



รายงานวิจัยฉบับสมบูรณ์

โครงการ : ทฤษฎีจุดตรึงสำหรับการส่งไม่เชิงเส้นและการประยุกต์
ไปยังปัญหาทางสมดุลและปัญหาการหาค่าที่เหมาะสม
Fixed Point Theory for Nonlinear Mappings and Applications to
Equilibrium and Optimization Problems

โดย ศาสตราจารย์ ดร.สุเทพ สวนใต้ และ คณะ
คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

สิงหาคม 2560

รายงานวิจัยฉบับสมบูรณ์

โครงการ : ทฤษฎีจุดตรึงสำหรับการส่งไม่เชิงเส้นและการประยุกต์
ไปยังปัญหาทางสมดุลและปัญหาการหาค่าที่เหมาะสม
Fixed Point Theory for Nonlinear Mappings and Applications to
Equilibrium and Optimization Problems

ศาสตราจารย์ ดร.สุเทพ สอนได้
คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย
และ มหาวิทยาลัยเชียงใหม่

(ความเห็นในรายงานนี้เป็นของผู้วิจัย
สกว. และ มหาวิทยาลัยเชียงใหม่ ไม่จำเป็นต้องเห็นด้วยเสมอไป)



**Fixed Point Theory for Nonlinear Mappings and Applications to Equilibrium
and Optimization Problems**

ทฤษฎีจุดตรึงสำหรับการส่งไม่เชิงเส้นและการประยุกต์ไปยังปัญหาทางสมดุล
และปัญหาการหาค่าเหมาะสม

Abstract : In this project, we first use the method which is different from the relaxed extragradient method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping, the set of solutions of equilibrium problems, and the set of solutions of a modified system of variational inequality. We construct an iterative process involving a hybrid pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of generalized nonexpansive multi-valued mappings and prove weak and strong convergence theorems of the proposed iterative process in Banach spaces. We prove some properties of a demicontractive mapping defined on a nonempty closed convex subset of a Hilbert space. By using these properties, we obtain strong convergence theorems of a hybrid shrinking projection method for finding a common element of the set of common fixed points of a finite family of demicontractive mappings and the set of common solutions of a finite family of variational inequality problems in a Hilbert space. We also study some fixed point iteration method in $CAT(0)$ spaces **and** prove a strong convergence theorem of a two-step viscosity iteration method for nonexpansive mappings in $CAT(0)$ spaces . Fixed point theory for multivalued mappings is another interesting topics which can be applied for many problems in applied science and economics. We study fixed point iteration methods for multi-valued nonexpansive mappings, quasi-nonexpansive mappings and lambda-hybrid mappings in Hilbert and Geodesic metric spaces. We propose a new proximal point algorithm for finding a common element of the set of fixed points of nonexpansive single-valued mappings, the set of fixed points of nonexpansive multi-valued mappings, and the set of minimizers of convex and lower semi-continuous functions. We obtain Δ -convergence and strong convergence of the proposed

algorithm to a common element of the three sets in $CAT(0)$ spaces. Moreover, we introduce a new class of set-valued mappings in a Hilbert space, called *Lambda-hybrid*, and prove some fixed point theorems for mappings in this class and we establish weak convergence of modified SP-iteration to a fixed point of *Lambda-hybrid* set-valued mappings in Hilbert spaces. We also propose a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in a $CAT(0)$ space and we prove strong convergence of the viscosity approximation method for multivalued nonexpansive mappings in $CAT(0)$ spaces. Fixed point theory with directed graphs is an interesting topic for many mathematicians. In this project, we study fixed point theorems and iteration methods for some contractive mappings and G -nonexpansive mappings in metric spaces and Banach spaces with directed graph. Split common fixed point problems, null point and equilibrium problems are very useful for applications in many inverse problems. In this direction, we consider a split common fixed point and null point problem which includes the split common fixed point problem, the split common null problem and other problems related to the fixed point problem and the null point problem. We introduce an algorithm for studying the split common fixed point and null problem for demicontractive operators and maximal monotone operators in real Hilbert spaces. Moreover, we also apply our main results to the split equilibrium problem. We also introduce and study iterative schemes for solving split equilibrium problems and fixed point problems of non-spreading multi-valued mappings in Hilbert spaces. We introduce and study iterative algorithms for solving split mixed equilibrium problems and fixed point problems of Λ -hybrid multivalued mappings in real Hilbert spaces and prove that the proposed iterative algorithm converges weakly to a common solution of the considered problems. We also use the hybrid method for the equilibrium problem and a family of generalized nonexpansive mappings in Banach spaces. We also discuss some algorithms for finding a common element of the set of common minimizers of a finite family of convex and lower semi-continuous functions and the set of common fixed points of a finite family of nonexpansive mappings in complete $CAT(0)$ spaces, and study two-stage stochastic quadratic symmetric programming to handle uncertainty in data defining (Deterministic) symmetric programs in which a quadratic function is minimized over the intersection of an affine set and a symmetric cone with finite event space. By using some fixed point theorems, we study the existence and uniqueness of solutions for impulsive multi-orders Caputo-Hadamard fractional differential equations equipped with boundary and integral conditions. We also study existence of solutions of some Fractional Differential Equations and Integro-difference Equations.

Keywords of the Project :

Fixed point problems, coincidence point theorems, coupled fixed point, Equilibrium problems, variational inequalities, minimization problems, nonexpansive mappings, multivalued mappings, monotone mappings, complete metric spaces with graphs, partially ordered sets, split null point problems, split common fixed point problems, fractional differential equations, Langevin fractional differential inclusions, fractional q -difference equations, integral boundary conditions, nonlocal boundary conditions.

Executive Summary

1. **Project Title :** **Fixed Point Theory for Nonlinear Mappings and Applications to Equilibrium and Optimization Problems**

ทฤษฎีจุดตรึงสำหรับการส่งไม่เชิงเส้นและการประยุกต์ไปยังปัญหาทางสมดุล และปัญหาการหาค่าที่เหมาะสม

2. **Research Team:**

(1) **Head of the Project :** Prof. Dr. Suthep Suantai
ศาสตราจารย์ ดร. สุเทพ สวนใต้

Researcher :

- (1) Assist. Prof. Dr. Attapol Kaewkoa
ผู้ช่วยศาสตราจารย์ ดร. อรรถพล แก้วขาว
- (2) Assist. Prof. Dr. Bancha panyanak
ผู้ช่วยศาสตราจารย์ ดร. บัญชา ปัญญานาค
- (3) Assist. Prof. Dr. Thanasak Muagthonhlang
ผู้ช่วยศาสตราจารย์ ดร. ธนะศักดิ์ หมวกทองกลาง
- (4) Assoc. Prof. Dr. Narin Petrot
รองศาสตราจารย์ ดร. นรินทร์ เพ็ชรโรจน์
- (5) Assoc. Prof. Dr. Jassada Teeraboo
รองศาสตราจารย์ ดร. เจษฎา ธีรบุญ
- (6) Dr. Pakdee Chareansawan
อ. ดร. ปกติ เจริญสวรรค์
- (7) Dr. Jakkrid Klin-Eam
อ. ดร. จักรกฤษณ์ กลิ่นเอี่ยม

(8) Asist. Prof. Dr. Atid Kangtunkarn

ผศ.ดร. อาทิตย์ แข็งธัญการ

(9) Asist. Prof. Dr. Withun Peungrattana

ผศ. ดร. วิฑูรย์ พึ่งรัตนา

(10) Dr. Watcharaporn Chalamjiak

อ. ดร. วัชรภรณ์ ช่อลำเจียก

Assistance of Researchers

10 Ph.D students , 10 Master Students

3. Importance of the research problem, Principles, Theory, Rationale and/or Hypotheses

Fixed point theory plays very important role in nonlinear analysis and applications. Most of important nonlinear problems in science and applied science reduce to solving a given equation or inequality which in turn may be reduced to finding the fixed points of a certain operator and fixed point theory plays an important role for solving solutions of those problems. There are mainly two important questions arising for solving those problems. The first one is the existence of solutions of those problems, and the second one is that how can we find or approximate those solutions of such problems. Many mathematicians are interested in studying and investigating those two problems. They can discover many new theorems which are very useful for solving many problems in science and applied science and economics.

Having in view that many of the most important nonlinear problems of science and applied science reduce to solving a given equation or inequality which in turn may be reduced to finding the fixed points of a certain operator and fixed point theory plays an important role in solving solutions of those problems. The metrical fixed point theory has developed significantly in the second part of the 20th century. There are big group of mathematicians who are interested in studying and investigating those two problems. They can discover many new theorems which are very useful for solving many problems in science and applied science, optimization and economics.

The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator

$T : X \rightarrow X$, in order to obtain results on;

1. the existence (and uniqueness) of fixed points;
2. the structure of the fixed point sets;
3. the approximation of fixed points.

The ambient spaces X involved in fixed point theorems cover a variety of spaces: metric space, normed linear space, generalized metric space, uniform space, linear topological space etc., while the conditions imposed on the operator T are generally metrical or compactness type conditions.

