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Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.

2. Main results

In this section, we prove weak and strong convergence theorems of the modified Noor iterations for asymptotically nonexpansive mapping in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *If $\{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n\to\infty}(b_n + c_n) < 1$ and $\{k_n\}$ is a sequence of real numbers with $k_n \geqslant 1$ for all $n \geqslant 1$ and $\lim_{n\to\infty} k_n = 1$, then there exist a positive integer N_1 and $\gamma \in (0, 1)$ such that $c_n k_n < \gamma$ for all $n \geqslant N_1$.*

Proof. By $\limsup_{n\to\infty}(b_n + c_n) < 1$, there exists a positive integer N_0 and $\eta \in (0, 1)$ such that

$$c_n \leqslant b_n + c_n < \eta, \quad \forall n \geqslant N_0.$$

Let $\eta' \in (0, 1)$ with $\eta' > \eta$. From $\lim_{n\to\infty} k_n = 1$, there exists a positive integer $N_1 \geqslant N_0$ such that

$$k_n - 1 < \frac{1}{\eta'} - 1, \quad \forall n \geqslant N_1,$$

from which we have $k_n < 1/\eta'$, $\forall n \geqslant N_1$. Put $\gamma = \eta/\eta'$, then we have $c_n k_n < \gamma$ for all $n \geqslant N_1$. □

The next lemma is crucial for proving the main theorems.

Lemma 2.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed, bounded, and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $[0, 1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geqslant 1$. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1).*

(i) *If q is a fixed point of T, then $\lim_{n\to\infty} \|x_n - q\|$ exists.*
(ii) *If $\liminf_{n\to\infty} \alpha_n > 0$ and $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$, then $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$.*
(iii) *If $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0$.*
(iv) *If $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$.*

Proof. From [5, Theorem 1], T has a fixed point $x^* \in C$. Choose a number $r > 0$ such that $C \subseteq B_r$ and $C - C \subseteq B_r$. By Lemma 1.2, there is a continuous, strictly increasing, and convex function $g_1 : [0, \infty) \to [0, \infty)$, $g_1(0) = 0$ such that

$$\left\|\lambda x + (1 - \lambda)y\right\|^2 \leqslant \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - w_2(\lambda)g_1\left(\|x - y\|\right) \tag{2.1}$$

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for all $x, y \in B_r$, $\lambda \in [0,1]$, where $w_2(\lambda) = \lambda(1-\lambda)^2 + \lambda^2(1-\lambda)$. It follows from (2.1) that

$$\begin{aligned}\|z_n - x^*\|^2 &= \big\|a_n(T^n x_n - x^*) + (1-a_n)(x_n - x^*)\big\|^2 \\ &\leqslant a_n\|T^n x_n - x^*\|^2 + (1-a_n)\|x_n - x^*\|^2 - w_2(a_n)g_1\big(\|T^n x_n - x_n\|\big) \\ &\leqslant a_n k_n^2\|x_n - x^*\|^2 + (1-a_n)\|x_n - x^*\|^2 \\ &\leqslant \big(1 + a_n k_n^2 - a_n\big)\|x_n - x^*\|^2.\end{aligned}$$

By Lemma 1.3, there exists a continuous strictly increasing convex function $g_2 : [0,\infty) \to [0,\infty)$, $g_2(0) = 0$, such that

$$\|\lambda x + \beta y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g_2\big(\|x - y\|\big), \tag{2.2}$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$. It follows from (2.2) that

$$\begin{aligned}\|y_n - x^*\|^2 &= \big\|b_n(T^n z_n - x^*) + (1 - b_n - c_n)(x_n - x^*) + c_n(T^n x_n - x^*)\big\|^2 \\ &\leqslant b_n\|T^n z_n - x^*\|^2 + (1 - b_n - c_n)\|x_n - x^*\|^2 + c_n\|T^n x_n - x^*\|^2 \\ &\quad - b_n(1 - b_n - c_n)g_2\big(\|T^n z_n - x_n\|\big) \\ &\leqslant b_n k_n^2\|z_n - x^*\|^2 + c_n k_n^2\|x_n - x^*\|^2 + (1 - b_n - c_n)\|x_n - x^*\|^2 \\ &\quad - b_n(1 - b_n - c_n)g_2\big(\|T^n z_n - x_n\|\big)\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \big\|\alpha_n(T^n y_n - x^*) + (1 - \alpha_n - \beta_n)(x_n - x^*) + \beta_n(T^n z_n - x^*)\big\|^2 \\ &\leqslant \alpha_n\|T^n y_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|x_n - x^*\|^2 + \beta_n\|T^n z_n - x^*\|^2 \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2\big(\|T^n y_n - x_n\|\big) \\ &\leqslant \alpha_n k_n^2\|y_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2\|z_n - x^*\|^2 \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2\big(\|T^n y_n - x_n\|\big) \\ &\leqslant \alpha_n k_n^2\big(b_n k_n^2\|z_n - x^*\|^2 + c_n k_n^2\|x_n - x^*\|^2 \\ &\qquad + (1 - b_n - c_n)\|x_n - x^*\|^2 - b_n(1 - b_n - c_n)g_2\big(\|T^n z_n - x_n\|\big)\big) \\ &\quad + \beta_n k_n^2\|z_n - x^*\|^2 + (1 - \alpha_n - \beta_n)\|x_n - x^*\|^2 \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2\big(\|T^n y_n - x_n\|\big) \\ &= \|x_n - x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1 - b_n - c_n) - \alpha_n - \beta_n\big)\|x_n - x^*\|^2 \\ &\quad + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2\big)\|z_n - x^*\|^2 \\ &\quad - \alpha_n k_n^2 b_n(1 - b_n - c_n)g_2\big(\|T^n z_n - x_n\|\big) \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2\big(\|T^n y_n - x_n\|\big) \\ &\leqslant \|x_n - x^*\|^2 + \big(\alpha_n k_n^2 c_n\big(k_n^2 - 1\big) \\ &\quad + \alpha_n\big(k_n^2 - 1\big) - \alpha_n k_n^2 b_n - \beta_n\big)\|x_n - x^*\|^2 \\ &\quad + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2\big)\big(1 + a_n k_n^2 - a_n\big)\|x_n - x^*\|^2\end{aligned}$$

$$
\begin{aligned}
&\quad - \alpha_n k_n^2 b_n(1 - b_n - c_n) g_2(\|T^n z_n - x_n\|) \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n) g_2(\|T^n y_n - x_n\|) \\
&= \|x_n - x^*\|^2 + \big(\alpha_n k_n^2 c_n(k_n^2 - 1) + \alpha_n(k_n^2 - 1) - \alpha_n k_n^2 b_n - \beta_n \\
&\quad + \alpha_n b_n k_n^4 + \beta_n k_n^2 + (\alpha_n b_n k_n^4 + \beta_n k_n^2) a_n(k_n^2 - 1)\big) \|x_n - x^*\|^2 \\
&\quad - \alpha_n k_n^2 b_n(1 - b_n - c_n) g_2(\|T^n z_n - x_n\|) \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n) g_2(\|T^n y_n - x_n\|) \\
&= \|x_n - x^*\|^2 + \big(\alpha_n k_n^2 c_n(k_n^2 - 1) + \alpha_n(k_n^2 - 1) + \alpha_n b_n k_n^2(k_n^2 - 1) \\
&\quad + \beta_n(k_n^2 - 1) + (\alpha_n b_n k_n^4 + \beta_n k_n^2) a_n(k_n^2 - 1)\big) \|x_n - x^*\|^2 \\
&\quad - \alpha_n k_n^2 b_n(1 - b_n - c_n) g_2(\|T^n z_n - x_n\|) \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n) g_2(\|T^n y_n - x_n\|) \\
&= \|x_n - x^*\|^2 + (k_n^2 - 1)\big(\alpha_n k_n^2 c_n + \alpha_n + \alpha_n b_n k_n^2 + \beta_n \\
&\quad + (\alpha_n b_n k_n^4 + \beta_n k_n^2) a_n\big) \|x_n - x^*\|^2 \\
&\quad - \alpha_n k_n^2 b_n(1 - b_n - c_n) g_2(\|T^n z_n - x_n\|) \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n) g_2(\|T^n y_n - x_n\|).
\end{aligned}
$$

Since $\{k_n\}$ and C are bounded, there exists a constant $M > 0$ such that

$$\big(\alpha_n k_n^2 c_n + \alpha_n + \alpha_n b_n k_n^2 + \beta_n + (\alpha_n b_n k_n^4 + \beta_n k_n^2) a_n\big) \|x_n - x^*\|^2 \leqslant M$$

for all $n \geqslant 1$. It follows that

$$
\begin{aligned}
&\alpha_n k_n^2 b_n(1 - b_n - c_n) g_2(\|T^n z_n - x_n\|) \\
&\quad \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M(k_n^2 - 1)
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
&\alpha_n(1 - \alpha_n - \beta_n) g_2(\|T^n y_n - x_n\|) \\
&\quad \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M(k_n^2 - 1).
\end{aligned}
\tag{2.4}
$$

(i) If $q \in F(T)$, by taking $x^* = q$ in the inequality (2.3) we have $\|x_{n+1} - q\|^2 \leqslant \|x_n - q\|^2 + M(k_n^2 - 1)$. Since $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, it follows from Lemma 1.1 that $\lim_{n\to\infty} \|x_n - q\|$ exists.

(ii) If $\liminf_{n\to\infty} \alpha_n > 0$ and $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$, then there exists a positive integer n_0 and $\eta, \eta' \in (0, 1)$ such that

$$0 < \eta < b_n, \qquad 0 < \eta < \alpha_n \quad \text{and} \quad b_n + c_n < \eta' < 1 \quad \text{for all } n \geqslant n_0.$$

This implies by (2.3) that

$$\eta^2(1 - \eta') g_2(\|T^n z_n - x_n\|) \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M(k_n^2 - 1) \tag{2.5}$$

for all $n \geqslant n_0$. It follows from inequality (2.5) that for $m \geqslant n_0$,

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$$
\begin{aligned}
\sum_{n=n_0}^{m} g_2\big(\|T^n z_n - x_n\|\big) &\leqslant \frac{1}{\eta^2(1-\eta')}\left(\sum_{n=n_0}^{m}\big(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\big)\right. \\
&\quad \left. + M\sum_{n=n_0}^{m}\big(k_n^2 - 1\big)\right) \\
&\leqslant \frac{1}{\eta^2(1-\eta')}\left(\|x_{n_0} - x^*\|^2 + M\sum_{n=n_0}^{m}\big(k_n^2 - 1\big)\right). \qquad (2.6)
\end{aligned}
$$

Since $0 \leqslant t^2 - 1 \leqslant 2t(t-1)$ for all $t \geqslant 1$, the assumption $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $m \to \infty$ in inequality (2.6) we get $\sum_{n=n_0}^{\infty} g_2(\|T^n z_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g_2(\|T^n z_n - x_n\|) = 0$. Since g_2 is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty}\|T^n z_n - x_n\| = 0$.

(iii) If $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then by using a similar method, together with inequality (2.4), it can be shown that $\lim_{n\to\infty}\|T^n y_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$ and $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, by (ii) and (iii) we have

$$
\lim_{n\to\infty}\|T^n y_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|T^n z_n - x_n\| = 0. \qquad (2.7)
$$

From $y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n$, we have

$$
\|y_n - x_n\| \leqslant b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\|.
$$

Thus

$$
\begin{aligned}
\|T^n x_n - x_n\| &\leqslant \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\
&\leqslant k_n\|x_n - y_n\| + \|T^n y_n - x_n\| \\
&\leqslant k_n\big(b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\|\big) + \|T^n y_n - x_n\| \\
&= k_n b_n\|T^n z_n - x_n\| + c_n k_n\|T^n x_n - x_n\| + \|T^n y_n - x_n\|. \qquad (2.8)
\end{aligned}
$$

By Lemma 2.1, there exists positive integer n_1 and $\gamma \in (0,1)$ such that $c_n k_n < \gamma$ for all $n \geqslant n_1$. This together with (2.8) implies that for $n \geqslant n_1$,

$$
\begin{aligned}
(1-\gamma)\|T^n x_n - x_n\| &< (1 - c_n k_n)\|T^n x_n - x_n\| \\
&\leqslant k_n b_n\|T^n z_n - x_n\| + \|T^n y_n - x_n\|.
\end{aligned}
$$

It follows from (2.7) that $\lim_{n\to\infty}\|T^n x_n - x_n\| = 0$. □

Theorem 2.3. *Let X be a uniformly convex Banach space, and C a nonempty closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geqslant 1$, and*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations (1.1). *Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

Proof. By Lemma 2.2, we have

$$\lim_{n\to\infty}\|T^n y_n - x_n\| = 0, \qquad \lim_{n\to\infty}\|T^n z_n - x_n\| = 0,$$
$$\lim_{n\to\infty}\|T^n x_n - x_n\| = 0. \tag{2.9}$$

Since $x_{n+1} - x_n = \alpha_n(T^n y_n - x_n) + \beta_n(T^n z_n - x_n)$, we have

$$\begin{aligned}
\|x_{n+1} - T^n x_{n+1}\| &\leqslant \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\
&\leqslant \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\
&= (1 + k_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\
&\leqslant (1 + k_n)\alpha_n\|T^n y_n - x_n\| + (1 + k_n)\beta_n\|T^n z_n - x_n\| \\
&\quad + \|T^n x_n - x_n\|.
\end{aligned}$$

This together with (2.9) implies that

$$\|x_{n+1} - T^n x_{n+1}\| \to 0 \quad (\text{as } n \to \infty).$$

Thus

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\| \\
&\leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|x_{n+1} - T^n x_{n+1}\| \to 0,
\end{aligned}$$

which implies

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{2.10}$$

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (2.10), $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k} = q$. By continuity of T and (2.10) we have that $Tq = q$, so q is a fixed point of T. By Lemma 2.2(i), $\lim_{n\to\infty}\|x_n - q\|$ exists. But $\lim_{k\to\infty}\|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty}\|x_n - q\| = 0$.

Since

$$\|y_n - x_n\| \leqslant b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\| \to 0 \quad \text{as } n \to \infty, \quad \text{and}$$
$$\|z_n - x_n\| \leqslant a_n\|T^n x_n - x_n\| \to 0 \quad \text{as } n \to \infty,$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. □

For $c_n = \beta_n \equiv 0$ in Theorem 2.3, we obtain the following result.

Theorem 2.4 [23, Theorem 2.1]. *Let X be a uniformly convex Banach space, and let C be a closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ be real sequences in $[0, 1]$ satisfying*

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(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, *define*

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + (1 - b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1. \end{aligned}$$

Then $\{x_n\}, \{y_n\}, \{z_n\}$ *converge strongly to a fixed point of* T.

When $a_n = c_n = \beta_n \equiv 0$ in Theorem 2.3, we can obtain Ishikawa-type convergence result which is a generalization of Theorem 3 in [14].

Theorem 2.5. *Let X be a uniformly convex Banach space, and let C be a closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{b_n\}$ and $\{\alpha_n\}$ be real sequences in* $[0, 1]$ *satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, *define*

$$\begin{aligned} y_n &= b_n T^n x_n + (1 - b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1. \end{aligned}$$

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a fixed point of* T.

For $a_n = b_n = c_n = \beta_n \equiv 0$, then Theorem 2.3 reduces to the following Mann-type convergence result, which is a generalization and refinement of [14, Theorem 2], [16, Theorem 1.5], and [17, Theorem 2.2].

Theorem 2.6. *Let X be a uniformly convex Banach space, and let C be a closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence in* $[0, 1]$ *satisfying*

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1.$$

For a given $x_1 \in C$, *define*

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \quad n \geqslant 1.$$

Then $\{x_n\}$ *converges strongly to a fixed point of* T.

In the next result, we prove weak convergence for the modified Noor iterations (1.1) for asymptotically nonexpansive mapping in a uniformly convex Banach space satisfying Opial's condition. To do this, we need a lemma.

Lemma 2.7. *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty}\|x_n-u\|$ and $\lim_{n\to\infty}\|x_n-v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u=v$.*

Proof. Suppose that $u \neq v$. Then, by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-u\| &= \lim_{k\to\infty}\|x_{n_k}-u\| < \lim_{k\to\infty}\|x_{n_k}-v\| \\ &= \lim_{n\to\infty}\|x_n-v\| = \lim_{k\to\infty}\|x_{m_k}-v\| \\ &< \lim_{k\to\infty}\|x_{m_k}-u\| = \lim_{n\to\infty}\|x_n-u\|,\end{aligned}$$

which is a contradiction. □

Theorem 2.8. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n-1)<\infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n+c_n \in [0,1]$ and $\alpha_n+\beta_n \in [0,1]$ for all $n \geqslant 1$, and*

(i) $0<\liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n+c_n)<1$, *and*
(ii) $0<\liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n+\beta_n)<1$.

Let $\{x_n\}$ be the sequence defined by the modified Noor iterations (1.1). *Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. It follows from Lemma 2.2(iv) that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n\to\infty$, without loss of generality. By Lemma 1.4, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.4, $u, v \in F(T)$. By Lemma 2.2(i), $\lim_{n\to\infty}\|x_n-u\|$ and $\lim_{n\to\infty}\|x_n-v\|$ exist. It follows from Lemma 2.7 that $u=v$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. □

When $c_n=\beta_n\equiv 0$ in Theorem 2.8, we obtain the following result.

Corollary 2.9. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n-1)<\infty$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be sequences of real numbers in $[0,1]$ and*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

Let $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be the sequences defined by*

$$z_n = a_n T^n x_n + (1 - a_n)x_n,$$
$$y_n = b_n T^n z_n + (1 - b_n)x_n,$$
$$x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1.$$

Then $\{x_n\}$ *converges weakly to a fixed point of* T.

When $a_n = c_n = \beta_n \equiv 0$ in Theorem 2.8, we obtain Ishikawa-type weak convergence theorem as follows:

Corollary 2.10. *Let* X *be a uniformly convex Banach space which satisfies Opial's condition, and* C *a nonempty closed, bounded, and convex subset of* X. *Let* T *be an asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \geqslant 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{b_n\}$ *and* $\{\alpha_n\}$ *be sequences of real numbers in* $[0, 1]$ *such that*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

Let $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by*

$$y_n = b_n T^n x_n + (1 - b_n)x_n,$$
$$x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1.$$

Then $\{x_n\}$ *converges weakly to a fixed point of* T.

When $a_n = b_n = c_n = \beta_n \equiv 0$ in Theorem 2.8, we obtain Mann-type weak convergence theorem as follows:

Corollary 2.11. *Let* X *be a uniformly convex Banach space which satisfies Opial's condition, and* C *a nonempty closed, bounded and convex subset of* X. *Let* T *be an asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \geqslant 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{\alpha_n\}$ *be a sequence of real numbers in* $[0, 1]$ *such that*

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1.$$

Let $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \quad n \geqslant 1.$$

Then $\{x_n\}$ *converges weakly to a fixed point of* T.

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Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings

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Abstract

In this paper, weak and strong convergence theorems of the modified Noor iterations with errors are established for asymptotically nonexpansive mappings in Banach spaces. The results obtained in this paper extend and improve the several recent results in this area.

Keywords: Asymptotically nonexpansive mapping; Completely continuous; Uniformly convex; Noor iterations; Opial's condition

1. Introduction

Fixed-point iteration processes for asymptotically nonexpansive mapping in Banach spaces including Mann and Ishikawa iterations processes have been studied extensively by many authors; see [1–19,22,23]. Many of them are used widely to study the approximate solutions of the certain problems; see [11,12,21]. In 2000, Noor [12] introduced a three-step iterative scheme and study the approximate solutions of variational inclusion

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in Hilbert spaces. In 2002, Xu and Noor [23] introduced and studied a three-step scheme to approximate fixed point of asymptotically nonexpansive mappings in a Banach space. Cho et al. [3] extended their schemes to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Suantai [17] defined a new three-step iterations which is an extension of Noor iterations and gave some weak and strong convergence theorems of the modified Noor iterations for asymptotically nonexpansive mappings in uniformly Banach space. Wangkeeree [20] gave a strong convergence theorem of Noor iterations with errors for asymptotically nonexpansive mappings in the intermediate sense. Inspired and motivated by research going on in this area, we consider and study the modified Noor iterations with errors. This scheme can be viewed as an extension for three-step and two-step iterative schemes of Noor [10,11], Xu and Noor [23], Suantai [17] and Ishikawa [6]. The scheme is defined as follows.

