

On the Residue of the Generalized Function P^λ

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These the residues of $\langle P^\lambda, \varphi \rangle$ at $\lambda = -\frac{n}{2m} - k$ is a functional concentrated on the vertex of the cone.

Now consider the singular point $\lambda = -k$. By (3.11) and (3.12) and also See [1, p255-256]. We obtain

$$\operatorname{res}_{\lambda=-k} \frac{(-1)^{k-1}}{(k-1)!} \langle \delta_1^{*(k-1)}(P), \varphi \rangle \quad (3.14)$$

where

$$\langle \delta_1^{*(k-1)}(P), \varphi \rangle = \frac{1}{4m^2} \int_0^\infty \left[\left(\frac{1}{2ms^{2m-1}} \frac{\partial}{\partial s} \right)^{k-1} \left\{ s^{q-2m} \frac{\psi(r, s)}{2m} \right\} \right]_{s=r} r^{p-1} dr. \quad (3.15)$$

Summarizing, we have the following. For odd n for even n if $k < \frac{1}{2m}n$, the generalized function P^λ has simple poles at $\lambda = -k$ for positive integral values of k , where the residues are

$$\operatorname{res}_{\lambda=-k} P^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{*(k-1)}(P). \quad (3.16)$$

Now consider the singular point at $\lambda = -\frac{n}{2m} - k$, from (3.13) we have

$$\operatorname{res}_{\lambda=-\frac{n}{2m}-k} \langle P^\lambda, \varphi \rangle = \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi\left(-\frac{n}{2m} - k, u\right) \right]_{u=0}$$

for n odd with p odd and q even. Thus, for $k = 0$ we have

$$\operatorname{res}_{\lambda=-\frac{n}{2m}} \langle P^\lambda, \varphi \rangle = \Phi\left(-\frac{n}{2m}, 0\right).$$

By (3.10), we obtain

$$\begin{aligned} \operatorname{res}_{\lambda=-\frac{n}{2m}} \langle P^\lambda, \varphi \rangle &= \frac{1}{4m^2} \int_0^1 (1-t)^{-\frac{n}{2m}} t^{\frac{q}{2m}-1} \psi_1(0, 0) dt \\ &= \frac{\psi_1(0, 0)}{4m^2} \int_0^1 (1-t)^{-\frac{n}{2m}} t^{\frac{q}{2m}-1} dt \\ &= \frac{\psi_1(0, 0)}{4m^2} \frac{\Gamma(\frac{q}{2m}) \Gamma(-\frac{n}{2m} + 1)}{\Gamma(-\frac{p}{2m} + 1)}, \end{aligned}$$

since

$$\int_0^1 (1-t)^{-\frac{n}{2m}} t^{\frac{q}{2m}-1} dt = \frac{\Gamma(\frac{q}{2m}) \Gamma(-\frac{n}{2m} + 1)}{\Gamma(-\frac{p}{2m} + 1)}.$$

Now

$$\begin{aligned} \psi_1(0, 0) &= \psi(0, 0) = \int \varphi(0) d\Omega_p d\Omega_q \quad \text{by (3.5)} \\ &= \Omega_p \Omega_q \varphi(0) \end{aligned}$$

and

$$\Omega_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \quad \text{and} \quad \Omega_q = \frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})}.$$

Thus

$$\text{res}_{\lambda=-\frac{n}{2m}} < P^\lambda, \varphi > = \frac{1}{4m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(-\frac{n}{2m}+1)}{\Gamma(-\frac{p}{2m}+1)} \left(\frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \right) \left(\frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})} \right) \varphi(0). \quad (3.17)$$

Now, for $p \geq 2m$ and p is even then $\Gamma(-\frac{p}{2m}-1) = \infty$. Thus $\text{res}_{\lambda=-\frac{n}{2m}} < P^\lambda, \varphi > = 0$. From (3.17),

$$\begin{aligned} \text{res}_{\lambda=-\frac{n}{2m}} < P^\lambda, \varphi > &= \frac{1}{4m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(-\frac{n}{2m}+1)}{\Gamma(-\frac{p}{2m}+1)} \frac{4\pi^{\frac{p+q}{2}}}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} < \delta(x), \varphi > \\ &= \frac{1}{m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(-\frac{n}{2m}+1)}{\Gamma(-\frac{p}{2m}+1)} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} < \delta(x), \varphi >. \end{aligned}$$

Thus

$$\text{res}_{\lambda=-\frac{n}{2m}} P^\lambda = \frac{1}{m^2} \frac{\Gamma(\frac{q}{2m})\Gamma(1-\frac{n}{2m})\pi^{\frac{n}{2}}}{\Gamma(1-\frac{p}{2m})\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \delta(x). \quad (3.18)$$

In particular, if $m = 1$ then (3.18) reduces to

$$\text{res}_{\lambda=-\frac{n}{2}} P^\lambda = \frac{(-1)^q \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \delta(x),$$

which appeared in [1,eq.(23),p258].

Moreover, for equations (3.1) and (3.2), if $m = 2$ we obtain

$$\text{res}_{\lambda=-k} (-1)^k \widehat{K_{2k,2k}}(x) = \frac{1}{(2\pi)^{n/2}} \text{res}_{\lambda=-k} P^\lambda = \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!} \delta_1^{*(k-1)}(P)$$

and

$$\text{res}_{\lambda=-\frac{n}{4}-k} < P^\lambda, \varphi > = \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi\left(-\frac{n}{4}-k, u\right) \right]_{u=0}$$

where $(-1)^k \widehat{K_{2k,2k}}(x)$ is the Fourier transform of the Diamond kernel, see [2, pages 715-723]. And if $m = 4$ we obtain

$$\text{res}_{\lambda=-k} (-1)^k \widehat{K_{2k,2k,2k,2k}}(x) = \text{res}_{\lambda=-k} P^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{*(k-1)}(P)$$

and

$$\text{res}_{\lambda=-\frac{n}{8}-k} < P^\lambda, \varphi > = \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi\left(-\frac{n}{8}-k, u\right) \right]_{u=0}$$

where $(-1)^k \widehat{K_{2k,2k,2k,2k}}(x)$ is the Fourier transform of the distributional kernel of the operator $\oplus^k$, defined by

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k, \quad p+q=n, \text{ see [3].} \square$$

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References

- [1] I.M. Gel'fand and G.E. Shilov, *Generalized Functions*, Vol.I, Academic Press, New York, 1964.
- [2] A. Kananthai, On the Spectrum of the Distributional Kernel Related to the Residue, *International Journal of Mathematics and Mathematical Sciences* (27):12(2001), 715-723.
- [3] A. Kananthai, S. Suantai, On the Residue of the Distributional Kernel Related to the Spectrum, In preparation.

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On the Residue of the Fourier Transform of the Distributional Kernel Related to the Spectrum

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Abstract

In this paper, we study the residue of the Fourier transform of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}(x)$ related to its spectrum, where $\alpha, \beta, \gamma, \nu$ are complex parameters and $x \in \mathbb{R}^n$ - the n -dimensional Euclidean space.

We found that the residue of $(-1)^k \widehat{K_{2k,2k,2k,2k}}(\xi)$ exists for any nonzero point ξ belongs to such a spectrum where $\alpha = \beta = \gamma = \nu = 2k$, k is a nonnegative integer and $\xi \in \mathbb{R}^n$.

1 Introduction

I. M. Gel'fand and G. E. Shilov (see [2, pp. 253-256]) have studied the generalized function P^λ , where

$$P = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \quad (1.1)$$

is a quadratic form, λ is a complex number and $p + q = n$ is the dimensional of the space $\mathbb{R}^n$.

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They found that P^λ has two sets of singularities, namely $\lambda = -1, -2, \dots, -k, \dots$ and $\lambda = -n/2, -n/2 - 1, \dots, -n/2 - k, \dots$, where k is a positive integer. For the singular point $\lambda = -k$, the generalized function P^λ has a simple pole with residue

$$\frac{(-1)^k}{(k-1)!} \delta_1^{(k-1)}(P) \quad \text{or} \quad \text{res}_{\lambda=-k} P^\lambda = \frac{(-1)^k}{(k-1)!} \delta_1^{(k-1)}(P) \quad (1.2)$$

for $p+q=n$ is an odd with p odd and q even.

Also, for the singular point $\lambda = -n/2 - k$, they obtained

$$\text{res}_{\lambda=-\frac{n}{2}-k} P^\lambda = \frac{(-1)^{q/2} L^k \delta(x)}{2^{2k} k! \Gamma(\frac{n}{2} + k)} \quad (1.3)$$

for $p+q=n$ is odd with p odd and q even.

The distributional kernel $K_{\alpha,\beta,\gamma,\nu}(x)$ is defined by the convolutions product of the tempered distribution, (see, for details, [8, p. 66, Theorem 3.1]; see also [7, p. 226, Theorem 3.1]). That is

$$K_{\alpha,\beta,\gamma,\nu}(x) = R_\alpha^H(u) * R_\beta^\ell(v) * S_\gamma(w) * T_\nu(z) \quad (1.4)$$

where $R_\alpha^H(u)$, $R_\beta^\ell(v)$, $S_\gamma(w)$ and $T_\nu(z)$ are defined by (2.2), (2.3), (2.6) and (2.7) respectively. Since $R_\alpha^H(u)$, $R_\beta^\ell(v)$, $S_\gamma(w)$ and $T_\nu(z)$ are all tempered distributions, thus $K_{\alpha,\beta,\gamma,\nu}(x)$ is also tempered distribution (see, for details, [3, p. 35, Lemma 2.2] and [1, pp. 156-159]).

In this paper, we use the idea of Gel'fand and Shilov to find the residue of Fourier transform $(-1)^k \widehat{K_{2k,2k,2k,2k}}(\xi)$, where $K_{2k,2k,2k,2k}(x)$ is defined by (1.4) with $\alpha = \beta = \gamma = \nu = 2k$ and k is a nonnegative integer. We found that for any nonzero point ξ that belongs to the spectrum of $(-1)^k K_{2k,2k,2k,2k}(x)$, there exists the residue of the Fourier transform $(-1)^k \widehat{K_{2k,2k,2k,2k}}(\xi)$.

Actually, $(-1)^k K_{2k,2k,2k,2k}(x)$ is an elementary solution of the operator $\oplus^k$ iterated k -times, that is, $\oplus^k [(-1)^k K_{2k,2k,2k,2k}(x)] = \delta$, where δ is the Dirac-delta distribution.

The operator $\oplus^k$ defined by

$$\oplus^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k \quad (1.5)$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$.

The operator $\oplus^k$ defined by (1.5) can be written in the form

$$\oplus^k = \diamond^k L_1^k L_2^k \quad (1.6)$$

where $\diamond^k$ is first introduced by Kananthai [4] and named as the *Diamond operator* iterated k -time and denoted by

$$\diamond^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \quad (1.7)$$

and L_1^k and L_2^k are the operators defined by

$$L_1^k = \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k, \quad (1.8)$$

$$L_2^k = \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \quad (1.9)$$

where $i = \sqrt{-1}$ and k is a nonnegative integer.

We obtained the convolution $(-1)^k R_{2k}^H(u) * R_{2k}^\ell(v)$ where $R_{2k}^H(u)$ and $R_{2k}^\ell(v)$ are defined by (2.2) and (2.3) with $\alpha = \beta = 2k$ is elementary solution of the operator $\diamond^k$, that is, $\diamond^k [(-1)^k R_{2k}^H(u) * R_{2k}^\ell(v)] = \delta$, where δ is the Dirac-delta distribution. Also, we obtain the functions $S_{2k}(w)$ and $T_{2k}(z)$ defined by (2.6) and (2.7) with $\gamma = \nu = 2k$ are elementary solutions of the operator L_1^k and L_2^k respectively, (see [2]).

2 Preliminaries

Now the following definitions and Lemmas are need for the purpose of this paper.

Let $x = (x_1, x_2, \dots, x_n)$ be a point in the space $\mathbb{R}^n$ and write

$$u = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 \quad (2.1)$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$.

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the set of an interior of the forward cone and $\bar{\Gamma}_+$ denotes it closure. For any complex number α , define

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \quad (2.2)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}.$$

The function $R_\alpha^H(u)$ is called the *ultra-hyperbolic kernel of Marcel Riesz* and was introduced by Y. Nozaki (see [9, p.72]).

It is well known that $R_\alpha^H(u)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let $\text{supp } R_\alpha^H(u)$ denote the support of $R_\alpha^H(u)$ and suppose $\text{supp } R_\alpha^H(u) \subset \bar{\Gamma}_+$, that is, $\text{supp } R_\alpha^H(u)$ is compact.

For any complex number β , define

$$R_\beta^\ell(v) = 2^{-\beta} \pi^{\frac{-n}{2}} \Gamma(\frac{n-\beta}{2}) \frac{v^{\frac{\beta-n}{2}}}{\Gamma(\frac{\beta}{2})}, \quad (2.3)$$

where

$$v = x_1^2 + x_2^2 + \cdots + x_n^2. \quad (2.4)$$

The function $R_\beta^\ell(v)$ is called the *elliptic kernel of Marcel Riesz* and is ordinary function for $Re(\beta) \geq n$ and is a distribution of β for $Re(\beta) < n$.

Let $x = (x_1, x_2, \dots, x_n)$ be a point of the space $\mathbb{R}^n$ and write

$$w = x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2) \quad (2.5)$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$ and $i = \sqrt{-1}$.

For any complex number γ , define

$$S_\gamma(w) = 2^{-\gamma} \pi^{\frac{-n}{2}} \Gamma(\frac{n-\gamma}{2}) \frac{w^{\frac{\gamma-n}{2}}}{\Gamma(\frac{\gamma}{2})}. \quad (2.6)$$

The function $S_\gamma(w)$ is an ordinary function if $Re(\gamma) \geq n$ and is a distribution of γ for $Re(\gamma) < n$. For any complex number ν , define

$$T_\nu(z) = 2^{-\nu} \pi^{\frac{-n}{2}} \Gamma(\frac{n-\nu}{2}) \frac{z^{\frac{\nu-n}{2}}}{\Gamma(\frac{\nu}{2})} \quad (2.7)$$

where

$$z = x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2), \quad (2.8)$$

We have $T_\nu(z)$ is an ordinary function if $\text{Re}(\nu) \geq n$ and is a distribution of ν for $\text{Re}(\nu) < n$.

Let $f(x)$ be continuous function and absolutely integrable on $\mathbb{R}^n$, then the Fourier transform of $f(x)$, denoted by $\mathfrak{F}f(x)$ or $\hat{f}(\xi)$, is defined by

$$\mathfrak{F}f(x) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx \quad (2.9)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, and $(\xi, x) = (\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n)$. From (2.9), the inverse Fourier transform of $\hat{f}(\xi)$ is defined by

$$f(x) = \mathfrak{F}^{-1} \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i(\xi, x)} \hat{f}(\xi) dx. \quad (2.10)$$

If $f(x)$ is a distribution with compact support, by [10, p. 187, Theorem 7.4.3] can be written as

$$\mathfrak{F}f(x) = \hat{f}(\xi) = \langle f(x), e^{-i(\xi, x)} \rangle. \quad (2.11)$$

Lemma 1. (Kananthai [4]; see also [1]) *The functions $R_\alpha^H(u)$, $R_\beta^\ell(v)$, $S_\gamma(w)$ and $T_\nu(z)$ defined by (2.2), (2.3), (2.6) and (2.7) respectively, are all tempered distributions.*

Lemma 2. (Kananthai and Suantai [6]) *The Fourier transform*

$$\begin{aligned} \widehat{K_{\alpha, \beta, \gamma, \nu}}(x) = & \frac{(\pi)^{2n} (i)^q 2^{\alpha+\beta+\gamma+\nu} \Gamma(\frac{\alpha}{2}) \Gamma(\frac{\beta}{2}) \Gamma(\frac{\gamma}{2}) \Gamma(\frac{\nu}{2}) \left(\sqrt{\sum_{r=1}^p \xi_r^2 - \sum_{j=p+1}^{p+q} \xi_j^2} \right)^{-\alpha}}{K_n(\alpha) H_n(\beta) H_n(\gamma) H_n(\nu)} \\ & \times \frac{\left(\sqrt{\sum_{r=1}^n \xi_r^2} \right)^{-\beta} \left(\sqrt{\sum_{r=1}^p \xi_r^2 + i \sum_{j=p+1}^{p+q} \xi_j^2} \right)^{-\gamma} \left(\sqrt{\sum_{r=1}^p \xi_r^2 - i \sum_{j=p+1}^{p+q} \xi_j^2} \right)^{-\nu}}{\Gamma(\frac{n-\alpha}{2}) \Gamma(\frac{n-\beta}{2}) \Gamma(\frac{n-\gamma}{2}) \Gamma(\frac{n-\nu}{2})} \end{aligned} \quad (2.12)$$

where $H_n(\ell) = \frac{\Gamma(\frac{\ell}{2}) 2^{\ell} \pi^{\frac{n}{2}}}{\Gamma(\frac{n-\ell}{2})}$ and $i = \sqrt{-1}$.

In particular, if $\alpha = \beta = \gamma = \nu = 2k$ then (2.12) reduces to

$$\mathfrak{F}K_{2k, 2k, 2k, 2k}(x) = \frac{(-1)^k}{\left[(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^4 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)^4 \right]^k}. \quad (2.13)$$

The spectrum of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}(x)$ is the support of the Fourier transform $\widehat{K_{\alpha,\beta,\gamma,\nu}}(\xi)$ or the spectrum of $K_{\alpha,\beta,\gamma,\nu}(x) = \text{supp} \widehat{K_{\alpha,\beta,\gamma,\nu}}(\xi)$. In particular, from (2.13) the spectrum of

$$(-1)^k K_{2k,2k,2k,2k}(x) = \text{supp} \left[\frac{1}{((\sum_{r=1}^p \xi_r^2)^4 - (\sum_{j=p+1}^{p+q} \xi_j^2)^4)^k} \right].$$

Lemma 3. Let $P(x_1, x_2, \dots, x_n)$ be a function of positive definite, and is defined by

$$P = P(x_1, x_2, \dots, x_n) = \left(\sum_{r=1}^p x_r^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^4, \quad (2.14)$$

then for any testing function $\varphi(x) \in \mathcal{D}$, the space of function of infinitely differentiable with compact support,

$$\langle \delta^{(k)}(P), \varphi \rangle = \int_0^\infty \left[\left(\frac{1}{8s^7} \frac{\partial}{\partial s} \right)^k \left\{ s^{q-8} \frac{\Psi(r, s)}{8} \right\} \right]_{s=r} r^{p-1} dr \quad (2.15)$$

and

$$\langle \delta^{(k)}(P), \varphi \rangle = (-1)^k \int_0^\infty \left[\left(\frac{1}{8r^7} \frac{\partial}{\partial r} \right)^k \left\{ r^{q-8} \frac{\Psi(r, s)}{8} \right\} \right]_{r=s} s^{p-1} ds \quad (2.16)$$

where $r^2 = x_1^2 + x_2^2 + \dots + x_p^2$, $s^2 = x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2$, $p+q = n$ is the dimension of the space $\mathbb{R}^n$, k is nonnegative integer, and

$$\Psi(r, s) = \int \varphi d\Omega_p d\Omega_q, \quad (2.17)$$

where $d\Omega_p$ and $d\Omega_q$ are elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively.

Proof. Assume that both $p > 1$ and $q > 1$ and let us transform to bipolar coordinates defined by

$$x_1 = r\omega_1, \dots, x_p = r\omega_p, \quad x_{p+1} = s\omega_{p+1}, \dots, x_{p+q} = s\omega_{p+q} \quad (2.18)$$

where $\sum_{i=1}^p \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q} \omega_j^2 = 1$.

In these coordinate the element of volume is given by

$$dx = dx_1 dx_2 \dots dx_n = r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q \quad (2.19)$$

where $d\Omega_p$ and $d\Omega_q$ are element of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Then (2.14) becomes $P = r^8 - s^8$.

Now let us choose the coordinates to be P, r and the ω_i , thus (2.19) becomes

$$dx = \frac{1}{8}(r^8 - P)^{\frac{1}{8}(q-8)} r^{p-1} dP dr d\Omega_p d\Omega_q.$$

By [2, p. 248, Eq.(3)], we have

$$< \delta^{(k)}(P), \varphi > = (-1)^k \int \left[\frac{\partial^k}{\partial P^k} \left(\frac{1}{8}(r^8 - P)^{\frac{1}{8}(q-8)} \varphi \right) \right]_{P=0} r^{p-1} dr d\Omega_p d\Omega_q. \quad (2.20)$$

Now, if we transform from P to $s = (r^8 - P)^{1/8}$ and $\partial/\partial P = -(1/8s^7)(\partial/\partial s)$. Then (2.20) becomes

$$< \delta^{(k)}(P), \varphi > = \int \left[\left(\frac{1}{8s^7} \frac{\partial}{\partial s} \right)^k \left\{ s^{q-8} \frac{\varphi}{8} \right\} \right]_{s=r} r^{p-1} dr d\Omega_p d\Omega_q. \quad (2.21)$$

Now, write

$$\Psi(r, s) = \int \varphi d\Omega_p d\Omega_q \quad (2.22)$$

which transforms (2.21) to be the form

$$< \delta^{(k)}(P), \varphi > = \int_0^\infty \left[\left(\frac{1}{8s^7} \frac{\partial}{\partial s} \right)^k \left\{ s^{q-8} \frac{\Psi(r, s)}{8} \right\} \right]_{s=r} r^{p-1} dr \quad (2.23)$$

similarly, if we transform from P to $r = (P + s^8)^{1/8}$, we obtain

$$< \delta^{(k)}(P), \varphi > = (-1)^k \int_0^\infty \left[\left(\frac{1}{8r^7} \frac{\partial}{\partial r} \right)^k \left\{ r^{p-8} \frac{\Psi(r, s)}{8} \right\} \right]_{r=s} s^{q-1} ds. \quad (2.24)$$

Thus, we obtain (2.15) and (2.16) as required.

Now we are still assuming that φ vanishes in a neighborhood of the origin, so that these integrals will converge for any k . If, further $(p-1) + (q-8) - 8k \geq 0$ or $k < \frac{1}{8}(p+q-8)$ then these integrals converge for any $\varphi(x) \in \mathcal{D}$. Thus we may take (2.23) and (2.24) to be the defining equations for $\delta^{(k)}(P)$ if $k < \frac{1}{8}(p+q-8)$. If, on the other hand, $k \geq \frac{1}{8}(p+q-8)$ we shall define $< \delta_1^{(k)}(P), \varphi >$ and $< \delta_2^{(k)}(P), \varphi >$ as the regularization of (2.23) and (2.24) respectively.

Moreover, if we put $u = r^8$, $v = s^8$ then (2.23) and (2.24) become

$$\langle \delta^{(k)}, \varphi \rangle = \frac{1}{64} \int_0^\infty \left[\frac{\partial^k}{\partial v^k} \{ v^{\frac{q-8}{8}} \Psi_1(u, v) \} \right]_{v=u} u^{\frac{p-8}{8}} du, \quad (2.25)$$

$$\langle \delta^{(k)}, \varphi \rangle = (-1)^k \frac{1}{64} \int_0^\infty \left[\frac{\partial^k}{\partial u^k} \{ u^{\frac{p-8}{8}} \Psi_1(u, v) \} \right]_{u=v} v^{\frac{q-8}{8}} dv \quad (2.26)$$

where $\Psi_1(u, v) = \Psi(r, s)$. It is obvious with the case in with either p or q is 1. $\square$

Lemma 4. (Kananthai [5]) Let $G_b = \{\xi \in \mathbb{R}^n : |\xi_1| \leq b_1, |\xi_2| \leq b_2, \dots, |\xi_n| \leq b_n\}$ be a parallelepiped in $\mathbb{R}^n$ and $b_i (1 \leq i \leq n)$ is a real constant and the inverse Fourier transform of $\widehat{K_{\alpha, \beta, \gamma, \nu}}(\xi)$ whose support contain in G_b is defined by

$$\begin{aligned} K_{\alpha, \beta, \gamma, \nu}(x) &= \mathfrak{F}^{-1} \widehat{K_{\alpha, \beta, \gamma, \nu}}(\xi) \\ &= \int_{G_b} e^{i(\xi, x)} \widehat{K_{\alpha, \beta, \gamma, \nu}}(\xi) d\xi \end{aligned} \quad (2.27)$$

where $K_{\alpha, \beta, \gamma, \nu}$ is defined by (1.1) and $x, \xi \in \mathbb{R}^n$, then $K_{\alpha, \beta, \gamma, \nu}(x)$ can be extended to the entire function $K_{\alpha, \beta, \gamma, \nu}(z)$ and analytic for all $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ where $\mathbb{C}^n$ is the n -tuple space of complex number and

$$|K_{\alpha, \beta, \gamma, \nu}(z)| \leq C \exp(b|Im(z)|) \quad (2.28)$$

where

$$\exp(b|Im(z)|) = \exp[b_1|Im(z_1)| + b_2|Im(z_2)| + \dots + b_n|Im(z_n)|]$$

and

$$C = \int_{G_b} |\widehat{K_{\alpha, \beta, \gamma, \nu}}(\xi)| d\xi$$

is a constant. Moreover, $K_{\alpha, \beta, \gamma, \nu}(x)$ has spectrum contained in G_b .

In particular for $\alpha = \beta = \gamma = \nu = 2k$, the spectrum of $(-1)^k K_{2k, 2k, 2k, 2k}(x)$ is also contained in G_b , that is $\text{supp}[(\widehat{(-1)^k K_{2k, 2k, 2k, 2k}}(\xi))] \subset G_b$ where $(-1)^k \widehat{K_{2k, 2k, 2k, 2k}}(\xi)$ is an elementary solution of the operator $\oplus^k$. The Fourier transform $(-1)^k \widehat{K_{2k, 2k, 2k, 2k}}(\xi)$ given by (2.13) can be defined as follows:

$$(-1)^k \widehat{K_{2k, 2k, 2k, 2k}}(\xi) = \begin{cases} \frac{1}{[(\sum_{r=1}^p \xi_r^2)^4 - (\sum_{j=p+1}^{p+q} \xi_j^2)^4]^k}, & \text{if } \xi \in G_b, \\ 0, & \text{if } \xi \in CG_b. \end{cases} \quad (2.29)$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ and CG_b is the complement of G_b .

3 Main Results

These are the following main results.

For any nonzero point $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in M$ where M is a spectrum of $(-1)^k K_{2k, 2k}$ then there exists the residue of the Fourier transform $(-1)^k \widehat{K_{2k, 2k, 2k, 2k}}(\xi)$ and such residue is

$$\frac{(-1)^{k-1}}{(k-1)!} \delta_1^{(k-1)} \quad \text{or} \quad \text{res}_{\lambda=-k} (-1)^k \widehat{K_{2k, 2k, 2k, 2k}}(\xi) = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{(k-1)}(P)$$

where

$$P = (\xi_1^2 + \dots + \xi_p^2)^4 - (\xi_{p+1}^2 + \dots + \xi_{p+q}^2)^4 \quad (3.1)$$

and $\delta_1^{(k-1)}(P)$ is defined by (2.25) with $\delta^{(k-1)}(P) = \delta_1^{(k-1)}(P)$ and $p+q = n$ is odd with p odd and q even.

In showing this main results; we use the technique of computation from I.M. Gel'fand and G.E. Shilov (see [2, p. 248]) by evaluating

$$\langle P^\lambda, \varphi \rangle = \int_{P>0} P^\lambda(\xi) \varphi(\xi) d\xi \quad (3.2)$$

where P^λ is the generalized function with λ is a complex number and P is given by (3.1), $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $d\xi = d\xi_1 d\xi_2 \dots d\xi_n$ and $\varphi(\xi) \in \mathcal{D}$, the space of continuous function of infinitely differentiable with compact support. Now

$$\langle P^\lambda, \varphi \rangle = \int_{P>0} \left((\xi_1^2 + \dots + \xi_p^2)^4 - (\xi_{p+1}^2 + \dots + \xi_{p+q}^2)^4 \right)^\lambda \varphi(\xi) d\xi.$$

Let us transform to bipolar coordinates defined by

$$x_1 = r\omega_1, \dots, x_p = r\omega_p, \quad x_{p+1} = s\omega_{p+1}, \dots, x_{p+q} = s\omega_{p+q},$$

where $\sum_{i=1}^p \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q} \omega_j^2 = 1$. Thus

$$r = \sqrt{\sum_{i=1}^p \xi_i^2}, \quad s = \sqrt{\sum_{j=p+1}^{p+q} \xi_j^2}.$$

From (3.2), we have

$$\langle P^\lambda, \varphi \rangle = \int [r^8 - s^8]^\lambda \varphi(\xi) d\xi.$$

Since the volume $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$ where $d\Omega_p$ and $d\Omega_q$ are the element of the surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Thus

$$\begin{aligned} \langle P^\lambda, \varphi \rangle &= \int_{P>0} (r^8 - s^8)^\lambda \varphi r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q \\ &= \int_0^\infty \int_0^r (r^8 - s^8)^\lambda \Psi(r, s) r^{p-1} s^{q-1} ds dr \end{aligned}$$

where $\Psi(r, s) = \int \varphi d\Omega_p d\Omega_q$.

Since $\varphi(\xi)$ is in $\mathcal{D}$, then $\Psi(r, s)$ is infinitely differentiable function of r^8 and s^8 with compact support.