A plethora of metrical fixed point theorems have been obtained, more or less important from a theoretical point of view, which establish usually the existence, or the existence and uniqueness of fixed points for a certain nonlinear mappings. Among these fixed point theorems, only a small number are important from a practical point of view, that, they offer a constructive method for finding the fixed points.

However, from a practical point of view it is important not only to know the fixed point exists (and, possible, is unique), but also to be able to construct that fixed points. As the constructive methods used in metrical fixed point theory are prevaillingly iterative procedures, that is, *approximate* methods, it is also of crucial importance to have a priori or/and a posteriori error estimates (or rate of convergence) for such method. For example, the Banach fixed point theorem concerns certain mappings (contractions) of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point and it also given a constructive procedure for obtaining better and better approximations to the fixed point (the solution of the practical problem). By definition, this is a method such that we choose an arbitrary x_0 in a given set and calculate recursively a sequence $x_0, x_1, x_2, \dots$ from a relation of the form

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, 3, \dots$$

That is, we choose an arbitrary x_0 and determine successively

$x_1 = Tx_0, x_2 = Tx_1, \dots$. It is also known as the *Picard iteration* starting at x_0 .

Iteration procedures are used in nearly every branch of applied mathematics, and convergence proofs and error estimates are very often obtained by an application of Banach fixed point theorem (or more difficult fixed point theorems). Construction of fixed point iteration processes of nonlinear mappings are an important subject in the theory of nonlinear mappings, and find application in a number of applied areas. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings, relatively nonexpansive mappings, hemi-relatively nonexpansive mappings, generalized nonexpansive mappings and maximal monotone operators in various space have been studied by many mathematicians. There are many new nonlinear mappings arising from various problems in science and applied science, optimization and economics. So the fixed point problems concerning those mappings are interesting for investigating. So we aim to investigate the existence problem, structure of fixed points set, and approximation method for finding fixed point of those new mappings. The mappings that we are interested in studying and have more applications are nonexpansive mappings,

quasi- nonexpansive mappings, asymptotically nonexpansive mappings, quasi-asymptotically nonexpansive mappings, pseudocontractive mappings, quasi-pseudocontractive mappings, relatively nonexpansive mappings, hemi-relatively nonexpansive mappings, generalized nonexpansive mappings, maximal monotone operators, and other generalizations of these mappings.

There are many problems in economics, finance, transportation, network and structural analysis, elasticity, and optimization which can be solved by using equilibrium theory. The equilibrium problems cover many important problems such as minimization problems, variational inequality problems. In 199, Combettes and Hirstoaga first studied and constructed an iterative method for finding a solution of the equilibrium problem in a Hilbert space. They proved strong convergence theorem of their iterative method under some sufficient conditions. Their main results can be applied for solving variational inequality problem and they showed that there is a close relation between the fixed point problem and the equilibrium problem. So, many techniques in fixed point theory can be applied for the equilibrium problems. Many problems in science and applies science, physics, economics and others areas are more complicated. The solutions of those problems concern intersection of fixed point problems, variational inequality and equilibrium problems. So, the problem of finding a common solutions of those problems are very useful for applications and widely studied by many mathematicians later.

Motivated by many previous works concerning the equilibrium problems, fixed point problems and other important problems in science, applied science and economics, we are interested to study and construct new methods for solving those problems.

4. Research Objectives

- 4.1 To investigate various properties of important classes of nonlinear mappings that related to fixed point problem and give some sufficient conditions in order to obtain new existence theorems for those nonlinear mappings in Hilbert spaces, Banach spaces, complete metric spaces, convex metric spaces, R-tree and CAT(k) spaces.
- 4.2 To construct new iterative methods for approximation fixed points of various kinds of nonlinear mappings and finding a common fixed point of a finite and infinite family of those nonlinear mappings studied in the objective (1), and give

some sufficient conditions for proving weak and strong convergence theorems of the studied iterative methods for those mappings. Moreover, we also study the rate of convergence and stability of the studied iterative methods.

- 4.3 To prove existence results concerning equilibrium and optimization problems and construct new iterative methods for finding a solution of those problems and a common element of the set of equilibrium problems, variational inequality problems and fixed point problems, and give some control conditions to prove weak and strong convergence theorems of those methods.
- 4.4 To construct and study new types of monotone multi-valued mappings in partially ordered metric spaces and find some sufficient conditions for which each types of the studied mappings has a fixed point and a maximal and minimal fixed points.
- 4.5 To construct and study new classes of graph-preserving multi-valued mappings in complete metric spaces with graphs and find some sufficient conditions for which each type of the studied mappings has a fixed point.
- 4.6 To construct and investigate new types mixed monotone multi-valued mappings in partially ordered metric spaces and find some sufficient conditions for which each type of mappings has a coupled fixed point, coupled coincidence point and a coupled common fixed point.
- 4.7 To construct and investigate new types mixed graph-preserving multi-valued mappings in complete metric spaces with graphs and find some sufficient conditions for which each type of mappings has a coupled fixed point, a coupled coincidence point and a coupled common fixed point.
- 4.8 To investigate existence of common fixed points for semigroup of nonlinear mappings in Hilbert spaces, Banach spaces, complete metric spaces, convex metric spaces, R-tree and $CAT(k)$ spaces.
- 4.9 To apply the new knowledge (new theorems) obtained from the project for solving some problems in game theory, Nash equilibria of non-cooperative games, existence problems for ordinary differential equations.
- 4.10 To build and develop new and young researchers in the area of fixed point theory and applications for developing human resource and high standard academic researches.
- 4.11. To create research network both inside and outside of the country for making cooperation in doing research in this area.

Chapter I

Some Fixed Point Iterative Methods for Some Nonlinear Mappings in Metric and Banach Spaces

In this chapter, we first use the method which is different from the relaxed extragradient method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping, the set of solutions of equilibrium problems, and the set of solutions of a modified system of variational inequality. We prove a strong convergence theorem involving a finite family of nonspreading mappings. We also give a numerical example to encourage our main theorem.

After that we construct an iterative process involving a hybrid pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of generalized nonexpansive multi-valued mappings and prove weak and strong convergence theorems of the proposed iterative process in Banach spaces. Our main results extend and generalize many results in the literature. In the third part of this chapter, we prove some properties of a demicontractive mapping defined on a nonempty closed convex subset of a Hilbert space. By using these properties, we obtain strong convergence theorems of a hybrid shrinking projection method for finding a common element of the set of common fixed points of a finite family of demicontractive mappings and the set of common solutions of a finite family of variational inequality problems in a Hilbert space. A numerical example is presented to illustrate the proposed method and convergence result. Our results improve and extend the corresponding results existing in the literature.

In the final part of the chapter, we study some fixed point iteration method in $CAT(0)$ spaces. We prove a strong convergence theorem of a two-step viscosity iteration method for nonexpansive mappings in $CAT(0)$ spaces without the condition on the property N .

1.1 A new technique for convergence theorem of fixed point problem of quasi-nonexpansive mapping

In this section, we use the method different from the relaxed extragradient method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping, the set of solutions of equilibrium problems, and the set of solutions of a modified system of variational inequalities without demiclosed condition of W and $W_\omega := (1 - \omega)I + \omega W$, where W is a quasi-nonexpansive mapping and $\omega \in (0, 1)$ in the framework of Hilbert space. By using our main result, we obtain a strong convergence theorem involving a finite family of nonspreading mappings and another corollary. Moreover, we give a numerical example to encourage our main theorem.

-

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H . Recall that the mapping $W : C \rightarrow C$ is called *quasi-nonexpansive* if

$$\|Wp - q\| \leq \|p - q\|,$$

for all $p \in C$ and $q \in F(W)$. We denote by $F(W)$ the set of fixed points of W . Fixed point problems have been widely studied and developed in the literature.

Let Ψ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $\Psi : C \times C \rightarrow \mathbb{R}$ is to find $p \in C$ such that

$$\Psi(p, \zeta) \geq 0, \quad \forall \zeta \in C. \tag{1.1}$$

We denote the set of solutions of (1.1) by $EP(\Psi)$. Equilibrium problems were introduced by Blum and Oettli [1] in 1994 and included many well-known problems such as the variational inequality problem, the optimization problem, and the nonexpansive mapping and fixed point problem.

A mapping $D_1 : C \rightarrow H$ is called d_1 -inverse strongly monotone if there exists a positive real number $d_1 > 0$ such that

$$\langle D_1 p - D_1 \zeta, p - \zeta \rangle \geq d_1 \|D_1 p - D_1 \zeta\|^2,$$

for all $p, \zeta \in C$.

Let $B : C \rightarrow H$. The *variational inequality* is to find a point $\phi \in C$ such that

$$\langle B\phi, \psi - \phi \rangle \geq 0, \quad (1.2)$$

for all $\psi \in C$. The set of solutions of (1.2) is denoted by $\text{VIP}(C, B)$. The variational inequalities were initially studied and introduced by Lions and Stampacchia [2].

The concept of quasi-nonexpansive mapping was investigated by Diaz and Metcalf [3]. In 2007, Su *et al.* [4] introduced strong convergence theorems for quasi-nonexpansive mappings, the monotone hybrid iteration method used to approximate the fixed point of quasi-nonexpansive mappings. In 2011, Tian and Jin [5] introduced an iterative method of a quasi-nonexpansive mapping in the framework of Hilbert space. They proved the strong convergence theorem of iterative scheme $\{p_n\}$ generated by (1.3) as follows.