Let X be a normed space, C be a nonempty convex subset of X, and $T : C \to C$ be a given mapping. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned}
z_n &= a_n T^n x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n,\\
y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n,\\
x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n, \quad n \geqslant 1,
\end{aligned} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C.

The iterative schemes (1.1) are called the modified Noor iterations with errors. Noor iterations include the Mann–Ishikawa iterations as spacial cases. If $\gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the modified Noor iterations defined by Suantai [17]

$$\begin{aligned}
z_n &= a_n T^n x_n + (1 - a_n)x_n,\\
y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n,\\
x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n)x_n, \quad n \geqslant 1,
\end{aligned} \tag{1.2}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

We note that the usual Ishikawa and Mann iterations are special cases of (1.1) and if $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the Noor iterations defined by Xu and Noor [23]:

$$\begin{aligned}
z_n &= a_n T^n x_n + (1 - a_n)x_n,\\
y_n &= b_n T^n z_n + (1 - b_n)x_n,\\
x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1,
\end{aligned} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

For $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, (1.1) reduces to the usual Ishikawa iterative scheme

$$\begin{aligned}
y_n &= b_n T^n x_n + (1 - b_n)x_n,\\
x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1,
\end{aligned} \tag{1.4}$$

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where $\{b_n\}$, $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (1.1) reduces to the usual Mann iterative scheme

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \quad n \geqslant 1, \tag{1.5}$$

where $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

The purpose of this paper is to establish several strong convergence results of the modified Noor iterations with errors for completely continuous asymptotically nonexpansive mappings in a uniformly convex Banach space, and weak convergence theorems of the modified Noor iterations with errors for asymptotically nonexpansive mappings in a uniformly convex Banach space with Opial's condition. Our results extend and improve the corresponding ones announced by Suantai [17], Xu and Noor [23] and others.

Now, we recall the well-known concepts and results.

Let X be normed space and C be a nonempty subset of X. A mapping $T: C \to C$ is said to be asymptotically nonexpansive on C if there exists a sequence $\{k_n\}$, $k_n \geqslant 1$, with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|,$$

for all $x, y \in C$ and each $n \geqslant 1$.

If $k_n \equiv 1$, then T is known as a nonexpansive mapping. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leqslant L\|x - y\|,$$

for all $x, y \in C$ and each $n \geqslant 1$.

It is easy to see that if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n: n \geqslant 1\}$.

Recall that a Banach space X is said to satisfy *Opial's condition* [13] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 [18, Lemma 1]. *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1 + \delta_n)a_n + b_n, \quad \forall n = 1, 2, \ldots.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ *exists.*
(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 1.2 [21, Theorem 2]. *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g: [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leqslant \lambda\|x\|^p + (1 - \lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|),$$

for all x, y *in* $B_r = \{x \in X\colon \|x\| \leqslant r\}$, $\lambda \in [0, 1]$, *where*

$$w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda).$$

Lemma 1.3 [3, Lemma 1.4]. *Let* X *be a uniformly convex Banach space and* $B_r = \{x \in X\colon \|x\| \leqslant r\}$, $r > 0$. *Then there exists a continuous, strictly increasing, and convex function* $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, *such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x-y\|),$$

for all $x, y, z \in B_r$, *and all* $\lambda, \beta, \gamma \in [0,1]$ *with* $\lambda + \beta + \gamma = 1$.

Lemma 1.4. *Let* X *be a uniformly convex Banach space and* $B_r = \{x \in X\colon \|x\| \leqslant r\}$, $r > 0$. *Then there exists a continuous, strictly increasing, and convex function* $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, *such that*

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leqslant \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x-y\|),$$

for all $x, y, z, w \in B_r$, *and all* $\alpha, \beta, \mu, \lambda \in [0,1]$ *with* $\alpha + \beta + \mu + \lambda = 1$.

Proof. We first observe that $(\mu/(1-\alpha-\beta))z + (\lambda/(1-\alpha-\beta))w \in B_r$ for all $z, w \in B_r$ and $\alpha, \beta, \mu, \lambda \in [0,1]$ with $\alpha + \beta + \mu + \lambda = 1$. It follows from Lemmas 1.3 and 1.2 that there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, such that

$$\begin{aligned}
&\|\alpha x + \beta y + \mu z + \lambda w\|^2 \\
&\quad = \left\|\alpha x + \beta y + (1-\alpha-\beta)\left[\frac{\mu}{(1-\alpha-\beta)}z + \frac{\lambda}{(1-\alpha-\beta)}w\right]\right\|^2 \\
&\quad \leqslant \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta g(\|x-y\|) \\
&\qquad + (1-\alpha-\beta)\left\|\frac{\mu}{(1-\alpha-\beta)}z + \frac{\lambda}{(1-\alpha-\beta)}w\right\|^2 \\
&\quad \leqslant \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta g(\|x-y\|) \\
&\qquad + (1-\alpha-\beta)\left[\frac{\mu}{(1-\alpha-\beta)}\|z\|^2 + \frac{\lambda}{(1-\alpha-\beta)}\|w\|^2\right] \\
&\quad = \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x-y\|). \qquad \square
\end{aligned}$$

Lemma 1.5 [3, Lemma 1.6]. *Let* X *be a uniformly convex Banach space,* C *a nonempty closed convex subset of* X, *and* $T : C \to C$ *be an asymptotically nonexpansive mapping. Then* $I - T$ *is demiclosed at 0, i.e., if* $x_n \to x$ *weakly and* $x_n - Tx_n \to 0$ *strongly, then* $x \in F(T)$, *where* $F(T)$ *is the set of fixed points of* T.

Lemma 1.6 [17, Lemma 2.7]. *Let* X *be a Banach space which satisfies Opial's condition and let* $\{x_n\}$ *be a sequence in* X. *Let* $u, v \in X$ *be such that* $\lim_{n\to\infty}\|x_n - u\|$ *and* $\lim_{n\to\infty}\|x_n - v\|$ *exist. If* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *are subsequences of* $\{x_n\}$ *which converge weakly to* u *and* v, *respectively, then* $u = v$.

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2. Main results

In this section, we prove weak and strong convergence theorems of modified Noor iterations with errors for asymptotically nonexpansive mapping in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *If $\{b_n\}$, $\{c_n\}$ and $\{\mu_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$ and $\{k_n\}$ is a sequence of real number with $k_n\geqslant 1$ for all $n\geqslant 1$ and $\lim_{n\to\infty}k_n=1$, then there exist a positive integer N_1 and $\gamma\in(0,1)$ such that $c_nk_n<\gamma$ for all $n\geqslant N_1$.*

Proof. By $\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$, there exist a positive integer N_0 and $\eta\in(0,1)$ such that

$$c_n\leqslant b_n+c_n+\mu_n<\eta \quad \forall n\geqslant N_0.$$

Let $\eta'\in(0,1)$ with $\eta'>\eta$. From $\lim_{n\to\infty}k_n=1$, there exists a positive integer $N_1\geqslant N_0$ such that

$$k_n-1<\frac{1}{\eta'}-1 \quad \forall n\geqslant N_1,$$

from which we have $k_n<\frac{1}{\eta'}$ $\forall n\geqslant N_1$. Put $\gamma=\frac{\eta}{\eta'}$, then we have $c_nk_n<\gamma$ for all $n\geqslant N_1$. □

The next lemma is crucial for proving the main theorems.

Lemma 2.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n\geqslant 1$ and $\sum_{n=1}^{\infty}(k_n-1)<\infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n+\gamma_n$, $b_n+c_n+\mu_n$ and $\alpha_n+\beta_n+\lambda_n$ are in $[0,1]$ for all $n\geqslant 1$, and $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$, and let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be the bounded sequences in C. For a given $x_1\in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1).*

(i) *If q is a fixed point of T, then $\lim_{n\to\infty}\|x_n-q\|$ exists.*
(ii) *If $0<\liminf_{n\to\infty}\alpha_n$ and $0<\liminf_{n\to\infty}b_n\leqslant\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$, then $\lim_{n\to\infty}\|T^nz_n-x_n\|=0$.*
(iii) *If $0<\liminf_{n\to\infty}\alpha_n\leqslant\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, then $\lim_{n\to\infty}\|T^ny_n-x_n\|=0$.*
(iv) *If $0<\liminf_{n\to\infty}b_n\leqslant\limsup_{n\to\infty}(b_n+c_n+\mu_n)<1$ and $0<\liminf_{n\to\infty}\alpha_n\leqslant\limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n)<1$, then $\lim_{n\to\infty}\|T^nx_n-x_n\|=0$.*

Proof. From [4, Theorem 1], T has a fixed point $x^*\in C$. Choose a number $r>1$ such that $C\subseteq B_r$ and $C-C\subseteq B_r$. By Lemma 1.3, there exists a continuous strictly increasing convex function $g_1:[0,\infty)\to[0,\infty)$, $g_1(0)=0$, such that

$$\|\lambda x+\beta y+\gamma z\|^2\leqslant\lambda\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\lambda\beta g_1(\|x-y\|), \tag{2.1}$$

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for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$. It follows from (2.1) that

$$\begin{aligned}
\|z_n - x^*\|^2 &= \big\|a_n(T^n x_n - x^*) + (1 - a_n - \gamma_n)(x_n - x^*) + \gamma_n(u_n - x^*)\big\|^2 \\
&\leqslant a_n\|T^n x_n - x^*\|^2 + (1 - a_n - \gamma_n)\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g_1\big(\|T^n x_n - x_n\|\big) \\
&\leqslant a_n k_n^2\|x_n - x^*\|^2 + (1 - a_n - \gamma_n)\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g_1\big(\|T^n x_n - x_n\|\big) \\
&= \big(a_n k_n^2 + (1 - a_n - \gamma_n)\big)\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g_1\big(\|T^n x_n - x_n\|\big). \qquad (2.2)
\end{aligned}$$

By Lemma 1.4, there is a continuous, strictly increasing, and convex function $g_2 : [0,\infty) \to [0,\infty)$, $g_2(0) = 0$, such that

$$\begin{aligned}
&\|\alpha x + \beta y + \mu z + \lambda w\|^2 \\
&\quad\leqslant \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g_2\big(\|x - y\|\big) \qquad (2.3)
\end{aligned}$$

and all $\alpha, \beta, \mu, \lambda \in [0,1]$ with $\alpha + \beta + \mu + \lambda = 1$, for all $x, y, z, w \in B_r$. It follows from (2.3) that

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&\quad= \big\|b_n(T^n z_n - x^*) + (1 - b_n - c_n - \mu_n)(x_n - x^*) \\
&\qquad + c_n(T^n x_n - x^*) + \mu_n(v_n - x^*)\big\|^2 \\
&\quad\leqslant b_n\|T^n z_n - x^*\|^2 + (1 - b_n - c_n - \mu_n)\|x_n - x^*\|^2 + c_n\|T^n x_n - x^*\|^2 \\
&\qquad + \mu_n\|v_n - x^*\|^2 - b_n(1 - b_n - c_n - \mu_n)g_2\big(\|T^n z_n - x_n\|\big) \\
&\quad\leqslant b_n k_n^2\|z_n - x^*\|^2 + (1 - b_n - c_n - \mu_n)\|x_n - x^*\|^2 + c_n k_n^2\|x_n - x^*\|^2 \\
&\qquad + \mu_n\|v_n - x^*\|^2 - b_n(1 - b_n - c_n - \mu_n)g_2\big(\|T^n z_n - x_n\|\big). \qquad (2.4)
\end{aligned}$$

It follows from (2.2)–(2.4) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad= \big\|\alpha_n(T^n y_n - x^*) + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - x^*) \\
&\qquad + \beta_n(T^n z_n - x^*) + \lambda_n(w_n - x^*)\big\|^2 \\
&\quad\leqslant \alpha_n\|T^n y_n - x^*\|^2 + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - x^*\|^2 + \beta_n\|T^n z_n - x^*\|^2 \\
&\qquad + \lambda_n\|w_n - x^*\|^2 - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g_2\big(\|T^n y_n - x_n\|\big) \\
&\quad\leqslant \alpha_n k_n^2\|y_n - x^*\|^2 + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - x^*\|^2 + \beta_n k_n^2\|z_n - x^*\|^2 \\
&\qquad + \lambda_n\|w_n - x^*\|^2 - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g_2\big(\|T^n y_n - x_n\|\big) \\
&\quad\leqslant \alpha_n k_n^2\big(b_n k_n^2\|z_n - x^*\|^2 + c_n k_n^2\|x_n - x^*\|^2 \\
&\qquad + (1 - b_n - c_n - \mu_n)\|x_n - x^*\|^2 + \mu_n\|v_n - x^*\|^2 + \beta_n k_n^2\|z_n - x^*\|^2 \\
&\qquad - b_n(1 - b_n - c_n - \mu_n)g_2\big(\|T^n z_n - x_n\|\big)\big)
\end{aligned}$$

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$$
\begin{aligned}
&\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n-x^*\|^2 + \lambda_n\|w_n-x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2\big(\|T^n y_n - x_n\|\big) \\
&= \|x_n-x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n - \lambda_n\big)\|x_n-x^*\|^2 \\
&\quad + \alpha_n\mu_n k_n^2\|v_n-x^*\|^2 + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2\big)\|z_n-x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2\big(\|T^n z_n - x_n\|\big) + \lambda_n\|w_n-x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2\big(\|T^n y_n - x_n\|\big) \\
&\leqslant \|x_n-x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n - \lambda_n\big)\|x_n-x^*\|^2 \\
&\quad + \alpha_n\mu_n k_n^2\|v_n-x^*\|^2 \\
&\quad + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2\big)\big(\big(a_n k_n^2 + (1-a_n-\gamma_n)\big)\|x_n-x^*\|^2 + \gamma_n\|u_n-x^*\|^2\big) \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2\big(\|T^n z_n - x_n\|\big) + \lambda_n\|w_n-x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2\big(\|T^n y_n - x_n\|\big) \\
&= \|x_n-x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n - \lambda_n\big)\|x_n-x^*\|^2 \\
&\quad + \alpha_n\mu_n k_n^2\|v_n-x^*\|^2 + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2\big)\big(a_n k_n^2 + (1-a_n-\gamma_n)\big)\|x_n-x^*\|^2 \\
&\quad + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2\big)\gamma_n\|u_n-x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2\big(\|T^n z_n - x_n\|\big) + \lambda_n\|w_n-x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2\big(\|T^n y_n - x_n\|\big) \\
&= \|x_n-x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n - \lambda_n\big)\|x_n-x^*\|^2 \\
&\quad + \alpha_n\mu_n k_n^2\|v_n-x^*\|^2 + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2 + a_n\alpha_n b_n k_n^6 + a_n\beta_n k_n^4 \\
&\quad - \alpha_n\gamma_n b_n k_n^4 - \gamma_n\beta_n k_n^2 - a_n\alpha_n b_n k_n^4 - a_n\beta_n k_n^2\big)\|x_n-x^*\|^2 \\
&\quad + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2\big)\gamma_n\|u_n-x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2\big(\|T^n z_n - x_n\|\big) + \lambda_n\|w_n-x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2\big(\|T^n y_n - x_n\|\big) \\
&\leqslant \|x_n-x^*\|^2 + \big(\alpha_n c_n k_n^2(k_n^2-1) + \alpha_n(k_n^2-1) + \alpha_n b_n k_n^2(k_n^2-1) + \beta_n(k_n^2-1) \\
&\quad + a_n\alpha_n b_n k_n^4(k_n^2-1) + a_n\beta_n k_n^2(k_n^2-1)\big)\|x_n-x^*\|^2 \\
&\quad + \alpha_n\mu_n k_n^2\|v_n-x^*\|^2 + \big(\alpha_n b_n k_n^4 + \beta_n k_n^2\big)\gamma_n\|u_n-x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2\big(\|T^n z_n - x_n\|\big) + \lambda_n\|w_n-x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2\big(\|T^n y_n - x_n\|\big) \\
&\leqslant \|x_n-x^*\|^2 + \big(\alpha_n c_n k_n^2 + \alpha_n + \alpha_n b_n k_n^2 + \beta_n \\
&\quad + a_n\alpha_n b_n k_n^4 + a_n\beta_n k_n^2\big)(k_n^2-1)\|x_n-x^*\|^2 + \big(k_n^4 + k_n^2\big)\gamma_n\|u_n-x^*\|^2 \\
&\quad + \mu_n k_n^2\|v_n-x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2\big(\|T^n z_n - x_n\|\big) + \lambda_n\|w_n-x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2\big(\|T^n y_n - x_n\|\big).
\end{aligned}
$$

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Since $\{k_n\}$ and C are bounded, there exists a constant $M > 0$ such that

$$\left(\alpha_n c_n k_n^2 + \alpha_n + \alpha_n b_n k_n^2 + \beta_n + a_n\alpha_n b_n k_n^4 + a_n\beta_n k_n^2\right)\|x_n - x^*\|^2 \leqslant M,$$

for all $n \geqslant 1$. It follows that

$$\begin{aligned}&\alpha_n b_n k_n^2(1 - b_n - c_n - \mu_n) g_2\left(\|T^n z_n - x_n\|\right)\\&\quad \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M\left(k_n^2 - 1\right) + L\gamma_n + A\mu_n + r^2\lambda_n\end{aligned}$$

and

$$\begin{aligned}&\alpha_n(1 - \alpha_n - \beta_n - \lambda_n) g_2\left(\|T^n y_n - x_n\|\right)\\&\quad \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M\left(k_n^2 - 1\right) + L\gamma_n + A\mu_n + r^2\lambda_n,\end{aligned}$$

where $L = \sup\{(k_n^4 + k_n^2)\|u_n - x^*\|^2 \colon n \geqslant 1\}$ and $A = \sup\{k_n^2\|v_n - x^*\|^2 \colon n \geqslant 1\}$.

Now, if we let $K = \max\{M, L, A, r^2\}$ then we get that

$$\begin{aligned}&\alpha_n b_n k_n^2(1 - b_n - c_n - \mu_n) g_2\left(\|T^n z_n - x_n\|\right)\\&\quad \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + K\left(\left(k_n^2 - 1\right) + \gamma_n + \mu_n + \lambda_n\right)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\alpha_n(1 - \alpha_n - \beta_n - \lambda_n) g_2\left(\|T^n y_n - x_n\|\right)\\&\quad \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + K\left(\left(k_n^2 - 1\right) + \gamma_n + \mu_n + \lambda_n\right).\end{aligned} \tag{2.6}$$

(i) If $q \in F(T)$, by taking $x^* = q$ in the inequality (2.5) we have $\|x_{n+1} - q\|^2 \leqslant \|x_n - q\|^2 + K((k_n^2 - 1) + \gamma_n + \mu_n + \lambda_n)$. Since $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, it follows from Lemma 1.1 that $\lim_{n\to\infty}\|x_n - q\|$ exists.

