_{k=1}^{\infty} f_k^n(x)|^{q_n} = \sup_n |T_n x|^{q_n} \le (1 + \|x\|)^{\alpha} \sup_n \|T_n\|^{q_n} < \infty$$

where $\alpha = \sup_n q_n$. This implies that $A \in (E, \ell_\infty(q))$ where $E = \{e(x) : x \in X\}$. By an application of Proposition 3.1(iii) we have $A \in (c(X,q), \ell_\infty(q))$. The proof is now complete. □

Theorem 4.8 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (c(X,p), c_0(q))$ if and only if*

(1) $m^{1/q_n} f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for every m, $k \in N$,

(2) $\sum_{k=1}^{\infty} m^{1/q_n} \|f_k^n\| r^{-1/p_k} \to 0$ as n, $r \to \infty$ for every $m \in N$ and

(3) $|\sum_{k=1}^{\infty} f_k^n(x)|^{q_n} \to 0$ as $n \to \infty$ for every $x \in X$.

Proof. Since $c(X,p) = c_0(X,p) + E$ where $E = \{e(x) : x \in X\}$, we have by Proposition 3.1 (iii), $A \in (c(X,p), c_0(q))$ if and only if $A \in (c_0(X,p), c_0(q))$ and $A \in (E, c_0(q))$. Clearly, $A \in (E, c_0(q))$ if and only if the condition (3) holds. By Theorem 4.1, we have $A \in (c_0(X,p), c_0(q))$ if and only if the conditions (1) and (2) hold. So, we have the theorem. □

Theorem 4.9 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (c(X,p), c(q))$ if and only if there is a sequence (f_k) with $f_k^n \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \|f_k^n\| M^{-1/p_k} < \infty$ for some $M \in N$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every m , $k \in N$,

(3) $\sum_{k=1}^{\infty} m^{1/q_n} \|f_k^n - f_k\| r^{-1/p_k} \to 0$ as n, $r \to \infty$ for every $m \in N$ and

(4) $\left(\sum_{k=1}^{\infty} f_k^n(x)\right)_{n=1}^{\infty} \in c(q)$ for all $x \in X$.

Proof. Since $c(X,p) = c_0(X,p) + E$, where $E = \{e(x) : x \in X\}$, it follows from Proposition 3.1 (iii) that $A \in (c(X,p), c(q))$ if and only if $A \in (c_0(X,p), c(q))$ and $A \in (E, c(q))$. By Theorem 4.3, we have $A \in (c_0(X,p), c(q))$ if and only if the conditions

(1) - (3) hold, and clearly, $A \in (E, c(q))$ if and only if (4) holds. Hence, the theorem is proved. □

Theorem 4.10 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k \leq 1$ for all $k \in N$, and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (\ell(X,p), c_0(q))$ if and only if*

(1) $m^{1/q_n} f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for every $m,k \in N$ and

(2) there exists $M \in N$ such that $\|m^{1/q_n} f_k^n\|^{p_k} \leq M$ for all $m,n,k \in N$

Proof. Since $c_0(q) = \cap_{n=1}^{\infty} c_{0(n^{1/q_k})}$, it follows from Proposition 3.1 (ii) and (vii) that $A \in (\ell(X,p), c_0(q))$ if and only if $\left(m^{1/q_n} f_k^n\right)_{n,k} \in (\ell(X,p), c_0)$ for all $m \in N$. By [8, Theorem 2.6], we have $\left(m^{1/q_n} f_k^n\right)_{n,k} \in (\ell(X,p), c_0)$ if and only if the conditions (1) and (2) hold. The proof is now complete. □

Wu and Liu [8, Theorem 2.7] have given a characterization of an infinite matrix A such that $A \in (\ell(X,p), c_0)$ when $p_k > 1$ for all $k \in N$. By using application of Proposition 3.1 (ii) and (iii) , we obtain the following result.

Theorem 4.11 *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in N$, and let $A = (f_k^n)$ be an infinite matrix. Then $A \in (\ell(X,p), c_0(q))$ if and only if*

(1) $m^{1/q_n} f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for all m, $k \in N$ and

(2) for each $m \in N$, $\left(\sum_{k=1}^{\infty} (m^{1/q_n} \|f_k^n\|)^{p_k/(p_k-1)} r^{-1/(p_k-1)}\right) \to 0$ as $r \to \infty$ uniformly on $n \in N$.

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Wilrijk, February 4, 1998.

Prof. A.Kananthai
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Dear Prof. Kananthai,

The Eighth International Congress on Computational and Applied Mathematics will take place at the University of Leuven (Belgium) on July 27 - Aug. 1, 1998.

This is to inform you that your paper 'A survey of distribution theory in solving differental equations' is accepted for being presented at this congress.

Unfortunately, due to financial limitations, we will not be able to pay your travel and local expenses. The participation fee can in your case be reduced to 15.000 BF (instead of the regular 25.000 BF). We hope that you will be able to find funds for making it possible to attend the Congress. Your presence will be very much appreciated.

We look forward to meet you in Leuven.

Yours Sincerely,

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Matrix Transformations of Some Vector-Valued Sequence Spaces

SUTHEP SUANTAI

ABSTRACT. **In this paper, we give the matrix characterizations from vector-valued sequence spaces $\ell_\infty(X,p)$, and $\underset{\sim}{c}_0(X,p)$ into the Orlicz sequence space ℓ_M where $p=(p_k)$ is a bounded sequences of positive real numbers.**

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1. INTRODUCTION

Let $(X, \|.\|)$ be a real Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. We write $x=(x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, and $\underset{\sim}{c}_0(X,p)$ are defined as

$$c_0(X,p) = \{x=(x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\},$$
$$c(X,p) = \{x=(x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\},$$
$$\ell_\infty(X,p) = \{x=(x_k) : \sup_k \|x_k\|^{p_k} < \infty\},$$
$$\ell(X,p) = \{x=(x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\}$$
$$\underset{\sim}{c}_0(X,p) = \{x=(x_k) : \sup_k \|\frac{x_k}{\delta_k}\|^{p_k} < \infty \text{ for some } (\delta_k) \in c_0 \text{ with } \delta_k \neq 0 \text{ for all } k \in N\}$$

When $X=R$, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$,$\ell(p)$, and $\underset{\sim}{c}_o(p)$ respectively. Each of the first four spaces are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7], Maddox [4, 5], and Nakano [6]. In [2] the structure of the spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ have been investigated.

Let $M : R \to [0,\infty)$ be convex, even, continuous and $M(u) = 0 \iff u = 0$. For a given real sequence $x = (x_n)$, define

$$\varrho_M(x) = \sum_{n=1}^{\infty} M(x_n) ,$$

$$\ell_M = \{x = (x_k) : \varrho_M(\lambda x) < \infty \text{ for some} \lambda > 0\}, \text{ and}$$

$$\|x\| = inf\{\lambda > 0 : \varrho_M(\frac{x}{\lambda}) \leq 1\} \text{ for } x \in \ell_M .$$

The sequence space $(\ell_M, \|.\|)$ is known as the Orlicz sequence space and it is a BK-space.

In this paper we consider the problem of characterizing those matrices that map an X-valued sequence spaces $\ell_\infty(X,p)$ and $\underline{c}_0(X,p)$ into the Orlicz sequence spaces. Wu and Liu [8] deal with the problem of characterization those infinite matrices mapping from $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$ and $\ell(X,p)$ into the scalar-sequence spaces of Maddox with some conditions on the sequences (p_k) and (q_k). Grosse-Erdmann [3] has given characterizations of matrix transformations between the scalar-valued sequence spaces of Maddox. Their characterizations are derived from functional analytic principles. Our approach here is different. We use a method of reduction introduced by Grosse-Erdmann [3]. In [2] it is pointed out that $c_0(p)$ is an echelon space of order 0 and that $\ell_\infty(p)$ is a co-echelon space of order ∞. In this paper we also show that $\underline{c}_0(X,p)$ and $\ell_\infty(X,p)$ is a co-echelon space of order ∞. Therefore these spaces are made up of simpler spaces. We will use certain auxiliary results(Section 3) to reduce our problem to the characterisations of matrix mapping between much simpler spaces.