Theorem 1.1 *Let H be a real Hilbert space, let F be a κ -Lipschitzian and η -strongly monotone operator on H with $\kappa > 0$, $\eta > 0$ and let W be a quasi-nonexpansive mapping on H , and f is a L -Lipschitzian mapping with coefficient $L > 0$ for all $p, \zeta \in H$. Assume the set $F(W)$ of fixed points of W is nonempty closed and convex. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L$ and start with an arbitrary chosen $p_0 \in H$, let the sequence $\{p_n\}$ be generated by*

$$p_{n+1} = \alpha_n \gamma f(p_n) + (I - \alpha_n \mu F) W_\omega p_n, \quad (1.3)$$

where the sequence $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Also $\omega \in (0, \frac{1}{2})$, $W_\omega := (1 - \omega)I + \omega W$ with two conditions on W :

1. $\|Wp - q\| \leq \|p - q\|$ for any $p \in H$, and $q \in F(W)$; this means that W is a quasi-nonexpansive mapping;
 2. W is demiclosed on H ; that is, if $\{\zeta_k\} \subset H$, $\zeta_k \rightarrow \xi$, and $(I - W)\zeta_k \rightarrow 0$, then $\xi \in F(W)$.
- Then $\{p_n\}$ converges strongly to the $p^* \in F(W)$ which is the unique solution of the VIP:

$$\langle (\mu F - \gamma f)p^*, p - p^* \rangle \leq 0, \quad \forall p \in F(W).$$

Many strong convergence theorems of quasi-nonexpansive mapping W were proved by assuming the following conditions:

1. $W_\omega := (1 - \omega)I + \omega W$ for all $\omega \in (0, \frac{1}{2})$,
2. W is demiclosed on H .

In 2012, Dong *et al.* [6] proved strong convergence theorem by using a relaxed extragradient method as follows.

Theorem 1.2 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let the mappings $D_1, D_2 : C \rightarrow H$ be d_1 -inverse strongly monotone and d_2 -inverse strongly monotone, respectively. Let Ψ be a bifunction from $C \times C \rightarrow \mathbb{R}$ satisfying (J1)-(J4) and let $\{W_n\}_{n=1}^{\infty} : C \rightarrow C$ be a countable family of nonexpansive mappings such that $\Omega :=$*

$\bigcap_{n=1}^{\infty} F(W_n) \cap \text{EP}(\Psi) \cap F(G) \neq \emptyset$. Let $f : C \rightarrow C$ be a contraction with coefficient $\rho \in (0, 1/2)$. Set $\beta_0 = 1$. For given $p_1 \in C$ arbitrarily, let the sequences $\{p_n\}$, $\{\zeta_n\}$, $\{\xi_n\}$, and $\{\phi_n\}$ be generated by

$$\begin{cases} \Psi(\phi_n, \zeta) + \frac{1}{g_n} \langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ \xi_n = P_C(\phi_n - \lambda_B D_2 \phi_n), \\ \zeta_n = \alpha_n f(p_n) + (1 - \alpha_n) P_C(\xi_n - \lambda_A D_1 \xi_n), \\ p_{n+1} = \beta_n p_n + \sigma_n \sum_{i=1}^{\infty} (\beta_{i-1} - \beta_i) W_i \zeta_n \\ \quad + (1 - \beta_n)(1 - \sigma_n) P_C(\xi_n - \lambda_A D_1 \xi_n), & \forall n \in \mathbb{N}, \end{cases} \quad (1.4)$$

where $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$, and the sequences $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$, $\{\sigma_n\} \subset [0, 1]$, and $\{g_n\} \subset (r, \infty)$, $r > 0$, are such that

- (i) $\{\beta_n\}$ is strictly decreasing,
- (ii) $0 < \liminf_{n \rightarrow \infty} \beta_n < \limsup_{n \rightarrow \infty} \beta_n < 1$,
- (iii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (iv) $\sigma_n > 1/2(1 - \rho)$, $\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty$,
- (v) $\sum_{n=1}^{\infty} |g_n - g_{n-1}| < \infty$.

Then the sequence $\{p_n\}$ generated by (1.4) converges strongly to $p^* = P_{\Omega} \cdot f(p^*)$, and (p^*, ξ^*) is a solution of the general system of variational inequalities (1.5) where $\xi^* = P_C(p^* - \lambda_B D_2 p^*)$.

Many authors used the extragradient method to prove fixed point theorem of nonlinear mappings.

Let $D_1, D_2 : C \rightarrow H$ be two mappings. In 2008, Ceng *et al.* [7] introduced a relaxed extragradient method for finding solutions of problem $(p^*, \xi^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_A D_1 \xi^* + p^* - \xi^*, p - p^* \rangle \geq 0, & \forall p \in C, \\ \langle \lambda_B D_2 p^* + \xi^* - p^*, p - \xi^* \rangle \geq 0, & \forall p \in C, \end{cases} \quad (1.5)$$

which is called a system of variational inequalities where $\lambda_A, \lambda_B > 0$.

In 2013, Kangtunyakarn [8] modified (1.5) for finding $(p^*, \xi^*) \in C \times C$ such that

$$\begin{cases} \langle p^* - (I - \lambda_A D_1)(ap^* + (1 - a)\xi^*), p - p^* \rangle \geq 0, & \forall p \in C, \\ \langle \xi^* - (I - \lambda_B D_2)p^*, p - \xi^* \rangle \geq 0, & \forall p \in C, \end{cases} \quad (1.6)$$

which is called a modification of system of variational inequalities, for every $\lambda_A, \lambda_B > 0$ and $a \in [0, 1]$. If $a = 0$, (1.6) reduces to (1.5). He introduced the relation between solutions of (1.6) and fixed point of the mapping G as follows.

Lemma 1.3 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \rightarrow H$ be mappings. For every $\lambda_A, \lambda_B > 0$ and $a \in [0, 1]$, the following statements are equivalent:*

1. $(p^*, \xi^*) \in C \times C$ is a solution of problem (1.6),
2. p^* is a fixed point of the mapping $G : C \rightarrow C$, i.e., $p^* \in F(G)$, defined by

$$G(p) = P_C(I - \lambda_A D_1)(ap + (1 - a)P_C(I - \lambda_B D_2)p),$$

where $\xi^* = P_C(I - \lambda_B D_2)p^*$.

After we investigated Theorem 1.1, Theorem 1.2 and researchers in the same direction, we have the questions as follows:

- (1) Can we prove strong convergence theorem without demiclosed condition and $W_\omega := (1 - \omega)I + \omega W$, where W is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$ in the framework of Hilbert space?
- (2) Can we prove strong convergence theorem without relaxed extragradient method?

In this paper, we give the answer for the mentioned questions and introduce the method of iterative scheme $\{p_n\}$ for finding a common element of the set of fixed points of a quasi-nonexpansive mapping, the set of solutions of equilibrium problems and the set of solutions of a modified system of variational inequalities. Applying our main result, we prove strong convergence theorem involving a finite family of nonspreading mappings and another corollary. Moreover, We also give a numerical example to support our main theorem.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. In this paper, we use the symbol of weak and strong convergence by ' $\rightharpoonup$ ' and ' $\rightarrow$ ', respectively. For every $p \in H$, there exists a unique nearest point $P_C p$ in C such that $\|p - P_C p\| \leq \|p - \zeta\|$ for all $\zeta \in C$. P_C is called the *metric projection* of H onto C .

Remark 2.1 It is well known that metric projection P_C has the following properties:

1. P_C is firmly nonexpansive, i.e.,

$$\|P_C p - P_C \zeta\|^2 \leq \langle P_C p - P_C \zeta, p - \zeta \rangle, \quad \forall p, \zeta \in H.$$

2. For each $p \in H$,

$$\xi = P_C(p) \Leftrightarrow \langle p - \xi, \xi - \zeta \rangle \geq 0, \quad \forall \zeta \in C.$$

Recall that H satisfies *Opial's condition* [9], i.e., for any sequence $\{p_n\}$ with $p_n \rightharpoonup p$, the inequality

$$\liminf_{n \rightarrow \infty} \|p_n - p\| < \liminf_{n \rightarrow \infty} \|p_n - \zeta\|$$

holds for every $\zeta \in H$ with $\zeta \neq p$.

Lemma 2.2 *Let H be a real Hilbert space. Then we have the following well-known results:*

1. $\|p \pm \zeta\|^2 = \|p\|^2 \pm 2\langle p, \zeta \rangle + \|\zeta\|^2$,
2. $\|p + \zeta\|^2 \leq \|p\|^2 + 2\langle \zeta, p + \zeta \rangle$,

for all $p, \zeta \in H$.