(ii) If $0 < \liminf_{n\to\infty}\alpha_n$ and $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, then there exist a positive integer n_0 and $\nu, \eta, \eta' \in (0, 1)$ such that

$$0 < \nu < \alpha_n \quad \text{and} \quad 0 < \eta < b_n \quad \text{and} \quad b_n + c_n + \mu_n < \eta' < 1, \quad \text{for all } n \geqslant n_0.$$

This implies by (2.5) that

$$\begin{aligned}&\nu\eta(1 - \eta') g_2\left(\|T^n z_n - x_n\|\right)\\&\quad \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + K\left(\left(k_n^2 - 1\right) + \gamma_n + \mu_n + \lambda_n\right),\end{aligned} \tag{2.7}$$

for all $n \geqslant n_0$. It follows from (2.7) that for $m \geqslant n_0$,

$$\begin{aligned}&\sum_{n=n_0}^{m} g_2\left(\|T^n z_n - x_n\|\right)\\&\quad \leqslant \frac{1}{\nu\eta(1 - \eta')}\left(\sum_{n=n_0}^{m}\left(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\right)\right.\\&\qquad \left. + K\sum_{n=n_0}^{m}\left(\left(k_n^2 - 1\right) + \gamma_n + \mu_n + \lambda_n\right)\right)\\&\quad \leqslant \frac{1}{\nu\eta(1 - \eta')}\left(\|x_{n_0} - x^*\|^2 + K\sum_{n=n_0}^{m}\left(\left(k_n^2 - 1\right) + \gamma_n + \mu_n + \lambda_n\right)\right).\end{aligned} \tag{2.8}$$

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Since $0 \leqslant t^2 - 1 \leqslant 2t(t-1)$ for all $t \geqslant 1$, the assumption $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $m \to \infty$ in inequality (2.8) we get $\sum_{n=n_0}^{\infty} g_2(\|T^n z_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g_2(\|T^n z_n - x_n\|) = 0$. Since g_2 is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$.

(iii) If $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then by using a similar method, together with inequality (2.6), it can be shown that $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, by (ii) and (iii) we have

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \tag{2.9}$$

From $y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n$, we have

$$\|y_n - x_n\| \leqslant b_n \|T^n z_n - x_n\| + c_n \|T^n x_n - x_n\| + \mu_n \|v_n - x_n\|.$$

Thus

$$\begin{aligned}
\|T^n x_n - x_n\| &\leqslant \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\
&\leqslant k_n \|x_n - y_n\| + \|T^n y_n - x_n\| \\
&\leqslant k_n \big(b_n \|T^n z_n - x_n\| + c_n \|T^n x_n - x_n\| + \mu_n \|v_n - x_n\|\big) \\
&\quad + \|T^n y_n - x_n\| \\
&= k_n b_n \|T^n z_n - x_n\| + c_n k_n \|T^n x_n - x_n\| + \mu_n k_n \|v_n - x_n\| \\
&\quad + \|T^n y_n - x_n\|.
\end{aligned} \tag{2.10}$$

By Lemma 2.1, there exists positive integer n_1 and $\gamma \in (0,1)$ such that $c_n k_n < \gamma$ for all $n \geqslant n_1$. This together with (2.10) implies that for $n \geqslant n_1$,

$$\begin{aligned}
(1-\gamma)\|T^n x_n - x_n\| &< (1 - c_n k_n)\|T^n x_n - x_n\| \\
&\leqslant k_n b_n \|T^n z_n - x_n\| + \mu_n k_n \|v_n - x_n\| + \|T^n y_n - x_n\|.
\end{aligned}$$

It follows from (2.9) that $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$. □

Theorem 2.3. *Let X be a uniformly convex Banach space, and C a nonempty closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n + \mu_n \in [0,1]$ and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \geqslant 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations with errors (1.1). Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. By Lemma 2.2, we have

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$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0,$$
$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0,$$
$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{2.11}$$

Since $x_{n+1} - x_n = \alpha_n(T^n y_n - x_n) + \beta_n(T^n z_n - x_n) + \lambda_n(w_n - x_n)$, we have

$$\begin{aligned}\|x_{n+1} - T^n x_{n+1}\| &\leqslant \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\ &\leqslant \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\ &= (1 + k_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\ &\leqslant (1 + k_n)\alpha_n\|T^n y_n - x_n\| + (1 + k_n)\beta_n\|T^n z_n - x_n\| \\ &\quad + (1 + k_n)\lambda_n\|w_n - x_n\| + \|T^n x_n - x_n\|.\end{aligned}$$

This together with (2.11) implies that

$$\|x_{n+1} - T^n x_{n+1}\| \to 0 \quad (\text{as } n \to \infty).$$

Thus

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\| \\ &\leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|x_{n+1} - T^n x_{n+1}\| \to 0,\end{aligned}$$

which implies

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{2.12}$$

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (2.12), $\{x_{n_k}\}$ converges. Let $\lim_{n\to\infty} x_{n_k} = q$. By continuity of T and (2.12) we have that $Tq = q$, so q is a fixed point of T. By Lemma 2.2(i), $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty} \|x_n - q\| = 0$.

Since

$$\|y_n - x_n\| \leqslant b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\| + \mu_n\|v_n - x_n\| \to 0 \quad \text{as } n \to \infty,$$

and

$$\|z_n - x_n\| \leqslant a_n\|T^n x_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0 \quad \text{as } n \to \infty,$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. □

For $\gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.3, we obtain the following result.

Theorem 2.4 [17, Theorem 2.3]. *Let X be a uniformly convex Banach space, and C a nonempty closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in $[0, 1]$ with $b_n + c_n \in [0, 1]$ and $\alpha_n + \beta_n \in [0, 1]$ for all $n \geqslant 1$, and*

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(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations (1.2). *Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.3, we obtain the following result.

Theorem 2.5 [23, Theorem 2.1]. *Let X be a uniformly convex Banach space, and let C be a closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$ be real sequences in* $[0, 1]$ *satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + (1 - b_n)x_n, \quad n \geqslant 1, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n. \end{aligned}$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

When $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.3, we can obtain Ishikawa-type convergence result which is a generalization of Theorem 3 in [13].

Theorem 2.6. *Let X be a uniformly convex Banach space, and let C be a closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{b_n\}$ and $\{\alpha_n\}$ be real sequences in* $[0, 1]$ *satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= b_n T^n z_n + (1 - b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1. \end{aligned}$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

For $a_n = b_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, Theorem 2.3 reduces to the following Mann-type convergence result, which is a generalization and refinement of Theorem 2 in [13], Theorem 1.5 in [15], and Theorem 2.2 in [16].

Theorem 2.7. *Let X be a uniformly convex Banach space, and let C be a closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying*

$$0 < \liminf_{n\to\infty}\alpha_n \leqslant \limsup_{n\to\infty}\alpha_n < 1.$$

For a given $x_1 \in C$, define

$$x_{n+1} = \alpha_n T^n x_n + (1-\alpha_n)x_n, \quad n \geqslant 1.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

In the next result, we prove weak convergence of the modified Noor iterations with errors for asymptotically nonexpansive mapping in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.8. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0, 1]$ with $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0, 1]$ for all $n \geqslant 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$.

Let $\{x_n\}$ be the sequence defined by modified Noor iterations with errors (1.1). *Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. It follows from Lemma 2.2(iv) that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.5, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.5, $u, v \in F(T)$. By Lemma 2.2(i), $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. It follows from Lemma 1.6 that $u = v$. Therefore $\{x_n\}$ converges weakly to fixed point of T. □

For $\gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.8, we obtain the following result.

Corollary 2.9 [17, Theorem 2.3]. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in $[0, 1]$ with $b_n + c_n \in [0, 1]$ and $\alpha_n + \beta_n \in [0, 1]$ for all $n \geqslant 1$, and*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(b_n + c_n) < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$.

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Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations (1.2). *Then $\{x_n\}$ converges weakly to a fixed point of T.*

For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.8, we obtain the following result.

Corollary 2.10. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$ be real sequences in* $[0, 1]$ *satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$z_n = a_n T^n x_n + (1 - a_n)x_n,$$
$$y_n = b_n T^n z_n + (1 - b_n)x_n, \quad n \geqslant 1,$$
$$x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n.$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

When $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.8, we can obtain Ishikawa-type convergence result which is a generalization of Theorem 3 in [13].

Corollary 2.11. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{b_n\}$ and $\{\alpha_n\}$ be real sequences in* $[0, 1]$ *satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$y_n = b_n T^n z_n + (1 - b_n)x_n,$$
$$x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geqslant 1.$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

For $a_n = b_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then Theorem 2.8 reduces to the following Mann-type convergence result, which is a generalization and refinement of Theorem 2 in [13], Theorem 1.5 in [15], and Theorem 2.2 in [16].

Corollary 2.12. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a closed, bounded and convex subset of X. Let T be an asymptotically*

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nonexpansive self-map of C with $\{k_n\}$ *satisfying* $k_n \geqslant 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{\alpha_n\}$ *be a real sequence in* $[0, 1]$ *satisfying*

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1.$$

For a given $x_1 \in C$, *define*

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \quad n \geqslant 1.$$

Then $\{x_n\}$ *converges weakly to a fixed point of* T.

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UNIFORM OPIAL PROPERTIES IN GENERALIZED CESÀRO SEQUENCE SPACES

NARIN PETROT[1] AND SUTHEP SUANTAI[2]

ABSTRACT. The main purpose of this paper is consider the generalized Cesàro sequence spaces defined by S. Suantai [12] by give some topological property and find condition for $Ces_{(p)}$ equipped with both the Amemiya norm and Luxemburg norm to possesses uniform Opial property.

Keyword: Generalized Cesàro sequence spaces, uniform Opial property, Amemiya norm, Luxemburg norm.

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1. Introductions.

In the whole paper $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural numbers and of real numbers, respectively. The space of all real sequence $x = (x(i))_{i=1}^{\infty}$ is denoted by ℓ^0. Let $(X, \|\cdot\|)$ be a real normed space and $B(X)(S(X))$ be the closed unit ball (the unit sphere) of X.

A Banach space $(X, \|\cdot\|)$ which is a subspace of ℓ^0 is said to be a Köthe sequence space, if :

(i) for any $x \in \ell^0$ and $y \in X$ such that $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, we have $x \in X$ and $\|x\| \leq \|y\|$,

(ii) there is $x \in X$ with $x(i) \neq 0$ for all $i \in \mathbb{N}$.

An element x from a Köthe sequence space X is called order continuous if for any sequence (x_n) in X_+ (the positive cone of X) such that $x_n \leq |x|$ for all $n \in \mathbb{N}$ and $x_n \to 0$ coordinatewise, we have $\|x_n\| \to 0$.

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A Köthe sequence space X is said to be order continuous if any $x \in X$ is order continuous. It is easy to see that X is order continuous if and only if $\|(0, 0, ..., 0, x(n+1), x(n+2), ...)\| \to 0$ as $n \to \infty$.

The Opial property is important because Banach spaces with this property have the weak fixed point property (see [3]). Opial has proved in [8] that the sequence spaces $\ell_p (1 < p < \infty)$ have this condition but $L_p[0, 2\pi] (p \neq 2, 1 < p < \infty)$ do not have it.

A Banach space X is said to have the *Opial property* (see [8]) if for any weakly null sequence (x_n) and every $x \neq 0$ in X there holds

$$\lim_{n\to\infty} inf \|x_n\| < \lim_{n\to\infty} inf \|x_n + x\|.$$

A Banach space X is said to have the *uniform Opial property* (see [10]) if for each $\varepsilon > 0$ there exists $\tau > 0$ such that for any weakly null sequence (x_n) in $S(X)$ and $x \in X$ with $\|x\| \geq \varepsilon$ there holds

$$1 + \tau \leq \lim_{n} inf \|x + x_n\|.$$

For a real vector space X, a function $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the following conditions :

(i) $\varrho(x) = 0$ if and only if $x = 0$;

(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar α with $|\alpha| = 1$;

(iii) $\varrho(\alpha x + \beta y) \leq \varrho(x) + \varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ϱ is called *convex* if

(iv) $\varrho(\alpha x + \beta y) \leq \alpha\varrho(x) + \beta\varrho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For any modular ϱ on X, the space

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\},$$

is called the *modular space.*

A sequence (x_n) of elements of X_ϱ is called *modular convergent* to $x \in X_\varrho$ if there exists a $\lambda > 0$ such that $\varrho(\lambda(x_n - x)) \to 0$, as $n \to \infty$.

If ϱ is a convex modular, the function

$$\|x\| = \inf\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\},$$

and

$$\|x\|_A = \inf_{k>0} \frac{1}{k}(1 + \varrho(kx)),$$

are two norms on X_ϱ, which is called the *Luxemburg norm* and the *Amemiya norm*, respectively. In addition, $\|x\| \leq \|x\|_A \leq 2\|x\|$ for all $x \in X_\varrho$ (see [7]).

Theorem 1.1 Let $(x_n) \subset X_\varrho$ then $\|x_n\| \to 0$ (or equivalently $\|x\|_A \to 0$) if and only if $\varrho(\lambda(x_n)) \to 0$, as $n \to \infty$, for every $\lambda > 0$.

Proof. See [6, Theorem 1.3(a)].

A modular ϱ is said to satisfy the $\Delta_2 -$ *condition* ($\varrho \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$$\varrho(2x) \leq K\varrho(x) + \varepsilon$$

for all $x \in X_\varrho$ with $\varrho(x) \leq a$.

If ϱ satisfies the Δ_2-condition for all $a > 0$ with $K \geq 2$ dependent on a, we say that ϱ satisfies the *strong* $\Delta_2 -$ *condition* ($\varrho \in \Delta_2^s$).

Theorem 1.2 Convergences in norm and in modular are equivalent in X_ϱ if $\varrho \in \Delta_2$.

Proof. See [1, Lemma 2.3].

Theorem 1.3 If $\varrho \in \Delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varrho(u + v) - \varrho(u)| < \varepsilon$$

whenever $u, v \in X_\varrho$ with $\varrho(u) \leq L$ and $\varrho(v) \leq \delta$.

Proof. See [1, Lemma 2.1].

Theorem 1.4 If $\varrho \in \Delta_2^s$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \geq 1 + \delta$ whenever $\varrho(x) \geq 1 + \varepsilon$.

Proof. See [1, Lemma 2.4].

For $1 \leq p < \infty$, the Cesàro sequence space (write ces_p , for short) is defined by

$$ces_p = \{x \in \ell^0 : \textstyle\sum_{j=1}^{\infty} \left(\frac{1}{j}\sum_{i=1}^{j} |x(i)|\right)^p < \infty\},$$

equipped with the norm

$$\|x\| = \left(\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p} \right)^{\frac{1}{p}}. \tag{*}$$

This space was first introduced by Shue [11]. It is useful in the theory of Matrix operator and others (see [4, 5]). Now, we introduce a generalized Cesàro sequence space.

Let $p = (p_j)$ be a sequences of positive real numbers with $p_j \geq 1$ for all $j \in \mathbb{N}$. The generalized Cesàro sequence space $ces_{(p)}$ and its subspace $ces^a_{(p)}$ are defined by

$$ces_{(p)} = \{x \in l^0 : \rho(\lambda x) < \infty, \text{for some } \lambda > 0\},$$

$$ces^a_{(p)} = \{x \in l^0 : \rho(\lambda x) < \infty, \text{for all } \lambda > 0\},$$

where

$$\rho(x) = \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p_j}$$

is a convex modular on $ces_{(p)}$ (see [12]). Assuming that the $ces_{(p)}$ is nontrivial it belongs to the class of Köthe sequence spaces. It is easy to see that if $\underset{j\to\infty}{lim\ sup}\, p_j < \infty$ then $\rho \in \Delta_2^s$, and $ces^a_{(p)} = ces_{(p)}$. To simplify notations, we put $ces_{(p)} = (ces_{(p)}, \|\cdot\|)$ and $ces^A_{(p)} = (ces_{(p)}, \|\cdot\|_A)$.

In the case when $p_j = p, 1 \leq p < \infty$ for all $j \in \mathbb{N}$ the generalized Cesàro sequence space $ces_{(p)}$ is nothing but the Cesàro sequence space ces_p and the Luxemburg norm is express by the formula (*).

In $ces^A_{(p)}$, if $p_j > 1$ for all $j \in \mathbb{N}$ then the set of all k's at which the infimum in the definition of $\|x\|_A$ for a fixed $x \in ces^A_{(p)}$ is attained, will be denoted by $K(x)$ (see [9, Theorem 2.1]).

Throughout this paper, for $x \in \ell^0, i \in \mathbb{N}$ we denote

$$e_i = (\overbrace{0, 0, ..., 0}^{i-1\ times}, 1, 0, 0, 0, ...),$$

$$x_{|_i} = (x(1), x(2), ..., x(i), 0, 0, ...),$$

$$x_{|_{\mathbb{N}-i}} = (0, 0, 0, ..., x(i+1), x(i+2), x(i+3), ...),$$

and

$$supp\ \ x = \{i \in \mathbb{N}; x(i) \neq 0\}.$$

2. Main Results

First we shall give some topological property of $ces^A_{(p)}$. For easy we denote $ces^{a,A}_{(p)} = (ces^a_{(p)}, \|\cdot\|_A)$.

Theorem 2.1 $ces^{a,A}_{(p)}$ is a closed subspace of $ces^A_{(p)}$.

Proof. It is easy to see that $ces^{a,A}_{(p)}$ is a subspace of $ces^A_{(p)}$. Next we will prove that $ces^{a,A}_{(p)}$ is closed in $ces^A_{(p)}$. We must show that if $x_n \in ces^{a,A}_{(p)}$ for each $n \in \mathbb{N}$ and $x_n \to x \in ces^A_{(p)}$, then $x \in ces^{a,A}_{(p)}$. Take any $k > 0$. Since $\|x_n - x\|_A \to 0$ we have by Theorem 1.1 that $\rho(t(x - x_n)) \to 0$, for all $t > 0$ hence, there exists $N \in \mathbb{N}$ such that $\rho(2k(x - x_N)) < 1$ and by $x_N \in ces^{a,A}_{(p)}$ we have $\rho(2kx_N) < \infty$. Thus

$$\begin{aligned}
\rho(kx) &= \sum_{j=1}^{\infty}\left(\frac{k}{j}\sum_{i=1}^{j}|x(i)|\right)^{p_j} \\
&= \sum_{j=1}^{\infty}\left(\frac{k}{j}\sum_{i=1}^{j}\left|\frac{2(x(i)-x_N(i))}{2} + \frac{2x_N(i)}{2}\right|\right)^{p_j} \\
&\leq \sum_{j=1}^{\infty}\left(\frac{1}{2}\frac{k}{j}\sum_{i=1}^{j}|2(x(i)-x_N(i))| + \frac{1}{2}\frac{k}{j}\sum_{i=1}^{j}|2x_N(i)|\right)^{p_j} \\
&\leq \frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{k}{j}\sum_{i=1}^{j}|2(x(i)-x_N(i))|\right)^{p_j} + \frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{k}{j}\sum_{i=1}^{j}|2x_N(i)|\right)^{p_j} \\
&= \frac{1}{2}\rho(2k(x - x_N)) + \frac{1}{2}\rho(2kx_N) < \infty.
\end{aligned}$$

Hence $x \in ces^{a,A}_{(p)}$.

Let E be the set of all finite sequences. The next lemma is a tool for showing that $ces^{a,A}_{(p)} = clE$.

Lemma 2.2 If $\rho(x) < \infty$, then the distance $d(x, E)$ from x to E is no more than 1.

Proof. Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$ define $x_n \in E$ by $x_n = (x(1), x(2), ..., x(n), 0, 0, ...$ Then $\rho(x_n) \to \rho(x)$ as $n \to \infty$, and $\rho(x - x_n) \leq \rho(x) - \rho(x_n)$. Choose $N \in \mathbb{N}$ such that

$\rho(x_N) > \rho(x) - \varepsilon$. Hence, by the definition of $\|\cdot\|_A$ we have

$$d(x, E) \leq \|x - x_N\|_A \leq 1 + \rho(x - x_N) \leq 1 + \rho(x) - \rho(x_N) < 1 + \varepsilon,$$

which implies, $d(x, E) \leq 1$ since ε is arbitrary.

Theorem 2.3 If $\liminf\limits_{j\to\infty} p_j > 1$, then following assertions are trues :

(i) $ces_{(p)}^{a,A} = clE$.

(ii) $ces_{(p)}^{a,A}$ is the subspace of all order continuous elements of $ces_{(p)}^{A}$.

