$$- \frac{4}{n(n-1)} \sum_{\substack{j,k \\ j \neq k}} E\hat{X}_{j\rho(k)}\hat{X}_{j\rho(j)} - \frac{4}{n(n-1)} \sum_{\substack{j,k \\ j \neq k}} E\hat{X}_{j\rho(k)}\hat{X}_{k\rho(k)}$$

$$= \frac{2}{n(n-1)} \sum_{i,j} E\hat{X}_{ij}^2 + \frac{2(n-2)}{n^2(n-1)} \sum_{i,j} E\hat{X}_{ij}^2$$

$$+ \frac{2}{n^2(n-1)^2} \sum_{\substack{j,k \\ j \neq k}} \sum_{\substack{l,m \\ l \neq m}} E\hat{X}_{jm}\hat{X}_{kl} + \frac{2}{n^2(n-1)^2} \sum_{\substack{j,k \\ j \neq k}} \sum_{\substack{l,m \\ l \neq m}} E\hat{X}_{jl}\hat{X}_{km}$$

$$- \frac{4}{n^2(n-1)} \sum_{j} \sum_{\substack{l,m \\ l \neq m}} E\hat{X}_{jm}\hat{X}_{jl} - \frac{4}{n^2(n-1)} \sum_{\substack{j,k \\ j \neq k}} \sum_{l} E\hat{X}_{jl}\hat{X}_{kl}$$

$$= \frac{4}{n^2} \sum_{i,j} E\hat{X}_{ij}^2 + \frac{4}{n^2(n-1)^2} \sum_{\substack{j,k \\ j \neq k}} \sum_{\substack{l,m \\ l \neq m}} E\hat{X}_{jm}\hat{X}_{kl}$$

$$- \frac{4}{n^2(n-1)} \sum_{j} \sum_{\substack{l,m \\ l \neq m}} E\hat{X}_{jm}\hat{X}_{jl} - \frac{4}{n^2(n-1)} \sum_{\substack{j,k \\ j \neq k}} \sum_{l} E\hat{X}_{jl}\hat{X}_{kl}$$

$$= \frac{4}{n-1}[1 - \frac{\sigma^2}{n(d^2+\sigma^2)}]$$

where we have used (2.6)-(2.8) in the last equality. □

Proposition 2.3 (Stein,(1986), p.9) Let (X, X') be an exchangeable pair and $F : \Omega^2 \to \mathbb{R}$ be an anti-symmetric measurable function in the sense that, $F(x, x') = -F(x', x)$ for all $x, x' \in \Omega$. If $E|F(X, X')| < \infty$, then $EF(X, X') = 0$.

Proposition 2.4 $\quad E[S(\rho)\sum_{i,j} \hat{X}_{ij}] = \frac{\sigma^2}{d^2+\sigma^2}.$

Proof.

Let $F(w, w') = (w')^2 - w^2$. By proposition 2.1, lemma 2.2(1) and proposition 2.3

we have

$$
\begin{aligned}
0 &= EF(S(\rho), \tilde{S}(\rho)) \\
&= E[\tilde{S}^2(\rho) - S^2(\rho)] \\
&= E\{2S(\rho)[\tilde{S}(\rho) - S(\rho)] + [\tilde{S}(\rho) - S(\rho)]^2\} \\
&= 2ES(\rho)[E^B\tilde{S}(\rho) - S(\rho)] + E[\tilde{S}(\rho) - S(\rho)]^2 \\
&= 2ES(\rho)[(1 - \frac{2}{n-1})S(\rho) + \frac{2}{n(n-1)}\sum_{i,j}\hat{X}_{ij} - S(\rho)] + E[\tilde{S}(\rho) - S(\rho)]^2 \\
&= 2ES(\rho)[(1 - \frac{2}{n-1})S(\rho) + \frac{2}{n(n-1)}\sum_{i,j}\hat{X}_{ij} - S(\rho)] + E[\tilde{S}(\rho) - S(\rho)]^2 \\
&= -\frac{4}{n-1}ES^2(\rho) + \frac{4}{n(n-1)}E[S(\rho)\sum_{i,j}\hat{X}_{ij}] + E[\tilde{S}(\rho) - S(\rho)]^2. \qquad (2.9)
\end{aligned}
$$

Hence, by (2.9), lemma 2.2(2) and the fact that $ES^2(\rho) = VarW = 1$, we have

$$
E[S(\rho)\sum_{i,j}\hat{X}_{ij}] = nES^2(\rho) - \frac{n(n-1)}{4}E[\tilde{S}(\rho) - S(\rho)]^2 \quad = \frac{\sigma^2}{d^2 + \sigma^2}. \qquad \square
$$

To prove lemma 2.5 and lemma 2.6 we need the following construction. Let $\overline{J}, \overline{K}, \overline{L}, \overline{M}$ be uniform distributed random variables on $\{1, 2, ..., n\}$ which satisfy the followings.

$(\overline{J}, \overline{K})$ and $(\overline{L}, \overline{M})$are uniformly distributed random variables on

$\{(j,k)|j,k = 1, 2, ..., n \ \text{ and } \ j \neq k\}$,

$[(\overline{J},\overline{K}),(\overline{L},\overline{M})]$ is uniformly on $\{[(j,k),(l,m)]|\ j,k,l,m=1,2,...,n$

and $j\neq k, l\neq m$ and $(j,k)\neq(l,m)\}$

and $[(\overline{J},\overline{K}),(\overline{L},\overline{M})]$ and ρ are mutually independent.

Hence

$$P([(\overline{J},\overline{K}),(\overline{L},\overline{M})]=[(j,k),(l,m)])=\frac{1}{n(n-1)[n(n-1)-1]}$$

for $j,k,l,m=1,2,...,n, j\neq k, l\neq m$ and $(j,k)\neq(l,m)$.

In what follows, we let $\delta=10\beta$ and also let

$Z_{[(j,k),(l,m)]}=|\hat{X}_{jl}+\hat{X}_{km}-\hat{X}_{jm}-\hat{X}_{kl}|\min(\delta,|\hat{X}_{jl}+\hat{X}_{km}-\hat{X}_{jm}-\hat{X}_{kl}|)$,

$\hat{Z}_{[(j,k),(l,m)]}=Z_{[(j,k),(l,m)]}-EZ_{[(j,k),(\rho(j),\rho(k))]}$, $Z(\rho)=\sum\limits_{\substack{j,k\\ j\neq k}}\hat{Z}_{[(j,k),(\rho(j),\rho(k))]}$

and

$$\begin{aligned}\tilde{Z}(\rho)=&Z(\rho)-\hat{Z}_{[(\overline{J},\overline{K}),(\rho(\overline{J}),\rho(\overline{K}))]}-\hat{Z}_{[(\overline{L},\overline{M}),(\rho(\overline{L}),\rho(\overline{M}))]}\\&+\hat{Z}_{[(\overline{J},\overline{K}),(\rho(\overline{L}),\rho(\overline{M}))]}+\hat{Z}_{[(\overline{L},\overline{M}),(\rho(\overline{J}),\rho(\overline{K}))]}.\end{aligned}$$

Observe that

1) $EZ^2(\rho)=Var\sum\limits_{\substack{j,k\\ j\neq k}}Z_{[(j,k),(\rho(j),\rho(k))]}$,

2) $\sum\limits_{\substack{l,m\\ l\neq m}}E\hat{Z}_{[(j,k),(l,m)]}=0$ for every (j,k) such that $j\neq k$, and

3) $(\tilde{Z}(\rho),Z(\rho))$ is an exchangeable pair.

Lemma 2.5

1) $E^{\mathcal{B}}(\tilde{Z}(\rho)) = (1 - \frac{2}{n(n-1)-1})Z(\rho) + \frac{2}{n(n-1)[n(n-1)-1]} \sum_{\substack{j,k \\ j \neq k}} \sum_{\substack{l,m \\ l \neq m}} \hat{Z}_{[(j,k),(l,m)]}$,

2) $E[\tilde{Z}(\rho) - Z(\rho)]^2 \leq \frac{9.143 n\beta}{n(n-1)-1}$ for $\beta < \frac{1}{350}$, and

3) $E \sum_{\substack{j,k \\ j \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} > 2.3n$ for $n \geq 32$.

Proof.

1) Using the same technique as in lemma 2.2(1).

2) $E[\tilde{Z}(\rho) - Z(\rho)]^2$

$$
\begin{aligned}
&= E[Z_{[(\mathcal{J},\overline{\mathcal{K}}),(\rho(\mathcal{J}),\rho(\overline{\mathcal{K}}))]} + Z_{[(\mathcal{L},\overline{\mathcal{M}}),(\rho(\mathcal{L}),\rho(\overline{\mathcal{M}}))]} - Z_{[(\mathcal{J},\overline{\mathcal{K}}),(\rho(\mathcal{L}),\rho(\overline{\mathcal{M}}))]} - Z_{[(\mathcal{L},\overline{\mathcal{M}}),(\rho(\mathcal{J}),\rho(\overline{\mathcal{K}}))]}]^2 \\
&\leq E[Z_{[(\mathcal{J},\overline{\mathcal{K}}),(\rho(\mathcal{J}),\rho(\overline{\mathcal{K}}))]} + Z_{[(\mathcal{L},\overline{\mathcal{M}}),(\rho(\mathcal{L}),\rho(\overline{\mathcal{M}}))]}]^2 + E[Z_{[(\mathcal{J},\overline{\mathcal{K}}),(\rho(\mathcal{L}),\rho(\overline{\mathcal{M}}))]} + Z_{[(\mathcal{L},\overline{\mathcal{M}}),(\rho(\mathcal{J}),\rho(\overline{\mathcal{K}}))]})]^2 \\
&\leq 2(EZ^2_{[(\mathcal{J},\overline{\mathcal{K}}),(\rho(\mathcal{J}),\rho(\overline{\mathcal{K}}))]} + EZ^2_{[(\mathcal{L},\overline{\mathcal{M}}),(\rho(\mathcal{L}),\rho(\overline{\mathcal{M}}))]} + EZ^2_{[(\mathcal{J},\overline{\mathcal{K}}),(\rho(\mathcal{L}),\rho(\overline{\mathcal{M}}))]} + EZ^2_{[(\mathcal{L},\overline{\mathcal{M}}),(\rho(\mathcal{J}),\rho(\overline{\mathcal{K}}))]}) \\
&= 4EZ^2_{[(\mathcal{J},\overline{\mathcal{K}}),(\rho(\mathcal{J}),\rho(\overline{\mathcal{K}}))]} + 4EZ^2_{[(\mathcal{J},\overline{\mathcal{K}}),(\rho(\mathcal{L}),\rho(\overline{\mathcal{M}}))]} \\
&= \frac{4}{n(n-1)} \sum_{\substack{j,k \\ j \neq k}} EZ^2_{[(j,k),(\rho(j),\rho(k))]} + \frac{4}{n(n-1)[n(n-1)-1]} \sum_{\substack{j,k \\ j \neq k}} \sum_{\substack{l,m \\ l \neq m \\ (l,m) \neq (j,k)}} EZ^2_{[(j,k),(l,m)]} \\
&\leq \frac{4}{n^2(n-1)^2} \sum_{\substack{j,k \\ j \neq k}} \sum_{\substack{l,m \\ l \neq m}} EZ^2_{[(j,k),(l,m)]} + \frac{4}{n(n-1)[n(n-1)-1]} \sum_{\substack{j,k \\ j \neq k}} \sum_{\substack{l,m \\ l \neq m}} EZ^2_{[(j,k),(l,m)]} \\
&\leq \frac{8}{n(n-1)[n(n-1)-1]} \sum_{\substack{j,k \\ j \neq k}} \sum_{\substack{l,m \\ l \neq m}} EZ^2_{[(j,k),(l,m)]}
\end{aligned}
$$

$$\leq \frac{8\delta^2}{n(n-1)[n(n-1)-1]} \sum_{\substack{j,k \\ j\neq k}} \sum_{\substack{l,m \\ l\neq m}} E|\hat{X}_{jl} + \hat{X}_{km} - \hat{X}_{jm} - \hat{X}_{kl}|^2$$

$$= \frac{8n(n-1)\delta^2}{n(n-1)-1} E[\tilde{S}(\rho) - S(\rho)]^2$$

$$\leq \frac{3200n\beta^2}{n(n-1)-1} \qquad \text{(by lemma 2.2 (2))}$$

$$\leq \frac{9.143n\beta}{n(n-1)-1}$$

where we have used the fact that $\beta < \frac{1}{350}$ in the last inequality.

3) By the fact that

$$E|\tilde{S}(\rho) - S(\rho)|^3 = E|\hat{X}_{J\rho(K)} + \hat{X}_{K\rho(J)} - \hat{X}_{J\rho(J)} - \hat{X}_{K\rho(K)}|^3$$

$$\leq 64E|\hat{X}_{IM}|^3$$

$$= \frac{64}{n^2} \sum_{i,j} E|\hat{X}_{ij}|^3$$

$$= \frac{64\beta}{n} \quad ,$$

$\min(a,b) \geq b - \frac{b^2}{4a}$ for $a, b > 0$ and lemma 2.2(2), we have

$$E|\tilde{S}(\rho) - S(\rho)| \min(\delta, |\tilde{S}(\rho) - S(\rho)|) \geq \{E|\tilde{S}(\rho) - S(\rho)|^2 - \frac{1}{4\delta} E|\tilde{S}(\rho) - S(\rho)|^3\}$$

$$\geq \{\frac{4}{n-1}[1 - \frac{\sigma^2}{n(d^2+\sigma^2)}] - \frac{64\beta}{4\delta n}\}$$

$$\geq \frac{12}{5n}.$$

Hence

$$E\sum_{\substack{j,k\\ j\neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} = n(n-1)E|\tilde{S}(\rho)-S(\rho)|\min(\delta,|\tilde{S}(\rho)-S(\rho)|)$$

$$\geq \frac{12(n-1)}{5}$$

$$\geq 2.3n$$

for $n \geq 32$. #

Lemma 2.6 Assume that $\beta < \frac{1}{350}$ and $n \geq 32$. Then

$$Var\sum_{\substack{j,k\\ j\neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} \leq 45.897n^2\beta.$$

Proof.

Let $T = \sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}} \hat{Z}_{[(j,k),(l,m)]}$. By lemma 2.5(1) and the same argument as in (2.9) we have

$$EZ^2(\rho) = \frac{1}{n(n-1)}E[TZ(\rho)] + \frac{n(n-1)-1}{4}E[\tilde{Z}(\rho)-Z(\rho)]^2. \tag{2.10}$$

Next, we will bound $E[TZ(\rho)]$. Let $\Lambda = \{(j,k,l,m,p,q,r,s)|j,k,l,m,p,q,r,s \in \{1,2,\ldots,n\}$ and $j\neq k, l\neq m, p\neq q, r\neq s\}$. Note that

$$E[TZ(\rho)] = E[(\sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}} \hat{Z}_{[(j,k),(l,m)]})(\sum_{\substack{p,q\\ p\neq q}} \hat{Z}_{[(p,q),(\rho(p),\rho(q))]})]$$

$$= \frac{1}{n(n-1)}\sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}\sum_{\substack{r,s\\ r\neq s}} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]})$$

$$= \frac{1}{n(n-1)}\sum_{(j,k,l,m,p,q,r,s)\in A} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \tag{2.11}$$

where $A = \bigcup_{i=1}^{7} A_i$ and

$$A_1 = \{(j,k,l,m,p,q,r,s) \in \Lambda | r = l, s = m\},$$

$$A_2 = \{(j,k,l,m,p,q,r,s) \in \Lambda | r = m, s = l\},$$

$$A_3 = \{(j,k,l,m,p,q,r,s) \in \Lambda | r = l, s \neq l, s \neq m\},$$

$$A_4 = \{(j,k,l,m,p,q,r,s) \in \Lambda | r = m, s \neq l, s \neq m\},$$

$$A_5 = \{(j,k,l,m,p,q,r,s) \in \Lambda | r \neq l, r \neq m, s = l\},$$

$$A_6 = \{(j,k,l,m,p,q,r,s) \in \Lambda | r \neq l, r \neq m, s = m\},$$

$$A_7 = \{(j,k,l,m,p,q,r,s) \in \Lambda | r \neq l, m, s \neq l, m\}.$$

Firstly, we consider the sum on A_1. By the facts that

$$\sum_{\substack{j,k \\ j\neq k}} \sum_{\substack{l,m \\ l\neq m}} \sum_{\substack{p,q \\ p\neq q}} E[Z_{[(j,k),(l,m)]} Z_{[(p,q),(l,m)]}]$$

$$\leq \delta \sum_{\substack{j,k \\ j\neq k}} \sum_{\substack{l,m \\ l\neq m}} \sum_{\substack{p,q \\ p\neq q}} E(|\hat{X}_{jl} + \hat{X}_{km} - \hat{X}_{jm} - \hat{X}_{kl}||\hat{X}_{pl} + \hat{X}_{qm} - \hat{X}_{pm} - \hat{X}_{ql}|^2)$$

$$\leq \delta \sum_{\substack{j,k \\ j\neq k}} \sum_{\substack{l,m \\ l\neq m}} \sum_{\substack{p,q \\ p\neq q}} (E|\hat{X}_{jl} + \hat{X}_{km} - \hat{X}_{jm} - \hat{X}_{kl}|^3)^{\frac{1}{3}}$$

$$\times (E|\hat{X}_{pl} + \hat{X}_{qm} - \hat{X}_{pm} - \hat{X}_{ql}|^3)^{\frac{2}{3}}$$

$$\leq 16\delta \sum_{\substack{j,k \\ j\neq k}} \sum_{\substack{l,m \\ l\neq m}} \sum_{\substack{p,q \\ p\neq q}} (E|\hat{X}_{jl}|^3 + E|\hat{X}_{km}|^3 + E|\hat{X}_{jm}|^3 + E|\hat{X}_{kl}|^3)^{\frac{1}{3}}$$

$$\times (E|\hat{X}_{pl}|^3 + E|\hat{X}_{qm}|^3 + E|\hat{X}_{pm}|^3 + E|\hat{X}_{ql}|^3)^{\frac{2}{3}}$$