2. Notation and Definitions

2.1 Let $(X, \|.\|)$ be a real Banach space, the space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ is denoted for the space of all finite sequences in X. When $X = R$, the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be any X-valued sequence space. For $x \in E$ and $k \in N$, we write x_k stands for the k^{th} term of x. For $k \in N$ denote by e_k the sequence $(0,0,...,0,1,0,...)$ with 1 in the k^{th} position and by e the sequence $(1,1,1,...)$. For $x \in X$ and $k \in N$, let $e^k(x)$ be the sequence $(0,0,...,0,x,0,...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x,x,x,...)$. For a fixed scalar sequence $\mu = (\mu_k)$ the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} .$$

The sequence space E is called *normal* if $x \in E$ and $y \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $y \in E$.

2.2 Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence

$Ax = (A_n(x)) \in F$. Let (E, F) denote for the set of all infinite matrices mapping from E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E, F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E, F)\}$$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (\frac{1}{u_k})$.

2.3 Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $k \in N$ the k^{th} coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E, τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK-(BK-, LFK-, LBK-) space*. Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

3. Some Auxiliary Results

In this section we give various useful results that can be used to reduce our problems into some simpler forms.

Proposition 3.1 *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar sequence spaces, and let u and v be sequences of real numbers with $u_k \neq 0$, $v_k \neq 0$ for all $k \in N$. Then we have*

(i) $\left(\cup_{n=1}^{\infty} E_n, F\right) = \cap_{n=1}^{\infty} (E_n, F)$

(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty} (E, F_n)$

(iii) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$

(iv) $(E, F_1) = (E, F_2) \cap (\Phi(X), F_1)$ *if E is an FK-space with AD, F_2 is an FK-space and F_1 is a closed subspace of F_2.*

(v) $(E_u, F_v) = {}_v(E, F)_{u^{-1}}$.

Proof. Assertions (i), (ii), (iii), and (v) are immediate. To show (iv), assume that E is an FK-space with AD, F_2 is an FK-space and F_1 is a closed subspace of F_2. Clearly, $(E, F_1) \subseteq (E, F_2) \cap (\Phi(X), F_1)$ is always the case. Now, assume that $A = (f_k^n) \in (E, F_2) \cap (\phi(X), F_1)$ and $x \in E$. By Zeller's theorem, $A : E \to F_2$ is continuous. Since E has AD, there is a sequence $(y^{(n)})$ with $y^{(n)} \in \Phi(X)$ for all $n \in N$ such that $y^{(n)} \to x$ in E as $n \to \infty$. By the continuity of A, we have $Ay^{(n)} \to Ax$ in F_2 as $n \to \infty$. Since $Ay^{(n)} \in F_1$ for all $n \in N$ and F_1 is a closed subspace of F_2, we obtain that $Ax \in F_1$. Hence $A \in (E, F_1)$, so that $(E, F_2) \cap (\Phi(X), F_1) \subseteq (E, F_1)$. This complete the proof. □

Proposition 3.2 *Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then*

(i) $c_0(X, p) = \cup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$. *Hence $c_0(X, p)$ is an echelon space of order 0.*

(ii) $\ell_\infty(X,p) = \cup_{n=1}^{\infty}\ell_\infty(X)_{(n^{-1/p_k})}$. *Hence* $\ell_\infty(X,p)$ *is a co-echelon space of order* ∞.

Proof. (i) Let $x = (x_k) \in \underline{c}_0(X,p)$. Then there is a sequence $(\delta_k) \in c_0$ with $\delta_k \neq 0$ for all $k \in N$ such that $\sup_k \|\frac{x_k}{\delta_k}\|^{p_k} < \infty$. Hence there exists $\alpha > 0$ such that $\|x_k\| \leq \alpha^{1/p_k}|\delta_k|$ for all $k \in N$. Choose $n_0 \in N$ with $n_0 > \alpha$. Then $\|x_k\|n_0^{-1/p_k} \leq (\frac{\alpha}{n_0})^{1/p_k}|\delta_k| < |\delta_k|$ which implies that $\lim_{k\to\infty} \|x_k\|n_0^{-1/p_k} = 0$, hence $x = (x_k) \in c_0(X)_{(n^{-1/p_k})} \subseteq \cup_{n=1}^{\infty}c_0(X)_{(n^{-1/p_k})}$. On the other hand, suppose that $x = (x_k) \in \cup_{n=1}^{\infty}c_0(X)_{(n^{-1/p_k})}$. Then $\lim_{k\to\infty} \|x_k\|n^{-1/p_k} = 0$ for some $n \in N$. Let $\delta = (\delta_k)$ be the sequence definded by

$$\delta_k = \begin{cases} \|x_k\|n^{-1/p_k} & \text{if } \|x_k\| \neq 0 \\ \frac{1}{k} & \text{otherwise.} \end{cases}$$

Clearly $(\delta_k) \in c_0$ and $\|\frac{x_k}{\delta_k}\|^{p_k} \leq n$ for all $k \in N$, hence $\sup_k \|\frac{x_k}{\delta_k}\|^{p_k} \leq n$, so $x = (x_k) \in \underline{c}_0(X,p)$.

Now we show (ii). If $x \in \ell_\infty(X,p)$, then there is some $n \in N$ with $\|x_k\|^{p_k} \leq n$ for all $k \in N$. Hence $\|x_k\|.n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \cup_{n=1}^{\infty}\ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\|x_k\|.n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\|x_k\|^{p_k} \leq n.M^{p_k} \leq n.M^{\alpha}$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$ □.

3. Main Results

We now turn to our main objective. We begin with giving characterisations of matrix transformations from $\ell_\infty(X)$ and $c_0(X)$ into ℓ_M. To do this we need a lemma.

Lemma 4.1 Let $E \in \{\ell_\infty(X), c_0(X)\}$ and (f_k) a sequence of continuous linear functionals on X. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in E$ if and only if $\sum_{k=1}^{\infty}\|f_k\| < \infty$

Proof. If $\sum_{k=1}^{\infty}\|f_k\| < \infty$, then for each $x = (x_k) \in E$, $\sum_{k=1}^{\infty}|f_k(x_k)| \leq \sum_{k=1}^{\infty}\|f_k\|\|x_k\| \leq \|x\|\sum_{k=1}^{\infty}\|f_k\| < \infty$, so that $\sum_{k=1}^{\infty} f_k(x_k)$ converges.

Conversley, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in E$. Define $T : E \to R$ by $Tx = \sum_{k=1}^{\infty} f_k(x_k)$. Clearly, T is linear. For each $n \in N$, let $s_n = \sum_{k=1}^{n} f_k \circ p_k$. Then $s_n \in E'$ since E is a K-space. It is clear that $s_n(x) \to Tx$ as $n \to \infty$ for all $x \in E$. It follows by Banach-Steinhaus theorem that $T \in E'$. Hence there is a positive real number α such that

$$\left|\sum_{k=1}^{\infty} f_k(x_k)\right| \leq \alpha \tag{4.1}$$

for all $x = (x_k) \in E$ with $\|x\| \leq 1$.

For each $x = (x_k) \in E$ with $\|x\| \leq 1$, we can choose a real sequence (t_k) with $|t_k| = 1$ for all $k \in N$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in N$. Clearly, $(t_k x_k) \in E$ and $\|(t_k x_k)\| \leq 1$. It follows by (4.1)

$$\sum_{k=1}^{\infty} |f_k(x_k)| = \left|\sum_{k=1}^{\infty} f_k(t_k x_k)\right| \leq \alpha \tag{4.2}$$

for all $x = (x_k) \in E$ with $\|x\| \leq 1$.
It implies by (4.2) that

$$\sum_{k=1}^{n} |f_k(x_k)| \leq \alpha \tag{4.3}$$

for all $n \in N$ and all $x_k \in X$ with $\|x_k\| \leq 1$.
It follows from (4.3) that $\sum_{k=1}^{n} \|f_k\| \leq \alpha$ for all $n \in N$, hence $\sum_{k=1}^{\infty} \|f_k\| \leq \alpha$. This complete the proof. □

Theorem 4.2 *Let $A = (f_k^n)$ be an infinite matrix and $E \in \{\ell_\infty(X), c_0(X)\}$. Then $A \in (E,\ \ell_M)$ if and only if*

(1) $\sum_{k=1}^{\infty} \|f_k^n\| < \infty$ for every $n \in N$, and

(2) There exists $K > 0$ such that $\sum_{n=1}^{\infty} M(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(x_k)) \leq 1$ for every $(x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$.

