and hence

$$c = \lim_{n \longrightarrow \infty} \|y_n - p\| = \lim_{n \longrightarrow \infty} \|\beta_n(x_n - p) + (1 - \beta_n)(T_n x_n - p)\|.$$

By J. Schu's Lemmma, we have

$$\lim_{n \longrightarrow \infty} \|x_n - T_n x_n\| = 0.$$

Hence, for all $l \in I$,

$$\begin{aligned} \|x_n - T_{n+l}x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \\ &\leq 2\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\|, \end{aligned}$$

which implies that

$$\lim_{n \longrightarrow \infty} \|x_n - T_{n+l}x_n\| = 0, \forall l \in I.$$

We note that any subsequence of a convergent number sequence converges to the same limit. Thus

$$\lim_{n \longrightarrow \infty} \|x_n - T_l x_n\| = 0, \forall l \in I. \tag{3.2}$$

We now suppose that there exists one member T in $\{T_i : i \in I\}$ to be semi-compact. Without loss of generality we may assume that T_1 is semi-compact. Therefore by (3.2), it follows that $\lim_{n \longrightarrow \infty} \|x_n - T_1 x_n\| = 0$ and by the definition of semi-compact there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \longrightarrow p \in C$ as $j \longrightarrow \infty$. By (3.2) again, we have

$$\|p - T_l p\| = \lim_{j \longrightarrow \infty} \|T_l x_{n_j} - x_{n_j}\| = 0, \forall 1 \leq l \leq N.$$

It show that $p \in F$. Furthermore, since $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists, we have $\lim_{n \longrightarrow \infty} \|x_n - p\| = 0$. This completes the proof. □

The next main theorem is prove the following convergent result for the process (1.4). First of all, we shall need the following results.

Lemma 3.3. *Let C be a nonempty closed convex subset of real Banach space X. Let $\{T_i : i \in I\}$ be N asymptotically quasi-nonexpansive self-mappings of C, i. e., $\|T_i^n x - q_i\| \leq (1 + u_{in})\|x - q_i\|$ for all $x \in C, q_i \in F(T_i), i \in I$. Suppose that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in C : T_i x = x\}$. Let $x_0 \in C, \{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$, such that $0 < \alpha \leq \alpha_n \leq 1$. Then the implicit iteration process $\{x_n\}$ generated by (1.4) satisfies the following;*

(1) *For each $p \in F$ and for each $n = (k-1)N + i \geq 1$, we have*

$$\|x_n - p\| \leq (1 + b_{ik})\|x_{n-1} - p\|,$$

where $\{b_{ik}\} \subseteq [0, 1]$ and $\sum_{k=1}^{\infty} b_{ik} < +\infty$ for all $i \in I$;

(2) *There exists a constant* $M > 0$ *such that* $\|x_{n+m} - p\| \leq M\|x_n - p\|, \forall p \in F, \forall n, m \geq 1$.

Proof. Let $p \in F$. Then, for each $n = (k-1)N + i \geq 1$, we have

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + (1-\alpha_n)T_i^k y_n - p\| \\ &\leq \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)\|T_i^k y_n - p\| \\ &\leq \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)(1+u_{ik})\|y_n - p\| \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)T_i^k x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|T_i^k x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)(1+u_{ik})\|x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n+u_{ik})\|x_n - p\| \\ &= (1+u_{ik})\|x_n - p\|. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3), it can be obtained that

$$\begin{aligned} \|x_n - p\| &\leq \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)(1+u_{ik})^2\|x_n - p\| \\ &\leq \alpha_n\|x_{n-1} - p\| + (1-\alpha_n+v_{ik})\|x_n - p\| \end{aligned}$$

where $v_{ik} = 2u_{ik} + u_{ik}^2$. Note that $\alpha_n > \alpha > 0$ for each $n \geq 1$. Then

$$\alpha_n\|x_n - p\| \leq \alpha_n\|x_{n-1} - p\| + v_{ik}\|x_n - p\| \leq \alpha_n\|x_{n-1} - p\| + v_{ik}\frac{\alpha_n}{\alpha}\|x_n - p\|$$

and so

$$\frac{\alpha - v_{ik}}{\alpha}\|x_n - p\| \leq \|x_{n-1} - p\|. \tag{3.5}$$

Since $\sum_{k=1}^{\infty} v_{ik} < \infty$ for all $i \in I$, $\lim_{k\longrightarrow\infty} v_{ik} = 0$. This implies that there exists a natural number n_0, as $k > \frac{n_0}{N} + 1$, i.e., $n > n_0$ such that

$$v_{ik} < \frac{\alpha}{2} \text{ and } \alpha - v_{ik} > 0.$$

Let

$$1 + b_{ik} = \frac{\alpha}{\alpha - v_{ik}} = 1 + \frac{v_{ik}}{\alpha - v_{ik}}.$$

Then

$$b_{ik} = \left(\frac{1}{\alpha - v_{ik}}\right) v_{ik} < \frac{2}{\alpha} v_{ik}.$$

Therefore

$$\sum_{k=1}^{\infty} b_{ik} < \frac{2}{\alpha}\sum_{k=1}^{\infty} v_{ik} < +\infty \text{ for all } i \in I$$

and (3.5) becomes

$$\|x_n - p\| \leq (1 + b_{ik})\|x_{n-1} - p\|, \forall p \in F.$$

We now to prove (2). Let $a_n = b_{ik}$ where $n = (k-1)N + i, i \in I$. Notice that when $x > 0, 1 + x \leq e^x$, and

$$\|x_n - p\| \leq (1 + a_n)\|x_{n-1} - p\|, \forall p \in F.$$

Then

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + a_{n+m})\|x_{n+m-1} - p\| \\
&\leq e^{a_{n+m}}\|x_{n+m-1} - p\| \\
&\leq e^{a_{n+m} + a_{n+m-1}}\|x_{n+m-2} - p\| \\
&\vdots \\
&\leq e^{\sum_{k=m}^{n+m} a_k}\|x_n - p\| \\
&\leq e^{\sum_{i=1}^{N}\sum_{k=1}^{\infty} b_{ik}}\|x_n - p\|, \forall p \in F
\end{aligned}$$

for all natural number m, n. Let $M = e^{\sum_{i=1}^{N}\sum_{k=1}^{\infty} b_{ik}} > 0$. Thus,

$$\|x_{n+m} - p\| \leq M\|x_n - p\|, \forall p \in F, \forall n, m \geq 1.$$

This completes the proof of (2). $\square$

Theorem 3.4. *Let C be a nonempty closed convex subset of real Banach space X. Let $\{T_i : i \in I\}$ be N asymptotically quasi-nonexpansive self-mapping of C, i.e., $\|T_i^n x - q_i\| \leq (1 + u_{in})\|x - q_i\|$ for all $x \in C, q_i \in F(T_i), i \in I$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in C : T_i x = x\}$. Let $x_0 \in C, \{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ such that $0 < \alpha \leq \alpha_n \leq 1$ and $\sum_{k=1}^{\infty} u_{ik} < +\infty$ for all $i \in I$. Then the implicit iteration sequence $\{x_n\}$ generated by (1.4) converges to a common fixed point in F if and only if $\liminf_{n\longrightarrow\infty} d(x_n, F) = 0$, where $d(x_n, F)$ denotes the distance of x to set F, i.e., $d(x, F) = \inf_{y\in F} d(x, y)$.*

Proof. The necessity of the conditions is obvious. Thus we will only prove the sufficiency. For any $p \in F$, and for each $n = (k-1)N + i \geq 1$, it follows by Lemma 3.3 that

$$\|x_n - p\| \leq (1 + b_{ik})\|x_{n-1} - p\|.$$

This implies that $d(x_n, F) \leq (1+b_{ik})d(x_{n-1}, F)$. From Lemma 2.4, we have $\lim_{n\longrightarrow\infty} d(x_n, F) = 0$. Hereafter, we will prove that $\{x_n\}$ is a Cauchy sequence. By Lemma 3.3, there exists a constant $M > 0$ such that

$$\|x_{n+m} - p\| \leq M\|x_n - p\|, \forall p \in F \tag{3.6}$$

for all $m, n \geq 1$. Let $\epsilon > 0$, since $\lim_{n\longrightarrow\infty} d(x_n, F) = 0$, there exists a natural number N_1 such that $d(x_n, F) \leq \frac{\epsilon}{2M}, \forall n \geq N_1$. In particular, we have $d(x_{N_1}, F) < \frac{\epsilon}{2M}$. This implies that there exists a point $p' \in F$ such that

$$\|x_{N_1} - p'\| < \frac{\epsilon}{2M}.$$

It follows, from (3.6), that when $n \geq N_1$, for all $m \geq 1$,

$$\|x_{n+m} - x_n\| \leq \|x_{m+n} - p'\| + \|x_n - p'\| \leq M\|x_{N_1} - p'\| + M\|x_{N_1} - p'\| < \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{x_n\}$ is convergent. Let $\lim_{n\longrightarrow\infty} x_n = p$. We note that

$$d(p, F) \leq d(x_n, F) + \|x_n - p\|, \forall n \geq 1.$$

Since $\lim_{n\longrightarrow\infty} d(x_n, F) = 0$ and the set F is closed, we have $p \in F$, i.e. p is a common point of $\{T_i : i \in I\}$. This completes the proof. □

Corollary 3.5. *Suppose that condition are as same as in Theorem 3.4. Then the implicit iteration sequence $\{x_n\}$ generated by (1.4) converges to a common fixed point in F if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.*

We now to prove the following convergent result for process (1.4).

Theorem 3.6. *Let X be a real uniformly convex Banach space, C a closed convex nonempty subset of X. Let $\{T_i : i \in I\}$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mapping of C. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, and there exists one member T in $\{T_i : i \in I\}$ to be semi-compact. Let $x_0 \in C, \{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ such that $0 < \alpha \leq \alpha_n, \beta_n \leq \beta < 1$ and $\sum_{k=1}^{\infty} u_{ik} < \infty$ for all $i \in I$. Then the implicit iteration sequence $\{x_n\}$ generated by (1.4) converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Proof. Let $p \in F$. From Lemma 3.3 , we have

$$\|x_n - p\| \leq (1 + b_{ik})\|x_{n-1} - p\|, \forall n = (k-1)N + i \geq 1.$$

Since $\sum_{k=1}^{\infty} b_{ik} < \infty$ for all $i \in I$, it follows by Lemma 2.4 that $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists. Let $\lim_{n\longrightarrow\infty} \|x_n - p\| = c$ for some $c \geq 0$. For all $n \geq 1$, Put $y_n =$

$\beta_n x_n + (1-\beta_n)T_i^k x_n$. Then

$$\begin{aligned}\|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)T_i^k x_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|T_i^k x_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)(1+u_{ik})\|x_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n+u_{ik})\|x_n - p\| \\ &= (1+u_{ik})\|x_n - p\|\end{aligned}$$

for all $n \ge 1$. Taking $\limsup_{n\longrightarrow\infty}$ in both sides, we obtain

$$\limsup_{n\longrightarrow\infty}\|y_n - p\| \le \limsup_{n\longrightarrow\infty}\|x_n - p\| = \lim_{n\longrightarrow\infty}\|x_n - p\| = c.$$

Note that

$$\limsup_{n\longrightarrow\infty}\|T_i^k y_n - p\| \le \limsup_{n\longrightarrow\infty}\|y_n - p\| \le \lim_{n\longrightarrow\infty}\|x_n - p\| = c,$$

and

$$\begin{aligned}c = \lim_{n\longrightarrow\infty}\|x_n - p\| &= \lim_{n\longrightarrow\infty}\|\alpha_n x_{n-1} + (1-\alpha_n)T_i^k y_n - p\| \\ &= \lim_{n\longrightarrow\infty}\|\alpha_n(x_{n-1} - p) + (1-\alpha_n)(T_i^k y_n - p)\|.\end{aligned}$$

We note that $n = (k-1)N + i$, $T_n = T_{n(monN)} = T_i, i \in I$. By J. Schu's Lemma, we have

$$\lim_{n\longrightarrow\infty}\|T_n^k y_n - x_{n-1}\| = \lim_{n\longrightarrow\infty}\|T_i^k y_n - x_{n-1}\| = 0.$$

Hence

$$\|x_n - x_{n-1}\| = (1-\alpha_n)\|T_n^k y_n - x_{n-1}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

as well as

$$\|x_n - x_{n+l}\| \longrightarrow 0 \text{ for all } l < N.$$

On the other hand, we observe that

$$\begin{aligned}\|x_n - p\| &\le \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)\|T_i^k y_n - p\| \\ &\le \alpha_n\|x_{n-1} - T_i^k y_n\| + \alpha_n\|T_i^k y_n - p\| + (1-\alpha_n)\|T_i^k y_n - p\| \\ &\le \alpha_n\|x_{n-1} - T_i^k y_n\| + \|T_i^k y_n - p\| \\ &\le \alpha_n\|x_{n-1} - T_n^k y_n\| + (1+u_{ik})\|y_n - p\|.\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}\|x_{n-1} - T_n^k y_n\| = 0$, we obtain that

$$c = \lim_{n\longrightarrow\infty}\|x_n - p\| \le \liminf_{n\longrightarrow\infty}\|y_n - p\|.$$

It follows that

$$c \le \liminf_{n\longrightarrow\infty}\|y_n - p\| \le \limsup_{n\longrightarrow\infty}\|y_n - p\| \le c,$$

and so

$$\lim_{n\longrightarrow\infty} \|y_n - p\| = c.$$

This implies that

$$c = \lim_{n\longrightarrow\infty} \|y_n - p\| = \lim_{n\longrightarrow\infty} \|\beta_n(x_n - p) + (1-\beta_n)(T_n^k x_n - p)\|.$$

By J. Schu's Lemmma, we have

$$\lim_{n\longrightarrow\infty} \|x_n - T_n^k x_n\| = 0.$$

We now to show that $\lim_{n\longrightarrow\infty} \|x_n - T_l x_n\| = 0, \forall l \in I$. Note that

$$\begin{aligned}
\|x_n - T_n x_n\| &\le \|x_n - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \\
&\le \|x_n - T_n^k x_n\| + L\|T_n^{k-1} x_n - x_n\| \\
&\le \|x_n - T_n^k x_n\| + L(\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| \\
&+ \|T_{n-N}^{k-1} x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|)
\end{aligned}$$

Since $n \equiv (n-N)(modN)$, we have $T_n = T_{n-N}$. The above inequality becomes

$$\begin{aligned}
\|x_n - T_n x_n\| &\le \|x_n - T_n^k x_n\| + L^2\|x_n - x_{n-N}\| + L\|T_{n-N}^{k-1} x_{n-N} - x_{n-N}\| + L\|x_n - x_{n-N}\| \\
&= \|x_n - T_n^k x_n\| + L(1+L)\|x_n - x_{n-N}\| + L\|T_{n-N}^{k-1} x_{n-N} - x_{n-N}\| \to 0.
\end{aligned}$$

Which implies that

$$lim_{n\longrightarrow\infty}\|x_n - T_n x_n\| = 0.$$

Hence, for all $l \in I$,

$$\begin{aligned}
\|x_n - T_{n+l} x_n\| &\le \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| + \|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\
&\le (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\|.
\end{aligned}$$

It follows that

$$lim_{n\longrightarrow\infty}\|x_n - T_{n+l} x_n\| = 0, \forall l \in I.$$

Thus

$$lim_{n\longrightarrow\infty}\|x_n - T_l x_n\| = 0, \forall l \in I. \tag{3.7}$$

which lies on the fact that any subsequence of a convergent number sequence converges to the same limit. By hypothesis that there exists T in $\{T_i : i \in I\}$ to be simi-compact, without loss of generality we may assume that T_1 is semi-compact, it follows that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \longrightarrow p$ as $j \longrightarrow \infty$. By (3.7) again, we have

$$\|p - T_l p\| = \lim_{j\longrightarrow\infty} \|T_l x_{n_j} - x_{n_j}\| = 0, \forall 1 \le l \le N.$$

It show that $p \in F$ and $\lim_{n\longrightarrow\infty} d(x_n, F) = 0$, therefore by Theorem 3.4 and Corollary 3.5 we have that $\{x_n\}$ converges to a common fixed point p in F. This completes the proof. □

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THE CHARACTERISTIC OF NONCOMPACT CONVEXITY AND RANDOM FIXED POINT THEOREM FOR SET-VALUED OPERATORS

SOMYOT PLUBTIENG AND POOM KUMAM

Abstract. Let (Ω, Σ) be a measurable space, X a Banach space whose characteristic of noncompact convexity is less than 1, C a bounded closed convex subset of X, $KC(C)$ the family of all compact convex subsets of C. We prove that a set-valued nonexpansive mappings $T : C \to KC(C)$ has a fixed point. Furthermore, if X is separable then we also prove that a set-valued nonexpansive operator $T : \Omega \times C \to KC(C)$ has a random fixed point.

Keywords : random fixed point, set-valued random operator, measure of noncompacness.
MSC 2000 : 47H10, 47H09, 47H04.

1. Introduction

The study of random fixed points has been a very active area of research in probabilistic operator theory in the last decade. In this direction, there have appeared various papers concerning random fixed point theorems for single-valued and set-valued random operators; see, for example, [6],[8],[9],[10],[11][14],[20] and reference therein.

In 2002, P. L. Ramírez [9] was proved the existence of a random fixed point theorems for a random nonexpansive operator in the framework of a Banach spaces with a characteristic of noncompact convexity $\varepsilon_\alpha(X)$ is less than 1. On the other hand, Domínguez Benavides and Ramírez [4] was proved a fixed point theorem for a set-valued nonexpansive and 1-λ-contractive mapping in the framework of a Banach spaces whose characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ is less than 1.

The purpose of the present paper is to prove a fixed point theorem for set-valued random nonexpansive operators in the framework of a Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ is less than 1. Moreover, we also prove a fixed point theorem foe set-valued nonexpansive mapping in a Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness $\varepsilon_\beta(X)$ is less than 1. Our results can also be seen as an extension of Theorem 6 in [9] and Theorem 4.2 in [4], respectively.

2. Preliminaries

Through this paper we will consider a measurable spaces (Ω, Σ) (where Σ is a σ−algebra of subset of Ω) and (X, d) will be a metric spaces. We denote by $CL(X)(resp.CB(X), KC(X))$ the family of all nonempty closed (resp. closed bounded, compact convex) subset of X, and by H the Hausdorff metric on $CB(X)$ induced by d, i.e.,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for $A, B \in CB(X)$, where $d(x,E) = \inf\{d(x,y)|y \in E\}$ is the distance from x to $E \subset X$.

Let C be a nonempty closed subset of a Banach space X. Recall now that a set-valued mapping $T : C \to 2^X$ is said to be upper semicontinuous on C if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^1(V) := \{x \in C : Tx \cap V \neq \emptyset\}$is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf.[2] and [3] for details). There is another defferent kind of continuity for multivalued operator: $T : C \to CB(X)$ is said to be continuous on C (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$ whenever $x_n \to x$. It is not hard to see (see Deimling [3])that both definitions of continuity are equivalent if Tx is compact for every $x \in C$.

A set-valued operator $T : \Omega \to 2^X$ is call $(\Sigma)-$ measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be a *measurable selector* of a measurable set-valued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. An operator $T : \Omega \times C \to 2^X$ is call a random operator if, for each fixed $x \in C$, the operator $T(\cdot, x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in C : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, \cdot) : C \to 2^X, F(\omega)$ may be nonempty. A measurable operator $x : \Omega \to C$ is said to be a *random fixed point of a operator* $T : \Omega \times C \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times C \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T : (\omega, \cdot) \to 2^X$ is continuous.

If C is a closed convex subset of a Banach spaces X, then a set-valued mapping $T : C \to CB(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in C,$$

and T is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in C,$$

A random operator $T : \Omega \times C \to 2^X$ is said to be *nonexpansive* if, for each fixed $\omega \in \Omega$ the map $T : (\omega, \cdot) \to C$ is nonexpansive.

For later convenience, we list the following results related to the concept of measurability.

Lemma 2.1. (Wagner cf.[13]) *Let (X, d) be a complete separable metric spaces and $F : \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

Lemma 2.2. (Itoh 1977, cf.[8]) *Suppose $\{T_n\}$ is a sequence of measurable set-valued operator from Ω to $CB(X)$ and $T : \Omega \to CB(X)$ is an operator. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \to 0$, then T is measurable.*

Lemma 2.3. (Tan and Yuan cf.[12]) *Let X be a separable metric spaces and Y a metric spaces. If $f : \Omega \times X \to Y$ is a measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y)$ is measurable.*

As an easy application of Proposition 3 of Itoh[8] we have the following result.