We now make the change of variable $u = r^8, v = s^8$ and writing $\Psi(r, s) = \Psi_1(u, v)$. Thus we obtain

$$\langle P^\lambda, \varphi \rangle = \frac{1}{64} \int_{u=0}^\infty \int_{v=0}^u (u-v)^\lambda \Psi_1(u, v) u^{\frac{p-8}{8}} v^{\frac{q-8}{8}} dv du$$

write $v = ut$, then

$$\langle P^\lambda, \varphi \rangle = \frac{1}{64} \int_0^\infty u^{\lambda + \frac{1}{8}(p+q)-1} \int_0^1 (1-t)^\lambda t^{\frac{q-8}{8}} \Psi_1(u, ut) dt. \quad (3.3)$$

Let the function

$$\Phi(\lambda, u) = \frac{1}{64} \int_0^1 (1-t)^\lambda t^{\frac{q-8}{8}} \Psi_1(u, ut) dt. \quad (3.4)$$

Thus $\Phi(\lambda, u)$ has singularity at $\lambda = -k$ where it has simple poles. By Gel'fand and Shilov [2, p.254, Eq.(12)] we obtain the residue of $\Phi(\lambda, u)$ at $\lambda = -k$, that is

$$\text{res}_{\lambda=-k} \Phi(\lambda, u) = \frac{1}{64} \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \{ t^{\frac{q-8}{8}} \Psi_1(u, ut) \} \right]_{t=1}. \quad (3.5)$$

Thus, $\text{res}_{\lambda=-k} \Phi(\lambda, u)$ is a functional concentrated on the surface $P = 0$ ($t = 1, v = u$ and $P = u - v = 0$). On the other hand, from (3.3) and (3.4) we have

$$\langle P^\lambda, \varphi \rangle = \int_0^\infty u^{\lambda + \frac{1}{8}(p+q)-1} \Phi(\lambda, u) du. \quad (3.6)$$

Thus $\langle P^\lambda, \varphi \rangle$ in (3.6) has singularities at $\lambda = -n/8, -n/8 - 1, \dots, -n/8 - k, \dots$. At these points

$$\text{res}_{\lambda=-\frac{n}{8}-k} \langle P^\lambda, \varphi \rangle = \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi(-\frac{n}{8} - k, u) \right]_{u=0}. \quad (3.7)$$

These the residue of $\langle P^\lambda, \varphi \rangle$ at $\lambda = -\frac{n}{8} - k$ is a functional concentrated on the origin of the surface $P = 0$, since $u = 0$ we have $v = ut = 0$. Thus $r = s = 0$, that implies $\xi = 0$ or $\xi_1 = \xi_2 = \dots = \xi_n = 0$.

Now consider the case singular point $\lambda = -k$. Write (3.4) in the neighborhood of $\lambda = -k$ in the form $\Phi(\lambda, u) = \frac{\Phi_0(u)}{\lambda + k} + \Phi_1(\lambda, u)$ where $\Phi_0(u) = \text{res}_{\lambda=-k} \Phi(\lambda, u)$ and $\Phi_1(\lambda, u)$ is regular at $\lambda = -k$ substitute $\Phi(\lambda, u)$ into (3.6), we obtain

$$\langle P^\lambda, \varphi \rangle = \frac{1}{\lambda + k} \int_0^\infty u^{\lambda + \frac{1}{8}(p+q)-1} \Phi_0(u) du + \int_0^\infty u^{\lambda + \frac{1}{8}(p+q)-1} \Phi_1(\lambda, u) du.$$

Thus

$$\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \int_0^\infty u^{-k + \frac{1}{8}(p+q)-1} \Phi_0(u) du.$$

By substituting $\Phi_0(u)$ and (3.5), we obtain

$$\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{1}{64} \frac{(-1)^{k-1}}{(k-1)!} \int_0^\infty \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \{ t^{\frac{q-8}{8}} \Psi_1(u, ut) \} \right]_{t=1} u^{-k + \frac{1}{8}(p+q)-1} du$$

since we put $v = ut$. Thus $\partial^{k-1}/\partial t^{k-1} = u^{k-1}(\partial^{k-1}/\partial v^{k-1})$, by substituting $\partial^{k-1}/\partial t^{k-1}$ we obtain

$$\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{(-1)^{k-1}}{64(k-1)!} \int_0^\infty \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \{ v^{\frac{q-8}{8}} \Psi_1(u, v) \} \right]_{u=v} u^{\frac{p}{8}-1} du.$$

Now, by (2.25) for $k \geq \frac{1}{8}(p+q-8)$, we obtain

$$\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{(k-1)}(P) \quad (3.8)$$

which is the residue of a functional concentrated on the surface $P = 0$.

Since $P = r^8 - s^8$, $u = r^8 = (\sum_{r=1}^p \xi_r^2)^4$ and $v = s^8 = (\sum_{j=p+1}^{p+q} \xi_j^2)^4$. Thus, for $P = 0$ we have $u = v$ and $u \neq 0, v \neq 0$. It follows that $\xi = (\xi_1, \xi_2, \dots, \xi_n) \neq 0$.

Since, by Eq.(2.29) we have

$$(-1)^k \widehat{K_{2k,2k,2k,2k}}(\xi) = P^\lambda, \quad \text{for } \lambda = -k \quad (3.9)$$

and $\xi \in G_b$. Let M be a spectrum of $(-1)^k K_{2k,2k,2k,2k}(x)$ and $M \subset G_b$ by Lemma 4.

Thus for any nonzero $\xi \in M$ we can find the residue of $(-1)^k \widehat{K_{2k,2k,2k,2k}}(\xi)$, that is,

$$\begin{aligned} \text{res}_{\lambda=-k} \langle (-1)^k \widehat{K_{2k,2k,2k,2k}}(\xi), \varphi(\xi) \rangle &= \text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle \\ &= \frac{(-1)^{k-1}}{(k-1)!} \langle \delta_1^{(k-1)}(P), \varphi \rangle \end{aligned} \quad (3.10)$$

or

$$\text{res}_{\lambda=-k}(-1)^k \widehat{K_{2k,2k,2k,2k}}(\xi) = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{(k-1)}(P).$$

for $\xi \in M$ and $\xi \neq 0$.

Now consider the case $\xi = 0$. We have from (3.6) and (3.7) that, the residue of $\langle P^\lambda, \varphi \rangle$ occurs at that point $\lambda = -n/8 - k$, that is, $\text{res}_{\lambda=-n/8-k} \langle P^\lambda, \varphi \rangle$ is functional concentrated on the vertex of surface P . Since $u = 0$ and $v = ut$, then $u = v = 0$, that implies

$$\sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2} = \sqrt{\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2} = 0. \quad (3.11)$$

It follows that $\xi_1 = \xi_2 = \cdots = \xi_{p+q} = 0, p+q = n$. Thus, the residue of $\langle P^\lambda, \varphi \rangle$ is concentrated on the point $\xi = 0$. But in our work, we consider the residue of $\langle P^\lambda, \varphi \rangle$ only at $\lambda = -k$. Thus for $\xi = 0$ in (3.6) and (3.7) not occur our the case.

In particular, for $k = 1$ and $k < \frac{1}{8}(p+q-8)$ we obtain

$$\text{res}_{\lambda=-k} - \widehat{K_{2,2,2,2}}(\xi) = \delta(P)$$

for any nonzero $\xi \in M$ where $K_{2,2,2,2}$ is given by (1.4) with $\alpha = \beta = \gamma = \nu = 2$ and $p+q = n$. $\square$

References

- [1] W.F. Donoghue, Distributions and Fourier Transforms, Academic Press, New York, 1969.
- [2] I.M. Gel'fand, G.E. Shilov, Generalized Functions. Vol. 1. Properties and Operations, Academic Press, New York, 1964.
- [3] A. Kananthai, On the convolution equation related to the Diamond kernel of Marcel Riezs, J. Comput. Appl. Math. 100 (1998) 33-39.
- [4] A. Kananthai, On the solutions of the n dimensional Diamond operator, Applied Mathematics and Computation 88 (1997) 27-37.
- [5] A. Kananthai, On the spectrum of the distributional kernel related to the residue, International Journal of Mathematics and Mathemaical Science 27(12) (2001) 715-723.
- [6] A. Kananthai, S. Suantai, On the Fourier transform of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$, Computational Technologies 7(5) (2002) 36-43.

- [7] A. Kananthai, S. Suantai, V. Longani, On the operator $\oplus^k$ related to the wave equation and Laplacian, *Applied Mathematics and Computation* 132 (2002) 219-229.
- [8] A. Kananthai, S. Suantai, V. Longani, On the weak solutions of the equation related to the Diamond operator, *Computational Technologies* 5(5) (2000) 61-67.
- [9] Y. Nozaki, On Riemann-Liouville integral of untra-hyperbolic type, *Kodai Mathematical Seminar Report* 6(2) (1964) 69-87.
- [10] A.H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw-Hill, New York, 1965.

On the Spectrum of the Generalized Heat Kernel

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Abstract

In this paper, we study the spectrum of the kernel of the generalized heat equation of the form

$$\frac{\partial}{\partial t} u(x, t) + c^2 (-\Delta)^k u(x, t) = f(x, t)$$

where $\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k$ is the Laplacian iterated k -times, n is the dimension of the Euclidean space $\mathbb{R}^n$, $f(x, t)$ is continuous function for $(x, t) \in \mathbb{R}^n \times (0, \infty)$, k is a nonnegative integer and c is a positive constant. We obtain the convolution $u(x, t) = E(x, t) * f(x, t)$ as a solution of such equation where $E(x, t)$ is the kernel. We study the spectrum of $E(x, t)$ and obtaining all properties which are interesting results.

1 Introduction

It is well known that the heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Delta u(x, t) \quad (1.1)$$

with the initial condition

$$u(x, 0) = f(x)$$

has $u(x, t) = E(x, t) * f(x)$ as the solution of (1.1) where

$$E(x, t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp \left[-\frac{|x|^2}{4c^2 t} \right]. \quad (1.2)$$

is called *the heat kernel* of the equation (1.1).

Moreover, A. Kananthai and K. Nonlaopon [5] extended (1.1) to the general form

$$\frac{\partial}{\partial t} u(x, t) + c^2 (-\Delta)^k u(x, t) = 0$$

with the initial condition $u(x, 0) = f(x)$ and also obtained the generalized heat kernel.

In this paper, we study the equation of the form

$$\frac{\partial}{\partial t} u(x, t) + c^2 (-\Delta)^k u(x, t) = f(x, t) \quad (1.3)$$

which is the generalized of (1.1) where $\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k$ is the Laplacian iterated k -times, $f(x, t)$ is continuous function for $(x, t) \in \mathbb{R}^n \times (0, \infty)$, k is a nonnegative integer and c is a positive constant.

We obtain $u(x, t) = E(x, t) * f(x, t)$ as a solution of (1.3) where

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k + i(\xi, x) \right] d\xi \quad (1.4)$$

is a kernel for (1.3) where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $i = \sqrt{-1}$.

If $k = 1$ in (1.3) then $E(x, t)$ in (1.4) of the integral form reduces to (1.2).

Now we study the spectrum of the kernel $E(x, t)$ of (1.4) by getting the idea from [5] which is the spectrum of kernel of the Diamond operator $\diamond^k$ related to the residue.

In this work, we apply the operator Δ^k to establish the equation (1.3) which is so called *the generalized heat kernel*. At first we study all properties of $E(x, t)$ in (1.4). Later we study the spectrum of $E(x, t)$ in the sense of n -dimensional Fourier transform and obtaining the interesting results.

2 Preliminaries

Definition 2.1 Let $f(x) \in \mathbb{L}_1(\mathbb{R}^n)$ -the space of integrable functions in $\mathbb{R}^n$. The n -dimensional Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx \quad (2.1)$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \dots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \widehat{f}(\xi) d\xi. \quad (2.2)$$

Definition 2.2 Let T be tempered distribution on $\mathbb{R}^n$, the closed set $\text{supp}(\widehat{T})$ is called *the spectrum of T* where $\text{supp}(\widehat{T})$ denotes the support of $\widehat{T}$ and $\widehat{T}$ is the Fourier transform of T .

Lemma 2.1 Given the function

$$f(x) = \exp \left[- \left(\sum_{i=1}^n x_i^2 \right)^k \right]$$

where $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then

$$\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \frac{\pi^{\frac{n}{2}}}{k} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})}. \quad (2.3)$$

Proof. We have

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} \exp \left[- \left(\sum_{i=1}^n x_i^2 \right)^k \right] dx.$$

Let us transform to bipolar coordinates defined by

$$x_1 = r\omega_1, x_2 = r\omega_2, \dots, x_n = r\omega_n$$

where $\sum_{i=1}^n \omega_i^2 = 1$.

Thus

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} e^{-r^{2k}} r^{n-1} dr d\Omega_n$$

where $dx = r^{n-1} dr d\Omega_n$, $d\Omega_n$ is the element of surface area on the unit sphere in $\mathbb{R}^n$. By direct computation, we obtain

$$\int_{\mathbb{R}^n} f(x) dx = \Omega_n \int_0^\infty e^{-r^{2k}} r^{n-1} dr$$

where $\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.

Put $u = r^{2k}$, then we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f(x) dx \right| &\leq \frac{\Omega_n}{2k} \int_0^\infty e^{-u} u^{\frac{n}{2k}-1} du \\ &= \frac{\Omega_n}{2k} \Gamma\left(\frac{n}{2k}\right) \\ &= \frac{\pi^{\frac{n}{2}}}{k} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})}. \end{aligned}$$

That is $\int_{\mathbb{R}^n} f(x) dx$ is bounded. □

Lemma 2.2 Given the equation

$$\frac{\partial}{\partial t} u(x, t) + c^2(-\Delta)^k u(x, t) = f(x, t). \quad (2.4)$$

Then we obtain $u(x, t) = E(x, t) * f(x, t)$ as a solution of (2.4) where

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k + i(\xi, x) \right] d\xi \quad (2.5)$$

is a kernel or the elementary solution of (2.4).

Proof. Let

$$\frac{\partial}{\partial t} E(x, t) + c^2(-\Delta)^k E(x, t) = \delta(x)\delta(t)$$

where $E(x, t)$ is the kernel or the elementary solution of the equation and δ is the Dirac-delta distribution. Take the Fourier transform to both sides of the equation, we obtain

$$\frac{\partial}{\partial t} \widehat{E(\xi, t)} + c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k \widehat{E(\xi, t)} = \frac{1}{(2\pi)^{n/2}} \delta(t).$$

Thus

$$\widehat{E(\xi, t)} = \frac{H(t)}{(2\pi)^{n/2}} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t \right]$$

where $H(t)$ is a Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$\begin{aligned} \widehat{E(\xi, t)} &= \frac{1}{(2\pi)^{n/2}} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t \right] \\ &= \frac{1}{(2\pi)^{n/2}} \exp \left[-c^2 |\xi|^{2k} t \right] \end{aligned} \quad (2.6)$$

where $|\xi|^2 = \sum_{i=1}^n \xi_i^2$. By taking the inverse Fourier transform we obtain

$$E(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{E(\xi, t)} e^{i(\xi, x)} d\xi$$

or

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t + i(\xi, x) \right] d\xi. \quad (2.7)$$

To obtain the solution $u(x, t)$ of (2.4), we convolve both sides of (2.4) by $E(x, t)$ defined by (2.7). Then we obtain $u(x, t) = E(x, t) * f(x, t)$ as required. $\square$

Definition 2.3 A sequence of functions $f_n(x)$, continuous and bounded on $\mathbb{R}^n$ converges narrowly to $f_0(x)$ having the same properties if and only if $f_n(x)$ converge to $f_0(x)$ uniform on bounded subset of $\mathbb{R}^n$ and the numbers $\|f_n\|_\infty$ converges to $\|f_0\|_\infty$ where

$$\|g(x)\|_\infty = \sup_{x \in \Omega} |g(x)|,$$

Ω is bounded subset of $\mathbb{R}^n$.

3 The properties of $E(x, t)$

Theorem 3.1 *The kernel $E(x, t)$ defined by (2.7) have the following properties :*

- (1) $E(x, t) \in C^\infty$ -the space of continuous function for $(x, t) \in \mathbb{R}^n \times (0, \infty)$ with infinitely differentiable.
- (2) $E(x, t) > 0$ for $t > 0$.
- (3) $(\frac{\partial}{\partial t} + c^2(-\Delta)^k) E(x, t) = 0$ for $t > 0$.

(4)

$$|E(x, t)| \leq \frac{1}{2^n \pi^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})}, \quad \text{for } t > 0$$

where Γ denotes the Gamma function. Thus $E(x, t)$ is bounded for any fixed t .

(5)

$$\int_{\mathbb{R}^n} |\widehat{E(\xi, t)}| d\xi \leq \frac{1}{2^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})}, \quad \text{for } t > 0$$

where $\widehat{E(\xi, t)}$ is the Fourier transform of $E(x, t)$.

(6) $\lim_{t \rightarrow 0} E(x, t) = \delta$.

Proof.

(1) From (2.7), since

$$\frac{\partial^n}{\partial x^n} E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\partial^n}{\partial x^n} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t + i(\xi, x) \right] d\xi.$$

Thus $E(x, t) \in C^\infty$ for $x \in \mathbb{R}^n$ and $t > 0$.

(2) $E(x, t) > 0$ for $t > 0$ is obvious by (2.7).

(3) By direct computation, we obtain

$$\left(\frac{\partial}{\partial t} + c^2(-\Delta)^k \right) E(x, t) = 0$$

for $t > 0$ where $E(x, t)$ is defined by (2.7).

(4) We have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t + i(\xi, x) \right] d\xi.$$

Thus

$$|E(x, t)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[-c^2 t \left(\sum_{i=1}^n \xi_i^2 \right)^k \right] d\xi.$$

Using the same procedure as Lemma 2.1, we obtain

$$|E(x, t)| \leq \frac{1}{2^n \pi^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})}.$$

Thus, for any fixed t and , $E(x, t)$ is bounded.

(5) From (2.6),

$$\widehat{E(\xi, t)} = \frac{1}{(2\pi)^{n/2}} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t \right].$$

Thus

$$\int_{\mathbb{R}^n} |\widehat{E(\xi, t)}| d\xi \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t \right] d\xi.$$

The same as (4),

$$\int_{\mathbb{R}^n} |\widehat{E(\xi, t)}| d\xi \leq \frac{1}{2^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})}, \quad \text{for } t > 0.$$

(6) Form (2.7),

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t + i(\xi, x) \right] d\xi$$

Thus

$$\begin{aligned} \lim_{t \rightarrow 0} E(x, t) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp [i(\xi, x)] d\xi \\ &= \delta \quad \text{see [3, p396]} \end{aligned}$$

□

Theorem 3.2 Let Ω be bounded subset of $\mathbb{R}^n \times (0, \infty)$ and is defined by

$$\Omega = \{(\xi, t) \in \mathbb{R}^n \times (0, \infty) : \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 \leq R^2 \quad \text{and } 0 < t \leq T\}$$

where R and T are real constants and let

$$\widehat{E(\xi, t)} = \begin{cases} \frac{1}{(2\pi)^{n/2}} \exp \left[-c^2 \left(\sum_{i=1}^n \xi_i^2 \right)^k t \right] & \text{for } (\xi, t) \in \Omega, \\ 0 & \text{for } (\xi, t) \notin \Omega. \end{cases} \quad (3.1)$$

Then $E(x, t)$ defined by (2.7) can be extended to the entire function $E(z, t)$ where $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ the n -dimensional complex space and satisfies the inequality

$$|E(z, t)| \leq \frac{1}{2^n \pi^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})} \exp (R |Im z|) \quad (3.2)$$

if and only if $E(x, t)$ has a spectrum contained in Ω with respect to t where $Im z$ denotes the imaginary part of z . Moreover, $E(x, t)$ is tempered distribution.

Proof. Now, we have the inverse of Fourier transform

$$E(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} \widehat{E(\xi, t)} e^{i(\xi, x)} d\xi \quad (3.3)$$

where $\widehat{E(\xi, t)}$ is defined by (3.1). Since the integral (3.3) converge uniformly over every bounded domain in complex plane. Then $E(x, t)$ can be extended to the entire function

$$E(z, t) = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} \widehat{E(\xi, t)} e^{i(\xi, z)} d\xi.$$

Thus

$$|E(z, t)| = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} |\widehat{E(\xi, t)}| |e^{i(\xi, z)}| d\xi.$$

Now

$$e^{i(\xi, z)} = e^{i[\xi_1 z_1 + \xi_2 z_2 + \dots + \xi_n z_n]}.$$

Let $z_j = x_j + iy_j$. Then

$$e^{i(\xi, z)} = e^{i\xi_1 x_1} e^{i\xi_2 x_2} \dots e^{i\xi_n x_n} e^{-\xi_1 y_1} e^{-\xi_2 y_2} \dots e^{-\xi_n y_n}$$

Therefore,

$$\begin{aligned} |e^{i(\xi, z)}| &\leq e^{(\xi_1 y_1 + \xi_2 y_2 + \dots + \xi_n y_n)} \\ &= e^{(\xi, y)} \\ &\leq e^{|\xi| |y|} \\ &\leq e^{\|\xi\| \|y\|} \\ &\leq e^{R|\operatorname{Im} z|} \end{aligned}$$

where $\|\xi\| = (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2)^{1/2}$ and $\|y\| = |\operatorname{Im} z|$. Therefore $|e^{i(\xi, z)}| \leq e^{R|\operatorname{Im} z|}$. Thus

$$|E(z, t)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\Omega} |\widehat{E(\xi, t)}| d\xi \cdot e^{R|\operatorname{Im} z|}.$$

Since

$$\frac{1}{(2\pi)^{n/2}} \int_{\Omega} \widehat{E(\xi, t)} d\xi \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{E(\xi, t)} d\xi$$

thus by (5) of Theorem 3.1, we obtain

$$|E(z, t)| \leq \frac{1}{(2\pi)^{n/2}} \frac{1}{2^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})} \exp(R|\operatorname{Im} z|)$$

or

$$|E(z, t)| \leq \frac{1}{2^n \pi^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma(\frac{n}{2k})}{\Gamma(\frac{n}{2})} \exp(R|\operatorname{Im} z|).$$

Suppose $E(z, t)$ satisfies the inequality (3.2) then $E(z, t)$ is analytic function and also is the entire function. Then by Paley-Wiener-Schwartz Theorem, see[2, p162], $\widehat{E(\xi, t)}$ has support contained in Ω that is $E(x, t)$ has the spectrum with respect to t contained in Ω . Since $\widehat{E(\xi, t)}$ given by (3.1) has compact support, then $\widehat{E(\xi, t)}$ is the tempered distribution see [4, p148]. It follows that $E(x, t)$ is also a tempered distribution. $\square$

4 The the spectrum of $E(x, t)$

Theorem 4.1 *A point (ξ_0, t) belongs to the spectrum of $E(x, t)$, that is $(\xi_0, t) \in \Omega = \text{supp } \widehat{E(\xi, t)}$ where $\widehat{E(\xi, t)}$ is given by (3.1) if and only if there exists a sequence of functions φ_n in the Schwartz space $\mathcal{S}$ such that φ_n converge narrowly to*

$$\frac{1}{(2\pi)^{n/2}} \exp \left[c^2 |\xi_0|^{2k} t + i(t^2 + (\xi_0, x)) \right]$$

where $(\xi_0, x) = \xi_{01}x_1 + \xi_{02}x_2 + \cdots + \xi_{0n}x_n$.

Proof. Let $(\xi_0, t) \in \text{supp } \widehat{E(\xi, t)}$, then by Beurling Theorem [4, p230–231] there exists a sequence $\varphi_n(x, t) \in \mathcal{S}$ such that

$$E(x, t) * \varphi_n(x, t) \longrightarrow \exp[i((x, t), (\xi_0, t))] \text{ narrowly}$$

where

$$\begin{aligned} ((x, t), (\xi_0, t)) &= ((x_1, x_2, \dots, x_n, t), (\xi_{01}, \xi_{02}, \dots, \xi_{0n}, t)) \\ &= x_1\xi_{01} + x_2\xi_{02} + \cdots + x_n\xi_{0n} + t^2. \end{aligned}$$

It follow that the Fourier transform

$$\begin{aligned} (2\pi)^{n/2} \widehat{E(\xi, t)} \widehat{\varphi_n(x, t)} &\longrightarrow \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp[-i(\xi, x)] \exp[i((x, t), (\xi_0, t))] dx \\ &= \frac{e^{it^2}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi_0 - \xi, x)} dx = e^{it^2} \delta(\xi - \xi_0). \end{aligned}$$

Thus

$$\begin{aligned} \widehat{\varphi_n(x, t)} &\longrightarrow \frac{e^{it^2}}{(2\pi)^{n/2} \widehat{E(\xi, t)}} \delta(\xi - \xi_0) \\ &= \frac{e^{it^2} (2\pi)^{n/2}}{(2\pi)^{n/2}} e^{c^2 |\xi|^{2k} t} \delta(\xi - \xi_0) \quad \text{by (2.6)} \end{aligned}$$

where $|\xi|^2 = \sum_{i=1}^n \xi_i^2$.

Thus, by the property of the Dirac-delta distribution $\delta(\xi - \xi_0)$ we obtain

$$\widehat{\varphi_n(x, t)} \longrightarrow e^{it^2} e^{c^2 |\xi_0|^{2k} t} \delta(\xi - \xi_0).$$

By the inverse of Fourier transform

$$\begin{aligned} \varphi_n(x, t) &\longrightarrow \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \left[e^{it^2} e^{c^2 |\xi_0|^{2k} t} \delta(\xi - \xi_0) \right] d\xi \\ &= \frac{1}{(2\pi)^{n/2}} e^{i(\xi_0, x)} e^{it^2} e^{c^2 |\xi_0|^{2k} t} \end{aligned}$$

Thus

$$\varphi_n(x, t) \longrightarrow \frac{1}{(2\pi)^{n/2}} \exp \left[c^2 |\xi_0|^{2k} t + i(t^2 + \xi_0, x) \right] \quad \text{narrowly.} \quad (4.1)$$

Suppose (4.1) hold, we have

$$\begin{aligned} \frac{1}{(2\pi)^{n/2}} \exp \left[c^2 |\xi_0|^{2k} t + i(t^2 + \xi_0, x) \right] &= \frac{1}{(2\pi)^{n/2}} e^{i(t^2 + \xi_0, x)} \frac{1}{e^{-c^2 |\xi_0|^{2k} t}} \\ &= \frac{1}{(2\pi)^n} \frac{e^{i(t^2 + \xi_0, x)}}{\widehat{E(\xi_0, t)}} \quad \text{by (2.6)} \end{aligned}$$

since $\widehat{E(\xi_0, t)}$ is defined by (3.1). It follows that (ξ_0, t) belongs to the spectrum of $E(x, t)$. $\square$

Theorem 4.2 Consider the solution $u(x, t) = E(x, t) * f(x, t)$ of the equation (2.4).

- (1) If $u(x, t) = 0$ for any fixed t and for every $x \in \mathbb{R}^n$ then the Fourier transform $\widehat{f(\xi, t)} = 0$ on the spectrum of $E(x, t)$.
- (2) If $\widehat{f(\xi, t)} = 0$ on the spectrum of $E(x, t)$ then the spectrum of $u(x, t)$ is a perfect subset of the boundary of the spectrum of $E(x, t)$.

Proof. (1) Suppose (ξ_0, t) belongs to the spectrum of $u(x, t)$, then by Beurling's theorem there exists a sequence $\varphi_n(x, t) \in \mathcal{S}$ so that $u(x, t) * \varphi_n(x, t)$ converge narrowly to $\exp[i((x, t), (\xi_0, x))]$.

Thus

$$\begin{aligned} u(x, t) * \varphi_n(x, t) &= (E(x, t) * f(x, t)) * \varphi_n(x, t) \\ &= 0 \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty). \end{aligned}$$

Now, for fixed t

$$\begin{aligned} (E(x, t) * f(x, t)) * \varphi_n(x, t) &= f(x, t) * (E(x, t) * \varphi_n(x, t)) \\ &= \int_{\mathbb{R}^n} f(y, t) (E * \varphi_n)(x - y, t) dy \end{aligned}$$

for all $x \in \mathbb{R}^n$ and t is fixed. Thus

$$0 = \int_{\mathbb{R}^n} f(y, t) (E * \varphi_n)(0 - y, t) dy$$

converges by the Lebesgue theorem to

$$\begin{aligned} \int_{\mathbb{R}^n} f(y, t) \exp[i(-y, t), (\xi_0, t)] dy &= e^{it^2} \int_{\mathbb{R}^n} f(y, t) e^{-i(y, \xi_0)} dy \\ &= (2\pi)^{n/2} e^{it^2} \widehat{f(\xi_0, t)} = 0 \end{aligned}$$

Thus $\widehat{f(\xi_0, t)} = 0$. It follows that $\widehat{f(\xi, t)} = 0$ on the spectrum of $E(x, t)$.

(2) Let (ξ_0, t) belongs to the spectrum of $u(x, t)$, by the Beurling theorem there exists a sequence $\varphi_n \in \mathcal{S}$ so that $u(x, t) * \varphi_n(x, t)$ converges narrowly to $\exp[i((x, t), (\xi_0, t))]$. For any function $\psi \in \mathcal{S}$, then the convolutions

$$(u * \varphi_n * \psi)(x, t) = \int_{\mathbb{R}^n} (E * f * \varphi_n)(x - y, t) \psi(y, t) dy$$

for every $x \in \mathbb{R}^n$ and t is fixed.