Proof. Assume that $A \in (E, \ell_\infty)$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in E$. Hence (1) holds by Lemma 4.1. Since E and ℓ_∞ are BK-spaces, by Zeller's theorem, A is continuous. It follows that there exists $K > 0$ such that

$$\|Ax\| \leq K \tag{4.4}$$

for every $x = (x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$.
Then we have $\|A(\frac{1}{K}x) \leq 1$ for all $x = (x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$. By [1, Theorem 1.38(1)], we have

$$\sum_{n=1}^{\infty} M(\frac{1}{K}\sum_{k=1}^{\infty} f_k(x_k)) \leq 1$$

for every $x = (x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$. Hence (2) holds.

Conversely, assume that (1) and (2) hold. By Lemma 4.1 , we have $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in E$. Let $K > 0$ be such that $\sum_{n=1}^{\infty} M(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(x_k)) \leq 1$ for every $x = (x_k) \in E$ with $\|x_k\| \leq 1$ for all $k \in N$. Then for $x = (x_k) \in E$ and $x \neq 0$, we have

$$\sum_{n=1}^{\infty} M(\frac{1}{K\|x\|}\sum_{k=1}^{\infty} f_k^n(x_k)) = \sum_{n=1}^{\infty} M(\frac{1}{K}\sum_{k=1}^{\infty} f_k^n(\frac{x_k}{\|x\|})) \leq 1$$

which implies that $Ax \in \ell_M$, hence we have $A \in (E,\ \ell_M)$. □

Corollary 4.3 *Let $A = (f_k^n)$ be an infinite matrix. If $(\sum_{k=1}^{\infty} \|f_k^n\|)_{n=1}^{\infty} \in \ell_M$, then $A \in (\ell_\infty(X),\ \ell_M)$.*

Proof. Assume that $(\sum_{k=1}^{\infty} \|f_k^n\|)_{n=1}^{\infty} \in \ell_M$. Then there exists $\lambda > 0$ and $\alpha > 1$ such that $\sum_{n=1}^{\infty} M(\lambda \sum_{k=1}^{\infty} \|f_k^n\|) \leq \alpha$. Let $x = (x_k) \in \ell_\infty(X)$ and $\|x\| \leq 1$. Then $\|x_k\| \leq 1$ for all $k \in N$, so $|f_k^n(x_k)| \leq \|f_k^n\|$ for all $n, k \in N$. Putting $K = \frac{\alpha}{\lambda}$. Since M is convex, even, and increasing on $[0, \infty)$, it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} M\Big(\frac{1}{K} \sum_{k=1}^{\infty} f_k^n(x_k)\Big) &= \sum_{n=1}^{\infty} M\left(\frac{\lambda}{\alpha}\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|\right) \\
&\leq \frac{1}{\alpha} \sum_{n=1}^{\infty} M\left(\lambda\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|\right) \\
&\leq \frac{1}{\alpha} \sum_{n=1}^{\infty} M\Big(\lambda \sum_{k=1}^{\infty} |f_k^n(x_k)|\Big) \\
&\leq \frac{1}{\alpha} \sum_{n=1}^{\infty} M\Big(\lambda \sum_{k=1}^{\infty} \|f_k^n\|\Big) \\
&\leq 1.
\end{aligned}$$

It follows by Theorem 4.2 that $A \in (\ell_\infty(X),\ \ell_M)$. □

Theorem 4.4 *Let $A = (f_k^n)$ be an infinite matrix and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $A \in (\ell_\infty(X, p),\ \ell_M)$ if and only if*

(1) $\sum_{k=1}^{\infty} m^{1/p_k} \|f_k^n\| < \infty$ for all $m, n \in N$, and

(2) There exists $K > 0$ such that

$$\sum_{n=1}^{\infty} M\Big(\frac{1}{K} \sum_{k=1}^{\infty} m^{1/p_k} f_k^n(x_k)\Big) \leq 1$$

for every sequence (x_k) with $\|x_k\| \leq 1$ for all $k \in N$.

Proof. By Proposition 3.2(ii), we have $\ell_\infty(X, p) = \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$. It implies by Proposition 3.1(i) that

$$A \in (\ell_\infty(X, p),\ \ell_M) \iff A \in (\ell_\infty(X)_{(m^{-1/p_k})},\ \ell_M) \text{ for all } m \in N$$

By Proposition 3.1(v), we have

$$A \in (\ell_\infty(X)_{(m^{-1/p_k})},\ \ell_M) \iff (m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X),\ \ell_M)$$

We have by Theorem 4.2 that

$$(m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X),\ \ell_M) \iff \text{ the conditions (1) and (2) hold.}$$

Hence the theorem is proved. □

Theorem 4.5 *Let $A = (f_k^n)$ be an infinite matrix and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $A \in (\underset{\sim}{c}_0(X,p),\ \ell_M)$ if and only if*

(1) $\sum_{k=1}^{\infty} m^{1/p_k}\|f_k^n\| < \infty$ for all $m, n \in N$, and

(2) There exists $K > 0$ such that

$$\sum_{n=1}^{\infty} M\Big(\frac{1}{K}\sum_{k=1}^{\infty} m^{1/p_k} f_k^n(x_k)\Big) \le 1$$

for every sequence $(x_k) \in c_0(X)$ with $\|x_k\| \le 1$ for all $k \in N$.

Proof. Since $\underset{\sim}{c}_0(X,p) = \cup_{n=1}^{\infty} c_0(X)_{(n^{-1/p_k})}$, we have by Proposition 3.1(i) that

$$A \in (\underset{\sim}{c}_0(X,p),\ \ell_M) \Longleftrightarrow A \in (c_0(X)_{(m^{-1/p_k})},\ \ell_M) \ \text{ for all } m \in N$$

By Proposition 3.1(v), we have

$$A \in (c_0(X)_{(m^{-1/p_k})},\ \ell_M) \Longleftrightarrow (m^{1/p_k} f_k^n)_{n,k} \in (c_0(X),\ \ell_M)$$

It follows by Theorem 4.2 that

$$(m^{1/p_k} f_k^n)_{n,k} \in (\ell_\infty(X),\ \ell_M) \Longleftrightarrow \ \text{the conditions (1) and (2) hold.}$$

Hence we have the theorem. □

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International Congress on Computational and Applied Mathematics

Prof. Dr. A. Kananthai
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Leuven, 31 maart 1998

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On convolution equation related to the n-dimentional ultra -hyperbolie operator.

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ON THE CONVOLUTION EQUATION RELATED TO THE N-DIMENTIONAL ULTRA-HYPERBOLIC OPERATOR

A. KANANTHAI

ABSTRACT. We introduce the distribution $e^{\alpha t}\Box^k\delta$ where $\Box^k$ is an ultra-hyperbolic operator iterated k times defined by $\Box^k \equiv \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial t_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial t_j^2}\right)^k$, $k = 0, 1, 2, \dots$, $p+q=n$ the dimension of the Euclidean space $\mathbb{R}^n$. Now δ is the Dirac-delta distribution with $\Box^0\delta = \delta$, $\Box^1\delta = \Box\delta$ and the variable $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$ and the constant $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha t = \alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n$. At first we study the property of $e^{\alpha t}\Box^k\delta$ and after that we study its application of the convolution equation $\left(e^{\alpha t}\Box^k\delta\right) * u(t) = e^{\alpha t}\sum_{r=0}^{m} C_r \overset{r}{\Box}\delta$ where $u(t)$ is the generalized function and C_r is a constant. The convolution equation is the main part of this work which is found that it is related to the ultra-hyperbolic equation. It is also found that the type of solutions of the convolution equation, such as the ordinary functions, the tempered distributions or the singular distributions depending on k, m and α.