Lemma 2.4. *Let C be s closed separable subset of a Banach space $X, T : \Omega \times C \to C$ a random continuous operator and $F : \Omega \to 2^C$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G : \Omega \to 2^C$ given by*

$$G(\omega) = \{x \in F(\omega) : \|x - T(\omega, x)\| < s\}, \quad \omega \in \Omega$$

is measureble and so is the operator $cl\{G(\omega)\}$ of the closure of $G(\omega)$.

Lemma 2.5. (Domínguez Benavidel and Lopez Acedo cf.[6]) *Suppose C is a weakly closed nonempty separable subset of a Banach space $X, F : \Omega \to 2^X$ a measurable with weakly compact values, $f : \Omega \times C \to \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \to \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x \in F(x)} f(\omega, x)$$

and the marginal map. $R : \Omega \to X$ defined by

$$R(\omega) := \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

Recall that the Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset B of X are respectively defined as the number

$$\alpha(B) = \inf\{r > 0 : B \text{ can be covered by finitely many sets of diameter } \leq r\},$$

$$\chi(B) = \inf\{r > 0 : B \text{ can be covered by finitely many ball of radius } \leq r\}.$$

The separation measure of noncompacness of a nonempty bounded subset B of X defined by

$$\beta(B) = \sup\{\varepsilon : \text{there exists a sequence } \{x_n\} \text{ in } B \text{ such that } sep(\{x_n\}) \geq \varepsilon\}$$

Let X be a Banach spaces and $\phi = \alpha, \beta$ or χ. The modulus of noncompact convexity associated to ϕ is defined in the following way:

$$\Delta_{X,\phi}(\varepsilon) = inf\{1 - d(0, A) : A \subset B_X \text{ is convex, } \phi(A) \geq \varepsilon\},$$

where B_X is the unit ball of X.

The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined by

$$\varepsilon_\phi(X) = \sup\{\varepsilon \geq 0 : \Delta_{X,\phi}(\varepsilon) = 0\}.$$

The following relationshops among the different moduli are easy to obtain

$$\Delta_{X,\alpha}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon), \tag{2.1}$$

and consequently

$$\varepsilon_\alpha(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X). \tag{2.2}$$

When X is a reflexive Banach spaces we have some alternative expressions for the moduli of noncompact convexity associated β and χ.

$$\Delta_{X,\beta}(\varepsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w -_n \lim x_n, sep(\{x_n\}) \geq \varepsilon\},$$

$$\Delta_{X,\chi}(\varepsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w -_n \lim x_n, \chi(\{x_n\}) \geq \varepsilon\}.$$

Let C be a nonempty bounded closed subset of Banach spaces X and $\{x_n\}$ bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$\begin{aligned} r(C, \{x_n\}) &= \inf\left\{\limsup_n \|x_n - x\| : x \in C\right\}, \\ A(C, \{x_n\}) &= \left\{x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\})\right\}. \end{aligned}$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf\{\sup\{\|x - y\| : y \in D\} : x \in C\}.$$

Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X. Then $\{x_n\}$ is called *regular* with respect to C if $r(C,\{x_n\}) = r(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Moreover, we also need the following Lemma

Lemma 2.6. (Benavides and Ramírez. Theorem 4.3 cf. [4].) *Let C be a closed convex subset of a reflexive Banach spaces X, and let x_n be a bounded sequence in C which is regular with respect to C. Then*

$$r_C(A(C,x_n)) \leq (1 - \Delta_{X,\beta}(1^-))r(C,\{x_n\}). \tag{2.3}$$

Moreover, if X satisfies the nonstrict Opial condition then

$$r_C(A(C,x_n)) \leq (1 - \Delta_{X,\chi}(1^-))r(C,\{x_n\}). \tag{2.4}$$

Lemma 2.7. (Deimling 1992, cf. [18]) *Let E be nonempty bouned closed closed convex subset f a Banach spaces and $T : E \to KC(X)$ a contraction. Assume $Tx \cap \overline{I_E(x)} \neq \emptyset$ for all $x \in E$. Then T has a fixed point.*

Proposition 2.8. (Kirk-Massa Theorem cf.[15]) *Let C be a nonempty weakly compact separable subset of a Banach space X. $T : C \to K(C)$ a nonexpansive mapping, and $\{x_n\}$ a sequence in C such that $\lim_n d(x_n - Tx_n) = 0$. Then, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \forall x \in A := A(C,\{z_n\})$$

3. The Results

We begin this section with an extension of Benavides-Ramírez's result by the 1-λ-contractive of T, can be remove.

Theorem 3.1. *Let C be a nonempty closed bounded convex subset of a Banach spaces X such that $\varepsilon_\beta(X) < 1$, and $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Proof Let $x_0 \in C$ be a fixed and, for each $n \geq 1$, define $T_n : C \to KC(C)$ by

$$T_n x = \frac{1}{n}x_0 + (1 - \frac{1}{n})Tx, \ \ \forall x \in C.$$

Then T_n is a set-valued contraction and hence has a fixed point x_n. It is easily see that $dist(x_n, Tx_n) \leq \frac{1}{n} diamC \to 0$ as $n \to \infty$. By Goebel and Kirk [7], we may assume that $\{x_n\}$ is regular with respect to C and using Proposition 2.8 we can also assume that

$$Tx \cap A \neq \emptyset, \ \ \forall x \in A := A(C\{x_n\}).$$

Since condition $\epsilon_\beta(X) < 1$ implies reflexivity [2], we apply Lemma 2.6 to obtain

$$r_C(A) \le \lambda r(C, \{x_n\}), \tag{3.1}$$

where $\lambda := (1 - \Delta_{X,\beta}(1^-)) < 1$.

It is clear that A is a weakly compact convex subset of C.

Now fixed $x_1 \in A$ and for each $n \ge 1$, we define the contraction $T_n^1 : A \to KC(C)$ defined by

$$T_n^1(x) = \frac{1}{n}x_1 + (1 - \frac{1}{n})T(x), \ \forall x \in A.$$

Since A is convex, each T_n^1 satisfies the same boundary condition as T does, that is, we have

$$T_n^1 x \cap \overline{I}_A(x) \neq \emptyset, \ \forall x \in A,$$

Hence by lemma 2.7, T_n^1 has a fixed point $z_n \in A$. Consequently, we can get a sequence $\{x_m^1\}$ in A satisfying $d(x_n^1, T(x_n^1)) \to 0$ as $n \to \infty$. as $n \to \infty$. Again, applying Lemma 2.6, we obtain

$$r_C(A^1) \le \lambda r(C, \{x_n^1\}), \tag{3.2}$$

where $A^1 := A(C, \{x_n^1\})$. Since, $\{x_n^1(\omega)\} \subset A.$, we have

$$r(C, \{x_n^1\}) \le r_C(A), \tag{3.3}$$

and then

$$r_C(A^1) \le \lambda^2 r_C(A). \tag{3.4}$$

By induction, for each $m \ge 1$, we construct A^m, and $\{x_n^m\}_n$ where $A^m = A(C, \{x_n^m\}), x_n^m \subset A^{m-1}$ such that $d(x_n^m, Tx_n^m) \to 0$ as $n \to \infty$ and

$$r_C(A^m) \le \lambda r_C(A) \le \lambda^m r(C, \{x_n\}). \tag{3.5}$$

By assumption $\epsilon_\beta(X) < 1$ and $diam A^m \le 2r_C(A^m)$ leads to $\lim_{m\to\infty} diam A^m = 0$. Since $\{A^m\}$ is a descending sequence of weakly compact subset of C, we have $\cap_m A^m = \{z\}$ for some $z \in C$. Finally, we will show that z is a fixed point of T. Indeed, for each $m \ge 1$, we have

$$\begin{aligned} d(z, Tz) &\le \|z - x_n^m\| + d(x_n^m, Tx_n^m) + H(Tx_n^m, Tz) \\ &\le 2\|z - x_n^m\| + d(x_n^m, Tx_n^m) \\ &\le 2diam A^m + d(x_n^m, Tx_n^m). \end{aligned}$$

Taking the upper limit as $n \to \infty$,

$$d(z, Tz) \le 2diam A^m.$$

Now taking the limit in m in both sides we obtain $z = Tz$. □

Corollary 3.2. (Theorem 4.2 in [4]) *Let C be a nonempty closed bounded convex subset of a Banach spaces X such that $\varepsilon_\beta(X) < 1$, and $T : C \to KC(C)$ be a nonexpansive and 1-λ-contractive nonexpansive mapping. Then T has a fixed point.*

Now we are ready to prove the main result of this paper.

Theorem 3.3. *Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\varepsilon_\beta(X) < 1$, and $T : \Omega \times C \to KC(C)$ be a set-valued nonexpansive random operator. Then T has a random fixed point.*

Proof For each $\omega \in \Omega$, and for every $n \geq 1$, we set

$$F(\omega) = \{x \in C : x \in T(\omega, x)\},$$

and

$$F_n(\omega) = \{x \in C : d(x, T(\omega, x)) \leq \frac{1}{n} diamC.$$

It follows from [Theorem 3.1] that $F(\omega)$ is nonempty. Clearly $F(\omega) \subseteq F_n(\omega)$, and $F_n(\omega)$ is closed and convex. Furthermore, by [8, Proposition 3], each F_n is measurable. Then, by Lemma 2.1, each F_n admits a measurable selector $x_n(\omega)$ and

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} diamC \to 0 \text{ as } n \to \infty.$$

Define a function $f_1 : \Omega \times C \to \mathbb{R}^+$ by

$$f_1(\omega, x) = \limsup_n \|x_n(\omega) - x\|, \ \ \forall \omega \in \Omega.$$

By Lemma 3.1, it is easily seen that for each $x \in C$, $f_1(\cdot, x) : \Omega \to \mathbb{R}^+$ is measurable and each $\omega \in \Omega$, $f_1(\omega, \cdot) : C \to \mathbb{R}^+$ is continuous and convex (and hence weakly lower semicontinuous (w-l.s.c.)). Note that, condition $\varepsilon_\beta(X) < 1$ implies reflexivity (see [2]) and so C is a weakly compact. Hence, by Lemma2.5 the marginal functions

$$r_1(\omega) := \inf_{x \in C} f_1(\omega, x),$$

and

$$R_1(\omega) := \{x \in C : f_1(\omega, x) = r_1(\omega)\}$$

are measurable. By Geoble [7], for any $\omega \in \Omega$ we may assume that the sequence $\{x_n(\omega)\}$ is regular with respect C. Observe that $R_1(\omega) = A(C, \{x_n(\omega)\})$ and $r_1(\omega) = r(C, \{x_n(\omega)\})$, thus we can apply Lemma 2.6 to obtain

$$r_C(R_1(\omega)) \leq \lambda r_1(\omega), \tag{3.6}$$

where $\lambda := 1 - \Delta_{X,\beta}(1^-) < 1$, since $\epsilon_\beta(X) < 1$. It is clear that $R_1(\omega)$ is a weakly compact and convex subset of C. By Lemma2.1 we can take $x_1(\omega)$ as a measurable selector of $R_1(\omega)$. For each $\omega \in \Omega$ and $n \geq 1$, we define the contraction $T_n^1(\omega, \cdot) : R_1(\omega) \to KC(C)$ defined by

$$T_n^1(\omega, x) = \frac{1}{n}x_1(\omega) + (1 - \frac{1}{n})T(\omega, x), \ \ \forall x \in R_1(\omega).$$

Since $R_1(\omega)$ is convex, each T_n satisfies the same boundary condition as T does, that is, we have

$$T_n^1(\omega, x) \cap \overline{I}_{R_1}(\omega)(x) \neq \emptyset, \ \ \forall x \in R_1(\omega),$$

Hence by lemma 2.7, $T_n(\omega, \cdot)$ has a fixed point $z_n(\omega) \in R_1(\omega), i.e. F(\omega) \cap R_1(\omega) \neq \emptyset$. Also it is easily seen that

$$dist(z_n(\omega), T(\omega, z_n(\omega))) \leq \frac{1}{n} diamC \to 0 \text{ as } n \to \infty.$$

Thus $F_n^1(\omega) = \{x \in R_1(\omega) : d(x, T(\omega, x)) \leq \frac{1}{n} diamC\} \neq \emptyset$ for each $n \geq 1$, closed and by Lemma 2.4, measurable. Hence, by Lemma 2.1, we can choose x_n^1 a measurable selector of F_n^1, and from definition of it we have $x_n^1(\omega) \in R_1(\omega)$ and $d(x_n^1(\omega), T(\omega, x_n^1(\omega))) \to 0$ as $n \to \infty$. as $n \to \infty$. Consider the function $f_2 : \Omega \times C \to \mathbb{R}^+$ defined by

$$f_2(\omega, x) = \limsup_n \|x_n^1(\omega) - x\|, \ \ \forall \omega \in \Omega.$$

As above, f_2 is a measurable function and weakly lower semicontunuous function. Then the marginal function

$$r_2(\omega) := \inf_{x \in R_1(\omega)} f_2(\omega, x)$$

and

$$R_2(\omega) := \{x \in R_1(\omega) : f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $R_2(\omega) = A(R_1(\omega), \{x_n^1(\omega)\})$, it follows that $R_2(\omega)$ is a weakly compact and convex. Also $r_2(\omega) = r(R_1(\omega), \{x_n^1(\omega)\})$. Again reasoning as above, for any $\omega \in \Omega$, we can assume that the sequence $\{x_n^1(\omega)\}$ is regular with respect to $R_1(\omega)$. Again, applying Lemma 2.6, we obtain

$$r_C(R_2(\omega)) \leq \lambda r_2(\omega). \tag{3.7}$$

Furthermore, $\{x_n^1(\omega)\} \subset R_1(\omega)$. Hence

$$r_2(\omega) \leq r_C(R_1(\omega)), \tag{3.8}$$

and thus

$$r_C(R_2(\omega)) \leq \lambda^2 r_1(\omega). \tag{3.9}$$

By induction, for each $m \geq 1$, we construct $R_m(\omega), r_m(\omega)$ and $\{x_n^m(\omega)\}_n$ where $x_n^m(\omega) \in R_m(\omega)$ such that $d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \to 0$ as $n \to \infty$ and

$$r_C(R_m(\omega)) \leq \lambda r_m(\omega) \leq \lambda^m r_1(\omega). \tag{3.10}$$

Since $diamR_m(\omega) \leq 2r_C(R_m(\omega))$ and $\lambda < 1$, it follows that $\lim_{m\to\infty} diamR_m(\omega) = 0$. Since $\{R_m(\omega)\}$ is a descending sequence of weakly compact subset of C for each $\omega \in \Omega$, we have $\cap_m R_m(\omega) = \{z(\omega)\}$ for some $z(\omega) \in C$. Furthermore, we see that

$$H(R_m(\omega), \{z(\omega)\}) \leq diamR_m(\omega) \to 0 \quad \text{as } n \to +\infty.$$

Therefore, by Lemma 2.2, $z(\omega)$ is measurable. Finally, we will show that $z(\omega)$ is a fixed point of T. Indeed, for each $m \geq 1$, we have

$$\begin{aligned} d(z(\omega), T(\omega, z(\omega)) &\leq \|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &+ H(T(\omega, x_n^m(\omega)), T(\omega, z(\omega))) \\ &\leq 2\|z(\omega) - x_n^m(\omega)\| + d(x_n^m(\omega), T(\omega, x_n^m(\omega))) \\ &\leq 2diamR_m(\omega) + d(x_n^m(\omega), T(\omega, x_n^m(\omega))). \end{aligned}$$

Taking the upper limit as $n \to \infty$,

$$d(z(\omega), T(\omega, z(\omega)) \leq 2diamR_m(\omega).$$

Finally, taking limit in m in both sides we obtain $z(\omega) \in T(\omega, z(\omega))$. □

Corollary 3.4. *Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\varepsilon_\beta(X) < 1$, and $T : \Omega \times C \to C$ be a random nonexpansive operator. Then T has a random fixed point.*

Corollary 3.5. (Ramírez, Theorem 6 in [9]) *Let C be a nonempty closed bounded convex separable subset of a Banach spaces X such that $\varepsilon_\alpha(X) < 1$, and $T : \Omega \times C \to C$ be a random nonexpansive operator. Then T has a random fixed point.*

Proof By (2.2) we have that $\varepsilon_\alpha(X) < 1$ implies $\varepsilon_\beta(X) < 1$.

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Ishikawa Iteration Sequences for Asymptotically Quasi-Nonexpansive Nonself-Mappings with Error Members*

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Abstract

Suppose C is a nonempty closed convex retract of a real uniformly convex Banach space X with P as a nonexpansive retraction of X onto C. Let $T : C \to X$ be an asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}_{n\geq 1} \subset [0,\infty), \lim k_n = 0, F(T) = \{x \in C : Tx = x\} \neq \emptyset$. Suppose $\{x_n\}_{n\geq 1}$ is generated iteratively by

$$\begin{aligned} x_1 \in C,\ x_{n+1} &= P((\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n), \\ y_n &= P(\alpha'_n x_n + \beta'_n T(PT)^{n-1} x_n + \gamma'_n v_n), n \geq 1 \end{aligned}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ and $0 < \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n < \beta < 1$. It is prove that if $\sum_{n=1}^{\infty} k_n < \infty$ and T is completely continuous and uniformly $L-$Lipschitzian, $\{x_n\}$ strongly converges to some fixed point $x^* \in F(T)$.

keywords: Asymptotically quasi-nonexpansive nonself-maps; Completely continuous; nonexpansive retraction; uniformly convex

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1 Introduction

Let C be a subset of real normed linear space X, and let T be a self-mapping on C. T is said to be *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that for each $x, y \in C$ and $n \geq 1$, $\|T^n x - T^n y\| \leq (1+k_n)\|x-y\|$. T is said to be an *asymptotically quasi-nonexpansive*, if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - x^*\| \leq (1 + k_n)\|x - x^*\|, \forall x \in C, x^* \in F(T), \tag{1.1}$$

for all $n \geq 1$, ($F(T)$ denotes the set of fixed points of T i.e. $F(T) = \{x \in C : Tx = x\}$). T is said to be an *uniformly L-Lipschitzian*, if there exists a constant $L > 0$ such that for each $x, y \in C$, $\|T^n x - T^n y\| \leq L\|x - y\|$, $\forall n \geq 1$.

From the above definitions, it follows that if $F(T)$ is nonempty then, nonexpansive mapping must be quasi-nonexpansive and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive. But the converse does not hold.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk [?] in 1972. The iterative approximation problems for nonexpansive mapping asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied extensively by Browder [?, ?], Goebel and Kirk [?],Ghosh and Debnath[?] and Liu[?, ?, ?].

In 1991[?] J. Schu introduced a modified Mann iteration process to approximate fixed point of asymptotically nonexpansive self-mappings defined on nonempty closed convex and bounded subsets of Hilbert space H. More precisely, he proved the following theorem:

Theorem JS ([??, Theorem 1.5]). *Let H be a Hilbert space, C closed convex bounded nonempty subset of H. Let $T : C \to C$ be completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^2+2k_n) < \infty$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1 - \epsilon, \forall n \geq 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$, by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, n \geq 1, \tag{1.2}$$

converges strongly to some fixed point of T.

Recently, Chidume, Ofoedu and Zegeye[?] have introduced the class of asymptotically nonexpansive nonself-maps and proved demiclosed principle for such maps. Moreover, they proved the strong and weak convergence theorems of a Mann iteration process for asymptotically nonexpansive nonself-mappings.

It is our purpose in this paper first to introduce the class of asymptotically quasi-nonexpansive nonself-mappings. Moreover, we prove the strong convergence theorem of an Ishikawa iteration sequence with error members for such maps. Our theorem improve and generalized important related results of Chidume, Ofoedu, and Zegeye[?], and Liu[?].

2 Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [?]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\} > 0$$

for all $0 < \epsilon \leq 2$(i.e., $\delta_X(\epsilon)$ is a function $(0,2] \to (0,1)$).

A subset C of X is called *retract* of X if there exists a continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to C$ is called *retraction* if $P^2 = P$. It follows that if a mapping P is a traction, then $Py = y$ for all y in the range of P.