$$
\begin{aligned}
&\leq \frac{16\delta}{3} \sum_{\substack{j,k \\ j\neq k}} \sum_{\substack{l,m \\ l\neq m}} \sum_{\substack{p,q \\ p\neq q}} (E|\hat{X}_{jl}|^3 + |\hat{X}_{km}|^3 + |\hat{X}_{jm}|^3 + |\hat{X}_{kl}|^3) \\
&\quad + \frac{32\delta}{3} \sum_{\substack{j,k \\ j\neq k}} \sum_{\substack{l,m \\ l\neq m}} \sum_{\substack{p,q \\ p\neq q}} (E|\hat{X}_{pl}|^3 + |\hat{X}_{qm}|^3 + |\hat{X}_{pm}|^3 + |\hat{X}_{ql}|^3) \\
&= 16n(n-1)\delta \sum_{\substack{j,k \\ j\neq k}} \sum_{\substack{l,m \\ l\neq m}} (E|\hat{X}_{jl}|^3 + |\hat{X}_{km}|^3 + |\hat{X}_{jm}|^3 + |\hat{X}_{kl}|^3) \\
&\leq 64n^2(n-1)^2\delta \sum_{i,j} E|\hat{X}_{ij}|^3 \\
&= 640n^3(n-1)^2\beta^2 \\
&\leq 1.829n^3(n-1)^2\beta
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big[\sum_{\substack{j,k \\ j\neq k}} EZ_{[(j,k),(\rho(j),\rho(k))]}\Big]^2 \\
&\leq \delta^2 \Big[\sum_{\substack{j,k \\ j\neq k}} E|\hat{X}_{j\rho(j)} + \hat{X}_{k\rho(k)} - \hat{X}_{j\rho(k)} - \hat{X}_{k\rho(j)}|\Big]^2 \\
&\leq \delta^2 [2(n-1) \sum_j E|\hat{X}_{j\rho(j)}| + 2\sum_{\substack{j,k \\ j\neq k}} E|\hat{X}_{j\rho(k)}|\Big]^2 \\
&\leq 4\delta^2 \Big[n \sum_j E|\hat{X}_{j\rho(j)}| + \sum_{j,k} E|\hat{X}_{j\rho(k)}|\Big]^2 \\
&\leq 8\delta^2 \Big[n^3 \sum_j E^2|\hat{X}_{j\rho(j)}| + n^2 \sum_{j,k} E^2|\hat{X}_{j\rho(j)}|\Big]
\end{aligned}
$$

$$\leq 8\delta^2\Big[n^3\sum_{j} E\hat{X}^2_{j\rho(j)} + n^2\sum_{j,k} E\hat{X}^2_{j\rho(k)}\Big]$$

$$= 8\delta^2\Big[n^3(1-\frac{1}{n}(\frac{d^2}{d^2+\sigma^2})) + n^2(n-\frac{d^2}{d^2+\sigma^2})\Big] \text{ (by (2.6))}$$

$$\leq 16n^3\delta^2$$

$$\leq 4.572n^3\beta$$

$$\leq 0.005n^3(n-1)^2\beta,$$

we have

$$\sum_{A_1} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]})$$

$$= \sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}} E[(Z_{[(j,k),(l,m)]} - EZ_{[(j,k),(\rho(j),\rho(k))]})(Z_{[(p,q),(l,m)]} - EZ_{[(p,q),(\rho(p),\rho(q))]})]$$

$$\leq \sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}} E[Z_{[(j,k),(l,m)]}Z_{[(p,q),(l,m)]}]$$

$$+ \sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}(EZ_{[(j,k),(\rho(j),\rho(k))]})(EZ_{[(p,q),(\rho(p),\rho(q))]}).$$

$$\leq 1.834n^3(n-1)^2\beta. \tag{2.12}$$

We can use the same technique of (2.12) to find the sums on $A_2 - A_6$ and get the bounds as follow:

$$\sum_{A_2} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \leq 1.834n^3(n-1)^2\beta, \tag{2.13}$$

$$\sum_{A_3} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \leq 1.834n^4(n-1)^2\beta, \tag{2.14}$$

$$\sum_{A_4} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \leq 1.834n^4(n-1)^2\beta, \tag{2.15}$$

$$\sum_{A_5} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \leq 1.834n^4(n-1)^2\beta, \tag{2.16}$$

$$\sum_{A_6} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) \leq 1.834n^4(n-1)^2\beta. \tag{2.17}$$

Next we find the sum on A_7. Note that

$$\begin{aligned}
\sum_{A_7} E(\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]}) &= \sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}}\sum_{\substack{p,q\\ p\neq q}}\sum_{\substack{r,s\\ r\neq s\\ r\neq l,m\\ s\neq l,m}} E\hat{Z}_{[(j,k),(l,m)]}\hat{Z}_{[(p,q),(r,s)]} \\
&= \sum_{\substack{j,k\\ j\neq k}}\sum_{\substack{l,m\\ l\neq m}} E\hat{Z}_{[(j,k),(l,m)]}\sum_{\substack{p,q\\ p\neq q}}\sum_{\substack{r,s\\ r\neq s}} E\hat{Z}_{[(p,q),(r,s)]} \\
&\quad - \sum_{\substack{A_i\\ i=1,2,\ldots,6}} E\hat{Z}_{[(j,k),(l,m)]}E\hat{Z}_{[(p,q),(r,s)]} \\
&= - \sum_{\substack{A_i\\ i=1,2,\ldots,6}} E\hat{Z}_{[(j,k),(l,m)]}E\hat{Z}_{[(p,q),(r,s)]}.
\end{aligned} \tag{2.18}$$

Using the same arguments as in (2.12)-(2.17) we have

$$\sum_{A_1} E(\hat{Z}_{[(j,k),(l,m)]}E\hat{Z}_{[(p,q),(r,s)]}) \geq -9.144n^4(n-1)\beta, \tag{2.19}$$

$$\sum_{A_2} E(\hat{Z}_{[(j,k),(l,m)]} E\hat{Z}_{[(p,q),(r,s)]}) \geq -9.144n^4(n-1)\beta, \tag{2.20}$$

$$\sum_{A_3} E(\hat{Z}_{[(j,k),(l,m)]} E\hat{Z}_{[(p,q),(r,s)]}) \geq -9.144n^5(n-1)\beta, \tag{2.21}$$

$$\sum_{A_4} E(\hat{Z}_{[(j,k),(l,m)]} E\hat{Z}_{[(p,q),(r,s)]}) \geq -9.144n^5(n-1)\beta, \tag{2.22}$$

$$\sum_{A_4} E(\hat{Z}_{[(j,k),(l,m)]} E\hat{Z}_{[(p,q),(r,s)]}) \geq -9.144n^5(n-1)\beta, \tag{2.23}$$

$$\sum_{A_6} E(\hat{Z}_{[(j,k),(l,m)]} E\hat{Z}_{[(p,q),(r,s)]}) \geq -9.144n^5(n-1)\beta. \tag{2.24}$$

Hence, by (2.11)-(2.24) and the fact that $n \geq 32$,

$$\begin{aligned} E[TZ(\rho)] &\leq 3.668n^2(n-1)\beta + 7.336n^3(n-1)\beta + 18.288n^3\beta + 36.576n^4\beta \\ &\leq 45.825n^3(n-1)\beta. \end{aligned} \tag{2.25}$$

So, by (2.10), (2.25) and lemma 2.5(2) we see that

$$Var \sum_{\substack{j,k \\ i \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} \leq 45.825n^2\beta + 2.286n\beta \leq 45.897n^2\beta. \qquad \square$$

Proposition 2.7(Concentration Inequality)

If $\beta < \dfrac{1}{350}$ and $n \geq 32$, then for $a < b$ we have

$$P(a \leq S(\rho) \leq b) \leq 2(b-a) + 78\beta.$$

Proof.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} -\frac{1}{2}(b-a) - \delta & if \quad t < a - \delta \\ t - \frac{1}{2}(b+a) & if \quad a - \delta \le t \le b + \delta \\ \frac{1}{2}(b-a) + \delta & if \quad t > b + \delta, \end{cases}$$

and

$$M(t) = (\frac{n-1}{4})(\tilde{S}(\rho) - S(\rho))[I(0 < t \le \tilde{S}(\rho) - S(\rho)) - I(\tilde{S}(\rho) - S(\rho) \le t < 0)]$$

where I is an indicator function.

Let $F(w, \tilde{w}) = (\tilde{w} - w)(f(\tilde{w}) + f(w))$. By the same tecnique of (2.9) we have

$$0 = -\frac{4}{n-1}E[S(\rho)f(S(\rho))] + E[\tilde{S}(\rho) - S(\rho)][f(\tilde{S}(\rho)) - f(S(\rho))] + \frac{4r}{(n-1)}.$$

where $r = \frac{1}{n}E[f(S(\rho))\sum_{i,j}\hat{X}_{ij}]$.

Then

$$\begin{aligned} & E[S(\rho)f(S(\rho))] \\ & = (\frac{n-1}{4})E[\tilde{S}(\rho) - S(\rho)][f(\tilde{S}(\rho)) - f(S(\rho))] + r \\ & = (\frac{n-1}{4})E[\tilde{S}(\rho) - S(\rho)][\int_0^{\tilde{S}(\rho)-S(\rho)} f'(S(\rho)+t)dt] + r \\ & = (\frac{n-1}{4})E[\tilde{S}(\rho) - S(\rho)]\int_{\mathbb{R}} f'(S(\rho)+t)[I(0 < t \le \tilde{S}(\rho) - S(\rho)) \\ & \quad - I(\tilde{S}(\rho) - S(\rho) < t \le 0)]dt + r \\ & = E[\int_{\mathbb{R}} f'(S(\rho)+t)M(t)dt] + r \end{aligned}$$

$$\geq E[I(a \leq S(\rho) \leq b)\int_{|t|<\delta} M(t)dt] + r$$

$$\geq (\frac{n-1}{4})E[I(a \leq S(\rho) \leq b)|\tilde{S}(\rho) - S(\rho)| \min(\delta, |\tilde{S}(\rho) - S(\rho)|)] + r$$

$$= \frac{1}{4n}E[I(a \leq S(\rho) \leq b)\sum_{\substack{j,k \\ j \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]}] + r$$

$$\geq \frac{1}{4n}E[I(a \leq S(\rho) \leq b)\sum_{\substack{j,k \\ j \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} I(\sum_{\substack{j,k \\ j \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} > \frac{6n}{5})] + r$$

$$\geq \frac{3}{10}E[I(a \leq S(\rho) \leq b) I(\sum_{\substack{j,k \\ j \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} > \frac{6n}{5})] + r$$

$$= \frac{3}{10}E[I(a \leq S(\rho) \leq b) - I(a \leq S(\rho) \leq b, \sum_{\substack{j,k \\ j \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} \leq \frac{6n}{5})] + r$$

$$\geq \frac{3}{10}E[I(a \leq S(\rho) \leq b) - I(\sum_{\substack{j,k \\ j \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} \leq \frac{6n}{5})] + r$$

$$= \frac{3}{10}[P(a \leq S(\rho) \leq b) - P(\sum_{\substack{j,k \\ j \neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} \leq \frac{6n}{5})] + r. \tag{2.26}$$

By proposition 2.4, $E(\sum_{i,j} \hat{X}_{ij})^2 = nE(S(\rho)\sum_{i,j} \hat{X}_{ij}) = \frac{n\sigma^2}{\sigma^2 + d^2}$ which implies

$$|r| = \frac{1}{n}|E[f(S(\rho))\sum_{i,j} \hat{X}_{ij}| \leq \frac{1}{n}(\frac{1}{2}(b-a)+\delta)\sqrt{E(\sum_{i,j} \hat{X}_{ij})^2} \leq \frac{1}{\sqrt{n}}(\frac{1}{2}(b-a)+\delta). \tag{2.27}$$

Hence, by (2.27),

$$
\begin{aligned}
&P(a \le S(\rho) \le b) \\
&\le \frac{10}{3}E[S(\rho)f(S(\rho))] + P(\sum_{\substack{j,k \\ j\neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} \le \frac{6n}{5}) - \frac{10r}{3} \\
&\le \frac{10}{3}E[S(\rho)f(S(\rho))] + P(E\sum_{\substack{j,k \\ j\neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} - \sum_{\substack{j,k \\ j\neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} \ge 1.1n) - \frac{10r}{3} \\
&\le \frac{10}{3}(\frac{1}{2}(b-a)+\delta) + \frac{1}{(1.1n)^2} Var\sum_{\substack{j,k \\ j\neq k}} Z_{[(j,k),(\rho(j),\rho(k))]} - \frac{10r}{3} \\
&\le 2(b-a) + 78\beta.
\end{aligned}
$$

where we have used lemma 2.5(3) in the second inequality and lemma 2.6 and (2.28) in the last inequality. □

3 Proof of the Main Result.

In this section, we will prove our main result by using Stein's method. In 1972, Stein gave a new method which is free from Fourier transform to find an error bound in normal approximation. His method based on the differential equation

$$f'(w) - wf(w) = h(w) - E(h(Z)) \tag{3.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function , $h : \mathbb{R} \to \mathbb{R}$ is a test function and Z is the standard normal random variable. We always call (3.1) Stein's equation for normal approximation. For any real number z, let h_z be an indicator function defined by

$$h_z(x) = \begin{cases} 1 & \text{if } x \le z, \\ \\ 0 & \text{if } x > z, \end{cases} \tag{3.2}$$

then Stein's equation (3.1) has the unique solution $f_z : \mathbb{R} \to \mathbb{R}$ defined by

$$f_z(w) = \begin{cases} \sqrt{2\pi} e^{\frac{1}{2}w^2} \Phi(w)[1 - \Phi(z)] & \text{if } w \le z, \\ \\ \sqrt{2\pi} e^{\frac{1}{2}w^2} \Phi(z)[1 - \Phi(w)] & \text{if } w > z. \end{cases} \tag{3.3}$$

By (3.1)-(3.3) we have $f_z'(W) - Wf_z(W) = h_z(W) - \Phi(z)$ which implies that

$$P(W \le z) - \Phi(z) = Ef_z'(W) - EWf_z(W). \tag{3.4}$$

So, we can find a bound of $Ef_z'(W) - EWf_z(W)$ instead of $P(W \le z) - \Phi(z)$. To do this, we also need the some propeties of the solution f_z of the Stein's equation (3.1) as follows. For $s, t, w \in \mathbb{R}$

$$f_z'(w+s) - f_z'(w+t) \le \begin{cases} 1 & \text{if } w+s < z, w+t > z, \\ \\ (|w| + \frac{\sqrt{2\pi}}{4})(|s| + |t|) & \text{if } s \ge t, \\ \\ 0 & \text{ortherwise}, \end{cases} \tag{3.5}$$

$$f_z'(w+s) - f_z'(w+t) \geq \begin{cases} -1 & \text{if } w+s > z, w+t < z, \\ -(|w| + \frac{\sqrt{2\pi}}{4})(|s|+|t|) & \text{if } s < t, \\ 0 & \text{ortherwise,} \end{cases} \tag{3.6}$$

(Chen and Shao (2001) p. 246-247)

$$|f_z'(w)| \leq 1, \tag{3.7}$$

(Stein(1986) p. 23)

$$|Ef_z'(S(\tau))E\int_{\mathbb{R}} M(t)dt - Ef_z'(S(\tau))\int_{\mathbb{R}} M(t)dt| < \frac{16}{n}, \tag{3.8}$$

and

$$|Ef_z(S(\rho))\sum_{i,j}\hat{X}_{ij}| \leq \frac{1}{n} \tag{3.9}$$

(Ho and Chen(1978) p.243-245).