1. Introduction

Consider the linear partial differential equation of the form

$$\Box^k u(t) = f(t) \tag{1.1}$$

where $\Box^k$ is the n-dimentional ultra-hyperbolic operator iterated k times, $f(t)$ is the generalized function for $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$. I.M.Gelfand and G.E.Shilov [1, pp. 279-282] have introduced the elementary solution of (1.1) and M.Aguirre Tellez [7, pp. 147-149] also proved that $R_{2k}(t)$ exists only for the case n is odd with p odd and q even or the case n is even with p odd and q odd where $p+q=n$ which are stated at the beginning.

Now, in this paper we consider the convolution equation

$$\left(e^{\alpha t}\Box^k\delta\right) * u(t) = e^{\alpha t}\sum_{r=0}^{m} C_r \overset{r}{\Box}\delta. \tag{1.2}$$

which is the extension of the equation $\left(e^{\alpha t}\Box^k\delta\right) * u(t) = \delta$ introduced by A.Kananthai [8]. The solution $u(t)$ of (1.2) can be obtained by using the method of convolution of the generalized functions. Before going to that point, the following definitions and some concepts are needed.

2. Some Definitions and Lemmas

Definition 2.1. *Let $t = (t_1, t_2, \dots, t_n)$ be a point of $\mathbb{R}^n$ and write $v = t_1^2 + t_2^2 + \cdots + t_p^2 - t_{p+1}^2 - t_{p+2}^2 - \cdots - t_{p+q}^2$, $p+q=n$. Define $\Gamma_+ = \{t \in \mathbb{R}^n : t_1 > 0 \text{ and } v > 0\}$ designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following function is introduce by Y.Nozaki [4, p.72] that is*

$$R_\gamma(t) = \begin{cases} \frac{v^{\frac{\gamma-n}{2}}}{K_n(\gamma)} & \text{if } t \in \Gamma_+ \\ 0 & \text{if } t \notin \Gamma_+ \end{cases} \tag{2.1}$$

$R_\gamma(t)$ is called the ultra-hyperbolic of Marcel Riesz. Here γ is a complex parameter and n the dimension of the space $\mathbb{R}^n$. The constant $K_n(\gamma)$ is defined by

$$K_n(\gamma) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\gamma-n}{2})\Gamma(\frac{1-\gamma}{2})\Gamma(\gamma)}{\Gamma(\frac{2+\gamma-p}{2})\Gamma(\frac{p-\gamma}{2})} \tag{2.2}$$

Let $SuppR_\gamma(t) \subset \overline{\Gamma}_+$. Now $R_\gamma(t)$ is an ordinary function if $R_\gamma(t) \geq n$ and is a distribution of γ if $\mathrm{Re}(\gamma) < n$.

Lemma 2.1. *$R_\gamma(t)$ is a homogeneous distribution of order $\gamma - n$ and also a tempered distribution.*

The proof of this Lemma is given by W.F.Donoghue [5, pp.154-155] which proved the theorem that every homogeneous distribution is a tempered. To prove a homogeneous is not difficult, it is only to show that $R_\gamma(t)$ satisfies the Euler equation

$$\sum_{i=1}^{n} t_i \frac{\partial R_\gamma(t)}{\partial t_i} = (\gamma - n)\, R_\gamma(t)$$

Definition 2.2. *The generalized function $u(t)$ is an elementary solution of the n-dimensional ultra-hyperbolic operator iterated k times if $u(t)$ satisfies the equation $\Box^k u(t) = \delta$ where $\Box^k$ is defined by*

$$\Box^k \equiv \left(\frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \cdots + \frac{\partial^2}{\partial t_p^2} - \frac{\partial^2}{\partial t_{p+1}^2} - \frac{\partial^2}{\partial t_{p+2}^2} - \cdots - \frac{\partial^2}{\partial t_{p+q}^2} \right)^k$$

where $p + q = n$.

Lemma 2.2. *From definition 2.2, if $\Box^k u(t) = \delta$, then $u(t) = R_{2k}(t)$ defined by (2.1) with $\gamma = 2k$ is the unique elementary solution of the equation.*

The proof of this Lemma is given by S.E.Trione [6] and also M.Aguirre Tellez [7, pp.147-149] has proved that $R_{2k}(t)$ exists only for the case n is odd with p odd and q even or the case n is even with p odd and q odd where $p + q = n$.

Lemma 2.3. *(The convolution of $R_{2k}(t)$) Let $R_\gamma(t)$ and $R_\beta(t)$ be defined by (2.1) and γ, β are positive even numbers with $\gamma + \beta = 2k$ where k is a nonnegative integer then $R_\gamma(t) * R_\beta(t) = R_{\gamma+\beta}(t)$.*

Proof. Since $R_\gamma(t)$ and $R_\beta(t)$ are tempered distributions by Lemma 2.1 and let $\mathrm{Supp}R_\gamma(t) = K \subset \overline{\Gamma}_+$ where K is a compact set and $\overline{\Gamma}_+$ appears in definition 2.1. Then $R_\gamma(t) * R_\beta(t)$ exists and well defined. To show that $R_\gamma(t) * R_\beta(t) = R_{\gamma+\beta}(t)$, by Lemma 2.2 $\Box^k u(t) = \delta$ we obtain $u(t) = R_{2k}(t)$. Now $\Box^k u(t) = \Box^r \Box^{k-r} u(t) = \delta$ for $r < k$, then by Lemma 2.2 $\Box^{k-r} u(t) = R_{2r}(t)$. Convolve both sides by $R_{2(k-r)}(t)$ we obtain $R_{2(k-r)}(t) * \Box^{k-r} u(t) = R_{2(k-r)}(t) * R_{2r}(t)$ or $\Box^{k-r} R_{2(k-r)}(t) * u(t) = R_{2(k-r)}(t) * R_{2r}(t)$. By Lemma 2.2 again $\delta * u(t) = R_{2(k-r)}(t) * R_{2r}(t)$. It follows that $u(t) = R_{2(k-r)}(t) * R_{2r}(t)$. Now $u(t) = R_{2k}(t)$ then $R_{2(k-r)}(t) * R_{2r}(t) = R_{2k}(t)$. Let $\gamma = 2k - 2r$ and $\beta = 2\gamma$, actually γ and β are positive even number. It follows that $R_\gamma(t) * R_\beta(t) = R_{\gamma+\beta}(t)$ as required. ∎

3. Properties of $e^{\alpha t}\Box^k\delta$

Lemma 3.1. *The distribution $e^{\alpha t}\Box^k\delta$ has the following properties.*

Properties 3.1.1 *For $k=1$*

$$e^{\alpha t}\Box\delta = \Box\delta - 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial\delta}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\delta}{\partial t_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\delta \qquad (3.1)$$

and $e^{\alpha t}\Box\delta$ is a tempered distribution of order 2 with support $\{0\}$.