Definition 2.2 (see [?]). Let X be a real normed linear space, C a nonempty subset of X. Let $P : X \to C$ be the nonexpansive retraction of X onto C. A map $T : C \to X$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1+k_n)\|x-y\|; \forall x, y \in C, n \geq 1. \tag{2.1}$$

T is called *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x-y\|; \forall x, y \in C, n \geq 1. \tag{2.2}$$

Theorem 2.3 ([??, Theorem 3.7]). *Let X be a real uniformly convex Banach space, C closed convex nonempty subset of X. Let $T : C \to X$ be completely continuous and asymptotically nonexpansive map with sequence $\{k_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 + 2k_n) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), n \geq 1, \tag{2.3}$$

where P is nonexpansive retraction of X onto C. Then $\{x_n\}$ converges strongly to some fixed point of T.

We shall make use of the following lemmas.

Lemma 2.4 ([??, Lemma 2]). *Let the nonnegative real number sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ satisfy that*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \forall n = 1, 2, ..., \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} c_n < \infty.$$

Then

(1) $\lim_{n\to\infty} a_n$ *exists;*

(2) If $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([?], **J. Schu's Lemma**). *Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1, x_n, y_n \in X, \limsup_{n\to\infty} \|x_n\| \leq a, \limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = a, a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3 Main results

In this section, we give new definition and prove our main theorems.

Definition 3.1. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to X$ is said to be *asymptotically quasi-nonexpansive* nonself-map if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that:

$$\|T(PT)^{n-1}x - x^*\| \leq (1 + k_n)\|x - x^*\|; \forall x \in C, x^* \in F(T), n \geq 1, \tag{3.1}$$

where P is a nonexpansive retraction of X onto C.

Remark 3.2. If T is a self-map, then $PT = T$, so that (??) coincide with (??).

Let C be a nonempty closed convex subset of a real uniformly convex Banach space X. The following iteration process is studied:

$$\begin{aligned} x_1 \in C, x_{n+1} &= P\left(\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n\right), \\ y_n &= P\left(\alpha'_n x_n + \beta'_n T(PT)^{n-1} x_n + \gamma'_n v_n\right) \end{aligned} \tag{3.2}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0,1]$ and P is a nonexpansive retraction of X onto C.

The following lemma is crucial in proving the main Theorem.

Lemma 3.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let $T : C \to X$ be an asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Then the sequence $\{x_n\}$ defined by (??) satisfies the following:*

1. *For each $x^* \in F(T)$ and each $n \geq 1$, we have $\|x_{n+1} - x^*\| \leq (1 + k_n)^2 \|x_n - x^*\| + d_n$, where $\{d_n\}$ is a nonnegative sequence with $\sum_{n=1}^{\infty} d_n < \infty$.*

2. *For each $m \geq 1$, there exists a constant $K > 0$ such that $\|x_{n+m} - x^*\| \leq K\|x_n - x^*\| + K \sum_{j=1}^{\infty} d_j; \forall x^* \in F(T), n \geq 1$.*

Proof. Let $x^* \in F(T)$ and $M = \sup_{n \geq 1}\{\|u_n - x^*\| \vee \|v_n - x^*\|\}$. Then, for each $n \geq 1$, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|P\left(\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n\right) - Px^*\| \\ &\leq \|\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n - x^*\| \\ &\leq \alpha_n \|x_n - x^*\| + \beta_n \|T(PT)^{n-1} y_n - x^*\| + \gamma_n \|u_n - x^*\| \\ &\leq \alpha_n \|x_n - x^*\| + \beta_n (1 + k_n) \|y_n - x^*\| + \gamma_n M \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|y_n - x^*\| &= \|P\left(\alpha'_n x_n + \beta'_n T(PT)^{n-1} x_n + \gamma'_n v_n\right) - Px^*\| \\ &\leq \|\alpha'_n x_n + \beta'_n T(PT)^{n-1} x_n + \gamma'_n v_n - x^*\| \\ &\leq \alpha'_n \|x_n - x^*\| + \beta'_n \|T(PT)^{n-1} x_n - x^*\| + \gamma'_n \|v_n - x^*\| \\ &\leq \alpha'_n \|x_n - x^*\| + \beta'_n (1 + k_n) \|x_n - x^*\| + \gamma'_n M. \end{aligned} \tag{3.4}$$

Substituting (??) into (??), it can be obtained that

$$
\begin{aligned}
\|x_{n+1}-x^*\| &\leq \alpha_n\|x_n-x^*\|+\beta_n\alpha_n'(1+k_n)\|x_n-x^*\| \\
&+ \beta_n\beta_n'(1+k_n)^2\|x_n-x^*\|+\beta_n(1+k_n)\gamma_n'M+M\gamma_n \\
&\leq \alpha_n\|x_n-x^*\|+(1-\alpha_n)\alpha_n'(1+k_n)^2\|x_n-x^*\| \\
&+ (1-\alpha_n)\beta_n'(1+k_n)^2\|x_n-x^*\|+d_n \\
&= \alpha_n\|x_n-x^*\|+(1-\alpha_n)(1+k_n)^2(\alpha_n'+\beta_n')\|x_n-x^*\|+d_n \\
&\leq \alpha_n\|x_n-x^*\|+(1-\alpha_n)(1+k_n)^2\|x_n-x^*\|+d_n \\
&\leq \alpha_n\|x_n-x^*\|+(1-\alpha_n+2k_n+k_n^2)\|x_n-x^*\|+d_n \\
&= (1+2k_n+k_n^2)\|x_n-x^*\|+d_n \\
&= (1+k_n)^2\|x_n-x^*\|+d_n
\end{aligned}
$$

where $d_n=(1+k_n)\gamma_n'M+M\gamma_n$. Since $\sum_{n=1}^{\infty}k_n<\infty$ and $\sum_{n=1}^{\infty}\gamma_n<\infty$, we have $\sum_{n=1}^{\infty}d_n<\infty$.

We now to prove (2). Notice that when $x\geq 0, 1+x\leq e^x$. For any $x^*\in F(T)$, it follows from (1) that

$$
\begin{aligned}
\|x_{n+m}-x^*\| &\leq (1+k_{n+m-1})^2\|x_{n+m-1}-x^*\|+d_{n+m-1} \\
&\leq e^{2k_{n+m-1}}\|x_{n+m-1}-x^*\|+d_{n+m-1} \\
&\leq e^{2(k_{n+m-1}+k_{n+m-2})}\|x_{n+m-2}-x^*\|+e^{2k_{n+m-1}}d_{n+m-2}+d_{n+m-1} \\
&\vdots \\
&\leq e^{2\sum_{j=n}^{n+m-1}k_j}\|x_n-x^*\|+e^{\sum_{j=n}^{n+m-1}k_j}\sum_{j=n}^{n+m-1}d_j \\
&\leq e^{2\sum_{j=1}^{\infty}k_j}\|x_n-x^*\|+e^{2\sum_{j=1}^{\infty}k_j}\sum_{j=1}^{\infty}d_j \\
&\leq K\|x_n-x^*\|+K\sum_{j=1}^{\infty}d_j, \forall n\geq 1,
\end{aligned}
$$

where $K=e^{2\sum_{j=1}^{\infty}k_j}>0$. Thus,

$$
\|x_{n+m}-x^*\|\leq K\|x_n-x^*\|+K\sum_{j=1}^{\infty}d_j, \forall n,m\geq 1, x^*\in F(T).
$$

This completes the proof of (2). □

We now to prove the following theorems.

Theorem 3.4. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let $T : C \to X$ be an asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C, \{\alpha_n\}, \{\beta_n\}$ $\{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Then the sequence $\{x_n\}$ defined by (??) strongly converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T))$ denote the distance of x to the set $F(T)$, i.e., $d(x, F(T)) = \inf_{y\in F(T)} d(x, y)$.*

Proof. The necessity of the conditions is obvious. Thus we will only prove the sufficiency. For any $x^* \in F(T)$, from (??), it follows by Lemma ?? that

$$\|x_{n+1} - x^*\| \leq (1 + k_n)^2 \|x_n - x^*\| + d_n, \ \forall n \geq 1.$$

This implies that $d(x_{n+1}, F(T)) \leq (1 + k_n)^2 d(x_n, F(T)) + d_n$. From Lemma ??, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hereafter, we will prove that $\{x_n\}$ is a Cauchy sequence. From Lemma ??, there exists a constant $K > 0$ such that

$$\|x_{n+m} - x^*\| \leq K\|x_n - x^*\| + K \sum_{j=1}^{\infty} d_j, \forall n, m \geq 1. \tag{3.5}$$

Let $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a natural number N_1 such that for each $n \geq N_1$

$$d(x_n, F(T)) < \frac{\epsilon}{3K} \text{ and } \sum_{k=n}^{\infty} d_k < \frac{\epsilon}{6K}.$$

In particular, we have $d(x_{N_1}, F(T)) < \frac{\epsilon}{3K}$. This implies that there exists a point $y' \in F(T)$ such that

$$\|x_{N_1} - y'\| < \frac{\epsilon}{3K}.$$

It follows, from (??), that when $n, m \geq N_1$,

$$\|x_{n+m} - x_n\| \leq \|x_{m+n} - y'\| + \|x_n - y'\| \leq K\|x_{N_1} - y'\| + K\|x_{N_1} - y'\| + 2K \sum_{j=1}^{\infty} d_j < \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{x_n\}$ is convergent. Let $\lim_{n\to\infty} x_n = y$. Moreover, we note that

$$d(y, F(T)) \leq d(x_n, F(T)) + \|x_n - y\|, \forall n \geq 1.$$

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and the set $F(T)$ is closed, we have $y \in F(T)$, i.e. y is a fixed point of T. This completes the proof. □

Corollary 3.5. *Suppose the conditions are as same as in Theorem ??. Then the sequence $\{x_n\}$ generated by (??) converges to a fixed point of T if and only if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to y.*

Theorem 3.6. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let $T : C \to X$ be an uniformly L-Lipschitzian completely continuous and asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}$ in $[0,\infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_1 \in C, \{\alpha_n\}, \{\beta_n\}$ $\{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0,1]$ such that $0 < \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n < \beta < 1, \alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n, \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$ Then the sequence $\{x_n\}$ defined by (??) strongly converges to a fixed point of T.*

Proof. Let $x^* \in F(T)$. From Lemma ??, we have

$$\|x_{n+1} - x^*\| \leq (1 + k_n)^2 \|x_n - x^*\| + d_n, \forall n \geq 1.$$

Since $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, it follows by Lemma ?? that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\| = c$ for some $c \geq 0$. From the proof of Lemma ??, we have that

$$\|y_n - x^*\| \leq (1 + k_n)\|x_n - x^*\| + \gamma'_n \|v_n - x^*\|, \forall n \geq 1.$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty} \|y_n - x^*\| \leq \limsup_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|x_n - x^*\| = c.$$

Note that

$$\limsup_{n\to\infty} \|T(PT)^{n-1} y_n - x^*\| \leq \limsup_{n\to\infty} \|y_n - x^*\| \leq \lim_{n\to\infty} \|x_n - x^*\| = c,$$

and by (??), we have

$$\begin{aligned}
c = \lim_{n\to\infty} \|x_{n+1} - x^*\| &\leq \lim_{n\to\infty} \|\alpha_n x_n + \beta_n T(PT)^{n-1} y_n + \gamma_n u_n - x^*\| \\
&= \lim_{n\to\infty} \|\alpha_n[(x_n - x^*) + \frac{\gamma_n}{2\alpha_n}(u_n - x^*)] \\
&\quad + \beta_n[(T(PT)^{n-1} y_n - x^*) + \frac{\gamma_n}{2\beta_n}(u_n - x^*)]\| \\
&\leq \lim_{n\to\infty} \alpha_n \|x_n - x^*\| + \lim_{n\to\infty} \beta_n \|T(PT)^{n-1} y_n - x^*\| \\
&\leq \lim_{n\to\infty} \alpha_n \|x_n - x^*\| + \lim_{n\to\infty} \beta_n (1 + k_n)\|y_n - x^*\| \\
&= \lim_{n\to\infty} (\alpha_n \|x_n - x^*\| + \beta_n (1 + k_n)\|y_n - x^*\|) \\
&\leq \lim_{n\to\infty} (\alpha_n \|x_n - x^*\| + \beta_n (1 + k_n)^2 \|x_n - x^*\| \\
&\quad + \gamma'_n \beta_n (1 + k_n)\|v_n - x^*\|)
\end{aligned}$$

$$\begin{aligned}
&\le \lim_{n\to\infty}(\alpha_n(1+k_n)^2\|x_n-x^*\|+(1-\alpha_n)(1+k_n)^2\|x_n-x^*\|)\\
&\le \lim_{n\to\infty}(1+k_n)^2\|x_n-x^*\|=c.
\end{aligned}$$

Hence

$$c=\lim_{n\to\infty}\|\alpha_n[(x_n-x^*)+\frac{\gamma_n}{2\alpha_n}(u_n-x^*)]+\beta_n[(T(PT)^{n-1}y_n-x^*)+\frac{\gamma_n}{2\beta_n}(u_n-x^*)]\|$$

By J. Schu's Lemma, we have

$$\lim_{n\to\infty}\|x_n-T(PT)^{n-1}y_n+(\frac{\gamma_n}{2\alpha_n}-\frac{\gamma_n}{2\beta_n})(u_n-x^*)\|=0.$$

Since $\lim_{n\to\infty}\|(\frac{\gamma_n}{2\alpha_n}-\frac{\gamma_n}{2\beta_n})(u_n-x^*)\|=0$. Then

$$\lim_{n\to\infty}\|x_n-T(PT)^{n-1}y_n\|=0.$$

It follows that

$$\|x_{n+1}-x_n\|\le\alpha_n\|x_n-T(PT)^{n-1}y_n\|\to 0 \text{ as } n\to\infty. \tag{3.6}$$

On the other hand, we obtain that

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le \|\alpha_n(x_n-x^*)+\beta_n(T(PT)^{n-1}y_n-x^*)+\gamma_n(u_n-x^*)\|\\
&\le \alpha_n\|x_n-x^*\|+\beta_n(1+k_n)\|y_n-x^*\|+\gamma_n\|u_n-x^*\|\\
&\le \alpha_n\|x_n-T(PT)^{n-1}y_n\|+\alpha_n\|T(PT)^{n-1}y_n-x^*\|\\
&\quad+\beta_n(1+k_n)\|y_n-x^*\|+\gamma_n\|u_n-x^*\|\\
&\le \alpha_n\|x_n-T(PT)^{n-1}y_n\|+\alpha_n(1+k_n)\|y_n-x^*\|\\
&\quad+\beta_n(1+k_n)\|y_n-x^*\|+\gamma_n\|u_n-x^*\|\\
&\le \alpha_n\|x_n-T(PT)^{n-1}y_n\|+(1-\beta_n)(1+k_n)\|y_n-x^*\|\\
&\quad+\beta_n(1+k_n)\|y_n-x^*\|+\gamma_n\|u_n-x^*\|\\
&\le \alpha_n\|x_n-T(PT)^{n-1}y_n\|+(1+k_n)\|y_n-x^*\|+\gamma_n\|u_n-x^*\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-T(PT)^{n-1}y_n\|=0=\lim_{n\to\infty}\gamma_n$, it follows that

$$c=\lim_{n\to\infty}\|x_n-x^*\|\le\liminf_{n\to\infty}\|y_n-x^*\|.$$

Hence

$$c\le\liminf_{n\to\infty}\|y_n-x^*\|\le\limsup_{n\to\infty}\|y_n-x^*\|\le c,$$

and so

$$\lim_{n\to\infty}\|y_n-x^*\|=c.$$

This implies that

$$
\begin{aligned}
c = \lim_{n\to\infty} \|y_n - x^*\| &\le \lim_{n\to\infty} \|\alpha'_n x_n + \beta'_n T(PT)^{n-1}x_n + \gamma'_n v_n - x^*\| \\
&= \lim_{n\to\infty} \|\alpha'_n[(x_n - x^*) + \frac{\gamma'_n}{2\alpha'_n}(v_n - x^*)] \\
&\quad + \beta'_n[(T(PT)^{n-1}x_n - x^*) + \frac{\gamma'_n}{2\beta'_n}(v_n - x^*)]\| \\
&\le \lim_{n\to\infty} \alpha'_n\|x_n - x^*\| + \lim_{n\to\infty} \beta'_n\|T(PT)^{n-1}x_n - x^*\| \\
&\le \lim_{n\to\infty} \alpha'_n\|x_n - x^*\| + \lim_{n\to\infty} \beta'_n(1+k_n)\|x_n - x^*\| \\
&\le \lim_{n\to\infty} (\alpha'_n\|x_n - x^*\| + (1-\alpha'_n)(1+k_n)\|x_n - x^*\|) \\
&\le \lim_{n\to\infty} (1+k_n)\|x_n - x^*\| = c.
\end{aligned}
$$

Then

$$c = \lim_{n\to\infty} \|\alpha'_n[(x_n - x^*) + \frac{\gamma'_n}{2\alpha'_n}(v_n - x^*)] + \beta'_n[(T(PT)^{n-1}x_n - x^*) + \frac{\gamma'_n}{2\beta'_n}(v_n - x^*)]\|.$$

By J. Schu's Lemma, we have

$$\lim_{n\to\infty} \|x_n - T(PT)^{n-1}x_n + (\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n})(v_n - x^*)\| = 0.$$

Since $\lim_{n\to\infty} \|(\frac{\gamma'_n}{2\alpha'_n} - \frac{\gamma'_n}{2\beta'_n})(v_n - x^*)\| = 0$, it follows that

$$\lim_{n\to\infty} \|x_n - T(PT)^{n-1}x_n\| = 0. \tag{3.7}$$

We now to show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. First, by (??) and (??), we note that

$$
\begin{aligned}
\|x_n - T(PT)^{n-2}x_n\| &\le \|x_n - x_{n-1}\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\| \\
&\quad + \|T(PT)^{n-2}x_{n-1} - T(PT)^{n-2}x_n\| \\
&\le \|x_n - x_{n-1}\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\| + L\|x_{n-1} - x_n\| \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Thus from the above inequality, we have

$$
\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{n-1}x_n - Tx_n\| \\
&= \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{1-1}(PT)^{n-1}x_n - T(PT)^{1-1}x_n\| \\
&\le \|x_n - T(PT)^{n-1}x_n\| + L\|PT(PT)^{n-2}x_n - x_n\| \\
&\le \|x_n - T(PT)^{n-1}x_n\| + L\|T(PT)^{n-2}x_n - x_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

It implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.8}$$

We note from Lemma ?? that $\{x_n\}$ is bounded. Since T is completely continuous, it follows that there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to q \in C$ as $k \to \infty$. Moreover, by (??), we have $\|Tx_{n_k} - x_{n_k}\| \to 0$ which implies that $x_{n_k} \to q$ as $k \to \infty$. By (??) again, we have

$$\|q - Tq\| = \lim_{k \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

It show that $q \in F(T)$. Furthermore, since $\lim_{n\to\infty} \|x_n - q\|$ exists. Therefore $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

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บทที่ 7
ทฤษฎีความน่าจะเป็น และ การแปลงเมทริกซ์ของปริภูมิลำดับ
(Probability theory and matrix transformations of sequence spaces)

ในบทนี้แบ่งออกเป็นสองหัวข้อ คือ หัวข้อแรกเป็นการศึกษาทฤษฎีต่างๆของความน่าจะเป็น และ หัวข้อที่สองเป็นการศึกษาเกี่ยวกับการแปลงเมทริกซ์ของปริภูมิลำดับ ในหัวข้อแรกนั้นเป็นการศึกษาเกี่ยวกับหา non-uniform bound สำหรับ Poisson binomial distribution และ การประมาณแบบ pointwise ของ poisson distribution นอกจากนั้นเป็นการหาค่าประมาณความคลาดเคลื่อนบนทฤษฎี combinatorial central limit theorem สำหรับในหัวข้อที่สองนั้นเป็นการหาเงื่อนไขที่จำเป็นและเพียงพอสำหรับเมทริกซ์อนันต์ ที่ส่งจากปริภูมิลำดับหนึ่งไปยังอีกปริภูมิลำดับหนึ่ง

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A NONUNIFORM BOUND FOR THE APPROXIMATION OF POISSON BINOMIAL BY POISSON DISTRIBUTION

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It is well known that Poisson binomial distribution can be approximated by Poisson distribution. In this paper, we give a nonuniform bound of this approximation by using Stein-Chen method.