From Chen and Shao (2001) p.246, we know that $\sup_{x\in\mathbb{R}}|P(W \leq x) - \Phi(x)| \leq 0.55$ for any random variable W such that $EW = 0$ and $VarW = 1$. Hence, it suffices to prove the theorem in case of $\beta \leq \frac{1}{350}$ and $n \geq 32$. Using the same argument as in proving (2.26) we can show that

$$E[S(\rho)f_z(S(\rho))] = E\int_{\mathbb{R}} f_z'(S(\rho)+t)M(t)dt + \frac{1}{n}E[f_z(S(\rho))\sum_{i,j}\hat{X}_{ij}] \tag{3.10}$$

and

$$1 = E[S^2(\rho)] = E\int_{\mathbf{R}} M(t)dt + \frac{1}{n}E[S(\rho)\sum_{i,j}\hat{X}_{ij}]. \tag{3.11}$$

Let $\Delta S = S(\rho) - S(\tau)$. Then

$$
\begin{aligned}
&P(W \le z) - \Phi(z)\\
&= Ef_z'(W) - EWf_z(W)\\
&= Ef_z'(S(\tau)) - E(S(\rho)f_z(S(\rho)))\\
&= Ef_z'(S(\tau))E\int_{\mathbf{R}} M(t)dt + \frac{1}{n}E[S(\rho)\sum_{i,j}\hat{X}_{ij}]Ef_z'(S(\tau)) - E[S(\rho)f_z(S(\rho))] \quad \text{(by (3.11))}\\
&\le E\int_{\mathbf{R}} f_z'(S(\tau))M(t)dt + \frac{17}{n} - E\int_{\mathbf{R}} f_z'(S(\rho)+t)M(t)dt\\
&\quad + \frac{1}{n}E[f_z(S(\rho))\sum_{i,j}\hat{X}_{ij}] \qquad \text{(by (3.8) and (3.10))}\\
&\le E\int_{\mathbf{R}}[f_z'(S(\tau)) - f_z'(S(\rho)+t)]M(t)dt + \frac{18}{n} \quad \text{(by (3.9))}\\
&= E\int_{\mathbf{R}}[f_z'(S(\tau)) - f_z'(S(\tau)+\Delta S+t)]M(t)dt + \frac{18}{n}\\
&\le E\int_{\substack{S(\tau)\le z\\ S(\tau)+\Delta S+t>z}} M(t)dt\\
&\quad + E\int_{\Delta S+t\le 0}(|S(\tau)| + \frac{\sqrt{2\pi}}{4})(|\Delta S| + |t|)M(t)dt + \frac{18}{n} \qquad \text{(by (3.5))}\\
&= E\int_{\Delta S+t>0} P(z-\Delta S-t \le S(\tau) \le z|\Delta S)M(t)dt\\
&\quad + E\int_{\Delta S+t\le 0}(|S(\tau)| + \frac{\sqrt{2\pi}}{4}(|\Delta S| + |t|)M(t)dt + \frac{18}{n}
\end{aligned}
$$

$$
\begin{aligned}
&= E\int_{\Delta S+t>0} P(z-t\le S(\rho)\le z+\Delta S|\Delta S)M(t)dt\\
&\quad + E\int_{\Delta S+t\le 0}(|S(\tau)|+\frac{\sqrt{2\pi}}{4})(|\Delta S|+|t|)M(t)dt+\frac{18}{n}\\
&\le 2E\int_{\mathbb{R}}(|\Delta S|+|t|)M(t)dt+78\beta E\int_{\Delta S+t>0}M(t)dt\\
&\quad + E\int_{\mathbb{R}}|S(\tau)|(|\Delta S|+|t|)M(t)dt+\frac{\sqrt{2\pi}}{4}E\int_{\mathbb{R}}(|\Delta S|+|t|)M(t)dt\\
&\quad + \frac{18}{n} \qquad \text{(by proposition 2.7)}\\
&\le 2E\int_{\mathbb{R}}|\Delta S|M(t)dt+2E\int_{\mathbb{R}}|t|M(t)dt+78\beta\\
&\quad + E\int_{\mathbb{R}}|S(\tau)||\Delta S|M(t)dt+E\int_{\mathbb{R}}|S(\tau)||t|M(t)dt+\frac{18}{n}. \qquad (3.12)
\end{aligned}
$$

Note that

$$E\int_{\mathbb{R}}|t|M(t)dt=(\frac{n-1}{8})E|\tilde{S}(\rho)-S(\rho)|^3\le(\frac{n-1}{8})(\frac{64\beta}{n})\le 8\beta, \qquad (3.13)$$

$$E\int_{\mathbb{R}}|S(\tau)||t|M(t)dt \;=\; E|S(\tau)|E\int_{\mathbb{R}}|t|M(t)dt \;\le\; 8\beta, \qquad (3.14)$$

and by the fact that

$$
\begin{aligned}
E|\Delta S|^3 &= E|\hat{X}_{\tau^{-1}(L)\tau(L)}+\hat{X}_{\tau^{-1}(M)\tau(K)}+\hat{X}_{IL}+\hat{X}_{KM}\\
&\qquad -\hat{X}_{I\tau(I)}-\hat{X}_{K\tau(K)}-\hat{X}_{\tau^{-1}(L)L}-\hat{X}_{\tau^{-1}(M)M}|^3\\
&\le \frac{512\beta}{n},
\end{aligned}
$$

Stochastic Modelling and Applications

An Error Estimation of Convergence of Random Sums with Finite Variances

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Abstract

Let (X_{nk}) be a double sequence of random variables with finite variances and (Z_n) a sequence of positive integral-valued random variables such that for each n, Z_n, $X_{n1}, X_{n2}, \ldots$ are independent. In this paper, we give an error estimation of the convergence of the distribution functions of random sums $X_{n1} + X_{n2} + \cdots + X_{nZ_n}$ and show that it tends to zero when the distribution function of random sums approaches a limit.

Keywords : random sums, infinitely divisible distribution function, Kolmogorov's formula, weakly convergence.

Mathematics Subject Classification(2000) : Primary 60F05; Secondary 60G50.

1. Introduction and main results

Let (X_n) be a sequence of independent random variables with $Var(X_n) < \infty$ and Z be a positive integral-valued random variable such that $Z, X_1, X_2, \ldots$ are independent. Many authors (for examples, Blum, Hanson and Rosenblatt (1962), Mogyoro di (1967), Orazov (1972), Pecinkin (1973), Kruglov(1976), Landers and Rogge(1976), Batirov, Manavic and Nagaev(1977), Krajka and Rychilk(1993) and Neammanee(1993)) found conditions under which the sequence of the distribution functions of random sums $X_1 + X_2 + \cdots + X_Z$ converges to the standard normal distribution function Φ. The error estimation of the convergence was then investigated by Nakas (1972), Korolev(1987), Korolev and Selivanovan(1991) and Neammanee(1992).

Azlarov and Dzamirzaev(1972), Szasz(1972), Belov and Recinkin(1979), Kubacki and Szynal(1985), Bethmann(1988), Finkelstein, Tucker and Veeh(1991), Neammanee(1993,2000) considered a more general case of double arrays.

Let (X_{nk}), $k=1,2,\ldots,n=1,2,\ldots$, be a double sequence of random variables with finite variances. Throughout this paper, we let $F_{nk}, \varphi_{nk}, \mu_{nk}$ and σ_{nk}^2 be the distribution function, characteristic function, mean, and variance of X_{nk} respectively. We also let (Z_n) be a sequence of positive integral-valued random variables such that for each $n, Z_n, X_{n1}, X_{n2},\ldots$ are independent.

Let

$$S_{Z_n} = X_{n1} + X_{n2} + \cdots + X_{nZ_n}$$

and F_n be the distribution function of S_{Z_n}.

For each $n \in \mathbf{N}$, we let $K_{nl_n(q)} : \mathbf{R} \to [0,\infty)$ be given by

$$K_{nl_n(q)}(x) = \sum_{k=1}^{l_n(q)} \int_{-\infty}^{x} u^2 dF_{nk}(u + \mu_{nk})$$

where the *q-quantile* $l_n : (0,1) \to \mathbf{N}$ is defined as

$$l_n(q) = \max\{k \in \mathbf{N} \mid P(Z_n < k) \le q\}.$$

Neammanee and Chaidee (2003) gave necessary and sufficient conditions of the convergence of random sums. The theorem states as follows:

Theorem 1.1 Assume that

(α) $(X_{nk} - \mu_{nk})$ is random infinitesimal with respect to (Z_n), i.e., for every $\varepsilon > 0$,

$$\max_{1 \le k \le Z_n} P(|X_{nk} - \mu_{nk}| \ge \varepsilon) \xrightarrow{P} 0 \text{ as } n \to \infty,$$

(β) there exists a constant $C > 0$ such that $\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 < C$ for a.e. $q \in (0,1)$,

(γ) for a.e. $q \in (0,1)$, $(K_{nl_n(q)})$ and $(\sum_{k=1}^{l_n(q)} \mu_{nk})$ are monotone.

Then

1. $F_n \xrightarrow{w} F$,
2. the sequence of the variances of S_{Z_n} converges to the variance σ^2 of F, i.e.,

$$E(\sum_{k=1}^{Z_n} \sigma_{nk}^2) \to \sigma^2$$

if and only if for a.e. $q \in (0,1)$ there exist a function $K^{(q)}$ in m and a constant μ_q such that

1^* $K_{nl_n(q)}(x) \to K^{(q)}(x)$ at all continuity point x of $K^{(q)}$,

2^* $K_{nl_n(q)}(\infty) \to K^{(q)}(\infty)$ and $\sigma^2 = \int_0^1 K^{(q)}(\infty) dq$,

An Error Estimation of Convergence of Random Sums with Finite Variances

$3^{\bullet}$ $\sum_{k=1}^{l_n(q)} \mu_{nk} \to \mu_q$

where m is the set of bounded, non-decreasing, right-continuous functions from $\mathbf{R}$ into $[0,\infty)$ that vanish at $-\alpha$.

Furthermore, we know that $F(x)=\int_0^1 F^{(q)}(x)dq$ is the distribution function whose characteristic function $\varphi^{(q)}$ is given by

$$\ln \varphi^{(q)}(t)=i\mu_q t+\int_{\mathbf{R}} f(t,x)dK^{(q)}(x)$$

and the function $f:\mathbf{R}\to\mathbf{C}$ is defined by

$$f(t,x)=\begin{cases}\dfrac{1}{x^2}(e^{itx}-1-itx), & x\neq 0\\ -\dfrac{1}{2}t^2, & x=0.\end{cases} \tag{1.1}$$

Let

$$M_n=\sup_{-\infty<x<\infty}\left|F_n(x)-F(x)\right|.$$

In 2004, Neammanee and Rattanawong gave a bound of M_n when $F=\Phi$. In this paper, we obtain bound of M_n for any F and show that whenever (F_n) converges to F the bound of M_n tends to zero as $n\to\infty$.

To state the main theorems, we need the following construction.

For each $q\in(0,1)$, we let $F^{(q)}$ be an infinitely divisible distribution function with mean, variance, and characteristic function being μ_q, σ_q^2, and $\varphi^{(q)}$ respectively. According to Kolmogorov's formula we have

$$\ln \varphi^{(q)}(t)=i\mu_q t+\int_{\mathbf{R}} f(t,x)dK^{(q)}(x)$$

and $K^{(q)}(\infty)=\sigma_q^2$.

For positive number A_q and δ_q such that $0<\delta_q\leq 2A_q$ and $\pm A_q$ are continue points of $K^{(q)}$, we define $g(n,A_q,\delta_q)$ as follows:

$$g(n,A_q,\delta_q)=g_1(n,A_q,\delta_q)+g_2(n,A_q,\delta_q)+g_3(n,A_q,\delta_q)+g_4(n,A_q,\delta_q) \text{ where}$$

$$g_1(n,A_q,\delta_q)=[\frac{1}{3}\max_{1\leq k\leq l_n(q)}\sigma_{nk}^2\sum_{k=1}^{l_n(q)}\sigma_{nk}^2]^{\frac{1}{5}},$$

$$g_2(n,A_q,\delta_q)=[\frac{5}{6}\delta_q(\sigma_q^2+\sum_{k=1}^{l_n(q)}\sigma_{nk}^2]^{\frac{1}{4}},$$

$$g_3(n,A_q,\delta_q)=[\frac{1}{2}|\ K_{nl_n(q)}(A_q)-K_{nl_n(q)}(-A_q)-K^{(q)}(A_q)+K^{(q)}(-A_q)\ |]^{\frac{1}{3}},\text{ and}$$

$$g_4(n,A_q,\delta_q)=[\ 2|\sum_{k=1}^{l_n(q)}\mu_{nk}-\mu_q\ |+\frac{4}{A_q}(K_{nl_n(q)}(\infty)-K_{nl_n(q)}(A_q)$$

$$+K^{(q)}(\infty)-K^{(q)}(A_q)+K_{nl_n(q)}(-A_q)+K^{(q)}(-A_q))\ |]^{\frac{1}{2}}.$$

Furthermore, we also let

$$S_n^{(q)}=X_{n1}+X_{n2}+\cdots+X_{nl_n(q)}$$

and let $F_n^{(q)},\varphi_n^{(q)}$ be the distribution function and the characteristic function of $S_n^{(q)}$, respectively. The following are our main results.

Theorem 1.2 Let F be a distribution function such that $F(x)=\int_0^1 F^{(q)}(x)dq$ where $F^{(q)}$ is the infinitely divisible distribution function with finite variance. Suppose that

1. for a.e. $q\in(0,1),\frac{d}{dx}F^{(q)}(x)$ exists and $|\ \frac{d}{dx}F^{(q)}(x)\ |\le B<\infty$ for all x,
2. $0<\sigma_{nk}^2\le 1$ for all $n,k\in\mathrm{N}$.

Then there exists a constant C such that

$$\sup_{-\infty<x<\infty}|\ F_n(x)-F(x)\ |\le C\int_0^1 g(n,A_q,\delta_q)dq.$$

Theorem 1.3 Let F be a distribution function such that $F(x)=\int_0^1 F^{(q)}(x)dq$ where $F^{(q)}$ is the infinitely divisible distribution function. Suppose that

1. for every $n\in\mathrm{N}$ and for a.e. $q\in(0,1)$, $F^{(q)}$ and $F_n^{(q)}$ are continuous everywhere except possibly at $x=x_s(x_s<x_{s+1},s=0,\pm1,\pm2,\ldots)$,
2. there exists a positive constant L such that $\min(x_{s+1}-x_s)\ge L$ for $s=0,\pm1,\pm2,\ldots$,
3. for a.e. $q\in(0,1),\frac{d}{dx}F^{(q)}(x)$ exists everywhere except at $x=x_s$ and $|\ \frac{d}{dx}F^{(q)}(x)\ |\le B<\infty$ for all $x\ne x_s$, $s=0,\pm1,\pm2,\ldots$,
4. $0<\sigma_{nk}^2\le 1$ for all $n,k\in\mathrm{N}$.

Then there exists a constant C such that

$$\sup_{-\infty<x<\infty} | F_n(x) - F(x) | \le C \int_0^1 g(n, A_q, \delta_q) dq.$$

Theorem 1.4 Under the assumptions of Theorem 1.1 and Theorem 1.2 (or Theorem 1.1 and Theorem 1.3), for a.e. $q \in (0,1)$ and $n \in \mathrm{N}$, there exist positive numbers $\delta_{n,q}, A_{n,q}$ and constant C such that

$$\sup_{-\infty<x<\infty} | F_n(x) - F(x) | \le C \int_0^1 g(n, A_{n,q}, \delta_{n,q}) dq$$

and

$$\int_0^1 g(n, A_{n,q}, \delta_{n,q}) dq \to 0 \text{ as } n \to \infty.$$

Note that the normal and Poisson distribution functions are examples of F in Theorem 1.2 and Theorem 1.3 respectively.

2. Proof of the Main Theorems

To prove the main theorems, we need the following lemma and proposition.

Lemma 2.1 (Bethmann (1988) p.336) For each n, let $(a_{nk})_{k=1}^{\infty}$ be a non-decreasing sequence of nonnegative numbers and $a \ge 0$. Then $a_{nZ_n} \xrightarrow{P} a$ if and only if $a_{nl_n(q)} \xrightarrow{P} a$ for all $q \in (0,1)$.

Proposition 2.1 Under the conditions $(\alpha), (\beta), (\gamma)$ and the necessary conditions of Theorem 1.1, for every $q \in (0,1)$, we have

1. $\max_{1\le k\le l_n(q)} \sigma_{nk}^2 \to 0$ as $n \to \infty$,
2. $(\sum_{k=1}^{l_n(q)} \sigma_{nk}^2)$ is bounded.

Proof. Fix $q \in (0,1)$.

1. From Theorem 1.1, there exists a bouded, non-decreasing, right-continuous function $K^{(q)}$ such that $K^{(q)}(-\infty) = 0$ and $K_{nl_n(q)} \xrightarrow{w} K^{(q)}$. Let $\varepsilon > 0$ and $c_q > 0$ be such that $\pm c_q$ are continuity points of $K^{(q)}$ and

$$| K^{(q)}(c_q) - K^{(q)}(\infty) | \le \varepsilon, \; K^{(q)}(-c_q) \le \varepsilon . \tag{2.1}$$

Let $n_0 \in \mathbf{N}$ be such that, for $n > n_0$,

$$\left| K_{nl_n(q)}(-c_q) - K^{(q)}(-c_q) \right| < \varepsilon, \left| K_{nl_n(q)}(c_q) - K^{(q)}(c_q) \right| < \varepsilon \text{ , and}$$

$$\left| K_{nl_n(q)}(\infty) - K^{(q)}(\infty) \right| < \varepsilon. \tag{2.2}$$

Hence for $n > n_0$, we have

$$\begin{aligned}
&\sum_{k=1}^{l_n(q)} \int_{|u|\ge c_q} u^2 dF_{nk}(u+\mu_{nk}) \\
&= K_{nl_n(q)}(-c_q) + K_{nl_n(q)}(\infty) - K^{(q)}_{nl_n(q)}(c_q) \\
&\le \left| K_{nl_n(q)}(-c_q) - K^{(q)}(-c_q) \right| + K^{(q)}(-c_q) + \left| K_{nl_n(q)}(\infty) - K^{(q)}(\infty) \right| \\
&\quad + \left| K^{(q)}(\infty) - K^{(q)}(c_q) \right| + \left| K^{(q)}(c_q) - K_{nl_n(q)}(c_q) \right| \\
&\le 5\varepsilon,
\end{aligned} \tag{2.3}$$

which implies that

$$\begin{aligned}
&\max_{1\le k\le l_n(q)} \sigma^2_{nk} \\
&= \max_{1\le k\le l_n(q)} \Big[\int_{|u|\le\sqrt{\varepsilon}} u^2 dF_{nk}(u+\mu_{nk}) + \int_{\sqrt{\varepsilon}<|u|<c_q} u^2 dF_{nk}(u+\mu_{nk}) + \int_{|u|\ge c_q} u^2 dF_{nk}(u+\mu_{nk}) \Big] \\
&\le \varepsilon + c_q^2 \max_{1\le k\le l_n(q)} P(\left| X_{nk} - \mu_{nk} \right| > \sqrt{\varepsilon}) + \sum_{k=1}^{l_n(q)} \int_{|u|\ge c_q} u^2 dF_{nk}(u+\mu_{nk}) \\
&\le 6\varepsilon + c_q^2 \max_{1\le k\le l_n(q)} P(\left| X_{nk} - \mu_{nk} \right| > \sqrt{\varepsilon}).
\end{aligned} \tag{2.4}$$

By (α) and Lemma 2.1, we know that

$$\max_{1\le k\le l_n(q)} P(\left| X_{nk} - \mu_{nk} \right| > \sqrt{\varepsilon}) \to 0 \text{ as } n \to \infty.$$

Hence $\max\limits_{1\le k\le l_n(q)} \sigma^2_{nk} \to 0$ as $n \to \infty$.

2. The second statement follows from the second condition of Theorem 1.1 and the fact that

$$E\left(\sum_{k=1}^{Z_n} \sigma^2_{nk}\right) \ge \sum_{l=l_n(q)}^{\infty} P(Z_n = l) \sum_{k=1}^{l} \sigma^2_{nk}$$

$$\geq P(Z_n \geq l_n(q)) \sum_{k=1}^{l_n(q)} \sigma_{nk}^2$$

$$\geq (1-q) \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 ,$$

where we have used the fact that $P(Z_n < l_n(q)) \leq q$ in the last inequality.