Proof. Let $\varphi(t) \in \mathfrak{D}$ the space of testing functions of infinitely differentiable with compact supports and $\mathfrak{D}'$ be the space of distributions. Now

$$\langle e^{\alpha t}\Box\delta, \varphi(t)\rangle = \langle\delta, \Box e^{\alpha t}\varphi(t)\rangle$$

for $e^{\alpha t}\Box\delta \in \mathfrak{D}'$. By computing directly we obtain

$$\begin{aligned}\Box e^{\alpha t}\varphi(t) &= \sum_{i=1}^{p}\frac{\partial^2(e^{\alpha t}\varphi(t))}{\partial t_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2(e^{\alpha t}\varphi(t))}{\partial t_j^2}\\ &= e^{\alpha t}\Box\varphi(t) + 2e^{\alpha t}\left(\sum_{i=1}^{p}\alpha_i\frac{\partial\varphi(t)}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\varphi(t)}{\partial t_j}\right)\\ &\quad + e^{\alpha t}\left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\varphi(t). \qquad (3.2)\end{aligned}$$

Then

$$\begin{aligned}\langle\delta, \Box e^{\alpha t}\varphi(t)\rangle &= \Box\varphi(0) + 2\sum_{i=1}^{p}\alpha_i\frac{\partial\varphi(0)}{\partial t_i} - 2\sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\varphi(0)}{\partial t_j}\\ &\quad + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\varphi(0)\\ &= \left\langle\Box\delta - 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial\delta}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\delta}{\partial t_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\delta, \varphi(t)\right\rangle. \qquad (3.3)\end{aligned}$$

By equality of distributions, we obtain (3.1) as required. To show that $e^{\alpha t}\Box\delta$ is a tempered, from (3.1) δ, $\frac{\partial\delta}{\partial t_i}$, $\frac{\partial\delta}{\partial t_j}$ and $\Box\delta$ have support $\{0\}$ which is compact, hence by L.Schwartz [2], they are tempered distributions. From (3.1), it follows that $e^{\alpha t}\Box\delta$ is also a tempered and by A.H. Zemanian [3, Theorem 3.5-2, p98] $e^{\alpha t}\Box\delta$ is of order 2 with point support $\{0\}$. ∎

Property 3.1.2 (A boundedness properties)

For every testing function $\varphi \in S$ a Schwartz space and $e^{\alpha t}\Box\delta \in S'$ a space of tempered distribution, then $|\langle e^{\alpha t}\Box\delta, \varphi\rangle| \le CM$ where C and M are constant with

$$M = \max\left\{|\varphi(0)|, \left|\frac{\partial\varphi(0)}{\partial t_i}\right|, \left|\frac{\partial\varphi(0)}{\partial t_j}\right|, |\Box\varphi(0)|\right\}$$

$$C = 1 + 2\sum_{i=1}^{p}|\alpha_i| + 2\sum_{j=p+1}^{p+q}|\alpha_j| + \sum_{i=1}^{p}\alpha_i^2 + \sum_{j=p+1}^{p+q}\alpha_j^2. \qquad (3.4)$$

Proof. Since $\langle e^{\alpha t}\Box\delta, \varphi(t)\rangle = \langle\delta, \Box e^{\alpha t}\varphi(t)\rangle$, hence by (3.2) we have

$$\begin{aligned}|\langle e^{\alpha t}\Box\delta, \varphi(t)\rangle| &\le |\Box\varphi(0)| + 2\sum_{i=1}^{p}|\alpha_i|\left|\frac{\partial\varphi(0)}{\partial t_i}\right| + 2\sum_{j=p+1}^{p+q}|\alpha_j|\left|\frac{\partial\varphi(0)}{\partial t_j}\right|\\ &\quad + \left(\sum_{i=1}^{p}\alpha_i^2 + \sum_{j=p+1}^{p+q}\alpha_j^2\right)|\varphi(0)|\end{aligned}$$

Let $M = \max\left\{|\varphi(0)|,\ \left|\frac{\partial\varphi(0)}{\partial t_i}\right|,\ \left|\frac{\partial\varphi(0)}{\partial t_j}\right|,\ |\Box\varphi(0)|\right\}$, then

$$\left|\langle e^{\alpha t}\Box\delta,\ \varphi(t)\rangle\right| \leq \left(1 + 2\sum_{i=1}^{p}|\alpha_i| + 2\sum_{j=p+1}^{p+q}|\alpha_j| + \sum_{i=1}^{p}\alpha_i^2 + \sum_{j=p+1}^{p+q}\alpha_j^2\right)M$$

It follows that $|\langle e^{\alpha t}\Box\delta,\ \varphi(t)\rangle| \leq CM$ where C is defined by (3.4).

Lemma 3.2. *Given $u(t)$ is any distribution in S', then*

$$(e^{\alpha t}\Box\delta) * u(t) = \Box u(t) - 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial u(t)}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial u(t)}{\partial t_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)u(t) \tag{3.5}$$

∎

Proof. Convolve both sides of (3.1) by $u(t)$, we obtain (3.5) .If L is the operator and is defined by

$$L \equiv \Box - 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial t_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right). \tag{3.6}$$

Then (3.5) can be written as

$$(e^{\alpha t}\Box\delta) * u(t) = Lu(t). \tag{3.7}$$

Lemma 3.3. *(The generalization of Lemma 3.2)*

$$(e^{\alpha t}\Box^k\delta) * u(t) = L^k u(t). \tag{3.8}$$

where L^k is the operator defined by (3.6) and is iterated k times ($k = 0, 1, 2, \dots$) with $L^0u(t) = u(t)$.

∎

Proof. We have $\langle e^{\alpha t}\Box^k\delta,\ \varphi(t)\rangle = \langle\Box^k\delta,\ e^{\alpha t}\varphi(t)\rangle$ for every $\varphi(t) \in \mathfrak{D}$ and $e^{\alpha t}\Box^k\delta \in \mathfrak{D}'$. So

$$\begin{aligned}\langle\Box^k\delta,\ e^{\alpha t}\varphi(t)\rangle &= \langle\Box^{k-1}\delta,\ \Box e^{\alpha t}\varphi(t)\rangle \\ &= \langle\Box^{k-1}\delta,\ e^{\alpha t}T\varphi(t)\rangle\end{aligned}$$

where T is the operator from (3.2) and is defined by

$$T \equiv \Box + 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial t_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right). \tag{3.9}$$

So

$$\begin{aligned}\langle\Box^{k-1}\delta,\ e^{\alpha t}T\varphi(t)\rangle &= \langle\Box^{k-2}\delta,\ \Box e^{\alpha t}T\varphi(t)\rangle \\ &= \langle\Box^{k-2}\delta,\ e^{\alpha t}T(T\varphi(t))\rangle \\ &= \langle\Box^{k-2}\delta,\ e^{\alpha t}T^2\varphi(t)\rangle.\end{aligned}$$

By keeping on operating $\Box$ with $k-2$ times, we obtain

$$\begin{aligned}\langle\Box^{k-2}\delta,\ e^{\alpha t}T^2\varphi(t)\rangle &= \langle\delta,\ e^{\alpha t}T^k\varphi(t)\rangle \\ &= T^k\varphi(0)\end{aligned}$$

where T^k is the operator of (3.9) iterated k times. Now

$$\begin{aligned}T^k\varphi(0) &= \langle\delta,\ T^k\varphi(t)\rangle \\ &= \langle L\delta,\ T^{k-1}\varphi(t)\rangle\end{aligned}$$

by the operator L in (3.6) and the derivatives of distribution. Continue this process, we obtain $T^k\varphi(0) = \langle L^k\delta, \varphi(t)\rangle$ or $\langle e^{\alpha t}\Box^k\delta, \varphi(t)\rangle = \langle L^k\delta, \varphi(t)\rangle$. It follows that

$$e^{\alpha t}\Box^k\delta = L^k\delta. \tag{3.10}$$

Convolve both sides of (3.10) by the distribution $u(t)$, then we obtain (3.8). ∎

4. Proof of Theorems

Theorem 4.1. *Let L be the partial differential operator defined by*

$$L \equiv \Box - 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial t_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)$$

where this operator appears in (3.4). Now L is an ultra-hyperbolic type. Consider the equation

$$Lu(t) = \delta \tag{4.1}$$

where $u(t)$ is any distribution in S' then $u(t) = e^{\alpha t}R_2(t)$ is a unique elementary solution of (4.1) where $R_2(t)$ is defined by (2.1) with $\gamma = 2$.

Proof. From (3.4) and (4.1) we can write $(e^{\alpha t}\Box\delta) * u(t) = Lu(t) = \delta$. Convolve both sides by $e^{\alpha t}R_2(t)$ we have

$$\begin{aligned}(e^{\alpha t}R_2(t)) * ((e^{\alpha t}\Box\delta) * u(t)) &= (e^{\alpha t}R_2(t)) * \delta \\ &= e^{\alpha t}R_2(t).\end{aligned}$$

Then

$$e^{\alpha t}(R_2(t) * \Box\delta) * u(t) = e^{\alpha t}R_2(t)$$

or

$$(e^{\alpha t}\Box R_2(t)) * u(t) = e^{\alpha t}R_2(t)$$

or

$$(e^{\alpha t}\delta) * u(t) = e^{\alpha t}R_2(t)$$

by Lemma 2.2 with $k = 1$. It follows that $u(t) = e^{\alpha t}R_2(t)$ since $e^{\alpha t}\delta = \delta$. We can check the solution $u(t)$ by computing directly from (4.1).