And such convolution converges by the Lebesgue theorem to

$$\begin{aligned} \int_{\mathbb{R}^n} \exp[i((x - y, t), (\xi_0, t))] \psi(y, t) dy &= e^{i(t^2 + (x, \xi_0))} \int_{\mathbb{R}^n} \exp[-i(y, \xi_0)] \psi(y, t) dy \\ &= (2\pi)^{n/2} \exp[i(t^2 + (x, \xi_0))] \widehat{\psi(\xi_0, t)}. \end{aligned} \quad (4.2)$$

We now show that (ξ_0, t) belongs to the boundary of the spectrum of $E(x, t)$ only not belongs to anywhere else.

If (ξ_0, t) belongs to the complement of the spectrum of $E(x, t)$, there exists a function $\psi(x, t) \in \mathcal{S}$ whose Fourier transform $\widehat{\psi(\xi)}$ is supported by a neighborhood of (ξ_0, t) lying wholly outside the spectrum of $E(x, t)$ with $\widehat{\psi(\xi_0, t)} = 1$. Thus in this case $E(x, t) * \psi(x, t)$ vanishes identically.

Therefore, $(E * f * \varphi_n * \psi)(x, t)$ also vanishes identically. Thus from (4.2), we obtain $\widehat{\psi(\xi_0, t)} = 0$ contradict $\widehat{\psi(\xi_0, t)} = 1$.

It follow that (ξ_0, t) does not belong to the complement of the spectrum of $E(x, t)$.

Similarly, if (ξ_0, t) is an interior point of the spectrum of $E(x, t)$, by the hypothesis there is a neighborhood of (ξ_0, t) for which $\widehat{f(\xi, t)} = 0$. We choose $\psi(x, t)$ so that $\widehat{\psi(\xi, t)}$ is supported by the neighborhood and satisfies $\widehat{\psi(\xi_0, t)} = 1$. Now $f(x, t) = 0$, since $\widehat{f(\xi, t)} = 0$. Thus $(f * \psi)(x, t) = 0$ and also $(E * f * \varphi_n * \psi)(x, t) = 0$.

Thus, from (4.2) we obtain $\widehat{\psi(\xi_0, t)} = 0$ contradiction. Therefore (ξ_0, t) is not an interior point of the set where $\widehat{f(\xi, t)} = 0$, nor of the complement of the spectrum of $E(x, t)$. It follows that the point (ξ_0, t) belongs to the boundary of the spectrum of $E(x, t)$ only.

To show that the spectrum of $u(x, t)$ is perfect, is to show that there is no isolated point in that spectrum. The details of proving see [4, p233-234]. $\square$

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References

- [1] A. Kananthai, *On the spectrum of the distributional kernel related to the residue*, International Journal of Mathematics and Mathematical Science, **27:12**(2001) : 715–723.
- [2] I.M. Gel'fand and G.E. Shilov, *Generalized Functions*, Vol.2, Academic Press, New York and London, 1968.
- [3] R. Haberman, *Elementary Applied Partial Differential Equations*, 2nd Edition, Prentice-Hall International, Inc. 1983.
- [4] W.F. Donoghue, *Distributions and Fourier Transforms*, Academic Press, New York and London, 1969.
- [5] A. Kananthai and K. Nonlaopon, *On the generalized heat kernel*, Computational Technologies, **9(1)**(2004) : 3–10.

บทที่ 5

สมบัติเรขาคณิตของปริภูมิบานาค

(Geometric Properties of Banach Spaces)

ในบทนี้เป็นการศึกษาสมบัติเรขาคณิตของปริภูมิลำดับเซวาโรที่วางนัยทั่วไปสองแบบ โดยที่แบบแรกเป็นการวางนัยทั่วไปโดยใช้ลำดับของจำนวนจริงบวก (p_k) และแบบที่สองเป็นการขยายแนวคิดจากปริภูมิลำดับออร์ลิค และ มูสิลัค (Orlicz , Musielak sequence spaces) ทั้งสองปริภูมินั้นพิจารณาภายใต้ทั้งนอร์มล็กเซมเบิร์กและนอร์มออร์ลิค สมบัติที่ศึกษาคือสมบัติ k -Nearly uniform convex property (k -NUC) สมบัติ Uniform Kadec Klee สมบัติ (H) สมบัติ Convexity แบบต่างๆ งานวิจัยที่ได้เป็นการสร้างองค์ความรู้ใหม่ในสาขาวิจัยนี้

ON PROPERTY (H) AND ROTUNDITY OF DIFFERENCE SEQUENCE SPACES

A. KANANTHAI, M. MUSARLEEN, W. SANHAN, AND S. SUANTAI*

ABSTRACT. In this paper, we define a modular on difference sequence space $\ell(\Delta, p)$ and consider it equipped with the Luxemburg norm induced by the modular, where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The main purpose of this paper is to show that $\ell(\Delta, p)$ has property (H) and we also show that $\ell(\Delta, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.

1. INTRODUCTION.

Convexity properties of Banach spaces is an important topic in functional analysis and plays an important role in infinite dimensional holomorphy. In order to study the geometric properties of Banach spaces, Clarkson [5] introduced the very important class of rotund (strictly convex) spaces. Since Clarkson's paper, many authors have defined and studied various convexity properties lying between uniform convexity and rotundity (see [2, 3, 5, 12, 14, 17].)

Among the geometrical properties of Banach spaces, property (H) has proved to be particularly important and has been studied by various authors. Criteria for property (H) in Orlicz spaces and Orlicz sequence spaces were given by S. Chen and Y. Wang [4] and C. Wu, S. Chen and Y. Wang [20]. R. Pluciennik, T. Wang and Y. Zhang [19] obtained necessary and sufficient conditions for H- points and denting points in Orlicz sequence spaces.

In [7], criteria are given for Musielak-Orlicz sequence spaces to have property (H).

In this paper, we introduce the difference sequence space $\ell(\Delta, p)$, when $p = (p_k)$ is a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$, and consider it equipped the Luxemburg norm. We show that $\ell(\Delta, p)$ has property (H) and establish criteria for rotundity.

We begin by introducing the basic notations and definitions. In the following, Let $\mathbb{R}$ be the real line and $\mathbb{N}$ the set of natural numbers.

For a Banach space X , we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X , respectively. A point $x_0 \in S(X)$ is called:

- a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;
- b) an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \rightarrow 1$ as $n \rightarrow \infty$, the weak convergence of (x_n) to x_0 (written $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$;

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A Banach space X is said to be *rotund* (R), if every point of $S(X)$ is an extreme point. X is said to possess property (H) provided every point of $S(X)$ is an H-point.

For these geometric notions and their role in Mathematics we refer to the monographs [2], [8], and [17]. Some of them were studied for Orlicz spaces in [3], [6], [9], [10], [11], [19], and [20].

Let X be a real vector space. A functional $\varrho : X \rightarrow [0, \infty]$ is called a *modular* if it satisfies the conditions

- (i) $\varrho(x) = 0$ if and only if $x = 0$;
 - (ii) $\varrho(\alpha x) = \varrho(x)$ for all scalars α with $|\alpha| = 1$;
 - (iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
- The modular ϱ is called *convex* if
- (iv) $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X , we define

$$X_\varrho = \{x \in X : \lim_{\lambda \rightarrow 0^+} \varrho(\lambda x) = 0\},$$

$$\text{and } X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, we define

$$(1.1) \quad \|x\| = \inf\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\}.$$

Orlicz [18] proved that if ϱ is a convex modular in X , then $X_\varrho = X_\varrho^*$ and $\|\cdot\|$ is a norm on X_ϱ for which it is a Banach space. The norm $\|\cdot\|$ defined as in (1.1) is called the Luxemburg norm.

A modular ϱ on X is called

- (a) *right-continuous* if $\lim_{\lambda \rightarrow 1^+} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$,
- (b) *left-continuous* if $\lim_{\lambda \rightarrow 1^-} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$,
- (c) *continuous* if it is both right-continuous and left-continuous.

The following known results gave some relationships between the modular ϱ and the Luxemburg norm $\|\cdot\|$ on X_ϱ .

Theorem 1.1 *Let ϱ be a convex modular on X and let $x \in X_\varrho$ and (x_n) a sequence in X_ϱ . Then $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\varrho(\lambda(x_n - x)) \rightarrow 0$ as $n \rightarrow \infty$ for every $\lambda > 0$.*

Proof. See [16, Theorem 1.3].

Theorem 1.2 *Let ϱ be a continuous convex modular on X . Then*

- (i) $\|x\| < 1$ if and only if $\varrho(x) < 1$.
- (ii) $\|x\| \leq 1$ if and only if $\varrho(x) \leq 1$.
- (iii) $\|x\| = 1$ if and only if $\varrho(x) = 1$.

Proof. See [16, Theorem 1.4].

Let us denote by ℓ^0 the space of all real sequences and let $p = (p_k)$ be a bounded sequence of positive real numbers. In [13], Kizmaz introduced the sequence spaces $\ell_\infty(\Delta)$, $c_0(\Delta)$ and $c(\Delta)$ by considering the difference sequence $\Delta x = (x_k - x_{k+1})_{k=1}^\infty$ for any sequence $x \in \ell^0$, where ℓ_∞ , c_0 and c are Banach spaces of bounded, null and

convergent sequences, respectively. In [1], these sequence spaces were extended to $\ell_\infty(\Delta, p)$, $c_0(\Delta, p)$ and $c(\Delta, p)$, where, for example

$$\ell_\infty(\Delta, p) = \{x \in l^0 : \Delta x \in \ell_\infty(p)\}$$

with

$$\ell_\infty(p) = \{x \in l^0 : \sup_k |x_k|^{p_k} < \infty\}.$$

In [1] and [13] the authors determined the Köthe-Töeplitz and generalized Köthe-Töeplitz duals of these spaces and consider various matrix transformations.

In this paper we introduce the space $\ell(\Delta, p)$ defined analogously as follows,

$$\ell(\Delta, p) = \{x \in l^0 : \Delta x \in \ell(p)\},$$

where

$$\ell(p) = \{x \in l^0 : \sum_{k=1}^{\infty} |x(k)|^{p_k} < \infty\}.$$

and study some of its geometric properties.

For details of the spaces $\ell_\infty(p)$ and $\ell(p)$, we refer to [15].

For $x \in \ell(\Delta, p)$, we define

$$\varrho_p(x) = |x(1)| + \sum_{k=1}^{\infty} |x(k) - x(k+1)|^{p_k}$$

If $p_k \geq 1$ for all $k \in \mathbb{N}$, we have, by the convexity of the functions $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, that ϱ_p is a convex modular on $\ell(\Delta, p)$. We consider $\ell(\Delta, p)$ equipped with the Luxemburg norm given by

$$\|x\| = \inf \{\varepsilon > 0 : \varrho_p(\frac{x}{\varepsilon}) \leq 1\}.$$

A normed sequence space S is said to be a K -space if each coordinate mapping P_k , defined by $P_k(x) = x_k$, is continuous. If S is both a Banach and a K -space, it is called a BK-space.

Throughout this paper we let $M = \sup_k p_k$ and assume that $p_k \geq 1$ for all $k \in \mathbb{N}$.

2. MAIN RESULTS

We begin by giving some basic properties of the modular on the space $\ell(\Delta, p)$.

Proposition 2.1 *For $x \in \ell(\Delta, p)$, the modular ϱ_p on $\ell(\Delta, p)$ satisfies the following:*

- (i) if $0 < a \leq 1$, then $a^M \varrho_p(\frac{x}{a}) \leq \varrho_p(x)$ and $\varrho_p(ax) \leq a \varrho_p(x)$,
- (ii) if $a \geq 1$, then $\varrho_p(x) \leq a^M \varrho_p(\frac{x}{a})$,
- (iii) if $a \geq 1$, then $\varrho_p(x) \leq a \varrho_p(x) \leq \varrho_p(ax)$.

Proof. It is obvious that (iii) is satisfied by the convexity of ϱ_p . It remains to prove (i) and (ii).

For $0 < a \leq 1$, we have

$$\varrho_p(x) = |x(1)| + \sum_{k=1}^{\infty} |x(k) - x(k+1)|^{p_k}$$

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$$\begin{aligned}
 &= a \left| \frac{x(1)}{a} \right| + \sum_{k=1}^{\infty} \left| \frac{a(x(k) - x(k+1))}{a} \right|^{p_k} \\
 &\geq a^M \left| \frac{x(1)}{a} \right| + a^M \sum_{k=1}^{\infty} \left| \frac{x(k) - x(k+1)}{a} \right|^{p_k} \\
 &= a^M \varrho_p\left(\frac{x}{a}\right).
 \end{aligned}$$

It follows by the convexity of ϱ that $\varrho_p(ax) \leq a\varrho_p(x)$, hence (i) is satisfied.

Now, suppose that $a \geq 1$. Then $\frac{1}{a} \leq 1$. It follows from (i) that

$$\left(\frac{1}{a}\right)^M \varrho_p(x) = \left(\frac{1}{a}\right)^M \varrho_p\left(\frac{x/a}{1/a}\right) \leq \varrho_p\left(\frac{x}{a}\right),$$

so that $\varrho_p(x) \leq a^M \varrho_p\left(\frac{x}{a}\right)$, hence (ii) is obtained. $\square$

Proposition 2.2 *The modular ϱ_p on $\ell(\Delta, p)$ is continuous.*

Proof. For $\lambda > 1$, by Proposition 2.1 (ii) and (iii), we have

$$(2.1) \quad \varrho_p(x) \leq \lambda \varrho_p(x) \leq \varrho_p(\lambda x) \leq \lambda^M \varrho_p(x)$$

By taking $\lambda \rightarrow 1^+$ in (2.1), we have $\lim_{\lambda \rightarrow 1^+} \varrho_p(\lambda x) = \varrho_p(x)$. Thus ϱ_p is right-continuous. If $0 < \lambda < 1$, by Proposition 2.1 (i), we have

$$(2.2) \quad \lambda^M \varrho_p(x) \leq \varrho_p(\lambda x) \leq \lambda \varrho_p(x)$$

By taking $\lambda \rightarrow 1^-$ in (2.2), we have that $\lim_{\lambda \rightarrow 1^-} \varrho_p(\lambda x) = \varrho_p(x)$, hence, ϱ_p is left-continuous. Thus ϱ_p is continuous. $\square$

Next, we give some relationships between the modular ϱ_p and the Luxemburg norm on $\ell(\Delta, p)$.

Proposition 2.3 *For any $x \in \ell(\Delta, p)$, we have*

- (i) if $\|x\| < 1$, then $\varrho_p(x) \leq \|x\|$,
- (ii) if $\|x\| > 1$, then $\varrho_p(x) \geq \|x\|$,
- (iii) $\|x\| = 1$ if and only if $\varrho_p(x) = 1$,
- (iv) $\|x\| < 1$ if and only if $\varrho_p(x) < 1$,
- (v) $\|x\| > 1$ if and only if $\varrho_p(x) > 1$,
- (vi) if $0 < a < 1$ and $\|x\| > a$, then $\varrho_p(x) > a^M$, and
- (vii) if $a \geq 1$ and $\|x\| < a$, then $\varrho_p(x) < a^M$.

Proof. (i) Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|\cdot\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ and $\varrho_p\left(\frac{x}{\lambda}\right) \leq 1$. From Proposition 2.1(i) and (iii), we have

$$\begin{aligned}
 \varrho_p(x) &\leq \varrho_p\left(\frac{(\|x\| + \epsilon)x}{\lambda}\right) \\
 &= \varrho_p\left((\|x\| + \epsilon)\frac{x}{\lambda}\right)
 \end{aligned}$$

$$\begin{aligned} &\leq (\|x\| + \epsilon) \varrho\left(\frac{x}{\lambda}\right) \\ &\leq \|x\| + \epsilon, \end{aligned}$$

which implies that $\varrho_p(x) \leq \|x\|$, so (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \frac{\|x\| - 1}{\|x\|}$, then $1 < (1 - \epsilon)\|x\| < \|x\|$. By definition of $\|\cdot\|$ and by Proposition 2.1 (i), we have

$$\begin{aligned} 1 &< \varrho_p\left(\frac{x}{(1 - \epsilon)\|x\|}\right) \\ &\leq \frac{1}{(1 - \epsilon)\|x\|} \varrho_p(x), \end{aligned}$$

so $(1 - \epsilon)\|x\| < \varrho_p(x)$ for all $\epsilon \in (0, \frac{\|x\| - 1}{\|x\|})$. This implies that $\|x\| \leq \varrho_p(x)$, hence (ii) is obtained.

Since ϱ_p is continuous (Proposition 2.2), (iii) and (iv) follow directly from Theorem 1.2.

(iv) follows directly from (i) and (iii).

(v) follows from (iii) and (iv).

(vi) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\left\|\frac{x}{a}\right\| > 1$. By (v), we have $\varrho_p\left(\frac{x}{a}\right) > 1$. Hence, by Proposition 2.1(i), we obtain that $\varrho_p(x) \geq a^M \varrho_p\left(\frac{x}{a}\right) > a^M$.

(vii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\left\|\frac{x}{a}\right\| < 1$. By (iv), we have $\varrho_p\left(\frac{x}{a}\right) < 1$. If $a = 1$, it is obvious that $\varrho_p(x) < 1 = a^M$. If $a > 1$, by Proposition 2.1(ii), we obtain that $\varrho_p(x) \leq a^M \varrho_p\left(\frac{x}{a}\right) < a^M$. $\square$

Proposition 2.4 *Let (x_n) be a sequence of elements of $\ell(\Delta, p)$.*

(i) *If $\|x_n\| \rightarrow 1$ as $n \rightarrow \infty$, then $\varrho_p(x_n) \rightarrow 1$ as $n \rightarrow \infty$.*

(ii) *$\|x_n\| \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\varrho_p(x_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. (i) Suppose $\|x_n\| \rightarrow 1$ as $n \rightarrow \infty$. Let $\epsilon \in (0, 1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < 1 + \epsilon$ for all $n \geq N$. By Proposition 2.3 (vi) and (vii), we have $(1 - \epsilon)^M < \varrho_p(x_n) < (1 + \epsilon)^M$ for all $n \geq N$, which implies that $\varrho_p(x_n) \rightarrow 1$ as $n \rightarrow \infty$.

(ii) The only part of (ii) is true by Theorem 1.1, so we need to show only the if part. Suppose $\|x_n\| \not\rightarrow 0$ as $n \rightarrow \infty$. Then there is an $\epsilon \in (0, 1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proposition 2.3 (vi), we have $\varrho_p(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho_p(x_n) \not\rightarrow 0$ as $n \rightarrow \infty$. $\square$

Next, we shall show that $\ell(\Delta, p)$ has property (H). To do this, we need two lemmas.

Lemma 2.5 *The space $\ell(\Delta, p)$ is a BK-space.*

Proof. Since $\ell(\Delta, p)$ equipped with the Luxemburg norm is Banach, we need only show that $\ell(\Delta, p)$ is a K -space. Suppose $(x_n) \subset \ell(\Delta, p)$ such that $x_n \rightarrow 0$ as $n \rightarrow \infty$. It follows by Proposition 2.4(ii) that $\varrho_p(x_n) \rightarrow 0$ as $n \rightarrow \infty$. This implies

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that

$$|x_n(1)| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{and} \\ |x_n(k) - x_n(k+1)| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for all } k \in \mathbb{N}.$$

By induction, we have $x_n(k) \rightarrow 0$ as $n \rightarrow \infty$ for all $k \in \mathbb{N}$. Hence $P_k(x_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $k \in \mathbb{N}$. This implies that P_k is continuous for all $k \in \mathbb{N}$.

Lemma 2.6 *Let $x \in \ell(\Delta, p)$ and $(x_n) \subseteq \ell(\Delta, p)$. If $\varrho_p(x_n) \rightarrow \varrho_p(x)$ as $n \rightarrow \infty$ and $x_n(i) \rightarrow x(i)$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$, then $x_n \rightarrow x$ as $n \rightarrow \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $\varrho_p(x) = |x(1)| + \sum_{k=1}^{\infty} |x(k) - x(k+1)|^{p_k} < \infty$, there is $k_0 \in \mathbb{N}$ such that

$$(2.3) \quad \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} < \frac{\epsilon}{3} \cdot \frac{1}{2^{M+1}}.$$

Since $\varrho_p(x_n) \rightarrow \varrho_p(x)$ as $n \rightarrow \infty$ and $x_n(i) \rightarrow x(i)$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$(2.4) \quad \varrho_p(x_n) - \left(|x_n(1)| + \sum_{k=1}^{k_0} |x_n(k) - x_n(k+1)|^{p_k} \right) \\ < \varrho_p(x) - \left(|x(1)| + \sum_{i=1}^{k_0} |x(k) - x(k+1)|^{p_k} \right) + \frac{\epsilon}{3 \cdot 2^M}$$

and

$$(2.5) \quad |x_n(1) - x(1)| + \sum_{k=1}^{k_0} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} < \frac{\epsilon}{3}.$$

It follows from (2.3), (2.4) and (2.5) that for $n \geq n_0$,

$$\begin{aligned} \varrho_p(x_n - x) &= |x_n(1) - x(1)| + \sum_{k=1}^{\infty} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\ &= |x_n(1) - x(1)| + \sum_{k=1}^{k_0} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\ &\quad + \sum_{k=k_0+1}^{\infty} |(x_n(k) - x(k)) - (x_n(k+1) - x(k+1))|^{p_k} \\ &< \frac{\epsilon}{3} + 2^M \left(\sum_{k=k_0+1}^{\infty} |x_n(k) - x_n(k+1)|^{p_k} + \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} \right) \\ &= \frac{\epsilon}{3} + 2^M (\varrho_p(x_n) - (|x_n(1)| + \sum_{k=1}^{k_0} |x_n(k) - x_n(k+1)|^{p_k})) \\ &\quad + \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} \end{aligned}$$

$$\begin{aligned}
 &< \frac{\varepsilon}{3} + 2^M (\varrho_p(x) - (|x(1)| + \sum_{k=1}^{k_0} |x(k) - x(k+1)|^{p_k}) + \frac{\varepsilon}{3 \cdot 2^M} \\
 &+ \sum_{k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k}) \\
 &= \frac{\varepsilon}{3} + 2^M (2 \sum_{k=k_0+1}^{\infty} |x(k) - x(k+1)|^{p_k} + \frac{\varepsilon}{3 \cdot 2^M}) \\
 &< \frac{\varepsilon}{3} + 2^{M+1} \frac{\varepsilon}{3 \cdot 2^{M+1}} + \frac{\varepsilon}{3} \\
 &= \varepsilon.
 \end{aligned}$$

This show that $\varrho_p(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$. Hence, by Proposition 2.4 (ii), we have $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 2.7 *The space $\ell(\Delta, p)$ has property (H).*

Proof. Let $x \in S(\ell(\Delta, p))$ and $(x_n) \subset \ell(\Delta, p)$ such that $\|x_n\| \rightarrow 1$ and $x_n \xrightarrow{w} x$ as $n \rightarrow \infty$. From Proposition 2.3 (iii), we have $\varrho_p(x) = 1$, so it follows from Proposition 2.4 (i) that $\varrho_p(x_n) \rightarrow \varrho_p(x)$ as $n \rightarrow \infty$. By Lemma 2.5, we have that the coordinate mapping $P_i : \ell(\Delta, p) \rightarrow \mathbb{R}$ is continuous, so it follows that $x_n(i) \rightarrow x(i)$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. Thus, we have by Lemma 2.6 that $x_n \rightarrow x$ as $n \rightarrow \infty$. $\square$

Theorem 2.8 *The space $\ell(\Delta, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Necessity. Suppose that there is $k_0 \in \mathbb{N}$ such that $p_{k_0} = 1$. Let $x = (1, 1, 1, \dots)$ and $y = (\underbrace{0, 0, 0, \dots, 0}_{k_0}, 1, 1, 1, \dots)$. Then $x \neq y$ and it is easy to see that

$$\varrho_p(x) = \varrho_p(y) = \varrho_p\left(\frac{x+y}{2}\right) = 1.$$

By Proposition 2.3(iii), we have x, y and $\frac{x+y}{2} \in S(\ell(\Delta, p))$, so that $\ell(\Delta, p)$ is not rotund.

Sufficiency. Suppose that $p_k > 1$ for all $k \in \mathbb{N}$. Let $x \in S(\ell(\Delta, p))$ and $y, z \in B(\ell(\Delta, p))$ with $x = \frac{y+z}{2}$. By convexity of ϱ_p and Proposition 2.3(iii), we have

$$1 = \varrho_p(x) \leq \frac{1}{2}(\varrho_p(y) + \varrho_p(z)) \leq \frac{1}{2} + \frac{1}{2} = 1.$$

This implies that

$$(2.6) \quad \varrho_p(y) = \varrho_p(z) = 1$$

$$(2.7) \quad \varrho_p(x) = \frac{1}{2}(\varrho_p(y) + \varrho_p(z)).$$

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By (2.7), we have

$$\begin{aligned} & \left| \frac{y(1) + z(1)}{2} \right| + \sum_{k=1}^{\infty} \left| \frac{(y(k) - y(k+1)) + (z(k) - z(k+1))}{2} \right|^{p_k} \\ &= \frac{1}{2} \left(|y(1)| + \sum_{k=1}^{\infty} |y(k) - y(k+1)|^{p_k} \right) + \frac{1}{2} \left(|z(1)| + \sum_{k=1}^{\infty} |z(k) - z(k+1)|^{p_k} \right) \\ &= \frac{1}{2} (|y(1)| + |z(1)|) + \frac{1}{2} \left(\sum_{k=1}^{\infty} |y(k) - y(k+1)|^{p_k} + \sum_{k=1}^{\infty} |z(k) - z(k+1)|^{p_k} \right), \end{aligned}$$

which implies that

$$(2.8) \quad |y(1) + z(1)| = |y(1)| + |z(1)| \quad \text{and}$$

$$(2.9) \quad \left| \frac{(y(k) - y(k+1)) + (z(k) - z(k+1))}{2} \right|^{p_k} = \frac{1}{2} (|y(k) - y(k+1)|^{p_k} + |z(k) - z(k+1)|^{p_k})$$

for all $k \in \mathbb{N}$.

Since the function $t \mapsto |t|^{p_k}$ is strictly convex for every $k \in \mathbb{N}$, we see that (2.9) implies,

$$(2.10) \quad y(k) - y(k+1) = z(k) - z(k+1) \quad \text{for all } k \in \mathbb{N}.$$

It follows from (2.6) and (2.10) that $|y(1)| = |z(1)|$. This implies by (2.8) that $y(1) = z(1)$. This, together with (2.10), yields by an inductive argument that $y(k) = z(k)$ for all $k \in \mathbb{N}$. Hence $y = z$.

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REFERENCES

- [1] Z.U. Ahmad and Mursaleen, *Köthe-Töeplitz duals of some new sequence spaces and thier matrix maps*, Publ. Inst. Math.(Beograd) **42(56)** (1987), 57-61.
- [2] K.W. Anderson, *Midpoint local uniform convexity, and other geometric properties of Banach spaces*, Dissertation University of Illonios, 1960.
- [3] S. Chen, *Geometry of Orlicz spaces*, *Dissertationes Math.*, 1996, pp. 356.
- [4] S. Chen and Y. Wang, *H property of Orlicz spaces*, Chinese Ann. Math. **8A** (1987), 61- 67.
- [5] J.A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396 - 414.
- [6] Y. A. Cui, H. Hudzik, and C. Meng, *On some lalcal geometry of Orlicz sequence spaces equipped the Luxemburg norms*, Acta Math. Hungar. **80 (1- 2)** (1998), 143-154.
- [7] Y.A. Cui, H. Hudzik, and R. Plicienik, *Banach-Saks property in some Banach sequence spaces*, Annales Math. Polonici **65** (1997), 193-202.
- [8] J. Diestel, *Geometry of Banach Spaces - Selected Topics*, Springer-Verlag, 1984.
- [9] R. Grzaslewicz, H. Hudzik, and W. Kurc, *Extremal and exposed points in Orlicz spaces*, Canadian J. Math. **44** (1992), 505-515.

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- [10] H. Hudzik, *Orlicz spaces without strongly extreme points and without H-points*, Canad. Math. Bull. **35** (1992), 1 - 5.
- [11] H. Hudzik and D. Pallaschke, *On some convexity properties of Orlicz sequence spaces*, Math. Nachr. **186** (1997), 167-185.
- [12] V.I. Istrătescu, *Lecture Note in Pure and Applied Math.* 87, Marcel Dekker, Inc., New York and Basel, 1984.
- [13] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull. **24** (1981), 169-175.
- [14] A. R. Lovaglia, *Locally uniformly convex Banach spaces*, Tran. Amer. Math. Soc **78** (1955), 225-238.
- [15] I.J. Maddox, *Continuous and Köthe-Töeplitz duals of certain sequence spaces*, Proc. Camb. phil. Soc. **65** (1967), 431-435.
- [16] L. Maligranda, *Orlicz Spaces and Interpolation*, Institute of Mathematics, Polish Academy of Science, Poznań, Poland, 1985.
- [17] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. 1034, Springer-Verlag, (1983).
- [18] W. Orlicz, *A note on modular space I*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9** (1961), 157 - 162.
- [19] R. Pluciennik, T. Wang, and Y. Zhang, *H-points and Denting Points in Orlicz Spaces*, Comment. Math. (Prace Mat.) **33** (1993), 135-151.
- [20] C. Wu, S. Chen, and Y. Wang, *H property of sequence Orlicz spaces*, J. Harbin Inst. Tech. Math. issue (1985), 6 - 11.