Lemma 2.3 ([10]) *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $p, \zeta, \xi \in E$ and $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, we have*

$$\begin{aligned} \|\alpha_1 p + \alpha_2 \zeta + \alpha_3 \xi\|^2 &= \alpha_1 \|p\|^2 + \alpha_2 \|\zeta\|^2 + \alpha_3 \|\xi\|^2 - \alpha_1 \alpha_2 \|p - \zeta\|^2 \\ &\quad - \alpha_1 \alpha_3 \|p - \xi\|^2 - \alpha_2 \alpha_3 \|\zeta - \xi\|^2. \end{aligned}$$

For solving the equilibrium problem, we assume that the bifunction $\Psi : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions:

- (J1) $\Psi(p, p) = 0$ for all $p \in C$;
- (J2) Ψ is monotone, i.e., $\Psi(p, \zeta) + \Psi(\zeta, p) \leq 0$ for all $p, \zeta \in C$;
- (J3) for each $p, \zeta, \xi \in C$,

$$\lim_{t \downarrow 0} \Psi(t\xi + (1-t)p, \zeta) \leq \Psi(p, \zeta);$$

- (J4) for each $p \in C$, $\zeta \mapsto \Psi(p, \zeta)$ is convex and lower semicontinuous.

Lemma 2.4 ([1]) *Let C be a nonempty closed convex subset of H and let Ψ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (J1)-(J4). Let $r > 0$ and $p \in H$. Then there exists $\xi \in C$ such that*

$$\Psi(\xi, \zeta) + \frac{1}{r} \langle \zeta - \xi, \xi - p \rangle \geq 0, \quad \forall \zeta \in C.$$

Lemma 2.5 ([11]) *Assume that $\Psi : C \times C \rightarrow \mathbb{R}$ satisfies (J1)-(J4). For $r > 0$, define a mapping $W_r : H \rightarrow C$ as follows:*

$$W_r(p) = \left\{ \xi \in C : \Psi(\xi, \zeta) + \frac{1}{r} \langle \zeta - \xi, \xi - p \rangle \geq 0, \forall \zeta \in C \right\},$$

for all $p \in H$. Then the following hold:

- (1) W_r is single-valued;
- (2) W_r is firmly nonexpansive, i.e., for any $p, \zeta \in H$,

$$\|W_r(p) - W_r(\zeta)\|^2 \leq \langle W_r(p) - W_r(\zeta), p - \zeta \rangle;$$

- (3) $F(W_r) = \text{EP}(\Psi)$;
- (4) $\text{EP}(\Psi)$ is closed and convex.

Lemma 2.6 ([12]) *Let $\{h_n\}$ be a sequence of nonnegative real numbers satisfying*

$$h_{n+1} \leq (1 - \alpha_n)h_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (2) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} h_n = 0$.

Lemma 2.7 ([13]) *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let D_1 be a mapping of C into H . Let $u \in C$. Then for $\lambda > 0$,*

$$u = P_C(I - \lambda D_1)u \quad \Leftrightarrow \quad u \in \text{VIP}(C, D_1),$$

where P_C is the metric projection of H onto C .

Lemma 2.8 ([14]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $W : C \rightarrow C$ be a quasi-nonexpansive mapping with $F(W) \neq \emptyset$. Then $\text{VIP}(C, I - W) = F(W)$.*

Remark 2.9 From Lemmas 2.7 and 2.8, we have

$$F(W) = \text{VIP}(C, I - W) = F(P_C(I - \lambda(I - W))),$$

for all $\lambda > 0$.

3 Main result

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H , let $\Psi_1, \Psi_2 : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (J1)-(J4) and let $W : C \rightarrow C$ be a quasi-nonexpansive mapping. Let $D_1, D_2 : C \rightarrow H$ be d_1, d_2 -inverse strongly monotone mappings, respectively. Define the mapping $G : C \rightarrow C$ by $G(p) = P_C(I - \lambda_A D_1)(ap + (1 - a)P_C(I - \lambda_B D_2)p)$ for all $p \in C$ and $a \in [0, 1]$. Assume $\mathcal{F} = \text{EP}(\Psi_1) \cap \text{EP}(\Psi_2) \cap F(G) \cap F(W) \neq \emptyset$. Suppose that $p_1, u \in C$ and let $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ be sequences generated by*

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n} \langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n} \langle \zeta - \psi_n, \psi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases} \quad (3.1)$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ for some $c, d > 0$ and for all $n \geq 1$,
- (iii) $0 < e \leq g_n, h_n \leq f$ for some $e, f > 0$ and for all $n \geq 1$,
- (iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
 $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F} \cup W}$ and (p_0, ξ_0) is a solution of (1.6) where $\xi_0 = P_C(I - \lambda_B D_2)p_0$.

Proof First, we show that G is a nonexpansive mapping. Let $p, \zeta \in C$. Since D_1, D_2 are d_1, d_2 -inverse strongly monotone, $\lambda_A \in (0, 2d_1)$, and $\lambda_B \in (0, 2d_2)$, we have

$$\begin{aligned} & \| (I - \lambda_A D_1)p - (I - \lambda_A D_1)\zeta \|^2 \\ &= \| p - \zeta \|^2 - 2\lambda_A \langle p - \zeta, D_1 p - D_1 \zeta \rangle + \lambda_A^2 \| D_1 p - D_1 \zeta \|^2 \\ &\leq \| p - \zeta \|^2 - 2d_1 \lambda_A \| D_1 p - D_1 \zeta \|^2 + \lambda_A^2 \| D_1 p - D_1 \zeta \|^2 \\ &= \| p - \zeta \|^2 + \lambda_A (\lambda_A - 2d_1) \| D_1 p - D_1 \zeta \|^2 \\ &\leq \| p - \zeta \|^2. \end{aligned}$$

Then $I - \lambda_A D_1$ is a nonexpansive mapping. Similarly $I - \lambda_B D_2$ is a nonexpansive mapping. Then G is a nonexpansive mapping.

Next, we show $\{p_n\}$ is bounded. Let $\xi \in \mathcal{F}$, then $\phi_n = W_{g_n} p_n$ and $\psi_n = W_{h_n} p_n$. It is clear that $\|\phi_n - \xi\| \leq \|p_n - \xi\|$ and $\|\psi_n - \xi\| \leq \|p_n - \xi\|$. By Remark 2.9, we have

$$\xi \in F(P_C(I - \lambda_n(I - W))). \quad (3.2)$$

Observe that

$$\begin{aligned} \|W\phi_n - \xi\|^2 &= \|(\phi_n - \xi) - (I - W)\phi_n\|^2 \\ &= \|\phi_n - \xi\|^2 - 2\langle \phi_n - \xi, (I - W)\phi_n \rangle + \|(I - W)\phi_n\|^2 \\ &\leq \|\phi_n - \xi\|^2. \end{aligned}$$

It implies that

$$\|(I - W)\phi_n\|^2 \leq 2\langle \phi_n - \xi, (I - W)\phi_n \rangle. \quad (3.3)$$

From (3.2) and (3.3), we have

$$\begin{aligned} \|P_C(I - \lambda_n(I - W))\phi_n - \xi\|^2 &= \|P_C(I - \lambda_n(I - W))\phi_n - P_C(I - \lambda_n(I - W))\xi\|^2 \\ &\leq \|(\phi_n - \xi) - \lambda_n((I - W)\phi_n - (I - W)\xi)\|^2 \\ &= \|\phi_n - \xi\|^2 - 2\lambda_n\langle \phi_n - \xi, (I - W)\phi_n \rangle \\ &\quad + \lambda_n^2 \|(I - W)\phi_n\|^2 \\ &\leq \|\phi_n - \xi\|^2 + \lambda_n(\lambda_n - 1)\|(I - W)\phi_n\|^2 \\ &\leq \|\phi_n - \xi\|^2. \end{aligned} \quad (3.4)$$

From the definition of p_n and (3.4), we have

$$\begin{aligned} \|p_{n+1} - \xi\| &= \|\alpha_n(u - \xi) + \beta_n(p_n - \xi) + \gamma_n(P_C(I - \lambda_n(I - W))\phi_n - \xi) \\ &\quad + \delta_n(G(\psi_n) - \xi)\| \\ &\leq \alpha_n\|u - \xi\| + \beta_n\|p_n - \xi\| + \gamma_n\|P_C(I - \lambda_n(I - W))\phi_n - \xi\| \\ &\quad + \delta_n\|G(\psi_n) - \xi\| \\ &\leq \alpha_n\|u - \xi\| + \beta_n\|p_n - \xi\| + \gamma_n\|\phi_n - \xi\| + \delta_n\|\psi_n - \xi\| \\ &\leq \alpha_n\|u - \xi\| + \beta_n\|p_n - \xi\| + \gamma_n\|p_n - \xi\| + \delta_n\|p_n - \xi\| \\ &= \alpha_n\|u - \xi\| + (1 - \alpha_n)\|p_n - \xi\|. \end{aligned}$$

By induction, we can conclude that

$$\|p_n - \xi\| \leq \max\{\|u - \xi\|, \|p_1 - \xi\|\},$$

for all $n \geq 1$. This implies that the sequence $\{p_n\}$ is bounded and so are $\{\phi_n\}$, $\{\psi_n\}$, $\{(I - W)\phi_n\}$, and $\{P_C(I - \lambda_n(I - W))\phi_n\}$.