(iii) $ces_{(p)}^{a,A}$ is separable.

Proof. (i) For any $x \in ces_{(p)}^{a,A}$ and $k \geq 1$, we have $kx \in ces_{(p)}^{a,A}$. Therefore, by Lemma 2.2 we get $d(kx, E) \leq 1$ or $d(x, E) \leq 1/k$. Since k is arbitrary, we find that $x \in clE$.

Conversely, since Theorem 2.1 asserts that $ces_{(p)}^{a,A}$ is a closed linear subspace of $ces_{(p)}^{A}$, hence to show $clE \subseteq ces_{(p)}^{a,A}$ it suffices to show that $e_i \in ces_{(p)}^{a,A}$ for each $i \in \mathbb{N}$. Write $\alpha = \liminf\limits_{j\to\infty} p_j > 1$. Fix $i \in \mathbb{N}$ and take any $k > 0$. Choose $j_o > max\{i, k\}$ such that $p_j \geq \alpha$ for all $j > j_o$. Thus,

$$\rho(ke_i) = \sum_{j=i}^{j_o} \left(\frac{k}{j}\right)^{p_j} + \sum_{j=j_o+1}^{\infty} \left(\frac{k}{j}\right)^{p_j} \leq \sum_{j=i}^{j_o} \left(\frac{k}{j}\right)^{p_j} + \sum_{j=j_o+1}^{\infty} \left(\frac{k}{j}\right)^{\alpha} < \infty.$$

Hence $e_i \in ces_{(p)}^{a,A}$.

(ii) Let $x \in ces_{(p)}^{a,A}$ we must show that $\|x - x_{|_i}\|_A \to 0$ as $i \to \infty$. Let $\varepsilon > 0$ be given. Since $x \in ces_{(p)}^{a,A}$ we have that there exists $i_o \in \mathbb{N}$ such that $\rho((x - x_{|_i})/\varepsilon) < \varepsilon$ for all $i > i_o$. Therefore, by the definition of $\|\cdot\|_A$ we have

$$\|\varepsilon^{-1}(x - x_{|_i})\|_A \leq 1 + \rho((x - x_{|_i})/\varepsilon) \leq 1 + \varepsilon$$

for all $i > i_o$ This yield $\|x - x_{|_i}\|_A \to 0$ as $i \to \infty$ since ε is arbitrary.

Let $x \in ces_{(p)}^{A}$ be an order continuous element. Since $\|x - x_{|_i}\|_A \to 0$ as $n \to \infty$, so it easy to see that $x \in clE$ and a proof is complete by (i).

(iii) By (ii), we obtain that for any $x \in ces_{(p)}^{a,A}, x = \sum a_i e_i$. This implies that (e_n) is a basis of $ces_{(p)}^{a,A}$ Hence $ces_{(p)}^{a,A}$ is separable.

Remark 2.4. Since $\|\cdot\|$ is equivalent to $\|\cdot\|_A$, thus Theorem 2.1 and Theorem 2.3 are also valid for $ces_{(p)}$.

Now, we give conditions for $ces^A_{(p)}$ and $ces_{(p)}$ to possess the uniform opial property.

Theorem 2.5 If $p_j > 1$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} \sup p_j < \infty$ then $ces^A_{(p)}$ has the uniform opial property.

Proof. Take any $\varepsilon > 0$ and $x \in ces^A_{(p)}$ with $\|x\|_A \geq \varepsilon$. Let (x_n) be weakly null sequence in $S(ces^A_{(p)})$. By $\lim_{j\to\infty} \sup p_j < \infty$, we have by Theorem 1.2 that there is $\delta \in (0,1)$ independent of x such that $\rho(\frac{x}{2}) > \delta$. Also, by $\lim_{j\to\infty} \sup p_j < \infty$, we have $ces^A_{(p)} = ces^{a,A}_{(p)}$ then as in the first part of a proof in Theorem 2.3(ii) assert that x is an order continuous element, this allows us to find $j_o \in \mathbb{N}$ be such that

$$\|x_{|\mathbb{N}-j_o}\|_A < \frac{\delta}{4}$$

and

$$\sum_{j=j_o+1}^{\infty} \left(\frac{1}{j}\sum_{i=1}^{j} \frac{|x(i)|}{2}\right)^{p_j} < \frac{\delta}{8}$$

It follows that,

$$\begin{aligned} \delta &\leq \sum_{j=1}^{j_o} \left(\frac{1}{j}\sum_{i=1}^{j} \frac{|x(i)|}{2}\right)^{p_j} + \sum_{j=j_o+1}^{\infty} \left(\frac{1}{j}\sum_{i=1}^{j} \frac{|x(i)|}{2}\right)^{p_j} \\ &\leq \sum_{j=1}^{j_o} \left(\frac{1}{j}\sum_{i=1}^{j} \frac{|x(i)|}{2}\right)^{p_j} + \frac{\delta}{8} \end{aligned}$$

which implies

$$\frac{7\delta}{8} \leq \sum_{j=1}^{j_o} \left(\frac{1}{j}\sum_{i=1}^{j} \frac{|x(i)|}{2}\right)^{p_j}. \tag{2.1}$$

From $x_n \xrightarrow{w} 0$, it implies that $x_n(i) \to 0$ for all $i \in \mathbb{N}$. So there exists $n_o \in \mathbb{N}$ such that

$$\|x_{n_{|j_o}}\|_A < \frac{\delta}{4} \quad \text{for all} \quad n > n_o.$$

Therefore,

$$
\begin{aligned}
\|x+x_n\|_A &= \|(x+x_n)_{|_{j_o}} + (x+x_n)_{|_{\mathbb{N}-j_o}}\|_A \\
&\geq \|x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}}\|_A - \|x_{|_{\mathbb{N}-j_o}}\|_A - \|x_{n|_{j_o}}\|_A \\
&\geq \|x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}}\|_A - \frac{\delta}{2}.
\end{aligned} \tag{2.2}
$$

Consider, $\|x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}}\|_A$. Since $p_j > 1$ for all $j \in \mathbb{N}$ we have that there exists $k_n > 0$ such that

$$
\|x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}}\|_A = \frac{1}{k_n}(1 + \rho(k_n(x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}})))
$$

combine with (2.2) and the fact $\rho(y+z) \geq \rho(y) + \rho(z)$ if *supp* $x \cap$ *supp* $y = \emptyset$ we get,

$$
\begin{aligned}
\|x+x_n\|_A &\geq \frac{1}{k_n} + \frac{1}{k_n}\rho(k_n x_{|_{j_o}}) + \frac{1}{k_n}\rho(k_n x_{n|_{\mathbb{N}-j_o}}) - \frac{\delta}{2} \\
&\geq \|x_{n|_{\mathbb{N}-j_o}}\|_A + \frac{1}{k_n}\rho(k_n x_{|_{j_o}}) - \frac{\delta}{2}.
\end{aligned} \tag{2.3}
$$

We may assume without loss of generality that $k_n \geq \frac{1}{2}$. Since $2k_n \geq 1$, we have by convexity of $t \mapsto |t|^p$ that $\rho(k_n x_{|_{j_o}}) \geq 2k_n\rho(x_{|_{j_o}})$, thus inequalities (2.1) and (2.3) implies that

$$
\begin{aligned}
\|x+x_n\|_A &\geq \|x_{n|_{\mathbb{N}-j_o}}\|_A + 2\rho\Big(\frac{x_{|_{j_o}}}{2}\Big) - \frac{\delta}{2} \\
&> \|x_{n|_{\mathbb{N}-j_o}}\|_A + 2\sum_{j=1}^{j_o}\left(\frac{1}{j}\sum_{i=1}^{j}\frac{|x(i)|}{2}\right)^{p_j} - \frac{\delta}{2} \\
&> 1 - \frac{\delta}{4} + \frac{14\delta}{8} - \frac{\delta}{2} \\
&= 1 + \delta,
\end{aligned}
$$

which deduce $\lim\limits_{n\to\infty} \inf \|x+x_n\|_A \geq 1+\delta$.

Theorem 2.6 If $\lim\limits_{j\to\infty} \sup p_j < \infty$ then $ces_{(p)}$ has the uniform opial property.

Proof. Take any $\varepsilon > 0$ and $x \in ces_{(p)}$ with $\|x\| \geq \varepsilon$. Let (x_n) be weakly null sequence in $S(ces_{(p)})$. By $\lim\limits_{j\to\infty} \sup p_j < \infty$, i.e., $\rho \in \Delta_2^s$, we have by Theorem 1.2 that there is

$\delta \in (0,1)$ independent of x such that $\rho(x) > \delta$. Also, by $\rho \in \Delta_2^s$, Theorem 1.3 assert that there exists $\delta_1 \in (0,\delta)$ such that

$$|\rho(y+z) - \rho(y)| < \frac{\delta}{4}, \tag{2.4}$$

whenever, $\rho(y) \leq 1$ and $\rho(z) \leq \delta_1$.

Choose $j_o \in \mathbb{N}$ such that

$$\sum_{j=j_o+1}^{\infty} \left(\frac{1}{j} \sum_{i=j_o+1}^{j} |x(i)| \right)^{p_j} < \sum_{j=j_o+1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p_j} < \frac{\delta_1}{4}, \tag{2.5}$$

this give

$$\begin{aligned} \delta &< \sum_{j=1}^{j_o} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p_j} + \sum_{j=j_o+1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p_j} \\ &\leq \sum_{j=1}^{j_o} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p_j} + \frac{\delta_1}{4}, \end{aligned}$$

which implies $\sum_{j=1}^{j_o} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p_j} > \delta - \frac{\delta_1}{4} > \delta - \frac{\delta}{4} = \frac{3\delta}{4}$. This together with an assumption that $x_n \xrightarrow{w} 0$, we have that there exists $n_o \in \mathbb{N}$ such that

$$\frac{3\delta}{4} \leq \sum_{j=1}^{j_o} \left(\frac{1}{j} \sum_{i=1}^{j} |x_n(i) + x(i)| \right)^{p_j}, \tag{2.6}$$

for all $n > n_o$, since weak convergence implies coordinatewise convergence. Again by $x_n \xrightarrow{w} 0$, there exists $n_1 > n_o$ such that $\|x_{n_{|j_o}}\| < 1 - (1 - \frac{\delta}{4})^{\frac{1}{M}}$ for all $n > n_1$, where $M \in \mathbb{N}$ be such that $p_j \leq M$ for all $j \in \mathbb{N}$. Hence, by the triangle inequality of norm we have that $\|x_{n_{|\mathbb{N}-j_o}}\| > (1 - \frac{\delta}{4})^{\frac{1}{M}}$ then it follows by the definition of $\|\cdot\|$ that

$$\begin{aligned} 1 \leq \rho\left(\frac{x_{n_{|\mathbb{N}-j_o}}}{(1-\frac{\delta}{4})^{\frac{1}{M}}} \right) &= \sum_{j=j_o+1}^{\infty} \left(\frac{\frac{1}{j}\sum_{i=j_o+1}^{j} |x_n(i)|}{(1-\frac{\delta}{4})^{\frac{1}{M}}} \right)^{p_j} \\ &\leq \left(\frac{1}{(1-\frac{\delta}{4})^{\frac{1}{M}}} \right)^{M} \sum_{j=j_o+1}^{\infty} \left(\frac{1}{j} \sum_{i=j_o+1}^{j} |x_n(i)| \right)^{p_j}, \end{aligned}$$

which give, $1-\frac{\delta}{4}\le\sum_{j=j_o+1}^{\infty}\left(\frac{1}{j}\sum_{i=j_o+1}^{j}|x_n(i)|\right)^{p_j}$ for all $n>n_1$. This together with (2.4), (2.5) and (2.6) we can obtain for any $n>n_1$ that

$$\begin{aligned}
\rho(x_n+x) &= \sum_{j=1}^{j_o}\left(\frac{1}{j}\sum_{i=1}^{j}|x_n(i)+x(i)|\right)^{p_j}+\sum_{j=j_o+1}^{\infty}\left(\frac{1}{j}\sum_{i=1}^{j}|x_n(i)+x(i)|\right)^{p_j}\\
&> \sum_{j=1}^{j_o}\left(\frac{1}{j}\sum_{i=1}^{j}|x_n(i)+x(i)|\right)^{p_j}+\sum_{j=j_o+1}^{\infty}\left(\frac{1}{j}\sum_{i=j_o+1}^{j}|x_n(i)+x(i)|\right)^{p_j}\\
&\ge \frac{3\delta}{4}+\sum_{j=j_o+1}^{\infty}\left(\frac{1}{j}\sum_{i=j_o+1}^{j}|x_n(i)|\right)^{p_j}-\frac{\delta}{4}\\
&\ge \frac{3\delta}{4}+(1-\frac{\delta}{4})-\frac{\delta}{4}\\
&= 1+\frac{\delta}{4}.
\end{aligned}$$

By $\rho\in\Delta_2^s$, we have by Theorem 1.4 that there is τ depending on δ only such that $\|x_n+x\|\ge 1+\tau$.

Corollary 2.7 [2, Theorem 2] For any $1<p<\infty$, the space ces_p has the uniform Opial property.

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บทที่ 6
ทฤษฎีจุดตรึงของปริภูมิบานาค
(Fixed Point Theory of Banach Spaces)

ในบทนี้เป็นการวิจัยที่ศึกษาการมีจุดตรึง (fixed point) ของ การส่งชนิดต่าง ๆ โดยเฉพาะการส่งแบบ nonexpansive และ asymptotically nonexpansive mappings นอกจากนั้นก็เป็นการสร้างทฤษฎีที่เกี่ยวกับระเบียบวิธีทำซ้ำ (Iterations) แบบต่างๆ ที่ใช้สำหรับประมาณ และ หารจุดตรึงการ mappings

ทฤษฎีที่ได้ต่างเป็นองค์ความรู้ใหม่เกี่ยวกับทฤษฎีจุดตรึงในสองแนวทางข้างต้น และ สามารถใช้ประยุกต์เพื่อตอบการมี และ การหาคำตอบของสมการต่างๆ ได้

DEMICLOSEDNESS PRINCIPLE AND FIXED POINT THEOREM FOR MAPPINGS OF ASYMPTOTICALLY NONEXPANSIVE TYPE

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Abstract

The object of the present paper to verify the demiclosedness principle for mappings of asymptotically nonexpansive type certain class of Banach space. In this paper, we proved the demiclosedness principle at zero for mappings of asymptotically nonexpansive type in some class of Banach space. Moreover, we investigated the behavior of the iterates $\{T^n x\}$ for mappings of asymptotically nonexpansive type. Finally, we shown that the uniformly Opial condition implies the fixed point property for mappings of asymptotically nonexpansive type defined on weakly compact convex subset.

1.Introduction

Let X be a real Banach space and let C be a nonempty closed convex subset of X. A mapping $T : C \to X$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C,$$

and *asymptotically nonexpansive* [Goebel and Kirk, 1972] if there exists a sequence (k_n) of real number with $k_n \to 1$ such that

$$\| T^n x - T^n y\| \leq k_n \|x - y\|, \text{ for all } x, y \in C \text{ and } n \in N.$$

More generally T is of *asymptotically nonexpansive type* [Kirk, 1974] if T^N is continuous for some integer $N \geq 1$ and, for each $x \in C$, there holds the inequality

$$\limsup_n [\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}] \leq 0.$$

In the sequel, we adopt the notations : $\overline{lim} = \limsup$, $\underline{lim} = \liminf$, B_X for the closed unit ball of X , S_X for the unit sphere of X, $\rightharpoonup$ for weak convergence, and $\to$ for strong convergence.

A mapping f :C $\to$ X is *demiclosed* (at y) if f (x) = y whenever $(x_n) \subset C$ with $x_n \rightharpoonup x$ and $f(x_n) \to y$.

One of the fundamental results in the theory of nonexpansive mappings is Browder's demiclosedness principle [Browder, 1968], which states that if X is a uniformly convex Banach space, C is a closed convex set and T :C$\to$X is nonexpansive, then I -T is demiclosed. This principle is also seen to be valid in spaces satisfying Opial's condition [Opial, 1967]:

If $x_n \rightharpoonup x_0$ and $x \neq x_0$ then $\overline{lim}_n \|x_n - x_0\| < \overline{lim}_n \|x_n - x\|$.

Given a Banach space X and sequence (x_n) in X let

$$r_X(c\,;\,x_n) := \inf\{\underline{lim}_n \|x_n - x\| - 1 : \|x\| \geq c\}.$$

We say X has the *locally uniform Opial condition* [Lin, Tan and Xu, 1993] if

$$r_X(c\,;\,x_n) > 0 \quad \text{whenever } c > 0,\ \underline{lim}\|x_n\| \geq 1, \text{ and } x_n \rightharpoonup 0,$$

and the *uniform Opial condition* [Prus, 1992] if

$$r_X(c) := \inf\{\underline{lim}_n \|x_n + x\| - 1 : \|x\| \geq c,\ \underline{lim}\|x_n\| \geq 1 \text{ and } x_n \rightharpoonup 0\} > 0,$$
whenever $c > 0$.

We observe that in the definition of Opial's modulus r_X of X, "$\underline{lim}$" can be replaced by "$\overline{lim}$", that is

$$r_X(c) := \inf\{\overline{lim}_n \|x_n + x\| - 1 : \|x\| \geq c,\ \overline{lim}\|x_n\| \geq 1 \text{ and } x_n \rightharpoonup 0\}.$$

The norm of X is said to be UKK (uniformly Kadec-Klee) if given $\epsilon > 0$ there exists $\delta(\epsilon)$>0 such that if $\{x_n\}$ is a sequence in B_X converging weakly to x and such that $\text{sep}(x_n) := \inf\{\|x_n - x_m\| : n \neq m\} \geq \epsilon$, then $\|x\| \leq 1 - \delta(\epsilon)$.

Recently the demiclosedness of I -T at 0 for T of asymptotically nonexpansive type has been established by Xu [1991] when x is uniformly convex and for asymptotically nonexpansive maps by Lin and Xu [1993] when X is a Banach space with the locally uniform Opial condition, and hence when X is UKK (uniformly Kadec-Klee, [Huff, 1990]) with Opial's condition. Demiclosedness of I - T at 0 when T is of asymptotically nonexpansive type and X satisfies the Generalized Gossez-Lami Dozo property (GGLD) and an Opial's condition studied by Garcia-Falset, Sims and Smyth [1996].

Moreover, in [1995] Lin, Tan and Xu was proved that the uniform Opial condition implies the fixed point property for asymptotically nonexpansive mappings defined on weakly compact convex subset.

2. Demiclosedness principle.

In this section we prove the demiclosedness principle for mapping of asymptotically nonexpansive type either in a Banach space with the locally uniform Opial condition or in a Banach space satisfying Opial's condition and whose norm is UKK. The following lemma was proved by Garcia-Falset, Sims and Smyth [1996]. We now give the another proof.

Lemma 2.1 Suppose X is Banach space satisfying Opial's condition and C is weakly compact convex subset of X and T:C $\to$ C is a uniformly continuous mapping of asymptotically nonexpansive type. Suppose also $\{x_n\}$ is a sequence in C converges weakly to x and for which the sequence $\{x_n - Tx_n\}$ converges strongly to 0. Then $\{T^n x\}$ converges weakly to x.