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ON k -NEARLY UNIFORM CONVEX PROPERTY IN GENERALIZED CESÀRO SEQUENCE SPACES

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We define a generalized Cesàro sequence space $\text{ces}(p)$, where $p = (p_k)$ is a bounded sequence of positive real numbers, and consider it equipped with the Luxemburg norm. The main purpose of this paper is to show that $\text{ces}(p)$ is k -nearly uniform convex (k -NUC) for $k \geq 2$ when $\lim_{n \rightarrow \infty} \inf p_n > 1$. Moreover, we also obtain that the Cesàro sequence space ces_p (where $1 < p < \infty$) is k -NUC, kR , NUC, and has a drop property.

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1. Introduction. Let $(X, \|\cdot\|)$ be a real Banach space and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X , respectively. For any subset A of X , we denote by $\text{conv}(A)$ (resp., $\overline{\text{conv}}(A)$) the convex hull (resp., the closed convex hull) of Clarkson [1] who introduced the concept of uniform convexity, and it is known that uniform convexity implies reflexivity of Banach spaces. There are different uniform geometric properties which have been defined between the uniform convexity and the reflexivity of Banach spaces. Huff [6] introduced the nearly uniform convexity of Banach spaces. He has proved that the class of nearly uniformly convexifiable spaces is strictly between superreflexive and reflexive Banach spaces.

A Banach space X is called *uniformly convex* (UC) if for each $\epsilon > 0$, there is $\delta > 0$ such that for $x, y \in S(X)$, the inequality $\|x - y\| > \epsilon$ implies that

$$\left\| \frac{1}{2}(x + y) \right\| < 1 - \delta. \quad (1.1)$$

For any $x \notin B(X)$, the *drop* determined by x is the set

$$D(x, B(X)) = \text{conv}(\{x\} \cup B(X)). \quad (1.2)$$

Rolewicz [12], basing on Daneš drop theorem [4], introduced the notion of drop property for Banach spaces.

A Banach space X has the *drop property* (D) if for every closed set C disjoint with $B(X)$, there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}. \quad (1.3)$$

A Banach space X is said to have the *Kadec-Klee property* (or *property (H)*) if every weakly convergent sequence on the unit sphere is convergent in norm.

In [13], Rolewicz proved that if the Banach space X has the drop property, then X is reflexive. Montesinos [11] extended this result by showing that X has the drop property if and only if X is reflexive and has the property (H).

Recall that a sequence $\{x_n\} \subset X$ is said to be ϵ -separated sequence for some $\epsilon > 0$ if

$$\text{sep}(x_n) = \inf \{\|x_n - x_m\| : n \neq m\} > \epsilon. \quad (1.4)$$

A Banach space X is said to be *nearly uniformly convex* (NUC) if for every $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $\text{sep}(x_n) > \epsilon$, we have

$$\text{conv}(x_n) \cap ((1 - \delta)B(X)) \neq \emptyset. \quad (1.5)$$

Huff [6] proved that every NUC Banach space is reflexive and it has property (H).

Kutzarova [7] has defined k -nearly uniformly convex Banach spaces. Let $k \geq 2$ be an integer. A Banach space X is said to be k -nearly uniformly convex (k -NUC) if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any sequence $(x_n) \subset B(X)$ with $\text{sep}(x_n) \geq \epsilon$, there are $n_1, n_2, \dots, n_k \in \mathbb{N}$ such that

$$\left\| \frac{x_{n_1} + x_{n_2} + x_{n_3} + \dots + x_{n_k}}{k} \right\| < 1 - \delta. \quad (1.6)$$

Clearly, k -NUC Banach spaces are NUC but the opposite implication does not hold in general (see [7]).

Fan and Glicksberg [5] have introduced fully k -convex Banach spaces. A Banach space X is said to be *fully k -rotund* (kR) if for every sequence $(x_n) \subset B(X)$, $\|x_{n_1} + x_{n_2} + \dots + x_{n_k}\| \rightarrow k$ as $n_1, n_2, \dots, n_k \rightarrow \infty$ implies that (x_n) is convergent.

It is well known that UC implies kR and kR implies $(k+1)R$, and kR spaces are reflexive and rotund, and it is easy to see that k -NUC implies kR .

Denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all natural and real numbers, respectively.

Let X be a real vector space. A functional $\varrho : X \rightarrow [0, \infty]$ is called a *modular* if it satisfies the following conditions:

- (i) $\varrho(x) = 0$ if and only if $x = 0$;
- (ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;
- (iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

- (iv) $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If ϱ is a modular in X , we define

$$\begin{aligned} X_\varrho &= \left\{ x \in X : \lim_{\lambda \rightarrow 0^+} \varrho(\lambda x) = 0 \right\}, \\ X_\varrho^* &= \{ x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \}. \end{aligned} \quad (1.7)$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If ϱ is a convex modular, for $x \in X_\varrho$, we define

$$\|x\| = \inf \left\{ \lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \right\}. \quad (1.8)$$

Orlicz [10] proved that if ϱ is a convex modular on X , then $X_\varrho = X_\varrho^*$ and $\|\cdot\|$ is a norm on X_ϱ for which X_ϱ is a Banach space. The norm $\|\cdot\|$, defined as in (1.8), is called the Luxemburg norm.

A modular ϱ is said to satisfy the δ_2 -condition ($\varrho \in \delta_2$) if for any $\epsilon > 0$, there exist constants $K \geq 2$ and $a > 0$ such that

$$\varrho(2u) \leq K\varrho(u) + \epsilon \quad (1.9)$$

for all $u \in X_\varrho$ with $\varrho(u) \leq a$.

If ϱ satisfies the δ_2 -condition for any $a > 0$ with $K \geq 2$ dependent on a , we say that ϱ satisfies the strong δ_2 -condition ($\varrho \in \delta_2^s$).

The following known results are very important for our consideration.

THEOREM 1.1. *If $\varrho \in \delta_2^s$, then for any $L > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|\varrho(u+v) - \varrho(u)| < \epsilon \quad (1.10)$$

whenever $u, v \in X_\varrho$ with $\varrho(u) \leq L$ and $\varrho(v) \leq \delta$.

PROOF. See [2, Lemma 2.1]. □

THEOREM 1.2. (1) *If $\varrho \in \delta_2^s$, then for any $x \in X_\varrho$, $\|x\| = 1$ if and only if $\varrho(x) = 1$.*

(2) *If $\varrho \in \delta_2$, then for any sequence (x_n) in X_ϱ , $\|x_n\| \rightarrow 0$ if and only if $\varrho(x_n) \rightarrow 0$.*

PROOF. See [2, Corollary 2.2 and Lemma 2.3]. □

THEOREM 1.3. *If $\varrho \in \delta_2^s$, then for any $\epsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that $\varrho(x) \leq 1 - \epsilon$ implies $\|x\| \leq 1 - \delta$.*

PROOF. Suppose that the theorem does not hold, then there exist $\epsilon > 0$ and $x_n \in X_\varrho$ such that $\varrho(x_n) < 1 - \epsilon$ and $1/2 \leq \|x_n\| < 1$. Let $a_n = 1/\|x_n\| - 1$. Then $a_n \rightarrow 0$ as $n \rightarrow \infty$. Let $L = \sup\{\varrho(2x_n); n \in \mathbb{N}\}$. Since $\varrho \in \delta_2^s$, there exists $K \geq 2$

such that

$$\varrho(2u) \leq K\varrho(u) + 1 \quad (1.11)$$

for every $u \in X_\varrho$ with $\varrho(u) < 1$.

By (1.11), we have $\varrho(2x_n) \leq K\varrho(x_n) + 1 < K + 1$ for all $n \in \mathbb{N}$. Hence, $0 < L < \infty$. By Theorem 1.2(1), we have

$$\begin{aligned} 1 &= \varrho\left(\frac{x_n}{\|x_n\|}\right) = \varrho(2a_n x_n + (1-a_n)x_n) \\ &\leq a_n \varrho(2x_n) + (1-a_n)\varrho(x_n) \\ &\leq a_n L + (1-\epsilon) \rightarrow 1-\epsilon, \end{aligned} \quad (1.12)$$

which is a contradiction. $\square$

Let l^0 be the space of all real sequences. For $1 < p < \infty$, the Cesàro sequence space (ces_p) is defined by

$$ces_p = \left\{ x \in l^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^p < \infty \right\} \quad (1.13)$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^p \right)^{1/p}. \quad (1.14)$$

This space was first introduced by Shiue [14], which is useful in the theory of Matrix operator and others (see [8, 9]). Some geometric properties of the Cesàro sequence space ces_p were studied by many authors. It is known that $(ces_p, \|\cdot\|)$ is locally uniformly rotund (LUR) and has property (H) (see [9]). Cui and Meng [3] proved that $(ces_p, \|\cdot\|)$ has property (β) .

Let $p = (p_n)$ be a sequences of positive real numbers with $p_n \geq 1$ for all $n \in \mathbb{N}$. The generalized Cesàro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}, \quad (1.15)$$

where

$$\rho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^{p_n} \quad (1.16)$$

is a convex modular on $ces(p)$.

We consider $ces(p)$ equipped with the Luxemburg norm:

$$\|x\| = \inf \left\{ \epsilon > 0 : \rho\left(\frac{x}{\epsilon}\right) \leq 1 \right\}. \quad (1.17)$$

When $p_n = q$ for all $n \in \mathbb{N}$, we see that $\text{ces}(p) = \text{ces}_q$ and the Luxemburg norm on $\text{ces}(p)$ given in (1.17) is equal to the norm $\|\cdot\|$ given in (1.14). In this paper, we show that $\text{ces}(p)$ equipped with the Luxemburg norm is k -NUC for $k \geq 2$, so it is kR and (NUC).

Throughout this paper, we assume that $p = (p_n)$ is bounded with

$$\liminf_{n \rightarrow \infty} p_n > 1 \quad (1.18)$$

and that $M = \sup_n p_n$.

2. Main results

PROPOSITION 2.1. *For $x \in \text{ces}(p)$, the modular ρ on $\text{ces}(p)$ satisfies the following properties:*

- (1) if $0 < a < 1$, then $a^M \rho(x/a) \leq \rho(x)$ and $\rho(ax) \leq a \rho(x)$,
- (2) if $a \geq 1$, then $\rho(x) \leq a^M \rho(x/a)$,
- (3) if $a \geq 1$, then $\rho(x) \leq a \rho(x) \leq \rho(ax)$.

PROOF. All assertions are clearly obtained by the definition and convexity of ρ . $\square$

PROPOSITION 2.2. *For any $x \in \text{ces}(p)$,*

- (1) if $\|x\| \leq 1$, then $\rho(x) \leq \|x\|$,
- (2) if $\|x\| > 1$, then $\rho(x) \geq \|x\|$,
- (3) $\|x\| = 1$ if and only if $\rho(x) = 1$.

PROOF. (1) Suppose that $\|x\| \leq 1$. If $x = 0$, then $\rho(x) = \|x\| = 0$. Suppose $x \neq 0$. By the definition of $\|\cdot\|$, there is a sequence (ϵ_n) with $\epsilon_n \downarrow \|x\|$ such that $\rho(x/\epsilon_n) \leq 1$. This implies that $\rho(x/\|x\|) \leq 1$. By Proposition 2.1(1), we have

$$\rho(x) = \rho\left(\frac{\|x\| \cdot x}{\|x\|}\right) \leq \|x\| \rho\left(\frac{x}{\|x\|}\right) \leq \|x\|. \quad (2.1)$$

(2) Suppose that $\|x\| > 1$. Then for $\epsilon \in (0, (\|x\| - 1)/\|x\|)$, we have $(1 - \epsilon)\|x\| > 1$. By Proposition 2.1(1), we have

$$1 < \rho\left(\frac{x}{(1 - \epsilon)\|x\|}\right) \leq \frac{\rho(x)}{(1 - \epsilon)\|x\|}, \quad (2.2)$$

so that $(1 - \epsilon)\|x\| < \rho(x)$. By taking $\epsilon \rightarrow 0$, we have $\rho(x) \geq \|x\|$.

- (3) It follows from Theorem 1.2(1) because $\rho \in \delta_2^s$. $\square$

PROPOSITION 2.3. *For any $L > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|\rho(u + v) - \rho(u)| < \epsilon \quad (2.3)$$

whenever $u, v \in \text{ces}(p)$ with $\rho(u) \leq L$ and $\rho(v) \leq \delta$.

PROOF. Since $p = (p_n)$ is bounded, it is easy to see that $\rho \in \delta_2^s$. Hence, the proposition is obtained directly from Theorem 1.1. $\square$

PROPOSITION 2.4. For every sequence $(x_n) \in \text{ces}(p)$, $\|x_n\| \rightarrow 0$ if and only if $\rho(x_n) \rightarrow 0$.

PROOF. It follows directly from Theorem 1.2(2) because $\rho \in \delta_2^s$. $\square$

THEOREM 2.5. For any $x \in \text{ces}(p)$ and $\epsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that $\rho(x) \leq 1 - \epsilon$ implies $\|x\| \leq 1 - \delta$.

PROOF. Since $\rho \in \delta_2^s$, the theorem is obtained directly from Theorem 1.3. $\square$

THEOREM 2.6. The space $\text{ces}(p)$ is k -NUC for any integer $k \geq 2$.

PROOF. Let $\epsilon > 0$ and $(x_n) \subset B(\text{ces}(p))$ with $\text{sep}(x_n) \geq \epsilon$. For each $m \in \mathbb{N}$, let

$$x_n^m = \left(\underbrace{0, 0, \dots, 0}_{m-1}, x_n(m), x_n(m+1), \dots \right). \quad (2.4)$$

Since for each $i \in \mathbb{N}$, $(x_n(i))_{n=1}^\infty$ is bounded, by using the diagonal method, we have that for each $m \in \mathbb{N}$, we can find a subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathbb{N}$, $1 \leq i \leq m$. Therefore, there exists an increasing sequence of positive integer (t_m) such that $\text{sep}((x_{n_j}^m)_{j > t_m}) \geq \epsilon$. Hence, there is a sequence of positive integers $(r_m)_{m=1}^\infty$ with $r_1 < r_2 < r_3 < \dots$ such that $\|x_{r_m}^m\| \geq \epsilon/2$ for all $m \in \mathbb{N}$. Then by Proposition 2.4, we may assume that there exists $\eta > 0$ such that

$$\rho(x_{r_m}^m) \geq \eta \quad \forall m \in \mathbb{N}. \quad (2.5)$$

Let $\alpha > 0$ be such that $1 < \alpha < \lim_{n \rightarrow \infty} \inf p_n$. For fixed integer $k \geq 2$, let $\epsilon_1 = ((k^{\alpha-1} - 1)/(k-1)k^\alpha)(\eta/2)$. Then by Proposition 2.3, there is a $\delta > 0$ such that

$$|\rho(u+v) - \rho(u)| < \epsilon_1 \quad (2.6)$$

whenever $\rho(u) \leq 1$ and $\rho(v) \leq \delta$.

Since by Proposition 2.2(1) $\rho(x_n) \leq 1$ for all $n \in \mathbb{N}$, there exist positive integers m_i ($i = 1, 2, \dots, k-1$) with $m_1 < m_2 < \dots < m_{k-1}$ such that $\rho(x_{m_i}^{m_i}) \leq \delta$ and $\alpha \leq p_j$ for all $j \geq m_{k-1}$. Define $m_k = m_{k-1} + 1$. By (2.5), we have $\rho(x_{r_{m_k}}^{m_k}) \geq \eta$. Let $s_i = i$ for $1 \leq i \leq k-1$ and $s_k = r_{m_k}$.

Then in virtue of (2.5), (2.6), and convexity of function $f_i(u) = |u|^{p_i}$ ($i \in \mathbb{N}$), we have

$$\begin{aligned} & \rho\left(\frac{x_{s_1} + x_{s_2} + \dots + x_{s_k}}{k}\right) \\ &= \sum_{n=1}^\infty \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_1}(i) + x_{s_2}(i) + \dots + x_{s_k}(i)}{k} \right| \right)^{p_n} \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{m_1} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_1}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} \\
&\leq \sum_{n=1}^{m_1} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_1}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} + \epsilon_1 \\
&\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^k \left(\frac{1}{n} \sum_{i=1}^n |x_{s_j}(i)| \right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} + \epsilon_1 \\
&\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^k \left(\frac{1}{n} \sum_{i=1}^n |x_{s_j}(i)| \right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_3}(i) + x_{s_4}(i) + \cdots + x_{s_k}(i)}{k} \right| \right)^{p_n} + 2\epsilon_1 \\
&\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^k \left(\frac{1}{n} \sum_{i=1}^n |x_{s_j}(i)| \right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2} \frac{1}{k} \sum_{j=2}^k \left(\frac{1}{n} \sum_{i=1}^n |x_{s_j}(i)| \right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{m_3} \frac{1}{k} \sum_{j=3}^k \left(\frac{1}{n} \sum_{i=1}^n |x_{s_j}(i)| \right)^{p_n} \\
&\quad + \cdots + \sum_{n=m_{k-1}+1}^{m_k} \frac{1}{k} \sum_{j=k-1}^k \left(\frac{1}{n} \sum_{i=1}^n |x_{s_j}(i)| \right)^{p_n} \\
&\quad + \sum_{n=m_k+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_k}(i)}{k} \right| \right)^{p_n} + (k-1)\epsilon_1
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{\rho(x_{s_1}) + \cdots + \rho(x_{s_{k-1}})}{k} + \frac{1}{k} \sum_{n=1}^{m_k} \left(\frac{1}{n} \sum_{i=1}^n |x_{s_k}(i)| \right)^{p_n} \\
& \quad + \sum_{n=m_k+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n \left| \frac{x_{s_k}(i)}{k} \right| \right)^{p_n} + (k-1)\epsilon_1 \\
& \leq \frac{k-1}{k} + \frac{1}{k} \sum_{n=1}^{m_k} \left(\frac{1}{n} \sum_{i=1}^n |x_{s_k}(i)| \right)^{p_n} \\
& \quad + \frac{1}{k^\alpha} \sum_{n=m_k+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x_{s_k}(i)| \right)^{p_n} + (k-1)\epsilon_1 \\
& \leq 1 - \frac{1}{k} + \frac{1}{k} \left[1 - \sum_{n=m_k+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x_{s_k}(i)| \right)^{p_n} \right] \\
& \quad + \frac{1}{k^\alpha} \sum_{n=m_k+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x_{s_k}(i)| \right)^{p_n} + (k-1)\epsilon_1 \\
& \leq 1 + (k-1)\epsilon_1 - \left(\frac{k^{\alpha-1}-1}{k^\alpha} \right) \sum_{n=m_k+1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x_{s_k}(i)| \right)^{p_n} \\
& \leq 1 + (k-1)\epsilon_1 - \left(\frac{k^{\alpha-1}-1}{k^\alpha} \right) \eta \\
& = 1 - \left(\frac{k^{\alpha-1}-1}{k^\alpha} \right) \left(\frac{\eta}{2} \right).
\end{aligned}$$

(2.7)

By Theorem 2.5, there exist $\gamma > 0$ such that $\|(x_{s_1} + x_{s_2} + \cdots + x_{s_k})/k\| < 1 - \gamma$. Therefore, $\text{ces}(p)$ is k -NUC. $\square$

Since k -NUC implies kR and kR implies R and reflexivity holds, and k -NUC implies NUC and NUC implies property (H) and reflexivity holds, by Theorem 2.6, the following results are obtained.

COROLLARY 2.7. *The space $\text{ces}(p)$ is kR , NUC, and has a drop property.*

COROLLARY 2.8. *For $1 < p < \infty$, the space ces_p is k -NUC.*

COROLLARY 2.9. *For $1 < p < \infty$, the space ces_p is kR and NUC.*

COROLLARY 2.10. *For $1 < p < \infty$, the space ces_p has the drop property.*

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REFERENCES

- [1] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396-414.
- [2] Y. Cui and H. Hudzik, *On the uniform Opial property in some modular sequence spaces*, Funct. Approx. Comment. Math. **26** (1998), 93-102.
- [3] Y. Cui, C. Meng, and R. Pluciennik, *Banach-Saks property and property (β) in Cesàro sequence spaces*, Southeast Asian Bull. Math. **24** (2000), no. 2, 201-210.
- [4] J. Daneš, *A geometric theorem useful in nonlinear functional analysis*, Boll. Un. Mat. Ital. (4) **6** (1972), 369-375.
- [5] K. Fan and I. Glicksberg, *Fully convex normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 947-953.
- [6] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), no. 4, 743-749.
- [7] D. Kutzarova, *k - β and k -nearly uniformly convex Banach spaces*, J. Math. Anal. Appl. **162** (1991), no. 2, 322-338.
- [8] P. Y. Lee, *Cesàro sequence spaces*, Math. Chronicle **13** (1984), 29-45.
- [9] Y. Q. Lui, B. E. Wu, and P. Y. Lee, *Method of Sequence Spaces*, Guangdong of Science and Technology Press, 1996 (Chinese).
- [10] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Mathematics, vol. 5, Polish Academy of Science, 1989.
- [11] V. Montesinos, *Drop property equals reflexivity*, Studia Math. **87** (1987), no. 1, 93-100.
- [12] S. Rolewicz, *On drop property*, Studia Math. **85** (1986), no. 1, 27-35 (1987).
- [13] ———, *On Δ -uniform convexity and drop property*, Studia Math. **87** (1987), no. 2, 181-191.
- [14] J.-s. Shiue, *On the Cesàro sequence spaces*, Tamkang J. Math. **1** (1970), no. 1, 19-25.

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Some Geometric Properties of Cesaro Sequence Space

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ABSTRACT. In this paper we define a modular on the Cesaro sequence space $ces(p)$ and consider it equipped with the Luxemburg norm. We give some relationships between the modular and the Luxemburg norm on this space and show that the space $ces(p)$ has property (H) but it is not rotund (R), where $p = (p_k)$ is a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space, and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X . A point $x \in S(X)$ is an *H-point* of $B(X)$ if for any sequence (x_n) in X such that $\|x_n\| \rightarrow 1$ as $n \rightarrow \infty$, the weak convergence of (x_n) to x (write $x_n \xrightarrow{w} x$) implies that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. If every point in $S(X)$ is an *H-point* of $B(X)$, then X is said to have the *property (H)*. A point $x \in S(X)$ is an *extreme point* of $B(X)$, if for any $y, z \in S(X)$ the equality $2x = y + z$ implies $y = z$. A point $x \in S(X)$ is an *locally uniformly rotund point* of $B(X)$ (LUR-point for short) if for any sequence (x_n) in $B(X)$ such that $\|x_n + x\| \rightarrow 2$ as $n \rightarrow \infty$ there holds $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. A Banach space X is said to be *rotund (R)*, if every point of $S(X)$ is an extreme point of $B(X)$. If every point of $S(X)$ is a LUR-point of $B(X)$, then X is said to be *locally uniformly rotund (LUR)*. It is known that if X is LUR, then it is (R) and possesses property (H). For these geometric notions and their role in Mathematics we refer to the monographs [1], [2], [6] and [13]. Some of them were studied for Orlicz spaces in [3], [7], [8], [9] and [14].

Let l^0 be the space of all real sequences. For $1 \leq p < \infty$, the Cesaro sequence space $(ces_p, \|\cdot\|_p)$ for short) is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty} (\frac{1}{n} \sum_{i=1}^n |x(i)|)^p < \infty\}$$

equipped with the norm

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$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^p \right)^{\frac{1}{p}}$$

and

$$\|x\|_0 = \left(\sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x(i)| \right)^p \right)^{\frac{1}{p}}$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq i < 2^{r+1}$.

It is known that these two norms are equivalent and ces_p is Banach with respect to each of the two norms. This space was introduced by J. S. Shue [15]. It is useful in the theory of matrix operator and others (see [10] and [12]). Some geometric properties of the Cesàro sequence space $(ces_p, \|\cdot\|)$ were studied by many mathematicians. It is known that $(ces_p, \|\cdot\|)$ is LUR and possesses property (H) (see [12]). Y. A. Cui and H. Hudzik [4] proved that $(ces_p, \|\cdot\|)$ has the Banach-Saks of type p if $p > 1$, and it was shown in [5] that $(ces_p, \|\cdot\|)$ has property (β) .

Now let $p = (p_k)$ be a bounded sequence of positive real number with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Cesaro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x(i)| \right)^{p_r}\}$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq i < 2^{r+1}$.

For $x \in ces(p)$, let $\rho(x) = \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x(i)| \right)^{p_r}$ and define the Luxemburg norm on $ces(p)$ by

$$\|x\| = \inf \left\{ \epsilon > 0 : \rho\left(\frac{x}{\epsilon}\right) \leq 1 \right\}, \quad x \in ces(p).$$

The main purpose of this paper is to show that the Cesaro sequence space $ces(p)$ equipped with the Luxemburg norm has property (H) but it is not rotund, so it is not LUR. Throughout this paper we let $M = \sup_r p_r$, and for $x \in l^0$ we put

$$x|_i = (x(1), x(2), \dots, x(i), 0, 0, \dots)$$

and

$$x|_{N-i} = (0, 0, \dots, 0, x(i+1), x(i+2), \dots).$$

2. Main results

First, we show that ρ is a convex modular on $ces(p)$.

Proposition 2.1. *The functional ρ is a convex modular on $ces(p)$.*

Proof. It is obvious that $\rho(x) = 0 \Leftrightarrow x = 0$ and $\rho(\alpha x) = \rho(x)$ for all scalar α with

$|\alpha| = 1$. Let $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $t \rightarrow |t|^{p_r}$ for every $r \in \mathbb{N}$, we have

$$\begin{aligned}\rho(\alpha x + \beta y) &= \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |\alpha x(i) + \beta y(i)| \right)^{p_r} \\ &\leq \sum_{r=0}^{\infty} \left(\alpha \frac{1}{2^r} \sum_r |x(i)| + \beta \frac{1}{2^r} \sum_r |y(i)| \right)^{p_r} \\ &\leq \alpha \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x(i)| \right)^{p_r} + \beta \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |y(i)| \right)^{p_r} \\ &= \alpha \rho(x) + \beta \rho(y).\end{aligned}$$

□

Proposition 2.2. For $x \in \text{ces}(p)$, the modular ρ on $\text{ces}(p)$ satisfies the following properties

- (i) if $0 < a < 1$, then $a^M \rho(\frac{x}{a}) \leq \rho(x)$ and $\rho(ax) \leq a\rho(x)$,
- (ii) if $a > 1$, then $\rho(x) \leq a^M \rho(\frac{x}{a})$,
- (iii) if $a \geq 1$, then $\rho(x) \leq a\rho(x) \leq \rho(ax)$.

Proof. (i) Let $0 < a < 1$. Then we have

$$\begin{aligned}\rho(x) &= \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x_n| \right)^{p_r} \\ &= \sum_{r=0}^{\infty} \left(\frac{a}{2^r} \sum_r \left| \frac{x_n}{a} \right| \right)^{p_r} \\ &= \sum_{r=0}^{\infty} a^{p_r} \left(\frac{1}{2^r} \sum_r \left| \frac{x_n}{a} \right| \right)^{p_r} \\ &\geq \sum_{r=0}^{\infty} a^M \left(\frac{1}{2^r} \sum_r \left| \frac{x_n}{a} \right| \right)^{p_r} \\ &= a^M \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r \left| \frac{x_n}{a} \right| \right)^{p_r} \\ &= a^M \rho\left(\frac{x}{a}\right).\end{aligned}$$

By convexity of ρ , we have $\rho(ax) \leq a\rho(x)$, so (i) is obtained

(ii) is an easy consequence of (i) when a is replaced by $\frac{1}{a}$.

(iii) follows from the convexity of ρ . □

Proposition 2.3. For any $x \in \text{ces}(p)$, we have

- (i) if $\|x\| < 1$, then $\rho(x) \leq \|x\|$,
- (ii) if $\|x\| > 1$, then $\rho(x) \geq \|x\|$,
- (iii) $\|x\| = 1$ if and only if $\rho(x) = 1$,
- (iv) $\|x\| < 1$ if and only if $\rho(x) < 1$ and
- (v) $\|x\| > 1$ if and only if $\rho(x) > 1$.

Proof. (i) Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \|x\|$, so $\|x\| + \epsilon < 1$. By definition of $\|\cdot\|$, there exists $\lambda > 0$ such that $\|x\| + \epsilon > \lambda$ and $\rho(\frac{x}{\lambda}) \leq 1$. By Proposition 2.2(i) and (iii), we have

$$\begin{aligned}\rho(x) &\leq \rho\left(\frac{(\|x\| + \epsilon)}{\lambda}x\right) \\ &= \rho\left((\|x\| + \epsilon)\frac{x}{\lambda}\right) \\ &\leq (\|x\| + \epsilon)\rho\left(\frac{x}{\lambda}\right) \\ &\leq \|x\| + \epsilon,\end{aligned}$$

which implies that $\rho(x) \leq \|x\|$. Hence (i) is satisfied.