Then we show that $\lim_{n \rightarrow \infty} \|p_{n+1} - p_n\| = 0$.

From the definition of p_n and nonexpansiveness of G , we have

$$\begin{aligned}
\|p_{n+1} - p_n\| &= \|(\alpha_n - \alpha_{n-1})u + \beta_n(p_n - p_{n-1}) + (\beta_n - \beta_{n-1})p_{n-1} \\
&\quad + \gamma_n(P_C(I - \lambda_n(I - W))\phi_n - P_C(I - \lambda_{n-1}(I - W))\phi_{n-1}) \\
&\quad + (\gamma_n - \gamma_{n-1})P_C(I - \lambda_{n-1}(I - W))\phi_{n-1} \\
&\quad + \delta_n(G(\psi_n) - G(\psi_{n-1})) + (\delta_n - \delta_{n-1})G(\psi_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}|\|p_{n-1}\| \\
&\quad + \gamma_n\|P_C(I - \lambda_n(I - W))\phi_n - P_C(I - \lambda_{n-1}(I - W))\phi_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|P_C(I - \lambda_{n-1}(I - W))\phi_{n-1}\| \\
&\quad + \delta_n\|G(\psi_n) - G(\psi_{n-1})\| + |\delta_n - \delta_{n-1}|\|G(\psi_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}|\|p_{n-1}\| \\
&\quad + \gamma_n\|(\phi_n - \phi_{n-1}) - (\lambda_n(I - W)\phi_n - \lambda_n(I - W)\phi_{n-1}) \\
&\quad - (\lambda_n(I - W)\phi_{n-1} - \lambda_{n-1}(I - W)\phi_{n-1})\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|P_C(I - \lambda_{n-1}(I - W))\phi_{n-1}\| + \delta_n\|\psi_n - \psi_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}|\|G(\psi_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}|\|p_{n-1}\| \\
&\quad + \gamma_n\|\phi_n - \phi_{n-1}\| + \lambda_n\|(I - W)\phi_n - (I - W)\phi_{n-1}\| \\
&\quad + |\lambda_n - \lambda_{n-1}|\|(I - W)\phi_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|P_C(I - \lambda_{n-1}(I - W))\phi_{n-1}\| + \delta_n\|\psi_n - \psi_{n-1}\| \\
&\quad + |\delta_n - \delta_{n-1}|\|G(\psi_{n-1})\|. \tag{3.5}
\end{aligned}$$

On the other hand, from $\phi_n = W_{g_n}p_n$ and $\phi_{n+1} = W_{g_{n+1}}p_{n+1}$, we have

$$\Psi_1(\phi_n, \zeta) + \frac{1}{g_n}\langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, \quad \forall \zeta \in C \tag{3.6}$$

and

$$\Psi_1(\phi_{n+1}, \zeta) + \frac{1}{g_{n+1}}\langle \zeta - \phi_{n+1}, \phi_{n+1} - p_{n+1} \rangle \geq 0, \quad \forall \zeta \in C. \tag{3.7}$$

Putting $\zeta = \phi_{n+1}$ in (3.6) and $\zeta = \phi_n$ in (3.7), we have

$$\Psi_1(\phi_n, \phi_{n+1}) + \frac{1}{g_n}\langle \phi_{n+1} - \phi_n, \phi_n - p_n \rangle \geq 0$$

and

$$\Psi_1(\phi_{n+1}, \phi_n) + \frac{1}{g_{n+1}}\langle \phi_n - \phi_{n+1}, \phi_{n+1} - p_{n+1} \rangle \geq 0.$$

From (J2), we have

$$\left\langle \phi_{n+1} - \phi_n, \frac{\phi_n - p_n}{g_n} - \frac{\phi_{n+1} - p_{n+1}}{g_{n+1}} \right\rangle \geq 0.$$

So

$$\left\langle \phi_{n+1} - \phi_n, \phi_n - \phi_{n+1} + \phi_{n+1} - p_n - \frac{g_n}{g_{n+1}}(\phi_{n+1} - p_{n+1}) \right\rangle \geq 0.$$

Then

$$\begin{aligned} \|\phi_{n+1} - \phi_n\|^2 &\leq \left\langle \phi_{n+1} - \phi_n, p_{n+1} - p_n + \phi_{n+1} - p_{n+1} - \frac{g_n}{g_{n+1}}(\phi_{n+1} - p_{n+1}) \right\rangle \\ &= \left\langle \phi_{n+1} - \phi_n, p_{n+1} - p_n + \left(1 - \frac{g_n}{g_{n+1}}\right)(\phi_{n+1} - p_{n+1}) \right\rangle \\ &\leq \|\phi_{n+1} - \phi_n\| \left(\|p_{n+1} - p_n\| + \left|1 - \frac{g_n}{g_{n+1}}\right| \|\phi_{n+1} - p_{n+1}\| \right), \end{aligned}$$

and hence

$$\begin{aligned} \|\phi_{n+1} - \phi_n\| &\leq \|p_{n+1} - p_n\| + \frac{1}{g_{n+1}} |g_{n+1} - g_n| \|\phi_{n+1} - p_{n+1}\| \\ &\leq \|p_{n+1} - p_n\| + \frac{1}{e} |g_{n+1} - g_n| \|\phi_{n+1} - p_{n+1}\|. \end{aligned} \quad (3.8)$$

We use $\psi_n = W_{h_n} p_n$ and $\psi_{n+1} = W_{h_{n+1}} p_{n+1}$. By using the same method as (3.8), we have

$$\|\psi_{n+1} - \psi_n\| \leq \|p_{n+1} - p_n\| + \frac{1}{e} |h_{n+1} - h_n| \|\psi_{n+1} - p_{n+1}\|. \quad (3.9)$$

From (3.5), (3.8), and (3.9), we have

$$\begin{aligned} \|p_{n+1} - p_n\| &\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| \|p_{n-1}\| \\ &\quad + \gamma_n \left(\|p_{n+1} - p_n\| + \frac{1}{e} |g_{n+1} - g_n| \|\phi_{n+1} - p_{n+1}\| \right) \\ &\quad + \lambda_n \|(I - W)\phi_n - (I - W)\phi_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|(I - W)\phi_{n-1}\| \\ &\quad + |\gamma_n - \gamma_{n-1}| \|P_C(I - \lambda_{n-1}(I - W))\phi_{n-1}\| \\ &\quad + \delta_n \left(\|p_{n+1} - p_n\| + \frac{1}{e} |h_{n+1} - h_n| \|\psi_{n+1} - p_{n+1}\| \right) \\ &\quad + |\delta_n - \delta_{n-1}| \|G(\psi_{n-1})\| \\ &\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|p_n - p_{n-1}\| + |\beta_n - \beta_{n-1}| \|p_{n-1}\| \\ &\quad + \gamma_n \|p_{n+1} - p_n\| + \frac{1}{e} |g_{n+1} - g_n| \|\phi_{n+1} - p_{n+1}\| \\ &\quad + \lambda_n \|(I - W)\phi_n - (I - W)\phi_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|(I - W)\phi_{n-1}\| \\ &\quad + |\gamma_n - \gamma_{n-1}| \|P_C(I - \lambda_{n-1}(I - W))\phi_{n-1}\| + \delta_n \|p_{n+1} - p_n\| \\ &\quad + \frac{1}{e} |h_{n+1} - h_n| \|\psi_{n+1} - p_{n+1}\| + |\delta_n - \delta_{n-1}| \|G(\psi_{n-1})\| \end{aligned}$$

$$\begin{aligned}
&\leq (1 - \alpha_n)\|p_n - p_{n-1}\| + |\alpha_n - \alpha_{n-1}|M + |\beta_n - \beta_{n-1}|M \\
&\quad + |\gamma_n - \gamma_{n-1}|M + |\delta_n - \delta_{n-1}|M + |\lambda_n - \lambda_{n-1}|M + \lambda_n M \\
&\quad + \frac{1}{e}|g_{n+1} - g_n|M + \frac{1}{e}|h_{n+1} - h_n|M,
\end{aligned}$$

where

$$\begin{aligned}
M := \max_{n \in \mathbb{N}} \{ &\|u\|, \|p_n\|, \|P_C(I - \lambda_n(I - W))\phi_n\|, \|G(\psi_n)\|, \|(I - W)\phi_n\|, \\
&\|(I - W)\phi_{n+1} - (I - W)\phi_n\|, \|\phi_n - p_n\|, \|\psi_n - p_n\| \}.
\end{aligned}$$

From the conditions (i), (iv), (v), and Lemma 2.6, we have

$$\lim_{n \rightarrow \infty} \|p_{n+1} - p_n\| = 0. \quad (3.10)$$

Since W_{g_n} is a firmly nonexpansive mapping, we obtain

$$\begin{aligned}
\|\phi_n - \xi\|^2 &= \|W_{g_n}p_n - W_{g_n}\xi\|^2 \\
&\leq \langle W_{g_n}p_n - W_{g_n}\xi, p_n - \xi \rangle \\
&\leq \langle \phi_n - \xi, p_n - \xi \rangle \\
&= \frac{1}{2}(\|\phi_n - \xi\|^2 + \|p_n - \xi\|^2 - \|\phi_n - p_n\|^2).
\end{aligned}$$