Theorem 4.2. *(The generalization of theorem 4.1)*
From Lemma 3.3, consider

$$(e^{\alpha t}\Box^k\delta) * u(t) = \delta \tag{4.2}$$

or

$$L^k u(t) = \delta \tag{4.3}$$

then $u(t) = e^{\alpha t}R_{2k}(t)$ is the unique elementary solution of (4.2) or (4.3)

Proof. We can prove by using the equation (4.2) or (4.3) as well. If we start with the equation (4.2), we convolve both sides of (4.2) by $e^{\alpha t}R_{2k}(t)$, we obtain

$$\begin{aligned}(e^{\alpha t}R_{2k}(t)) * ((e^{\alpha t}\Box^k\delta) * u(t)) &= e^{\alpha t}R_{2k}(t) * \delta \\ &= e^{\alpha t}R_{2k}(t)\end{aligned}$$

or $e^{\alpha t}\left(\Box^k R_{2k}(t)\right) * u(t) = e^{\alpha t} R_{2k}(t)$. Since $\Box^k R_{2k}(t) = \delta$ by Lemma 2.2, we have $(e^{\alpha t}\delta) * u(t) = \delta * u(t) = u(t) = e^{\alpha t} R_{2k}(t)$ as required. Or if we use the equation (4.3), we convolve both sides of (4.3) by $e^{\alpha t} R_2(t)$ then we obtain

$$e^{\alpha t} R_2(t) * L^k u(t) = e^{\alpha t} R_2(t) * \delta = e^{\alpha t} R_2(t)$$

or $L(e^{\alpha t} R_2(t)) * L^{k-1} u(t) = e^{\alpha t} R_2(t)$. By theorem 4.1, we obtain $\delta * L^{k-1} u(t) = e^{\alpha t} R_2(t)$ or $L^{k-1} u(t) = e^{\alpha t} R_2(t)$. By keeping on convolving $e^{\alpha t} R_2(t)$ $k-1$ times, we obtain

$$\begin{aligned} u(t) &= e^{\alpha t}\left(R_2(t) * R_2(t) * \cdots * R_2(t)\right) \\ &= e^{\alpha t} R_{2k}(t) \end{aligned}$$

by Lemma 2.3. ∎

Theorem 4.3. *Given the convolution equation*

$$\left(e^{\alpha t}\Box^k \delta\right) * u(t) = e^{\alpha t} \sum_{r=0}^{m} C_r \Box^r \delta \qquad (4.4)$$

where $\Box^k$ *is the ultra-hyperbolic operator iterated* k *times defined by*

$$\Box^k \equiv \left(\frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \cdots + \frac{\partial^2}{\partial t_p^2} - \frac{\partial^2}{\partial t_{p+1}^2} - \frac{\partial^2}{\partial t_{p+2}^2} - \cdots - \frac{\partial^2}{\partial t_{p+q}^2}\right)^k$$

where $p+q=n$ *the dimension of the space* $\mathbb{R}^n$ *with* p *odd and* q *odd or* p *odd and* q *even, the variable* $t=(t_1, t_2, \ldots t_n) \in \mathbb{R}^n$, *the constant* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ *and* δ *is the Dirac-delta distribution with* $\Box^0\delta = \delta$, $\Box^1\delta = \Box\delta$ *and* C_r *is a constant. Then the type of solutions* $u(t)$ *of (4.4) depend on* k, m *and* α *as the following cases.*

(1) If $m < k$ *and* $m = 0$, *then the solution of (4.4) is* $u(t) = c_0 e^{\alpha t} R_{2k}(t)$ *which is the elementary solution of the operator* $\Box^k$. *Now* $R_{2k}(t)$ *is defined by (2.1) with* $\gamma = 2k$. *If* $2k \geq n$ *and for any* α, *then* $e^{\alpha t} R_{2k}(t)$ *is the ordinary function. If* $2k < n$ *and for some* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *with* $\alpha_i < 0$ $(i = 1, 2, \ldots, n)$, *then* $e^{\alpha t} R_{2k}(t)$ *is a tempered distribution.*

(2) If $0 < m < k$, *then the solution of (4.4) is* $u(t) = e^{\alpha t} \sum_{r=1}^{m} C_r R_{2k-2r}(t)$ *which is the ordinary function for* $2k - 2r \geq n$ *with any arbitrary constant* α *and is a tempered distribution if* $2k - 2r < n$ *for some* α *with* $\alpha_i < 0$ $(i = 1, 2, \ldots, n)$.

(3) If $m \geq k$ *and for any* α *and suppose that* $k \leq m \leq M$, *then (4.4) has* $u(t) = e^{\alpha t} \sum_{r=k}^{M} C_r \Box^{r-k}\delta$ *as a solution which is only the singular distribution.*

Proof. (1) For $m < k$ and $m = 0$, then (4.4) becomes

$$\left(e^{\alpha t}\Box^k \delta\right) * u(t) = c_0 e^{\alpha t}\delta = c_0\delta$$

and by Theorem 4.2 we obtain $u(t) = c_0 e^{\alpha t} R_{2k}(t)$. Now $R_{2k}(t)$ is defined by (2.1) with $\gamma = 2k$. If $2k \geq n$ we obtain $R_{2k}(t)$ is an analytic function for every $t \in \Gamma_+$ where Γ_+ appears in the definition 2.1 and so $R_{2k}(t)$ is the ordinary function. Now $e^{\alpha t}$ is a continuous function and is infinitely differentiable for every $t \in \Gamma_+$ and every α. It follows that $e^{\alpha t} R_{2k}(t)$ is the ordinary function. Now if $2k < n$ then R_{2k} is an analytic function except at the origin and by Lemma 2.1 $R_{2k}(t)$ is a tempered distribution and for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i < 0$ $(i = 1, 2, \ldots, n)$ we have $e^{\alpha t}$ is a slow growth function and also its partial derivative is a slow growth. It follows that $c_0 e^{\alpha t} R_{2k}(t)$ is a tempered distribution.

(2) For $0 < m < k$, then we have

$$\left(e^{\alpha t}\Box^k\delta\right) * u(t) = c_1 e^{\alpha t}\Box\delta + c_2 e^{\alpha t}\Box^2\delta + \cdots + c_m e^{\alpha t}\Box^m\delta.$$

Convolve both sides by $e^{\alpha t}R_{2k}(t)$ and by Lemma 2.2 we obtain

$$u(t) = c_1 e^{\alpha t}\Box R_{2k}(t) + c_2 e^{\alpha t}\Box^2 R_{2k}(t) + \cdots + c_m e^{\alpha t}\Box^m R_{2k}(t).$$

Now $\Box^k R_{2k}(t) = \delta$, then $\Box^{k-r}\Box^r R_{2k}(t) = \delta$ for $r < k$. Convolve both sides by $R_{2k-2r}(t)$ we obtain

$$R_{2k-2r}(t) * \Box^{k-r}\Box^r R_{2k}(t) = R_{2k-2r}(t) * \delta = R_{2k-2r}(t)$$

or

$$\Box^{k-r} R_{2k-2r}(t) * R_{2k}(t) = R_{2k-2r}(t)$$

or

$$\delta * \Box^r R_{2k}(t) = \Box^r R_{2k}(t) = R_{2k-2r}(t)$$

for $r < k$. It follows that

$$u(t) = c_1 e^{\alpha t} R_{2k-2}(t) + c_2 e^{\alpha t} R_{2k-4}(t) + \cdots + c_m e^{\alpha t} R_{2k-2m}(t)$$

or $u(t) = e^{\alpha t}\sum_{r=1}^{m} C_r R_{2k-2r}(t)$. Similarly, as in the case (1), $e^{\alpha t}R_{2k-2r}(t)$ is the ordinary function for $2k-2r \geq n$ and for any α. It follows that $u(t) = e^{\alpha t}\sum_{r=1}^{m} C_r R_{2k-2r}(t)$ is also the ordinary dunction. For the case $2k - 2r < n$ and for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i < 0$ $(i = 1, 2, \ldots, n)$ we obtain $e^{\alpha t}R_{2k-2r}(t)$ is a tempered distribution. It follows that $u(t) = e^{\alpha t}\sum_{r=1}^{m} C_r R_{2k-2r}(t)$ is also a tempered distribution.