(ii) Let $\epsilon > 0$ be such that $0 < \epsilon < \frac{\|x\| - 1}{\|x\|}$, then $1 < (1 - \epsilon)\|x\| < \|x\|$. By definition of $\|\cdot\|$ and by Proposition 2.2(i), we have $1 < \rho\left(\frac{x}{(1-\epsilon)\|x\|}\right) \leq \frac{1}{(1-\epsilon)\|x\|}\rho(x)$,

so $(1 - \epsilon)\|x\| < \rho(x)$ for all $\epsilon \in (0, \frac{\|x\| - 1}{\|x\|})$, which implies that $\|x\| \leq \rho(x)$.

(iii) Assume that $\|x\| = 1$. Let $\epsilon > 0$, then there exists $\lambda > 0$ such that $1 + \epsilon > \lambda > \|x\|$ and $\rho(\frac{x}{\lambda}) \leq 1$. By Proposition 2.2(ii), we have $\rho(x) \leq \lambda^M \rho(\frac{x}{\lambda}) \leq \lambda^M < (1 + \epsilon)^M$, so $(\rho(x))^{\frac{1}{M}} < 1 + \epsilon$ for all $\epsilon > 0$ which implies that $\rho(x) \leq 1$. If $\rho(x) < 1$, let $a \in (0, 1)$ such that $\rho(x) < a^M < 1$. From Proposition 2.2(i), we have $\rho(\frac{x}{a}) \leq \frac{1}{a^M} \rho(x) < 1$, hence $\|x\| \leq a < 1$, which is a contradiction. Thus, we have $\rho(x) = 1$.

Conversely, assume that $\rho(x) = 1$. By definition of $\|\cdot\|$, we conclude that $\|x\| \leq 1$. If $\|x\| < 1$, then we have by (i) that $\rho(x) \leq \|x\| < 1$, which contradicts to our assumption, so we obtain that $\|x\| = 1$.

(iv) follows from (i) and (iii).

(v) follows from (iii) and (iv). □

Proposition 2.4. For $x \in \text{ces}(p)$ we have

- (i) if $0 < a < 1$ and $\|x\| > a$, then $\rho(x) > a^M$ and

(ii) if $a \geq 1$ and $\|x\| < a$, then $\rho(x) < a^M$.

Proof. (i) Suppose $0 < a < 1$ and $\|x\| > a$. Then $\|\frac{x}{a}\| > 1$. By Proposition 2.3(ii), we have $\rho(\frac{x}{a}) > 1$. Hence, by Proposition 2.2(i), we obtain that $\rho(x) \geq a^M \rho(\frac{x}{a}) > a^M$.

(ii) Suppose $a \geq 1$ and $\|x\| < a$. Then $\|\frac{x}{a}\| < 1$. By Proposition 2.3(i), we have $\rho(\frac{x}{a}) < 1$. If $a = 1$, we have $\rho(x) < 1 = a^M$. If $a > 1$, by Proposition 2.2(ii), we obtain that $\rho(x) < a^M \rho(\frac{x}{a}) < a^M$. $\square$

Proposition 2.5. Let (x_n) be a sequence in $\text{ces}(p)$.

(i) If $\lim_{n \rightarrow \infty} \|x_n\| = 1$, then $\lim_{n \rightarrow \infty} \rho(x_n) = 1$.

(ii) If $\lim_{n \rightarrow \infty} \rho(x_n) = 0$ then $\lim_{n \rightarrow \infty} \|x_n\| = 0$.

Proof. (i) Suppose $\lim_{n \rightarrow \infty} \|x_n\| = 1$. Let $\epsilon \in (0, 1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < 1 + \epsilon$ for all $n \geq N$. By Proposition 2.4, $(1 - \epsilon)^M < \rho(x_n) < (1 + \epsilon)^M$ for all $n \geq N$, which implies that $\lim_{n \rightarrow \infty} \rho(x_n) = 1$.

(ii) Suppose $\|x_n\| \not\rightarrow 0$. Then there is an $\epsilon \in (0, 1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proposition 2.4 (i), we obtain $\rho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\rho(x_n) \not\rightarrow 0$ as $n \rightarrow \infty$. $\square$

Lemma 2.6. Let (x_n) be a sequence in $\text{ces}(p)$. If $\rho(x_n) \rightarrow \rho(x)$ and $x_n(k) \rightarrow x(k) \forall k$, then $x_n \rightarrow x$ as $n \rightarrow \infty$.

Proof. Suppose that $x_n \not\rightarrow x$. By Proposition 2.5 (ii), we have $\rho(\frac{x_n - x}{2}) \not\rightarrow 0$. Without loss of generality we may assume that there exists $\epsilon \in (0, 1)$ such that $\rho(\frac{x_n - x}{2}) > \epsilon$ for all $n \in \mathbb{N}$. Since $(\rho(\frac{x_n - x}{2}))_{n=1}^\infty$ is a bounded sequence, it must have a convergent subsequence. Passing through a subsequence, if necessary we can assume $\rho(\frac{x_n - x}{2}) \rightarrow \epsilon_0$ for some $\epsilon_0 \geq \epsilon$. Since $\rho(x) = \lim_{i \rightarrow \infty} \rho(x|_{2^i})$ and $(\rho(x|_{2^i}))_{i=0}^\infty$ is nondecreasing, we have $\rho(x) = \sup\{\rho(x|_{2^i}) : i \in \mathbb{N}\}$. So there exists $i \in \mathbb{N}$ such that $\rho(x|_{2^i}) > \rho(x) - \epsilon/2$. Thus

$$(2.1) \quad \rho(x|_{N-2^i}) < \epsilon/2.$$

Since $x_n(k) \rightarrow x(k)$ for all $k \in \mathbb{N}$, we have

$$(2.2) \quad \rho(x_n|_{2^i}) \rightarrow \rho(x|_{2^i}) \text{ and } \rho(\frac{x_n - x}{2}|_{2^i}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By the convexity of ρ together with (2.1) and (2.2), we have

$$\begin{aligned} \epsilon_0 &= \lim_{n \rightarrow \infty} \rho(\frac{x_n - x}{2}) \\ &= \lim_{n \rightarrow \infty} [\rho(\frac{x_n - x}{2}|_{2^i}) + \rho(\frac{x_n - x}{2}|_{N-2^i})] \\ &= \lim_{n \rightarrow \infty} \rho(\frac{x_n - x}{2}|_{2^i}) + \lim_{n \rightarrow \infty} \rho(\frac{x_n - x}{2}|_{N-2^i}) \end{aligned}$$

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$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \rho\left(\frac{x_n - x}{2} \Big|_{2^i}\right) + \lim_{n \rightarrow \infty} \rho\left(\frac{x_n - x}{2} \Big|_{N-2^i}\right) \\
&= 0 + \lim_{n \rightarrow \infty} \rho\left(\frac{x_n - x}{2} \Big|_{N-2^i}\right) \\
&\leq \frac{1}{2} \lim_{n \rightarrow \infty} \rho(x_n \Big|_{N-2^i}) + \frac{1}{2} \rho(x \Big|_{N-2^i}) \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} (\rho(x_n) - \rho(x_n \Big|_{2^i})) + \frac{1}{2} \rho(x \Big|_{N-2^i}) \\
&= \frac{1}{2} (\rho(x) - \rho(x \Big|_{2^i})) + \frac{1}{2} \rho(x \Big|_{N-2^i}) \\
&= \frac{1}{2} \rho(x \Big|_{N-2^i}) + \frac{1}{2} \rho(x \Big|_{N-2^i}) \\
&= \rho(x \Big|_{N-2^i}) \\
&< \epsilon/2 \\
&< \epsilon_0,
\end{aligned}$$

which is a contradiction. Therefore $x_n \rightarrow x$ as $n \rightarrow \infty$. $\square$

Theorem 2.7. *The space $ces(p)$ has the property (H).*

Proof. Let $x \in S(ces(p))$, $x_n \in B(ces(p))$ for all $n \in \mathbb{N}$ such that $x_n \xrightarrow{\omega} x$ and $\|x_n\| \rightarrow 1$ as $n \rightarrow \infty$. By Proposition 2.3(iii), we have $\rho(x) = 1$. By Proposition 2.5(i), we obtain that $\rho(x_n) \rightarrow 1$ as $n \rightarrow \infty$. So $\rho(x_n) \rightarrow \rho(x)$ as $n \rightarrow \infty$. Since $x_n \xrightarrow{\omega} x$ and the i^{th} coordinate mapping $\pi_i : ces(p) \rightarrow \mathbb{R}$, defined by $\pi_i(x) = x_i$, is continuous, it implies that $x_n(i) \rightarrow x(i)$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. It follows from Lemma 2.6 that $x_n \rightarrow x$ as $n \rightarrow \infty$. $\square$

The following result is obtained directly from Theorem 2.7.

Corollary 2.8. *For $1 \leq p < \infty$, $(ces_p, \|\cdot\|_0)$ has property (H).*

Remark 2.9. For a bounded sequence of positive real numbers $p = (p_k)$ with $p_k \geq 1$ for all $k \in \mathbb{N}$, the space $ces(p)$ equipped the Luxemburg norm is not rotund, so it is not LUR. To see this we put

$$x = (0, 1, 1, 0, 0, \dots) \text{ and } y = (0, 2, 0, 0, \dots)$$

Then $x, y \in S(ces(p))$ because $\rho(x) = \rho(y) = 1$. Since $\rho\left(\frac{x+y}{2}\right) = 1$, we have by Proposition 2.3 (iii) that $\left\|\frac{x+y}{2}\right\| = 1$. This shows that $ces(p)$ is not rotund, so it is not LUR.

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References

- [1] S. T. Chen, Geometry of Orlicz spaces, *Dissertationes Math.*, (1996), 356.
- [2] Y. A. Cui and H. Hudzik, *On the Banach-Saks and weak Banach-Saks properties of some Banach sequence spaces*, *Acta Sci. Math. (Szeged)*, **65**(1999), 179-187.
- [3] Y. A. Cui, H. Hudzik and C. Meng, *On some local geometry of Orlicz sequence spaces equipped the Luxemburg norms*, *Acta Math. Hungar.*, **80**(1-2)(1998), 143-154.
- [4] Y. A. Cui, H. Hudzik and R. Płociennik, *Banach-Saks property in some Banach sequence spaces*, *Annales Math. Polonici*, **65**(1997), 193-202.
- [5] Y. A. Cui and C. Meng, *Banach-Sak property and property (β) in Cesaro sequence spaces*, *SEA. Bull. Math.*, **24**(2000), 201-210.
- [6] J. Diestel, *Geometry of Banach Spaces - Selected Topics*, Springer-Verlag, (1984).
- [7] R. Grzaslewicz, H. Hudzik and W. Kurc, *Extreme and exposed points in Orlicz spaces*, *Canadian J. Math.*, **44**(1992), 505-515.
- [8] H. Hudzik, *Orlicz spaces without strongly extreme points and without H-points*, *Canad. Math. Bull.*, **35**(1992), 1-5.
- [9] H. Hudzik and D. Pallaschke, *On some convexity properties of Orlicz sequence spaces*, *Math. Nachr.*, **186**(1997), 167-185.
- [10] P. Y. Lee, *Cesáro sequence spaces*, *Math. Chronicle, New Zealand*, **13**(1984), 29-45.
- [11] B. L. Lin, P. K. Lin and S. L. Troyanski, *Characterization of denting points*, *Proc. Amer. Math. Soc.*, **102**(1988), 526-528.
- [12] Y. Q. Liu, B. E. Wu and Y. P. Lee, *Method of sequence spaces*, Guangdong of Science and Technology Press, (1996)(in Chinese).
- [13] J. Musielak, *Orlicz spaces and modular spaces*, *Lecture Notes in Math.* 1034, Springer-Verlag, (1983).
- [14] R. Płociennik, T. F. Wang and Y. L. Zhang, *H-points and Denting Points in Orlicz Spaces*, *Comment. Math. (Prace Mat.)*, **33**(1993), 135-151.
- [15] J. S. Shue, *Cesáro sequence spaces*, *Tamkang J. Math.*, **1**(1970), 143-150.

ON UNIFORM KADEC-KLEE PROPERTIES AND ROTUNDITY IN GENERALIZED CESÀRO SEQUENCE SPACES

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We consider the generalized Cesàro sequence spaces defined by Suantai (2003) and consider it equipped with the Amemiya norm. The main purpose of this paper is to show that $\text{ces}_{(\mu)}$ equipped with the Amemiya norm is rotund and has uniform Kadec-Klee property.

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1. Introduction. In the whole paper, $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural numbers and real numbers, respectively. Let $(X, \|\cdot\|)$ be a real normed space and $B(X)$ ($S(X)$) the closed unit ball (the unit sphere) of X .

A point $x \in S(X)$ is called an *extreme point* if for any $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$.

A Banach space X is said to be *rotund* (abbreviated as (R)) if every point of $S(X)$ is an extreme point.

A Banach space X is said to have the *Kadec-Klee property* (or H-property) if every weakly convergent sequence on the unit sphere is convergent in norm.

Recall that a sequence $\{x_n\} \subset X$ is said to be ε -*separated sequence* for some $\varepsilon > 0$ if

$$\text{sep}(x_n) = \inf \{\|x_n - x_m\| : n \neq m\} > \varepsilon. \quad (1.1)$$

A Banach space is said to have the *uniform Kadec-Klee property* (abbreviated as (UKK)) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence (x_n) in $S(X)$ with $\text{sep}(x_n) > \varepsilon$ and $x_n \xrightarrow{w} x$, we have $\|x\| < 1 - \delta$. Every (UKK) Banach space has H-property (see [3]).

A Banach space is said to be *nearly uniformly convex* (abbreviated as (NUC)) if for every $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that for every sequence $(x_n) \subseteq B(X)$ with $\text{sep}(x_n) > \varepsilon$, we have

$$\text{conv}(x_n) \cap ((1 - \delta)B(X)) \neq \emptyset. \quad (1.2)$$

Huff [3] proved that every (NUC) Banach space is reflexive and has H-property and he also proved that X is NUC if and only if X is reflexive and UKK.

A Banach space X is said to be *locally uniform rotund* (abbreviated as (LUR)) if for each $x \in S(X)$ and each sequence $(x_n) \subset S(X)$ such that $\lim_{n \rightarrow \infty} \|x_n + x\| = 2$ there holds $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$.

A continuous function $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ is called *convex* if

$$\Phi\left(\frac{u+v}{2}\right) \leq \frac{\Phi(u) + \Phi(v)}{2} \quad (1.3)$$

for all $u, v \in \mathbb{R}$. If, in addition, the two sides of inequality (1.3) are not equal for all $u \neq v$, then we call Φ *strictly convex*.

For a real vector space X , a function $\varrho: X \rightarrow [0, \infty]$ is called a *modular* if it satisfies the following conditions:

- (i) $\varrho(x) = 0$ if and only if $x = 0$;
- (ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;
- (iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

- (iv) $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$ for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular ϱ on X , the space

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\} \quad (1.4)$$

is called the *modular space*. If ϱ is a convex modular, the functions

$$\begin{aligned} \|x\| &= \inf \left\{ \lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1 \right\}, \\ \|x\|_0 &= \inf_{k > 0} \frac{1}{k} (1 + \varrho(kx)) \end{aligned} \quad (1.5)$$

are two norms on X_ϱ , which are called the *Luxemburg norm* and the *Amemiya norm*, respectively. In addition, $\|x\| \leq \|x\|_0 \leq 2\|x\|$ for all $x \in X_\varrho$ (see [6]).

A modular ϱ is said to satisfy the Δ_2 -condition ($\varrho \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$$\varrho(2x) \leq K\varrho(x) + \varepsilon \quad (1.6)$$

for all $x \in X_\varrho$ with $\varrho(x) \leq a$.

If ϱ satisfies the Δ_2 -condition for all $a > 0$ with $K \geq 2$ dependent on a , we say that ϱ satisfies the *strong* Δ_2 -condition ($\varrho \in \Delta_2^s$).

Let ℓ^0 be the space of all real sequences. The Musielak-Orlicz sequence space ℓ_Φ , where $\Phi = (\phi_i)_{i=1}^\infty$ is a sequence of Orlicz functions, is defined as

$$\ell_\Phi = \{x = (x(i))_{i=1}^\infty \in \ell^0 : \varrho_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}, \quad (1.7)$$

where $\varrho_\Phi(x) = \sum_{i=1}^\infty \phi_i(x(i))$ is a convex modular on ℓ_Φ . Then ℓ_Φ is a Banach space with both Luxemburg norm $\|\cdot\|_{\ell_\Phi}$ and Amemiya norm $\|\cdot\|_{\ell_\Phi^0}$ (see [6]). In [2], Hudzik and Zbąszczyński proved that in the space ℓ_Φ endowed with the Amemiya norm, there exists $k \in \mathbb{R}$ such that

$$\|x\|_{\ell_\Phi^0} = \frac{1}{k} (1 + \varrho_\Phi(kx)) \quad (x \in \ell_\Phi) \quad (1.8)$$

if $\phi_i(u)/u \rightarrow \infty$ as $u \rightarrow \infty$ for any $i \in \mathbb{N}$.

For $1 < p < \infty$, the Cesàro sequence space (ces_p) is defined by

$$ces_p = \left\{ x \in \ell^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^p < \infty \right\} \quad (1.9)$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^p \right)^{1/p}. \quad (1.10)$$

This space was first introduced by Shue [8]. It is useful in the theory of matrix operator and others (see [4, 5]). Some geometric properties of the Cesàro sequence space ces_p were studied by many authors. Now, we introduce a generalized Cesàro sequence space.

Let $p = (p_n)$ be a sequence of positive real numbers with $p_n \geq 1$ for all $n \in \mathbb{N}$. The generalized Cesàro sequence space $ces_{(p)}$ is defined by

$$ces_{(p)} = \{x \in \ell^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}, \quad (1.11)$$

where

$$\rho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^{p_n} \quad (1.12)$$

is a convex modular on $ces_{(p)}$. To simplify the notations, we put $ces_{(p)} = (ces_{(p)}, \|\cdot\|)$ and $ces_{(p)}^0 = (ces_{(p)}, \|\cdot\|_0)$.

For $ces_{(p)}$, Suantai [9] proved that $ces_{(p)}$ is LUR, hence it is R and has H-property where $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$.

In $ces_{(p)}^0$, the set of all k 's, at which the infimum in the definition of $\|x\|_0$ for a fixed $x \in ces_{(p)}^0$ is attained, will be denoted by $K(x)$.

Throughout this paper, we let $p = (p_k)$ be a bounded sequence of positive real numbers.

2. Main results. We first give an important fact for $\|x\|_0$ on $ces_{(p)}^0$.

PROPOSITION 2.1. For each $x \in ces_{(p)}^0$, there exists $k \in \mathbb{R}$ such that

$$\|x\|_0 = \frac{1}{k} (1 + \rho(kx)). \quad (2.1)$$

PROOF. First, we note that $\phi(t) = |t|^r$ ($r > 1$) is an Orlicz function which satisfies $\phi_i(u)/u \rightarrow \infty$ as $u \rightarrow \infty$.

Now, observe that for each $x = (x(i))_{i=1}^{\infty} \in ces_{(p)}^0$ we have

$$x' = \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)_{n=1}^{\infty} \in \ell_{\phi}, \quad (2.2)$$

where $\Phi = (\phi_i)_{i=1}^\infty$ and $\phi_i(t) = |t|^{p_i}$ for each $i \in \mathbb{N}$. Moreover, $\|x\|_0 = \|x'\|_{\ell_\Phi^0}$ and by (1.8) there exists $k \in \mathbb{R}$ such that

$$\begin{aligned}\|x\|_0 &= \|x'\|_{\ell_\Phi^0} = \frac{1}{k}(1 + \varrho_\Phi(kx')) \\ &= \frac{1}{k} \left(1 + \sum_{n=1}^\infty \left(\frac{k}{n} \sum_{i=1}^n |x(i)| \right)^{p_n} \right) = \frac{1}{k}(1 + \rho(kx)).\end{aligned}\quad (2.3)$$

PROPOSITION 2.2. *For a modular space X_ϱ , convergence in norm and convergence in modular are equivalent if and only if $\varrho \in \Delta_2$.*

PROOF. See [1].

PROPOSITION 2.3. *Suppose that $\{x_n\}$ is a bounded sequence in $\text{ces}_{(p)}^0$ with $p_k > 1$ for all $k \in \mathbb{N}$ and $x_n \xrightarrow{w} x$ for some $x \in \text{ces}_{(p)}^0$. If $k_n \in K(x_n)$ and $k_n \rightarrow \infty$, then $x = 0$.*

PROOF. For each $n \in \mathbb{N}$, $\eta > 0$, put $G_{(n,\eta)} = \{i \in \mathbb{N} : (1/i) \sum_{j=1}^i |x_n(j)| \geq \eta\}$. First, we claim that for each $\eta > 0$, $G_{(n,\eta)} = \emptyset$ for all large $n \in \mathbb{N}$. If not, without loss of generality, we may assume that $G_{(n,\eta)} \neq \emptyset$ for all $n \in \mathbb{N}$ for some $\eta > 0$. Then,

$$\|x_n\|_0 = \frac{1}{k_n}(1 + \rho(k_n x_n)) \geq \frac{(k_n \eta)^{p_i}}{k_n} \quad (i \in G_{(n,\eta)}). \quad (2.4)$$

Applying the fact $|t|^r/t \rightarrow \infty$ as $t \rightarrow \infty$, where $r > 1$, we obtain $\|x_n\|_0 \rightarrow \infty$ which contradicts the fact that $\{x_n\}$ is bounded, hence we have the claim. By the claim, we have $(1/i) \sum_{j=1}^i |x_n(j)| \rightarrow 0$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. Hence, we obtain by induction that $x_n(i) \rightarrow 0$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. Since $x_n \xrightarrow{w} x$, we have $x_n(i) \rightarrow x(i)$ for all $i \in \mathbb{N}$, so it must be $x(i) = 0$ for all $i \in \mathbb{N}$. $\square$

THEOREM 2.4. *The space $\text{ces}_{(p)}^0$ is R if each $p_k > 1$.*

PROOF. Let $x \in S(\text{ces}_{(p)}^0)$ and suppose $y, z \in S(\text{ces}_{(p)}^0)$ with $y + z = 2x$. Take $k' \in K(y)$, $k'' \in K(z)$ and define $k = k'k''/(k' + k'')$. Then by convexity of $u \mapsto |u|^{p_n}$ for every $n \in \mathbb{N}$, we have

$$\begin{aligned}2 &= \|y\|_0 + \|z\|_0 = \frac{k' + k''}{k'k''} \left[1 + \frac{k''}{k' + k''} \rho(k'y) + \frac{k'}{k' + k''} \rho(k''z) \right] \\ &\geq \frac{1}{k} [1 + \rho(ky + kz)] = \frac{2}{2k} [1 + \rho(2kx)] \geq 2\|x\|_0 = 2.\end{aligned}\quad (2.5)$$

This implies

$$\frac{k''}{k' + k''} \left(\frac{k'}{n} \sum_{i=1}^n |y(i)| \right)^{p_n} + \frac{k'}{k' + k''} \left(\frac{k''}{n} \sum_{i=1}^n |z(i)| \right)^{p_n} = \left(\frac{2k}{n} \sum_{i=1}^n |x(i)| \right)^{p_n} \quad (2.6)$$

for all $n \in \mathbb{N}$.

Since the function $u \mapsto |u|^{p_n}$ is strictly convex function for all $n \in \mathbb{N}$, it implies that

$$k' \left[\frac{1}{n} \sum_{i=1}^n |y(i)| \right] = k'' \left[\frac{1}{n} \sum_{i=1}^n |z(i)| \right] = 2k \left[\frac{1}{n} \sum_{i=1}^n |x(i)| \right] \quad (2.7)$$

for each $n \in \mathbb{N}$. This gives $k'|y(i)| = k''|z(i)|$ for all $i \in \mathbb{N}$, and it follows that $k' = \|k'y\|^\circ = \|k''z\|^\circ = k''$, hence $|y(i)| = |z(i)|$ for all $i \in \mathbb{N}$. To complete the proof, it suffices to show that $y(i) = z(i)$ for all $i \in \mathbb{N}$. If not, let $i_0 \in \mathbb{N}$ be the first coordinate such that $y(i_0) \neq z(i_0)$, so $y(i_0) = -z(i_0)$ and hence $2x(i_0) = y(i_0) + z(i_0) = 0$. Since $k' = k'' = 2k$, we have

$$\begin{aligned} \left[\frac{1}{i_0-1} \sum_{i=1}^{i_0-1} |z(i)| \right] &= \left[\frac{1}{i_0-1} \sum_{i=1}^{i_0-1} |x(i)| \right], \\ \left[\frac{1}{i_0} \sum_{i=1}^{i_0} |z(i)| \right] &= \left[\frac{1}{i_0} \sum_{i=1}^{i_0} |x(i)| \right] \end{aligned} \quad (2.8)$$

which implies $z(i_0) = 0$, which is a contradiction. Hence $y = z$. $\square$

THEOREM 2.5. *The space $\text{ces}_{(p)}^0$ is UKK if each $p_k > 1$.*

PROOF. For a given $\varepsilon > 0$, by Proposition 2.2 there exists $\delta \in (0, 1)$ such that $\|y\|_0 \geq \varepsilon/4$ implies $\rho(y) \geq 2\delta$. Given $x_n \in B(\text{ces}_{(p)}^0)$, $x_n \rightarrow x$ weakly, and $\|x_n - x_m\|_0 \geq \varepsilon$ ($n \neq m$), we will complete the proof by showing that $\|x\|_0 \leq 1 - \delta$. Indeed, if $x = 0$, then we have nothing to show. So, we assume that $x \neq 0$. In this case, by Proposition 2.3 we have that $\{k_n\}$ is bounded, where $k_n \in K(x_n)$. Passing to a subsequence, if necessary we may assume that $k_n \rightarrow k$ for some $k > 0$. Next, we select a finite subset I of $\mathbb{N}$ such that $\|x_{I_I}\|_0 \geq \|x\|_0 - \delta$, say $I = \{1, 2, 3, \dots, j\}$; since the weak convergence of $\{x_n\}$ implies that $x_n \rightarrow x$ coordinatewise, we deduce that $x_n \rightarrow x$ uniformly on I . Consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\|(x_n - x_m)_{I_I}\|_0 \leq \frac{\varepsilon}{2} \quad \forall n, m \geq n_0, \quad (2.9)$$

which implies

$$\|(x_n - x_m)_{I_{N \setminus I}}\|_0 \geq \frac{\varepsilon}{2} \quad \forall n, m \geq n_0, \quad m \neq n. \quad (2.10)$$

This gives $\|x_{n_{I_{N \setminus I}}}\|_0 \geq \varepsilon/4$ or $\|x_{m_{I_{N \setminus I}}}\|_0 \geq \varepsilon/4$, for all $m, n \geq n_0$, $m \neq n$, which yields that $\|x_{n_{I_{N \setminus I}}}\|_0 \geq \varepsilon/4$ for infinitely many $n \in \mathbb{N}$, hence $\rho(x_{n_{I_{N \setminus I}}}) \geq 2\delta$. Without loss of generality, we may assume that $\|x_{n_{I_{N \setminus I}}}\|_0 \geq \varepsilon/4$, for all $n \in \mathbb{N}$. By using the inequality $(a+b)^t \geq a^t + b^t$ ($a, b \geq 0$, $t \geq 1$) combined with the fact that $k_n \geq 1$ and the convexity

of function $t \mapsto |t|^{p_n}$, we have

$$\begin{aligned}
 1 - 2\delta &\geq \|x_n\|_0 - \rho(x_n|_{N_U}) \\
 &\geq \|x_n\|_0 - \frac{1}{k_n} \rho(k_n x_n|_{N_U}) \\
 &= \frac{1}{k_n} + \frac{1}{k_n} \left[\sum_{i=1}^{\infty} \left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)| \right)^{p_i} \right] - \frac{1}{k_n} \left[\sum_{i=j+1}^{\infty} \left(\frac{k_n}{i} \sum_{r=1}^{i-j} |x_n(j+r)| \right)^{p_i} \right] \\
 &= \frac{1}{k_n} + \frac{1}{k_n} \left[\sum_{i=1}^j \left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)| \right)^{p_i} \right] \\
 &\quad + \frac{1}{k_n} \left[\sum_{i=j+1}^{\infty} \left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)| \right)^{p_i} - \sum_{i=j+1}^{\infty} \left(\frac{k_n}{i} \sum_{r=1}^{i-j} |x_n(j+r)| \right)^{p_i} \right] \\
 &= \frac{1}{k_n} + \frac{1}{k_n} \left[\sum_{i=1}^j \left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)| \right)^{p_i} \right] \\
 &\quad + \frac{1}{k_n} \left[\sum_{i=j+1}^{\infty} \left(\frac{k_n}{i} \sum_{r=1}^j |x_n(r)| + \frac{k_n}{i} \sum_{r=1}^{i-j} |x_n(j+r)| \right)^{p_i} \right. \\
 &\quad \left. - \sum_{i=j+1}^{\infty} \left(\frac{k_n}{i} \sum_{r=1}^{i-j} |x_n(j+r)| \right)^{p_i} \right] \\
 &\geq \frac{1}{k_n} + \frac{1}{k_n} \left[\sum_{i=1}^j \left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)| \right)^{p_i} \right] + \frac{1}{k_n} \left[\sum_{i=j+1}^{\infty} \left(\frac{k_n}{i} \sum_{r=1}^j |x_n(r)| \right)^{p_i} \right] \\
 &= \frac{1}{k_n} + \frac{1}{k_n} \rho(k_n x_n|_U) = \frac{1}{k} + \frac{1}{k} \rho(k x_n|_U) \geq \|x_n\|_0 - \delta,
 \end{aligned}$$

(2.11)

hence $\|x\|_0 \leq 1 - \delta$. □

Since every (UKK) Banach space has H-property, the following result is obtained.