It implies that

$$\|\phi_n - \xi\|^2 \leq \|p_n - \xi\|^2 - \|\phi_n - p_n\|^2. \quad (3.11)$$

By using the same method as (3.11), we have

$$\|\psi_n - \xi\|^2 \leq \|p_n - \xi\|^2 - \|\psi_n - p_n\|^2. \quad (3.12)$$

From the definition of p_n , (3.4), (3.11), and (3.12), we have

$$\begin{aligned}
\|p_{n+1} - \xi\|^2 &= \|\alpha_n(u - \xi) + \beta_n(p_n - \xi) + \gamma_n(P_C(I - \lambda_n(I - W))\phi_n - \xi) \\
&\quad + \delta_n(G(\psi_n) - \xi)\|^2 \\
&\leq \alpha_n\|u - \xi\|^2 + \beta_n\|p_n - \xi\|^2 + \gamma_n\|P_C(I - \lambda_n(I - W))\phi_n - \xi\|^2 \\
&\quad + \delta_n\|G(\psi_n) - \xi\|^2 - \beta_n\gamma_n\|P_C(I - \lambda_n(I - W))\phi_n - p_n\|^2 \\
&\quad - \beta_n\delta_n\|G(\psi_n) - p_n\|^2 \\
&\leq \alpha_n\|u - \xi\|^2 + \beta_n\|p_n - \xi\|^2 + \gamma_n\|\phi_n - \xi\|^2 + \delta_n\|\psi_n - \xi\|^2 \\
&\quad - \beta_n\gamma_n\|P_C(I - \lambda_n(I - W))\phi_n - p_n\|^2 - \beta_n\delta_n\|G(\psi_n) - p_n\|^2 \\
&\leq \alpha_n\|u - \xi\|^2 + \beta_n\|p_n - \xi\|^2 + \gamma_n(\|p_n - \xi\|^2 - \|\phi_n - p_n\|^2) \\
&\quad + \delta_n(\|p_n - \xi\|^2 - \|\psi_n - p_n\|^2) - \beta_n\delta_n\|G(\psi_n) - p_n\|^2
\end{aligned}$$

$$\begin{aligned}
& -\beta_n \gamma_n \|P_C(I - \lambda_n(I - W))\phi_n - p_n\|^2 \\
& = \alpha_n \|u - \xi\|^2 + (1 - \alpha_n) \|p_n - \xi\|^2 - \gamma_n \|\phi_n - p_n\|^2 \\
& \quad - \delta_n \|\psi_n - p_n\|^2 - \beta_n \gamma_n \|P_C(I - \lambda_n(I - W))\phi_n - p_n\|^2 \\
& \quad - \beta_n \delta_n \|G(\psi_n) - p_n\|^2 \\
& \leq \alpha_n \|u - \xi\|^2 + \|p_n - \xi\|^2 - \gamma_n \|\phi_n - p_n\|^2 - \delta_n \|\psi_n - p_n\|^2 \\
& \quad - \beta_n \gamma_n \|P_C(I - \lambda_n(I - W))\phi_n - p_n\|^2 - \beta_n \delta_n \|G(\psi_n) - p_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\gamma_n \|\phi_n - p_n\|^2 & \leq \alpha_n \|u - \xi\|^2 + \|p_n - \xi\|^2 - \|p_{n+1} - \xi\|^2 \\
& \leq \alpha_n \|u - \xi\|^2 + \|p_n - p_{n+1}\| (\|p_n - \xi\| + \|p_{n+1} - \xi\|).
\end{aligned}$$

From the conditions (i), (ii), and (3.10), we have

$$\lim_{n \rightarrow \infty} \|\phi_n - p_n\| = 0. \quad (3.13)$$

By using the same method as (3.13), we can imply that

$$\lim_{n \rightarrow \infty} \|\psi_n - p_n\| = \lim_{n \rightarrow \infty} \|P_C(I - \lambda_n(I - W))\phi_n - p_n\| = \lim_{n \rightarrow \infty} \|G(\psi_n) - p_n\| = 0. \quad (3.14)$$

From (3.13) and (3.14), we have

$$\lim_{n \rightarrow \infty} \|\phi_n - \psi_n\| = 0. \quad (3.15)$$

Afterwards, we show that $\limsup_{n \rightarrow \infty} \langle u - p_0, p_n - p_0 \rangle \leq 0$, where $p_0 = P_{\mathcal{F}}u$. To show this inequality, take a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u - p_0, p_n - p_0 \rangle = \lim_{j \rightarrow \infty} \langle u - p_0, p_{n_j} - p_0 \rangle.$$

Without loss of generality, we may assume that $u_{n_j} \rightharpoonup \omega$ as $j \rightarrow \infty$. From (3.15), we have $v_{n_j} \rightharpoonup \omega$ as $j \rightarrow \infty$. By using the same method as [15] in Theorem 3.2, we have

$$\omega \in \text{EP}(\Psi_1) \quad (3.16)$$

and

$$\omega \in \text{EP}(\Psi_2). \quad (3.17)$$

Furthermore, we show that $\omega \in F(W)$. From Remark 2.9, we have $F(W) = F(P_C(I - \lambda_{n_j}(I - W)))$. Assume that $\omega \notin F(W)$, we have $\omega \neq P_C(I - \lambda_{n_j}(I - W))\omega$. From (3.13), we have $p_{n_j} \rightharpoonup \omega$ as $j \rightarrow \infty$. By (3.13), (3.14), the condition (iv), and Opial's property, we have

$$\begin{aligned}
\liminf_{j \rightarrow \infty} \|p_{n_j} - \omega\| & < \liminf_{j \rightarrow \infty} \|p_{n_j} - P_C(I - \lambda_{n_j}(I - W))\omega\| \\
& \leq \liminf_{j \rightarrow \infty} (\|p_{n_j} - P_C(I - \lambda_{n_j}(I - W))u_{n_j}\|
\end{aligned}$$

$$\begin{aligned}
& + \|P_C(I - \lambda_{n_j}(I - W))u_{n_j} - P_C(I - \lambda_{n_j}(I - W))p_{n_j}\| \\
& + \|P_C(I - \lambda_{n_j}(I - W))p_{n_j} - P_C(I - \lambda_{n_j}(I - W))\omega\| \\
& \leq \liminf_{j \rightarrow \infty} (\|u_{n_j} - p_{n_j}\| + \lambda_{n_j} \|(I - W)u_{n_j} - (I - W)p_{n_j}\| \\
& \quad + \|p_{n_j} - \omega\| + \lambda_{n_j} \|(I - W)p_{n_j} - (I - W)\omega\|) \\
& = \liminf_{j \rightarrow \infty} \|p_{n_j} - \omega\|.
\end{aligned}$$

It is a contradiction. So we have

$$\omega \in F(W). \quad (3.18)$$

After that, we show that $\omega \in F(G)$. Assume that $\omega \notin F(G)$, that is, $\omega \neq G(\omega)$. Since $p_{n_j} \rightharpoonup \omega$ as $j \rightarrow \infty$, (3.14), the condition (iv), and Opial's property, we have

$$\begin{aligned}
\liminf_{j \rightarrow \infty} \|p_{n_j} - \omega\| & < \liminf_{j \rightarrow \infty} \|p_{n_j} - G(\omega)\| \\
& \leq \liminf_{j \rightarrow \infty} (\|p_{n_j} - G(\psi_{n_j})\| + \|G(\psi_{n_j}) - G(p_{n_j})\| \\
& \quad + \|G(p_{n_j}) - G(\omega)\|) \\
& \leq \liminf_{j \rightarrow \infty} (\|\psi_{n_j} - p_{n_j}\| + \|p_{n_j} - \omega\|) \\
& = \liminf_{j \rightarrow \infty} \|p_{n_j} - \omega\|.
\end{aligned}$$

It is a contradiction. So we have

$$\omega \in F(G). \quad (3.19)$$

Therefore $\omega \in \mathcal{F}$. Since $p_{n_j} \rightharpoonup \omega$ as $j \rightarrow \infty$, we have

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \langle u - p_0, p_n - p_0 \rangle & = \lim_{j \rightarrow \infty} \langle u - p_0, p_{n_j} - p_0 \rangle \\
& = \langle u - p_0, \omega - p_0 \rangle \leq 0.
\end{aligned} \quad (3.20)$$

Finally, we show that the sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$. From the definition of p_n , (3.4), and $p_0 = P_{\mathcal{F}}u$, we have

$$\begin{aligned}
\|p_{n+1} - p_0\|^2 & = \|\alpha_n(u - p_0) + \beta_n(p_n - p_0) + \gamma_n(P_C(I - \lambda_n(I - W))\phi_n - p_0) \\
& \quad + \delta_n(G(\psi_n) - p_0)\|^2 \\
& \leq \|\beta_n(p_n - p_0) + \gamma_n(P_C(I - \lambda_n(I - W))\phi_n - p_0) \\
& \quad + \delta_n(G(\psi_n) - p_0)\|^2 + 2\alpha_n \langle u - p_0, p_{n+1} - p_0 \rangle \\
& \leq (1 - \alpha_n) \|p_n - p_0\|^2 + 2\alpha_n \langle u - p_0, p_{n+1} - p_0 \rangle.
\end{aligned}$$