Proof Showing the weakly convergence of $\{T^n x\}$ to x is equivalent to showing $\bigcap_{m=1}^{\infty} \overline{co}\{T^i x : i \geq m\} = \{x\}$. Let the functional f be defined by $f(y) = \overline{lim}_n \|x_n - y\|$, y$\in$X. If there exists $y_0 \in \bigcap_{m=1}^{\infty} \overline{co}\{T^i x : i \geq m\}$ such that $y_0 \neq x$, then by Opial's condition, $f(y_0) > f(x)$. Write R $:= f(y_0) - f(x)$. By the definition of asymptotically nonexpansive type, there exist $n_0 \in$ N such that

$$\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} < R/2 ,$$

for all $n \geq m_0$. Since $y_0 \in \overline{co}\{T^i x : i \geq m_0+1\}$, there exist an integer $p \geq 1$ and nonnegative numbers $t_1, t_2, ..., t_p$ with $\sum_{j=1}^{p} t_j = 1$ such that

$$\| y_0 - \sum_{j=1}^{p} t_j T^{m_0+j} x\| < R/2.$$

It follows that

$$\begin{aligned} f(y_0) &= \overline{lim}_n \|x_n - y_0\| \\ &\leq \overline{lim}_n (\|x_n - \textstyle\sum_{j=1}^{p} t_j T^{m_0+j} x\| + \| \sum_{j=1}^{p} t_j T^{m_0+j} x - y_0\|) \end{aligned}$$

$< R/2 + \overline{lim}_n\|\sum_{j=1}^{p} t_j x_n - \sum_{j=1}^{p} t_j T^{m_0+j}x\|$

$\le R/2 + \sum_{j=1}^{p} t_j[\overline{lim}_n\|x_n - T^{m_0+j}x_n\| + \overline{lim}_n\|T^{m_0+j}x_n - T^{m_0+j}x\|]$

$\le R/2 + \sum_{j=1}^{p} t_j[\overline{lim}_n(\sup\{\|T^{m_0+j}x_n - T^{m_0+j}x\| - \|x_n - x\|\})$

$+ \overline{lim}_n\|x_n - x\|]$

$\le R/2 + R/2 + f(x) = f(y_0).$

This contradiction shows that we must have $\bigcap_{m=1}^{\infty} \overline{co}\{T^i x : i \ge m\} = \{x\}$.

Theorem 2.2 Suppose X is a Banach space satisfying the locally uniform Opial condition, C is a nonempty weakly compact convex subset of X, and T:C $\to$ C is a uniformly continuous mapping of asymptotically nonexpansive type. Then $I - T$ is demiclosedness at zero.

Proof Suppose we are given a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$. Since T: C$\to$C is uniformly continuous, it follow that $x_n - T^m x_n \to 0$ for each fixed $m \in N$. By lemma 2.1, we have $T^n x \rightharpoonup x$. It follows from the definition of asymptotically nonexpansive type that , for each x

$$\overline{lim}_n[\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}] \le 0.$$

Then

$$\overline{lim}_m\, \overline{lim}_n\|T^n x - T^m x\| = \overline{lim}_m \overline{lim}_n\|T^m x - T^m(T^{n-m} x)\|$$

$$= \overline{lim}_m \overline{lim}_n(\|T^m x - T^m(T^{n-m}x)\| - \|x - T^{n-m}x\| + \|x - T^{n-m}x\|)$$

$$\le \overline{lim}_m(\sup\{\|T^m x - T^m(T^{n-m}x)\| - \|x - T^{n-m}x\|\}$$

$$+ \overline{lim}_n\|x - T^{n-m}x\|)$$

$$\le 0 + \overline{lim}_n\|x - T^n x\|.$$

Hence, by [Lin,Tan and Xu, Proposition 2.2 p 931], we have $T^n x \to x$. Therefore $Tx = x$ by the uniformly continuity of T. The proof is complete.

Theorem 2.3 Suppose X is a Banach space satisfying Opial's condition and whose norm is UKK and C is weakly compact convex subset of X, and T: C$\to$C is a uniformly continuous mapping of asymptotically nonexpansive type. Then I-T is demiclosed at zero.

Proof Suppose that $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$. It follows by Lemma 2.1 that $T^n x \rightharpoonup x$. Let

$$r = \lim_n\|T^n x - x\| \text{ and } r_m = \overline{\lim}_n\|T^n x - T^m x\| \text{ for all } m\geq 1.$$

By the definition of Opial's condition, we have $r \leq r_m$ for all $m\geq 1$. We now show that $\lim_{m\to\infty} r_m = r$.
Let $\epsilon > 0$. By the definition of asymptotically nonexpansive type, there exist $m_0 \in N$ such that, for each $n \geq m_0$,

$$\sup\{\|T^n x - T^n y\| - \|x - y\|: y\in C\} < \epsilon.$$

Thus, for all $m\geq m_0$, we obtain that

$$\begin{aligned} r_m &= \overline{\lim}_n\|T^n x - T^m x\| \\ &\leq \sup_{n\geq m}\{\|T^m(T^{n-m}x) - T^m x\| - \|T^{n-m}x - x\|\} + \overline{\lim}_n\|T^{n-m}x - x\|. \\ &< \epsilon + \overline{\lim}_n\|T^n x - x\| = \epsilon + r. \end{aligned}$$

Hence $\lim_{m\to\infty} r_m = r$. Suppose $r > 0$. Then $\{T^n x\}$ does not contain any strongly convergent subsequence and, therefore, $\{T^n x\}$ has a subsequence $\{T^{n_k}x\}$ such that $sep(T^{n_k}x) > 0$. Set $\epsilon_0 = sep(T^{n_k}x)/2r$. By the definition of UKK, there exist a $\delta_0 > 0$ such that $\|v\| \leq 1 - \delta_0$ for any sequence $\{v_n\}$ in B_X converging weakly to v and such that $sep(v_n) \geq \epsilon_0$. Choose $0< \eta <1$ such that $(1 + \eta)(1 - \delta_0) < 1$. Since T is an asymptotically nonexpansive type, there exist $N_1 \geq m_0$ such that

$$\sup\{\|T^n x - T^n y\| - \|x - y\|: y\in C\} < \eta r/2 \text{ , for all } n \geq N_1.$$

For $m \geq N_1$, we have

$$\begin{aligned} r_m &= \overline{\lim}_n\|T^n x - T^m x\| \\ &\leq \overline{\lim}_n[\|T^m x - T^m(T^{n-m}x)\| - \|x - T^{n-m}x\|]+\overline{\lim}_n\|T^{n-m}x - x\| \\ &\leq \sup\{\|T^m x - T^m u\| - \|x - u\|: u\in C\} + \overline{\lim}_n\|T^{n-m}x - x\|. \\ &< (\eta r/2) + \overline{\lim}_n\|T^n x - x\| \\ &< (\eta r/2) + r < (1 + \eta)r. \end{aligned}$$

It implies that there exist $j_0 \in N$ such that

$$\|T^{n_j}x - T^m x\| <(1 + \eta)r \text{ for all } j \geq j_0.$$

Let $y_j = (T^{n_j}x - T^m x)/(1 + \eta)r$ for all $j \geq j_0$. Then $\|y_j\| \leq 1$, $y_j \rightharpoonup (x - T^m x)/(1 + \eta)r$, and $sep(y_j) \geq \epsilon_0$. By the definition of UKK, we have

$$\|x - T^m x\| \leq (1 + \eta)(1 - \delta_0)r$$

for all $m \geq N_1$. By passing to the limit as $m\to \infty$, we obtain

$$r = \overline{\lim}_m\|x - T^m x\| \leq (1 + \eta)(1 - \delta_0)r.$$

This is a contradiction to the fact that $(1 + \eta)(1 - \delta_0)r < 1$. Hence $r = 0$, and therefore $x = Tx$ by the uniformly continuity of T.

3. Weak convergence of iterates.

In this section we investigate the asymptotic behavior of the iterates $\{T^n x\}$ for a mappings of asymptotically nonexpansive type T.

Theorem 3.1 Suppose X is a Banach space satisfying the uniform Opial's condition and C is weakly compact convex subset of X,and $T: C \to C$ is an asymptotically nonexpansive type. The given an converge to a fixed point of T if and only if T is weakly asymptotically regular at x, i.e.($T^n x - T^{n+1} x) \rightharpoonup 0$.

Proof Assume that $T^n x - T^{n+1} x \rightharpoonup 0$. Consider $\omega_w(x)$ the set of all weak subsequence limit of (x_n).
$\omega_w(x) := \{y \in X : T^{n_j} x \rightharpoonup y, \text{for some increasing subsequence} (n_j) \subseteq N\}$. We shall show that $\omega_w(x) \subseteq Fix(T)$. Let y be in $\omega_w(x)$. Then we have a subsequence $\{T^{n_j} x\}$ of $\{T^n x\}$ such that $T^{n_j} x \rightharpoonup y$. It follows by our hypothesis that, for all integers $m \geq 0$, $T^{n_j+m} x \rightharpoonup y$. Set $b_m = \overline{lim}_{j \to \infty} \|T^{n_j+m} x - y\|$. By the definition of Opial's condition, we have

$$\overline{lim}_{j \to \infty} \|T^{n_j+m+k} x - y\| \leq \overline{lim}_{j \to \infty} \|T^{n_j+m+k} x - T^k y\| \quad \text{for all } m, k \geq 0.$$

Let $\epsilon > 0$ and let $b = \inf\{b_m : m \geq 0\}$. Then there exists an $m_0 \in N$ such that $b_{m_0} < b + \epsilon/2$. Since T is a mapping of asymptotically nonexpansive type, there exist an integers $N_1 \geq m_0$ such that, for all $n \geq N_1$,

$$\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} < \epsilon/2.$$

This implies that, for all integers $j \geq N_1$,

$$\begin{aligned} b_{m_0+j} &\leq \overline{lim}_{j \to \infty} \|T^{n_j+m_0+k} x - T^k y\| \\ &\leq \sup\{\|T^k u - T^k y\| - \|u - y\| : u \in C\} + \overline{lim}_{\to \infty} \|T^{n_j+m_0} x - y\|. \\ &\leq \epsilon/2 + b_{m_0}. \end{aligned}$$

Hence $\lim_{m \to \infty} b_m = \inf\{b_m : m \geq 0\}$.
For all $m \geq N_1$, we note that

$$\begin{aligned} \|T^m x - y\| &\leq \overline{lim}_n \|T^m x - T^{n_j+2m} x\| + \overline{lim}_n \|T^{n_j+2m} x - y\| \\ &\leq \sup\{\|T^m y - T^m u\| - \|y - u\| : u \in C\} + \overline{lim}_{n \to \infty} \|y - T^{n_j+m} x\|) \\ &\quad + \overline{lim}_n \|T^{n_j+2m} x - y\| \\ &\leq \epsilon/2 + b_m + b_{2m}. \end{aligned}$$

If $b = 0$, then $\lim_{m \to \infty} \|T^m x - y\| = 0$, and therefore $Ty = y$ by continuity of

T^N for some N $\geq$1. Suppose now b>0. For m $\geq N_1$. Let
$z_n^{(m)} = (T^{n_j+m}x - y)/b_m$.

Then for each fixed m ≥ 0, $z_n^{(m)} \rightharpoonup 0$ and $\overline{lim}_n\|z_n^{(m)}\| = 1$. By the definition of Opial's modulus r_X of X, we obtain $\overline{lim}_n\|z_n^{(m)} - z\| \geq 1 + r_X(c)$ for all $z \in X$ with $\|z\| \geq c$.

Taking $z = (y\text{-}T^m y)/b_{2m}$, we get

$$\overline{lim}_n\|z_n^{(2m)} + z\| = \overline{lim}_n\|((T^{n_j+2m}x - y)/b_{2m}) + ((y\text{-}T^m y)/b_{2m})\|$$
$$\leq (1/b_{2m})[\sup\{\|(T^m u - T^m y\| - \|u-y\|: u \in C\}\}$$
$$+ \overline{lim}_n\|T^{n_j+m}x - y\|]$$
$$\leq (1/b_{2m})[\,\epsilon/2 + b_m]$$

for all $n \geq N_1$. It follows by the definition of the Opial modulus that $b_{2m}(r_X(\|(y\text{-}T^m y)/b_m\|) + 1) \leq \epsilon/2 + b_m$.

Taking the limit as $m \to \infty$ we get $b(r_X(\overline{lim}_m\|(y\text{-}T^m y)/b\|) + 1) \leq \epsilon/2 + b$ and $r_X(\overline{lim}_m\|(y\text{-}T^m y)/b\|) \leq \epsilon/2b$.

Since r_X is nondecreasing and continuous, we have $\overline{lim}_m\|y\text{-}T^m y\| = \epsilon/2$
Hence $T^m y \to y$, and therefore Ty = y by the continuity of T^N for some $N \geq 1$. Thus $\omega_w(x) \subseteq Fix(T)$.

4. Existence result.

In this section we show that the uniform Opial's condition implies the fixed point property for mappings of asymptotically nonexpansive type defined on weakly compact convex subsets. To prove the mean result we use the following lemmas.

Lemma 4.1 [Kim and Xu,1998] Let T be an asymptotically nonexpansive type on a nonempty weakly compact convex subset of a Banach space X. Then there are a closed convex nonempty subset K of C and $\rho \geq 0$ such that

(i) if $x \in K$, then every weak limit point of $\{T^n x\}$ is contained in K,

(ii) $\rho_x(y) = \rho$ for all $x,y \in K$, where ρ_x is the function define by

$$\rho_x(y) = \overline{lim}_n\|T^n x - y\|, \quad y \in X.$$

Lemma 4.2 Let C be a nonempty weakly compact convex subset of a Banach space X satisfying Opial's condition and let T be a mapping of asymptotically nonexpansive type on C. Let $\{ x_n \}$ be a sequence in C which satisfies the following condition

$$w\text{ - }\lim_n T^n x_n = z_m, \quad \forall\, m \geq 0.$$

Then $\quad \lim_n b_m = \inf\{b_m : m \geq 0\}$. where $b_m = \overline{lim}_n \|T^m x_n - z_m\|$.

Proof Let $\epsilon > 0$, and let $b = \inf\{b_m : m \geq 1\}$. Then there exist natural number m_0 such that $b_{m_0} - b < \epsilon/2$. By the definition of asymptotically nonexpansive type, there exist a natural number N such that

$$\sup\{\|T^n(z_{m_0}) - T^n y\| - \|z_{m_0} - y\| : y \in C\} < \epsilon/2\ , \forall\, n \geq N.$$

By the definition of Opial's condition, we have

$$\begin{aligned}
b_{m_0+j} &= \overline{lim}_n \|T^{m_0+j} x_n - z_{m_0+j}\| \\
&\leq \overline{lim}_n \|T^{m_0+j} x_n - T^j z_{m_0}\| \\
&\leq \overline{lim}_n (\|T^{m_0+j} x_n - T^j z_{m_0}\| - \|T^{m_0} x_n - z_{m_0}\| + \|T^{m_0} x_n - z_{m_0}\|) \\
&\leq \sup\{\|T^j u - T^j z_{m_0}\| - \|u - z_{m_0}\| : u \in C\} + \overline{lim}_n \|T^{m_0} x_n - z_{m_0}\| \\
&< (\epsilon/2) + b_m, \quad \text{for all } j \geq N.
\end{aligned}$$

This implies that $\|b - b_{m_0+j}\| < \epsilon \quad \forall\, j \geq N$.

Therefore $\lim_{m\to\infty} b_m = \inf\{b_m : m \geq 1\}$.

Theorem 4.3 Suppose X is a Banach space satisfying the uniform Opial condition, C is weakly compact convex subset of X, and T: $C \to C$ is an asymptotically nonexpansive type. Then T has a fixed point.

Proof Let K,ρ_x and ρ be as in lemma 4.1. Let $x \in K$ and let$\{T^{n_j} x\}$ be a weakly convergence subsequence of $\{T^n x\}$. Passing to subsequence, we may assume that $\{T^{n_j+m} x\}$ converge weakly for every $m \geq 0$, say

$$T^{n_j+m} x \rightharpoonup z_m. \text{ Let } b_m = \overline{lim}_j \|T^{n_j+m} x - z_m\|.$$

By lemma 4.2, $\{b_m\}$ converge to $b = \inf\{b_m : m \geq 0\} \geq 0$. We note by Lemma 4.1, that $z_m \in K$ for each $m \geq 0$. By weak lower semi-continuous of the norm implies

$$\|z_m - z_{m'}\| \leq \overline{lim}_{j\to\infty} \|z_m - T^{n_j+m'}\| \leq \rho, \quad \text{for all } m, m' \geq 0.$$

Hence, $\quad \operatorname{diam}(\{z_m : m \geq 0\}) \leq \rho$.

We claim that

(i).for any $\epsilon > 0$ there exist $y \in K$, $m' > 0$ and $N > 0$ such that

$$\|T^n y - z_{n+m'}\| \leq \epsilon, \quad \text{whenever } n > N, \text{and}$$

(ii). $\rho = 0$.

Case I. $\lim_{m\to\infty} b_m = 0$. For any $\epsilon>0$, there is $m'>0$ such thus if $m>m'$, then $b_m< \epsilon/3$. By the definition of asymptotically nonexpansive type, there exist $m_0\geq m'$ such that

$$\sup\{\|T^{n_j+m'+k}x - T^k z_{m'}\| - \|T^{n_j+m'}x - z_{m'}\|\}<\epsilon/3 \text{ for all } k\geq m_0.$$

Thus, for all $k\geq m_0$, we have

$$\|z_{m'+k} - T^k z_{m'}\| \leq \overline{lim}_j\|z_{m'+k} - T^{n_j+m'+k}x\| + \overline{lim}_j\|T^{n_j+m'+k}x - T^k z_{m'}\|$$

$$\leq b_{m'+k}+\sup\{\|T^{n_j+m'+k}x - T^k z_{m'}\|-\|T^{n_j+m'}x - z_{m'}\|\}$$

$$+\overline{lim}_{j\to\infty}\|T^{n_j+m'}x - z_{m'}\|$$

$$\leq b_{m'+k}+\epsilon/3 + b_{m'}< \epsilon.$$

If choose $y = z_{m'}$ and $N = 1$,Then (1) is hold in this case.

Case II. $\lim_{m\to\infty} b_m = b > 0$. If follow by the uniform Opial property of X that for any $\epsilon>0$ there is $\delta>0$ and an integer $N>1$ such that for all integer $m\geq N$ and $z\in X$,

$$\overline{lim}_{j\to\infty}\|T^{n_j+m}x - z\| \leq b + \delta \Rightarrow \|z - z_m\| \leq \epsilon. \quad (*)$$

Next, we choose an integer N so large that for all $m\geq N$,

$$b_m < b +\delta/2.$$

Since T is asymptotically nonexpansive type, there exist an integer $m_0\geq N$ such that

$$\sup\{\|T^n x - T^n y\|-\|x - y\|:y\in C\} < \delta/2, \text{ for all } n\geq m_0.$$

so if $k>m_0$, then

$$\overline{lim}_j\|T^{n_j+m_0+k}x - T^k z_{m_0}\| \leq \sup\{\|T^{n_j+m_0+k}x - T^j z_{m_0}\|-\|T^{n_j+m_0}x - z_{m_0}\|\}$$

$$+ \overline{lim}_j\|T^{n_j+m_0}x - z_{m_0}\|.$$

$$< \delta/2 + b_{m_0}< b + \delta.$$

If we choose $m'=m_0$ and $y=z_{m_0}$, then for any $j\geq N$, we have $\|T^j y - z_{m'+j}\| < \epsilon$ by (*).

Finally, it follows by use as argument as in the proof of Theorem 5.1[Lin,Tan and Xu, p 944] we can show that $\rho = 0$. Hence $K=\{x\}$ and $\lim_{n\to\infty}\|T^n x - x\|=0$. Therefore $Tx = x$ by continuity of T^N for some $N\geq1$.

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Fixed point theorems of asymptotically nonexpansive type mappings

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1. Introduction

Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping. Then T is said to be asymptotically nonexpansive [6] if there exists a sequence (k_n) of real numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \text{ for all } x, y \in C \text{ and } n = 1, 2, 3... \tag{1.1}$$

If (1.1) is valid for all $k_n = 1$, then T is said to be nonexpansive. If for each x in C, there holds the inequality

$$\limsup_n [\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}] \leq 0, \tag{1.2}$$

then T is said to be of asymptotically nonexpansive type [8].