2.1 Proof of Theorem 1.2

We shall first find a bound of $\left|\varphi_n^{(q)}(t) - \varphi^{(q)}(t)\right|$ and then apply Berry-Esseen Theorem (p.196 of Gnedenko and Kolmogorov(1954)) to obtain a bound of $\left|F_n^{(q)}(x) - F^{(q)}(x)\right|$.

Let $F_{nk}^*(x) = F_{nk}(x + \mu_{nk})$ and let φ_{nk}^* be its characteristic function. Let $a_{nk}(t) = \varphi_{nk}^*(t) - 1$. So

$$\begin{aligned}
|a_{nk}(t)| &= \left| \int_{\mathbf{R}} (e^{itx} - 1) dF_{nk}^*(x) \right| \\
&= \left| \int_{\mathbf{R}} (e^{itx} - 1 - itx) dF_{nk}(x + \mu_{nk}) \right| \\
&\leq \int_{\mathbf{R}} \left| \frac{1}{2} \theta t^2 x^2 \right| dF_{nk}(x + \mu_{nk}) \quad \text{for some } |\theta| \leq 1 \\
&\leq \frac{1}{2} t^2 \sigma_{nk}^2 ,
\end{aligned} \tag{2.5}$$

where we have used the fact that $e^{itx} = 1 + itx + \frac{1}{2}\theta t^2 x^2$ in the first inequality. For each $q \in (0,1)$ and $n \in \mathbf{N}$, let $T_q = \dfrac{1}{g(n, A_q, \delta_q)}$. From (2.5), we see that, for $|t| \leq T_q$ and $k = 1,2,3,\ldots,l_n(q)$,

$$\begin{aligned}
|a_{nk}(t)| &\leq \frac{\sigma_{nk}^2}{2g^2(n, A_q, \delta_q)} \\
&\leq \frac{\sigma_{nk}^2}{2g_1^2(n, A_q, \delta_q)} \\
&\leq \frac{4}{5} \frac{\max\limits_{1 \leq k \leq l_n(q)} \sigma_{nk}^2}{[(\max\limits_{1 \leq k \leq l_n(q)} \sigma_{nk}^2)^2]^{\frac{2}{5}}}
\end{aligned}$$

$$\leq \frac{4}{5}[\max_{1\leq k\leq l_n(q)} \sigma_{nk}^2]^{\frac{1}{5}}$$

$$\leq \frac{4}{5}.$$

Because of this and the fact that $|\ln(1+Z)-Z| \leq |Z|^2$ for $|Z| \leq 1$, we have

$$\begin{aligned}|\ln\varphi_{nk}^*(t) - a_{nk}(t)| &= |\ln(1+a_{nk}(t)) - a_{nk}(t)| \\ &\leq |a_{nk}(t)|^2. \end{aligned} \tag{2.6}$$

Note that for all $n \in \mathbf{N}$, $t \in \mathbf{R}$, and $q \in (0,1)$,

$$\begin{aligned}\int_{\mathbf{R}} f(t,x)dK_{nl_n(q)}(x) &= \sum_{k=1}^{l_n(q)} \int_{\mathbf{R}} (e^{itx} - 1 - itx)dF_{nk}(x+\mu_{nk}) \\ &= \sum_{k=1}^{l_n(q)} a_{nk}(t) \end{aligned} \tag{2.7}$$

where f defined in (1.1).

In order to obtain a bound of $|\ln\varphi_n^{(q)}(t) - \ln\varphi^{(q)}(t)|$, we will find a bound of $|\ln\varphi_n^{(q)}(t) - \psi_n^{(q)}(t)|$ and $|\psi_n^{(q)}(t) - \ln\varphi^{(q)}(t)|$, where

$$\psi_n^{(q)}(t) = it\sum_{k=1}^{l_n(q)} \mu_{nk} + \int_{\mathbf{R}} f(t,x)dK_{nl_n(q)}(x) = \sum_{k=1}^{l_n(q)} (it\mu_{nk} + a_{nk}(t)). \tag{2.8}$$

Observe that

$$\begin{aligned}\ln\varphi_n^{(q)}(t) &= \sum_{k=1}^{l_n(q)} \ln\varphi_{nk}(t) \\ &= it\sum_{k=1}^{l_n(q)} \mu_{nk} + \sum_{k=1}^{l_n(q)} (\ln\varphi_{nk}^*(t)), \end{aligned} \tag{2.9}$$

where the last equality comes from the fact that

$$\varphi_{nk}(t) = e^{it\mu_{nk}} \int_{\mathbf{R}} e^{it(x-\mu_{nk})}dF_{nk}(x) = e^{it\mu_{nk}}\varphi_{nk}^*(t).$$

Hence, from (2.8) and (2.9), we have

$$\begin{aligned}|\ln\varphi_n^{(q)}(t) - \psi_n^{(q)}(t)| &= \left|\sum_{k=1}^{l_n(q)} (\ln\varphi_{nk}^*(t) - a_{nk}(t))\right| \\ &\leq \sum_{k=1}^{l_n(q)} |\ln\varphi_{nk}^*(t) - a_{nk}(t)| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{l_n(q)} |a_{nk}(t)|^2 \quad \text{(by (2.6))} \\
&\leq \sum_{k=1}^{l_n(q)} \frac{(\sigma_{nk}^2)^2 t^4}{4} \quad \text{(by (2.5))} \\
&\leq \frac{1}{4} t^4 \max_{1\leq k\leq l_n(q)} \sigma_{nk}^2 \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 . \qquad (2.10)
\end{aligned}
$$

From Lemma 3 p.619 in Shapiro(1955) we know that

$$
\left| \int_{\mathbf{R}} f(t,x) d(K_{nl_n(q)}(x) - K^{(q)}(x)) \right| \leq D(t,n,A_q,\delta_q),
$$

where

$$
\begin{aligned}
D(t,n,A_q,\delta_q) \;=\; & \frac{5}{4}\delta_q |t^3| \left(\sum_{k=1}^{l_n(q)} \sigma_{nk}^2 + \sigma_q^2\right) \\
& + \frac{1}{2} t^2 \left| K_{nl_n(q)}(A_q) - K_{nl_n(q)}(-A_q) - K^{(q)}(A_q) + K^{(q)}(-A_q) \right| \\
& + \frac{2|t|}{A_q} (K_{nl_n(q)}(\infty) - K_{nl_n(q)}(A_q) + K^{(q)}(\infty) - K^{(q)}(A_q) \\
& + K_{nl_n(q)}(-A_q) + K^{(q)}(-A_q)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left| \psi_n^{(q)}(t) - \ln\varphi^{(q)}(t) \right| &= \sum_{k=1}^{l_n(q)} \mu_{nk} + \left| \int_{\mathbf{R}} f(t,x) dK_{nl_n(q)}(x) - i\mu_q t - \int_R f(t,x) dK^{(q)}(x) \right| \\
&\leq |t| \left| \sum_{k=1}^{l_n(q)} \mu_{nk} - \mu_q \right| + \left| \int_{\mathbf{R}} f(t,x) d(K_{nl_n(q)}(x) - K^{(q)}(x)) \right| \\
&\leq |t| \left| \sum_{k=1}^{l_n(q)} \mu_{nk} - \mu_q \right| + D(t,n,A_q,\delta_q). \qquad (2.11)
\end{aligned}
$$

So, by (2.10) and (2.11),

$$
\begin{aligned}
\left| \ln\varphi_n^{(q)}(t) - \ln\varphi^{(q)}(t) \right| &\leq |\ln\varphi_n^{(q)}(t) - \psi_n^{(q)}(t)| + |\psi_n^{(q)}(t) - \ln\varphi^{(q)}(t)| \\
&\leq \frac{1}{4} t^4 \max_{1\leq k\leq l_n(q)} \sigma_{nk}^2 \sum_{k=1}^{l_n(q)} \sigma_{nk}^2 + |t| \left| \sum_{k=1}^{l_n(q)} \mu_{nk} - \mu_q \right| + D(t,n,A_q,\delta_q)
\end{aligned}
$$

$$\equiv \ h(t,n,A_q,\delta_q).$$

From the fact that $|Z_1 - Z_2| \leq |\ln Z_1 - \ln Z_2|$ whenever $|Z_1| \leq 1$ and $|Z_2| \leq 1$, we have

$$\left|\varphi_n^{(q)}(t) - \varphi^{(q)}(t)\right| \leq h(t,n,A_q,\delta_q) \text{ for } |t| \leq T_q.$$

So

$$\begin{aligned}
\int_{-T_q}^{T_q} \left|\frac{\varphi_n^{(q)}(t) - \varphi^{(q)}(t)}{t}\right| dt &\leq \int_{-T_q}^{T_q} \frac{h(t,n,A_q,\delta_q)}{|t|} dt \\
&\leq 2\int_0^{T_q} \frac{h(t,n,A_q,\delta_q)}{|t|} dt \\
&\leq \frac{g_1^5(n,A_q,\delta_q)}{g^4(n,A_q,\delta_q)} + \frac{g_2^4(n,A_q,\delta_q)}{g^3(n,A_q,\delta_q)} + \frac{g_3^3(n,A_q,\delta_q)}{g^2(n,A_q,\delta_q)} + \frac{g_4^2(n,A_q,\delta_q)}{g(n,A_q,\delta_q)} \\
&\leq \frac{g_1^5(n,A_q,\delta_q)}{g_1^4(n,A_q,\delta_q)} + \frac{g_2^4(n,A_q,\delta_q)}{g_2^3(n,A_q,\delta_q)} + \frac{g_3^3(n,A_q,\delta_q)}{g_3^2(n,A_q,\delta_q)} + \frac{g_4^2(n,A_q,\delta_q)}{g_4(n,A_q,\delta_q)} \\
&= g(n,A_q,\delta_q).
\end{aligned}$$

From Theorem 1 p.196 of Gnedenko and Kolmogorov(1954), for every $q \in (0,1)$ and $a > 1$, we have

$$|F_n^{(q)}(x) - F^{(q)}(x)| \leq C(a,B)g(n,A_q,\delta_q),$$

where $C(a,B)$ is a constant depending on a and B. From Neammanee and Chaidee(2003), we see that $F_n(x) = \int_0^1 F_n^{(q)}(x)dq$ for every x. So

$$\begin{aligned}
\sup_{-\infty<x<\infty} |F_n(x) - F(x)| &\leq \sup_{-\infty<x<\infty} \int_0^1 \left|F_n^{(q)}(x) - F^{(q)}(x)\right| dq \\
&\leq C(a,B)\int_0^1 g(n,A_q,\delta_q)dq. \qquad \#
\end{aligned}$$

2.2 Proof of Theorem 1.3

The proof follows the same line of arguments as that of Theorem 1.2 except that the discontinuous version of Berry-Esseen Theorem (p.200 of Gnedenko and Kolmogorov(1954)) is used instead. #

2.3 Proof of Theorem 1.4

For each $q \in (0,1)$, let $0 < \delta_{q,n} < 1$ be such that $\pm\frac{1}{\sqrt{\delta_{n,q}}}$ are continuity points of $K^{(q)}$ and

$\delta_{q,n} \to 0$ as $n \to \infty$. Let $A_{n,q} = \dfrac{1}{\sqrt{\delta_{n,q}}}$. By Theorem 1.2 we have

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \le C(a,B) \int_0^1 g(n, A_{n,q}, \delta_{n,q})dq.$$

By 1* and 2* of Theorem 1.1, Proposition 2.1 and the fact that $\delta_{n,q} \to 0$ as $n \to \infty$, we see that

$$g(n, A_{n,q}, \delta_{n,q}) \to 0 \text{ as } n \to \infty$$

for every $q \in (0,1)$. Since $(\sum_{k=1}^{l_n(q)} \sigma_{nk}^2)$ and $K^{(q)}(\infty)$ are bounded for all $q \in (0,1)$, then $g(n, A_{n,q}, \delta_{n,q})$ is bounded. By Lebesgue dominated convergence theorem,

$$\int_0^1 g(n, A_{n,q}, \delta_{n,q})dq \to 0 \text{ as } n \to \infty. \qquad \#$$

3. Examples

The followings are some examples of the estimations when the limit distribution functions are standard normal and Poisson.

Example 3.1 Let Z_n and X_{nk}, $k = 1,2,\dots,n$ be defined by

$$P(Z_n = n^3) = \frac{1}{n^3} \text{ and } P(Z_n = n^3+1) = 1 - \frac{1}{n^3},$$

$$P(X_{nk} = \frac{-1}{\sqrt{n^3}}) = P(X_{nk} = \frac{1}{\sqrt{n^3}}) = \frac{1}{2}.$$

Assume that, for each $n, Z_n, X_{n1}, X_{n2}, \dots$ are independent. Then there exists a constant C

$$\sup_{-\infty<x<\infty} |F_n(x) - \Phi(x)| \le \frac{C}{\sqrt{n}}$$

where Φ is the standard normal distribution function.

Proof. Note that

$$l_n(q) = \begin{cases} n^3 & \text{if } 0 < q < \dfrac{1}{n^3}, \\ n^3+1 & \text{if } \dfrac{1}{n^3} \le q < 1, \end{cases}$$

$\mu_{nk} = 0, \sigma_{nk}^2 = \dfrac{1}{n^3}$ and, for large n,

$$K_{nl_n(q)}(u) = \begin{cases} 0 & \text{if } u < 0, \\ \dfrac{n^3+1}{n^3} & \text{if } u > 0. \end{cases}$$

From Lukacs(1960) page 93 we know that the function K in Kolmogorov's formula of Φ is defined by

$$K(u) = \begin{cases} 0 & \text{if } u < 0, \\ 1 & \text{if } u \geq 0. \end{cases}$$

Since $\Phi(x) = \int_0^1 \Phi(x)dq$, by Theorem 1.2 with $\delta_q = \frac{1}{n^2}$ we have that

$$\sup_{-\infty<x<\infty} |F_n(x) - \Phi(x)| \leq \frac{C}{\sqrt{n}}$$

for some constant C. #

Example 3.2 For $\lambda > 0$, let $Poi(\lambda)$ be a Poisson random variable with parameter λ, i.e., $P(Poi(\lambda) = \omega_0) = \frac{e^{-\lambda}\lambda^{\omega_0}}{\omega_0!}$. Fix $\lambda \in (0,1)$ and let Z_n and $X_{nk}, k = 1,2,\ldots,n$, be defined by

$$P(Z_n = n^3) = 1 - \frac{n^2}{2n^2+1} \text{ and } P(Z_n = 2n^3) = \frac{n^2}{2n^2+1},$$

$$P(X_{nk} = 1) = \frac{\lambda}{n^3} \text{ and } P(X_{nk} = 0) = 1 - \frac{\lambda}{n^3}.$$

Assume that, for each $n, Z_n, X_{n1}, X_{n2}, \ldots$ are independent. Then

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leq \frac{C}{\sqrt{n}}$$

where $F(x) = \int_0^1 F^{(q)}(x)dq$ and $F^{(q)}$ is the distribution function of $Poi(\lambda)$ for $0 < q \leq \frac{1}{2}$ and $Poi(2\lambda)$ for $\frac{1}{2} < q \leq 1$ and C is a constant.

Proof. Note that

$$l_n(q) = \begin{cases} n^3 & \text{if } 0 < q \leq 1 - \dfrac{n^2}{2n^2+1}, \\ 2n^3 & \text{if } 1 - \dfrac{n^2}{2n^2+1} \leq q < 1, \end{cases}$$

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$\mu_{nk} = \frac{\lambda}{n^3}, \sigma^2_{nk} = \frac{\lambda}{n^3}(1 - \frac{\lambda}{n^3})$, and for large n

$$K_{nl_n(q)}(u) = \begin{cases} \frac{\lambda}{n^3}\left(1-\frac{\lambda}{n^3}\right) & \text{if } u < 1 \text{ and } 0 < q \le 1 - \frac{n^2}{2n^2+1}, \\ \lambda\left(1-\frac{\lambda}{n^3}\right) & \text{if } u > 1 \text{ and } 0 < q \le 1 - \frac{n^2}{2n^2+1}, \\ \frac{2\lambda}{n^3}\left(1-\frac{\lambda}{n^3}\right) & \text{if } u < 1 \text{ and } 1 - \frac{n^2}{2n^2+1} < q < 1, \\ 2\lambda\left(1-\frac{\lambda}{n^3}\right) & \text{if } u > 1 \text{ and } 1 - \frac{n^2}{2n^2+1} < q < 1. \end{cases}$$

From Lukacs (1960) page 93 we know that the function K_λ and $K_{2\lambda}$ in Kolmogorov's formula of $Poi(\lambda)$ and $Poi(2\lambda)$ are defined by

$$K_\lambda(u) = \begin{cases} 0 & \text{if } u < 1, \\ \lambda & \text{if } u \ge 1, \end{cases}$$

and

$$K_{2\lambda}(u) = \begin{cases} 0 & \text{if } u < 1, \\ 2\lambda & \text{if } u \ge 1, \end{cases}$$

respectively. By Theorem 1.3 for $A_{n,q} = 2n$, we have that

$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \le C\left[\left(\frac{1}{2n^3}\right)^{\frac{1}{5}} + \left(\frac{10}{3}\int_0^1 \delta_{n,q}\,dq\right)^{\frac{1}{4}} + \left(\frac{3}{2n^3}\right)^{\frac{1}{3}} + \left(\frac{6}{n^2}\right)^{\frac{1}{2}}\right].$$

If we choose $\delta_{n,q} = \frac{q}{n^2}$ it is clear that the bound is of the form $\frac{C}{\sqrt{n}}$ for some constant C. #

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A POINTWISE APPROXIMATION OF ISOLATED TREES IN A RANDOM GRAPH

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ABSTRACT. In this paper, we give a pointwise approximation of the number of isolated trees of order k in a random graph by Poisson distribution. The technique we used here is the Stein's method.