From the condition (i), (3.20), and Lemma 2.6, we can conclude that the sequence $\{p_n\}$ converges strongly to $p_0 = P_{\mathcal{F}}u$. Consequently, we see that $\{\phi_n\}$ and $\{\psi_n\}$ also converge strongly to $p_0 = P_{\mathcal{F}}u$. This completes the proof. $\square$

From our main result, if we take $a = 0$, we have the following corollary.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H , let $\Psi_1, \Psi_2 : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (J1)-(J4) and let $W : C \rightarrow C$ be a quasi-nonexpansive mapping. Let $D_1, D_2 : C \rightarrow H$ be d_1, d_2 -inverse strongly monotone mappings, respectively. Define the mapping $G : C \rightarrow C$ by $G(p) = P_C(I - \lambda_A D_1)(P_C(I - \lambda_B D_2)p)$ for all $p \in C$. Assume $\mathcal{F} = \text{EP}(\Psi_1) \cap \text{EP}(\Psi_2) \cap F(G) \cap F(W) \neq \emptyset$. Suppose that $p_1, u \in C$ and let $\{p_n\}, \{\phi_n\}$, and $\{\psi_n\}$ be sequences generated by*

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n} \langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n} \langle \zeta - \psi_n, \psi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases} \quad (3.21)$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ for some $c, d > 0$ and for all $n \geq 1$,
- (iii) $0 < e \leq g_n, h_n \leq f$ for some $e, f > 0$ and for all $n \geq 1$,
- (iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
 $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}, \{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$ and (p_0, ξ_0) is a solution of (1.5) where $\xi_0 = P_C(I - \lambda_B D_2)p_0$.

4 Application

In this section, we use our main result to obtain Theorem 4.7 and Theorem 4.8. Before we prove these theorems, we need the following definition and lemma. A mapping $W : C \rightarrow C$ is said to be nonspreading if

$$2\|Wp - W\zeta\|^2 \leq \|Wp - \zeta\|^2 + \|W\zeta - p\|^2, \quad \forall p, \zeta \in C. \quad (4.1)$$

Such a mapping is defined by Kohsaka and Takahashi [16].

In 2009, Iemoto and Takahashi [17] proved that (4.1) is equivalent to

$$\|Wp - W\zeta\|^2 \leq \|p - \zeta\|^2 + 2\langle p - Wp, \zeta - W\zeta \rangle, \quad \forall p, \zeta \in C. \quad (4.2)$$

Remark 4.1 A nonspreading mapping W with $F(W) \neq \emptyset$ is quasi-nonexpansive mapping.

Example 4.2 Let $W : [-5, \infty) \rightarrow [-5, \infty)$ be defined by

$$Wp = \frac{p-5}{2}, \quad \forall p \in [-5, \infty).$$

Since W is a nonspreading mapping and $F(W) = \{-5\}$, we have W is a quasi-nonexpansive mapping.

The following lemmas and definition are used to prove the results in this section.

Lemma 4.3 ([8]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \rightarrow H$ be d_1, d_2 -inverse strongly monotone mappings, respectively, with $\text{VIP}(C, D_1) \cap \text{VIP}(C, D_2) \neq \emptyset$. Define a mapping $G : C \rightarrow C$ by*

$$G(p) = P_C(I - \lambda_A D_1)(ap + (1 - a)P_C(I - \lambda_B D_2)p),$$

for every $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$ and $a \in (0, 1)$. Then $F(G) = \text{VIP}(C, D_1) \cap \text{VIP}(C, D_2)$.

Lemma 4.4 ([16]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H , and let W be a nonspreading mapping of C into itself. Then $F(W)$ is closed and convex.*

In 2009, Kangtunyakarn and Suantai [18] introduced the S -mapping generated by $W_1, W_2, W_3, \dots, W_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$ as follows.

Definition 4.5 Let C be a nonempty convex subset of a real Banach space. Let $\{W_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \rightarrow C$ as follows:

$$\begin{aligned} U_0 &= I, \\ U_1 &= \alpha_1^1 W_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\ U_2 &= \alpha_1^2 W_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\ U_3 &= \alpha_1^3 W_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\ &\dots, \\ U_{N-1} &= \alpha_1^{N-1} W_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\ S = U_N &= \alpha_1^N W_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned}$$

This mapping is called an S -mapping generated by $W_1, W_2, \dots, W_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

For every $i = 1, 2, \dots, N$, put $\alpha_3^i = 0$ in Definition 4.5, then the S -mapping is reduced to the K -mapping generated by $\alpha_1^1, \alpha_1^2, \dots, \alpha_1^N$ where the K -mapping is defined by Kangtunyakarn and Suantai [19] as follows.

Lemma 4.6 ([20]) *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{W_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(W_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \dots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0, 1)$, $\alpha_3^N \in [0, 1)$, $\alpha_2^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by $W_1, W_2, \dots, W_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(W_i)$ and S is a quasi-nonexpansive mapping.*

By using these results, we obtain the following theorems.

Theorem 4.7 *Let C be a nonempty closed convex subset of a real Hilbert space H , let $\Psi_1, \Psi_2 : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (J1)-(J4) and let $W : C \rightarrow C$ be a quasi-nonexpansive mapping. Let $D_1, D_2 : C \rightarrow H$ be d_1, d_2 -inverse strongly monotone mappings,*

respectively. Assume $\mathcal{F} = \text{EP}(\Psi_1) \cap \text{EP}(\Psi_2) \cap F(W) \cap \text{VIP}(C, D_1) \cap \text{VIP}(C, D_2) \neq \emptyset$. Suppose that $p_1, u \in C$ and let $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ be sequences generated by

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n} \langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n} \langle \zeta - \psi_n, \psi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n \\ \quad + \delta_n P_C(I - \lambda_A D_1)(ap_n + (1-a)P_C(I - \lambda_B D_2)p_n), & \forall n \in \mathbb{N}, \end{cases} \quad (4.3)$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, for all $n \in \mathbb{N}$, and $a \in (0, 1)$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ for some $c, d > 0$ and for all $n \geq 1$,
- (iii) $0 < e \leq g_n, h_n \leq f$ for some $e, f > 0$ and for all $n \geq 1$,
- (iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
 $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$ and (p_0, ξ_0) be a solution of (1.6) where $\xi_0 = P_C(I - \lambda_B D_2)p_0$.

Proof By using Theorem 3.1 and Lemma 4.3, we obtain the conclusion. $\square$

Theorem 4.8 Let C be a nonempty closed convex subset of a real Hilbert space H , let $\Psi_1, \Psi_2 : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (J1)-(J4). Let $\{W_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, \dots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0, 1]$, $\alpha_3^N \in [0, 1]$, $\alpha_2^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by $W_1, W_2, \dots, W_N$, and $\alpha_1, \alpha_2, \dots, \alpha_N$. Let $D_1, D_2 : C \rightarrow H$ be d_1, d_2 -inverse strongly monotone mappings, respectively. Define the mapping $G : C \rightarrow C$ by $G(p) = P_C(I - \lambda_A D_1)(ap + (1-a)P_C(I - \lambda_B D_2)p)$ for all $p \in C$ and $a \in [0, 1]$. Assume $\mathcal{F} = \text{EP}(\Psi_1) \cap \text{EP}(\Psi_2) \cap F(G) \cap \bigcap_{i=1}^N F(W_i) \neq \emptyset$. Suppose that $p_1, u \in C$ and let $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ are sequences generated by

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n} \langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n} \langle \zeta - \psi_n, \psi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - S))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases} \quad (4.4)$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ for some $c, d > 0$ and for all $n \geq 1$,
- (iii) $0 < e \leq g_n, h_n \leq f$ for some $e, f > 0$ and for all $n \geq 1$,
- (iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
 $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$ and (p_0, ξ_0) is a solution of (1.6) where $\xi_0 = P_C(I - \lambda_B D_2)p_0$.

Proof By using Theorem 3.1 and Lemma 4.6, we obtain the conclusion. $\square$

The following result is directly proven from Theorem 4.8. Therefore, we omit the proof.

Corollary 4.9 *Let C be a nonempty closed convex subset of a real Hilbert space H , let $\Psi_1, \Psi_2 : C \times C \rightarrow \mathbb{R}$ be bifunctions satisfying (J1)-(J4). Let W be a nonspreading mappings of C into itself with $F(W) \neq \emptyset$. Let $D_1, D_2 : C \rightarrow H$ be d_1, d_2 -inverse strongly monotone mappings, respectively. Define the mapping $G : C \rightarrow C$ by $G(p) = P_C(I - \lambda_A D_1)(ap + (1-a)P_C(I - \lambda_B D_2)p)$ for all $p \in C$ and $a \in [0, 1]$. Assume $\mathcal{F} = \text{EP}(\Psi_1) \cap \text{EP}(\Psi_2) \cap F(G) \cap F(W) \neq \emptyset$. Suppose that $p_1, u \in C$ and let $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ be sequences generated by*

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n} \langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n} \langle \zeta - \psi_n, \psi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases} \quad (4.5)$$

where the sequences $\lambda_A \in (0, 2d_1)$, $\lambda_B \in (0, 2d_2)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N}$. Suppose the following conditions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (ii) $0 < c \leq \beta_n, \gamma_n, \delta_n \leq d < 1$ for some $c, d > 0$ and for all $n \geq 1$,
- (iii) $0 < e \leq g_n, h_n \leq f$ for some $e, f > 0$ and for all $n \geq 1$,
- (iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,
- (v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$,
 $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |g_{n+1} - g_n| < \infty$, $\sum_{n=1}^{\infty} |h_{n+1} - h_n| < \infty$.