In 1965, Kirk [7] proved that if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping T

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of C has a fixed point. A nonempty convex subset C of a normed linear space is said to have normal structure if each convex subset of C consisting of more than one point contains a nondiametral point. That is, a point $x \in K$ such that $\sup\{\|x-y\| : y \in K\} < \sup\{\|u-v\| : u, v \in K\} = \operatorname{diam} K$. Seven years later, in 1972, Goebel and Kirk [6] prove that if the space X is assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping T of C has a fixed point. This was extended to mappings of asymptotically nonexpansive type by Kirk in [8]. More recently these results have been extended to wider classes of space, see for example [3, 5, 9, 10, 16, 17]. In particular, Lin, Tan and Xu [10] have demomstrated the existence of fixed point for asymptotically nonexpansive mappings in Banach space with uniform Opial condition. It is know [11] that if the Maluta's constant $D(X) < 1$, then X is reflexive and has normal structure and hence the fixed point property for nonexpansive mappings. However, it is not clear if $D(X) < 1$ implies the fixed point property for asymptotically nonexpansive mappings. In 1994, Lim and Xu [9] proved two partial answers to this question.

The present paper answer a question raised by Lim and Xu in [9]. It extends results in their paper to mappings of asymptotically nonexpansive type. Precisely, we prove in section 2 the strong convergence (under certain assumptions) of an approximating fixed points for an asymptotically nonexpansive type mapping in a Banach space with a uniformly Gâteaux differentiable norm. Finally, we extends results in paper [10] to mappings of asymptotically nonexpansive type.

2. Maluta's constant.

Let X be a Banach space. Then recall that Maluta's constant $D(X)$ [11] of X is defined by

$$D(X) = \sup\{(\limsup_n d(x_{n+1}, co(x_1, ..., x_n))/diam(x_n)\} \tag{2.1}$$

where the supremum is taken over all bounded nonconstant sequence $\{x_n\}$ in X. Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each x and y in $S(X)$. It is also said to be uniformly Gâteaux differentiable if for each $y \in S(X)$, the limit (2.2) attained uniformly for x in $S(X)$. With each $x \in X$, we associate the set

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|\}.$$

Then $J : X \to X^*$ is said to be the duality mapping. It is well know if X is smooth, then the duality mapping J is single-valued and strong-*weak** continuous. It is also know that if X has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded sets when X has its strong topology while X^* has its weak star topology; see Diestel [4].

In this section we provide a fixed point theorem for asymptotically non-expansive type mappings which connect with Maluta's constant for a Banach space. Moreover, we prove the strong convergence of an approximating fixed points for an asymptotically nonexpansive type mapping in a Banach space X (whose norm is uniformly Gâteaux differentiable) such that $D(X) < 1$.

Theorem 2.1. Suppose that X is a Banach space such that $D(X) < 1$, that C is a nonempty bounded subset of X, and $T : C \to C$ is an asymptotically nonexpansive type mapping such that T is continuous on C. Further, suppose that there exist a nonempty closed convex subset K of C with the following property (ω):

$$x \in K \text{ implies } \quad \omega_w(x) \subset K,$$

where $\omega_w(x)$ is the weak ω-limit set of T at x; that is, the set

$$\{y \in X : y = weak - \lim_i T^{n_i}x \text{ for some } n_i \uparrow \infty\}.$$

Then T has a fixed point in E.

To prove the theorem we use the following two lemmas.

Lemma 2.1. [17]. Let C be a nonempty subset of a Banach space X and let T be a mapping of asymptotically nonexpansive type on C. Suppose there exists a nonempty bounded closed convex subset E of C with the property (ω). Then there is a closed convex subset K of C and a $\rho \geq 0$ such that:

(*i*) if $x \in K$, then every weak limit of $\{T^n x\}$ is contained in K;

(*ii*) $\rho_x(y) = \rho$ for all $x, y \in K$ where ρ_x is the functional defined by

$$\rho_x(y) = \limsup_{n\to\infty} \|T^n x - y\|, \quad y \in X. \tag{2.3}$$

Lemma 2.2. Suppose that X is a Banach space such that $D(X) < 1$, that K is closed bounded convex subset of X, and that $T : K \to K$ is an asymptotically nonexpansive type mapping. If $\{T^n x\}$ is a sequence in K converging weakly to $x \in K$, then $\{T^n x\}$ converges (strongly) to x.

Proof. Assume that $\{T^n x\}$ does not converges to x. Then there is a subsequence $\{T^{n_k} x\}$ of $\{T^n x\}$ such that

$$\limsup_{k\to\infty} \|T^{n_k} x - x\| > 0.$$

Take a real number $q > 0$ small enough so that

$$0 < q < \limsup_{k\to\infty} \|T^{n_k} x - x\| \quad \text{and} \quad (1+q)D(X) < 1.$$

It then follow from the definition of $D(X)$ that

$$\limsup_{k\to\infty} \|T^{n_k} x - x\| \leq D(X) diam(\{T^{n_k} x\}).$$

By the definition of asymptotically nonexpansive type, there exists a natural number N such that

$$\sup(\{\|T^n x - T^n y\| - \|x - y\| : y \in K\}) < q^2/2,$$

for all $n \geq N$. Now we show that $\limsup_k \|T^{n_k} - x\| = 0$. Clearly we may assume that

$$\sup(\{\|T^{n_k} x - T^{n_k} y\| - \|x - y\| : y \in K\}) < q^2/2,$$

for all $k \geq 1$. However, for any fixed $i > j$, notting the fact that $T^{n_k+(n_i-n_j)}x \to x$ weakly as $k \to \infty$ and the weakly lower semi-continuous of the norm $\|\cdot\|$, we have

$$\begin{aligned}
\|T^{n_i} x - T^{n_j} x\| &= \|T^{n_j} x - T^{n_j}(T^{n_i - n_j} x)\| \\
&\leq \sup(\{\|T^{n_j} x - T^{n_j} y\| - \|x - y\| : y \in K\}) + \|T^{n_i - n_j} x - x\| \\
&\leq q^2/2 + \limsup_{k\to\infty} \|T^{n_i - n_j} x - T^{n_k + (n_i - n_j)} x\| \\
&< q^2/2 + \limsup_{k\to\infty} (\|T^{n_i - n_j} x - T^{n_i - (n_j} (T^{n_k} x)\| - \|T^{n_k} x - x\|) \\
&\quad + \limsup_{k\to\infty} \|T^{n_k} x - x\|
\end{aligned}$$

$$< (1+q)\limsup_{k\to\infty}\|T^{n_k}x - x\|.$$

We thus obtain

$$\limsup_k \|T^{n_k}x - x\| < (1+q)D(X)\limsup_k \|T^{n_k}x - x\|$$

which implies that $\limsup_{k\to\infty}\|T^{n_k}x - x\| = 0$ since $(1+q)D(X) < 1$. Therefore $\{T^n x\}$ converges strongly to x. □

Proof of theorem 2.1. Let K, ρ_x and ρ be as in lemma 2.1. Let $\Im$ be a free Ultrafilter on the set of positive integer. We then define a mapping S on K by

$$S(x) = w - \lim_{\Im} T^n x, \quad x \in K.$$

Since K is weakly compact, $S(x)$ is well define for all $x \in K$. By the definition of asymptotically nonexpansive type, we obtain S is nonexpansive mapping on K. Hence, S has a fixed point $x \in K$, that is,

$$w - \lim_{\Im} T^n x = x. \tag{2.4}$$

This yields a subsequence $T^{n_i}x$ of $T^n x$ converge weakly to x. Now we show that x is a fixed point of T. Passing to subsequences and using the diagonal method, we may assume that $\{T^{n_i+m}x\}$ converges weakly to every $m \geq 0$, say $w - \lim_{i\to\infty} T^{n_i+m}x = z_m$. By lemma 2.2, we have $\lim_{i\to\infty}\|T^{n_i+m}x - z_m\| = 0$ for all $m \geq 0$. We note by lemma 2.1 that $z_m \in K$ each $m \geq 0$. By weak lower semi-continuous of the norm $\|\cdot\|$ implies

$$\|z_m - z_{m'}\| \leq \liminf_{i\to\infty}\|z_m - T^{n_i+m'}x\|$$

$$\leq \limsup_{i\to\infty}\|z_m - T^{n_i+m'}x\| \leq \rho_x$$

for all $m, m' \geq 0$. Hence, $\operatorname{diam}(\overline{co}\{z_m : m \geq 0\}) \leq \rho$. We claim that;

(1) for any $\epsilon > 0$, there exists $y \in K, m' > 0$ and $N > 0$ such that $\|T^n y - z_{n+m'}\| \leq \epsilon$ whenever $n > N$;

(2) $\rho = 0$.

To prove (1). For any $\epsilon > 0$, there is $m' > 0$ such that if $n \geq m'$, then

$$\sup\{\|T^n x - T^n u\| - \|x - u\| : u \in K\} < \epsilon.$$

If we choose $N = m'$ and $y = z_{m'}$, then for any $j \geq N$, we have

$$\|z_{m'+j} - T^j y\| \leq \limsup_{i\to\infty}\|z_{m'+j} - T^{n_i+m'+j}x\|$$

$$+ \limsup_{i\to\infty} \|T^{n_i+m'+j}x - T^j y\|$$

$$\leq \limsup_{i\to\infty}(\|T^{n_i+m'+j}x - T^j y\| - \|T^{n_i+m'}x - z_{m'}\|)$$

$$+ \limsup_{i\to\infty} \|T^{n_i+m'}x - z_{m'}\|$$

$$< \epsilon.$$

To prove (2), we distinguish two cases.

Case I. There is $N_0 > 0$ such that $\mathrm{diam}(\overline{co}\{z_m : m \geq N_0\}) = \rho' < \rho$. By (1), there are $y \in K, m', N \in \mathbb{N}$ such that $\|T^n y - z_{n+m'}\| \leq (\rho - \rho')/2$, for all $n > N$. So if $n > \max\{N, N_0\}$, then

$$\|z_{N_0} - T^n y\| \leq \|z_{N_0} - z_{n+m'}\| + \|z_{n+m'} - T^n y\| \leq (\rho + \rho')/2.$$

Case II. diam $(\overline{co}\{z_m : m \geq N\}) = \rho$ for all $N \in \mathbb{N}$. Since X has normal structure and K is weakly compact, K has normal structure. Hence, there exists $z_0 \in \overline{co}\{z_m : m \in \mathbb{N}\}$, such that

$$\rho' = \sup_{m\in\mathbb{N}} \|z_0 - z_m\| < diam(\overline{co}\{z_m : m \in \mathbb{N}\}) = \rho.$$

By (1), there are $y \in K, m', N \in \mathbb{N}$ such that $\|T^n y - z_{n+m'}\| \leq (\rho - \rho')/2$ whenever $n \geq N$. So if $n > N$, then

$$\|z_0 - T^n y\| \leq \|z_0 - z_{n+m'}\| + \|z_{n+m'} - T^n y\| \leq (\rho + \rho')/2.$$

This prove (2).
By (2), $\lim_{n\to\infty} \|T^n x - x\| = 0$. Therefore, $Tx = x$ by continuity of T. □

As a direct consequence of Theorem 2.1 we have the following:

Corollary 2.1. Let C and X be as in Theorem 2.1 and let $T : C \to C$ be an asymptotically nonexpansive mappings. Suppose there exists a nonempty bounded closed convex subset of K of C with the property(ω). Then T has a fixed point in K.

Corollary 2.2. Let X be a Banach space such that $D(X) < 1$, let C be a bounded closed convex subset of X, and suppose $T : C \to C$ is a continuous mappings of asymptotically nonexpansive type. Then T has a fixed point.

Suppose now C is a bounded closed convex subset of a Banach space X and $T : C \to C$ is an asymptotically nonexpansive mapping (we may always asumme $k_n \geq 1$ for all $n \geq 1$). For any $n \geq 1$, we lake $t_n = \min\{1 - (k_n - 1)^{\frac{1}{2}}, 1 - \frac{1}{n}\}$. Fix a u in C and define for each integer $n \geq 1$ the contraction $S_n : C \to C$ by

$$S_n(x) = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x.$$

Then the Banach Contraction Principle yields a unique point $x_n \in C$ that is fixed by S_n, that is, we have

$$x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x_n. \tag{2.5}$$

Theorem 2.2. Let X be a Banach space with a uniformly Gâteaux differentiable norm such that $D(X) < 1$. Suppose in addition the following condition

$$\lim_n \|x_n - Tx_n\| = 0 \tag{2.6}$$

holds. Then the sequence x_n defined by (2.5) converges strongly to a fixed of T.

Proof. Suppose that the sequence $\{x_n\}$ defined by (2.5). From corollary 2.1, the fixed point set $F(T)$ of T is nonempty. We now show that (x_n) converge strongly to a fixed point of T. Now let μ be a Banach limit and define $f : C \to [0, \infty)$ by

$$f(z) = \mu_n\|x_n - z\| \quad \text{for every } z \in C.$$

Then, since the function f on C is convex and continuous, $f(z) \to \infty$ as $\|z\| \to \infty$, and X is reflexive it follows from [1, p. 79] that there exists $u \in C$ with $f(u) = \inf_{z\in C} f(z)$. Define the set

$$M = \{v \in C : f(v) = \inf_{z\in C} f(z)\}.$$

Then M is a nonempty, closed and convex. We further claim that M has the property (ω). If x is in M and $y = w - \lim_j T^{m_j}x$ belong to the weak ω-limit set $\omega_w(x)$ of T at x, then from the weakly lower semicintinuous of f, we have

$$f(y) = \liminf f(T^{m_j}x) \leq \limsup_{m\to\infty} f(T^m x)$$

$$\leq \limsup_m(\mu_n\|x_n - T^m x\|).$$

Since T is uniformly continuous it follows from $x_n - Tx_n \to 0$ that $x_n - T^m x_n \to 0$ for each fixed $m \in \mathbb{N}$. Thus

$$f(z) \leq \limsup_m(\mu_n\|T^m x_n - T^m x\|)$$

$$\leq \limsup_m(k_m\mu_n\|x_n - x\|) = \mu_n\|x_n - x\|$$

$$= \inf_{z\in C} f(z) \leq f(y).$$

This show that y belongs to M and hence M satisfies the property (ω). It follows from corollary 2.1, that T has a fixed $z_0 \in M$. Next, to show that (x_n) converges strongly to a fixed point of T. We note that, for any $w \in F(T)$,

$$\langle x_n - T^n x_n, J(x_n - w)\rangle = \langle x_n - w, J(x_n - w)\rangle + \langle w - T^n x_n, J(x_n - w)\rangle$$

$$\geq \|x_n - w\|^2 - \|w - T^n x_n\|\|x_n - w\|$$

$$\geq -(k_n - 1)\|x_n - w\|^2$$

$$\geq -(k_n - 1)d^2$$

where $d = \operatorname{diam} C$. Since x_n is a fixed point of S_n, it follows that

$$x_n - T^n x_n = \frac{k_n - t_n}{t_n}(u - x_n)$$

and from last inequality above,we get

$$\langle x_n - u, J(x_n - w)\rangle \leq s_n d^2, \tag{2.7}$$

where $s_n = \frac{t_n(k_n-1)}{(k_n-t_n)} \to 0$ as $n \to \infty$. So, putting $w = z_o$, we have

$$\langle x_n - u, J(x_n - z_0)\rangle \leq s_n d^2. \tag{2.8}$$

On the other hand, since z_0 is the minimizer of the function f on C, by [14, Lemma 3], we have

$$\mu_n\langle z - z_0, J(x_n - z_0)\rangle \leq 0$$

for all $z \in C$. In particular, we have

$$\mu_n\langle u - z_0, J(x_n - z_0)\rangle \leq 0. \tag{2.9}$$

Combining (2.8) and (2.9), we get

$$\mu_n\langle x_n - z_0, J(x_n - z_0)\rangle = \mu_n\|x_n - z_0\|^2 \leq 0.$$

Therefore, there is a subsequence (x_{n_k}) of (x_n) which converges strongly to z_0. To show that (x_n) converges strongly to a fixed point of T, let (x_{n_j}) and

(x_{m_j}) be subsequences of (x_n) such that $x_{n_j} \to z$ and $x_{m_j} \to z'$. Then z and z' are fixed points of T by hypothesis (2.6). It follows from (2.7) that

$$\langle z - u, J(z - z')\rangle \leq 0$$

and

$$\langle z' - u, J(z' - z)\rangle \leq 0.$$

Adding these two inequalities yields

$$\langle z - z', J(z - z')\rangle = \|z - z'\|^2 = 0.$$

So we have $z = z'$. Therefore (x_n) converges strongly to a fixed point of T.□

3. Uniform Opial condition.

A Banach space X is said to satisfy Opial's condition [12] if each sequence $\{x_n\}$ in X the condition $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{3.1}$$

for all $y \neq x$. A Banach space X is said to satisfy the uniform Opial condition [13] if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x + x_n\| \tag{3.2}$$

for each $x \in X$ with $\|x\| \geq c$ and each $\{x_n\}$ in X such that $w - \lim x_n = 0$ and $\liminf_{n\to\infty} \|x_n\| \geq 1$. We now define Opial's modulus of X, denote by r_X, as follows

$$r_X(c) = \inf\{\liminf_{n\to\infty} \|x + x_n\| - 1\}, \tag{3.3}$$

where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and sequence $\{x_n\}$ in X such that $w - \lim x_n = 0$ and $\liminf_{n\to\infty} \|x_n\| \geq 1$. It is easy to see that the function r_X is nondecreasing and that X satisfies the uniform Opial condition if and only if $r_X(c) > 0$ for all $c > 0$.

In this section we show that the uniform Opial condition implies the fixed point property for mappings of asymptotically nonexpansive type defined on weakly compact subset.

Theorem 3.1. Suppose that X is a Banach space satisfying the uniform Opial condition, C is a nonempty weakly compact convex subset of X, and

$T : C \to C$ is an asymptotically nonexpansive type mapping such that T is continuous on C. Then T has a fixed point.

To prove the theorem we use the following lemma.

Lemma 3.1. Let C be a nonempty weakly compact convex subset of a Banach space X satisfying Opial's condition and let T be a mapping of asymptotically nonexpansive type on C. Let $\{x_n\}$ be a sequence in C which satisfies the following condition

$$w - \lim_n T^m x_n = z_m, \quad \text{for all } m \geq 0.$$

Then $\lim_m b_m = \inf\{b_m : m \geq 0\}$, where $b_m = \limsup_n \|T^m x_n - z_m\|$.
Proof Let $\epsilon > 0$, and let $b = \inf\{b_m : m \geq 1\}$. Then there exist natural number m_0 such that $b_{m_0} - b < \epsilon/2$. By the definition of asymptotically nonexpansive type, there exist a natural number N such that

$$\sup\{\|T^n(z_{m_0}) - T^n y\| - \|z_{m_0} - y\| : y \in C\} < \epsilon/3, \forall\, n \geq N.$$

Using Opial's condition, we have for $j \geq N$,

$$\begin{aligned} b_{m_0+j} &= \limsup_n \|T^{m_0+j} x_n - z_{m_0+j}\| \\ &\leq \limsup_n \|T^{m_0+j} x_n - T^j z_{m_0}\| \\ &\leq \limsup_n (\|T^{m_0+j} x_n - T^j z_{m_0}\| - \|T^{m_0} x_n - z_{m_0}\| + \|T^{m_0} x_n - z_{m_0}\|) \\ &\leq (\epsilon/3) + \limsup_n \|T^{m_0} x_n - z_{m_0}\| \\ &< (\epsilon/2) + b_m. \end{aligned}$$

This implies that $b_{m_0+j} - b < \epsilon$, for all $j \geq N$.
Therefore $\lim_{m\to\infty} b_m = \inf\{b_m : m \geq 0\}$. □

Proof of theorem 3.1 Let K, ρ_x and ρ be as in lemma 3.1. Let $x \in K$ and let $\{T^{n_j} x\}$ be a weakly convergence subsequence of $\{T^n x\}$. Passing to subsequence, we may assume that $\{T^{n_j+m} x\}$ converge weakly for every $m \geq 0$, say $w - \lim_n T^m x_n = z_m$. Let $b_m = \limsup_j \|T^{n_j+m} x - z_m\|$. By lemma 3.1, $\{b_m\}$ converge to $b = \inf\{b_m : m \geq 0\} \geq 0$. We note by lemma 2.1, that $z_m \in K$ for each m ≥ 0. By weak lower semi-continuous of the norm implies

$$\|z_m - z_{m'}\| \le \limsup_{j\to\infty} \|z_m - T^{n_j+m'}x\| \le \rho,$$

for all $m, m' \ge 0$. Hence, $diam(\{z_m : m \ge 0\}) \le \rho$. We claim that:

($*$) for any $\epsilon > 0$ there exist $y \in K$, $m' > 0$ and $N > 0$ such that $\|T^n y - z_{n+m'}\| \le \epsilon$ whenever $n > N$.