1. INTRODUCTION

A random graph is a collection of points, or vertices, with lines, or edges, connecting pairs of them at random. The study of random graphs has a long history. Starting with the influential work of Erdős and Rényi in the 1950s and 1960s [7-9], random graph theory has developed into one of the mainstays of modern discrete mathematics, and has produced a prodigious number of results, many of them highly ingenious, describing statistical properties of graphs, such as distribution of component sizes, existence and size of a giant component, and typical vertex-vertex distances.

Random graphs are not merely a mathematical toy; they have been employed extensively as models of real world networks of various types, particularly in epidemiology. The passage of a disease through a community depends strongly on the pattern of contacts between those infected with the disease and those susceptible to it. This pattern can be depicted as a network, with individuals represented by vertices and contacts capable of transmitting the disease by edges. A large class of epidemiological models known as susceptible/infectious/recovered (or SIR) model [4,17,19] makes frequent use of the so-called fully mixed approximation, which is the assumption that contacts are random and uncorrelated, i.e., that they form a random graph.

Random graphs however turn out to have severe shortcomings as models of such real world phenomena. Although it is difficult to determine experimentally the structure of the network of contacts by which a disease is spread [14], studies have been performed of other social networks such as networks of friendships within a variety of communities [5,10,13], networks of telephone calls [1,2], airline timetables [3], the power grid [22], the structure and conformation space of polymers [16], metabolic pathways [11,15], and food webs [23]. There are many situations in which

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the theory tells us that the distribution of a random variable may be approximated by Poisson distribution. In the random graph theory, one application of the approximation by Poisson distribution arises naturally when counting the number of occurrences of individually rare and unrelated events within a large ensemble. In this paper, we choose to count the number of isolated trees of order k in a random graph with n vertices and give a non-uniform bound of the Poisson approximation to this number.

Let $G(n,p)$ be a random graph with n vertices $1,2,\ldots,n$, in which each possible edge $\{i,j\}$ is present independently with probability p. A tree is, by definition, a connected graph containing no cycles and a tree in $G(n,p)$ is isolated if there is no edge in $G(n,p)$ with one vertex in the tree and the other outside of the tree. Let

$$D_{n,k} = \{(i_1,i_2,\cdots,i_k) | 1 \le i_1 < i_2 < \ldots < i_k \le n\}$$

be the set of all possible combinations of k vertices. For each $i \in D_{n,k}$, we define

$$X_i = \begin{cases} 1 & \text{if there is an isolated tree in } G(n,p) \text{ that spans the vertices} \\ & i = (i_1, i_2 \ldots, i_k), \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$W_{n,k} = \sum_{i \in D_{n,k}} X_i.$$

Clearly $W_{n,k}$ is the number of isolated trees of order k in $G(n,p)$ and the X_i's are not independent unless $k = 1$. For the small value of probability p, that is when $k^2 p \to 0$ and $\frac{k^2}{n} \to 0$, Stein ([21], chapter 13) proved that the distribution of $W_{n,k}$ can be approximated by Poisson distribution with parameter

$$\lambda = EW_{n,k} = \binom{n}{k} P(X_i = 1) = k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-k+1}$$

and the uniform error bound is given by

$$(1.1) \qquad |P(W_{n,k} \in A) - P(Poi_\lambda \in A)| \le \frac{B}{\sqrt{k}} (1 + c_n) e^{1-c_n} (c_n e^{1-c_n})^{k-1}$$

for all $A \subseteq \mathbb{N} \cup \{0\}, n \in \mathbb{N}$, and $k \le n$, where Poi_λ is a Poisson random variable with parameter λ, B is a constant independent of A, and

$$c_n = -n \log(1-p).$$

It is evident from (1.1) that the error bound tend to zero as c_n decreases to zero provided $k \ge 2$. Observe that the bound in (1.1) is a uniform bound that works for any possible number of trees in the graph. In this paper, we shall introduce a non-uniform bound of the Poisson approximation, i.e. a better error bound once the number of trees is specified.

Throughout the paper, let us fix the number of trees $w_0 \in \{1, 2, \ldots, \binom{n}{k}\}$ and denote for convenience

$$\Delta(n,k,w_0) \equiv \left| P(W_{n,k} = w_0) - \frac{e^{-\lambda} \lambda^{w_0}}{w_0!} \right|$$

where $\lambda = EW_{n,k}$. The following theorems are our main results.

Theorem 1.1. *Suppose* $2k < n$ *and* $w_0 \neq 0$. *Then*

1. $\Delta(n,k,w_0) \le \lambda \min\left\{\frac{1}{w_0}, \frac{1}{\lambda}\right\} \min\left\{2, \frac{\lambda k^2}{n}\left(1 + c_n e^{\frac{k^2}{n}(c_n - 1)}\right)\right\}$, *and*
2. $\Delta(n,k,0) \le \min\left\{1, \lambda\right\} \min\left\{2, \lambda\right\}$.

In order to gain a better understanding of these results, some asymptotic behaviors of $\Delta(n,k,w_0)$ and λ as n goes to infinity are summarized in the following two theorems.

Theorem 1.2. *Let $\delta > 1$, $k \le O(n^{\frac{\delta}{2}})$, and $p = O\left(\frac{1}{n^\delta}\right)$. Then, for $\delta^* \in (1,\delta)$, we have*

1. $\lambda \le \frac{1}{k^{\frac{5}{2}}} O\left(\frac{1}{n^{(\delta^*-1)(k-1)-1}}\right)$,
2. $\Delta(n,k,w_0) \le \frac{1}{w_0 k^3} O\left(\frac{1}{n^{2(\delta^*-1)(k-1)-1}}\right)$, *and*
3. $\Delta(n,k,0) \le \frac{1}{k^3} O\left(\frac{1}{n^{2(\delta^*-1)(k-1)-1}}\right)$,

where $\lim_{n\to\infty} \frac{O(g(n))}{g(n)} = c$ *for some* $c > 0$.

Theorem 1.3. *Let $\beta \in \left(0, \frac{1}{2}\right)$, $k \le O(n^\beta)$, and $p = O(\frac{1}{n^\delta})$ for some $\delta > 0$. Then*

1. *for $w_0 > 2$ and $\delta > 1$, $\frac{\Delta(n,k,w_0)}{\lambda^{w_0}} \to 0$ as $n \to \infty$ and*
2. *for $\delta > 2$, $w_0!\Delta(n,k,w_0) \to 0$ as $w_0 \to \infty$.*

Some remarks are in order.
1. When the probability p is small compared to a positive power of n, i.e., $p = \mathrm{O}\left(\frac{1}{n^\delta}\right)$ for $\delta > 1$, both the error bound and λ tend to zero as n approaches infinity.
2. If $p = \mathrm{O}\left(\frac{1}{n^\delta}\right)$ for some $\delta > 1$, then we are dealing with a Poisson distribution with parameter λ smaller than $\mathrm{O}\left(\frac{1}{n^{(\delta^*-1)(k-1)-1}}\right)$ for all $\delta^* \in (1,\delta)$. Theorem 1.3(1) says that as n increases without limit, the error bound $\Delta(n,k,w_0)$ tend to zero faster than the Poisson probability $\frac{e^{-\lambda}\lambda^{w_0}}{w_0!}$.
3. Theorem 1.3(2) confirms that the Poisson probability, $\frac{e^{-\lambda}\lambda^{w_0}}{w_0!}$, tends to zero slower than the error bound $\Delta(w_0,k,w_0)$ as w_0 goes to infinity.

Throughout the paper, C stands for an absolute constant with possibly different values at different places.

2. Proof of the Main Results

The main result in Theorem 1.1 will be proved by Stein's method for Poisson distribution. Stein [20] introduced a new technique of computing a bound in normal approximation by using differential equation and Chen [6] applied Stein's idea to the Poisson case. The Stein's equation for Poisson distribution with parameter λ is given by

$$\lambda f(w+1) - w f(w) = h(w) - \mathcal{P}_\lambda(h) \tag{2.1}$$

where f and h are real-valued functions defined on $\mathbb{N}\cup\{0\}$ and $\mathcal{P}_\lambda(h) = E[h(Poi_\lambda)]$.

For each subset A of $\mathbb{N} \cup \{0\}$, define $h_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ by

$$h_A(w) = \begin{cases} 1 & \text{if } w \in A, \\ 0 & \text{if } w \notin A. \end{cases}$$

For convenience, we shall write h_ω for $h_{\{\omega\}}$ and denote $C_\omega = \{0, 1, 2, \ldots, \omega\}$. For each $\omega_0 \in \mathbb{N} \cup \{0\}$, it is well known [21, p.87] that the solution $U_\lambda h_{\omega_0}$ of (2.1) is of the form

$$U_\lambda h_{w_0}(w) = \begin{cases} \dfrac{(w-1)!}{w_0!}\lambda^{w_0-w}\mathcal{P}_\lambda(1-h_{C_{w-1}}) & \text{if } w_0 < w, \\ -\dfrac{(w-1)!}{w_0!}\lambda^{w_0-w}\mathcal{P}_\lambda(h_{C_{w-1}}) & \text{if } 0 < w \le w_0, \\ 0 & \text{if } w = 0. \end{cases} \tag{2.2}$$

Some properties of $U_\lambda h_{w_0}$ needed to prove Theorem 1.1.

Lemma 2.1. *Let $w_0 \in \mathbb{N}$ and $U_\lambda h_{w_0}$ be the solution of (2.1) with $h = h_{w_0}$. Then*

1. $|U_\lambda h_{w_0}| \le \min\left\{\dfrac{1}{w_0}, \dfrac{1}{\lambda}\right\}$ *and*
2. $|V_\lambda h_{w_0}| \le \min\left\{\dfrac{1}{w_0}, \dfrac{1}{\lambda}\right\}$

where $V_\lambda h_{w_0}(w) = U_\lambda h_{w_0}(w+1) - U_\lambda h_{w_0}(w)$.

Proof. To prove 1., we shall first derive that $|U_\lambda h_{w_0}(w)| \le \dfrac{1}{w_0}$ by splitting w into two cases according to the definition of $U_\lambda h_{w_0}(w)$.

When $w > w_0$, it follows straightforwardly that

$$\begin{aligned} 0 < U_\lambda h_{w_0}(w) &= \frac{(w-1)!}{w_0!}\lambda^{w_0-w}e^{-\lambda}\sum_{k=w}^{\infty}\frac{\lambda^k}{k!} \\ &= \frac{(w-1)!}{w_0!}e^{-\lambda}\left(\frac{\lambda^{w_0}}{w!} + \frac{\lambda^{w_0+1}}{(w+1)!} + \frac{\lambda^{w_0+2}}{(w+2)!} + \cdots\right) \\ &= \frac{(w-1)!}{w_0!}\frac{e^{-\lambda}}{w!}\left(\lambda^{w_0} + \frac{\lambda^{w_0+1}}{(w+1)} + \frac{\lambda^{w_0+2}}{(w+1)(w+2)} + \cdots\right) \\ &= \frac{1}{w}e^{-\lambda}\left(\frac{\lambda^{w_0}}{w_0!} + \frac{\lambda^{w_0+1}}{w_0!(w+1)} + \frac{\lambda^{w_0+2}}{w_0!(w+1)(w+2)} + \cdots\right) \\ &\le \frac{1}{w_0}e^{-\lambda}\left(\frac{\lambda^{w_0}}{w_0!} + \frac{\lambda^{w_0+1}}{(w_0+1)!} + \frac{\lambda^{w_0+2}}{(w_0+2)!} + \cdots\right) \\ &\le \frac{1}{w_0}. \end{aligned}$$

For $w \le w_0$, the bound of $U_\lambda h_{w_0}(w)$ is obtained as follows:

$$\begin{aligned} 0 &< \frac{(w-1)!}{w_0!}\lambda^{w_0-w}\mathcal{P}_\lambda(h_{C_{w-1}}) \\ &= \frac{(w-1)!}{w_0!}e^{-\lambda}\left(\frac{\lambda^{w_0-w}}{0!} + \frac{\lambda^{w_0-w+1}}{1!} + \cdots + \frac{\lambda^{w_0-1}}{(w-1)!}\right) \\ &= \frac{(w-1)!}{w_0!}e^{-\lambda}\left\{\frac{[(w_0-1)-w+1]!\lambda^{(w_0-1)-w+1}}{[(w_0-1)-w+1]!}\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{2\cdot 3\cdots[(w_0-1)-w+2]!\lambda^{(w_0-1)-w+2}}{[(w_0-1)-w+2]!}\\
&+\cdots+\frac{w(w+1)\cdots(w_0-1)\lambda^{(w_0-1)}}{(w_0-1)!}\Bigg\}\\
&\le\frac{(w-1)!}{w_0!}e^{-\lambda}\frac{(w_0-1)!}{(w-1)!}\Bigg\{\frac{\lambda^{(w_0-1)-w+1}}{[(w_0-1)-w+1]!}\\
&+\frac{\lambda^{(w_0-1)-w+2}}{[(w_0-1)-w+2]!}+\cdots+\frac{\lambda^{(w_0-1)}}{(w_0-1)!}\Bigg\}\\
&=\frac{1}{w_0}e^{-\lambda}\Bigg\{\frac{\lambda^{(w_0-1)-w+1}}{[(w_0-1)-w+1]!}+\frac{\lambda^{(w_0-1)-w+2}}{[(w_0-1)-w+2]!}\\
&+\cdots+\frac{\lambda^{(w_0-1)}}{(w_0-1)!}\Bigg\}\\
&\le\frac{1}{w_0}.
\end{aligned}
$$

Combining the two cases gives

$$
|U_\lambda h_{w_0}(w)|\le\frac{1}{w_0}. \tag{2.3}
$$

Similarly, we show that $|U_\lambda h_{w_0}|\le\frac{1}{\lambda}$ by considering two cases. If $w>w_0$,

$$
\begin{aligned}
0<U_\lambda h_{w_0}(w)&=\frac{1}{\lambda}\frac{(w-1)!}{w_0!}e^{-\lambda}\left(\frac{\lambda^{w_0+1}}{w!}+\frac{\lambda^{w_0+2}}{(w+1)!}+\frac{\lambda^{w_0+3}}{(w+2)!}+\cdots\right)\\
&=\frac{1}{\lambda}e^{-\lambda}\left(\frac{\lambda^{w_0+1}}{w_0!w}+\frac{\lambda^{w_0+2}}{w_0!w(w+1)}+\frac{\lambda^{w_0+3}}{w_0!w(w+1)(w+2)}+\cdots\right)\\
&\le\frac{1}{\lambda}e^{-\lambda}\left(\frac{\lambda^{w_0+1}}{(w_0+1)!}+\frac{\lambda^{w_0+2}}{(w_0+2)!}+\frac{\lambda^{w_0+3}}{(w_0+3)!}+\cdots\right)\\
&\le\frac{1}{\lambda}.
\end{aligned}
$$

For $w\le w_0$,

$$
\begin{aligned}
0&<\frac{(w-1)!}{w_0!}\lambda^{w_0-w}\mathcal{P}_\lambda(h_{C_{w-1}})\\
&=\frac{1}{\lambda}\frac{(w-1)!}{w_0!}e^{-\lambda}\Bigg\{\frac{[w_0-w+1]!\lambda^{w_0-w+1}}{[w_0-w+1]!}+\frac{2\cdot 3\cdots[w_0-w+2]\lambda^{w_0-w+2}}{[w_0-w+2]!}\\
&+\cdots+\frac{w(w+1)\cdots w_0\lambda^{w_0}}{w_0!}\Bigg\}\\
&\le\frac{1}{\lambda}\frac{(w-1)!}{w_0!}e^{-\lambda}\frac{w_0!}{(w-1)!}\Bigg\{\frac{\lambda^{w_0-w+1}}{[w_0-w+1]!}+\frac{\lambda^{w_0-w+2}}{[w_0-w+2]!}+\cdots+\frac{\lambda^{w_0}}{w_0!}\Bigg\}
\end{aligned}
$$

$$= \frac{1}{\lambda} e^{-\lambda} \left\{ \frac{\lambda^{w_0 - w + 1}}{[w_0 - w + 1]!} + \frac{\lambda^{w_0 - w + 2}}{[w_0 - w + 2]!} + \cdots + \frac{\lambda^{w_0}}{w_0!} \right\}$$
$$\leq \frac{1}{\lambda}.$$

The above two inequalities demonstrate that

$$|U_\lambda h_{w_0}(w)| \leq \frac{1}{\lambda}. \tag{2.4}$$

Therefore, by (2.3) and (2.4), 1. is proved.