2000 Mathematics Subject Classification: 60F05, 60G50.

1. Introduction and main result. Let $X_1, X_2, \dots, X_n$ be independent, possibly not identically distributed, Bernoulli random variables with $P(X_i = 1) = 1 - P(X_i = 0) = p_i$ and let $S_n = X_1 + X_2 + \cdots + X_n$. The sum of this kind is often called a Poisson binomial random variable. In the case where the "success" probabilities are all identical, $p_i = p$, S is the binomial random variable $\mathcal{B}(n,p)$. Let $\lambda = \sum_{i=1}^{n} p_i$ and let $\mathcal{P}_\lambda$ be the Poisson random variable with parameter λ, that is, $P(\mathcal{P}_\lambda = \omega) = e^{-\lambda}\lambda^\omega/\omega!$ for all nonnegative integers ω. It has long been known that if p_i's are small, then the distribution of S_n can be approximated by a distribution of $\mathcal{P}_\lambda$ (see, e.g., Chen [2]).

In this paper, we investigate the bound of this approximation. As an illustration, we look at the case of $p_1 = p_2 = \cdots = p_n = p$. There are at least three known uniform bounds: Kennedy and Quine [6] showed that, for $0 < \lambda \le 2 - \sqrt{2}$,

$$\left|P(S_n \le \omega) - P(\mathcal{P}_\lambda \le \omega)\right| \le 2\lambda\left[(1-p)^{n-1} - e^{-np}\right], \tag{1.1}$$

Barbour and Hall [1] showed that

$$\left|P(S_n \le \omega) - P(\mathcal{P}_\lambda \le \omega)\right| \le \min(p, \lambda p), \tag{1.2}$$

and Deheuvels and Pfeifer [5] proved that

$$\begin{aligned}\left|P(S_n \le \omega) - P(\mathcal{P}_\lambda \le \omega)\right| \\ \le \lambda p e^{-\lambda}\left\{\frac{(np)^{(a-1)}(a-np)}{a!} - \frac{(np)^{(b-1)}(b-np)}{b!}\right\} + R\end{aligned} \tag{1.3}$$

with $a = [np + 1/2 + \sqrt{np + 1/4}]$, $b = [np + 1/2 - \sqrt{np + 1/4}]$, and $|R| \le (1/2)(2p)^{3/2}/(1 - \sqrt{2p})$, for $0 < p < 1/2$, and $[x]$ is understood to be the integer part of x.

For the general case, Le Cam [7] investigated and showed that

$$\sum_{\omega=0}^{\infty}\left|P(S_n=\omega)-\frac{e^{-\lambda}\lambda^{\omega}}{\omega!}\right|\le\frac{16}{\lambda}\sum_{i=1}^{n}p_i^2. \tag{1.4}$$

It can be observed that the constant $16/\lambda$ will be large when λ is small. Stein [10] used the method of Chen [3] to improve the bound and showed that

$$|P(S_n\le\omega)-P(\mathcal{P}_\lambda\le\omega)|\le(\lambda^{-1}\wedge 1)\sum_{i=1}^{n}p_i^2 \tag{1.5}$$

for $\omega=0,1,2,\dots,n$ and $\lambda^{-1}\wedge 1=\min(\lambda^{-1},1)$. In case when λ tends to 0, one can see that (1.5) becomes

$$|P(S_n\le\omega)-P(\mathcal{P}_\lambda\le\omega)|\le\sum_{i=1}^{n}p_i^2. \tag{1.6}$$

In this paper, we consider a nonuniform bound when λ is small, that is, $\lambda\in(0,1]$ and $\omega\in\{1,2,\dots,n-1\}$. Note that, when $\omega\notin\{1,2,\dots,n-1\}$, we can compute the exact probabilities, that is,

$$\begin{gathered}P(S_n=0)=\prod_{i=1}^{n}(1-p_i),\qquad P(S_n=n)=\prod_{j=1}^{n}p_j,\\ P(S_n=\omega)=0,\quad \omega=n+1,n+2,\dots.\end{gathered} \tag{1.7}$$

In finding the uniform bound, there are several techniques which can be used; for example,

(i) the operator method initiated in Le Cam [7],
(ii) the semigroup approach due to Deheuvels and Pfeifer [4],
(iii) the Chen-Stein technique, see Chen [3] and Stein [10],
(iv) direct computations as in Kennedy and Quine [6],
(v) the coupling method, see Serfling [8] and Stein [10].

In the present paper, our argument closely follows the Chen-Stein technique in Chen [3] and Stein [10]. The following theorem is our main result.

Theorem 1.1. *Let $\lambda\in(0,1]$ and $\omega_0\in\{1,2,\dots,n-1\}$. Then*

$$|P(S_n=\omega_0)-P(\mathcal{P}_\lambda=\omega_0)|\le\frac{1}{\omega_0}\sum_{i=1}^{n}p_i^2. \tag{1.8}$$

2. Proof of the main result. Stein [9] gave a new technique to find a bound in the normal approximation to a distribution of a sum of dependent random variables. His technique was free from Fourier methods and relied instead on the elementary differential equation

$$f'(\omega)-wf(\omega)=h(\omega)-N(h), \tag{2.1}$$

where h is a function that is used to test convergence and $N(h) = E[h(Z)]$ where Z is the standard normal. Chen [3] applied Stein's ideas in the Poisson setting. Corresponding to the differential equation in the normal case above, one has an analogous difference equation

$$\lambda f(\omega+1) - \omega f(\omega) = h(\omega) - \mathcal{P}_\lambda(h), \tag{2.2}$$

where $\mathcal{P}_\lambda(h) = E[h(\mathcal{P}_\lambda)]$ and f and h are real-valued functions defined on $\mathbb{Z}^+ \cup \{0\}$. Let $\omega_0 \in \{1,2,\dots,n-1\}$ and define $h, h_{\omega_0} : \mathbb{Z}^+ \cup \{0\} \to \mathbb{R}$ by

$$h(\omega) = \begin{cases} 1, & \text{if } \omega = \omega_0, \\ 0, & \text{if } \omega \neq \omega_0, \end{cases} \qquad h_{\omega_0}(\omega) = \begin{cases} 1, & \text{if } \omega \le \omega_0, \\ 0, & \text{if } \omega > \omega_0. \end{cases} \tag{2.3}$$

Then we see that the solution f of (2.2) can be expressed in the form

$$f_{\omega_0}(\omega) = \begin{cases} \dfrac{(\omega-1)!}{\omega_0!}\lambda^{\omega_0-\omega}\mathcal{P}_\lambda(1-h_{\omega-1}), & \text{if } \omega_0 < \omega, \\ -\dfrac{(\omega-1)!}{\omega_0!}\lambda^{\omega_0-\omega}\mathcal{P}_\lambda(h_{\omega-1}), & \text{if } \omega_0 \ge \omega > 0, \\ 0, & \text{if } \omega = 0, \end{cases} \tag{2.4}$$

$$\lambda E[f_{\omega_0}(S_n+1)] - E[S_n f_{\omega_0}(S_n)] = P(S_n = \omega_0) - P(\mathcal{P}_\lambda = \omega_0). \tag{2.5}$$

Let $S_n^{(i)} = S_n - X_i$ for $i = 1,2,\dots,n$. By using the facts that each X_j takes on values 0 and 1 and that X_j's are independent, we have

$$\begin{aligned} E[S_n f_{\omega_0}(S_n)] &= \sum_{i=1}^{n} p_i E[f(S_n^{(i)}+1)] \\ &= \lambda E[f_{\omega_0}(S_n+1)] + \sum_{i=1}^{n} p_i E[f_{\omega_0}(S_n^{(i)}+1) - f_{\omega_0}(S_n+1)] \\ &= \lambda E[f_{\omega_0}(S_n+1)] + \sum_{i=1}^{n} p_i E\{X_i[f_{\omega_0}(S_n^{(i)}+1) - f_{\omega_0}(S_n^{(i)}+2)]\} \\ &= \lambda E[f_{\omega_0}(S_n+1)] + \sum_{i=1}^{n} p_i^2 E[f_{\omega_0}(S_n^{(i)}+1) - f_{\omega_0}(S_n^{(i)}+2)], \end{aligned} \tag{2.6}$$

which implies, by (2.5), that

$$P(S_n = \omega_0) - P(\mathcal{P}_\lambda = \omega_0) = \sum_{i=1}^{n} p_i^2 E[f_{\omega_0}(S_n^{(i)}+2) - f_{\omega_0}(S_n^{(i)}+1)]. \tag{2.7}$$

From (2.4), it follows that

$$f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)$$
$$=\begin{cases}-\lambda^{\omega_0-\omega-2}\dfrac{\omega!}{\omega_0!}[(\omega+1)\mathcal{P}_\lambda(h_{\omega+1})-\lambda\mathcal{P}_\lambda(h_\omega)], & \text{if } \omega\le\omega_0-2,\\ \lambda^{\omega_0-\omega-2}\dfrac{\omega!}{\omega_0!}[(\omega+1)\mathcal{P}_\lambda(1-h_{\omega+1})+\lambda\mathcal{P}_\lambda(h_\omega)], & \text{if } \omega=\omega_0-1,\\ \lambda^{\omega_0-\omega-2}\dfrac{\omega!}{\omega_0!}[(\omega+1)\mathcal{P}_\lambda(1-h_{\omega+1})-\lambda\mathcal{P}_\lambda(1-h_\omega)], & \text{if } \omega\ge\omega_0.\end{cases} \tag{2.8}$$

CASE 1 ($\omega\le\omega_0-2$). Since

$$(\omega+1)\mathcal{P}_\lambda(h_{\omega+1})-\lambda\mathcal{P}_\lambda(h_\omega)=e^{-\lambda}\sum_{k=0}^{\omega+1}\frac{\lambda^k}{k!}(\omega+1-k), \tag{2.9}$$

we have

$$\begin{aligned}|f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)| &= \lambda^{(\omega_0-2)-\omega}\frac{\omega!}{\omega_0!}\left[e^{-\lambda}\sum_{k=0}^{\omega+1}\frac{\lambda^k}{k!}(\omega+1-k)\right]\\ &\le\frac{(\omega+1)!}{\omega_0!}\left[e^{-\lambda}\sum_{k=0}^{\omega+1}\frac{\lambda^k}{k!}\right]\\ &\le\frac{(\omega_0-1)!}{\omega_0!}\\ &=\frac{1}{\omega_0},\end{aligned} \tag{2.10}$$

where we have used the facts that $\lambda\in(0,1]$ and $0\le\omega+1-k\le\omega+1$ in the first inequality and the conditions $\omega\le\omega_0-2$ and $e^{-\lambda}\sum_{k=0}^{\omega+1}(\lambda^k/k!)\le1$ in the second inequality.

CASE 2 ($\omega=\omega_0-1$). We have

$$\begin{aligned}|f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)| &= \frac{\lambda^{-1}}{\omega_0}\left[\omega_0e^{-\lambda}\sum_{k=\omega_0+1}^{\infty}\frac{\lambda^k}{k!}+\lambda e^{-\lambda}\sum_{k=0}^{\omega_0-1}\frac{\lambda^k}{k!}\right]\\ &\le\frac{\lambda^{-1}}{\omega_0}\left[e^{-\lambda}\sum_{k=\omega_0+1}^{\infty}k\frac{\lambda^k}{k!}+e^{-\lambda}\sum_{k=0}^{\omega_0-1}(k+1)\frac{\lambda^{k+1}}{(k+1)!}\right]\\ &=\frac{\lambda^{-1}}{\omega_0}E[\mathcal{P}_\lambda]\\ &=\frac{1}{\omega_0}.\end{aligned} \tag{2.11}$$

CASE 3 ($\omega \geq \omega_0$). Since

$$\begin{aligned}
&\frac{1\lambda^{\omega+2}}{(\omega+2)!}+\frac{2\lambda^{\omega+3}}{(\omega+3)!}+\frac{3\lambda^{\omega+4}}{(\omega+4)!}+\cdots \\
&\quad\leq \lambda^{\omega-\omega_0+2}\left[\frac{\omega_0\lambda^{\omega_0}}{\omega_0!(\omega_0+1)\cdots(\omega+2)}+\frac{(\omega_0+1)\lambda^{\omega_0+1}}{(\omega_0+1)!(\omega_0+2)\cdots(\omega+3)}+\cdots\right] \\
&\quad\leq \frac{\lambda^{\omega-\omega_0+2}}{(\omega_0+1)(\omega_0+2)\cdots(\omega+2)}\left[\sum_{k=\omega_0}^{\infty}\frac{k\lambda^k}{k!}\right] \\
&\quad\leq \frac{e^{\lambda}\lambda^{\omega-\omega_0+2}E[\mathcal{P}_\lambda]}{(\omega_0+1)(\omega_0+2)\cdots(\omega+2)} \\
&\quad= \frac{e^{\lambda}\lambda^{\omega-\omega_0+3}}{(\omega_0+1)(\omega_0+2)\cdots(\omega+2)}, \\
&(\omega+1)\mathcal{P}_\lambda(1-h_{\omega+1})-\lambda\mathcal{P}_\lambda(1-h_\omega)=-e^{-\lambda}\sum_{k=\omega+2}^{\infty}\frac{\lambda^k}{k!}(k-(\omega+1))<0,
\end{aligned} \tag{2.12}$$

we have

$$\begin{aligned}
|f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)| &= \lambda^{\omega_0-\omega-2}\frac{\omega!}{\omega_0!}e^{-\lambda}\sum_{k=\omega+2}^{\infty}\frac{\lambda^k}{k!}(k-(\omega+1)) \\
&\leq \frac{\lambda\omega!}{(\omega+2)!}\leq\frac{1}{(\omega+1)(\omega+2)}.
\end{aligned} \tag{2.13}$$

From Cases 1, 2, and 3, we conclude that

$$|f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)| \leq \frac{1}{\omega_0}. \tag{2.14}$$

By (2.7) and (2.14), we have

$$\begin{aligned}
&|P(S_n=\omega_0)-P(\mathcal{P}_\lambda=\omega_0)| \\
&\quad\leq\left(\sum_{i=1}^{n}p_i^2\right)E[|f_{\omega_0}(S_n^{(i)}+2)-f_{\omega_0}(S_n^{(i)}+1)|]\leq\frac{1}{\omega_0}\sum_{i=1}^{n}p_i^2.
\end{aligned} \tag{2.15}$$

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Pointwise Approximation of Poisson Binomial by Poisson Distribution

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Abstract

It is well known that Poisson binomial distribution can be approximated by Poisson distribution. In this paper, we improve a non-uniform bound of this approximation by using Stein's method. Our result is better than Neammanee (2003).

Keywords and Phrases : Poisson binomial distribution, Poisson distribution and Stein's method.

AMS2000 Subject Classification: Primary 60F05, Secondary 60G05.

1. Introduction and Main Results

Let $X_1, X_2, \ldots, X_n$ be independent, possibly not identically distributed, Bernoulli random variables with $P(X_j = 1) = 1 - P(X_j = 0) = p_j$ and let $S_n = X_1 + X_2 + \cdots + X_n$. The sum of this kind is often called a Poisson binomial random variable. In the case where the "success" are all identical.

$p_j = p$, S_n is the binomial random variable $B(n,p)$. Let $\lambda = \sum_{j=1}^{n} p_j$ and P_λ be the Poisson random variable with parameter λ, i.e., $P(P_\lambda = \omega) = \frac{e^{-\lambda}\lambda^\omega}{\omega!}$ for all non-negative integer ω. It has long been known that if p_j's are small then the distribution of S_n has approximately the distribution of P_λ (see for example Chen(1974)). Many authors (see for examples, Barbour and Hall (1984), Kennedy and Quine (1989), Deheuvels and Pfeifer (1986,1988)) investigated the case of $p_1 = p_2 = \cdots = p_n$. But in this work, we give a non-uniform bound when the p_j's are not equal. This case was first investigated by Le Cam (1960). For a uniform bound, Stein (1986) used the Stein's method to find a bound and show that

$$|P(S_n \le \omega) - P(P_\lambda \le \omega)| \le \min(\lambda^{-1}, 1)\sum_{j=1}^{n} p_j^2. \tag{1.1}$$

In case of pointwise approximation, Neammanee (2003) gave a bound in the form of

$$|P(S_n = \omega_0) - P(P_\lambda = \omega_0)| \le \frac{1}{\omega_0}\sum_{j=1}^{n} p_j^2 \tag{1.2}$$

for $\omega_0 \in \{1,2,\ldots,n-1\}$ and $\lambda \in (0,1]$. In this paper, we will improve (1.2) for the case of all positive λ. Here is our main result.

Theorem 1. *Let* $\omega_0 \in \{1,2,\ldots,n-1\}$. *Then*

$$|P(S_n = \omega_0) - P(P_\lambda = \omega_0)| \le \min(\frac{1}{\omega_0}, \lambda^{-1})\sum_{j=1}^{n} p_j^2. \tag{1.3}$$

We note that in case of $\omega_0 = 0$ and $\omega_0 \ge n$, one can compute the exact probability of S_n.

These are $P(S_n = 0) = \prod_{j=1}^{n}(1-p_j)$, $P(S_n = n) = \prod_{j=1}^{n} p_j$ and $P(S_n = \omega) = 0$ for $\omega > n$.

2. Proof of Main Result

We will prove our result by using Stein's method for Poisson distribution. In 1972, Stein gave a new technique to find a bound in the normal approximation by using differential equation and Chen(1975) applied Stein's idea to the Poisson case. The Stein's equation for Poisson distribution with parameter λ is

$$\lambda f(\omega+1) - \omega f(\omega) = h(\omega) - P_\lambda(h) \tag{2.1}$$

when $P_\lambda(h) = E[h(P_\lambda)]$ and f and h are real valued functions defined on $Z^+ \cup \{0\}$. For any given function h, a solution of (2.1) is

$$U_\lambda h(\omega) = \sum_{l=0}^{\omega-1}\frac{(\omega-1)!}{l!}\lambda^{l-\omega}[h(l) - P_\lambda(h)] \tag{2.2}$$

where $\omega = 1,2,\ldots$ and $U_\lambda h(0) = 0$. We also need the following proposition to prove the main result (1.3).

Proposition 2.1

$$Eh(S_n) = P_\lambda(h) + \sum_{j=1}^{n} p_j^2 E[V_\lambda h(S_n^{(j)})] \tag{2.3}$$

where $V_\lambda h(\omega) = U_\lambda h(\omega+2) - U_\lambda h(\omega+1)$ *and* $S_n^{(j)} = S_n - X_j$.