To prove our claim, we distinguish two cases.

Case I. $\lim_{m\to\infty} b_m = 0$. For any $\epsilon > 0$, there is $m' > 0$ such thus if $m > m'$, then $b_m < \epsilon/3$. By the definition of asymptotically nonexpansive type, there exist $m_0 \ge m'$ such that

$$\sup\{\|T^{n_j+m'+k}x - T^k u\| - \|T^{n_j+m'}x - u\| \ : u \in K\} < \epsilon/3,$$

for all $k \ge m_0$. Thus for $k \ge m_0$,

$$\begin{aligned}
\|z_{m'+k} - T^k z_{m'}\| &\le \limsup_j \|z_{m'+k} - T^{n_j+m'+k}x\| \\
&\quad + \limsup_j \|T^{n_j+m'+k}x - T^k z_{m'}\| \\
&\le b_{m'+k} + \limsup_j \{\|T^{n_j+m'+k}x - T^k z_{m'}\| - \|T^{n_j+m'}x - z_{m'}\|\} \\
&\quad + \limsup_{j\to\infty} \|T^{n_j+m'}x - z_{m'}\| \\
&\le b_{m'+k} + \epsilon/3 + b_{m'} < \epsilon.
\end{aligned}$$

If we choose $y = z_{m'}$ and $N = 1$, then ($*$) is holds in this case.

Case II. $\lim_{m\to\infty} b_m = b > 0$. It follows by the uniform Opial property of X that for any $\epsilon > 0$ there is $\delta > 0$ and an integer $N > 1$ such that for all integer $m \ge N$ and $z \in X$,

$$\limsup_{j\to\infty} \|T^{n_j+m}x - z\| \le b + \delta \Rightarrow \|z - z_m\| \le \epsilon. \tag{3.4}$$

We may ssume that N is chosen so large that for all $m \ge N$,

$$b_m < b + \delta/2.$$

Since T is asymptotically nonexpansive type, there exist an integer $m_0 \ge N$ such that

$$\sup\{\|T^n x - T^n u\| - \|x - u\| : u \in C\} < \delta/2,$$

for all $n \geq m_0$. So if $k > m_0$, then
$\limsup_{j\to\infty} \|T^{n_j+m_0+k}x - T^k z_{m_0}\|$

$$\leq \limsup_{j\to\infty}\{\|T^{n_j+m_0+k}x - T^j z_{m_0}\| - \|T^{n_j+m_0}x - z_{m_0}\|\}$$

$$+ \limsup_{j\to\infty} \|T^{n_j+m_0}x - z_{m_0}\|.$$

$$< \delta/2 + b_{m_0} < b + \delta.$$

If we choose $m' = m_0$ and $y = z_{m_0}$, then for any $j \geq N$, we have $\|T^j y - z_{m'+j}\| \leq \epsilon$ by (3.4). This proves $(*\ )$.

Finally, it follows by use as argument as in the proof of Theorem 2.1 we can show that $\rho = 0$. Hence $\lim_{n\to\infty} \|T^n x - x\| = 0$. Therefore $Tx = x$ by continuity of T . □

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Fixed point theorems in spaces with a weakly continuous duality map

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1. Introduction.

In 1965, Kirk [5] proved that if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping T of C has a fixed point. A nonempty convex subset C of a normed linear space is said to have normal structure if each convex subset of C consisting of more than one point contains a nondiametral point. That is, a point $x \in K$ such that $\sup\{\|x-y\| : y \in K\} < \sup\{\|u-v\| : u, v \in K\} = \operatorname{diam} K$. Seven years later, in 1972, Goebel and Kirk [4] prove that if the space X is assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping T of C has a fixed point. In 1994,Lim and Xu [7] verify that the existence and weak convergence of fixed point of asymptotically nonexpansive mapping in a space with a weakly continuous duality map. However, whether a weakly continuous duality map implies the existence of fixed points for mappings of asymptotically nonexpansive type is a natural quesion that remains open.

In this paper we present the existence and weak convergence of fixed point of asymptotically nonexpansive type mapping in a space with weakly continuous duality map. Precisely, we prove the strong convergence (under certain assumption) of an approximating fixed point for as asymptotically nonexpansive mapping in space with uniformly Gâteaux differentiable norm.

2. Preliminaries.

Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping. Then T is said to be asymptotically nonexpansive [4] if there exists a sequence (k_n) of real numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x-y\|, \text{ for all } x, y \in C \text{ and } n = 1,2,3... \tag{2.1}$$

If (1.1) is valid for all $k_n = 1$, then T is said to be nonexpansive. If for each x in C, and T^N is continuous for some $N \geq 1$ and there holds the inequality

$$\limsup_n [\sup\{\|T^n x - T^n y\| - \|x-y\| : y \in C\}] \leq 0, \tag{2.2}$$

then T is said to be of asymptotically nonexpansive type [6].

Let μ be mean on positive integers N, i.e. a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(a) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, ...) \in l^\infty$. According to time and circumstance, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, ...) \in l^\infty$.Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the exists of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, ...) \in l^\infty$. So, if $a = (a_1, a_2, ...) \in l^\infty$ and $a_n \to c$, as $n \to \infty$ we have $\mu_n(a_n) = \mu(a) = c$.

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each x and y in $S(X)$. It is also said to be uniformly Gâteaux differentiable if for each $y \in S(X)$, the limit (2.2) attained uniformly for x in $S(X)$.

Recall that a Banach space X is said to satisfy Opial's condition [8] if, for any sequence $\{x_n\}$ in X, the condition that $\{x_n\}$ converges weakly to $x \in X$ implies that

$$\limsup_n \|x_n - x\| < \limsup_n \|x_n - y\|$$

for all $y \in X, y \neq x$. It has been proven that Opial's condition implies weakly normal structure and, hence, the fixed point property for nonexpansive mappings. And Opial's condition implies the fixed point property for asymptotically nonexpansive mappings provide by Lim and Xu in 1994.However, it is not clear whether Opial's condition implies the fixed point property for asymptotically nonexpansive type mappings. Theorem 3 of section 3 will provide a partial answer to this question.

By a gauge we mean a continuous strictly increasing function φ defined $R^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. We associate with a gauge φ a (generally multivalued) duality map $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) = \{x^* \in X^* :< x, x^* >= \|x\|\varphi(\|x\|) and \|x^*\| = \varphi(\|x\|)\}.$$

Clearly the (normalized) duality map J corresponds to the gauge $\varphi(t) = t$. Browder [1] initiated the study of certain classes of nonlinear operators by means of a duality map J_φ. Set for $t \geq 0$,

$$\Phi(t) = \int_0^t \varphi(r)dr.$$

Then it is known that $J_\varphi(x)$ is the convex function $\Phi(\|\cdot\|)$ at x. Now recall that X is said to have a weakly continuous duality map if there exists a gauge φ such that the duality map J_φ is single-valued and continuous from X with the weak topology to X^* with the weak* topology. A space with a weakly continuous duality map is easily seen to satisfy Opial's condition (*cf*.[1]). Every $l^p(1 < p < \infty)$ space has a weakly continuous duality map with the gauge $\varphi(t) = t^{p-1}$.

Lemma 1 [10, lemma 1] Let C be a nonempty weakly compact convex subset of Banach space X. For each x in closed convex nonempty subset K of C, defined the functional

$$\rho_x(y) = \limsup_{n\to\infty} \|T^n x - y\|, \quad y \in X.$$

Then the function $\rho_x(\cdot)$ is a constant on K and this constant is independent of $x \in K$.

Lemma 2 [10, lemma 2] Let C be a nonempty weakly compact convex subset of Banach space X and let $T : C \to C$ be a mapping of asymptotically nonexpansive type such that T^N is continuous for some integer $N \geq 1$.

Suppose there exist a closed convex nonempty subset K of C which has properties:

(i) $x \in K \Rightarrow \omega_w(x) \subseteq K$

(ii) for each x in K and each subsequence $\{n_i\}$ of the positive integers $\{n\}$,$\{T^{n_i}x\}$ admits a norm-convergent subsequence; and

(iii) for each $x \in K$, $\omega(x)$ is norm-compact, where $\omega(x) = \{y \in X : y = \|\cdot\| - \lim_{i\to\infty} T^{n_i}x$ for some $n_i \uparrow \infty\}$ is the ω-set of T at x. Then T has a fixed point.

3. Main result.

This section we prove that the existence and weak convergence of fixed point of asymptotically nonexpansive type mapping in a space with weakly continuous duality map. Precisely, we prove the strong convergence (under certain assumption) of an approximating fixed point for as asymptotically nonexpansive mapping in space with uniformly Gâteaux differentiable norm.

Theorem 3 Suppose X is a Banach space with a weakly continuous duality map J_φ, C is a weakly compact convex subset of X, and $T : C \to C$, is an asymptotically nonexpansive type. Then we have following conditions:

(i) T has a fixed point in C, and

(ii) if T is weakly asymptotically regular at $x \in C$ that is $w-\lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$, then $\{T^n x\}$ converges weakly to a fixed point of T.

Proof. (i) Let F be the family of subset K of C which are nonempty, closed, convex, and satisfy the following property,

$$(\omega) \quad x \in K \text{ implies } \omega_w(x) \subseteq K.$$

F is then ordered by inclusion. The weak compactness of C now allows one to use Zorn's lemma to obtain a minimal element say K in F. We defined $r : X \to \mathbf{R}$, by for each $x \in C$

$$r_x(y) = \limsup_{n\to\infty} \|T^n x - y\|.$$

Then by lemma 1,2(i), when x lies in K, r_x is a constant over $y \in K$ and this constant is independent of $x \in K$; that is

$$\limsup_{n\to\infty} \|T^n x - y\| = r \quad \text{for all } \ x, y \in K.$$

Now fixed $x \in K$ and $\{T^{n_i}x\}$ be a subsequence of $\{T^n x\}$ converging weakly to same $y \in K$ by property (ω) and such that $\limsup_{i\to\infty} \|T^{n_i}x - y\|$ exists. Say $r' = \limsup_{i\to\infty} \|T^{n_i}x - y\|$. For any integers $n, m \geq 1$, noting the identity

$$\Phi(\|x+y\|) = \Phi(\|x\|) + \int_0^1 < y, J_\varphi(x+ty) > dt$$

for all $x, y \in X$.

Claim that $r' = 0$. We must show that $\Phi(r') = 0$.
Consider,
$\Phi(\|T^n x - T^m x\|) = \Phi(\|(T^n x - y) + (y - T^m x)\|)$

$$= \Phi(\|T^n x - y\|) + \textstyle\int_0^1 < (y - T^m x), J_\varphi((T^n x - y) + t(y - T^m x)) > dt$$

For subsequence n_i of n letting $i \to \infty$, we get
$\lim_{i\to\infty} \Phi(\|T^{n_i}x - T^m x\|) = \Phi(r') + \int_0^1 < (y - T^m x), J_\varphi(t(y - T^m x)) > dt$

$$= \Phi(r') + \textstyle\int_0^1 \|y - T^m x\|\varphi(t\|y - T^m x\|)dt$$

$$= \Phi(r') + \Phi(\|y - T^m x\|)$$

It follows that
$\limsup_{m\to\infty}(\lim_{i\to\infty} \Phi(\|T^{n_i}x - T^m x\|)) = \limsup_{m\to\infty}(\Phi(r') + \Phi(\|y - T^m x\|))$

$$= \limsup_{m\to\infty} \Phi(r') + \limsup_{m\to\infty} \Phi(\|y - T^m x\|)$$

$$= \Phi(r') + \Phi(r)$$

Then
$\Phi(r') + \Phi(r) = \limsup_{m\to\infty}(\lim_{i\to\infty} \Phi(\|T^{n_i}x - T^m x\|))$

$$\leq \limsup_{m\to\infty}(\limsup_{n\to\infty} \Phi(\|T^n x - T^m x\|))$$

$$\leq \limsup_{m\to\infty}(\limsup_{n\to\infty} \Phi(\|T^m x - T^m(T^{n-m}x)\| - \|x - T^{n-m}x\| + \|x - T^{n-m}x\|)$$

$$= \limsup_{m\to\infty}(\Phi(\limsup_{n\to\infty}(\|T^m x - T^m(T^{n-m}x)\| - \|x - T^{n-m}x\| + \|x - T^{n-m}x\|)))$$

$$\leq \limsup_{m\to\infty}(\Phi(\limsup_{n\to\infty}(\|T^m x - T^m(T^{n-m}x)\| - \|x - T^{n-m}x\|) + \limsup_{n\to\infty} \|x - T^{n-m}x\|))$$

$$\leq \Phi(\limsup_{m\to\infty} \sup\{\|T^m x - T^m(T^{n-m}x)\| - \|x - T^{n-m}x\|\} + \limsup_{m\to\infty} \limsup_{n\to\infty} \|x - T^{n-m}x\|)$$

$\leq \Phi(0 + \limsup_{m\to\infty} \limsup_{n\to\infty} \|x - T^n x\|)$

$= \Phi(\limsup_{n\to\infty} \|x - T^n x\|) = \Phi(r)$

Thus $\Phi(r') + \Phi(r) \leq \Phi(r)$. Hence $\Phi(r') = 0 \Rightarrow r' = 0$. Then $\{T^n x\}$ strongly convergence to y.

This proves that, for each $x \in K$, the strong ω -limit set $\omega(x) := \{y \in X : y - \text{strong} - \lim_i T^{n_i} x \text{ for some } n_i \uparrow \infty\}$ of T at x is nonempty. It is clearly closed. We further claim that $\omega(x)$ is norm-compact. In fact, given any sequence $\{u_j\}$ in $\omega(x)$. It is easy to construct a subsequence $\{T^{m_j} x\}$ of $\{T^n x\}$ such that $\|T^{m_j} x - u_j\| < \frac{1}{j}$ for all $j \geq 1$. Repeating the argument above, we get a subsequence $\{T^{m_{j'}} x\}$ of $\{T^{m_j} x\}$ converging strongly to some $z \in \omega(x)$. Hence, $u_j \to z$ strongly indicating the norm-compactness of $\omega(x)$. Now bylemma 2(iii), T has a fixed point and (i) is thus proven.

Now, we turn to proof of (ii). First observe that for any $p \in F(T)$, the $\lim_{n\to\infty} \|T^n x - p\|$ exist.

In fact, for all integers $n, m \geq 1$, we have

$\|T^{n+m} x - p\| \leq \limsup_{n\to\infty} \|T^{n+m} x - p\|$

$= \limsup_{n\to\infty}(\|T^n(T^m x) - T^n p\| - \|T^m x - p\| + \|T^m x - p\|)$

$\leq \limsup_{n\to\infty}(\sup\{\|T^n u - T^n p\| - \|u - p\| : u \in C\})$

$+ \limsup_{n\to\infty} \|T^m x - p\|$

$\leq \limsup_{n\to\infty} \|T^m x - p\| = \|T^m x - p\|$

It follows that for all integers $m \geq 1$,

$$\limsup_{n\to\infty} \|T^n x - p\| = \limsup_{n\to\infty} \|T^{n+m} x - p\| \leq \|T^m x - p\|$$

Which implies that $\limsup_{n\to\infty} \|T^n x - p\| \leq \liminf_{m\to\infty} \|T^m x - p\|$.

Hence $\lim_{n\to\infty} \|T^n x - p\|$ exists.

To show that $\{T^n x\}$ converges weakly to a fixed point of T. It suffics to show that

$$\omega_w(x) \subset F(T)$$

where $\omega_w(x) := \{y \in X : T^{n_j} x \rightharpoonup y, \text{ for some } n_j \uparrow \infty\}$.

First, claim that $\omega_w(x)$ is singleton set. Let $p_1, p_2 \in \omega_w(x)$ and $p_1 \neq p_2$, there exist subsequence $\{T^{n_i} x\}, \{T^{n_j} x\}$ of $\{T^n x\}$ such that $T^{n_i} x \rightharpoonup p_1, T^{n_j} x \rightharpoonup p_2$. Since X is weakly continuous duality map and by [1] that weakly continuous duality map satisfies Opial's condition, then

$$\begin{aligned}\lim_{n\to\infty}\|T^nx-p_1\| &= \lim_{i\to\infty}\|T^{n_i}x-p_1\|\\ &< \lim_{i\to\infty}\|T^{n_i}x-p_2\|\\ &= \lim_{j\to\infty}\|T^{n_j}x-p_2\|\\ &< \lim_{j\to\infty}\|T^{n_j}x-p_1\|\\ &= \lim_{n\to\infty}\|T^nx-p_1\|\end{aligned}$$

Then $\lim_{n\to\infty}\|T^nx-p_1\| < \lim_{n\to\infty}\|T^nx-p_1\|$ a contradiction.

Hence $\omega_w(x)$ is singleton set. Thus we have $T^nx \rightharpoonup x$.

To show that $\omega_w(x) \subset F(T)$. Let $y \in \omega_w(x)$, there exist subsequence $\{T^{n_j}x\}$ of $\{T^nx\}$ such that $T^{n_j}x \rightharpoonup y$. By weakly asymptotically regular of T at x, we have

$$T^nx - T^{n+1}x \rightharpoonup 0.$$

And we have $T^{n_j}x \rightharpoonup y$. For all integers $m > 0$,
$(T^{n_j}x - T^{n_j+m}x) \leq (T^{n_j}x - T^{n_j+1}x) + (T^{n_j+1}x - T^{n_j+2}x) + \ldots$

$$+(T^{n_j+(m-1)}x - T^{n_j+m}x) \rightharpoonup 0$$

Thus $T^{n_j}x - (T^{n_j}x - T^{n_j+m}x) \rightharpoonup y - 0 \Rightarrow T^{n_j+m}x \rightharpoonup y$.
Let $r_m = \limsup_{j\to\infty}\|T^{n_j+m}x - y\|$ and $r = \inf\{r_m : m > 0\}$.
Claim that $\lim_{m\to\infty} r_m = r$. Let $\epsilon > 0$, by the definition of asymptotically nonexpansive type,

$$\limsup_{n\to\infty}(\sup\{\|T^nx - T^nt\| - \|x-y\| : y \in C\}) \leq 0 < \frac{\epsilon}{2}$$

then there exist $N_1 \in \mathrm{N}$ such that,

$$\sup\{\|T^nx - T^nt\| - \|x-y\| : y \in C\}) < \frac{\epsilon}{2}$$

for all $n \geq N_1$. And since $r + \frac{\epsilon}{2}$ is not lower bound of r, there exist $m_0 \in \mathrm{N}$ such that $r < r_{m_0} < r + \frac{\epsilon}{2}$. For $l \geq N_1$,
$r_{m_0+l} = \limsup_{j\to\infty}\|T^{n_j+m_0+l}x - y\| \leq \limsup_{j\to\infty}\|T^{n_j+m_0+l}x - T^ly\|$

$$\begin{aligned}&= \limsup_{j\to\infty}(\|T^{n_j+m_0+l}x - T^ly\| - \|T^{n_j+m_0}x - y\| + \|T^{n_j+m_0}x - y\|)\\ &\leq \limsup_{j\to\infty}(\sup\{\|T^lu - T^ly\| - \|u-y\| : u \in C\})\\ &\quad + \limsup_{j\to\infty}\|T^{n_j+m_0}x - y\|\\ &< \limsup_{j\to\infty}(\tfrac{\epsilon}{2}) + \limsup_{j\to\infty}\|T^{n_j+m_0}x - y\|\end{aligned}$$

$= \frac{\epsilon}{2} + r_{m_0}$

Then $\lim_{m\to\infty} r_m = r$ exist.