COROLLARY 2.6. *The space $\text{ces}_{(p)}^0$ possesses H-property if each $p_k > 1$.*

COROLLARY 2.7. *The space $\text{ces}_{(p)}^0$ possesses the property NUC if each $p_k > 1$ and $\lim_{k \rightarrow \infty} \inf p_k$.*

PROOF. By [7], $\text{ces}_{(p)}$ is NUC, so it is reflexive. Since a Banach space X is NUC if and only if X is reflexive and UKK, the corollary follows from Theorem 2.5. □

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REFERENCES

- [1] Y. Cui and H. Hudzik, *On the uniform Opial property in some modular sequence spaces*, Funct. Approx. Comment. Math. 26 (1998), 93-102.
- [2] H. Hudzik and Z. Zbąszyniak, *Smooth points of Musielak-Orlicz sequence spaces equipped with the Luxemburg norm*, Colloq. Math. 65 (1993), no. 2, 157-164.
- [3] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. 10 (1980), no. 4, 743-749.
- [4] Y. Q. Lui, B. E. Wu, and P. Y. Lee, *Method of Sequence Spaces*, Guangdong of Science and Technology Press, 1996 (Chinese).
- [5] P. Y. Lee, *Cesàro sequence spaces*, Math. Chronicle 13 (1984), 29-45.
- [6] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, Berlin, 1983.
- [7] W. Sanhan and S. Suantai, *On k -nearly uniform convex property in generalized Cesàro sequence spaces*, Int. J. Math. Math. Sci. 2003 (2003), no. 57, 3599-3607.
- [8] J. S. Shue, *Cesàro sequence spaces*, Tamkang J. Math. 1 (1970), 143-150.
- [9] S. Suantai, *On some convexity properties of generalized Cesàro sequence spaces*, Georgian Math. J. 10 (2003), no. 1, 193-200.

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EXTREME POINTS AND ROTUNDITY OF ORLICZ DIFFERENCE SEQUENCE SPACES

NARIN PETROT* AND SUTHEP SUANTAI**

ABSTRACT. In this paper we define a new sequence space $\ell_M(\Delta)$, generated by Orlicz function M , and equipped with the Luxemburg norm. Extreme points of the unit ball in this space are characterized and criteria for rotundity are given.

1. INTRODUCTIONS

Convexity properties in Banach space are an important topic in functional analysis and play an important role in infinite dimensional holomorphy. In order to study the geometric properties of Banach spaces, Clarkson [5] introduced the very important class of rotund (strictly convex) spaces. Since Clarkson's paper many authors have defined and studied classes of Banach spaces lying between uniformly convex and rotund spaces see [2], [12], and [16]. The criteria for rotundity in Orlicz spaces were given in [4].

In this paper we introduce the Orlicz difference sequence space $\ell_M(\Delta)$, where M is an Orlicz function and consider it equipped with the Luxemburg norm. The main purpose of this paper is to give criteria for extreme points and rotundity in $\ell_M(\Delta)$.

Now we introduce the basic notation and definitions. In the following, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and real numbers, respectively. For a finite subset $A \subset \mathbb{N}$ we denoted $\sharp A$ the number of elements in A . Let $(X, \|\cdot\|)$ be a real Banach space, and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X . A point $x \in S(X)$ is called an *extreme point* if for every $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$. Let $ExtB(X)$ denote for the set of all extreme points of $B(X)$. A Banach space X is said to be *rotund* (write **(R)** for short), if $ExtB(X) = S(X)$. For its geometric implications and role in mathematics we refer to the monographs [3], [7], [10], [14], and [15].

A function $\varrho : X \rightarrow [0, \infty]$ is called a *modular* if it satisfies the conditions

- (i) $\varrho(x) = 0$ if and only if $x = 0$;
- (ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;
- (iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

- (iv) $\varrho(\alpha x + \beta y) \leq \alpha \varrho(x) + \beta \varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular ϱ on X , the space

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

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is called the *modular space* (generated by ϱ). It is easy to show that X_ϱ is a vector space. If ϱ is a convex modular, the function

$$\|x\| = \inf\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\},$$

is a norm on X_ϱ , which is called the *Luxemburg norm* (see [15]).

The subspace X_ϱ^* of X_ϱ is defined by

$$X_\varrho^* = \{x \in X : \varrho(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

A modular ϱ is said to satisfy the Δ_2 -condition ($\varrho \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$$\varrho(2x) \leq K\varrho(x) + \varepsilon$$

for all $x \in X_\varrho$ with $\varrho(x) \leq a$. It is well known that $X_\varrho = X_\varrho^*$ if $\varrho \in \Delta_2$ (see [8]).

If ϱ satisfies the Δ_2 -condition for all $a > 0$ with $K \geq 2$ dependent on a , we say that ϱ satisfies the *strong Δ_2 -condition* ($\varrho \in \Delta_2^s$).

A map $M : \mathbb{R} \rightarrow [0, \infty]$ is said to be an *Orlicz function* if M vanishes only at 0, and M is even, convex and continuous on the whole of $\mathbb{R}$. In addition, an Orlicz function N is called the *complementary function* of an Orlicz function M if

$$N(v) = \sup\{|v|u - M(u) : u \geq 0\}.$$

We say an Orlicz function M satisfies the δ_2 -condition if there exist constants $K \geq 2, u_0 > 0$ such that the inequality

$$M(2u) \leq KM(u)$$

holds for every $u \in \mathbb{R}$ satisfying $|u| \leq u_0$.

We say that $u \in \mathbb{R}$ is a *point of strict convexity* of M if $M\left(\frac{v+w}{2}\right) < \frac{M(v)+M(w)}{2}$, whenever $u = \frac{v+w}{2}$ and $v \neq w$. We denoted by S_M the set of *strictly convex points* of M .

An interval $[a, b]$ is called a *structurally affine interval* for an Orlicz function M , or simply, *SAI* of M , provided that M is affine on $[a, b]$ and it is not affine either on $[a - \varepsilon, b]$ or on $[a, b + \varepsilon]$ for any $\varepsilon > 0$. Let $\{[a_i, b_i]\}_i$ be all the SAIs of M , it is obvious that

$$S_M = \mathbb{R} \setminus \bigcup_i [a_i, b_i].$$

In [9], Kizmaz introduced the sequence spaces $\ell_\infty(\Delta)$, $c_0(\Delta)$, and $c(\Delta)$ by considering the difference sequence $\Delta x = (x(i+1) - x(i))_{i=1}^\infty$ for any $x \in \ell^0$, where ℓ^0 is the space of all real sequences and ℓ_∞, c_0, c are the Banach spaces of bounded, null and convergent sequences, respectively. In [1], these sequence spaces were extended to $\ell_\infty(\Delta, p)$, $c_0(\Delta, p)$, and $c(\Delta, p)$. For example,

$$\ell_\infty(\Delta, p) = \{x \in \ell^0 : \Delta x \in \ell_\infty(p)\},$$

where $p = (p_k)$ is a sequence of positive real numbers; and

$$\ell_\infty(p) = \{x \in \ell^0 : \sup_k |x_k|^{p_k} < \infty\}.$$

In [1] and [13] the authors determined the Köthe-Töeplitz and generalized Köthe-Töeplitz duals of these spaces and considered various matrix transformations.

In this paper we introduced the space $\ell_M(\Delta)$, where M is an Orlicz function. The Orlicz difference sequence space $\ell_M(\Delta)$, is defined by

$$\ell_M(\Delta) := \{x \in \ell^0 : \varrho_M(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where

$$\varrho_M(x) = |x(1)| + \sum_{i=1}^{\infty} M(x(i+1) - x(i)).$$

By using the convexity of M it is easy to see that ϱ_M is a convex modular on $\ell_M(\Delta)$, and it is useful to note that if $M \in \delta_2$ then $\varrho_M \in \Delta_2^s$. In this work we consider $\ell_M(\Delta)$ equipped with the Luxemburg norm given by

$$\|x\| = \inf\{\varepsilon > 0 : \varrho_M(\frac{x}{\varepsilon}) \leq 1\}.$$

Moreover, we define the subspace $h_M(\Delta)$ of $\ell_M(\Delta)$, by

$$h_M(\Delta) := \{x \in \ell^0 : \varrho_M(\lambda x) < \infty \text{ for all } \lambda > 0\}.$$

2. MAIN RESULTS

In order to establish our new results, we start by recalling some auxiliary lemmas.

Lemma 2.1 Let ϱ be a convex modular on X_ϱ and $x \in X_\varrho$. Then

- (1) $\|x\| \leq 1 \Rightarrow \varrho(x) \leq \|x\|$,
- (2) $\|x\| > 1 \Rightarrow \varrho(x) > \|x\|$, and
- (3) $\varrho(x) = 1 \Rightarrow \|x\| = 1$, and the converse of (3) is true if $\varrho \in \Delta_2^s$.

Proof. See [8].

Lemma 2.2 An Orlicz function M satisfies the δ_2 -condition if and only if there exist $l > 1, u_0 > 0$, and $K > 1$ such that $M(lu) \leq KM(u)$ for all $|u| \leq u_0$.

Proof. See [3].

Lemma 2.3 If $M \notin \delta_2$, then there exists $x \in \ell_M(\Delta) \setminus h_M(\Delta)$ such that $\|x\| = 1$ and $\varrho_M(x) < 1$.

Proof. Since $M \notin \delta_2$, by Lemma 2.2, we can choose a sequence (α_k) of positive real numbers such that $\alpha_k \downarrow 0$, $M(\alpha_k) < \frac{1}{2^k}$, and

$$M((1 + \frac{1}{k})\alpha_k) > 2^{k+1}M(\alpha_k) \text{ for all } k \in \mathbb{N}.$$

Choose an integer m_k such that

$$\frac{1}{2^{k+1}} \leq m_k M(\alpha_k) < \frac{1}{2^k}.$$

Let $x \in \ell^0$ be the sequence defined as follows : Put $x(1) = 0$, and for $2 \leq i \leq m_1 + 1$, let

$$x(i) = \begin{cases} \alpha_1 & ; \text{if } i \text{ is even} \\ 2\alpha_1 & ; \text{if } i \text{ is odd.} \end{cases}$$

If m_1 is odd, for $m_1 + 2 \leq i \leq m_1 + m_2 + 1$, we let

$$x(i) = \begin{cases} \alpha_1 + \alpha_2 & ; \text{if } i \text{ is odd} \\ \alpha_1 & ; \text{if } i \text{ is even,} \end{cases}$$

and if m_1 is even, for $m_1 + 2 \leq i \leq m_1 + m_2 + 1$, we let

$$x(i) = \begin{cases} 2\alpha_1 + \alpha_2 & ; \text{if } i \text{ is even} \\ 2\alpha_1 & ; \text{if } i \text{ is odd.} \end{cases}$$

If m_2 is odd and $2 \leq j \leq m_3 + 1$, we let

$$x(m_1 + m_2 + j) = \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 & ; \text{if } m_1 \text{ is odd and } j \text{ is even} \\ 2\alpha_1 + \alpha_2 + \alpha_3 & ; \text{if } m_1 \text{ is even and } j \text{ is even} \\ \alpha_1 + \alpha_2 & ; \text{if } m_1 \text{ is odd and } j \text{ is odd} \\ 2\alpha_1 + \alpha_2 & ; \text{if } m_1 \text{ is even and } j \text{ is odd} \end{cases}$$

and if, m_2 is even and $2 \leq j \leq m_3 + 1$, we let

$$x(m_1 + m_2 + j) = \begin{cases} \alpha_1 + \alpha_3 & ; \text{if } m_1 \text{ is odd and } j \text{ is even} \\ 2\alpha_1 + \alpha_3 & ; \text{if } m_1 \text{ is even and } j \text{ is even} \\ \alpha_1 & ; \text{if } m_1 \text{ is odd and } j \text{ is odd} \\ 2\alpha_1 & ; \text{if } m_1 \text{ is even and } j \text{ is odd} \end{cases}$$

Observe that if m_2 is odd and $2 \leq j \leq m_3 + 1$, we have

$$x(m_1 + m_2 + j) = \begin{cases} x(m_1 + 2) + \alpha_3 & ; \text{if } j \text{ is even} \\ x(m_1 + 2) & ; \text{if } j \text{ is odd,} \end{cases}$$

and if, m_2 is even and $2 \leq j \leq m_3 + 1$, we have

$$x(m_1 + m_2 + j) = \begin{cases} x(m_1 + 1) + \alpha_3 & ; \text{if } j \text{ is even} \\ x(m_1 + 1) & ; \text{if } j \text{ is odd.} \end{cases}$$

By continuing in this way, for $2 \leq j \leq m_k + 1$, if m_{k-1} is odd, we let

$$x(m_1 + m_2 + \cdots + m_{k-1} + j) = \begin{cases} x(m_1 + m_2 + \cdots + m_{k-2} + 2) + \alpha_k & ; \text{if } j \text{ is even} \\ x(m_1 + m_2 + \cdots + m_{k-2} + 2) & ; \text{if } j \text{ is odd,} \end{cases}$$

and if, m_{k-1} is even, we let

$$x(m_1 + m_2 + \cdots + m_{k-1} + j) = \begin{cases} x(m_1 + m_2 + \cdots + m_{k-2} + 1) + \alpha_k & ; \text{if } j \text{ is even} \\ x(m_1 + m_2 + \cdots + m_{k-2} + 1) & ; \text{if } j \text{ is odd.} \end{cases}$$

Then we have

$$\begin{aligned}
 \varrho_M(x) &= |x(1)| + \sum_{i=1}^{m_1} M(x(i+1) - x(i)) + \sum_{i=m_1+1}^{m_1+m_2+1} M(x(i+1) - x(i)) + \cdots \\
 &\quad + \sum_{i=m_1+m_2+\cdots+m_{k-1}+1}^{m_1+m_2+\cdots+m_k+1} M(x(i+1) - x(i)) + \cdots \\
 &= 0 + m_1 M(\alpha_1) + m_2 M(\alpha_2) + \cdots + m_k M(\alpha_k) + \cdots \\
 &= \sum_{k=1}^{\infty} m_k M(\alpha_k) < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.
 \end{aligned}$$

This shows that $x \in \ell_M(\Delta)$ and $\varrho_M(x) < 1$. But, for any $l > 1$, let $k_o \in \mathbb{N}$ satisfy $l > 1 + \frac{1}{k_o}$. Then

$$\begin{aligned}
 \varrho_M(lx) &= \sum_{k=1}^{\infty} m_k M(l\alpha_k) > \sum_{k=k_o}^{\infty} m_k M\left(\left(1 + \frac{1}{k}\right)\alpha_k\right) \\
 &> \sum_{k=k_o}^{\infty} m_k 2^{k+1} M(\alpha_k) \geq \sum_{k=k_o}^{\infty} 1 = \infty,
 \end{aligned}$$

which implies that $\|x\|_M = 1$ and $x \notin h_M$.

Corollary 2.4 $M \in \delta_2$ if and only if $\ell_M(\Delta) = h_M(\Delta)$.

Proof. If $M \in \delta_2$ then $\varrho_M \in \Delta_2$ and hence $\ell_M(\Delta) = h_M(\Delta)$. If $M \notin \delta_2$ then by Lemma 2.3 we can find an element $x \in \ell_M(\Delta)$ but $x \notin h_M(\Delta)$.

Now we give a characterization of an extreme point of the closed unit ball of $\ell_M(\Delta)$.

Theorem 2.5 For an Orlicz function M and $x \in S(\ell_M(\Delta))$, $x \in \text{Ext}B(\ell_M(\Delta))$ if and only if

(1) $\varrho_M(x) = 1$, and

(2) (a) If $x(1) \neq 0$ then $\#\{i \in \mathbb{N} : x(i+1) - x(i) \in \mathbb{R} \setminus S_M\} = 0$, or (b) if $x(1) = 0$ then $\#\{i \in \mathbb{N} : x(i+1) - x(i) \in \mathbb{R} \setminus S_M\} \leq 1$.

Proof. *Sufficiency.* Let $x = \frac{y+z}{2}$, where $y, z \in S(\ell_M(\Delta))$. By Lemma 2.1(1), we have $\varrho_M(y) \leq 1$ and $\varrho_M(z) \leq 1$. Then $1 = \varrho_M(x) = \varrho_M\left(\frac{y+z}{2}\right) \leq \frac{1}{2}[\varrho_M(y) + \varrho_M(z)] \leq 1$, hence

$$\varrho_M(x) = \varrho_M(y) = \varrho_M(z) = 1, \quad (2.1)$$

$$|x(1)| = \left| \frac{y(1) + z(1)}{2} \right| = \frac{|y(1)| + |z(1)|}{2}, \quad (2.2)$$

and

$$\begin{aligned}
 M\left(\frac{y(i+1) - y(i) + z(i+1) - z(i)}{2}\right) \\
 = \frac{1}{2} [M(y(i+1) - y(i)) + M(z(i+1) - z(i))] \quad (2.3)
 \end{aligned}$$

for all $i \in \mathbb{N}$.

It follows from (2.2) that $\operatorname{sgn} y(1) = \operatorname{sgn} z(1)$.

If $x(1) \neq 0$, by $\#\{i : x(i+1) - x(i) \in \mathbb{R} \setminus S_M\} = 0$ and (2.3) we have

$$y(i+1) - y(i) = z(i+1) - z(i) \quad \text{for all } i \in \mathbb{N}. \quad (2.4)$$

By (2.1) and (2.4), we have

$$\begin{aligned} |y(1)| &= 1 - \sum_{i=1}^{\infty} M(y(i+1) - y(i)) \\ &= 1 - \sum_{i=1}^{\infty} M(z(i+1) - z(i)) \\ &= |z(1)|, \end{aligned}$$

this together with the fact that $\operatorname{sgn} y(1) = \operatorname{sgn} z(1)$ gives $y(1) = z(1)$. By (2.4), we obtain that

$$0 = y(1) - z(1) = y(2) - z(2) = y(3) - z(3) = \cdots = y(i) - z(i) = \cdots$$

for all $i \in \mathbb{N}$. Hence, we have $y(i) = z(i)$ for all $i \in \mathbb{N}$, that is $y = z$.

If $x(1) = 0$, by 2(b), we have

$$\#\{i : x(i+1) - x(i) \in \mathbb{R} \setminus S_M\} \leq 1. \quad (2.5)$$

Then there exists a most one $j \in \mathbb{N}$ such that $x(i+1) - x(i) \in S_M$ for all $i \neq j$. It implies by (2.3) that

$$y(i+1) - y(i) = z(i+1) - z(i) \quad \text{for all } i \in \mathbb{N} \text{ which } i \neq j. \quad (2.6)$$

Since $y(1) = z(1) = 0$, we have

$$\begin{aligned} M(y(j+1) - y(j)) &= 1 - \sum_{i \neq j} M(y(i+1) - y(i)) \\ &= 1 - \sum_{i \neq j} M(z(i+1) - z(i)) \\ &= M(z(j+1) - z(j)), \end{aligned}$$

which implies that

$$|y(j+1) - y(j)| = |z(j+1) - z(j)|. \quad (2.7)$$

Since $0 \in S_M$, we have $y(j+1) - y(j), z(j+1) - z(j)$ are in the same SAI of M . This implies by (2.7) that $y(j+1) - y(j) = z(j+1) - z(j)$. This together with (2.6), gives

$$0 = y(1) - z(1) = y(2) - z(2) = y(3) - z(3) = \cdots = y(i) - z(i) = \cdots$$

for all $i \in \mathbb{N}$. Hence, we have $y(i) = z(i)$ for all $i \in \mathbb{N}$, that is $y = z$.

Necessity. If condition (1) does not holds, then there is $0 < c < 1$ such that $\varrho_M(x) = c$. Let $d > 0$ be such that $c + d \leq 1$. Since $\lim_{i \rightarrow \infty} M(x(i+1) - x(i)) = 0$ and

M is continuous, there exist $i_o \in \mathbb{N}$ and $\varepsilon > 0$ such that $M(x(i_o) - x(i_o + 1) \pm \varepsilon) \leq d$. Let y and z be the sequences defined by

$$y(i) = \begin{cases} x(i) & ; \text{if } 1 \leq i \leq i_o, \\ x(i) + \varepsilon & ; \text{otherwise,} \end{cases}$$

and

$$z(i) = \begin{cases} x(i) & ; \text{if } 1 \leq i \leq i_o, \\ x(i) - \varepsilon & ; \text{otherwise.} \end{cases}$$

Then $2x = y + z$ and $y \neq z$, and we have that

$$\begin{aligned} \varrho_M(y) &= |x(1)| + \sum_{i \neq i_o} M(x(i+1) - x(i)) + M(x(i_o+1) - x(i_o) + \varepsilon) \\ &\leq \varrho_M(x) + d \leq c + d \leq 1. \end{aligned}$$

Similarly, we have that $\varrho_M(z) \leq 1$. Thus $y, z \in B(\ell_M(\Delta))$, so $x \notin \text{Ext}B(\ell_M(\Delta))$.

If condition 2(a) does not hold, then we may assume that there exist $j \in \mathbb{N}$ such that $x(j+1) - x(j)$ belong to an affine interval (a_j, b_j) of M .

Let $M(u) = A_j u + B_j, u \in (a_j, b_j)$, and choose $\varepsilon > 0$ such that $|x(1)| > A_j \varepsilon$ and $x(j+1) - x(j) \pm \varepsilon \in (a_j, b_j)$.

Let y and z be the sequences defined as follows:

If $x(1) - A_j \varepsilon > 0$, let

$$y(i) = \begin{cases} x(i) - A_j \varepsilon & ; \text{if } 1 \leq i \leq j, \\ x(i) - A_j \varepsilon + \varepsilon & ; \text{otherwise,} \end{cases}$$

and

$$z(i) = \begin{cases} x(i) + A_j \varepsilon & ; \text{if } 1 \leq i \leq j, \\ x(i) + A_j \varepsilon - \varepsilon & ; \text{otherwise,} \end{cases}$$

If $x(1) - A_j \varepsilon < 0$, let

$$y(i) = \begin{cases} x(i) - A_j \varepsilon & ; \text{if } 1 \leq i \leq j, \\ x(i) - A_j \varepsilon - \varepsilon & ; \text{otherwise,} \end{cases}$$

and

$$z(i) = \begin{cases} x(i) + A_j \varepsilon & ; \text{if } 1 \leq i \leq j, \\ x(i) + A_j \varepsilon + \varepsilon & ; \text{otherwise.} \end{cases}$$

It is clear that $y \neq z$ and $2x = y + z$. By the definition of y , we have

$$\begin{aligned} \varrho_M(y) &= |x(1) - A_j \varepsilon| + \sum_{i=1}^{j-1} M(x(i+1) - x(i)) + M(x(j+1) - x(j) \pm \varepsilon) \\ &\quad + \sum_{i=j+1}^{\infty} M(x(i+1) - x(i)) \end{aligned}$$

$$\begin{aligned}
&= |x(1)| \mp A_j \varepsilon + \sum_{i=1}^{j-1} M(x(i+1) - x(i)) + A_j(x(j+1) - x(j)) \pm A_j \varepsilon + B_j \\
&\quad + \sum_{i=j+1}^{\infty} M(x(i+1) - x(i)) \\
&= |x(1)| + \sum_{i=1}^{\infty} M(x(i+1) - x(i)) \\
&= \varrho_M(x) = 1.
\end{aligned}$$

Similarly, we have that $\varrho_M(z) = 1$. Thus $y, z \in S(\ell_M(\Delta))$, hence $x \notin \text{Ext}B(\ell_M(\Delta))$.

If condition 2(b) does not hold, then we may assume that there are $j, k \in \mathbb{N}$, $j < k$, such that $x(j+1) - x(j)$ and $x(k+1) - x(k)$ belong to some affine intervals $(a_j, b_j), (a_k, b_k)$ of M , respectively. Let $M(u) = A_j u + B_j, u \in (a_j, b_j)$ and $M(u) = A_k u + B_k, u \in (a_k, b_k)$.

Select $\varepsilon_j, \varepsilon_k > 0$ such that $A_j \varepsilon_j = A_k \varepsilon_k$ and $x(j+1) - x(j) \pm \varepsilon_j \in (a_j, b_j)$ and $x(k+1) - x(k) \pm \varepsilon_k \in (a_k, b_k)$.

Let y and z be the sequences defined by

$$y(i) = \begin{cases} x(i) & ; \text{if } 1 \leq i \leq j, \\ x(i) + \varepsilon_j & ; \text{if } j+1 \leq i \leq k, \\ x(i) + \varepsilon_j - \varepsilon_k & ; \text{otherwise,} \end{cases}$$

and

$$z(i) = \begin{cases} x(i) & ; \text{if } 1 \leq i \leq j, \\ x(i) - \varepsilon_j & ; \text{if } j+1 \leq i \leq k, \\ x(i) - \varepsilon_j + \varepsilon_k & ; \text{otherwise.} \end{cases}$$

It is clear that $y \neq z$ and $2x = y + z$. By the definition of y , we have

$$\begin{aligned}
\varrho_M(y) &= |x(1)| + \sum_{i=1}^{j-1} M(x(i+1) - x(i)) + M(x(j+1) - x(j) + \varepsilon_j) \\
&\quad + \sum_{i=j+1}^{k-1} M(x(i+1) - x(i)) + M(x(k+1) - x(k) - \varepsilon_k) \\
&\quad + \sum_{i=k+1}^{\infty} M(x(i+1) - x(i)) \\
&= |x(1)| + \sum_{i=1}^{j-1} M(x(i+1) - x(i)) + A_j(x(j+1) - x(j)) + A_j \varepsilon_j + B_j \\
&\quad + \sum_{i=j+1}^{k-1} M(x(i+1) - x(i)) + A_k(x(k+1) - x(k)) - A_k \varepsilon_k + B_k \\
&\quad + \sum_{i=k+1}^{\infty} M(x(i+1) - x(i))
\end{aligned}$$

$$\begin{aligned}
 &= |x(1)| + \sum_{i=1}^{\infty} M(x(i+1) - x(i)) \\
 &= \varrho_M(x) = 1.
 \end{aligned}$$

Similarly, we have that $\varrho_M(z) = 1$. Thus $y, z \in S(\ell_M(\Delta))$, hence $x \notin \text{Ext}B(\ell_M(\Delta))$.

Theorem 2.6 $\ell_M(\Delta)$ is Rotund if and only if (1) $M \in \delta_2$ and (2) M is strictly convex on $[0, M^{-1}(1)]$.

Proof. *Sufficiency.* Let $x \in S(\ell_M(\Delta))$. Since $M \in \delta_2$, we have by Lemma 2.1(3) that $\varrho_M(x) = 1$. Let $I = \{i \in \mathbb{N} : x(i) - x(i+1) \in \mathbb{R} \setminus S_M\}$.

If $x(1) \neq 0$, and there exist $j \in \mathbb{N}$ with $j \in I$, we have $|x(j+1) - x(j)| > M^{-1}(1)$. This implies that $\varrho_M(x) > 1$, a contradiction. Hence $I = \emptyset$.

If $x(1) = 0$, then by (2), we have $|x(i) - x(i+1)| > M^{-1}(1)$ for all $i \in I$. So, $I = \emptyset$.

Therefore, we conclude by Theorem 2.5 that $x \in \text{Ext}B(\ell_M(\Delta))$.

Necessity. If $M \notin \delta_2$, by Lemma 2.3, there exists $x \in S(\ell_M(\Delta))$ such that $\varrho_M(x) < 1$, hence, $x \notin \text{Ext}B(\ell_M(\Delta))$ by Theorem 2.5.

If (2) does not hold, then M is affine on some interval $[a, b]$ in $[0, M^{-1}(1)]$. Since $M(b) \leq 1$, we can find $c \in (a, b)$ and $d > 0$ such that $M(c) + d = 1$.

Define

$$x = (d, d+c, d+c, d+c, d+c, d+c, \dots).$$

Then $\varrho_M(x) = |d| + M(c) = 1$, but $x(1) = d \neq 0$ and $x(2) - x(1) = c \notin S_M$. This yields that $x \notin \text{Ext}B(\ell_M(\Delta))$ by Theorem 2.5.