Formula of $V_\lambda h_{w_0}$ is easily derived from that of $U_\lambda h_{w_0}$ in (2.2), that is

$$V_\lambda h_{w_0}(w) = \begin{cases} \lambda^{w_0 - w - 1} \frac{(w-1)!}{w_0!} [w\mathcal{P}_\lambda(1 - h_{C_w}) - \lambda \mathcal{P}_\lambda(1 - h_{C_{w-1}})] & \text{if } w \geq w_0 + 1, \\ \lambda^{w_0 - w - 1} \frac{(w-1)!}{w_0!} [w\mathcal{P}_\lambda(1 - h_{C_w}) + \lambda \mathcal{P}_\lambda(1 - h_{C_{w-1}})] & \text{if } w = w_0, \\ -\lambda^{w_0 - w - 1} \frac{(w-1)!}{w_0!} [w\mathcal{P}_\lambda(h_{C_w}) - \lambda \mathcal{P}_\lambda(h_{C_{w-1}})] & \text{if } w \leq w_0 + 1. \end{cases}$$

Similar arguments as in Neammanee [18] produce the desired bound for $V_\lambda h_{w_0}$ in 2. □

Proof of Theorem 1.1. Proof of 1. is divided into two steps.
Step 1. We claim that

$$\Delta(n, k, w_0) \leq \lambda \min\left\{\frac{1}{w_0}, \frac{1}{\lambda}\right\} \min\{2, E|W_{n,k} - W_{n-k,k}|\}.$$

In fact, for each $i \in D_{n,k}$,

$$\begin{aligned} E[X_i f(W_{n,k})] &= E\{E[X_i f(W_{n,k})|X_i]\} \\ &= E[X_i f(W_{n,k})|X_i = 0]P(X_i = 0) \\ &+ E[X_i f(W_{n,k})|X_i = 1]P(X_i = 1) \\ &= E[f(W_{n,k})|X_i = 1]P(X_i = 1) \\ &= P(X_i = 1)E[f(W^*_{n-k,k} + 1)], \end{aligned}$$

where $W^*_{n-k,k} \sim (W_{n,k} - X_i)|X_i = 1$ is the number of isolated trees of order k in the graph $G(n-k, p)$ obtained from $G(n, p)$ by dropping the vertices $i_1, i_2 \ldots i_k$ and all the edges containing any of these vertices. By the fact that $W^*_{n-k,k}$ has identical distribution as $W_{n-k,k}$, we easily deduce

$$\begin{aligned} E[W_{n,k} f(W_{n,k})] &= \sum_{i \in D_{n,k}} E[X_i f(W_{n,k})] \\ &= \sum_{i \in D_{n,k}} P(X_i = 1)E[f(W_{n-k,k} + 1)] \\ &= \lambda E[f(W_{n-k,k} + 1)]. \end{aligned} \tag{2.5}$$

Once we set $h = h_{w_0}$ in (2.1) and apply (2.5) to the left hand side of (2.1), it follows immediately that

$$\begin{aligned} \left| P(W_{n,k} = w_0) - e^{-\lambda} \frac{\lambda^{w_0}}{w_0!} \right| &= |E[\lambda U_\lambda h_{w_0}(W_{n,k} + 1) - W_{n,k} U_\lambda h_{w_0}(W_{n,k})]| \\ &\leq \lambda E|U_\lambda h_{w_0}(W_{n,k} + 1) - U_\lambda h_{w_0}(W_{n-k,k} + 1)| \end{aligned}$$

$$\leq \lambda[2\sup_w |U_\lambda h_{w_0}(w+1)||$$
$$\leq 2\lambda \min\left\{\frac{1}{w_0}, \frac{1}{\lambda}\right\}$$

where Lemma 2.1(1) was used in the last inequality.

By writing $U_\lambda h_{w_0}(W_{n,k}+1) - U_\lambda h_{w_0}(W_{n-k,k}+1)$ as the sum of 1-step increments and applying Lemma 2.1(2),

$$\begin{aligned}\left|P(W_{n,k}=w_0) - e^{-\lambda}\frac{\lambda^{w_0}}{w_0!}\right| &\leq \lambda E|U_\lambda h_{w_0}(W_{n,k}+1) - U_\lambda h_{w_0}(W_{n-k,k}+1)| \\ &\leq \lambda E|\sup_w[U_\lambda h_{w_0}(w+1) - U_\lambda h_{w_0}(w)] \\ &\quad \times [(W_{n,k}+1) - (W_{n-k,k}+1)]| \\ &\leq \lambda \min\{\frac{1}{w_0}, \frac{1}{\lambda}\} E|W_{n,k} - W_{n-k,k}|.\end{aligned}$$

Hence $\Delta(n,k,w_0) \leq \lambda \min\{\frac{1}{w_0}, \frac{1}{\lambda}\} \min\{2, E|W_{n,k} - W_{n-k,k}|\}$.

Step 2. It now suffices to find a bound of $E|W_{n,k} - W_{n-k,k}|$ for $n \geq 2k$.

By [21] (p. 140, 142), this expectation can be estimated by

$$\begin{aligned}E|W_{n,k} - W_{n-k,k}| &= E(W_{n,k} - W_{n-k,k})^+ + E(W_{n-k,k} - W_{n,k})^+ \\ &\leq \frac{k^2}{n} EW_{n,k} + \left[1 - (1-p)^{k^2}\right] EW_{n-k,k} \\ &= \left(\frac{k^2}{n} + \left[1-(1-p)^{k^2}\right]\frac{EW_{n-k,k}}{\lambda}\right)\lambda\end{aligned}$$

and for $n > 2k$, we have

$$\frac{EW_{n-k,k}}{\lambda} < e^{\frac{k^2}{n}(c_n-1)}.$$

Therefore

$$\begin{aligned}E|W_{n,k} - W_{n-k,k}| &\leq \left(\frac{k^2}{n} + \left[1-(1-p)^{k^2}\right]e^{\frac{k^2}{n}(c_n-1)}\right)\lambda \\ &= \left(\frac{k^2}{n} + e^{\frac{k^2}{n}(c_n-1)} - e^{-\frac{k^2}{n}}\right)\lambda \\ &= \left(\frac{k^2}{n} + (e^{\frac{k^2 c_n}{n}} - 1)e^{-\frac{k^2}{n}}\right)\lambda \\ &\leq \frac{\lambda k^2}{n}\left(1 + c_n e^{\frac{k^2}{n}(c_n-1)}\right)\end{aligned}$$

where we have used the fact that $e^x - 1 \leq xe^x$ for $x \geq 0$ in the last inequality.

It follows readily from step 1. and step 2. that

$$\Delta(n,k,w_0) \leq \lambda\min\left\{\frac{1}{w_0}, \frac{1}{\lambda}\right\}\min\left\{2, \frac{\lambda k^2}{n}\left(1 + c_n e^{\frac{k^2}{n}(c_n-1)}\right)\right\}.$$

The bound of $\Delta(n,k,0)$ in 2. is obtained in the same fashion as that of $\Delta(n,k,w_0)$ except that the inequalities

1. $|U_\lambda h_0| \leq \min\left\{1, \frac{1}{\lambda}\right\}$
2. $|V_\lambda h_0| \leq \min\left\{1, \frac{1}{\lambda}\right\}$

are used instead of Lemma 2.1. With a few obvious adjustments, proof of Lemma 2.1 work equally well for $U_\lambda h_0$ and $V_\lambda h_0$. □

Proof of Theorem 1.2. From Theorem 1.1 and the fact that

$$c_n = \mathrm{O}\Big(\frac{1}{n^{\delta-1}}\Big) \text{ when } p = \mathrm{O}\Big(\frac{1}{n^{\delta}}\Big) \tag{2.6}$$

we see that

$$\Delta(n,k,w_0) \le C\frac{\lambda^2 k^2}{n w_0} \tag{2.7}$$

and

$$\Delta(n,k,0) \le C\frac{\lambda^2 k^2}{n}. \tag{2.8}$$

Under the settings, we would naturally like an estimate of λ in terms of n and k. By (19)–(22) in p. 141 of [21], λ can be factored as

$$\lambda = \alpha(k)\beta(k,p)\gamma(n,k,p) \tag{2.9}$$

where $\alpha(k) = \dfrac{k^{k+\frac{1}{2}}e^{-k}}{k!}$, $\beta(k,p) = k^{-\frac{5}{2}}e^k p^{k-1}(1-p)^{-(\frac{k^2+3k}{2})+1}$ and

$$\gamma(n,k,p) = n^k e^{-kc_n}\prod_{j=1}^{k-1}\Big(1-\frac{j}{n}\Big).$$

The Stirling's formula ([12] p.54),

$$\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}e^{\frac{1}{12k}},$$

easily derives the inequalities

$$\frac{1}{\sqrt{2\pi}e^{\frac{1}{12k}}} < \alpha(k) < \frac{1}{\sqrt{2\pi}e^{\frac{1}{12k+1}}}. \tag{2.10}$$

Finally, by (2.6), (2.9)–(2.10) and the fact that $k \le \mathrm{O}(n^{\frac{\delta}{2}})$,

$$\begin{aligned}
\lambda &\le \frac{Cn^k}{k^{\frac{5}{2}}}e^{k(1-c_n)}p^{k-1} \\
&= \frac{Cn}{k^{\frac{5}{2}}}(c_n e^{1-c_n})^{k-1}e^{1-c_n}\Big(\frac{p}{-\log(1-p)}\Big)^{k-1} \\
&\le \frac{Cn}{k^{\frac{5}{2}}}(c_n e^{1-c_n})^{k-1}e^{1-c_n} \\
&\le \frac{n}{k^{\frac{5}{2}}}\mathrm{O}\Big(\frac{e}{n^{\delta-1}}\Big)^{k-1} \\
&\le \frac{1}{k^{\frac{5}{2}}}\mathrm{O}\Big(\frac{1}{n^{(\delta^*-1)(k-1)-1}}\Big)
\end{aligned} \tag{2.11}$$

where we have used the facts that $\lim\limits_{n\to\infty}\Big(\dfrac{p}{-\log(1-p)}\Big)^{k-1} = 1$ in the second inequality and $\lim\limits_{n\to\infty}\dfrac{e}{n^{\beta}} = 0$ for all $\beta > 0$ in the last inequality. And the theorem follows from (2.7)–(2.8) and (2.11). □

Proof of Theorem 1.3. Let us first observe that

$$\prod_{j=1}^{k-1}\Big(1-\frac{j}{n}\Big)=\exp\Big[-\frac{k(k-1)}{2n}-\frac{\theta k^3}{3n^2}\Big] \tag{2.12}$$

for $k<\frac{n}{2}$ with $0<\theta<1$. So, after substituting (2.12) into (2.9) and noting the fact that $\lim_{n\to\infty}\frac{k^2}{n}=0$, we obtain

$$\begin{aligned}\lambda &\geq \frac{Cn}{k^{\frac{5}{2}}}(c_n e^{1-c_n})^{k-1}e^{1-c_n}\\ &=\frac{C}{k^{\frac{5}{2}}}e^{k-1}O\Big(\frac{1}{n^{(\delta-1)(k-1)-1}}\Big).\end{aligned}$$

With this lower bound of λ, (2.7) becomes

$$\begin{aligned}0\leq\frac{\Delta(n,k,w_0)}{\lambda^{w_0}}&\leq\frac{1}{w_0e^{(w_0-2)(k-1)}}O\Big(\frac{1}{n^{(w_0-2)[(\delta-1)(k-1)-1+\frac{5\beta}{2}]+1-2\beta}}\Big)\\ &=\Big(\frac{n^{(w_0-2)-1+2\beta}}{w_0e^{(w_0-2)(k-1)}}\Big)O\Big(\frac{1}{n^{(w_0-2)[(\delta-1)(k-1)+\frac{5\beta}{2}]}}\Big)\\ &\leq\frac{1}{w_0}O\Big(\frac{1}{n^{(w_0-2)[(\delta-1)(k-1)+\frac{5\beta}{2}]}}\Big).\end{aligned} \tag{2.13}$$

Obviously, the right hand side converges to 0 as n goes to infinity. Again by Stirling's formula ([12],p.52), $k!\sim\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}$, an upper bound of the number of isolated trees can be computed. That is,

$$\begin{aligned}w_0\leq\binom{n}{k}&=\frac{n!}{(n-k)!k!}\\ &\sim\sqrt{2\pi}\Big(\frac{n}{n-k}\Big)^{n+\frac{1}{2}}\Big(\frac{n-k}{k}\Big)^k\\ &\leq Cn^{(1-\beta)k}\\ &\leq Cn^{(\delta-1)(k-1)}\end{aligned}$$

for k sufficiently large. This immediately implies that, for k large enough,

$$(\omega_0-1)!\leq Cn^{(\omega_0-2)(\delta-1)(k-1)}.$$

Thus, we conclude from this bound and (2.13) that

$$\begin{aligned}0\leq w_0!\Delta(n,k,w_0)&\leq w_0!\frac{\Delta(n,k,w_0)}{\lambda^{w_0}}\\ &\leq(w_0-1)!O\Big(\frac{1}{n^{(w_0-2)[(\delta-1)(k-1)+\frac{5\beta}{2}]}}\Big)\\ &\leq\frac{1}{n^{\frac{5}{2}\beta(w_0-2)}}\end{aligned}$$

which converges to zero as ω_0 increases to infinity. □

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A POINTWISE APPROXIMATION OF ISOLATED TREES IN A RANDOM GRAPH

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On the AIDS Incubation Distribution

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Abstract

This paper derives the distribution of the AIDS incubation period, by considering two models which are more complicated than the ones usually studied. Model 1 is a multistate Markovian model with probabilities of moving back and forth from one state to the next, with the possible onset of AIDS at any stages. Model 2, of a similar kind, has an added state B for the onset of AIDS related illnesses, which may also lead to AIDS. There follows a good deal of matrix algebra, ending with distributions of the AIDS incubation period which are (different) weighted sums of gamma distributions for both Models.
Keywords: HIV incubation distribution, homogeneous Markov process, generator matrix and first passage probability distribution .
Mathematical Subject Classification 2002: 60G07, 15A21

1 Introduction and main results.

In the studies of incubation of Acquired Immunodeficiency Syndrome (AIDS), an important problem is the characterization of the probability distribution of the random time between infection with Human Immunodeficiency Virus (HIV) to AIDS onset. This probability distribution has been referred to as the AIDS incubation distribution. We also know that the mean value of this distribution is usually very large taking a value of about 10 years for people between ages 20–50 (Anderson et al., 1989).

In this paper, we will derive this probability distribution in two models as follows.

$$\begin{tikzcd}
S_1 \arrow[r, shift right, "\gamma_1"'] \arrow[ddrr, "\mu_1"'] & S_2 \arrow[l, shift right, "\beta_2"'] \arrow[r, shift right, "\gamma_2"'] \arrow[ddr, "\mu_2"] & \cdots\cdots \arrow[l, shift right, "\beta_3"'] \arrow[r, shift right, "\gamma_{k-2}"'] & S_{k-1} \arrow[l, shift right, "\beta_{k-1}"'] \arrow[r, shift right, "\gamma_{k-1}"'] \arrow[ddl, "\mu_{k-1}"'] & S_k \arrow[l, shift right, "\beta_k"'] \arrow[ddll, "\gamma_k=\mu_k"] \\
 & & & & \\
 & & \text{AIDS} & &
\end{tikzcd}$$

Figure 1: *Model 1*

The model given in Figure 1 is for the HIV epidemic in which β_i, $\gamma_i(>0)$ and μ_i denote the transition rates. In this model, S_1 corresponds to the exposure stage of the Walter-Reed staging system (WRO stage, see Redfield and Burke, 1988) and is the HIV infected but antibody negative stage defined in Longini et al.(1989a, 1989b, 1991, 1992). In Tan and Hsu (1989), S_1 was referred to as the latent stage (L-stage) to account for the latency of HIV provirus. In Figure 1, AIDS denotes the AIDS stage and S_i denotes the $(i-1)$st substage of the infective stage $(i = 2, 3, \ldots, k)$. The Walter-Reed staging system assumed $k = 5$; Hethcote et al.(1991) assumed $k = 6$ whereas Longini et al.(1991) assumed $k = 7$. The model in Figure 1 is more general than most of the models in the literature in the following two aspects:

(a) We assume that it is possible to have backward transition from S_i to S_{i-1}. The data reported by Nagelkerke et al.(1990) suggests that this is possible and hence should be taken into account.

(b) We assume also that it is possible to develop AIDS from any substages of infective stages, i.e., $S_i \rightarrow$ AIDS for $i \leq k$. The MACS data (Multicenter AIDS Cohort Studies) reported by Zhou et al.(1993) and the new revised 1993 AIDS defined by CDC (Center for Disease Control, Atlanta, GA) suggest that it is possible to have $S_i \rightarrow$ AIDS for $i < k$ and hence should be taken into account.

Notice that if $\beta_i = 0$ for $2 \leq i \leq k$ and if $\mu_j = 0$ for $1 \leq j \leq k-1$, then the model in Figure 1 reduces to the model considered by Longini et al.(1989a, 1989b, 1991, 1992).

Figure 1 is assumed that AIDS can be developed directly from any substages of the infective stages. Since the average incubation period is usually very long, intuitively it is difficult to imagine that AIDS would be developed directly from early substages of the infective stage. Because of this consideration, we will postulate a state B for AIDS-related illness and consider a modified model in Figure 2. Notice that in the CDC staging system (see CDC Report, 1986), the stage B has been suggested as a stage between the infective stage and the AIDS stage. Hence the model in Figure 2 is in essence a reformulation of the CDC staging system of the HIV epidemic.

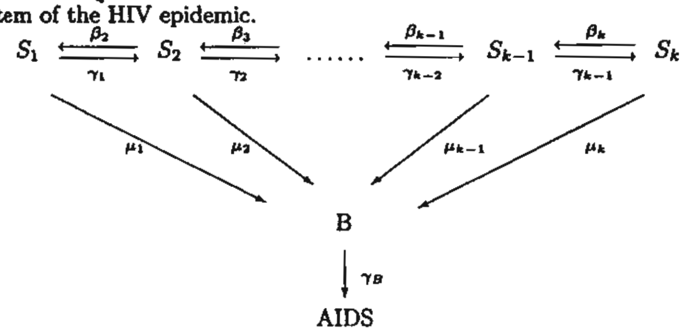

Figure 2: *Model 2*

In Model 1 and Model 2, let X_t $(t \geq 0)$ be a random variable which value is the stage of HIV epidemic at the time t and v_{ij} is the transition rate at which X_t jumps from i to j. We assume that (X_t) is a homogeneous Markov process and the transition matrix $P(t) = [p_{ij}(t)]$ satisfies (1.1) as follows.

For $\Delta t \to 0$,

$$\left.\begin{aligned} p_{ij}(\Delta t) &= v_{ij}\Delta t + o(\Delta t) \quad \text{where } i \neq j, \\ p_{ii}(\Delta t) &= 1 - v_{ii}\Delta t + o(\Delta t), \\ v_{ij} \geq 0, v_{ii} &= \sum_{i \neq j} v_{ij} \quad \text{and} \quad \lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0. \end{aligned}\right\} \tag{1.1}$$

Our main results can be stated as follows :

Theorem 1.1 (Model 1) For $i_0 = 1, 2, \dots, k$, let T_{i_0} be the first passage time of state S_{i_0} to AIDS stage in Model 1 and f_{i_0} be its probability function. Then

1. f_{i_0} can be written in the weighted sums of Gamma density function, $f_{i_0}(t) = \sum_{j=0}^{k-1}\sum_{i=1}^{k} c_{ij}^{i_0} t^j e^{-b_i t}$ where $c_{ij}^{i_0}$ and b_i are constants, and
2. the average time of AIDS incubation from stage S_{i_0} is $E(T_{i_0}) = \sum_{j=0}^{k-1}\sum_{i=1}^{k} c_{ij}^{i_0} \frac{(j+1)!}{b_i^{j+2}}$.