(3) For $m \geq k$ and for any α and suppose that $k \leq m \leq M$, we have

$$\left(e^{\alpha t}\Box^k\delta\right) * u(t) = c_k e^{\alpha t}\Box^k\delta + c_{k+1} e^{\alpha t}\Box^{k+1}\delta + \cdots + c_M e^{\alpha t}\Box^M\delta.$$

Convolve both sides by $e^{\alpha t}R_{2k}(t)$ and by Lemma 2.2 again we have

$$u(t) = c_k e^{\alpha t}\Box^k R_{2k}(t) + c_{k+1} e^{\alpha t}\Box^{k+1} R_{2k}(t) + \cdots + c_M e^{\alpha t}\Box^M R_{2k}(t).$$

Now

$$\Box^m R_{2k}(t) = \Box^{m-k}\Box^k R_{2k}(t) = \Box^{m-k}\delta$$

for $k \leq m \leq M$. So

$$\begin{aligned} u(t) &= c_k e^{\alpha t}\delta + c_{k+1} e^{\alpha t}\Box\delta + c_{k+2} e^{\alpha t}\Box^2\delta + \cdots + c_M e^{\alpha t}\Box^{M-k}\delta \\ &= e^{\alpha t}\sum_{r=k}^{M} C_r \Box^{r-k}\delta. \end{aligned}$$

Now, by (3.10) $e^{\alpha t}\Box^{r-k}\delta = \Box^{r-k}\delta +$ (the terms of lower order of partial derivative of δ). Since all terms of the right hand side of the above equation are singular distributions. It follows that $u(t) = e^{\alpha t}\sum_{r=k}^{M} C_r \Box^{r-k}\delta$ is only a singular distribution. That completes the proof. ∎

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APPENDIX 7

ASEANIP Council Meeting & Activities Report

ASEANIP Council Meeting

ASEANIP (Asean Institute of Physics)† held a Council Meeting during 5-6 October at the Chulalongkorn University, Bangkok, Thailand. The Meeting was sponsored by ROSTSEA, UNESCO Office at Jakarta. Participants of the Meeting were Prof. Virulh Sa-yakanit. (President, Thai Institute of Physics), Dr. Jong-Orn Berananda (observer, Treasurer of TIP), Dr. Wichit Sritrakool (observer, Secretary of ITP), Prof. S. Parangtopo (President, Indonesian Physical Society), Prof. S. P. Chia (President, Malaysian Institute of Physics, Prof. Danilo M. Yanga (President, Philippines Physical Society), Professor Dao Vong Duc (observer, President of Vietnam National Institute of Physics) Dr. F. Zhang (observer, Program Specialist, ROTSEA, UNESCO) and S. C. Lim (observer, AAPPS Council Member).

The main item of the agenda was the election of the President to fill the vacancy left by Dr. B. C. Tan who resigned in 1992. Prof. Virulh Sa-yakanit was unanimously elected as the new President with Prof. S. C. Lim as the Hon. Secretary and Dr. Jong-Orn Berananda as the Hon. Treasurer.

Some amendments to the Constitution were adopted by the Council after a lengthy deliberation. One notable change is that election of President shall take place once every three years; and the term of office of the President shall not be more than two consecutive terms.

The Meeting welcomed Vietnam National Institute of Physics as a new member of ASEANIP, thus making the total number of members to six. It was also agreed that efforts should be made to encourage physicists from Burma, Cambodia and Laos to participate in ASEANIP activities, and to get the physical societies of these countries to join ASEANIP.

Finally, the Council came up with the following tentative plan for the ASEANIP Regional Workshops:

- Regional Workshop on Applications of Synchrotron Radiation (December 1995, Bangkok, Thailand)
- Regional Workshop on Surface Physics (1996, Kuala Lumpur, Malaysia)
- Regional Workshop on Physics of Metals and Alloys (December 1996/January 1997, Indonesia)
- Regional Workshop on Condensed Matter Physics (April 1997, Hanoi, Vietnam)

† ASEANIP has five members: Indonesian Physical Society, Malaysian Institute of Physics, Philippines Society, Singapore Institute of Physics, Thai Institute of Physics. Vietnam National Institute of Physics joined as the sixth member during the Meeting.

Participants of the ASEANIP Meeting: (upper row left to right) Wichit Sritrakool, Parangtopo, S. C. Lim, Dao Vong Duc; (lower row left to right) F. Zhong, Jong-Orn Berananda, Virulh Ja-yakanit, S. P. Chia and Danilo M. Yanga.

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members were 42 scientists. In 1979 the members reached 191 and it increased to 400 members in 1990. In 1991 the total number of members was over 800. Currently there are approximately 2000 physicists in Indonesia.

For the time being HFI already published regular Physics Journal in Indonesian language and English. HFI will continue to develop as self-sustainable organization for fostering scientific communication and cooperation among the physicists in Indonesia as well as in the South-East Asian countries.

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Report on the Activities and
Achievments of ASEANIP

submitted to
ROSTSEA/UNESCO
for the support of
the ASEANIP Activities

History

The ASEANIP was founded in Kuala Lumpur Malaysia in 1980. At the time of its founding an international Conference on physics and technology for the 80's was taking place at the Malaysian Institute of Physics.

On the third of September, a group of physicists who were attending the conference met together and decided that an institute for physics in ASEAN countries was needed and now was the time to inaugurate it. At that time, ASEAN consisted of only five countries: Indonesia, Malaysia, Philippines, Singapore and Thailand. So it was these five who were the first members. Since Mynmar and Vietnam have become members of ASEAN and only Vietnam has joined ASEANIP but we expect the others to follow suit soon.

The objectives that the founding members established are listed below.

1) To promote the advancement and status of physics in ASEAN countries.

2) To facilitate cooperation among physicists in ASEAN countries through information exchange on current research interests as well as teaching methods and curriculum development.

3) To encourage joint research projects in ASEAN countries to maximize usage of research facilities and manpower.

4) To develop and maintain relations between organizations of physicists in the ASEAN countries and other regional and international physics related organization in all aspects of their activities.

5) To publish newsletters, bulletins, journals and other publications of the institute.

6) The hold annual regional conferences as a means to achieve the objectives of the institute.

7) To organize lecture tours by prominent physicists to stimulate active discussion of important developments in physics.

8) To connect with agencies such as UNESCD, IAEA, IUPAP, ICTP, APPS and COSTED in order to obtain support, financially and otherwise.

A pro-tem committee was elected to organize the formation as well as to define a program of activities of ASEANIP and also to seek financial support from various international agencies. An ambitious program was drawn up by the committee covering the 1981-1986 period.

Past Activities and Past Presidents

Although ambitious plans were made for the period 1981-1986 not all programs were realized because funding was not available until 1983. So our first project took place from 6 December 1983 to 14 January 1984. This was the First Tropical College on Applied Physics: Laser and Plasma Technology. It took place in Kuala Lumpur and was very much of a hands-on learning experience for the forty participants from ASEAN and neighboring countries. Participants learned how to construct laser power systems including both power supplies and associated control electronics. Thus they could start research in their own countries immediately on returning home and indeed many did so. Most of the lecturers were local so the costs were minimized. However, there were scientists from developed countries to add depth to the college.

The second program of the 1981-1985 time period was The Workshop on Microcomputer Applications and Measurement : Techniques in Physics Education held in Bangkok from 28 October - 9 November 1985. The driving force for the workshop was a proposal to UNESCO from ASEANIP who provided the funds to make it possible. As in the Tropical College, the concept was to make the workshop a hands-on experience for participants. This time the focus was on microcomputers. There were twenty-five participants from ASEAN and neighboring countries. The equipment needed in this course was cheaper than the Tropical College and the knowledge gained was so important that this workshop was very cost effective. It was agreed by all involved in the workshop that such programs involving microcomputers and their applications should be held regularly throughout the region. The workshop was an ASEAN idea however, it was conducted under the auspices of ASPEN.