REFERENCES

- [1] Z. U. Ahmad and Mursaleen, *Köthe-Töeplitz duals of some new sequence spaces and their matrix maps*, Publ. Inst. Math(Beograd) **42**(56) (1987), 57-61.
- [2] K. W. Anderson, *Midpoint local uniform convexity and other geometric properties of Banach spaces*, Dissertation, University of Illinois, 1960.
- [3] S. T. Chen, *Geometry of Orlicz spaces—Dissertationes Math.*, The institute of Mathematics, Polish Academy of Sciences 1996, pp. 356.
- [4] S. T. Chen and H. Sun, *Extreme points and rotundity of sequence Orlicz spaces* Natur. Sci. J. Harbin Normal Univ. **1**(2) (1985), 1-6 (in Chinese).
- [5] J. A. Clarkson, *Uniformly convex spaces*, Tran. Amer. Math. Soc. **40** (1936), 396-414.
- [6] Y. A. Cui, H. Hudzik and C. Meng, *On some local geometry of Orlicz sequence spaces equipped the Luxemburg norms*, Acta Math. Hungar. **80**(1-2) (1998), 143-154.
- [7] J. Diestel, *Geometry of Banach Spaces—Selected Topics*, Springer-Verlag, 1984.
- [8] W. A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, 2001, pp. 339-389.
- [9] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull. **24** (1981), 169-175.
- [10] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, Springer-Verlag Berlin Heidelberg, 1996.
- [11] Y. Q. Liu, B. E. Wu. and Y. P. Lee, *Method of sequence spaces*, Guangdong of Science and Technology Press, 1996 (in Chinese).
- [12] A. R. Lovaglia, *Locally uniformly convex Banach spaces* Tran. Amer. Math. Soc. **29** (1977), 963-970.
- [13] I. J. Maddox, *Continuous and Köthe-Töeplitz duals of certain sequence spaces* Proc. camb. Phil. Soc. **65** (1967), 431-435.
- [14] R. E. Megginson, *An Introduction to Banach space Theory*, Springer-Verlag New York, Inc., 1998.

- [15] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. 1034, Springer-Verlag, 1983.
- [16] M. A. Smith, *Banach spaces that are uniformly rotund in weakly compact set of direction*, *Canad. J. Math.* **29** (1977), 963–970.

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LOCALLY UNIFORM ROTUNDITY IN METRIC LINEAR SPACES

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Abstract. A new notion of locally uniform rotundity (**LUR**) in a metric linear space, which is a generalization of uniform rotundity introduced in [1] is defined and studied. The main purpose of this paper is to show that in a metric linear space, uniform rotundity implies locally uniform rotundity and locally uniform rotundity implies property (**H**) and we also give some examples showing that the converse of above implications are not true.

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1. Introductions

The notion of rotundity (**R**) or strictly convexity in metric linear spaces was introduced by Ahuja, Narang and Trehan [1] and the concept of uniform rotundity (**UR**) in metric linear spaces was introduced by Sastry and Naidu [8]. These notions are generalizations of the corresponding concepts in normed linear spaces. T.D. Narang [6] gave a characterization of strictly convex metric linear spaces. W. Junde and C. Lianchang [2] showed that in a complete metric linear space uniform rotundity implies reflexivity.

In [3], W. Junde and T.D. Narang showed that if a metric linear space (X, d) is (**UR**) then (X, d) has property (**H**). In this paper we introduce a new notion of locally uniform rotundity (**LUR**) which is a generalization of uniform rotundity

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and give characterization of **(LUR)**. We show that if a metric linear space X is **(LUR)**, then X has property **(H)**. Moreover, examples showing that property **(H)** does not imply **(LUR)** and **(LUR)** does not imply **(UR)** are given.

Let X be a vector space over the scalar field of real numbers $\mathbb{R}$ and d an invariant linear metric on X . We denote $B_d(0, r) = \{x \in X \mid d(x, 0) \leq r\}$ and $S_d(0, r) = \{x \in X \mid d(x, 0) = r\}$. A metric linear space (X, d) is called

- a) *uniformly rotund* (written **(UR)** for short) if, for each $r > 0$ and $\varepsilon > 0$ there exists $\delta = \delta_{(r, \varepsilon)} > 0$ such that $x, y \in B_d(0, r)$ and $d(x, y) \geq \varepsilon$ imply

$$d\left(\frac{x+y}{2}, 0\right) < r - \delta.$$

- b) *locally uniformly rotund* (written **(LUR)** for short) if, for each $r > 0, \varepsilon > 0$ and for each $x \in S_d(0, r)$ there exists $\delta = \delta_{(r, \varepsilon, x)} > 0$ such that $y \in B_d(0, r)$ and $d(x, y) \geq \varepsilon$ imply

$$d\left(\frac{x+y}{2}, 0\right) < r - \delta.$$

- c) *rotund* (written **(R)** for short) if, for each $x, y \in B_d(0, r)$ and $x \neq y$ imply

$$d\left(\frac{x+y}{2}, 0\right) < r.$$

A metric linear space (X, d) is said to have *property (H)* if for each $x \in S_d(0, r)$ and $(x_n) \subset S_d(0, r)$ such that $x_n \xrightarrow{w} x$, implies $x_n \rightarrow x$.

The subset A of X is *absorbing* if, for each x in X , there is a positive number t_x such that $x \in tA$ whenever $t > t_x$. For an absorbing subset A of X , the *Minkowski functional or gauge functional of A* is the function $M_A : X \rightarrow \mathbb{R}$ defined by $M_A(x) = \inf\{t \mid t > 0, x \in tA\}$ for each x in X .

The following known results are many useful in our consideration.

Proposition 1.1. [8] *Let (X, d) be a strictly convex metric linear space. Then for each $\varepsilon > 0$, $\{x \in X \mid d(x, 0) \leq \varepsilon\}$ is an absolutely convex absorbing neighbourhood of 0 in (X, d) .*

Proposition 1.2. [2] *Let (X, d) be a strictly convex metric linear space. If $A = \{x \in X \mid d(x, 0) \leq r\} \neq X$, then the Minkowski gauge functional M_A of A is a strictly convex norm and $M_A(x) = 1$ if and only if $d(x, 0) = r$.*

Proposition 1.3. [2] *Let (X, d) be a strictly convex metric linear space. Then for any $x \in E, x \neq 0$ the function $f_x : [0, \infty) \rightarrow [0, \infty)$ defined by*

$$f_x(t) = d(tx, 0)$$

is strictly increasing and continuous function.

2. Main results

First, we give a characterization of uniform rotundity in a metric linear space.

Theorem 2.1. *A metric linear space (X, d) is **(UR)** if and only if (X, d) is **(R)** and for each $r > 0$ if $(x_n), (y_n) \subset B_d(0, r)$ are such that $d\left(\frac{x_n + y_n}{2}, 0\right) \rightarrow r$, then $d(x_n, y_n) \rightarrow 0$.*

Proof. *Necessity.* Let $r > 0$ be given. Assume that (X, d) is **(UR)**. The implication that (X, d) is **(R)** is obvious so it suffices to show that for each $(x_n), (y_n) \subset B_d(0, r)$ such that $d\left(\frac{x_n + y_n}{2}, 0\right) \rightarrow r$, we must have $d(x_n, y_n) \rightarrow 0$. If not, it would be exist $\varepsilon > 0$ and subsequences $(x_{n_j}), (y_{n_j})$ of (x_n) and (y_n) , respectively, such that $d(x_{n_j}, y_{n_j}) > \varepsilon$ for all $j \in \mathbb{N}$. Since (X, d) is **(UR)**, there exists $\delta = \delta_{(\varepsilon, r)} > 0$ such that

$$d\left(\frac{x_{n_j} + y_{n_j}}{2}, 0\right) < r - \delta \text{ for all } j \in \mathbb{N}.$$

This implies

$$d\left(\frac{x_{n_j} + y_{n_j}}{2}, 0\right) \not\rightarrow r,$$

which is a contradiction.

Sufficiency. If (X, d) is not **(UR)** and it is **(R)**, then there exists $\varepsilon > 0$ and sequences $(x_n), (y_n) \subset B_d(0, r)$ such that $d(x_n, y_n) > \varepsilon$ and $d\left(\frac{x_n + y_n}{2}, 0\right) > r - \frac{1}{n}$ for each $n \in \mathbb{N}$. This together with the hypothesis that (X, d) is **(R)**, we have

$$\left| r - d\left(\frac{x_n + y_n}{2}, 0\right) \right| < \frac{1}{n} \text{ for each } n \in \mathbb{N},$$

that is $d\left(\frac{x_n + y_n}{2}, 0\right) \rightarrow r$ as $n \rightarrow \infty$, and $d(x_n, y_n) \not\rightarrow 0$.

By using analogy proof as in Theorem 2.1, characterization of **(LUR)** metric linear spaces is given.

Theorem 2.2. *A metric linear space (X, d) is **(LUR)** if and only if (X, d) is **(R)** and for each $r > 0$ if $x \in S_d(0, r)$ and $(y_n) \subset B_d(0, r)$ are such that $d\left(\frac{x + y_n}{2}, 0\right) \rightarrow r$, then $d(x, y_n) \rightarrow 0$.*

The following result deals with the property **(H)** in a metric linear space.

Theorem 2.3. *If (X, d) is a (LUR) metric linear space, then (X, d) possesses property (H).*

Proof. Let $r > 0$ be given. Fixed $x \in S_d(0, r)$ and let $x_n \in S_d(0, r)$ ($n = 1, 2, 3, \dots$) be such that $x_n \xrightarrow{w} x$. If $x_n \not\rightarrow x$, then there exists $\varepsilon > 0$ such that $d(x_n, x) > \varepsilon$, $\forall n \in \mathbb{N}$ (passing to a subsequence, if necessary). Since (X, d) is (LUR), there exists $\delta = \delta_{(\varepsilon, r, x)} > 0$ such that

$$d\left(\frac{x_n + x}{2}, 0\right) \leq r - \delta \quad \text{for all } n \in \mathbb{N}.$$

Note that $B_d(0, r) \neq (X, d)$. Indeed, if not, we have $2x_n - x \in B_d(0, r)$ for all $n \in \mathbb{N}$ and it follows by Proposition 1.3 that $\varepsilon < d(x_n, x) < d(2(x_n - x), 0) = d(2x_n - x, x)$ which implies that

$$r - \delta \geq d\left(\frac{2x_n - x + x}{2}, 0\right) = d(x_n, 0) = r,$$

a contradiction. Let $A = B_d(0, r)$. Since (X, d) is strictly convex we have by Proposition 1.1 that A is an absorbing balance subset of X , this allows us to define the Minkowski guage functional M_A of A . Moreover, from Proposition 1.2 we know that (E, M_A) is a normed space and $M_A(x) = 1$. By Hahn-Banach Theorem there exists $f \in (E, M_A)'$, the dual space of (E, M_A) , such that $\|f\|_{M_A} = 1$ and $f(x) = M_A(x) = 1$.

Again, by Proposition 1.1 we have $B_d(0, \delta)$ is an absorbing subset of (X, d) so it must be exists $t > 0$ such that $tx \in B_d(0, \delta)$, observe that $f(tx) = tf(x) = t > 0$. Let $z = tx \in B_d(0, \delta)$, hence $f(z) > 0$, and

$$d\left(\frac{x_n + x}{2} + z, 0\right) \leq d\left(\frac{x_n + x}{2}, 0\right) + d(0, z) \leq r - \delta + \delta = r.$$

This give $M_A\left(\frac{x_n + x}{2} + z\right) \leq 1$. Thus we have

$$\left|\frac{f(x_n) + f(x)}{2} + f(z)\right| = \left|f\left(\frac{x_n + x}{2} + z\right)\right| \leq \|f\|_{M_A} \cdot M_A\left(\frac{x_n + x}{2} + z\right) \leq 1.$$

Since $f(x_n) \rightarrow f(x)$, we have from above inequality that $1 + f(z) = f(x) + f(z) \leq 1$, this contradicts the choice of z .

Let (p_i) be a bounded sequence of positive real numbers which $p_i \geq 1$ for all $i \in \mathbb{N}$. Let $M = \sup\{p_i \mid i \in \mathbb{N}\}$. The Nakano sequence space $\ell_{(p_i)}$ is defined by

$$\ell_{(p_i)} = \left\{ x \in \ell^0 \mid \sum_{i=1}^{\infty} |x(i)|^{p_i} < \infty \right\}.$$

We consider $\ell_{(p_i)}$ under the metric given by

$$d(x, y) = \left(\sum_{i=1}^{\infty} |x(i) - y(i)|^{p_i} \right)^{\frac{1}{M}}$$

Theorem 2.4. *Let (p_i) be a bounded sequence of positive real numbers which $p_i \geq 1$ for all $i \in \mathbb{N}$. Then*

- (i) $(\ell_{(p_i)}, d)$ possesses property (H).
- (ii) $(\ell_{(p_i)}, d)$ is (LUR) space if each $p_i > 1$.

Proof. Denote $M = \sup\{p_i \mid i \in \mathbb{N}\}$.

(i) Let $r > 0$ be given. Let $x \in S_d(0, r)$ and let $x_n \in S_d(0, r)$ ($n = 1, 2, 3, \dots$) be such that $x_n \xrightarrow{w} x$ we must show that $x_n \rightarrow x$. If $x_n \not\rightarrow x$ we have that there exist $\varepsilon \in (0, 1)$ such that $d\left(\frac{x_n - x}{2}, 0\right) > \varepsilon$ for all $n \in \mathbb{N}$ (passing to a subsequence, if necessary). Since $d\left(\frac{x_n - x}{2}, 0\right) \leq r^M$ for all $n \in \mathbb{N}$, the sequence $\left(d\left(\frac{x_n - x}{2}, 0\right)\right)_{n=1}^{\infty}$ is bounded, it has a convergent subsequence. Without loss of generality we may assume that $d\left(\frac{x_n - x}{2}, 0\right) \rightarrow \varepsilon_o$ for some $\varepsilon_o \geq \varepsilon$. Choose $i_o \in \mathbb{N}$ such that

$$(2.1) \quad \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} < \left(\frac{\varepsilon_o}{2}\right)^M.$$

By assumption $x_n \xrightarrow{w} x$ and since the i^{th} coordinate mapping $\pi_i : (\ell_{(p_i)}, d) \rightarrow \mathbb{R}$, defined by $\pi_i(x) = x(i)$ is continuous, it implies that $x_n(i) \rightarrow x(i)$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. This give

$$(2.2) \quad \sum_{i=1}^{i_o} |x_n(i)|^{p_i} \rightarrow \sum_{i=1}^{i_o} |x(i)|^{p_i},$$

$$(2.3) \quad \sum_{i=1}^{i_o} \left| \frac{x_n(i) - x(i)}{2} \right|^{p_i} \rightarrow 0,$$

as $n \rightarrow \infty$. By convexity of the mapping $t \mapsto |t|^p$ together with (2.1), (2.2), and (2.3), we have

$$\begin{aligned} \varepsilon_o &= \lim_{n \rightarrow \infty} d\left(\frac{x_n - x}{2}, 0\right) = \\ &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^{i_o} \left| \frac{x_n(i) - x(i)}{2} \right|^{p_i} + \sum_{i=i_o+1}^{\infty} \left| \frac{x_n(i) - x(i)}{2} \right|^{p_i} \right)^{\frac{1}{M}} = \\ &= \left(\lim_{n \rightarrow \infty} \sum_{i=1}^{i_o} \left| \frac{x_n(i) - x(i)}{2} \right|^{p_i} + \lim_{n \rightarrow \infty} \sum_{i=i_o+1}^{\infty} \left| \frac{x_n(i) - x(i)}{2} \right|^{p_i} \right)^{\frac{1}{M}} = \\ &= \left(0 + \lim_{n \rightarrow \infty} \sum_{i=i_o+1}^{\infty} \left| \frac{x_n(i) - x(i)}{2} \right|^{p_i} \right)^{\frac{1}{M}} \leq \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1}{2} \lim_{n \rightarrow \infty} \sum_{i=i_o+1}^{\infty} |x_n(i)|^{p_i} + \frac{1}{2} \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} \right)^{\frac{1}{M}} = \\
&= \left(\frac{1}{2} \lim_{n \rightarrow \infty} \left(r^M - \sum_{i=1}^{i_o} |x_n(i)|^{p_i} \right) + \frac{1}{2} \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} \right)^{\frac{1}{M}} = \\
&= \left(\frac{1}{2} \left(r^M - \sum_{i=1}^{i_o} |x(i)|^{p_i} \right) + \frac{1}{2} \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} \right)^{\frac{1}{M}} = \\
&= \left(\frac{1}{2} \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} + \frac{1}{2} \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} \right)^{\frac{1}{M}} = \\
&= \left(\sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} \right)^{\frac{1}{M}} < \\
&< \left(\left(\frac{\varepsilon_o}{2} \right)^M \right)^{\frac{1}{M}} = \\
&= \frac{\varepsilon_o}{2},
\end{aligned}$$

which is a contradiction. Therefore $x_n \rightarrow x$ as $n \rightarrow \infty$.

(ii) Let $r > 0$ be given. Let $x \in S_d(0, r)$ and $x_n \in B_d(0, r)$ ($n = 1, 2, 3, \dots$) be such that $d\left(\frac{x+x_n}{2}, 0\right) \rightarrow r$, we must show that $d(x, x_n) \rightarrow 0$. First, we shall show that $x_n(i) \rightarrow x(i)$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. Suppose on the contrary that there exists $i \in \mathbb{N}$ which $x_n(i) \not\rightarrow x(i)$ as $n \rightarrow \infty$. Without loss of generality we may assume that $i = 1$, and there exists $\eta > 0$ such that

$$|x_n(1) - x(1)|^{p_1} > \eta \text{ for all } n \in \mathbb{N},$$

(passing to a subsequence if necessary), this give

$$2^{p_1}(|x_n(1)|^{p_1} + |x(1)|^{p_1}) > \eta \text{ for all } n \in \mathbb{N}.$$

Since $t \mapsto |t|^{p_1}$ is a uniformly convex function so there exists $\delta > 0$ such that

$$\left| \frac{x_n(1) + x(1)}{2} \right|^{p_1} < (1 - \delta) \left(\frac{|x_n(1)|^{p_1} + |x(1)|^{p_1}}{2} \right) \text{ for all } n \in \mathbb{N}.$$

Thus,

$$\begin{aligned}
 d\left(\frac{x_n + x}{2}, 0\right) &= \left(\sum_{i=1}^{\infty} \left|\frac{x_n(i) + x(i)}{2}\right|^{p_i}\right)^{\frac{1}{M}} = \\
 &= \left(\left|\frac{x_n(1) + x(1)}{2}\right|^{p_1} + \sum_{i=2}^{\infty} \left|\frac{x_n(i) + x(i)}{2}\right|^{p_i}\right)^{\frac{1}{M}} < \\
 &< \left(1 - \delta\right) \left(\frac{|x_n(1)|^{p_1} + |x(1)|^{p_1}}{2}\right) + \frac{1}{2} \sum_{i=2}^{\infty} |x_n(i)|^{p_i} + \frac{1}{2} \sum_{i=2}^{\infty} |x(i)|^{p_i} \Big)^{\frac{1}{M}} = \\
 &= \left(\frac{1}{2} \sum_{i=1}^{\infty} |x_n(i)|^{p_i} + \frac{1}{2} \sum_{i=1}^{\infty} |x(i)|^{p_i} - \delta \left(\frac{|x_n(1)|^{p_1} + |x(1)|^{p_1}}{2}\right)\right)^{\frac{1}{M}} \leq \\
 &\leq \left(\frac{r^M}{2} + \frac{r^M}{2} - \frac{\delta\eta}{2^{p_1+1}}\right)^{\frac{1}{M}} = \\
 &= \left(r^M - \frac{\delta\eta}{2^{p_1+1}}\right)^{\frac{1}{M}} < \\
 &< r - a,
 \end{aligned}$$

where $a \in \left(0, r - \left(r^M - \frac{\delta\eta}{2^{p_1+1}}\right)^{\frac{1}{M}}\right)$.

This implies $d\left(\frac{x_n + x}{2}, 0\right) \not\rightarrow r$ as $n \rightarrow \infty$, a contradiction.

Next, let $\varepsilon > 0$ be given. Since $x_n(i) \rightarrow x(i)$ as $n \rightarrow \infty$ there exist $\varepsilon_o \in (0, \varepsilon^M)$ and $i_o, n_o \in \mathbb{N}$ such that

$$\begin{aligned}
 \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} &< \frac{\varepsilon_o}{3 \cdot 2^{M+1}}, \\
 \sum_{i=1}^{i_o} (|x_n(i) - x(i)|)^{p_i} &< \frac{\varepsilon_o}{3} \text{ for all } n > n_o, \text{ and} \\
 \sum_{i=1}^{i_o} |x_n(i)|^{p_i} &> \sum_{i=1}^{i_o} |x(i)|^{p_i} - \frac{\varepsilon_o}{3 \cdot 2^M} \text{ for all } n > n_o.
 \end{aligned}$$

Thus for all $n > n_o$, we have

$$\begin{aligned}
 d(x_n - x, 0) &= \left(\sum_{i=1}^{\infty} |x_n(i) - x(i)|^{p_i}\right)^{\frac{1}{M}} = \\
 &= \left(\sum_{i=1}^{i_o} |x_n(i) - x(i)|^{p_i} + \sum_{i=i_o+1}^{\infty} |x_n(i) - x(i)|^{p_i}\right)^{\frac{1}{M}} < \\
 &< \left(\frac{\varepsilon_o}{3} + 2^M \left(\sum_{i=i_o+1}^{\infty} |x_n(i)|^{p_i} + \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i}\right)\right)^{\frac{1}{M}} \leq
 \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{\varepsilon_o}{3} + 2^M \left(r^M - \sum_{i=1}^{i_o} |x_n(i)|^{p_i} + \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} \right) \right)^{\frac{1}{M}} < \\
&< \left(\frac{\varepsilon_o}{3} + 2^M \left(r^M - \sum_{i=1}^{i_o} |x(i)|^{p_i} + \frac{\varepsilon_o}{3 \cdot 2^M} + \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} \right) \right)^{\frac{1}{M}} = \\
&= \left(\frac{\varepsilon_o}{3} + 2^M \left(\sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} + \frac{\varepsilon_o}{3 \cdot 2^M} + \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} \right) \right)^{\frac{1}{M}} = \\
&= \left(\frac{\varepsilon_o}{3} + 2^M \left(2 \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} + \frac{\varepsilon_o}{3 \cdot 2^M} \right) \right)^{\frac{1}{M}} = \\
&= \left(\frac{\varepsilon_o}{3} + 2^{M+1} \sum_{i=i_o+1}^{\infty} |x(i)|^{p_i} + \frac{\varepsilon_o}{3} \right)^{\frac{1}{M}} < \\
&< \left(\frac{\varepsilon_o}{3} + \frac{\varepsilon_o}{3} + \frac{\varepsilon_o}{3} \right)^{\frac{1}{M}} = \\
&= \varepsilon_o^{\frac{1}{M}} < \varepsilon.
\end{aligned}$$

Hence $d(x_n - x, 0) \longrightarrow 0$ as $n \longrightarrow \infty$.

In the following we give an example of **(LUR)** metric linear space which is not **(UR)**.

Example 2.5. By Theorem 2.4(ii) we have that if $(p_i) = \left(\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, \frac{n+2}{n+1}, \dots \right)$ then $(\ell_{(p_i)}, d)$ is **(LUR)**. We shall show that $(\ell_{(p_i)}, d)$ is not **(UR)**. Choose $r = 1$ and define $(x_n), (y_n) \subset B_d(0, 1)$ by

$$x_n(i) = \begin{cases} 1 & ; \text{ if } i = n \\ 0 & ; \text{ otherwise,} \end{cases}$$

$$y_n(i) = \begin{cases} 1 & ; \text{ if } i = n + 1 \\ 0 & ; \text{ otherwise.} \end{cases}$$

Then

$$d\left(\frac{x_n + y_n}{2}, 0\right) = \left(\left(\frac{1}{2}\right)^{\frac{n+2}{n+1}} + \left(\frac{1}{2}\right)^{\frac{n+3}{n+2}} \right)^{\frac{2}{3}} \longrightarrow 1 \text{ as } n \longrightarrow \infty.$$

But $d(x_n, y_n) = d(x_n - y_n, 0) = 2^{\frac{2}{3}}$ for all $n \in \mathbb{N}$, so $x_n - y_n \not\rightarrow 0$ as $n \longrightarrow \infty$. By Theorem 2.1, $(\ell_{(p_i)}, d)$ is not **(UR)**.

The next example show that the converse of Theorem 2.3 is not true.

Example 2.6. Let $(p_i) = \left(1, 1, \frac{4}{3}, \frac{5}{4}, \dots, \frac{n+1}{n}, \dots\right)$. By Theorem 2.4(i), $(\ell_{(p_i)}, d)$ has property **(H)**, we shall show that it is not **(LUR)**. To show this we shall show that $(\ell_{(p_i)}, d)$ is not **(R)**. Choose $r = 2^{\frac{3}{4}}$. Let $x = (1, 1, 0, 0, 0, \dots)$ and $y = (2, 0, 0, 0, \dots)$. Then $x, y \in B_d(0, 2^{\frac{3}{4}})$ and

$$d\left(\frac{x+y}{2}, 0\right) = \left(\frac{3}{2} + \frac{1}{2} + 0 + 0 + \dots\right)^{\frac{3}{4}} = 2^{\frac{3}{4}}.$$

Hence, $(\ell_{(p_i)}, d)$ is not **(R)**.

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References

- [1] AHUJA, G.C., NARANG, T.D. and TREHAN, S., *Best approximation on convex sets in metric linear spaces*, Math. Nachr, vol. 78, 1977, 125–130.
- [2] JUNDE, W. and LIANCHANG, C., *Reflexivity of uniform convexity in metric linear spaces and its application*, Advances in Math. (China), vol. 23, 1994, 439–444.
- [3] JUNDE, W. and NARANG, T.D., *H-property, Normal structure and fixed points of nonexpensive mappings in metric linear spaces*, Acta Mathematica Vietnamica, vol. 25 (1), 2000, 13–18.
- [4] KOTHE, G., *Topological Vector Spaces*. I, Springer-Verlag, New York, 1983.
- [5] MADDOX, I.J., *Elements of functional analysis*, Cambridge University Press, London, New York, Melbourne, 1970.
- [6] NARANG, T.D., *A characterization of strictly convex metric linear spaces* Publ. de L' inst. Math., Nouvelle series, tome vol. 39 (53), 1986, 149–151.

- [7] ROLEWICZ, S., *Metric linear spaces*, Polish Scientific Publishers, Warszawa, 1985.
- [8] SASTRY, K.P.R. and NAIDU, S.V.R., *Convexity conditions in metric linear spaces*, Math. Seminar Notes 7, 1979, 235-251.

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Some Geometric properties in Orlicz- Cesaro Spaces

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ABSTRACT: On the Orlicz- Cesaro sequence spaces (ces_{Φ}) which are defined by using Orlicz function Φ we show that the space equipped with both Amemiya and Luxemburg norms possesses uniform Opial property and uniform Kadec-Klee property if satisfy the condition.

KEYWORDS: Orlicz- Cesaro sequence spaces, uniform Kadec-Klee property, uniform Opial property, Amemiya norm, Luxemburg norm.

INTRODUCTION

In the whole paper $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural numbers and of real numbers, respectively. The space of all real sequences is denoted by ℓ^0 . Let $(X, \|\cdot\|)$ be a real normed space and $B(X)(S(X))$ be the closed unit ball (the unit sphere) of X .

A Banach space $(X, \|\cdot\|)$ which is a subspace of ℓ^0 is said to be a *Kothe sequence space*, if :

(i) for any $x \in \ell^0$ and $y \in X$ such that $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, we have $x \in X$ and $\|x\| \leq \|y\|$,

(ii) there is $x \in X$ with $x(i) \neq 0$ for all $i \in \mathbb{N}$

An element x from a Kothe sequence space X is called *order continuous* if for any sequence (x_n) in X_+ (the positive cone of X) such that $x_n \leq |x|$ for all $n \in \mathbb{N}$ and $x_n \rightarrow 0$ coordinatewise, we have $\|x_n\| \rightarrow 0$.

A Kothe sequence space X is said to be *order continuous* if any $x \in X$ is order continuous. It is easy to see that $x \in X$ is order continuous if and only if $\|(0, 0, \dots, 0, x(n+1), x(n+2), \dots)\| \rightarrow 0$ as $n \rightarrow \infty$.

A Banach space X is said to have the *Kadec-Klee property* (or *H-property*) if every weakly convergent sequence on the unit sphere is convergent in norm.

Recall that a sequence $(x_n) \subset X$ is said to be ε -*separated sequence* for some $\varepsilon > 0$ if

$$sep(x_n) = \inf \{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

A Banach space is said to have the *uniform Kadec-Klee property* (write **(UKK)** for short) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence (x_n) in $S(X)$ with $sep(x_n) > \varepsilon$ and $x_n \xrightarrow{w} x$, we have $\|x\| < 1 - \delta$. Every **(UKK)** Banach space has *H-property* (see [1]).

The Opial property is important because Banach spaces with this property have the weak fixed point property (see [2]). Opial has proved in [3] that the sequence spaces ℓ_p ($1 < p < \infty$) have this condition but $L_p[0, 2\pi]$ ($p \neq 2$, $1 < p < \infty$) do not.