Theorem 1.2 (Model 2) Let T_{i_0}, f_{i_0} and $E(T_{i_0})$ be defined as in Theorem 1.1 for Model 2. Then $f_{i_0}(t) = \sum_{j=0}^{k}\sum_{i=1}^{k+1} d_{ij}^{i_0} t^j e^{-b_i t}$ and $E(T_{i_0}) = \sum_{j=0}^{k}\sum_{i=1}^{k+1} d_{ij}^{i_0} \frac{(j+1)!}{b_i^{j+1}}$ where $d_{ij}^{i_0}$ and b_i are constants.

Furthermore, if we follow the procedure in Section 4, the constants in Theorem 1.1 and Theorem 1.2 can be found.

2 Proof of Theorem 1.1.

In the first four theorems, we give some auxiliary results on matrix algebra in order to prove our main results.

Theorem 2.1. *(Curtis 1984, Chapter 7) Let A be a $k \times k$ matrix with eigenvalues $b_1, b_2, \dots, b_k$. Then A is similar to a $k \times k$ matrix J which of the form*

$$J = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & J_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & J_n \end{bmatrix} \tag{2.1}$$

for some $n \in \mathbb{N}$ and for $l = 1, 2, \dots, n$ there exists b_i $(i = 1, 2, \dots, k)$ such that

$$J_l = \begin{bmatrix} b_i & 1 & 0 & \cdots & 0 \\ 0 & b_i & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_i & 1 \\ 0 & \cdots & \cdots & 0 & b_i \end{bmatrix} \quad \textit{or} \quad [b_i].$$

The matrix J is called the Jordan canonical form corresponding to A.

Furthermore, we know that the Jordan canonical form J corresponding to A can be represented in the form

$$\begin{bmatrix} b_1 & d_1 & 0 & \cdots & 0 \\ 0 & b_2 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{k-1} & d_{k-1} \\ 0 & \cdots\cdots & & 0 & b_k \end{bmatrix}$$

where $d_i \in \{0, 1\}$.

Theorem 2.2. *Let A_k be a tri-diagonal $k \times k$ matrix of the form*

$$A_k = \begin{bmatrix} \lambda_1 & -\gamma_1 & 0 & \cdots\cdots & & & 0 \\ -\beta_2 & \lambda_2 & -\gamma_2 & 0 & \cdots\cdots & & 0 \\ 0 & -\beta_3 & \lambda_3 & -\gamma_3 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots\cdots & & 0 & -\beta_{k-1} & \lambda_{k-1} & -\gamma_{k-1} \\ 0 & \cdots\cdots & & & 0 & -\beta_k & \lambda_k \end{bmatrix},$$

where $\beta_i \geq 0$, $\gamma_i > 0$ and $\lambda_i \geq \gamma_i + \beta_i$ for $i = 1, 2, \ldots, k$. Then A_k has a positive determinant.

Proof. Let $\lambda_i = \gamma_i + \beta_i + \epsilon_i$ for some $\epsilon_i \geq 0$. It is clear that $\det A_1 > 0$. To prove the theorem by induction, we suppose that

$$\det A_1, \det A_2, \ldots, \det A_k \text{ have positive determinants for } k \in \mathbb{N}. \tag{2.2}$$

In order to show $\det A_{k+1} > 0$, it suffices to prove the following:

$$\det A_l \geq (\beta_1 + \epsilon_1) \begin{vmatrix} \lambda_2 & -\gamma_2 & 0 & \cdots\cdots & & & 0 \\ -\beta_3 & \lambda_3 & -\gamma_3 & 0 & \cdots\cdots & & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots\cdots & & -\beta_{l-1} & \lambda_{l-1} & -\gamma_{l-1} \\ 0 & \cdots\cdots & & 0 & -\beta_l & \lambda_l \end{vmatrix} + \gamma_1\gamma_2\cdots\gamma_l > 0 \tag{2.3}$$

for $l = 2, 3, \ldots, k+1$.

We also prove (2.3) by induction on $\{2, 3, \ldots, k+1\}$. Since $\det A_2 = (\gamma_1 + \beta_1 + \epsilon_1)(\gamma_2 + \beta_2 + \epsilon_2) - \gamma_1\beta_2 \geq (\beta_1 + \epsilon_1)\lambda_2 + \gamma_1\gamma_2 > 0$, the basis step of induction holds. Next we assume that (2.3) holds for $2, 3, \ldots, m$ where $m \in \{2, 3, \ldots, k\}$. Then

$\det A_{m+1}$

$$= (\gamma_1 + \beta_1 + \epsilon_1) \begin{vmatrix} \lambda_2 & -\gamma_2 & 0 & \cdots\cdots & 0 \\ -\beta_3 & \lambda_3 & -\gamma_3 & 0 \cdots\cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots\cdots & & \lambda_m & -\gamma_m \\ 0 & \cdots\cdots & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix} + \gamma_1 \begin{vmatrix} -\beta_2 & -\gamma_2 & 0 & \cdots\cdots & 0 \\ 0 & \lambda_3 & -\gamma_3 & 0 \cdots\cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots\cdots & & \lambda_m & -\gamma_m \\ 0 & \cdots\cdots & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix}$$

$$= (\beta_1+\epsilon_1)\begin{vmatrix} \lambda_2 & -\gamma_2 & 0 & \cdots & & 0 \\ -\beta_3 & \lambda_3 & -\gamma_3 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & \lambda_m & -\gamma_m \\ 0 & \cdots & & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix} - \gamma_1\beta_2 \begin{vmatrix} \lambda_3 & -\gamma_3 & 0 & \cdots & & 0 \\ -\beta_4 & \lambda_4 & -\gamma_4 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & \lambda_m & -\gamma_m \\ 0 & \cdots & & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix}$$

$$+ \gamma_1 \begin{vmatrix} \lambda_2 & -\gamma_2 & 0 & \cdots & & 0 \\ -\beta_3 & \lambda_3 & -\gamma_3 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & \lambda_m & -\gamma_m \\ 0 & \cdots & & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix}$$

$$\geq (\beta_1+\epsilon_1)\begin{vmatrix} \lambda_2 & -\gamma_2 & 0 & \cdots & & 0 \\ -\beta_3 & \lambda_3 & -\gamma_3 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & \lambda_m & -\gamma_m \\ 0 & \cdots & & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix} - \gamma_1\beta_2 \begin{vmatrix} \lambda_3 & -\gamma_3 & 0 & \cdots & & 0 \\ -\beta_4 & \lambda_4 & -\gamma_4 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & \lambda_m & -\gamma_m \\ 0 & \cdots & & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix}$$

$$+ \gamma_1 \left((\beta_2+\epsilon_2) \begin{vmatrix} \lambda_3 & -\gamma_3 & 0 & \cdots & & 0 \\ -\beta_4 & \lambda_4 & -\gamma_4 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & \lambda_m & -\gamma_m \\ 0 & \cdots & & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix} + \gamma_2\gamma_3\cdots\gamma_{m+1} \right)$$

$$\geq (\beta_1+\epsilon_1)\begin{vmatrix} \lambda_2 & -\gamma_2 & 0 & \cdots & & 0 \\ -\beta_3 & \lambda_3 & -\gamma_3 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & \lambda_m & -\gamma_m \\ 0 & \cdots & & & -\beta_{m+1} & \lambda_{m+1} \end{vmatrix} + \gamma_1\gamma_2\gamma_3\cdots\gamma_{m+1} > 0$$

where we have used the induction hypothesis in the first inequality and (2.2) in the second inequality.

Theorem 2.3. *(Curtis 1984, Chapter 10) Let A and B be $k \times k$ matrices such that $AB = BA$. Then $e^A e^B = e^B e^A$ where $e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j$ and X^0 is the identity matrix with the same dimension of X.*

In Model 1, the state space of $\{X_t | t \geq 0\}$ is $\{S_1, S_2, \ldots, S_{k+1}\}$ where S_{k+1} is the AIDS state. Hence the state $k+1$ is an absorbing state and by the fact that $\mu_i > 0$ for $i = 1, 2, \ldots, k$, we see that $S_1, S_2, \ldots, S_k$ are transient states. Starting at time $t = 0$, let T_i be the random time that S_i is absorbed into S_{k+1}, $i = 1, 2, \ldots, k$. Then T_i is referred to as the first passage time of S_i and $f_i(t)$, the probability function, the first passage probability of S_i. In what follows, we put $f(t) = [f_1(t), \ldots, f_k(t)]^T$, where T denotes the transpose.

Theorem 2.4. *1) The generator matrix of (X_t) is the $(k+1)\times(k+1)$ matrix $-Q$ such that*

$$Q = \begin{bmatrix} \lambda_1 & -\gamma_1 & \ldots & 0 & 0 & -\mu_1 \\ -\beta_2 & \lambda_2 & -\gamma_2 & \ldots & 0 & -\mu_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_{k-1} & -\gamma_{k-1} & -\mu_{k-1} \\ 0 & 0 & \ldots & -\beta_k & \lambda_k & -\gamma_k \\ 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \tag{2.4}$$

where $\lambda_i = \beta_i + \gamma_i + \mu_i$ for $i = 1, 2, \ldots, k-1$, $\lambda_k = \beta_k + \gamma_k$ and $\beta_1 = 0$.

2) Let $A = A_k$ in Theorem 2.2. If A is invertible, then $f(t) = e^{-At}A1_k$ where $1_k = \underbrace{[1, 1, \ldots, 1]}_{k}^T$.

Proof.

1) From Model 1 and (1.1), for every $i, j = 1, 2, \ldots, k$ we see that $v_{ij} = 0$ for $|i-j| > 1$, $v_{i(i+1)} = \gamma_i$, $v_{i(k+1)} = \mu_i$ and $v_{i(i-1)} = \beta_i$ $(i \neq 1)$. Since the $(k+1)$st state is an absorbing state, $v_{(k+1)j} = 0$ for $j = 1, 2, \ldots, k$. So the generator matrix of $\{X_t | t \geq 0\}$ is $-Q$ where Q is defined by (2.4).

2) We observe that $Q = \begin{bmatrix} A & -\mu \\ 0_k & 0 \end{bmatrix}_{(k+1)\times(k+1)}$ where $\mu = [\mu_1, \mu_2, \ldots, \mu_k]^T$ and $0_k = \underbrace{[0, 0, \cdots, 0]}_{k}$.

Hence, by inductive reasoning, we have that $Q^j = \begin{bmatrix} A^j & -A^{j-1}\mu \\ 0_k & 0 \end{bmatrix}$ for $j = 1, 2, \cdots$. From this fact and the facts that $P(t) = e^{-Qt}$ (see Sydney (2002) p.388) and

$$A^{-1}[I_k - e^{-At}]\mu = -A^{-1}\{\sum_{j=0}^{\infty}(-1)^j \frac{t^j}{j!}A^j - I_k\}\mu = -\sum_{j=1}^{\infty}(-1)^j \frac{t^j}{j!}A^{j-1}\mu,$$

we have

$$\begin{aligned} P(t) &= \sum_{j=0}^{\infty}(-1)^j \frac{t^j}{j!}Q^j \\ &= I_{k+1} + \sum_{j=1}^{\infty}(-1)^j \frac{t^j}{j!} \begin{bmatrix} A^j & -A^{j-1}\mu \\ 0_k & 0 \end{bmatrix} \\ &= I_{k+1} + \begin{bmatrix} \sum_{j=1}^{\infty}(-1)^j \frac{t^j}{j!}A^j & -A^{-1}\sum_{j=1}^{\infty}(-1)^j \frac{t^j}{j!}A^j\mu \\ 0_k & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{-At} & A^{-1}[I_k - e^{-At}]\mu \\ 0_k & 1 \end{bmatrix}. \end{aligned} \tag{2.5}$$

Let $h(t) = \begin{bmatrix} p_{1(k+1)}(t) & p_{2(k+1)}(t) & \cdots & \cdots & p_{k(k+1)}(t) \end{bmatrix}^T$. Hence, by (2.5), we have

$$h(t) = A^{-1}[I_k - e^{-At}]\mu. \tag{2.6}$$

For $t \geq 0$ and $i = 1, 2, \ldots, k$ we note that

$$\begin{aligned}
&[0, ..., 0, \underbrace{1}_{i-th}, 0, ..., 0](h(t)) \\
&= p_{i(k+1)}(t) \\
&= Prob(\{X_t = k+1, T_i > t | X_0 = i\}) + P(\{X_t = k+1, T_i \leq t | X_0 = i\}) \\
&= Prob(\{X_t = k+1, T_i \leq t | X_0 = i\}) \\
&= Prob(T_i \leq t) \\
&= F_i(t) \qquad \text{where } F_i(t) \text{ is the distribution function of } T_i.
\end{aligned}$$

Hence $f(t) = \frac{d}{dt}(h(t))$. From this fact, (2.6) and the fact that

$$A1_k = \begin{bmatrix} \lambda_1 & -\gamma_1 & 0 & \cdots\cdots & 0 \\ -\beta_2 & \lambda_2 & -\gamma_2 & \cdots\cdots & 0 \\ 0 & -\beta_3 & \lambda_3 & \cdots\cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots\cdots & 0 & -\beta_k & \lambda_k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \end{bmatrix}^T = \mu, \tag{2.7}$$

we have, $f(t) = \frac{d}{dt}[A^{-1}[I_k - e^{-At}]A1_k] = \frac{d}{dt}[1_k - e^{-At}1_k] = e^{-At}A1_k$. #

Proof of Theorem 1.1 Let A be defined as in Theorem 2.4. By Theorem 2.2, and Theorem 2.4 we have

$$f(t) = e^{-At}A1_k. \tag{2.8}$$

From Theorem 2.1, there exist the $k \times k$ Jordan Canonical form

$$J = \begin{bmatrix} b_1 & d_1 & 0 & \cdots\cdots & & 0 \\ 0 & b_2 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & d_3 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots\cdots & & 0 & b_{k-1} & d_{k-1} \\ 0 & \cdots\cdots & & & 0 & b_k \end{bmatrix}$$

where $b_1, b_2, \ldots, b_k$ are eigenvalues of A and $d_i \in \{0, 1\}$ for $i = 1, 2, \ldots, k$ and an invertible matrix V such that $A = VJV^{-1}$. Moreover, we know that there exists $n \in \mathbb{N}$ such that J can be written in the form of (2.1). For each $l = 1, 2, \ldots, n$, if

$$J_l = \begin{bmatrix} b_i & 1 & 0 & \cdots & 0 \\ 0 & b_i & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_i & 1 \\ 0 & \cdots\cdots & & 0 & b_i \end{bmatrix}, \text{ let } S_l = \begin{bmatrix} b_i & 0 & 0 & \cdots & 0 \\ 0 & b_i & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_i & 0 \\ 0 & \cdots\cdots & & 0 & b_i \end{bmatrix} \quad \text{and} \quad N_l = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots\cdots & & 0 & 0 \end{bmatrix}$$

and if $J_l = [b_i]$, let $S_l = [b_i]$ and $N_l = [0]$. Then

$$J_l = S_l + N_l \quad \text{and} \quad S_l N_l = N_l S_l \quad \text{for all } l = 1, 2, \dots, n.$$

Let

$$N = \begin{bmatrix} N_1 & 0 & 0 & \cdots & 0 \\ 0 & N_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & N_{n-1} & 0 \\ 0 & \cdots\cdots & & 0 & N_n \end{bmatrix} = \begin{bmatrix} 0 & d_1 & 0 & \cdots & 0 \\ 0 & 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & d_{k-1} \\ 0 & \cdots\cdots & & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} S_1 & 0 & 0 & \cdots & 0 \\ 0 & S_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-1} & 0 \\ 0 & \cdots\cdots & & 0 & S_n \end{bmatrix} = \begin{bmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{k-1} & 0 \\ 0 & \cdots\cdots & & 0 & b_k \end{bmatrix},$$

$\overline{N} = VNV^{-1}$ and $\overline{S} = VSV^{-1}$.

So, $J = N + S$ and $NS = SN$. These imply

$$A = \overline{N} + \overline{S} \qquad \text{and} \qquad \overline{N}\,\overline{S} = \overline{S}\,\overline{N}. \tag{2.9}$$

Note that

$$e^{-St} = \sum_{n=0}^{\infty} \frac{(-tS)^n}{n!} = \begin{bmatrix} e^{-b_1 t} & 0 & \cdots & 0 \\ 0 & e^{-b_2 t} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots\cdots\cdots & & e^{-b_k t} \end{bmatrix}$$

and for natural number m we have $N^m = \left[n_{ij}^{(m)}\right]$ where

$$n_{ij}^{(m)} = \begin{cases} \prod_{l=i}^{i+m-1} d_l, & \text{if } j = i + m \text{ and } j < k, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $N^m = 0$ for $m \geq k$ and $e^{-Nt} = \sum_{m=0}^{k-1} \frac{(-tN)^m}{m!} = [\beta_{ij}]$ where

$$\beta_{ij} = \begin{cases} n_{ij}^{(p)} \frac{(-t)^p}{p!}, & \text{if } j > i \text{ and } j = i + p, \\ 1, & \text{if } j = i, \\ 0, & \text{if } j < i. \end{cases}$$

By (2.7) and (2.8) we have

$$\begin{aligned}
f_{i_0}(t) &= e_{i_0}^T f(t) \quad \text{where } e_{i_0}^T = (0, \ldots, 0, \underbrace{1}_{i_0-\text{th}}, 0, \ldots, 0) \\
&= e^{-At} \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \end{bmatrix}^T; \quad \text{which by (2.9)} \\
&= e_{i_0}^T e^{-\overline{S}t} e^{-\overline{N}t} \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \end{bmatrix}^T \\
&= e_{i_0}^T V e^{-St} e^{-Nt} V^{-1} \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \end{bmatrix}^T \\
&= \begin{bmatrix} e^{-b_1 t} v_{i_0 1} & e^{-b_2 t} v_{i_0 2} & \cdots & e^{-b_k t} v_{i_0 k} \end{bmatrix} [\beta_{ij}] \begin{bmatrix} \sum_{i=1}^{k} v'_{1i}\mu_i & \sum_{i=1}^{k} v'_{2i}\mu_i & \cdots & \sum_{i=1}^{k} v'_{ki}\mu_i \end{bmatrix}^T \\
&\qquad \text{where } V = (v_{ij}) \text{ and } V^{-1} = (v'_{ij}) \\
&= \begin{bmatrix} e^{-b_1 t} v_{i_0 1} & \sum_{l=1}^{2} (-1)^{2-l} e^{-b_l t} v_{i_0 l} n_{l2}^{(2-l)} \frac{t^{2-l}}{(2-l)!} & \cdots & \sum_{l=1}^{k} (-1)^{k-l} e^{-b_l t} v_{i_0 l} n_{lk}^{(k-l)} \frac{t^{k-l}}{(k-l)!} \end{bmatrix} \\
&\qquad \begin{bmatrix} \sum_{i=1}^{k} v'_{1i}\mu_i & \sum_{i=1}^{k} v'_{2i}\mu_i & \cdots & \sum_{i=1}^{k} v'_{ki}\mu_i \end{bmatrix}^T \\
&= \sum_{j=0}^{k-1} \sum_{i=1}^{k} c_{ij}^{i_0} t^j e^{-b_i t}
\end{aligned}$$

where

$$c_{ij}^{i_0} = \begin{cases} \frac{(-1)^j}{j!} v_{i_0 i} \prod_{l=i}^{i+j-1} d_l \sum_{r=1}^{k} \mu_r v'_{(i+j)r}, & \text{if } i + j \leq k, \\ 0, & \text{otherwise,} \end{cases} \tag{2.10}$$

and $\prod_{l=i}^{j} d_l = 1$ for $j < i$.