The final program of the first period was the second Tropical College on Laser and Plasma Technology which was held at the physics department of the University of Malaya in Kuala Lumpur. Sixty participants from eleven different countries attended this intensive three week course which again focused on practical hands-on experience and it had the same goals as the First Tropical College. The college was funded by UNESCO and ICTP. The participants were given the proceedings of the First Tropical College which was used as a basic text. The Second College took place from 17 March to 5 April 1986.

After the end of the first five year period, it was decided that the second five year (1987-1991) should contain more activities. The Tropical College series was found to be very productive and cost-effective so it was decided they should certainly be continued once every two years. Also, the Workshop on Microcomputers produced such good results at such a low cost that it should be held once a year in each ASEAN country. Other important areas of physics such as semiconductor physics and applications of radioisotopes would only be held as conferences. The first program of the second period was a Workshop on Topics in Semiconductor Physics held in Bangkok from 5-9 January 1987. A number of research institutions in the Southeast Asian region were involved in research on semiconductors and it was considered worthwhile for them to meet together along with other Asian researchers to exchange ideas and results. It was sponsored by ASEANIP as well as UNESCO, ICTP, CIDA, TIP, Uppsala University and Chulalongkorn University. The first program of period was a Workshop on Topics in Semiconductor Physics held in Bangkok from 5 - 9 January 1987. A number of research institutions in the Southeast Asian region were involved in research on semiconductors and it was considered worthwhile for them to meet together along with other Asian researchers to exchange ideas and results. It was sponsored by ASEANIP as well as UNESCO, ICTP, CIDA, TIP Uppsala University and Chulalongkorn University. In the interest of brevity we shall list the other programs that took place during the second five-year period without elaborating on the details. We shall just say that they were all worthwhile and they were all successful in promoting physics in Southeast Asia..

- Workshop on the Design and Construction of Microcomputer-aided and Microcomputer-controlled Physics Experiments, Kuala Lumpur (1987).
- Third Tropical College on Applied Physics : Laser and Plasma Technology, Kuala Lumpur (1988).
- Microcomputer Applications in Physics, Singapore (1988).
- Workshop on Radioisotopes and Applications, Indonesia (1989).
- Microcomputer Applications in Physics, Philippines (1989).
- Fourth Tropical College on Applied Physics : Laser and Plasma Technology May - June 1990, Kuala Lumpur (1990).
- Conference on Physics and Technologies in the Nineties, Singapore (1990).
- Microcomputer Applications in Physics, Indonesia (1991).
- Workshop on Solar Physics, Philippines (1991).
- Microcomputer Applications in Physics, Thailand (1991).

In addition, a quaterly journal published by ASEANIP was proposed and seriously considered.

Things were progressing very satisfactorily as ASEANIP held its third council meeting on 26 July 1989 at the University of Malaya in Kuala Lumpur. The presidents of all the National Institutes of ASEAN countries were present except for Singapore. The vice president of the Malaysian Institute was present as well as Mr. T. Kuroda of the ROSTSEA/Unesco.

At this meeting the constitution of ASEANIP was passed. Prof. B. C. Tan was unanimously elected to be the chairman of ASEANIP for a further two years. In addition, the council decided that ASEANIP should join AAPPS as a founding member and it also encouraged member societies to join AAPPS as well. In addition, seven tentative programs were accepted to be carried out by national physics societies under the auspices of ASEANIP.

After 1991, ASEANIP disappeared from the world of physics. The reason is that its dynamic president, Prof. B. C. Tan (who was also president of the Malaysian Institute of Physics), abruptly resigned both presidencies. The cause of his resignation had nothing to do with his scientific ability. He was so disgusted that he left science completely and is today a businessman.

Prof. Virulh Sa-yakanit, the president of the Thai Institute of Physics, was determined to resurrect ASEANIP and make it once again a leader in promoting science in the ASEAN comtries. To accomplish this, he organized an ASEANIP council meeting under the auspices of the Thai Institute of Physics and sponsored by ROSTSEA/UNESCO. It was held at Chulalongkorn University in Bangkok from 6 - 7 October 1994. Present at the meeting were the presidents of all the national institutes of physics of the ASEAN countries except Singapore. The president of the Malaysian Institute of Physics, Prof. S. P. Chia was at that time the acting president of ASEANIP.

Also present were the Vice president and secretary general of the Thai Institute of Physics, Prof. F. Zhang of ROSTSEA/UNESCO, S. C. Lim of AAPPS and Dao Van Duc the president of the Vietnam Institute of Physics.

Prof. Dao Van Duc's presence was of enormous importance because it was at this meeting that Vietnam was formally admitted to ASEANIP. Vietnam is certainly the most scientifically advanced country in Southeast Asia. Having Vietnam as a member, ASEANIP's prestige and stature in physics grew immensely. With Vietnamese physicists participating in and planning physics programs, the quality of all our events improved greatly. With Vietnam ASEANIP gained greater recognition in the international scientific community.

A pro-tem committee was elected to organize, formulate and define activites that would activate the moribund organization. Several proposed amendments of ASEANIP's constitution that improved its effectiveness and efficiency were made and passed.

One of the new amendments was that the term of the president would be three years and that no one president could serve more than two consecutive terms. It was also resolved that ASEANIP would strive to persuade Cambodia, Laos and Myanmar to organize institute of physics and to have these institutes become members of ASEANIP.

Prof. Virulh Sa-yakanit, the president of the Thai Institute of Physics, was elected as the new president of ASEANIP for three years. Immediately, the committee decided that the first two main activities of ASEANIP will be Regional Workshops on the Applications of Synchrotron Light Sources to be held in Bangkok in 1995 and 1996.

The other main activities for the 1994-1997 period of Prof.Sa-yakanit's Presidency are:

- A Regional Workshop on Surface Physics in Kuala Lumpur, 1996.
- A Regional Workshop on the Physics of Metals and Alloys from December 1996 to January 1997 in Indonesia.
- A Regional Workshop on Condensed Matter Physics in April 1997 to be held in Hanoi, Vietnam.

This historic meeting in 1994, inspired by Prof. Sa-yakanit, definitely showed that ASEANIP was alive and active and was definitely here to stay as a force for developing physics in Southeast Asia.

As Prof. Sa-yakanit's term in office drawing to a close in 1997, it was necessary to hold a new meeting of ASEANIP to elect a new president and decide which activities to plan in the next three year period.

As a result, the fifth ASEANIP Council meeting was held in Korat Thailand on 25 February 1997 at the end of the Regional School on the Applications in Korat during the entire month of February. All the presidents of the institutes of physics of the ASEAN countries were present except for Brunci and Singapore.

As the prime mover of ASEANIP, Prof. Sa-yakanit was re-elected to another three year term. His election showed the appreciation that the council felt for the success that ASEANIP achieved under his leadership. The following four activities were agreed upon for the future:

- High Energy Physics Conference to be held in Vietnam (1997/1998).
- School for Theoretical Physics to be held in Malaysia (1998).
- School on Synchrotron Radiation Physics to be held in Thailand (1998).
- Condensed Matter Physics to be held in the Philippines (1999).

Conclusion

ASEANIP is here to stay. Its future is bright. We are determined to see all ASEAN countries grow together in unison. In the future we expect that Cambodia, Laos and Myanmar will gain the most as they are not at the moment as developed as the other ASEAN countries.

The manpower, talent and will are all there to move forward quickly into the new century. Financial support to ASEAN from international agencies will be an excellent investment in terms of the return that they will get. We at ASEANIP all do believe that we can become fully developed countries scientifically by 2020 at the latest as long as we can gain the necessary funding.

APPENDIX 8

Related Documents
(Attached Separately)

1. APCTP BULLETIN
2. AAPPS BULLETIN
3. Proceeding of the Regional Workshop on Applications of Synchrotron Radiation, 3-7 January 1996, Chulalongkorn University, Bangkok