A Banach space X is said to have the *Opial property* (see [3]) if for any weakly null sequence (x_n) and every $x \neq 0$ in X , we have

$$\liminf_{n \rightarrow \infty} \|x_n\| < \liminf_{n \rightarrow \infty} \|x_n + x\|.$$

A Banach space X is said to have the *uniform Opial property* (see [4]) if for each $\varepsilon > 0$ there exists $\tau > 0$ such that for any weakly null sequence (x_n) in $S(X)$ and $x \in X$ with $\|x\| \geq \varepsilon$ the following inequality holds:

$$1 + \tau \leq \liminf_{n \rightarrow \infty} \|x_n + x\|.$$

For a real vector space X a function $\mathfrak{M} : X \rightarrow [0, \infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\mathfrak{M}(x) = 0$ if and only if $x = 0$,

(ii) $\mathfrak{M}(\alpha x) = \mathfrak{M}(x)$ for all scalar α with $|\alpha| = 1$,

(iii) $\mathfrak{M}(\alpha x + \beta y) \leq \mathfrak{M}(x) + \mathfrak{M}(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular $\mathfrak{M}$ is called *convex* if

(iii) $\mathfrak{M}(\alpha x + \beta y) \leq \alpha \mathfrak{M}(x) + \beta \mathfrak{M}(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular $\mathfrak{M}$ on X , the space

$$X_{\mathfrak{M}} = \{x \in X : \mathfrak{M}(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\},$$

is called the *modular space*.

A sequence (x_n) of elements of $X_{\mathfrak{M}}$ is called *modular convergent* to $x \in X_{\mathfrak{M}}$ if there exists a $\lambda > 0$ such that $\mathfrak{M}(\lambda(x_n - x)) \rightarrow 0$, as $n \rightarrow \infty$.

If $\mathfrak{M}$ is a convex modular, the function

$$\|x\| = \inf \left\{ \lambda > 0 : \mathfrak{M}\left(\frac{x}{\lambda}\right) \leq 1 \right\},$$

and

$$\|x\|_{\lambda} = \inf_{k > 0} \frac{1}{k} (1 + \mathfrak{M}(kx)),$$

are two norms on $X_{\mathfrak{M}}$, which are called the *Luxemburg norm* and the *Amemiya norm*, respectively. In addition, $\|x\| \leq \|x\|_{\lambda} \leq 2\|x\|$ for all $x \in X_{\mathfrak{M}}$ (see [5]).

Theorem 1.1 Let $(x_n) \subset X_{\mathfrak{M}}$ then $\|x_n\| \rightarrow 0$ (or equivalently $\|x_n\|_{\lambda} \rightarrow 0$) if and only if $\mathfrak{M}(\lambda(x_n)) \rightarrow 0$, as $n \rightarrow \infty$.

Proof. See [6, Theorem 1.3(a)].

A modular $\mathfrak{M}$ is said to satisfy the Δ_2 -condition ($\mathfrak{M} \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$$\mathfrak{M}(2x) \leq K\mathfrak{M}(x) + \varepsilon$$

$$\text{for all } x \in X_{\mathfrak{M}} \text{ with } \mathfrak{M}(x) \leq a.$$

If $\mathfrak{M}$ satisfies the Δ_2 -condition for all $a > 0$ with $K \geq 2$ dependent on a , we say that $\mathfrak{M}$ satisfies the strong Δ_2 -condition ($\mathfrak{M} \in \Delta_2^s$).

Theorem 1.2 Convergences in norm and in modular are equivalent in $X_{\mathfrak{M}}$ if $\mathfrak{M} \in \Delta_2$.

Proof. See [7, Lemma 2.3].

Theorem 1.3 If $\mathfrak{M} \in \Delta_2^s$ then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\mathfrak{M}(u+v) - \mathfrak{M}(u)| < \varepsilon$$

$$\text{whenever } u, v \in X_{\mathfrak{M}} \text{ with } \mathfrak{M}(u) \leq L \text{ and } \mathfrak{M}(v) \leq \delta.$$

Proof. See [7, Lemma 2.1].

Theorem 1.4 If $\mathfrak{M} \in \Delta_2^s$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \geq 1 + \delta$ whenever $\mathfrak{M}(x) \geq 1 + \varepsilon$.

Proof. See [7, Lemma 2.4].

A map $\Phi: \mathbb{R} \rightarrow [0, \infty]$ is said to be an Orlicz function if it is even, convex, continuous and vanishing at 0 and $\Phi(u) \rightarrow \infty$ as $u \rightarrow \infty$. Furthermore, we say that an Orlicz function Φ is an N' -function if $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \infty$. The Orlicz sequence space, ℓ_{Φ} , where Φ is an Orlicz function is defined as

$$\ell_{\Phi} = \{x \in \ell^0 : I_{\Phi}(\lambda x) < \infty \exists \lambda > 0\},$$

where $I_{\Phi}(x) = \sum_{i=1}^{\infty} \Phi(x(i))$ is a convex modular on ℓ_{Φ} . Then ℓ_{Φ} is a Banach space with both Luxemburg norm $\|\cdot\|_{\ell_{\Phi}}$ and Amemiya norm $\|\cdot\|_{\ell_{\Phi}^A}$ (see [5]). Denoted by $K(x)$ the set of all $k > 0$ such that $\|x\|_A = \frac{1}{k}(1 + \mathfrak{M}(kx))$, it is well known that $K(x) \neq \emptyset$ for all $x \in \ell_{\Phi}$ whenever Φ is an N' -function (see [8]).

An Orlicz function Φ is said to satisfy the δ_2 -condition (we will write $\Phi \in \delta_2$ for short) if there exist constants $K \geq 2$ and $u_0 > 0$ such that the inequality $\Phi(2u) \leq K\Phi(u)$ holds for every $u \in \mathbb{R}$ satisfying $|u| \leq u_0$.

For $1 < p < \infty$, the Cesaro sequence space (write ces_p , for short) is defined by

$$ces_p = \left\{ x \in \ell^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^p < \infty \right\},$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)^p \right)^{\frac{1}{p}} \quad (1.1)$$

This space was first introduced by Shiue [9]. It is useful in the theory of Matrix operators and others (see [10] and [11]). Some geometric properties of the Cesaro sequence spaces ces_p were studied by many authors.

For an Orlicz function Φ the Orlicz-Cesaro sequence space, ces_{Φ} , is defined by

$$ces_{\Phi} = \{x \in \ell^0 : \rho_{\Phi}(\lambda x) < \infty, \exists \lambda > 0\},$$

where

$$\rho_{\Phi}(x) = \sum_{n=1}^{\infty} \Phi \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right),$$

is a convex modular on ces_{Φ} . The subspace E_{Φ} of ces_{Φ} is defined by

$$E_{\Phi} = \{x \in \ell^0 : \rho_{\Phi}(\lambda x) < \infty, \forall \lambda > 0\}.$$

It is worth noting that if $\Phi \in \delta_2$, then $\rho_{\Phi} \in \Delta_2^s$ and $ces_{\Phi} = E_{\Phi}$.

To simplify notations, we put $ces_{\Phi}^L = (ces_{\Phi}, \|\cdot\|_L)$ and $ces_{\Phi}^A = (ces_{\Phi}, \|\cdot\|_A)$. In the case when $\Phi(t) = |t|^p$, ($p > 1$) the Orlicz-Cesaro sequence space ces_{Φ} becomes the Cesaro sequence space ces_p and the Luxemburg norm is that one defined by (1.1).

From now on, for $x \in \ell^0$ and $i \in \mathbb{N}$ we let

$$x_i = (x(1), x(2), \dots, x(i), 0, 0, \dots),$$

$$x_{i \rightarrow \infty} = (0, 0, \dots, x(i+1), x(i+2), x(i+3), \dots),$$

and

$$\text{supp } x = \{i \in \mathbb{N} : x(i) \neq 0\}.$$

RESULTS

We first give an important fact for $\|x\|_A$ on ces_{Φ}^A .

Lemma 2.1 If Φ is an N' -function, then for each $x \in ces_{\Phi}^A$ there exists $k \in \mathbb{R}$ such that

$$\|x\|_A = \frac{1}{k}(1 + \rho_{\Phi}(kx)).$$

Proof. For each $x = (x(i))_{i=1}^{\infty} \in ces_{\Phi}$ we have $\bar{x} = \left(\frac{1}{n} \sum_{i=1}^n |x(i)| \right)_{n=1}^{\infty} \in \ell_{\Phi}$. Observe that $\|x\|_{ces_{\Phi}^A} = \|\bar{x}\|_{\ell_{\Phi}^A}$,

and Φ is an N' -function, by [8, Corollary 2.3] there exists $k \in \mathbb{R}$ such that

$$\|x\|_{ces_{\Phi}^A} = \|\bar{x}\|_{\ell_{\Phi}^A} = \frac{1}{k}(1 + I_{\Phi}(k\bar{x}))$$

$$= \frac{1}{k} \left(1 + \sum_{n=1}^{\infty} \Phi \left(\frac{k}{n} \sum_{i=1}^n |x(i)| \right) \right) = \frac{1}{k}(1 + \rho_{\Phi}(kx)).$$

This completes the proof of our Lemma.

Proposition 2.2 Suppose that Φ is an N' -function and let $\{x_n\}$ be a bounded sequence in ces_{Φ}^A such that $x_n \xrightarrow{w} x$ for some $x \in ces_{\Phi}^A$. If $k_n \in K(x_n)$

and $k_n \rightarrow \infty$, then $x = 0$.

Proof. For each $n \in \mathbb{N}, \eta > 0$, put $G_{(n,\eta)} = \left\{ i \in \mathbb{N} : \frac{1}{i} \sum_{j=1}^i |x_n(j)| \geq \eta \right\}$. First, we claim that for each $\eta > 0, G_{(n,\eta)} = \emptyset$ for all large $n \in \mathbb{N}$. Otherwise, without loss of generality, we may assume that $G_{(n,\eta)} \neq \emptyset$ for all $n \in \mathbb{N}$ and for some $\eta > 0$. Then

$$\|x_n\|_A = \frac{1}{k_n} (1 + \rho_\Phi(k_n x_n)) \geq \frac{\Phi(k_n x_n)}{k_n} \quad (i \in G_{(n,\eta)}).$$

By applying the assumption that Φ is an N' -function, we obtain $\|x_n\|_A \rightarrow \infty$, which contradicts to the fact that $\{x_n\}$ is bounded, hence we have the claim. By the claim, we have $\frac{1}{i} \sum_{j=1}^i |x_n(j)| \rightarrow 0$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. This implies that $x_n(i) \rightarrow 0$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. Since $x_n \xrightarrow{w} x$, we have $x_n(i) \rightarrow x(i)$ for all $i \in \mathbb{N}$, so it follows that $x(i) = 0$ for all $i \in \mathbb{N}$.

Lemma 2.3 For any Orlicz function Φ , we have $E_\Phi \subseteq \{x \in \text{ces}_\Phi : \|x - x_i\|_A \rightarrow 0\}$.

Proof. Write $A = \{x \in \text{ces}_\Phi : \|x - x_i\|_A \rightarrow 0\}$. Let $x \in E_\Phi$ and $\varepsilon > 0$ be given. Since $x \in E_\Phi$, there exists $i_0 \in \mathbb{N}$ such that $\rho_\Phi((x - x_i)/\varepsilon) < \varepsilon$ for all $i > i_0$. Therefore, by the definition of $\| \cdot \|_A$ we have

$$\|\varepsilon^{-1}(x - x_i)\|_A \leq 1 + \rho_\Phi((x - x_i)/\varepsilon) < 1 + \varepsilon$$

for all $i > i_0$. This yields $\|(x - x_i)\|_A \rightarrow 0$ as $i \rightarrow \infty$ since ε is arbitrary. Hence $x \in A$, proving the Lemma.

Theorem 2.4 The space ces_Φ^A is (UKK) if Φ is an N' -function which satisfies the δ -condition.

Proof. For a given $\varepsilon > 0$, by Theorem 1.2 there exists $\delta \in (0, 1)$ such that $\|y\|_A \geq \varepsilon/4$ implies $\rho_\Phi(y) \geq 2\delta$. Given $x_n \in B(\text{ces}_\Phi^A)$, $x_n \rightarrow x$ weakly and $\|x_n - x_m\|_A \geq \varepsilon (n \neq m)$, we shall complete the proof by showing that $\|x\|_A \leq 1 - \delta$. Indeed, if $x = 0$, then it is clear. So, we assume $x \neq 0$. In this case, by Proposition 2.2 we have that $\{k_n\}$ is bounded, where $k_n \in K(x_n)$. Passing to a subsequence if necessary we may assume that $k_n \rightarrow k$ for some $k > 0$. Since $\Phi \in \delta_2$, Lemma 2.3 assures that there exists $j \in \mathbb{N}$ such that $\|x_j\|_A \geq \|x\|_A - \delta$. Since the weak convergence of $\{x_n\}$ implies that $x_n \rightarrow x$ coordinatewise, we deduce that $x_n(i) \rightarrow x(i)$ uniformly on $\{1, 2, \dots, j\}$. Consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\|(x_n - x_m)_j\|_A \leq \varepsilon/2 \quad \text{for all } n, m \geq n_0,$$

which implies

$$\|(x_n - x_m)_{n_0-j}\|_A \geq \varepsilon/2 \quad \text{for all } n, m \geq n_0, m \neq n.$$

This gives or $\|x_{n_0-j}\|_A \geq \varepsilon/4$ for all $n, m \geq n_0, m \neq n$, which yields $\|x_{n_0-j}\|_A \geq \varepsilon/4$ for infinitely many $n \in \mathbb{N}$, hence $\rho_\Phi(x_{n_0-j}) \geq 2\delta$. Without loss of generality we may assume that $\|x_{n_0-j}\|_A \geq \varepsilon/4$ for all $n \in \mathbb{N}$. By using the convexity of Φ and the inequality $\Phi(a+b) \geq \Phi(a) + \Phi(b)$, $a, b \in \mathbb{R}^+$ together with the fact that $k_n \geq 1$, we have

$$\begin{aligned} 1 - 2\delta &\geq \|x_n\|_A - \rho_\Phi(x_{n_0-j}) \\ &\geq \|x_n\|_A - \frac{1}{k_n} \rho_\Phi(k_n x_{n_0-j}) \\ &= \frac{1}{k_n} + \frac{1}{k_n} \sum_{i=1}^{\infty} \Phi\left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)|\right) - \frac{1}{k_n} \sum_{i=j+1}^{\infty} \Phi\left(\frac{k_n}{i} \sum_{r=1}^{i-j} |x_n(j+r)|\right) \\ &= \frac{1}{k_n} + \frac{1}{k_n} \sum_{i=1}^j \Phi\left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)|\right) + \frac{1}{k_n} \left[\sum_{i=j+1}^{\infty} \Phi\left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)|\right) - \sum_{i=j+1}^{\infty} \Phi\left(\frac{k_n}{i} \sum_{r=1}^{i-j} |x_n(j+r)|\right) \right] \\ &= \frac{1}{k_n} + \frac{1}{k_n} \sum_{i=1}^j \Phi\left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)|\right) + \frac{1}{k_n} \left[\sum_{i=j+1}^{\infty} \Phi\left(\frac{k_n}{i} \sum_{r=1}^j |x_n(r)|\right) + \frac{k_n}{i} \sum_{r=1}^{i-j} |x_n(j+r)| \right] - \sum_{i=j+1}^{\infty} \Phi\left(\frac{k_n}{i} \sum_{r=1}^{i-j} |x_n(j+r)|\right) \\ &\geq \frac{1}{k_n} + \frac{1}{k_n} \sum_{i=1}^j \Phi\left(\frac{k_n}{i} \sum_{r=1}^i |x_n(r)|\right) + \frac{1}{k_n} \sum_{i=j+1}^{\infty} \Phi\left(\frac{k_n}{i} \sum_{r=1}^j |x_n(r)|\right) \\ &= \frac{1}{k_n} + \frac{1}{k_n} \rho_\Phi(k_n x_{n_0-j}) \rightarrow \frac{1}{k_n} + \frac{1}{k_n} \rho_\Phi(k_n x_j) \geq \|x_j\|_A \geq \|x\|_A - \delta, \end{aligned}$$

hence $\|x\|_A \leq 1 - \delta$.

Theorem 2.5 If Φ is an N' -function which satisfies δ -condition, then ces_Φ^A has the uniform opial property.

Proof. Take any $\varepsilon > 0$ and $x \in ces_\Phi^A$ with $\|x\|_A \geq \varepsilon$. Let (x_n) be weakly null sequence in $S(ces_\Phi^A)$. By $\Phi \in \delta_2$, and Theorem 1.2, there is $\xi \in (0, 1)$ independent of x such that $\rho\left(\frac{x}{2}\right) > \xi$. Also, by $\Phi \in \delta_2$, we have $ces_\Phi^A = E_\Phi$. By Lemma 2.3, x is an order continuous element, this allows us to find $j_0 \in \mathbb{N}$ such that

$$\|x_{n_{k_n}}\|_A < \frac{\xi}{4}$$

and

$$\sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=1}^j \frac{|x(i)|}{2}\right) < \frac{\xi}{8}.$$

It follows that

$$\xi \leq \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j \frac{|x(i)|}{2}\right) + \sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=1}^j \frac{|x(i)|}{2}\right)$$

$$\leq \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j \frac{|x(i)|}{2}\right) + \frac{\xi}{8},$$

which implies

$$\frac{7\xi}{8} \leq \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j \frac{|x(i)|}{2}\right). \quad (2.1)$$

From $x_n \xrightarrow{w} 0$, we have $x_n(i) \rightarrow 0$ for all $i \in \mathbb{N}$, which implies that $\rho_\Phi(x_{n_{k_n}}) \rightarrow 0$. By Theorem 1.2 we have $\|x_{k_n}\|_A \rightarrow 0$, so there exists $n_0 \in \mathbb{N}$ such that

$$\|x_{n_{k_n}}\|_A < \frac{\xi}{4} \quad \text{for all } n > n_0.$$

Therefore,

$$\begin{aligned} \|x + x_n\|_A &= \|(x + x_n)_{k_n} + (x + x_n)_{n_{k_n}}\|_A \\ &\geq \|x_{k_n} + x_{n_{k_n}}\|_A - \|x_{k_n}\|_A - \|x_{n_{k_n}}\|_A \\ &\geq \|x_{k_n} + x_{n_{k_n}}\|_A - \frac{\xi}{2}. \end{aligned} \quad (2.2)$$

Since Φ is an N' -function, by Lemma 2.1 there exists $k_n > 0$ such that

$$\|x_{k_n} + x_{n_{k_n}}\|_A = \frac{1}{k_n} \left(1 + \rho_\Phi(k_n(x_{k_n} + x_{n_{k_n}})) \right).$$

This together with (2.2) and the fact that $\rho_\Phi(y+z) \geq \rho_\Phi(y) + \rho_\Phi(z)$ if $\text{supp } y \cap \text{supp } z = \emptyset$, we have

$$\begin{aligned} \|x + x_n\|_A &\geq \frac{1}{k_n} + \frac{1}{k_n} \rho_\Phi(k_n x_{k_n}) + \frac{1}{k_n} \rho_\Phi(k_n x_{n_{k_n}}) - \frac{\xi}{2} \\ &\geq \|x_{n_{k_n}}\|_A + \frac{1}{k_n} \rho_\Phi(k_n x_{k_n}) - \frac{\xi}{2} \end{aligned} \quad (2.3)$$

We may assume without loss of generality that $k_n \geq \frac{1}{2}$. Since $2k_n \geq 1$, by convexity of Orlicz

function Φ we have that $\rho_\Phi(k_n x_{k_n}) \geq 2k_n \rho_\Phi(x_{k_n})$. Thus inequalities (2.1) and (2.3) imply that

$$\begin{aligned} \|x + x_n\|_A &\geq \|x_{n_{k_n}}\|_A + 2\rho_\Phi\left(\frac{x_{k_n}}{2}\right) - \frac{\xi}{2} \\ &> \|x_{n_{k_n}}\|_A + 2 \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j \frac{|x(i)|}{2}\right) - \frac{\xi}{2} \\ &> 1 - \frac{\xi}{4} + \frac{14\xi}{8} - \frac{\xi}{2} \\ &= 1 + \xi \end{aligned} \quad \text{for all } n > n_0,$$

which deduces $\liminf_{n \rightarrow \infty} \|x + x_n\|_A \geq 1 + \xi$.

Theorem 2.6 If Φ is an Orlicz function which satisfies δ_2 -condition, then ces_Φ^L has the uniform opial property.

Proof. Take any $\varepsilon > 0$ and $x \in ces_\Phi^L$ with $\|x\|_L \geq \varepsilon$. Let (x_n) be weakly null sequence in $S(ces_\Phi^L)$. By $\Phi \in \delta_2$, we have $\rho_\Phi \in \Delta_2^L$. Thus by Theorem 1.2, there is $\eta \in (0, 1)$ independent of x such that $\eta < \rho_\Phi(x) < \infty$. Also, by $\rho_\Phi \in \Delta_2^L$, Theorem 1.3 asserts that there exists $\eta_1 \in (0, \eta)$ such that

$$|\rho_\Phi(y+z) - \rho_\Phi(y)| < \frac{\eta}{4}, \quad (2.4)$$

whenever $\rho_\Phi(y) \leq 1$ and $\rho_\Phi(z) \leq \eta_1$.

Since $\rho_\Phi(x) < \infty$, we choose $j_0 \in \mathbb{N}$ such that

$$\sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=1}^j |x(i)|\right) < \sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=1}^j |x(i)|\right) < \frac{\eta_1}{4}. \quad (2.5)$$

This gives

$$\begin{aligned} \eta &< \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j |x(i)|\right) + \sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=1}^j |x(i)|\right) \\ &\leq \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j |x(i)|\right) + \frac{\eta_1}{4}, \end{aligned}$$

which implies

$$\sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j |x(i)|\right) > \eta - \frac{\eta_1}{4} > \eta - \frac{\eta}{4} = \frac{3\eta}{4}.$$

This together with the assumption that $x_n \xrightarrow{w} 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{3\eta}{4} \leq \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j |x_n(i) + x(i)|\right). \quad (2.6)$$

for all $n > n_0$, since the weak convergence implies the coordinatewise convergence. Again by $x_n \xrightarrow{w} 0$, there exists $n_1 > n_0$ such that $\rho_\Phi(x_{n_{k_n}}) < \eta_1$ for all $n > n_1$, so from (2.4) obtain

$$|\rho_\Phi(x_{n_{k_n}} + x_{n_{k_n}}) - \rho_\Phi(x_{n_{k_n}})| < \frac{\eta}{4},$$

since $\rho_\Phi(x_n) = 1$. Hence,

$$1 - \frac{\eta}{4} = \rho_\Phi(x_n) - \frac{\eta}{4} < \rho_\Phi(x_{n_{k_n}}) = \sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=1}^j |x_n(i)|\right),$$

for all $n > n_1$. This together with (2.4), (2.5) and (2.6) imply that for any $n > n_1$,

$$\begin{aligned} \rho_{\Phi}(x_n + x) &= \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j |x_n(i) + x(i)|\right) + \sum_{j=k+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=1}^j |x_n(i) + x(i)|\right) \\ &> \sum_{j=1}^k \Phi\left(\frac{1}{j} \sum_{i=1}^j |x_n(i) + x(i)|\right) + \sum_{j=k+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=k+1}^j |x_n(i) + x(i)|\right) \\ &\geq \frac{3\eta}{4} + \sum_{j=k+1}^{\infty} \Phi\left(\frac{1}{j} \sum_{i=k+1}^j |x_n(i)|\right) - \frac{\eta}{4} \\ &\geq \frac{3\eta}{4} + \left(1 - \frac{\eta}{4}\right) - \frac{\eta}{4} = 1 + \frac{\eta}{4}. \end{aligned}$$

By $\rho_{\Phi} \in \Delta_2^s$, and by Theorem 1.4, there is τ depending on η only such that $\|x_n + x\|_{\Phi} \geq 1 + \tau$.

Corollary 2.7 ([12, Theorem 2]) For any $1 < p < \infty$, the space $\mathcal{C}es_p$ has the uniform Opial property.

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REFERENCES

1. Huff R (1985) Banach spaces which are nearly uniformly convex. *Rocky Mountain J. Math.* **10**, 743-9.
2. Gossez JP and Lamu Dozo E (1972) Some geometric properties related to fixed point theory for nonexpensive mappings. *Pacific J. Math.* **40**, 565-73.
3. Opial Z (1967) Weak convergence of the sequence of successive approximations for non expensive mappings. *Bull. Amer. Math. Soc.* **73**, 591-7.
4. Prus S (1992) Banach spaces with uniform Opial property. *Math. Chronicle, New Zealand* **18(8)**, 697-704.
5. Musielak J (1983) *Orlicz spaces and modular spaces*. Lecture Notes in Math. 1034. Springer-Verlag.
6. Mahgranda L (1985) *Orlicz spaces and Interpolation*. Seminars in Mathematics, Campinas Vol. 5.
7. Cui YA and Hudzik H (1998) On the uniform opial property in some modular sequence spaces. *Funct. Approx. Comment.* **26**, 93-102.
8. Cui YA, Hudzik H, Nowak M and Pluciennik R (1999) Some geometric properties in Orlicz sequence spaces equipped with Orlicz norm. *J. Convex Anal.* **6(1)**, 91-113.
9. Shiue JS (1970) Cesaro sequence spaces. *Tamkang J. Math.* **1**, 143-50.
10. Lee PY (1984) Cesaro sequence spaces. *Math. Chronicle, New Zealand* **13**, 29-45.
11. Liu YQ, Wu BE and Lee PY (1996) *Method of sequence spaces*. Guangdong of Science and Technology Press (in Chinese).
12. Cui YA and Hudzik H (1999) Some geometric properties related to fixed point theory in Cesaro spaces. *Collect. Math.* **50(3)**, 277-88.

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Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings

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Abstract

In this paper, weak and strong convergence theorems are established for a three-step iterative scheme for asymptotically nonexpansive mappings in Banach spaces. Mann-type and Ishikawa-type iterations are included by the new iterative scheme. The results obtained in this paper extend and improve the recent ones announced by Xu and Noor, Glowinski and Le Tallec, Noor, Ishikawa, and many others.

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1. Introduction

Fixed-point iteration processes for asymptotically nonexpansive mapping in Banach spaces including Mann and Ishikawa iterations processes have been studied extensively by many authors; see [1–19,21,22]. Many of them are used widely to study the approximate solutions of the certain problems; see [4,6,11,12,21]. In 2000, Noor [12] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [4] applied three-step iterative schemes for

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finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory.

Recently, Xu and Noor [23] introduced and studied a three-step scheme to approximate fixed point of asymptotically nonexpansive mappings in a Banach space, and Cho, Zhou and Guo [3] extended their schemes to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Wangkeeree [20] gave a strong convergence theorem of Noor iterations with errors for asymptotically nonexpansive mapping in the intermediate sense. Inspired and motivated by these facts, a new class of three-step iterative scheme is introduced and studied in this paper. This scheme can be viewed as an extension for three-step and two-step iterative schemes of Glowinski and Le Tallec [4], Noor [12], Xu and Noor [23], and Ishikawa [8]. The scheme is defined as follows.

Let X be a normed space, C be a nonempty convex subset of X , and $T : C \rightarrow C$ be a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n) x_n, \quad n \geq 1, \end{aligned} \quad (1.1)$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

The iterative schemes (1.1) are called the modified Noor iterations. Noor iterations include the Mann-Ishikawa iterations as special cases. If $c_n = \beta_n \equiv 0$, then (1.1) reduces to Noor iterations defined by Xu and Noor [23]:

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n, \\ y_n &= b_n T^n z_n + (1 - b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1, \end{aligned} \quad (1.2)$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

For $a_n = c_n = \beta_n \equiv 0$, then (1.1) reduces to the usual Ishikawa iterative scheme

$$\begin{aligned} y_n &= b_n T^n x_n + (1 - b_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1, \end{aligned} \quad (1.3)$$

where $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = \beta_n \equiv 0$, then (1.1) reduces to the usual Mann iterative scheme

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad n \geq 1, \quad (1.4)$$

where $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

The purpose of this paper is to establish several strong convergence theorems of the modified Noor iterations for completely continuous asymptotically nonexpansive mappings in a uniformly convex Banach space, and weak convergence theorems of the modified Noor iterations for asymptotically nonexpansive mappings in a uniformly convex Banach space with Opial's condition. Our results extend and improve the corresponding ones announced by Xu and Noor [23], and others.

Now, we recall the well-known concepts and results.

Let X be normed space and C be a nonempty subset of X . A mapping $T : C \rightarrow C$ is said to be asymptotically nonexpansive on C if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and each $n \geq 1$.

If $k_n \equiv 1$, then T is known as a nonexpansive mapping. The mapping T is called *uniformly L -Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L \|x - y\|,$$

for all $x, y \in C$ and each $n \geq 1$.

It is easy to see that if T is asymptotically nonexpansive, then it is uniformly L -Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

Recall that a Banach space X is said to satisfy *Opial's condition* [13] if $x_n \rightarrow x$ weakly as $n \rightarrow \infty$ and $x \neq y$ imply that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 [18, Lemma 1]. Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall n = 1, 2, \dots$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

- (i) $\lim_{n \rightarrow \infty} a_n$ exists.
- (ii) $\lim_{n \rightarrow \infty} a_n = 0$ whenever $\liminf_{n \rightarrow \infty} a_n = 0$.

Lemma 1.2 [21, Theorem 2]. Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$, $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda) \|y\|^p - w_p(\lambda)g(\|x - y\|),$$

for all x, y in $B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0, 1]$, where

$$w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda).$$

Lemma 1.3 [3, Lemma 1.4]. Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$, $g(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \lambda \beta g(\|x - y\|),$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 1.4 [3, Lemma 1.6]. Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X , and $T : C \rightarrow C$ be an asymptotically nonexpansive mapping.