By the fact that $\int_0^\infty t^n e^{bt}\, dt = \frac{n!}{b^{n+1}}$, we see that the average time of AIDS incubation from stage S_{i_0} is

$$E(T_{i_0}) = \int_0^\infty t f_{i_0}(t)\, dt = \sum_{j=0}^{k-1} \sum_{i=1}^{k} c_{ij}^{i_0} \frac{(j+1)!}{b_i^{j+2}}.$$

3 Proof of Theorem 1.2.

Let A be the $(k+1)\times(k+1)$ matrix defined by $A = \begin{bmatrix} R & -\mu \\ 0_k & \gamma_B \end{bmatrix}$ where

$$R = \begin{bmatrix} \lambda_1 & -\gamma_1 & 0 & \cdots & 0 \\ -\beta_2 & \lambda_2 & -\gamma_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & -\beta_{k-1} & \lambda_{k-1} & -\gamma_{k-1} \\ 0 & \cdots\cdots\cdots\cdots & & -\beta_k & \lambda_k \end{bmatrix}, \mu = [\mu_1, \mu_2, \ldots, \mu_k]^T \quad \text{and} \quad 0_k = \underbrace{[0, 0, \ldots, 0]}_{k}.$$

Since R has the positive determinant, we have $\det A = \gamma_B \det R > 0$ which implies that A has the positive determinant. By the argument in Theorem 2.4 we can show that $f_{i_0}(t) = e^{-Rt}R1_{k+1}$. Let

$$J = \begin{bmatrix} b_1 & d_1 & \cdots\cdots & 0 \\ 0 & b_2 & \cdots\cdots & 0 \\ & & \ddots & \\ \cdots\cdots\cdots\cdots & & b_k & d_k \\ \cdots\cdots\cdots\cdots\cdots & & & b_{k+1} \end{bmatrix}$$

be the Jordan canonical form corresponding to A where $b_1, \ldots, b_{k+1}$ are eigenvalues of A and $d_i \in \{0,1\}$ and $U = [u_{ij}]$, $U^{-1} = [u'_{ij}]$ be such that $A = UJU^{-1}$. By following the argument in the proof of Theorem 1.1 we have

$$f_{i_0}(t) = e_{i_0}^T e^{-Rt} R 1_{k+1} = \sum_{j=0}^{k}\sum_{i=1}^{k+1} d_{ij}^{i_0} t^j e^{-b_i t}$$

where

$$d_{ij}^{i_0} = \begin{cases} \dfrac{(-1)^j}{j!}\left(\prod_{l=i}^{i+j-1} d_l\right) u_{i_0 i} u'_{(i+j)(k+1)}\gamma_B, & \text{if } i+j \le k+1, \\ 0, & \text{otherwise.} \end{cases}$$

We also know that the mean-time of Model 2 is $E(T_{i_0}) = \sum_{j=0}^{k}\sum_{i=1}^{k+1} d_{ij}^{i_0}\dfrac{(j+1)!}{b_i^{j+1}}$. #

4 Procedure to find AIDS incubation and examples.

From the proof of Theorem 1.1, we have the procedure to find AIDS incubation as follows :

Step 1. Find all eigenvalues $b_1, b_2, \ldots, b_k$ of the matrix

$$A = \begin{bmatrix} \lambda_1 & -\gamma_1 & \cdots & 0 & 0 \\ -\beta_2 & \lambda_2 & -\gamma_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k-1} & -\gamma_{k-1} \\ 0 & 0 & \cdots & -\beta_k & \lambda_k \end{bmatrix}$$

Step 2. Find the invertible matrix P and the Jordan canonical form

$$J = \begin{bmatrix} b_1 & d_1 & 0 & \cdots\cdots & & 0 \\ 0 & b_2 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & d_3 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots\cdots & & 0 & b_{k-1} & d_{k-1} \\ 0 & \cdots\cdots & & & 0 & b_k \end{bmatrix}$$

where $d_i \in \{0, 1\}$ (see Bronson (1969), Chapter 8) which satisfy the condition $A = PJP^{-1}$.

Step 3. By substitution all of constants from Step 1 and Step 2 in formulas (2.10) we can solve the AIDS incubation distribution.

For Model 2, we also have the same procedure of Model 1.

Example 4.1. (Model 1 of Tan and Byers 1993) Assume that $\mu_i = 0$ for $i = 1, 2, \ldots, k-1$ and $\beta_i = 0$, $\gamma_i = \gamma$ for $i = 1, 2, \ldots, k$.

$$S_1 \xrightarrow{\gamma} S_2 \xrightarrow{\gamma} S_3 \longrightarrow \cdots \longrightarrow S_k \xrightarrow{\gamma} \text{AIDS}$$

Model 1 of Tan and Byers 1993

Step 1 Since

$$A = \begin{bmatrix} \gamma & -\gamma & 0 & \cdots\cdots & & 0 \\ 0 & \gamma & -\gamma & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots\cdots & & 0 & \gamma & -\gamma \\ 0 & \cdots\cdots & & & 0 & \gamma \end{bmatrix},$$

we can see that γ is the eigenvalue of matrix A in this model.

Step 2 We can find

$$P = [v_{ij}] = \begin{bmatrix} (-\gamma)^{k-1} & 0 & \cdots\cdots & & 0 \\ 0 & (-\gamma)^{k-2} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & -\gamma & 0 \\ 0 & \cdots\cdots & & 0 & 1 \end{bmatrix}, \quad P^{-1} = [v'_{ij}] = \begin{bmatrix} \frac{1}{(-\gamma)^{k-1}} & 0 & \cdots\cdots & & 0 \\ 0 & \frac{1}{(-\gamma)^{k-2}} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \frac{1}{-\gamma} & 0 \\ 0 & \cdots\cdots & & 0 & 1 \end{bmatrix}$$

and

$$J = \begin{bmatrix} \gamma & 1 & 0 & \cdots\cdots & & 0 \\ 0 & \gamma & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots\cdots & & 0 & \gamma & 1 \\ 0 & \cdots\cdots\cdots & & & 0 & \gamma \end{bmatrix}.$$

Hence $d_i = 1$ for $i = 1, 2, \ldots, k-1$.

Step 3 From Step 1, Step 2 and (2.10),

$$c_{ij}^{i_0} = \begin{cases} \dfrac{\gamma^{k-i_0+1}}{(k-i_0)!}, & \text{if } i = i_0 \text{ and } j = k - i_0, \\ 0, & \text{otherwise.} \end{cases}$$

So $f_{i_0}(t) = \dfrac{\gamma^{k-i_0+1}}{(k-i_0)!} t^{k-i_0} e^{-\gamma t}$ which is the Gamma density.

Example 4.2. (Model 1 of Longini et al. (1989a, 1989b, 1991) and Anderson et al. (1989)) Assume that $\mu_i = 0$ for $i = 1, 2, \ldots, k-1$, $\beta_i = 0$ for $i = 1, 2, \ldots, k$ and γ_i are distinct.

$$S_1 \xrightarrow{\gamma_1} S_2 \xrightarrow{\gamma_2} S_3 \longrightarrow \cdots \longrightarrow S_k \xrightarrow{\gamma_k} \text{AIDS}$$

Model 1 of Longini et al. (1989a, 1989b, 1991) and Anderson et al. (1989)

Follow the procedure, we have

$$A = \begin{bmatrix} \gamma_1 & -\gamma_1 & 0 & \cdots\cdots & & 0 \\ 0 & \gamma_2 & -\gamma_2 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots\cdots & & 0 & \gamma_{k-1} & -\gamma_{k-1} \\ 0 & \cdots\cdots\cdots & & & 0 & \gamma_k \end{bmatrix}, \quad J = \begin{bmatrix} \gamma_1 & 0 & \cdots\cdots & & 0 \\ 0 & \gamma_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \gamma_{k-1} & 0 \\ 0 & \cdots\cdots & & 0 & \gamma_k \end{bmatrix},$$

$P = [p_{ij}]$ and $P^{-1} = [p'_{ij}]$ where

$$p_{ij} = \begin{cases} \prod_{t=i}^{j-1} \dfrac{\gamma_t}{\gamma_t - \gamma_j}, & \text{if } i \le j, \\ 0, & \text{if } i > j, \end{cases} \quad \text{and} \quad p'_{ij} = \begin{cases} \dfrac{\prod_{t=i}^{j-1} \gamma_t}{\prod_{t=i+1}^{j} (\gamma_t - \gamma_i)}, & \text{if } i \le j, \\ 0, & \text{if } i > j. \end{cases}$$

Hence $d_i = 0$ for $i = 1, 2, \ldots, k-1$ which implies that

$$f_{i_0}(t) = \sum_{i=1}^{k} c_{i0}^{i_0} e^{-\gamma_i t} = \sum_{i=1}^{k} \left(\prod_{j=1}^{i-1} \frac{\gamma_j}{\gamma_j - \gamma_i} \right) \left(\frac{\gamma_i \gamma_{i+1} \cdots \gamma_k}{\prod_{j=i+1}^{k} (\gamma_j - \gamma_i)} \right) e^{-\gamma_i t} = \sum_{i=1}^{k} \frac{\prod_{j=1}^{k} \gamma_j}{\prod_{\substack{j=1 \\ j \ne i}}^{k} (\gamma_j - \gamma_i)} e^{-\gamma_i t}. \qquad \#$$

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•Output ที่ได้จากโครงการ

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• 1. Publication Papers จำนวน 28 papers ดังต่อไปนี้คือ

1. K. Nonlaopon, A. Kananthai, On the generalized heat kernel, Computational Technologies, Vol. 9(1),2004:3-10.
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2. ผลงานที่ส่งไปตีพิมพ์ในวารสารนานาชาติ (Submitted papers) 17 papers ดังต่อไปนี้

1. K. Nonlaopon, A. Kananthai, On the Generalized ultra-hyperbolic heat kernel ได้ส่งไปตีพิมพ์ในวารสาร Science Asia
2. A. Kananthai, On the Generalized nonlinear heat equation ได้ส่งไปตีพิมพ์ในวารสาร Nonlinear Analysis : Real World Applications
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5. A. Kananthai, On the spectrum of the generalized heat kernel ได้ส่งไปตีพิมพ์ในวารสาร Proceeding of American
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3 ความก้าวหน้าในการสร้างนักวิจัย

(1) นักศึกษาระดับปริญญาเอกจำนวน 5 คน ได้จบการศึกษาภายใต้หัวข้อที่เกี่ยวข้องกับโครงการนี้ คือ

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10. นางสาวจิราพรรณ สุนทรโชติ จุฬาลงกรณ์มหาวิทยาลัย

(3) นักศึกษาระดับปริญญาโทจำนวน 9 คน ได้จบการศึกษาภายใต้หัวข้อที่เกี่ยวข้องกับโครงการนี้ คือ

1. นางสาว เบญจวรรณ โรจน์ดิษฐ์ มหาวิทยาลัยเชียงใหม่
2. นางสาว กรรณิการ์ ขำพึ่งสน มหาวิทยาลัยเชียงใหม่
3. นาย ธนดล ชาวบ้านเกาะ มหาวิทยาลัยเชียงใหม่
4. นายพีระพงษ์ สืบแสน มหาวิทยาลัยเชียงใหม่
5. นางสาว มัชฌิมา โชติพันธุ์วิทยากุล มหาวิทยาลัยเชียงใหม่
6. นายธวัชชัย ปัญญาติ๊บ มหาวิทยาลัยเชียงใหม่
7. นางสาวมนต์ธิรา สุวรรณประภา มหาวิทยาลัยเชียงใหม่

8. นายสมคิด อินเทพ มหาวิทยาลัยเชียงใหม่
9. นายสมยศ อุดมปรัชญาภรณ์ มหาวิทยาลัยเชียงใหม่

(4) กำลังสร้างนักศึกษาระดับปริญญาโท อีกจำนวน 3 คน คือ

1. นางสาว ราชกุล พูนช่วย มหาวิทยาลัยเชียงใหม่
2. นายนเรศ รัตนพรหมรินท์ มหาวิทยาลัยเชียงใหม่
3. นายธนา นันธิกาญจนะ มหาวิทยาลัยเชียงใหม่

4 กิจกรรมทางวิชาการอื่น ๆ

4.1 หัวหน้าโครงการ ได้รับเชิญเป็นวิทยากรบรรยายกงานวิจัยสาขาทางคณิตศาสตร์ ในการประชุมวิชาการวิทยาศาสตร์ และ เทคโนโลยีแห่งประเทศไทยครั้งที่ 29 ช่วงวันที่ 20 – 22 ตุลาคม 2546 ณ. ศูนย์ประชุมเอนกประสงค์กาญจนาภิเษก จังหวัด ขอนแก่น ในงานนี้มีนักคณิตศาสตร์ และ ผู้สนใจเข้ารับฟังการบรรยายพอสมควร ในการบรรยายครั้งนี้ประสบผลสำเร็จเป็นอย่างดีเพราะมีนักวิจัยรุ่นใหม่ให้ความสนใจเป็นอย่างมาก ทำให้เกิดความคิดในการทำวิจัยเป็นเครือข่ายในอนาคตต่อไป นอกจากนี้ยังได้แลกเปลี่ยนความคิดเห็นในด้านงานวิจัยเกี่ยวกับปัญหาต่างๆ

4.2 หัวหน้าโครงการได้รับเชิญให้เป็น Reviewer ประจำวารสาร Zentralblatt Math ซึ่งเป็นวารสารที่มีชื่อเสียงระดับนานาชาติที่มีหน้าที่ review ผลงานทางคณิตศาสตร์ที่ได้รับการตีพิมพ์

4.3 หัวหน้าโครงการ ได้รับการแต่งตั้งจากศูนย์ส่งเสริมการวิจัยทางคณิตศาสตร์แห่งประเทศไทย ให้เป็น Chief Editor ของ Thai Journal of Mathematics มีหน้าที่กลั่นกรองผลงานของนักคณิตศาสตร์ที่ส่งผลงานมาเพื่อลงตีพิมพ์ และที่สำคัญคือมีเป้าหมายที่จะสนับสนุนนักวิจัยคนไทยรุ่นใหม่ให้ทำวิจัย และส่งผลงานลงตีพิมพ์ในวารสาร Thai Journal of Mathematics

4.4 ผู้ร่วมวิจัยคือ รศ.ดร.สุเทพ สวนใต้ ได้รับเชิญให้เป็นวิทยากรบรรยายงานวิจัยในที่ประชุมนานาชาติ International Conference on Mathematics and Its Applications ณ. Gadjah Mada University Yogyakarta ประเทศ อินโดนิเซีย ระหว่างวันที่ 14 - 17 กรกฎาคม 2546 ทั้งนี้ได้รับการสนับสนุนค่าเดินทางจาก สกว.

4.5 หัวหน้าโครงการได้รับเชิญจากสภาวิจัยแห่งชาติให้ร่วมงานแถลงข่าวผลงานวิจัยที่ได้รับรางวัลจากสภาวิจัยแห่งชาติ ในฐานะที่ได้เป็นผู้ที่ได้รับการคัดเลือกให้เป็นนักวิจัยดีเด่นแห่งชาติประจำปี 2547 ณ โรงแรมโซฟิเทล เซนทรัล ลาดพร้าว วันที่ 3 ธันวาคม 2547

4.6 ทีมงานวิจัย รศ.ดร. สุเทพ สวนใต้ ได้เข้าร่วมเสนอผลงานวิจัยในการประชุมนานาชาติ The Chinese Mathematician Conference 2004 ระหว่างวันที่ 17 – 23 ธันวาคม 2547 ณ. Chinese University of Hong Kong , Hong Kong

4.7 หัวหน้าโครงการได้ประชุมมอบหมายนโยบายในการทำงานวิจัยต่อหัวหน้าโครงการวิจัยย่อย ๆ และ ผู้ร่วมวิจัยในแต่ละโครงการย่อยทั้งหมด 6 โครงการ โดยได้กำหนดวัตถุประสงค์ของงานวิจัยในแต่ละรอบปีของแต่ละโครงการย่อย ตลอดจนกำหนดระเบียบวิธีวิจัย และ output ที่คาดว่าจะได้รับจากโครงการวิจัยแต่ละโครงการฯ และ หัวหน้าโครงการของแต่ละโครงการย่อยได้ดำเนินการวิจัยตามแผนที่กำหนด