Now, for all integers $m, j \geq N_1$, we have

$\Phi(\|T^{n_j+2m}x - y\|) = \Phi(\|(T^{n_j+2m}x - (T^m y) + (T^m y - y)\|)$

$$= \Phi(\|T^{n_j+2m}x - T^m y\|)$$

$$+ \int_0^1 < (T^m y - y), J_\varphi((T^{n_j+2m}x - T^m y), t(T^m y - y)) > dt$$

we can take $j \to \infty, T^{n_j+2m} \rightharpoonup y$ and

$\limsup_{j\to\infty} \Phi(\|T^{n_j+2m}x - y\|) = \limsup_{j\to\infty}(\Phi(\|T^{n_j+2m}x - T^m y\|)$

$$+ \int_0^1 < (T^m y - y), J_\varphi((T^{n_j+2m}x - T^m y), t(T^m y - y)) > dt)$$

and then,

$\Phi(\limsup_{j\to\infty} \|T^{n_j+2m}x - y\|) = \Phi(\limsup_{j\to\infty} \|T^{n_j+2m}x - T^m y\|)$

$$- \int_0^1 \|(T^m y - y\|\varphi(t\|T^m y - y\|)dt$$

$$\begin{aligned}
\Phi(r_{2m}) &\leq \Phi(\limsup_{j\to\infty} \|T^{n_j+2m}x - T^m y\|) - \Phi(\|T^m y - y\|) \\
&= \Phi(\limsup_{j\to\infty}(\|T^m(T^{n_j+m}x) - T^m y\| - \|T^{n_j+m}x - y\| \\
&\quad + \|T^{n_j+m}x - y\|) - \Phi(\|T^m y - y\|) \\
&\leq \Phi(\limsup_{j\to\infty}(\sup\{\|T^m(T^{n_j+m}x) - T^m y\| - \|T^{n_j+m}x - y\| : \\
&\quad (T^{n_j+m}x) \subseteq C\}) + \limsup_{j\to\infty} \|T^{n_j+m}x - y\|) - \Phi(\|T^m y - y\|) \\
&\leq \Phi(\tfrac{\epsilon}{2} + r_m) - \Phi(\|T^m y - y\|)
\end{aligned}$$

Then

$$\Phi(\|T^m y - y\|) \leq \Phi(\frac{\epsilon}{2} + r_m) - \Phi(r_{2m})$$

Take $m \to \infty$, we get

$$\Phi(\lim_{m\to\infty} \|T^m y - y\|) = \lim_{m\to\infty} \Phi(\|T^m y - y\|) \leq \Phi(\frac{\epsilon}{2})$$

By Φ is increasing, $\|T^m y - y\| \leq \frac{\epsilon}{2} < \epsilon$ for all $m \geq N_1$.

This implies that $T^m y \to y$ strongly and, hence $Ty = y$ by continuity of T^N for some $N \geq 1$, then $y \in F(T)$. Hence $\omega_w(x) \subset F(T)$.

Corollary 4 Suppose X is a Banach space with a weakly continuous duality map J_φ, C is a weakly compact convex subset of X, and $T : C \to C$, is an asymptotically nonexpansive. Further, suppose that there exists a nonempty closed convex subset K of C with the following property (ω):

$$x \in K \ \text{ implies } \ \omega_w(x) \subseteq K$$

where $\omega_w(x)$ is the weak $\omega - limit$ set of T at x; that is, the set

$$\{y \in k : y = weak - \lim_i T^{n_i}x \quad \text{for some } n_i \uparrow \infty\}.$$

Then T has a fixed point in K.

Suppose now C is a bounded closed convex subset of a Banach space X and $T : C \to C$ is an asymptotically nonexpansive mapping (we may always asumme $k_n \geq 1$ for all $n \geq 1$). For any $n \geq 1$, we lake $t_n = \min\{1 - (k_n - 1)^{\frac{1}{2}}, 1 - \frac{1}{n}\}$. Fix a u in C and define for each integer $n \geq 1$ the contraction $S_n : C \to C$ by

$$S_n(x) = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x. \tag{3.1}$$

Then the Banach Contraction Principle yields a unique point $x_n \in C$ that is fixed by S_n, that is, we have

$$x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x_n. \tag{3.2}$$

Theorem 5 Suppose that X is a reflexive Banach space with weakly continuous duality map and uniformly Gâteaux differentiable norm. Suppose in addition the following condition

$$\lim_n \|x_n - Tx_n\| = 0 \tag{3.3}$$

Then (x_n) converge strongly to a fixed point of T.

Proof. Suppose that the sequence $\{x_n\}$ defined by (3.2). From corollary 4, the fixed point set $F(T)$ of T is nonempty. We now show that (x_n) converge strongly to a fixed point of T. Now let μ be a Banach limit and define $f : C \to [0, \infty)$ by

$$f(z) = \mu_n\|x_n - z\| \quad \text{for every } z \in C.$$

Then, since the function f on C is convex and continuous, $f(z) \to \infty$ as $\|z\| \to \infty$, and X is reflexive it follows from [2, p. 79] that there exists $u \in C$ with $f(u) = \inf_{z\in C} f(z)$. Define the set

$$M = \{v \in C : f(v) = \inf_{z\in C} f(z)\}.$$

Then M is a nonempty, closed and convex. We further claim that M has the property (ω). If x is in M and $y = w - \lim_j T^{m_j}x$ belong to the weak ω-limit set $\omega_w(x)$ of T at x, then from the weakly lower semicintinuous of f, we have

$$f(y) = \liminf f(T^{m_j}x) \leq \limsup_{m\to\infty} f(T^m x)$$

$$\leq \limsup_m (\mu_n\|x_n - T^m x\|).$$

Since T is uniformly continuous it follows from $x_n - Tx_n \to 0$ that $x_n - T^m x_n \to 0$ for each fixed $m \in \mathbb{N}$. Thus

$$f(z) \leq \limsup_m (\mu_n\|T^m x_n - T^m x\|)$$

$$\leq \limsup_m (k_m \mu_n\|x_n - x\|) = \mu_n\|x_n - x\|$$

$$= \inf_{z\in C} f(z) \leq f(y).$$

This show that y belongs to M and hence M satisfies the property (ω). It follows from corollary 4, that T has a fixed $z_0 \in M$. Next, to show that (x_n) converges strongly to a fixed point of T. We note that, for any $w \in F(T)$,

$$\langle x_n - T^n x_n, J(x_n - w)\rangle = \langle x_n - w, J(x_n - w)\rangle + \langle w - T^n x_n, J(x_n - w)\rangle$$

$$\geq \|x_n - w\|^2 - \|w - T^n x_n\|\|x_n - w\|$$

$$\geq -(k_n - 1)\|x_n - w\|^2$$

$$\geq -(k_n - 1)d^2$$

where $d = \operatorname{diam} C$. Since x_n is a fixed point of S_n, it follows that

$$x_n - T^n x_n = \frac{k_n - t_n}{t_n}(u - x_n)$$

and from last inequality above,we get

$$\langle x_n - u, J(x_n - w)\rangle \leq s_n d^2, \tag{3.4}$$

where $s_n = \frac{t_n(k_n-1)}{(k_n-t_n)} \to 0$ as $n \to \infty$. So, putting $w = z_o$, we have

$$\langle x_n - u, J(x_n - z_0)\rangle \leq s_n d^2. \tag{3.5}$$

On the other hand, since z_0 is the minimizer of the function f on C, by [9, Lemma 3], we have

$$\mu_n\langle z - z_0, J(x_n - z_0)\rangle \leq 0$$

for all $z \in C$. In particular, we have

$$\mu_n \langle u - z_0, J(x_n - z_0) \rangle \leq 0. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\mu_n \langle x_n - z_0, J(x_n - z_0) \rangle = \mu_n \|x_n - z_0\|^2 \leq 0.$$

Therefore, there is a subsequence (x_{n_k}) of (x_n) which converges strongly to z_0. To show that (x_n) converges strongly to a fixed point of T, let (x_{n_j}) and (x_{m_j}) be subsequences of (x_n) such that $x_{n_j} \to z$ and $x_{m_j} \to z'$. Then z and z' are fixed points of T by hypothesis (3.3). It follows from (3.4) that

$$\langle z - u, J(z - z') \rangle \leq 0$$

and

$$\langle z' - u, J(z' - z) \rangle \leq 0.$$

Adding these two inequalities yields

$$\langle z - z', J(z - z') \rangle = \|z - z'\|^2 = 0.$$

So we have $z = z'$. Therefore (x_n) converges strongly to a fixed point of T.□

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AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we prove that the new modified implicit iteration sequence for finite family of nonexpansive (asymptotically quasi-nonexpansive) self-mappings converges strongly to a common fixed point of the family in a uniformly convex Banach space, requiring one member T in the family to be semi-compact. Our results extend and improve some recent results of Xu and Ori[17] and Sun[14], respectively.

keywords: Implicit iteration process; Finite family of asymptotically quasi-nonexpansive mappings; semi-compact; Common Fixed point

1. INTRODUCTION

Let C be a subset of normed space X, and let T be a self-mapping on C. T is said to be *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence (u_n) in $[0, \infty)$ with $\lim_{n\longrightarrow\infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$ for all $x, y \in C$ and $n \geq 1$. T is said to be an *asymptotically quasi-nonexpansive map*, if there exists a sequence (u_n) in $[0, \infty)$ with $\lim_{n\longrightarrow\infty} u_n = 0$ such that $\|T^n x - p\| \leq (1 + u_n)\|x - p\|$ for all $x \in C$ and $p \in F(T)$ and $n \geq 1$ ($F(T)$ denotes the set of fixed points of T i.e. $F(T) = \{x \in C : Tx = x\}$).

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The class of quasi-nonexpansiveness was introduced by Daiz and Metcalf [5] in 1967, the concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk [6] in 1972. The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied extensively by Browder [1, 2], Goebel and Kirk [6] and Liu [8], Wittmann [15], Chang et al [3] in the settings of Hilbert spaces and uniformly convex Banach spaces.

Let C be a nonempty convex subset of X, and let $T_1, T_2, ..., T_N$ be N nonexpansive self-mappings of C. We will denote the index set $\{1, 2, ..., N\}$ by I. In [17], Xu and Ori have introduced the following implicit iteration process. For an initial point $x_0 \in C$ and $\{\alpha_n\}_{n\geq 1}$ a real sequence in $(0, 1)$ the sequence $\{x_n\}_{n\geq 1}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1,\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2,\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1},\\
&\vdots
\end{aligned}$$

The scheme is expressed in a compact from as:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \ \ n \geq 1, \tag{1.1}$$

where $T_k = T_{k \, mod \, N}$.

Using this iteration process, they proved the following convergence theorem for nonexpansive maps in Hilbert spaces.

Theorem XO [17]. Let H be a Hilbert space and let C a nonempty closed convex subset of H. Let $\{T_i : i \in I\}$ be N nonexpansive self-mappings of C such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in C : T_i x = x\}$. Let $x_0 \in C$, and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $(0, 1)$ such that $\lim_{n\longrightarrow\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ defined implicitly by (1.1) converges weakly to a common fixed point of the mappings $\{T_i : i \in I\}$.

Recently, Sun [14] was extended the process (1.1) to process for a finite family of asymptotically quasi-nonexpansive self-mappings on the nonempty bounded closed convex subset C of X, with $\{\alpha_n\}$ a real sequence in $(0,1)$, and an initial point $x_0 \in C$, which is defined as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1 \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2 \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1} \\ &\vdots \\ x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N} \\ x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1} \\ &\vdots \end{aligned}$$

which can be written in the following compact from:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_n \text{ forall } n \geq 1, \tag{1.2}$$

where $n = (k-1)N + i, i \in \{1, 2, ..., N\} = I$.

Furthermore, Sun[14] was study the implicit iteration process (1.2) in the general setting of a uniformly convex Banach space and prove the strong convergence of the process to a common fixed point.

In this paper, we will extend the process (1.1) to a process for a finite family of nonexpansive mappings, with $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0,1]$, and an initial point $x_0 \in C$, which is defined as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n(\beta_n x_n + (1-\beta_n)T_n x_n), \ n \geq 1, \tag{1.3}$$

where $T_k = T_{k \, mod \, N}$. It is easy to see that the sequence $\{x_n\}$ generated by process (1.2) always exists by Banach's contraction principle. Moreover, we will extend the process (1.2) to a process for a finite family of asymptotically quasi-nonexpansive self-mappings on the nonempty closed convex subset C of X, with $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0,1]$, and an initial point $x_0 \in C$:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^n(\beta_n x_n + (1-\alpha_n)T_i^n x_n), \ n \geq 1, \tag{1.4}$$

where $n = (k-1)N + i, i \in I$.

Throughout this paper, we always suppose that the sequence $\{x_n\}$ generated by process (1.4) exists. Our purpose in this paper is to study the implicit iteration process (1.3) and (1.4) in the general setting of a uniformly convex Banach space and prove the strong convergence of the process to a common fixed point, requiring only one member T in the family $\{T_i : i \in I\}$ to be semi-compact. The results presented in this paper generalize and extend the corresponding main results of Xu and Ori [17], and Z. h. Sun [14].

2. Preliminaries

We first recall some definitions.

Definition 2.1. (see [6]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\} > 0$$

for all $0 < \epsilon \leq 2$(i.e., $\delta_X(\epsilon)$ is a function $(0,2] \longrightarrow (0,1)$).

Definition 2.2. (see [3]). Let C be a closed subset of a Banach space X. A mapping $T : C \longrightarrow C$ is said to be *semi-compact* if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \longrightarrow x^* \in C$.

Definition 2.3. A mapping $T : C \longrightarrow C$ is called *uniformly L-Lipschitzian* if there exits a constant $L > 0$ such that $\forall x, y \in C$,

$$\|T^n x - T^n y\| \leq L\|x - y\|, \text{ for all } n \geq 1.$$

In what follows, we shall make use the following lemmas.

Lemma 2.4. (see [12]). *Let the nonnegative number sequences $\{a_n\}, \{b_n\}$ satisfy that*

$$a_{n+1} \leq (1 + b_n)a_n, \forall n = 1, 2, ..., \sum_{n=1}^{\infty} b_n < \infty.$$

Then

(1) $\lim_{n\longrightarrow\infty} a_n$ *exists, and*

(2) If $\liminf_{n\longrightarrow\infty} a_n = 0$, *then* $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.5 (J. Schu's Lemma[13]). *Let ba a uniformly convex Banach space,* $0 < \alpha \leq t_n \leq \beta < 1, x_n, y_n \in X, \limsup_{n\longrightarrow\infty} \|x_n\| \leq a, \limsup_{n\longrightarrow\infty} \|y_n\| \leq a$, *and* $\lim_{n\longrightarrow\infty} \|t_n x_n + (1 - t_n)y_n\| = a, a \geq 0$. *Then* $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$.

3. Main results

In this section, we study the convergence properties of the sequences (1.3) and (1.4). First of all, we shell need the following results.

Lemma 3.1. *Let C be a nonempty closed convex subset of Banach space X. Let $\{T_i : i \in I\}$ be N nonexpansive self-mapping of C such that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in C : T_i x = x\}$. Let $x_0 \in C$, and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$, and $\{x_n\}$ be the sequence generated by process (1.3) Then for each $p \in F$, $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists.*

Proof. Let $p \in F$.Then, from (1.3), we have that

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n x_{n-1} + (1-\alpha_n)T_n(\beta_n x_n + (1-\beta_n)T_n x_n) - (1-\alpha_n)p - \alpha_n p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\|T_n(\beta_n x_n + (1-\beta_n)T_n x_n) - T_n p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\beta_n \|x_n - p\| + (1-\alpha_n)(1-\beta_n)\|x_n - p\| \\
&= \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\|x_n - p\|.
\end{aligned}$$

Hence

$$\|x_n - p\| \leq \|x_{n-1} - p\| \leq \|x_{n-2} - p\| \leq \cdots \leq \|x_1 - p\|. \tag{3.1}$$

So from (3.1), we get that $\{x_n\}$ is bounded and decreasing, we have $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists. This completes the proof. □

The purpose of the next main theorem is to prove the following convergent result for the process (1.3).

Theorem 3.2. *Let X be a real uniformly convex Banach space, C a closed convex nonempty subset of X. Let $\{T_i : i \in I\}$ be N nonexpansive self-mapping of C such that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in C : T_i x = x\}$. Let $x_0 \in C, \{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$, such that $0 < \alpha \leq \alpha_n, \beta_n \leq \beta < 1$ for some α, β in $(0,1)$ and $\{x_n\}$ be the sequence generated by process (1.3). Then, for each $l \in I$, $\lim_{n \longrightarrow \infty} \|x_n - T_l x_n\| = 0$. Moreover, if there exists one member T in $\{T_i : i \in I\}$ to be semi-compact, then the sequence $\{x_n\}$ strongly converges to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Proof. Let $p \in F$. Then, by Lemma 3.1, we obtain $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists. Let $\lim_{n \longrightarrow \infty} \|x_n - p\| = c$ for some real number $c \geq 0$. For each $n \geq 1$, putting

$y_n = \beta_n x_n + (1-\beta_n)T_n x_n$. Then

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)T_n x_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\|, \end{aligned}$$

for each $n \ge 1$. Taking $\limsup_{n\longrightarrow\infty}$ in both sides, we obtain

$$\limsup_{n\longrightarrow\infty} \|y_n - p\| \le \limsup_{n\longrightarrow\infty} \|x_n - p\| = \lim_{n\longrightarrow\infty} \|x_n - p\| = c.$$

Moreover, we note that

$$\limsup_{n\longrightarrow\infty} \|T_n y_n - p\| \le \limsup_{n\longrightarrow\infty} \|y_n - p\| \le \lim_{n\longrightarrow\infty} \|x_n - p\| = c,$$

and

$$\begin{aligned} c = \lim_{n\longrightarrow\infty} \|x_n - p\| &= \lim_{n\longrightarrow\infty} \|\alpha_n x_{n-1} + (1-\alpha_n)T_n y_n - p\| \\ &= \lim_{n\longrightarrow\infty} \|\alpha_n(x_{n-1} - p) + (1-\alpha_n)(T_n y_n - p)\|. \end{aligned}$$

Then, by J. Schu's Lemma, we have

$$\lim_{n\longrightarrow\infty} \|T_n y_n - x_{n-1}\| = 0,$$

and hence

$$\|x_n - x_{n-1}\| = (1-\alpha_n)\|T_n y_n - x_{n-1}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This implies that

$$\|x_n - x_{n+l}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty, \text{ for all } l < N.$$

On the other hand, we have

$$\begin{aligned} \|x_n - p\| &\le \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)\|T_n y_n - p\| \\ &\le \alpha_n\|x_{n-1} - T_n y_n\| + \alpha_n\|T_n y_n - p\| + (1-\alpha_n)\|y_n - p\| \\ &\le \alpha_n\|x_{n-1} - T_n y_n\| + \alpha_n\|y_n - p\| + (1-\alpha_n)\|y_n - p\| \\ &= \alpha_n\|x_{n-1} - T_n y_n\| + \|y_n - p\|, \end{aligned}$$

for each $n \ge 1$. Since $\lim_{n\longrightarrow\infty} \|x_{n-1} - T_n y_n\| = 0$, we obtain that

$$c = \lim_{n\longrightarrow\infty} \|x_n - p\| \le \liminf_{n\longrightarrow\infty} \|y_n - p\|.$$

It follows that

$$c \le \liminf_{n\longrightarrow\infty} \|y_n - p\| \le \limsup_{n\longrightarrow\infty} \|y_n - p\| \le c.$$

Thus

$$\lim_{n\longrightarrow\infty} \|y_n - p\| = c$$