Proof Let $X_1^*, X_2^*, \ldots, X_n^*$ be independent random variables independent of the X_j's. Assume that for each j, X_j^* has the same distribution as X_j. With J uniformly distributed on

$\{1,2,\ldots,n\}$ independent of the X_j's and X_j^*'s, let $S_n^* = S_n - X_J + X_J^*$. Then (S_n, S_n^*) is an exchangeable pair, i.e., $P(S_n \in A, S_n^* \in B) = P(S_n \in B,\ S_n^* \in A)$ for all events A and B. For arbitrary $f: Z^+ \cup \{0\} \to R$, let $F: \Omega^2 \to R$ be defined by

$$F(x,x^*) = f(x^*)I(x^* = x+1) - f(x)I(x = x^* + 1)$$

where I is an indicator function. Then F is anti-symmetric in the sense of $F(x,x^*) = -F(x,x^*)$. By p.10 in Stein (1986), we have $E[F(S_n, S_n^*)] = 0$. So

$$\begin{aligned}
0 &= EE^{S_n}[f(S_n^*)I(S_n^* = S_n + 1) - f(S_n)I(S_n = S_n^* + 1)] \\
&= E[f(S_n + 1)E^{S_n}(S_n^* = S_n + 1) - f(S_n)E^{S_n}(S_n = S_n^* - 1)] \\
&= \frac{1}{n}E[f(S_n + 1)\sum_{j=1}^{n} E^{S_n} I(X_j^* - X_j = 1) - f(S_n)\sum_{j=1}^{n} E^{S_n} I(X_j - X_j^* = 1)] \\
&= \frac{1}{n}E[f(S_n + 1)\sum_{j=1}^{n} p_j I(X_j = 0) - f(S_n)\sum_{j=1}^{n}(1 - p_j)I(X_j = 1)] \\
&= \frac{1}{n}E[f(S_n + 1)\sum_{j=1}^{n} p_j(1 - I(X_j = 1)) - f(S_n)\sum_{j=1}^{n}(1 - p_j)I(X_j = 1)],
\end{aligned}$$

which implies

$$\begin{aligned}
0 &= E[f(S_n + 1)\sum_{j=1}^{n} p_j(1 - I(X_j = 1)) - f(S_n)\sum_{j=1}^{n}(1 - p_j)I(X_j = 1)] \\
&= E[f(S_n + 1)\sum_{j=1}^{n} p_j] - E[f(S_n + 1)\sum_{j=1}^{n} p_j I(X_j = 1)] \\
&\quad - E[f(S_n)\sum_{j=1}^{n} I(X_j = 1)] + E[f(S_n)\sum_{i=1}^{n} p_j I(X_j = 1)] \\
&= E[\lambda f(S_n + 1)] - E[f(S_n + 1)\sum_{j=1}^{n} p_j I(X_j = 1)] \\
&\quad - E[S_n f(S_n)] + E[f(S_n)\sum_{j=1}^{n} p_j I(X_j = 1)].
\end{aligned}$$

That is

$$E[\lambda f(S_n + 1) - S_n f(S_n)] = E[\sum_{j=1}^{n} p_j I(X_j = 1)(f(S_n + 1) - f(S_n))]. \tag{2.4}$$

Substituting $f = U_\lambda h$ in (2.4), we obtain

$$E[h(S_n) - P_\lambda(h)] = E[\lambda(U_\lambda h)(S_n + 1) - S_n(U_\lambda h)(S_n)]$$

$$= E[\sum_{j=1}^{n} p_j I(X_j = 1)(U_\lambda h(S_n+1) - U_\lambda h(S_n))]$$

$$= E[\sum_{j=1}^{n} p_j I(X_j = 1)(U_\lambda h(S_n^{(j)} + X_j + 1) - U_\lambda h(S_n^{(j)} + X_j))]$$

$$= E[\sum_{j=1}^{n} p_j I(X_j = 1)(U_\lambda h(S_n^{(j)} + 2) - U_\lambda h(S_n^{(j)} + 1))]$$

$$= \sum_{j=1}^{n} p_j^2 E[U_\lambda h(S_n^{(j)} + 2) - U_\lambda h(S_n^{(j)} + 1)].$$

Hence $\quad Eh(S_n) = P_\lambda(h) + \sum_{j=1}^{n} p_j^2 E[V_\lambda h(S_n^{(j)})].$

Now, we ready to prove our main result (1.3). Let $\omega_0 \in \{1,2,\ldots,n-1\}$. For any subset A of $Z^+ \cup \{0\}$, let $h_A : Z^+ \cup \{0\} \to R$ defined by

$$h_A(\omega) = \begin{cases} 1 & if \quad \omega \in A \\ 0 & if \quad \omega \notin A. \end{cases}$$

For convenience, we will write h_s for $h_{\{s\}}$ and let $C_\omega = \{0,1,2,\ldots,\omega\}$. In case of h in (2.1), using h_s, we see that the solution $U_\lambda h_s$ of (2.2) can be expressed in the form

$$U_\lambda h_s(\omega) = \begin{cases} \frac{(\omega-1)!}{s!}\lambda^{s-\omega} P_\lambda(1-h_{C_{\omega-1}}) & if \quad s<\omega \\ -\frac{(\omega-1)!}{s!}\lambda^{s-\omega} P_\lambda(h_{C_{\omega-1}}) & if \quad s\geq\omega>0 \\ 0 & if \quad \omega=0. \end{cases}$$

So

$$V_\lambda h_s(\omega) = \begin{cases} -\lambda^{s-\omega-2}\frac{\omega!}{s!}[(\omega+1)P_\lambda(h_{C_{\omega+1}}) - \lambda P_\lambda(h_{C_\omega})] & if \quad \omega\leq s-2 \\ \lambda^{s-\omega-2}\frac{\omega!}{s!}[(\omega+1)P_\lambda(1-h_{C_{\omega+1}}) + \lambda P_\lambda(h_{C_\omega})] & if \quad \omega=s-1 \\ \lambda^{s-\omega-2}\frac{\omega!}{s!}[(\omega+1)P_\lambda(1-h_{C_{\omega+1}}) - \lambda P_\lambda(1-h_{C_\omega})] & if \quad \omega\geq s. \end{cases}$$

From (2.3), when $h = h_{\omega_0}$, we see that

$$|P(S_n = \omega_0) - P(P_\lambda = \omega_0)| \leq \sum_{j=1}^{n} p_j^2 |E[V_\lambda h_{\omega_0}(S_n^{(j)})]|. \tag{2.5}$$

So, to prove (1.3) it suffices to prove that $|V_\lambda h_{\omega_0}| \leq \min(\frac{1}{\omega_0}, \lambda^{-1})$.

Step 1. We will show that $|V_\lambda h_{\omega_0}| \le \frac{1}{\omega_0}$.

To do this, we improve the result in Neammanee (2003) which is done in the case of $\lambda \in (0,1]$.

Case 1 $\omega \le \omega_0 - 2$.

Note that

$$\lambda^{(\omega_0-2)-\omega}[(\omega+1)P_\lambda(h_{C_{\omega+1}}) - \lambda P_\lambda(h_{C_\omega})]$$

$$= \lambda^{(\omega_0-2)-\omega} e^{-\lambda} \sum_{k=0}^{\omega+1} \frac{\lambda^k}{k!}(\omega+1-k) \tag{2.6}$$

$$\le (\omega+1)e^{-\lambda}\{\frac{\lambda^{(\omega_0-2)-\omega}}{0!} + \frac{\lambda^{(\omega_0-2)-\omega+1}}{1!} + \cdots + \frac{\lambda^{(\omega_0-2)}}{\omega!}\}$$

$$\le (\omega_0-1)e^{-\lambda}\{[(\omega_0-2)-\omega]!\frac{\lambda^{(\omega_0-2)-\omega}}{[(\omega_0-2)-\omega]!} + 2\cdot 3\cdots[(\omega_0-2)-\omega+1]\frac{\lambda^{(\omega_0-2)-\omega+1}}{[(\omega_0-2)-\omega+1]!}$$

$$+\cdots+(\omega+1)(\omega+2)\cdots(\omega_0-2)\frac{\lambda^{(\omega_0-2)}}{(\omega_0-2)!}\}$$

$$\le \frac{(\omega_0-1)!}{\omega!}e^{-\lambda}\{\frac{\lambda^{(\omega_0-2)-\omega}}{[(\omega_0-2)-\omega]!} + \frac{\lambda^{(\omega_0-2)-\omega+1}}{[(\omega_0-2)-\omega+1]!} + \cdots + \frac{\lambda^{(\omega_0-2)}}{(\omega_0-2)!}\}$$

$$\le \frac{(\omega_0-1)!}{\omega!}. \tag{2.7}$$

Hence, by (2.6) and (2.7)

$$0 > V_\lambda h_{\omega_0}(\omega) \ge -\frac{1}{\omega_0}. \tag{2.8}$$

Case 2 $\omega = \omega_0 - 1$.

It follows from Neammanee (2003) that

$$0 < V_\lambda h_{\omega_0}(\omega) \le \frac{1}{\omega_0}. \tag{2.9}$$

Case 3 $\omega \ge \omega_0$.

By the fact that

$$(\omega+1)P_\lambda(1-h_{C_{\omega+1}}) - \lambda P_\lambda(1-h_{C_\omega}) = e^{-\lambda}\sum_{k=\omega+2}^{\infty}\frac{\lambda^k}{k!}(\omega+1-k) < 0$$

and

$$\sum_{k=\omega+2}^{\infty}\frac{\lambda^k}{k!}(k-\omega-1) = \frac{\lambda^{\omega+2}}{(\omega+2)!} + \frac{2\lambda^{\omega+3}}{(\omega+3)!} + \frac{3\lambda^{\omega+4}}{(\omega+4)!} + \cdots$$

$$= \lambda^{\omega-\omega_0+2}[\frac{\lambda^{\omega_0}}{(\omega_0+1)!(\omega_0+2)\cdots(\omega+2)} + \frac{2\lambda^{\omega_0+1}}{(\omega_0+2)!(\omega_0+3)\cdots(\omega+3)}$$

$$+\frac{3\lambda^{\omega_0+2}}{(\omega_0+3)!(\omega_0+4)\cdots(\omega+3)}+\cdots]$$

$$\leq \lambda^{\omega-\omega_0+2}[\frac{\lambda^{\omega_0}}{(\omega_0)!(\omega_0+2)\cdots(\omega+2)}+\frac{\lambda^{\omega_0+1}}{(\omega_0+1)!(\omega_0+3)\cdots(\omega+3)}$$

$$+\frac{\lambda^{\omega_0+2}}{(\omega_0+2)!(\omega_0+3)\cdots(\omega+3)}+\cdots]$$

$$\leq \frac{\lambda^{\omega-\omega_0+2}}{(\omega_0+2)(\omega_0+3)\cdots(\omega+2)}[\frac{\lambda^{\omega_0}}{(\omega_0)!}+\frac{\lambda^{\omega_0+1}}{(\omega_0+1)!}+\frac{\lambda^{\omega_0+2}}{(\omega_0+2)!}+\cdots]$$

$$\leq e^{\lambda}\lambda^{\omega-\omega_0+2}\frac{(\omega_0+1)!}{(\omega+2)!},$$

we have

$$0>V_\lambda h_{\omega_0}(\omega)\geq -\frac{\omega_0+1}{(\omega+1)(\omega+2)}\geq -\frac{1}{\omega+2}\geq -\frac{1}{\omega_0}. \tag{2.10}$$

From cases 1-3, step 1 is proved.

Step 2. We will show that $|V_\lambda h_{\omega_0}|\leq \lambda^{-1}$.

Note from (2.8)-(2.10) that

$$V_\lambda h_{\omega_0}(\omega)\begin{cases} >0 & if \quad \omega=\omega_0-1 \\ <0 & if \quad \omega\neq\omega_0-1. \end{cases}$$

Hence

$$\begin{aligned} V_\lambda h_{\omega+1}(\omega) &\geq V_\lambda h_{\omega_0}(\omega) \\ &\geq V_\lambda h_{\{\omega+1\}^c}(\omega) \\ &= V_\lambda 1(\omega)-V_\lambda h_{\omega+1}(\omega) \\ &= -V_\lambda h_{\omega+1}(\omega), \end{aligned}$$

which implies

$$|V_\lambda h_{\omega_0}(\omega)| \leq V_\lambda h_{\omega+1}(\omega).$$

Since

$$V_\lambda h_{\omega+1}(\omega)=\frac{\lambda^{-1}}{\omega+1}[(\omega+1)(1-P_\lambda(h_{C_{\omega+1}}))+\lambda P_\lambda(h_{C_\omega})]$$

$$=\lambda^{-1}+\frac{\lambda^{-1}}{\omega+1}[\lambda P_\lambda(h_{C_\omega})-(\omega+1)P_\lambda(h_{C_{\omega+1}})]$$

$$=\lambda^{-1}-\frac{\lambda^{-1}}{\omega+1}[e^{-\lambda}\sum_{l=0}^{\omega+1}\frac{\lambda^l}{l!}(\omega+1-l)]$$

$$\le \lambda^{-1},$$

we have $|V_\lambda h_{\omega_0}| \le \lambda^{-1}$.

From step 1, step 2 and (2.5) we have (1.3).

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Matrix Transformations on the Bounded Variation Vector-Valued Sequence Space

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Abstract

In this paper, the characterizations of infinite matrices which transform bounded variation vector-valued sequence space into Maddox sequence spaces, where the sequence $p = (p_k)$ are bounded sequences of positive real numbers such that $p_k \leq 1$ for all $k \in N$ is presented.

Keywords: Bounded variation vector-valued; Maddox; sequence space; infinite matrix transformation

1 Introduction

Let$(X,\|.\|)$ be a Banach space and $p = (p_k)$ are bounded sequences of positive real numbers. We write $x = (x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces $bv(X,p), c_0(X,p), c(X,p), l_\infty(X,p), \underline{l_\infty}(X,p)$ and $F_r(X,p)$ are defined by

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$$bv(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k - x_{k+1}\|^{p_k} < \infty\},$$
$$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\},$$
$$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \quad \text{for some} \quad a \in X\},$$
$$l_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\},$$
$$l(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\},$$
$$\underline{l_\infty}(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|\delta_k x_k\|^{p_k} = 0 \quad \text{for each } (\delta_k) \in c_0\},$$
$$F_r(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} k^r \|x_k\|^{p_k} < \infty\}.$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $bv(p), c_0(p), c(q), l_\infty(p), l(p), \underline{l_\infty}(p)$ and $F_r(p)$ respectively. For $bv(p)$ when $p_k = 1$ for all $k \in N$, it becomes bv which S.M Srirajudeen [7] gave characterizations of infinite matrice to transform it into Maddox sequence spaces in 1992. The spaces $c_0(p), c(p)$ and $l_\infty(p)$ are known as the sequence spaces of Maddox. These spaces were introduced and studied by Maddox [3,4] and Simons[5].The space $l(p)$ was first define by Nakano sequence sequence space, and the space $l(X,p)$ is known as Nakano vector-valued sequence space. Grosse-Erdmann [2] investigated the structure of the space $c_0(p), c(p), l_\infty(p)$. The problem of characterizing a matrix that maps a sequence space of Maddox into another such space are studied by them in [3]. Suantai [8,9,10] gave the matrix characterizations from $l(X,p)$ into the space $c_0(Y,p), c(X,p), \underline{l_\infty}(q)$ and F_r in the case $p_k \leq 1$ for all $k \in N$ and $r \geq 0$, where Y is a Banach space. Wu and Lui [11] gave the matrix characterizations mapping from X-valued sequence spaces $c_0(X,p), l_\infty(X,p)$ and $l(X,p)$ into $c_0(q)$ and $l_\infty(q)$. In [1] Chanan Sudsukh characterized an infinite matrix transformations from Maddox vector-valued sequence space into Nakano sequence space and from Nakano vector-valued sequence space into Maddox sequence space.

However, the matrix transformations from bounded variation vector-valued sequence space into Maddox sequence space is also an open problem, so the objective of this paper is to present the characterizations of infinite matrices mapping $bv(X,p)$ into Maddox sequence spaces when $p_k \leq 1$ for all $k \in N$. Furthermore, the results in [7] must be covered and generalized by this paper .

2 Notations and Definitions.

Let $(X, \|\cdot\|)$ be a Banach space with a scalar field K, the space of all sequence in X is denoted by $W(X)$ and $\phi(X)$ is denoted for the space of all finite sequences in X. Let E be an X-valued sequence space, for $x \in E$ and $k \in N$, we write x_k stand for the k^{th} term of x. For $k \in N$, denote e_k by the sequence $(0,0,0,...,0,1,0,...)$ with 1 in the k^{th} position and e by the sequence $(1,1,1,...)$. For $z \in X$ and $k \in N$, let $e^k(z)$ be the sequence $(0,0,0,...,z,0,..)$ with z in the k^{th} position and let $e(z)$be the sequence $(z,z,z,...)$. For a fix scalar sequence $u = (u_k)$, the sequence space E_u is defined as

$$E_u = \{x = (x_k) \in W(X) : (u_k x_k) \in E\}.$$

The X-valued sequence space E is called a *K-space* if for each $n \in N$, the n^{th} coordinate mapping $p_n : E \to X$, defined by $p_n(x) = x_n$ is continuous on E.If the X-valued sequence space$E$ is an *fre'chet* and a $K-space$ then E is called $FK-space$. Suppose that E contains $\phi(X)$, the space of all finite sequences in X, then E is said to have *property* AB if the set $\{\sum_{k=1}^{n} e^k(x) : n \in N\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have $AK-property$ if $\{\sum_{k=1}^{n} e^k(x_k) \to x\}$ in E as $n \to \infty$ for every $x = (x_k) \in E$. Let $A = (f_k^n)$ with $f_k^n \in X'$, the topologygical dual of X. Suppose that E is a space of X-valued sequence space and F a space of scalar-valued sequence, then A is said to map E into F, written by $A : E \to F$, if for each $x = (x_k) \in E$,$A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$converges for each $n \in N$ and the sequence $Ax = (A_n(x)) \in F$. Let (E, F) is denoted for the set of all infinite matrices mapping E into F. When $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$${}_u(E,F)_v = \{A = (f_k^n) : (u_n v_k f_k^n \in (E,F)\}.$$

If $u_k \neq 0$ for all $k \in N$ we write $u^{-1} = (\frac{1}{u^k})$

3 Some Auxiliary Results.

In this part we discuss about about $FK-space$, $AK-property$ and $\beta-dual$ of $bv(X,p)$, These results will reduce our problems into some simpler forms.

Proposition 1. *Let (f_k) be a sequence of continuous linear functional on X and $p=(p_k)$ a bounded sequence of positive real numbers with $p_k \le 1$. Then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x=(x_k)\in bv(X,p)$ if and only if there exists $M\in N$ such that $\sup_j \|g_j\| M^{\frac{-1}{p_j}} < \infty$, where g_j is the bounded linear functional on X and g_j is defined by $g_j(x)=\sum_{k=j}^{\infty} f_k(x)$ for all $x\in X$.*

Proof. Suppose that there exist $M\in N$ such that $\sup_j \|g_j\| M^{\frac{-1}{p_j}}$, then there exist $K>0$ such that $\|g_j\| \le KM^{\frac{1}{p_j}}$ for all $j\in N$, and for each $x=(x_k)\in bv(X,p)$, we know that $(z_j)=(x_j-x_{j-1}\in l(X,p)$ then there exist $j_0\in N$ such that $M^{\frac{1}{p_j}}\|z_j\|\le 1, \forall j\ge j_0$. By $p_j\le 1, \forall j\ge j_0$, we have

$$M^{\frac{1}{p_j}}\|z_j\| \le (M^{\frac{1}{p_j}}\|z_j\|)^{p_j} = M\|z_j\|^{p_j}.$$

Considers,

$$\begin{aligned}
\sum_{k=1}^{\infty} f_k(x_k) &= \lim_{J\to\infty}\sum_{k=1}^{J} f_k(x_k) \\
&= \lim_{J\to\infty}\Big\{\sum_{j=1}^{J} g_j(x_j-x_{j-1}) - g_{J+1}(x_J)\Big\} \\
&= \sum_{j=1}^{\infty} g_j(x_j-x_{j-1}) \qquad ; g_{J+1}(x_J)\to 0, J\to\infty \\
&= \sum_{j=1}^{\infty} g_j(z_j) \qquad ; z_j = x_j - x_{j-1}
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{\infty}|g_j(z_j)| &\le \sum_{j=1}^{\infty}\|g_j\|\|z_j\| \\
&= \sum_{j=1}^{j_0}\|g_j\|\|z_j\| + \sum_{j_0+1}^{\infty}\|g_j\|\|z_j\| \\
&\le \sum_{j=1}^{j_0}\|g_j\|\|z_j\| + \sum_{j_0+1}^{\infty}KM^{\frac{1}{p_j}}\|z_j\| \\
&= \sum_{j=1}^{j_0}\|g_j\|\|z_j\| + K\sum_{j_0+1}^{\infty}M^{\frac{1}{p_j}}\|z_j\| \\
&< \infty.
\end{aligned}$$

This implies $\sum_{j=1}^{\infty} g_j(z_j)$ converges, since we have $\sum_{j=1}^{\infty} f_k(x_k) = \sum_{j=1}^{\infty} g_j(z_j)$, so we obtain that $\sum_{j=1}^{\infty} f_k(x_k)$ converges for all $(x_k) \in bv(X)$.

Convesely, assume that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $(x_k) \in bv(X,p)$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $(x_k) \in l(X,p)$ because $l(X,p) \subset bv(X,p)$. For each $x = (x_k) \in l(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_kx_k) = |f_k(x_k)|$ for all $k \in N$. Since $(t_kx_k) \in l(X,p)$, by our assumtion, thus we have that $\sum_{k=1}^{\infty} f_k(t_kx_k)$ converges, and hence

$$\sum_{k=1}^{\infty}|f_k(x_k)| < \infty \qquad \text{for all } x = (x_k) \in l(X,p). \tag{1}$$

Now suppose that $\sup_j \|g_j\|m^{\frac{-1}{p_j}} = \infty$ for all $m \in N$. For each $i \in N$, choose the sequence (m_i) and (j_i) of positive integers with $m_1 < m_2 < m_3 ...$ and $j_1 < j_2 < j_3 < ...$ such that $m_i > 2^i$ and $\|g_{j_i}\|m_i^{\frac{-1}{p_{j_i}}} > 1$. Choose $x_{j_i} \in X$ with $\|x_{j_i}\| = 1$ such that

$$|g_{j_i}(x_{j_i})|m_i^{\frac{-1}{p_{j_i}}} = |\sum_{k=j_i}^{\infty} f_k(x_{j_i})|m_i^{\frac{-1}{p_{j_i}}} > 1$$

with also $\qquad |f_{j_i}(x_{j_i})| m_i^{\frac{-1}{p_{j_i}}} > 1.$ (2)

Let $y = (y_k), y_k = m_i^{\frac{-1}{p_j}}$ if $k = j_i$ for some i, and 0 otherwise, then $\sum_{k=1}^{\infty} \|y_k\|^{p_k} = \sum_{i=1}^{\infty} \frac{1}{m_i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ so that $(y_k) \in l(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| = \sum_{i=1}^{\infty} |f_{j_i}(m_i^{\frac{-1}{p_j}} x_{j_i})|$$
$$= \sum_{i=1}^{\infty} m_i^{\frac{-1}{p_{j_i}}} |f_i(x_{j_i})| = \infty \qquad by \quad (2),$$

and this contradiction with (1). Therefore, there exists $M \in N$ such that $\sup_j \|g_j\| M^{\frac{-1}{p_j}} < \infty$.The proof is complete.

□

It is not difficult to show that $bv(X,p)$ contains $\phi(X)$ and has property AK when $p = (p_k)$ is a bounded sequence of positive real numbers and $p_k \leq 1$ and also $bv(X,p)$ is $FK-$ space . In addition, we also already know from [1] that the space $l(q)$ is a $FK-$ space with property AK under the paranorm $g(x) = \sum_{k=1}^{\infty} \|x_k\|^{\frac{1}{M}}$ where M =max $\{1, \sup p_k\}$. Now let us quote a known result in [1]

Lemma 1. *Let $E \subseteq WX)$ be an FK-space. with AKproperty of scalar sequences. Then, for an infinite matrix $A = (f_k^n), A : E \to F$ if and only if*

(1) for each $n \in N, \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x = (x_k) \in E$,

(2) for each $k \in N, (f_k^n(z))_{n=1}^{\infty} \in E$ for all $z \in X$, and

(3) $A : \phi(X) \to F$ is continuous when $\phi(X)$ is considered as a subspace of E.

4 Main Results.

Now we start to give characterizations of matrix transformation from $bv(X,p)$ into $l(q)$ by using lemma A.

Theorem 1. *Let $A = (f_k^n)$ be an infinnite matrix of bounded linear funtional on X. Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k \leq 1$. Then $A : bv(X,p) \to l(q)$ if and only if*

(1) for each $n \in N$, there is $M_n \in N$ such that $\sup_j \|g_j^n\| M_n^{\frac{-1}{p_j}} < \infty$, where g_j is the bounded linear functional on X define by $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x), \forall x \in X$.
(2) for every $k \in N, \sum_{n=1}^{\infty} |f_k^n(x)|^{q_n} < \infty$ for all $x \in X$ and
(3) for each $r \in N$ there exists $M_r \in N$ such that

$$\| \lim x\| + \sum_{k \in K} \|x_k - x_{k-1}\|^{p_k} < \frac{1}{M_r} \Longrightarrow \sum_{n=1}^{\infty} |\sum_{k \in K} f_k^n(x_k)|^{q_n} < \frac{1}{r}$$

for all $x = (x_k) \in \phi(X)$ and for all subsets K of N.

Proof. Assume that $A : bv(X,p) \to l(q)$. Since $bv(X,p)$ and $l(q)$ are FK-space, by Lemma 1 and proposition 1, thus the condition (1)and (2) are obtained. Now we have to show that (3) holds. Since $bv(X,p)$ and $l(q)$ are FK-space and $bv(X,p)$ has AK property, we have by Lemma 1 that $A : \phi(X) \to l(q)$ is continuous when $\phi(X)$ considers as a subspace of $bv(X,p)$. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$\sum_{n=1}^{\infty} |\sum_{k \in K} f_k^n(x_k)|^{q_n} < \epsilon \qquad \text{for all} \quad x = (x_k) \in \phi(X),$$
$$\|x\| = \| \lim x\| + \sum_{k \in K} \|x_k - x_{k-1}\|^{p_k} < \delta.$$

That is, for each $\epsilon > 0$, there exist $r \in N$ with $r \geq \frac{1}{\epsilon}$, for each r there exist $M_r \in N$ with $M_r \geq \frac{1}{\delta}$ such that

$$\sum_{n=1}^{\infty} | \sum_{k\in K} f_k^n(x_k)|^{q_n} < \frac{1}{r} \qquad \text{for all} \quad x = (x_k) \in \phi(X)$$

$$\text{when} \quad \|x\| = \|\lim x\| + \sum_{k\in K} \|x_k - x_{k-1}\|^{p_k} < \frac{1}{M_r}.$$

Thus the condition (3) holds.

Conversely, assume that the condition (1), (2) and (3) hold. The condition (1), by proposition 1, implies that $\sum_{k=1}^{\infty} f_k^n(x_k)$ converges for all $x = (x_k) \in bv(X,p)$. We have $(f_k^n(x))_{n=1}^{\infty} \in l(q)$, for all $k \in N$ and for all $x \in X by condition (2)$. Thus $A : \phi(X) \to l(q)$. Now we shall show that $A : \phi(X) \to l(q)$ is continuous when $\phi(X)$ is considered as a subspace of $bv(X,p)$. Let $\varepsilon > 0$ be any given, there exists $r \in N$ such that $r \geq \frac{1}{\varepsilon}$. For each $r \in N$, there exists $M_r \in N$ such that $\|\lim x\| + \sum_{k\in K} \|x_k - x_{k-1}\|^{p_k} < \frac{1}{M_r} \Longrightarrow \sum_{n=1}^{\infty} | \sum_{k\in K} f_k^n(x_k)|^{q_n} < \frac{1}{r} \leq \varepsilon$ for all $x = (x_k) \in \phi(X)$ and for all subsets K of N. By Lemma 1, $A : bv(X,p) \to l(q)$ is obtained.

$\square$

Theorem 2. *Let $A = (f_k^n)$ be an infinite matrix of bounded linear functional on X. Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k \leq 1$. Then $A : bv(X,p) \to l(q)$ if*

$$\sup_j \sum_{n=1}^{\infty} \|g_j^n\|^{q_n} < \infty, \tag{3}$$

when g_j^n is the bounded linear functional on X and g_j^n is defined by $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ for all $x \in X$ and for all $n \in N$.

Proof. Let $\sup_j \sum_{n=1}^{\infty} \|g_j^n\|^{q_n} < \infty$, that is for each $n \in N, \sup_j \|g_j^n\| < \infty$. Let $x = (x_k) \in bv(X,p)$, so that we have $\sum_{k=1}^{\infty} \|x_k - x_{k-1}\|^{p_k} < \infty$, say converges to L. For each n considers,

$$
\begin{aligned}
A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k) &= \lim_{J\to\infty} \sum_{k=1}^{J} f_k^n(x_k) \\
&= \lim_{J\to\infty} [\sum_{j=1}^{J} g_j^n(x_j - x_{j-1}) - g_{J+1}^n(x_J)] \\
&= \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1}) \qquad ; g_{J+1}(x_J) \to 0, J \to \infty.
\end{aligned}
$$

Let us write $H = \max\{1, \sup q_n\}$ then we observe that $|\lambda|^{q_n} \leq \max\{1, |\lambda|^H\}$ and not difficult to show that

$\sum_{j=1}^{\infty} \|x_j - x_{j-1}\| \leq L_1 \sum_{j=1}^{\infty} \|x_j - x_{j-1}\|^{p_j}$ for $p_j \leq 1$,

considers

$$
\begin{aligned}
|A_n(x)|^{q_n} &= |\sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1})|^{q_n} \\
&\leq (\sum_{j=1}^{\infty} \|g_j^n\| \|x_j - x_{j-1}\|)^{q_n} \\
&\leq (\sup_j \|g_j^n\| \sum_{j=1}^{\infty} \|x_j - x_{j-1}\|)^{q_n} \quad ; \|g_j^n\| \leq \sup_j \|g_j^n\|, \forall n \in N \\
&\leq (\sup_j \|g_j^n\| L_1 \sum_{j=1}^{\infty} \|x_j - x_{j-1}\|^{p_j})^{q_n} \qquad ; p_j \leq 1 \\
&= \sup_j \|g_j^n\|^{q_n} L_2^{q_n} ; L_2 = L_1 L
\end{aligned}
$$

So that
$$
\begin{aligned}
\sum_{n=1}^{\infty} |A_n(x)|^{q_n} &\leq \sum_{n=1}^{\infty} \sup_j \|g_j^n\|^{q_n} L_2^{q_n} \\
&\leq M \sum_{n=1}^{\infty} \sup_j \|g_j^n\|^{q_n}
\end{aligned}
$$

where $M = \max(1, L_2^H)$. From $\sup_j \sum_{n=1}^{\infty} \|g_j^n\|^{q_n} < \infty$ so we have $\sum_{n=1}^{\infty} |A_n(x)|^{q_n} < \infty$. Thus $A : bv(X,p) \to l(q)$. □

The following from here we shall characterize infinite matrices mapping from $bv(X,p)$ into $l_\infty(q), c_0(q)$ and $c(q)$. For $(x_k) \in bv(X,p)$ and let $z = x_j - x_{j-1}$. Since we have

$$\sum_{k=1}^{\infty} f_k^n(x_k) = \sum_{j=1}^{\infty} g_j^n(x_j - x_{j-1}),$$

so that we must have $(z_j) \in l(X,p)$. When the infinite matrix $A = (f_k^n)$ and $B = (g_j^n)$and since we already have theorems about $B : l(X,p) \to l_\infty(q), B : l(X,p) \to c_0(q), B : l(X,p) \to c(q)$ from [1]. So we can characterize infinite matrix $A : bv(X,p) \to l_\infty(q), A : bv(X,p) \to c_0(q), A : bv(X,p) \to c(q)$.

Theorem 3. *Let $A = (f_k^n)$ be an infinite matrix of bounded linear functional on X. Let $p = (p_k)$ an $q = (q_k)$ be bounded sequence of positive real numbers with $p_k \leq 1$.Then $A : bv(X,p) \to l_\infty(q)$ if and only if there exists $M \in N$ such that*

$$\|g_j^n\| \leq M^{\frac{1}{p_j}+\frac{1}{q_n}}; \qquad \text{for all} \quad n,j \in N, \tag{4}$$

when g_j^n is the bounded linear functional on X and g_j^n is defined by $g_j^n(x) = \sum f_k^n(x)$ for all $x \in X$ and , for all $n \in N$.

Proof. By [1,theorem 1.3], thus we have $A : bv(X,p) \to l_\infty(q)$ if and only if $\|g_j^n\| \leq M_1^{\frac{1}{p_j}+\frac{1}{q_n}} \forall n,j \in N$. The proof is complete. □

Theorem 4. *Let $A = (f_k^n)$ be an infinite matrix of bounded linear functional on X. Let $p = (p_k)$ and $q = (q_k)$ be bounded sequence of positive real numbers with $p_k \leq 1$. Then $A : bv(X,p) \to c_0(q)$ if and only if*

(1) for all $m,k \in N$, $m^{\frac{1}{q_n}} g_j^n\, x \xrightarrow{w^*} 0 \quad$ as $\quad n \to \infty$,

(2) for each $m \in N$ there exists $M_m \in N$ such that $m^{\frac{p_j}{q_n}} \|g_j^n\|^{p_j} \leq M_m$ for all $n,j \in N$,

when g_j^n is the bounded linear functional on X and g_j^n is defined by $g_j^n(x) = \sum_{k=j}^{\infty} f_k^n(x)$ for all $x \in X$ and for all $n \in N$.

Proof. Since

$$
\begin{aligned}
A : bv(X,p) \to c_0(q) \quad &\text{if and only if} \quad B : l(X) \to c_0(q). \\
&\Leftrightarrow B : l(X) \to \cap_{m=1}^{\infty} c_0(m^{\frac{1}{q_n}}), \\
&\Leftrightarrow (n_{\iota}^{\frac{1}{q_n}} g_j^n) : l(X) \to c_0, \quad \text{for all} \quad m \in M.
\end{aligned}
$$

By [1,theorem 1.5 and proposition2.3(i)], we have $A : bv(X,p) \to c_0(q)$ if and on ly if the conditions (1) and (2) hold.

□

Theorem 5. *Let $A = (f_k^n)$ be an infinite matrix of bounded linear functional on X. Let $p = (p_k)$ and $q = (q_k)$ be bounded sequence of positive real numbers with $p_k \leq 1$. Then $A : bv(X,p) \to c(q)$ if and only if there is a sequence (g_j) with $g_j \in X'$ for all $j \in N$ such that*

(1) *for some* $M \in N, \sup_j \|g_j\| M^{\frac{-1}{p_j}} < \infty,$

(2) *for all* $m, j \in N, m^{\frac{1}{q_n}} (g_j^n - g_j) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *and*

(3) *for each* $m \in N, \sup_j m^{\frac{p_j}{q_n}} \|g_j^n - g_j\|^{p_j} < \infty$ *for all* $n, j \in N.$

Proof. By [1,theorem 3.1.8], we have $A : bv(X,p) \to c(q)$ if and only if the conditions (1), (2) and (3) hold. □

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Error Estimation of Convergence of Distributions of Average of Reciprocals of Sine to the Cauchy Distribution

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ABSTRACT. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables such that $\sin X_n \neq 0$ and F_n the distribution function of $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sin X_k}$. Neammanee ([7]) showed that $\lim_{n\to\infty} F_n(x) = F(x)$ where F is the Cauchy distribution. In this paper, we investigate the accuracy of the approximation of $F(x)$ to $F_n(x)$. Under general conditions, we show that $\sup_{-\infty<x<\infty} |F_n(x) - F(x)|$ is bounded by $\frac{C}{n^d}$ for $0 < d < \frac{1}{9}$ where C is a constant.

1. Introduction and main result

Let $X_1, X_2, \cdots$ be a sequence of independent continuous random variables. Many authors (for examples, Shapiro ([9]-[11]), Termwuttipong ([12]), and Neammanee ([4]-[7])) investigated the limit distribution of reciprocals of the random variables. Neammanee ([7]) showed that under general conditions the distribution function F_n of the average of the reciprocals of sine of the random variables weakly converges to the Cauchy distribution function F. In this paper the error involved in using F as an approximation to F_n is investigated. More specifically we shall find a bound on $\sup_{-\infty<x<\infty} |F_n(x) - F(x)|$ which converges to zero as $n \to \infty$.

It will be observed that the specific form of the random variables X_n is not specified. In this sense our result will resemble the Berry-Esseen estimates for the case of convergence to the normal distribution function (Feller, [2]). To obtain the order of the bound, we also assume that $X_1, X_2, \cdots$ are identically distributed random variables with common probability density function f. Suppose that $L = \sum_{j=-\infty}^{\infty} f(j\pi)$ is positive. Neammanee ([7]) gave the conditions to show that

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$$\lim_{n\to\infty} P\Big(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sin X_k} - A_n \le x\Big) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \tan^{-1}\frac{x}{\pi L}\Big)$$

where $A_n = \displaystyle\int_{|u|>\frac{1}{n}} \frac{f(u)}{u}\,du$. If f is symmetric it follows that $A_n = 0$ and

(1.1) $$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{for all } x$$

where F_n is the distribution function of $\displaystyle\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sin X_k}$ and F is the Cauchy distribution function defined by

$$F(x) = \frac{1}{\pi}\Big(\frac{\pi}{2} + \tan^{-1}\frac{x}{\pi L}\Big).$$

The following theorem gives a bound of (1.1).

Theorem 1.1. *Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed continuous random variables with common symmetric probability density function f and $\sin X_n \neq 0$ for every n. Assume that*

(a) $\displaystyle\sum_{j=-\infty}^{\infty} f(j\pi + u)$ *is uniformly bounded on $|u| < \pi$*

(b) $\displaystyle\sum_{j=-\infty}^{\infty} f'(j\pi + u)$ *is uniformly bounded in some neighborhood of 0.*

Then for any fixed $0 < d < \frac{1}{9}$ there exists a constant C such that

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| < \frac{C}{n^d}.$$

We also give an example which satisfies Theorem 1.1 in section 3. Throughout this paper, C stands for an absolute constant with possibly different values in different places.

2. Proof of main result

For each n and k, let $X_{nk} = \dfrac{1}{n\sin X_k}$ and let H be the common distribution function of X_k. For $a \ge 1$ we define $X_{nk}^a = X_{nk}$ if $-a \le X_{nk} \le a$ and otherwise let $X_{nk}^a = 0$.
That is

$$X_{nk}^a = \begin{cases} \dfrac{1}{n\sin X_k} & \text{if } X_k \in A \\ 0 & \text{if } X_k \notin A \end{cases}$$

where

$$A = \bigcup_{j=-\infty}^{\infty} [2j\pi + \sin^{-1}(\frac{1}{na}), (2j+1)\pi - \sin^{-1}(\frac{1}{na})] \cup [(2j-1)\pi + \sin^{-1}(-\frac{1}{na}), 2j\pi + \sin^{-1}(-\frac{1}{na})].$$

Let F^a_{nk}, φ^a_{nk}, $\mu_{nk}(a)$ and $\sigma^2_{nk}(a)$ be the distribution function, characteristic function, mean and variance of X^a_{nk} respectively. Let $S_n = X_{n1} + X_{n2} + \cdots + X_{nn}$ and $S^a_n = X^a_{n1} + X^a_{n2} + \cdots + X^a_{nn}$.
To bound $\sup\limits_{-\infty\le x<\infty} |F_n(x) - F(x)|$, we note that

$$|F_n(x) - F(x)| \le |F_n(x) - F^a_n(x)| + |F^a_n(x) - F(x)| \tag{2.1}$$

where F^a_n is the distribution function of S^a_n.

Hence, in order to prove Theorem 1.1 we need to bound both terms on the right side of (2.1). The following lemma will give a bound on the first term.

Lemma 2.1. *For $a \ge 1$, there exists a positive constant C such that*

$$\sup_{-\infty<x<\infty} |F_n(x) - F^a_n(x)| \le \frac{C}{a}.$$

Proof. Boonyasombat and Shapiro ([1]) showed in their paper that

$$|F_n(x) - F^a_n(x)| \le n\{F_{nk}(-a) + 1 - F_{nk}(a)\} \tag{2.2}$$

and Neammanee ([7]) showed that
(2.3)

$$F_{nk}(x) = \begin{cases} \sum\limits_{j=-\infty}^{\infty} [H(2j\pi) - H((2j-1)\pi)], & \text{if } x = 0 \text{ or } |\frac{1}{nx}| \ge 1 \\ \sum\limits_{j=-\infty}^{\infty} [H((2j-1)\pi - \sin^{-1}(\frac{1}{nx})) - H((2j-1)\pi)] & \\ \quad + \sum\limits_{j=-\infty}^{\infty} [H(2j\pi) - H(2j\pi + \sin^{-1}(\frac{1}{nx}))], & \text{if } -1 < \frac{1}{nx} < 0 \\ \sum\limits_{j=-\infty}^{\infty} [H(2j\pi) - H((2j-1)\pi)] + 1 & \\ \quad - \sum\limits_{j=-\infty}^{\infty} [H(2j\pi + \sin^{-1}(\frac{1}{nx})) & \\ -H((2j-1)\pi - \sin^{-1}(\frac{1}{nx}))] & \text{if } 0 < \frac{1}{nx} < 1. \end{cases}$$

Hence, by (2.2) and (2.3) we have

$$
\begin{aligned}
&|F_n(x) - F_n^a(x)| \\
\leq\ & n \sum_{j=-\infty}^{\infty} [H(2j\pi + \sin^{-1}(\frac{1}{na})) - H(2j\pi + \sin^{-1}(-\frac{1}{na})) \\
&+ H((2j-1)\pi - \sin^{-1}(-\frac{1}{na})) - H((2j-1)\pi - \sin^{-1}(\frac{1}{na}))] \\
=\ & n \sum_{j=-\infty}^{\infty} [\int_{2j\pi+\sin^{-1}(-\frac{1}{na})}^{2j\pi+\sin^{-1}(\frac{1}{na})} f(x)\,dx + \int_{(2j-1)\pi-\sin^{-1}(\frac{1}{na})}^{(2j-1)\pi-\sin^{-1}(-\frac{1}{na})} f(x)\,dx] \\
=\ & n \sum_{j=-\infty}^{\infty} [\int_{-\frac{1}{na}}^{\frac{1}{na}} \frac{f(2j\pi + \sin^{-1} u)}{\sqrt{1-u^2}}\,du + \int_{-\frac{1}{na}}^{\frac{1}{na}} \frac{f((2j-1)\pi - \sin^{-1} u)}{\sqrt{1-u^2}}\,du] \\
\leq\ & \frac{n}{\sqrt{1-(\frac{1}{na})^2}} \sum_{j=-\infty}^{\infty} \int_{-\frac{1}{na}}^{\frac{1}{na}} f(j\pi + \sin^{-1} u)\,du \\
\leq\ & \frac{nC}{\sqrt{(na)^2 - 1}} \\
\leq\ & \frac{nC}{\sqrt{(na)^2 - \frac{(na)^2}{2}}} \\
=\ & \frac{C}{a}
\end{aligned}
$$

where we have used the fact that $\sum_{j=-\infty}^{\infty} f(j\pi + \sin^{-1} u)$ is uniform bounded in third inequality. □

To give a bound of the second term on the right of (2.1) we need the following construction.

According to Lukacs ([3]) p.93, we know that Levy-Khinchine formula of the characteristic function φ of F be defined by

$$\log \varphi(t) = \int_{-\infty}^{\infty} (e^{itu} - 1 - \frac{itu}{1+u^2}) \frac{1+u^2}{u^2} dG(u)$$

where $G(u) = L \tan^{-1} u + \frac{\pi L}{2}$. Let $G^a : \mathbb{R} \to [0,1]$ defined by

$$G^a(u) = \begin{cases} 0 & \text{if} \quad u \leq -a \\ G(u) - G(-a) & \text{if} \quad -a < u \leq a \\ G(a) - G(-a) & \text{if} \quad u > a\ , \end{cases}$$

$$\begin{aligned}
\gamma^a &= -\int_{|u|>a} \frac{1}{u} dG(u), \\
\mu(a) &= \gamma^a + \int_{-\infty}^{\infty} \frac{1}{u} dG^a(u) \\
\text{and } K^a(u) &= \int_{-\infty}^{u} (1+x^2) dG^a(x).
\end{aligned}$$

We also let F^a be an infinitely divisible distribution function whose logarithm of its characteristic function given by

$$\log \varphi^a(t) = i\mu(a)t + \int_{-\infty}^{\infty} f(t,x) dK^a(x)$$

where

$$f(t,x) = \begin{cases} (e^{itx} - 1 - itx)\dfrac{1}{x^2} & \text{if } x \neq 0 \\ -\dfrac{t^2}{2} & \text{if } x = 0 \end{cases}$$

and

$$K_n^a(u) = n\int_{-\infty}^{u} x^2 dF_{nk}^a(x + \mu_{nk}(a)).$$

For $0 < \delta \leq 2a$, define

$$m \equiv m(a,\delta) = \left[\frac{2a}{\delta}\right] + 1$$

where $[x]$ is the greatest integer that is less than or equal to x.
Let $-a = x_0 < x_1 < \cdots < x_m = a$ be such that $\max\limits_{1\leq i\leq m}(x_i - x_{i-1}) < \delta$. Boonyasombat and Shapiro ([1]) show that

$$\sup_{-\infty < x < \infty} |F_n^a(x) - F(x)| \leq Cg^a(n, m(a,\delta), r) \tag{2.4}$$

when $0 \leq \sigma_{nk}^2(a) \leq 1$ for large n and $k = 1, 2, \cdots, n$ and

$$\begin{aligned}
&g^a(n, m(a,\delta), r) \\
= &\left[\sigma_n^2(a) \max_{1\leq k\leq n} \sigma_{nk}^2(a)\right]^{\frac{1}{5}} + \left[\sum_{i=0}^{m} |K_n^a(x_i) - K^a(x_i)|\right]^{\frac{1}{3}} + \left[\delta(\sigma_n^2(a) + \sigma^2(a))\right]^{\frac{1}{4}} \\
&+ \{\frac{1}{a}|K_n^a(\infty) - K_n^a(a) + K^a(\infty) - K^a(a) + K_n^a(-a) + K^a(-a)| \\
&+ |\mu_n(a) - \mu(a)|\}^{\frac{1}{2}} + [\frac{1}{r}\int_{|u|>a} |u|^r dG(u)]^{\frac{1}{1+r}}
\end{aligned}$$

for any $r \in (0, 1)$.

Lemma 2.2. *For large n and $r \in (0,1)$ we have a constant C such that*

$$\sup_{-\infty<x<\infty} |F_n^a(x) - F(x)| \;\leq\; Ch^a(n, m(a,\delta), r)$$

where

$$\begin{aligned} h^a(n, m(a,\delta), r) &= \left[\frac{a^2}{n}\right]^{\frac{1}{5}} + \left[\sum_{i=1}^{m} |K_n^a(x_i) - K^a(x_i)|\right]^{\frac{1}{3}} + |\delta a|^{\frac{1}{4}} \\ &\quad + \left\{\frac{1}{a}\left[K_n^a(\infty) - K_n^a(a) + K_n^a(-a)\right]\right\}^{\frac{1}{2}} + \left[\frac{1}{r(1-r)a^{1-r}}\right]^{\frac{1}{1+r}}. \end{aligned}$$

Proof. First, we will show that (2.4) holds. To do this, it suffices to prove that $0 \leq \sigma_{nk}^2(a) \leq 1$ for large n and $k = 1, 2, \cdots, n$. Since f is symmetric, we have

$$\begin{aligned} \mu_{nk}(a) &= \sum_{j=-\infty}^{\infty} \int_{2j\pi+\sin^{-1}(\frac{1}{na})}^{(2j+1)\pi-\sin^{-1}(\frac{1}{na})} \frac{f(x)}{n\sin x}\,dx + \sum_{j=-\infty}^{\infty} \int_{(2j-1)\pi-\sin^{-1}(-\frac{1}{na})}^{2j\pi+\sin^{-1}(-\frac{1}{na})} \frac{f(x)}{n\sin x}\,dx \\ &= \sum_{j=-\infty}^{\infty} \int_{2j\pi+\sin^{-1}(\frac{1}{na})}^{(2j+1)\pi-\sin^{-1}(\frac{1}{na})} \frac{f(x)}{n\sin x}\,dx + \int_{(-2j-1)\pi-\sin^{-1}(-\frac{1}{na})}^{-2j\pi+\sin^{-1}(-\frac{1}{na})} \frac{f(x)}{n\sin x}\,dx \\ &= \sum_{j=-\infty}^{\infty} \int_{2j\pi+\sin^{-1}(\frac{1}{na})}^{(2j+1)\pi-\sin^{-1}(\frac{1}{na})} \frac{f(x)}{n\sin x}\,dx - \int_{2j\pi+\sin^{-1}(\frac{1}{na})}^{(2j+1)\pi-\sin^{-1}(\frac{1}{na})} \frac{f(-x)}{n\sin x}\,dx \\ &= 0. \end{aligned}$$

From the above fact, the fact that

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} \int_{(2j-1)\pi-\sin^{-1}(-\frac{1}{na})}^{2j\pi+\sin^{-1}(-\frac{1}{na})} \frac{f(x)}{(n\sin x)^2}\,dx \\ &= \sum_{j=-\infty}^{\infty} \left[\int_{(2j-1)\pi-\sin^{-1}(-\frac{1}{na})}^{2j\pi-\frac{\pi}{2}} \frac{f(x)}{(n\sin x)^2}\,dx + \int_{2j\pi-\frac{\pi}{2}}^{2j\pi+\sin^{-1}(-\frac{1}{na})} \frac{f(x)}{(n\sin x)^2}\,dx\right] \\ &= \frac{1}{n^2}\sum_{j=-\infty}^{\infty} \left[\int_{-\frac{1}{na}}^{-1} \frac{-f((2j-1)\pi - \sin^{-1}u)}{u^2\sqrt{1-u^2}} + \int_{-1}^{-\frac{1}{na}} \frac{f(2j\pi + \sin^{-1}u)}{u^2\sqrt{1-u^2}}\right] \\ &= \frac{1}{n^2}\int_{-1}^{-\frac{1}{na}} \sum_{j=-\infty}^{\infty} \frac{[f((2j-1)\pi - \sin^{-1}u) + f(2j\pi + \sin^{-1}u)]}{u^2\sqrt{1-u^2}}\,du \end{aligned}$$

$$
\begin{aligned}
&< \frac{C}{n^2}\int_{-1}^{-\frac{1}{na}} \frac{1}{u^2\sqrt{1-u^2}}\,du \\
&= \frac{C}{n^2}\left[na\sqrt{1-(\frac{1}{na})^2}\right] \\
&< \frac{Ca}{n}
\end{aligned}
$$

and $\displaystyle\sum_{j=-\infty}^{\infty}\int_{2j\pi+\sin^{-1}(\frac{1}{na})}^{(2j+1)\pi-\sin^{-1}(\frac{1}{na})} \frac{f(x)}{(n\sin x)^2}\,dx < \frac{Ca}{n}$, we have

$$
\begin{aligned}
&\sigma_{nk}^2(a) \\
= &\sum_{j=-\infty}^{\infty}\int_{2j\pi+\sin^{-1}(\frac{1}{na})}^{(2j+1)\pi-\sin^{-1}(\frac{1}{na})} \frac{f(x)}{(n\sin x)^2}\,dx + \sum_{j=-\infty}^{\infty}\int_{(2j-1)\pi-\sin^{-1}(-\frac{1}{na})}^{2j\pi+\sin^{-1}(-\frac{1}{na})} \frac{f(x)}{(n\sin x)^2}\,dx \\
\le &\frac{Ca}{n},
\end{aligned}
$$

i.e., $0 \le \sigma_{nk}^2(a) \le 1$ for large n and $k = 1, 2, \cdots, n$. So (2.4) holds.

In our case,we see that

$$
\begin{aligned}
&\gamma^a = 0, \quad \mu(a) = 0, \quad \mu_n(a) = \mu_{n1}(a) + \cdots + \mu_{nn}(a) = 0, \\
&K^a(u) \;=\; \begin{cases} 0 & \text{if } u < -a \\ L(a+u) & \text{if } -a \le u < a \\ 2La & \text{if } u \ge a, \end{cases} \\
&\sigma^2(a) = K^a(\infty) = 2La \ \text{ and} \\
&\int_{|u|>a} |u|^r dG(u) \;=\; 2L\int_a^{\infty} \frac{u^r}{1+u^2}\,du \;\le\; 2L\int_a^{\infty} u^{r-2}\,du \;=\; \frac{2L}{(1-r)a^{1-r}}.
\end{aligned}
$$

Hence $g^a(n, m(a,\delta), r) \;\le\; h^a(n, m(a,\delta), r)$. The lemma follows from this fact and (2.4). □

To bound $h^a(n, m(a,\delta), r)$ we need the formula of F_{nk}^a in the following lemma.

Lemma 2.3.

$$F^a_{nk}(x) = \begin{cases} 0, & \text{if } \ x \leq -a \\ \sum\limits_{j=-\infty}^{\infty} [H((2j-1)\pi - \sin^{-1}(\frac{1}{nx})) - H((2j-1)\pi \\ -\sin^{-1}(-\frac{1}{na}))] + \sum\limits_{j=-\infty}^{\infty} [H(2j\pi + \sin^{-1}(-\frac{1}{na})) \\ -H(2j\pi + \sin^{-1}(\frac{1}{nx}))], & \text{if } \ -a < x < 0 \\ \sum\limits_{j=-\infty}^{\infty} [H(2j\pi + \sin^{-1}(\frac{1}{na})) - H((2j-1)\pi - \sin^{-1}(\frac{1}{na}))] & \text{if } \ x = 0 \\ \sum\limits_{j=-\infty}^{\infty} [H(2j\pi + \sin^{-1}(\frac{1}{na})) - H((2j-1)\pi \\ -\sin^{-1}(-\frac{1}{na}))] + 1 - \sum\limits_{j=-\infty}^{\infty} [H(2j\pi + \sin^{-1}(\frac{1}{nx})) \\ -H((2j-1)\pi - \sin^{-1}(\frac{1}{nx}))], & \text{if } \ 0 < x \leq a \\ 1 & \text{if } \ x > a. \end{cases}$$

Proof. Case 1. $x \leq -a$.

$$F^a_{nk}(x) \ = \ P(X^a_{nk} \leq x) \ = \ P[(\frac{1}{nx} \leq \sin X_k) \cap A] \ = \ P(\phi) \ = \ 0.$$

Case 2. $-a < x < 0$.

$$\begin{aligned} F^a_{nk}(x) &= P(X^a_{nk} \leq -a) + P(-a < X^a_{nk} \leq x) \\ &= 0 + P\left((\frac{1}{nx} \leq \sin X_k < -\frac{1}{na}) \cap A\right) \\ &= P\Big(\bigcup_{j=-\infty}^{\infty} [(2j-1)\pi - \sin^{-1}(-\frac{1}{na}) < X_k < (2j-1)\pi - \sin^{-1}(\frac{1}{nx}) \end{aligned}$$

$$
\begin{aligned}
& \quad or \;\; (2j\pi + \sin^{-1}(\frac{1}{nx}) \le X_k < 2j\pi + \sin^{-1}(-\frac{1}{na})]\Big) \\
= & \sum_{j=-\infty}^{\infty} [H((2j-1)\pi - \sin^{-1}(\frac{1}{nx})) - H((2j-1)\pi - \sin^{-1}(-\frac{1}{na}))] \\
& + \sum_{j=-\infty}^{\infty} [H(2j\pi + \sin^{-1}(-\frac{1}{na})) - H(2j\pi + \sin^{-1}(\frac{1}{nx}))].
\end{aligned}
$$

Case 3. $x = 0$.

$$
\begin{aligned}
F_{nk}^a(x) &= P(X_{nk}^a \le 0) \\
&= P(X_{nk}^a \le -a) + P(-a < X_{nk}^a < 0) + P(X_{nk}^a = 0) \\
&= P\Big((\sin X_k < -\frac{1}{na}) \cap A\Big) \\
&\quad + P\Big[\bigcup_{j=-\infty}^{\infty} ((2j-1)\pi - \sin^{-1}(\frac{1}{na})) < X_k < (2j-1)\pi - \sin^{-1}(-\frac{1}{na}))\Big] \\
&\quad + P\Big[\bigcup_{j=-\infty}^{\infty} (2j\pi + \sin^{-1}(-\frac{1}{na})) < X_k < 2j\pi + \sin^{-1}(\frac{1}{na}))\Big] \\
&= P\Big[\bigcup_{j=-\infty}^{\infty} ((2j-1)\pi - \sin^{-1}(-\frac{1}{na})) < X_k < 2j\pi + \sin^{-1}(-\frac{1}{na}))\Big] \\
&\quad + P\Big[\bigcup_{j=-\infty}^{\infty} ((2j-1)\pi - \sin^{-1}(\frac{1}{na})) < X_k < (2j-1)\pi - \sin^{-1}(-\frac{1}{na}))\Big] \\
&\quad + P\Big[\bigcup_{j=-\infty}^{\infty} (2j\pi + \sin^{-1}(-\frac{1}{na})) < X_k < 2j\pi + \sin^{-1}(\frac{1}{na}))]\Big] \\
&= \sum_{j=-\infty}^{\infty} [H(2j\pi + \sin^{-1}(\frac{1}{na}))) - H((2j-1)\pi - \sin^{-1}(\frac{1}{na}))].
\end{aligned}
$$

For the other cases we can prove in the same argument as in case 3. □

Lemma 2.4.

1. $K_n^a(\infty) - K_n^a(a) + K_n^a(-a) = 0$ *and*

2. $|K_n^a(x_i) - K^a(x_i)| \le \frac{C}{n} \ln na$ *for any* $i = 1, 2, \cdots, m$.

Proof. By Lemma 2.3 and the fact that $\mu_{nk}(a) = 0$, we have

$$K_n^a(v)=\begin{cases}0, & \text{if } \ v\le -a\\ \displaystyle\sum_{j=-\infty}^{\infty}\int_{-a}^{v}\Big[\frac{f((2j-1)\pi-\sin^{-1}(\frac{1}{nx}))}{\sqrt{1-(\frac{1}{nx})^2}} \\ \qquad +\frac{f(2j\pi+\sin^{-1}(\frac{1}{nx}))}{\sqrt{1-(\frac{1}{nx})^2}}\Big]\,dx, & \text{if } \ -a<v<a\\ \displaystyle\sum_{j=-\infty}^{\infty}\int_{-a}^{a}\Big[\frac{f((2j-1)\pi-\sin^{-1}(\frac{1}{nx}))}{\sqrt{1-(\frac{1}{nx})^2}} \\ \qquad +\frac{f(2j\pi+\sin^{-1}(\frac{1}{nx}))}{\sqrt{1-(\frac{1}{nx})^2}}\Big]\,dx & \text{if } \ v\ge a.\end{cases}$$

Hence 1. follows immediately from the formula of K_n^a. To prove 2., we divide into 3 cases as follows.

Case 1. $x_i\le -\dfrac{1}{n\sin\varepsilon}$.

By Mean-Value Theorem, for all $u\in[\sin^{-1}(\frac{1}{nx_i}),\sin^{-1}(-\frac{1}{na})]$ there exists $\eta_u^j\in(2j\pi+u,2j\pi)$ such that

$$|f(2j\pi+u)-f(2j\pi)|=|f'(\eta_u^j)(u)|$$

and there exists $\xi_u^j\in((2j-1)\pi,(2j-1)\pi-u)$ such that

$$|f((2j-1)\pi-u)-f((2j-1)\pi)|=|f'(\xi_u^j)(u)|.$$

By conditions (a) and (b),

$$\begin{aligned}A_n &\equiv \int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})}\sum_{j=-\infty}^{\infty}\frac{|f(2j\pi+u)-f(2j\pi)|}{n(\sin u)^2}\,du\\ &= \int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})}\sum_{j=-\infty}^{\infty}\frac{-u|f'(\eta_u^j)|}{n(\sin u)^2}\,du\\ &< \frac{C}{n}\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})}\frac{-u}{(\sin u)^2}\,du\\ &= \frac{C}{n}\Big[-na\sin^{-1}(-\frac{1}{na})\cos(\sin^{-1}(-\frac{1}{na}))-\ln\Big|-\frac{1}{na}\Big|+\ln\Big|\frac{1}{nx_i}\Big|\Big]\\ &< \frac{C}{n}\ln na,\\ B_n &\equiv \int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})}\sum_{j=-\infty}^{\infty}\frac{|f((2j-1)\pi-u)-f((2j-1)\pi)|}{n(\sin u)^2}\,du\\ &< \frac{C}{n}\ln na,\end{aligned}$$

$$
\begin{aligned}
C_n &\equiv \int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} |\sum_{j=-\infty}^{\infty} f((2j-1)\pi)|\frac{|\cos 0 - \cos u|}{n(\sin u)^2}\,du \\
&= \int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} |\sum_{j=-\infty}^{\infty} f((2j-1)\pi)|\frac{|-\sin\xi|(-u)}{n(\sin u)^2}\,du \quad \text{for some } \xi \in (u,0) \\
&\le C\int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} \frac{-u}{n(\sin u)^2}du \\
&< \frac{C}{n}\ln na
\end{aligned}
$$

and

$$
\begin{aligned}
D_n &\equiv \int_{\sin^{-1}(\frac{1}{nx_i})}^{\sin^{-1}(-\frac{1}{na})} |\sum_{j=-\infty}^{\infty} f(2j\pi)|\frac{|\cos 0 - \cos u|}{n(\sin u)^2}\,du \\
&< \frac{C}{n}\ln na.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(2.5)\qquad & |K_n^a(x_i) - K^a(x_i)| \\
&= \Big|\sum_{j=-\infty}^{\infty}\int_{-a}^{x_i} \frac{f((2j-1)\pi) - \sin^{-1}(\frac{1}{nx})) + f(2j\pi + \sin^{-1}(\frac{1}{nx}))}{\sqrt{1-(\frac{1}{nx})^2}} \\
&\qquad - (f((2j-1)\pi) + f(2j\pi))\,dx\Big| \\
&= \Big|\sum_{j=-\infty}^{\infty}\int_{\sin^{-1}(-\frac{1}{na})}^{\sin^{-1}(\frac{1}{nx_i})} \Big\{\frac{f((2j-1)\pi - u) + f(2j\pi + u)}{\cos u} \\
&\qquad + (f((2j-1)\pi) + f(2j\pi))\Big\}\frac{\cos u}{n(\sin u)^2}\,du\Big| \\
&\le A_n + B_n + C_n + D_n \\
&< \frac{C}{n}\ln na.
\end{aligned}
$$

Case 2. $-\dfrac{1}{n\sin\varepsilon} < x_i \le \dfrac{1}{n\sin\varepsilon}$.
We observe from (2.5) that

$$
|K_n^a(-\frac{1}{n\sin\varepsilon}) - K^a(-\frac{1}{n\sin\varepsilon})| \le \frac{C}{n}\ln na
$$

and, by the fact that $|x_i| < \dfrac{1}{n\sin\varepsilon}$,

$$
\begin{aligned}
&\Big|\sum_{j=-\infty}^{\infty}\int_{-\frac{1}{n\sin\varepsilon}}^{x_i}\frac{f((2j-1)\pi-\sin^{-1}(\frac{1}{nx}))+f(2j\pi+\sin^{-1}(\frac{1}{nx}))}{\sqrt{1-(\frac{1}{nx})^2}}-Ldx\Big| \\
\le\;& \int_{-\frac{1}{n\sin\varepsilon}}^{\frac{1}{n\sin\varepsilon}}\sum_{j=-\infty}^{\infty}\frac{|f((2j-1)\pi-\sin^{-1}(\frac{1}{nx}))+f(2j\pi+\sin^{-1}(\frac{1}{nx}))|}{\sqrt{1-(\frac{1}{nx})^2}}dx+\frac{C}{n} \\
\le\;& \frac{C}{n}.
\end{aligned}
$$

So

$$
\begin{aligned}
&|K_n^a(x_i)-K^a(x_i)| \\
\le\;& \Big|K_n^a(-\frac{1}{n\sin\varepsilon})-K^a(-\frac{1}{n\sin\varepsilon})\Big| \\
&+\Big|\sum_{j=-\infty}^{\infty}\int_{-\frac{1}{n\sin\varepsilon}}^{x_i}\frac{f((2j-1)\pi)-\sin^{-1}(\frac{1}{nx}))+f(2j\pi+\sin^{-1}(\frac{1}{nx}))}{\sqrt{1-(\frac{1}{nx})^2}}-Ldx\Big| \\
\le\;& \frac{C}{n}\ln na.
\end{aligned}
$$

Case 3. $\quad x_i>\dfrac{1}{n\sin\varepsilon}$.
Using the same arguments of case 1 and case 2, we can show that

$$|K_n^a(x_i)-K^a(x_i)|\;\le\;\frac{C}{n}\ln na.$$

□

Proof of Theorem 1.1.
From Lemma 2.2 and Lemma 2.4 we see that

$$h^a(n,m(a,\delta),r)\le C\Big[(\frac{a^2}{n})^{\frac{1}{3}}+\Big\{(\frac{m+1}{n})\ln na\Big\}^{\frac{1}{3}}+(\delta a)^{\frac{1}{4}}+C(r)(\frac{1}{a})^{\frac{1-r}{1+r}}\Big]$$

which implies

$$
\begin{aligned}
(2.6)\qquad & \sup_{-\infty<x<\infty}|F_n(x)-F(x)| \\
\le\;& C\Big[\frac{1}{a}+(\frac{a^2}{n})^{\frac{1}{3}}+\Big\{(\frac{m+1}{n})\ln na\Big\}^{\frac{1}{3}}+(\delta a)^{\frac{1}{4}}+C(r)(\frac{1}{a})^{\frac{1-r}{1+r}}\Big]
\end{aligned}
$$

where $C(r)$ are constants depending on r.
We set $a=n^{k_1}$ and $\delta=n^{-k_2}$ where k_1 and k_2 are positive numbers. Then we see that the orders in (2.6) are

$$\frac{1}{n^{k_1}},\ \frac{1}{n^{\frac{1-2k_2}{3}}},\ \frac{(1+k_1)\ln n}{n^{(\frac{1-k_1-k_2}{3})}},\ \frac{1}{n^{\frac{k_2-k_1}{4}}},\ C(r)\frac{1}{n^{\frac{k_1(1-r)}{1+r}}}.$$

We determine k_1 and k_2 such that the function

$$\min(k_1, \frac{1-2k_1}{5}, \frac{1-k_1-k_2}{3}, \frac{k_2-k_1}{4}) \text{ is maximized.} \tag{2.7}$$

By maximin criterion, we consider the six lines corresponding to six combinations of equalities among the linear functions involved in the definition of the function in (2.7) and by computing the value of (2.7) at the intersection of these lines, we obtain that $k_1 = \frac{1}{9}$ and $k_2 = \frac{5}{9}$ gives the maximum value of $\frac{1}{9}$ for the function in (2.7). For $0 < \alpha < \frac{1}{9}$ the choice of $r = \frac{9\alpha}{2-9\alpha}$ implies that $k_1 \cdot \frac{1-r}{1+r} = \frac{1}{9} - \alpha$.We have that for any constant $0 < d < \frac{1}{9}$, there exists a constant C such that

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| < \frac{C}{n^d}.$$

3. Example

For each n, let X_n be a Cauchy random variable with parameter 1 such that $\sin X_n \neq 0$. If (X_n) is independent, by Theorem 1.1 we know that a sequence of distribution functions F_n of $\frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sin X_k}$ converges to the Cauchy distribution

$$F(x) = \frac{1}{\pi}(\frac{\pi}{2} + \tan^{-1}\frac{x}{\pi L})$$

where $L = \frac{1}{\pi}\sum_{j=-\infty}^{\infty} \frac{1}{1+j^2}$ and for $0 < d < \frac{1}{9}$ there exists a constant C such that

$$\sup_{x\in\mathbf{R}} |F_n(x) - F(x)| < \frac{C}{n^d}.$$

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A uniform bound on a Combinatorial Central Limit Theorem

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Abstract: This paper establishes a combinatorial central limit theorem for an array of independent random variables (X_{ij}), $1 \le i, j \le n$, $(n \to \infty)$ with finite third moments. Let $\pi = (\pi(1), \pi(2), ..., \pi(n))$ be a permutation of $\{1, 2, ..., n\}$, and define $W_n = \sum_i X_{i\pi(i)}$. Then the authors prove the following uniform central limit property: $\sup_{\mathbb{R}} |F_n(x) - \Phi(x)| \le 198\beta + \frac{18}{n}$, where F_n is the distribution of $\frac{W_n - EW_n}{\sqrt{VarW_n}}$, Φ is the strandrad normal distribution, and $\beta = \frac{1}{n}\sum_{i,j} E|\hat{X}_{ij}|^3$ with $\hat{X}_{ij}$ is a suitable normalization of X_{ij}. The proof uses the Stein's method and the result generalizes and improves a number of known results.

Keywords: combinatorial central limit theorem, Stein's method, concentration inequality and random permutation.

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1 Introduction and main results.

Let (X_{ij}) be an $n \times n$ matrix of independent random variables with finite third moment and $\pi = (\pi(1), \pi(2), ..., \pi(n))$ be a random permutation of $\{1, 2, ..., n\}$ such that π and $X_{ij}'s$ are independent. This paper is concerned with the normal approximation to the distribution function of $W_n = \sum_i X_{i\pi(i)}$. The special cases of W_n are the statistics $\eta_n = \sum_i a_i b_{i\pi(i)}$ and $\xi_n = \sum_i c_{i\pi(i)}$ where a_i, b_{ij} and c_{ij} $(i, j = 1, 2, ..., n)$ are real numbers. Both statistics η_n and ξ_n arise in permutation tests in nonparametric inference.(see,for examples Fraser(1957),Puri and Sen(1971), Does(1982)). The literature concerning the limit behavior of W_n dates back to 1944 when Wald and Wolfowitz(1944) first established the asymptotic normality of η_n with some strong sufficient conditions. After that, a theorem has been proved under various conditions by Hoeffding(1951), Matoo(1957), Hájek(1961), Robinson(1972), Kolchim and Chistyakov (1973),Ho and Chen(1978), Does(1982),Bolthausen(1984), Schneller(1988) and Loh(1996).Almost all of the literatures gave a bound when X_{ij} are constants and the best bound of order $\frac{1}{n}$ is given by Bolthausen(1984) and Chen and Neammanee (2003). In case of $X_{ij}'s$ are any random variables, the estimations have been obtained by Von Bahr(1976) and Ho and Chen (1978) but they yield the rate $\frac{1}{\sqrt{n}}$ only under some boundedness condition like $\sup_{i,j} |X_{ij}| = O(n^{-\frac{1}{2}})$. In this paper we give the rate $\frac{1}{n}$ by using Stein's method and the idea from Chen and Neammanee(2003).

Stein(1972) originally introduced his method for obtainning rate of convergence on a central limit theorem for sums of nearly independent random variables. There are at least 3 approaches to use Stein's method when the limit distribution is normal, i.e. namely a concentration inequality approach (see for examples, Ho and Chen(1978) and Chen and Shao(2001)), an inductive approach (see for example, Bolthausen(1984)) and a coupling approach (see for examples, Stein(1986)). In this work we use the concentration inequality approach.

For each $i, j \in \{1, 2, ..., n\}$, let μ_{ij} and σ_{ij}^2 be the mean and variance of X_{ij}, respectively and

$$\begin{aligned} \mu_{i.} &= \frac{1}{n}\sum_j \mu_{ij}\ , \quad \mu_{.j} = \frac{1}{n}\sum_i \mu_{ij}, \quad \mu_{..} = \frac{1}{n^2}\sum_{i,j} \mu_{ij} \\ d^2 &= \frac{1}{(n-1)}\sum_{i,j}(\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..})^2 \quad \text{and} \quad \sigma^2 = \frac{1}{n}\sum_{i,j}\sigma_{ij}^2. \end{aligned}$$

From Ho and Chen(1978) we know that $VarW_n = d^2 + \sigma^2$.

Define

$$W = \frac{1}{\sqrt{d^2 + \sigma^2}}\sum_i (X_{i\pi(i)} - \mu_{..}).$$

So $EW = 0, VarW = 1$ and

$$W = \frac{W_n - EW_n}{\sqrt{VarW_n}} = \sum_i \hat{X}_{i\pi(i)}$$

where $\hat{X}_{ij} = \frac{1}{\sqrt{d^2+\sigma^2}}(X_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..})$.

Our main result is the following theorem.

Theorem
$$\sup_{\mathbf{R}} |F_n(x) - \Phi(x)| \leq 198\beta + \frac{18}{n}$$

where F_n is the distribution function of $\frac{W_n - EW_n}{\sqrt{VarW_n}}$, Φ is the standard normal distribution function and $\beta = \frac{1}{n}\sum_{i,j} E|\hat{X}_{ij}|^3$.

2 Concentration inequality.

In this section, we will prove the concentration inequality (proposition 2.7) which is the important tool for proving the main result in section 3. In order to prove the concentration inequality, we need the following construction from Ho and Chen(1978).

Let J, K, L, M be random variables which uniformly distribution on $\{1, 2, ..., n\}$ and $\pi = (\pi(1), \pi(2), ..., \pi(n))$, $\rho = (\rho(1), \rho(2), ..., \rho(n))$ and $\tau = (\tau(1), \tau(2), ..., \tau(n))$ are random permutations of $\{1, 2, ..., n\}$. Assume that

$$\{J, K, L, M, \pi, \rho, \tau\} \text{ is independent of } X_{ij}'s, \tag{2.1}$$

$$(J, K) \text{ and } (L, M) \text{ are uniformly distributed on } \{(j,k)|j,k = 1, 2, ..., n \text{ and } j \neq k\}, \tag{2.2}$$

$$(J, K), (L, M) \text{ and } \tau \text{ are mutually independent}, \tag{2.3}$$

(J, K) and ρ are mutually independent, and (2.4)

$$\rho(\alpha) = \begin{cases} \tau(\alpha) & \text{if } \alpha \neq J, K, \tau^{-1}(L), \tau^{-1}(M), \\ L & \text{if } \alpha = J, \\ M & \text{if } \alpha = K, \\ \tau(J) & \text{if } \alpha = \tau^{-1}(L), \\ \tau(K) & \text{if } \alpha = \tau^{-1}(M), \end{cases} \tag{2.5}$$

where $\rho(\rho^{-1}(\alpha)) = \rho^{-1}(\rho(\alpha)) = \alpha$.

We note that there exists a system which satisfies (2.1)-(2.5) (see for example, Ho(1975)). Let

$$S(\rho) = \sum_i \hat{X}_{i\rho(i)} \text{ and } \tilde{S}(\rho) = S(\rho) - \hat{X}_{J\rho(J)} - \hat{X}_{K\rho(K)} + \hat{X}_{J\rho(K)} + \hat{X}_{K\rho(J)}.$$

Proposition 2.1 *$(S(\rho), \tilde{S}(\rho))$ is an exchangeable pair, in the sense of*

$$P(S(\rho) \in B, \tilde{S}(\rho) \in \tilde{B}) = P(S(\rho) \in \tilde{B}, \tilde{S}(\rho) \in B)$$

for every Borel sets B and $\tilde{B}$.

Proof.

Let $a, b \in \mathbb{R}$ and S_n be the set of all permutations of $\{1, 2, ..., n\}$. Then the proposition follows from the following fact.

$$P(S(\rho) \le a, \tilde{S}(\rho) \le b)$$

$$= \sum_{\substack{j,k \\ j \neq k}} \sum_{(l_1, l_2, \ldots, l_n) \in S_n} P(\sum_i \hat{X}_{il_i} \le a, \hat{X}_{1l_1} + \cdots + \hat{X}_{jl_k} + \cdots + \hat{X}_{kl_j} + \cdots + \hat{X}_{nl_n} \le b,$$

$$(J, K) = (j, k), \rho = (l_1, l_2, \ldots, l_n))$$

$$= \sum_{\substack{j,k \\ j \neq k}} \sum_{(l_1, l_2, \ldots, l_n) \in S_n} P(\sum_i \hat{X}_{il_i} \le a, \hat{X}_{1l_1} + \cdots + \hat{X}_{jl_k} + \cdots + \hat{X}_{kl_j} + \cdots + \hat{X}_{nl_n} \le b,$$

$$(J, K) = (j, k), \rho = (l_1, \ldots, l_{j-1}, l_k, l_{j+1}, \ldots, l_{k-1}, l_j, l_{k+1}, \ldots, l_n))$$

$$= \sum_{\substack{j,k \\ j \neq k}} \sum_{(l_1, l_2, \ldots, l_n) \in S_n} P(\hat{X}_{1l_1} + \cdots + \hat{X}_{jl_k} + \cdots + \hat{X}_{kl_j} + \cdots + \hat{X}_{nl_n} \le a, \sum_i \hat{X}_{il_i} \le b,$$

$$(J, K) = (j, k), \rho = (l_1, l_2, \ldots, l_n))$$

$$= P(\tilde{S}(\rho) \le a, S(\rho) \le b).$$ □

Lemma 2.2 *Let $\mathcal{B}$ be a σ-algebra generated by ρ and $X'_{ij}s$. Then*

1. $E^{\mathcal{B}} \tilde{S}(\rho) = (1 - \frac{2}{n-1}) S(\rho) + \frac{2}{n(n-1)} \sum_{i,j} \hat{X}_{ij}$ *, and*

2. $E[\tilde{S}(\rho) - S(\rho)]^2 = \frac{4}{n-1}[1 - \frac{\sigma^2}{n(d^2 + \sigma^2)}]$

where $E^{\mathcal{B}}(X)$ is the conditional expectation of X with respect to $\mathcal{B}$.

Proof.

1. $$E^{\mathcal{B}}\tilde{S}(\rho) = E^{\mathcal{B}}[S(\rho) - \hat{X}_{J\rho(J)} - \hat{X}_{K\rho(K)} + \hat{X}_{J\rho(K)} + \hat{X}_{K\rho(J)}]$$
$$= S(\rho) - E^{\mathcal{B}}[\hat{X}_{J\rho(J)} + \hat{X}_{K\rho(K)} - \hat{X}_{J\rho(K)} - \hat{X}_{K\rho(J)}]$$
$$= S(\rho) - \frac{1}{n}\sum_j E^{\mathcal{B}}\hat{X}_{j\rho(j)} - \frac{1}{n}\sum_k E^{\mathcal{B}}\hat{X}_{k\rho(k)}$$
$$+ \frac{1}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}} E^{\mathcal{B}}\hat{X}_{j\rho(k)} + \frac{1}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}} E^{\mathcal{B}}\hat{X}_{k\rho(j)}$$
$$= S(\rho) - \frac{2}{n}\sum_j E^{\mathcal{B}}\hat{X}_{j\rho(j)} + \frac{2}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}} E^{\mathcal{B}}\hat{X}_{j\rho(k)}$$
$$= S(\rho) - \frac{2}{n}\sum_j \hat{X}_{j\rho(j)} + \frac{2}{n(n-1)}[\sum_{j,k} \hat{X}_{j\rho(k)} - \sum_j \hat{X}_{j\rho(j)}]$$
$$= [1 - \frac{2}{n} - \frac{2}{n(n-1)}]S(\rho) + \frac{2}{n(n-1)}\sum_{i,j}\hat{X}_{ij}$$
$$= (1 - \frac{2}{n-1})S(\rho) + \frac{2}{n(n-1)}\sum_{i,j}\hat{X}_{ij}.$$

2. For each $i, j = 1, 2, ..., n$, let $\hat{\mu}_{ij}$ be the mean of $\hat{X}_{ij}$. We note that

$$\sum_j \hat{\mu}_{ij} = 0 \text{ for every } i \text{ and } \sum_i \hat{\mu}_{ij} = 0 \text{ for every } j,$$

and $\sum_{i,j}\hat{\mu}_{ij}^2 = \frac{1}{d^2+\sigma^2}\sum_{i,j}(\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..})^2 = (n-1)(\frac{d^2}{d^2+\sigma^2}).$

From these facts, we have

$$\sum_{i,j} E\hat{X}_{ij}^2 = \frac{1}{d^2+\sigma^2}(\sum_{i,j}\sigma_{ij}^2) + \sum_{i,j}\hat{\mu}_{ij}^2 = n - \frac{d^2}{d^2+\sigma^2}, \tag{2.6}$$

$$\sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}} E\hat{X}_{jm}\hat{X}_{kl} = \sum_{\substack{j,k\\ j\neq k}}\Big(\sum_{l,m}\hat{\mu}_{jm}\hat{\mu}_{kl} - \sum_{l}\hat{\mu}_{jl}\hat{\mu}_{kl}\Big)$$

$$= -\sum_{j,k,l}\hat{\mu}_{jl}\hat{\mu}_{kl} + \sum_{j,l}\hat{\mu}_{jl}^2$$

$$= -\sum_{j,l}\hat{\mu}_{jl}\sum_{k}\hat{\mu}_{kl} + (n-1)\frac{d^2}{d^2+\sigma^2}$$

$$= (n-1)\frac{d^2}{d^2+\sigma^2}, \tag{2.7}$$

and
$$\sum_{j}\sum_{\substack{l,m\\ l\neq m}} E\hat{X}_{jm}\hat{X}_{jl} = \sum_{\substack{j,k\\ j\neq k}}\sum_{l} E\hat{X}_{jl}\hat{X}_{kl} = -(n-1)\frac{d^2}{d^2+\sigma^2}. \tag{2.8}$$

Hence

$$E[\tilde{S}(\rho) - S(\rho)]^2$$

$$= \frac{1}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}} E[\hat{X}_{j\rho(k)} + \hat{X}_{k\rho(j)} - \hat{X}_{j\rho(j)} - \hat{X}_{k\rho(k)}]^2$$

$$= \frac{2}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}}[E\hat{X}_{j\rho(j)}^2 + E\hat{X}_{j\rho(k)}^2 + E\hat{X}_{j\rho(k)}\hat{X}_{k\rho(j)} + E\hat{X}_{j\rho(j)}\hat{X}_{k\rho(k)}]$$

$$- \frac{4}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}}[E\hat{X}_{j\rho(k)}\hat{X}_{j\rho(j)} + E\hat{X}_{j\rho(k)}\hat{X}_{k\rho(k)}]$$

$$= \frac{2}{n(n-1)}[(n-1)\sum_{j} E\hat{X}_{j\rho(j)}^2 + \sum_{j,k} E\hat{X}_{j\rho(k)}^2 - \sum_{j} E\hat{X}_{j\rho(j)}^2]$$

$$+ \frac{2}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}} E\hat{X}_{j\rho(k)}\hat{X}_{k\rho(j)} + \frac{2}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}} E\hat{X}_{j\rho(j)}\hat{X}_{k\rho(k)}$$