Then $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to $p_0 = P_{\mathcal{F}}u$ and (p_0, ξ_0) is a solution of (1.6) where $\xi_0 = P_C(I - \lambda_B D_2)p_0$.

5 Example and numerical results

In this section, we give an example supporting Theorem 3.1.

Example 5.1 Let $\mathbb{R}$ be the set of real numbers and let the mapping $D_1, D_2 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $D_1 p = \frac{p-2}{3}$ and $D_2 p = \frac{p-2}{5}$, $\forall p \in \mathbb{R}$, respectively. Let the mapping $W : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $Wp = \frac{p+2}{2}$, $\forall p \in \mathbb{R}$, let $\Psi_1, \Psi_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\Psi_1(p, \zeta) = -(p - \zeta)(-4 + p + \zeta), \quad \forall p, \zeta \in \mathbb{R}$$

and

$$\Psi_2(p, \zeta) = -2(p - 2)^2 + (p - 2)(\zeta - 2) + (\zeta - 2)^2, \quad \forall p, \zeta \in \mathbb{R}.$$

By the definition of Ψ_1 , we have

$$\begin{aligned} 0 &\leq \Psi_1(\phi_n, \zeta) + \frac{1}{g_n} \langle \zeta - \phi_n, \phi_n - p_n \rangle \\ &= -(\phi_n - \zeta)(-4 + \phi_n + \zeta) + \frac{1}{g_n} (\zeta - \phi_n)(\phi_n - p_n) \\ &= -(\phi_n - \zeta)(-4 + \phi_n + \zeta) + \frac{1}{g_n} (\zeta \phi_n - \zeta p_n - \phi_n^2 + \phi_n p_n) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \quad 0 &\leq -g_n(\phi_n - \zeta)(-4 + \phi_n + \zeta) + (\zeta\phi_n - \zeta p_n - \phi_n^2 + \phi_n p_n) \\ &= 4g_n\phi_n - \phi_n^2 - g_n\phi_n^2 + \phi_n p_n + (-4g_n + \phi_n - p_n)\zeta + g_n\zeta^2. \end{aligned}$$

Let $G(\zeta) = g_n\zeta^2 + (-4g_n + \phi_n - p_n)\zeta + 4g_n\phi_n - \phi_n^2 - g_n\phi_n^2 + \phi_n p_n$, which is a quadratic function of ζ with coefficient $a = g_n$, $b = -4g_n + \phi_n - p_n$, and $c = 4g_n\phi_n - \phi_n^2 - g_n\phi_n^2 + \phi_n p_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-4g_n + \phi_n - p_n)^2 - 4g_n(4g_n\phi_n - \phi_n^2 - g_n\phi_n^2 + \phi_n p_n) \\ &= 16g_n^2 - 8g_n\phi_n - 16g_n^2\phi_n + \phi_n^2 + 4g_n\phi_n^2 + 4g_n^2\phi_n^2 + 8g_n p_n - 2\phi_n p_n - 4g_n\phi_n p_n + p_n^2 \\ &= (-4g_n + \phi_n + 2g_n\phi_n - p_n)^2. \end{aligned}$$

We know that $G(\zeta) \geq 0$, $\forall \zeta \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$. So we obtain

$$\phi_n = \frac{4g_n + p_n}{1 + 2g_n}. \quad (5.1)$$

By using the same method as (5.1), we have

$$\psi_n = \frac{6h_n + p_n}{1 + 3h_n}. \quad (5.2)$$

Let $p_1, u \in \mathbb{R}$, and $\{p_n\}$ be generated by (3.1) as follows:

$$\begin{cases} \Psi_1(\phi_n, \zeta) + \frac{1}{g_n} \langle \zeta - \phi_n, \phi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ \Psi_2(\psi_n, \zeta) + \frac{1}{h_n} \langle \zeta - \psi_n, \psi_n - p_n \rangle \geq 0, & \forall \zeta \in C, \\ p_{n+1} = \alpha_n u + \beta_n p_n + \gamma_n P_C(I - \lambda_n(I - W))\phi_n + \delta_n G(\psi_n), & \forall n \in \mathbb{N}, \end{cases}$$

where $a = 0.5$, $\lambda_A = 1$, $\lambda_B = 1$, $g_n = \frac{n}{3n+1}$, $h_n = \frac{n}{4n+1}$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{3n-1}{16n}$, $\gamma_n = \frac{10n-3}{16n}$, $\delta_n = \frac{3n-4}{16n}$, and $\lambda_n = \frac{1}{2n^2}$ for all $n \in \mathbb{N}$. By the definitions of Ψ_1 , Ψ_2 , G , and W , we have $\text{EP}(\Psi_1) \cap \text{EP}(\Psi_2) \cap F(G) \cap F(W) = \{2\}$. From Theorem 3.1, we can conclude that the sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ converge strongly to 2. From (5.1) and (5.2), we can rewrite (3.1) as follows:

$$\begin{cases} \phi_n = \frac{4g_n + p_n}{1 + 2g_n}, \\ \psi_n = \frac{6h_n + p_n}{1 + 3h_n}, \\ p_{n+1} = \frac{1}{2n} u + \frac{3n-1}{16n} p_n + \frac{10n-3}{16n} (I - \frac{1}{2n^2} (I - W))\phi_n + \frac{3n-4}{16n} G(\psi_n), & \forall n \geq 1. \end{cases} \quad (5.3)$$

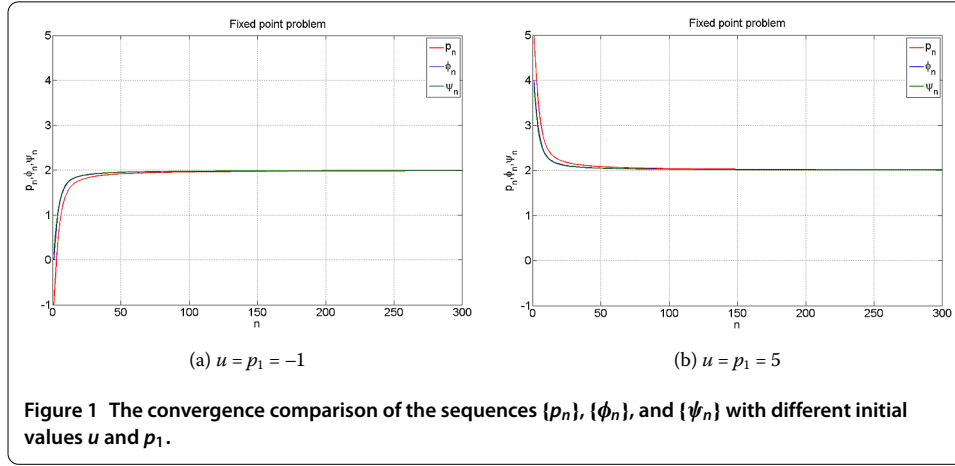
Table 1 shows the values of the sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ where $u = p_1 = -1$ and $u = p_1 = 5$ and $n = 300$.

Conclusion

1. The sequences $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ in Table 1 and Figure 1 converge to 2, where $\{2\} = \text{EP}(\Psi_1) \cap \text{EP}(\Psi_2) \cap F(G) \cap F(W)$.
2. Theorem 3.1 ensures the convergence of $\{p_n\}$, $\{\phi_n\}$, and $\{\psi_n\}$ in Example 5.1.

Table 1 The values of $\{\phi_n\}$, $\{\psi_n\}$, and $\{p_n\}$ where $n = 300$

n	$u = p_1 = -1$			$u = p_1 = 5$		
	ϕ_n	ψ_n	p_n	ϕ_n	ψ_n	p_n
1	0.0000	0.1250	-1.0000	4.000000	3.8750	5.0000
2	0.4339	0.5234	-0.4609	3.5661	3.4766	4.4609
3	0.7688	0.8360	0.0301	3.2312	3.1640	3.9699
4	1.0254	1.0771	0.4256	2.9746	3.9229	3.5744
5	1.2188	1.2595	0.7306	2.7812	2.7405	3.2694
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
150	1.9837	1.9845	1.9728	2.0163	2.0155	2.0272
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
296	1.9918	1.9922	1.9863	2.0082	2.0078	2.0137
297	1.9918	1.9922	1.9864	2.0082	2.0078	2.0136
298	1.9918	1.9922	1.9864	2.0082	2.0078	2.0136
299	1.9919	1.9923	1.9865	2.0081	2.0077	2.0135
300	1.9919	1.9923	1.9865	2.0081	2.0077	2.0135



References

1. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**(14), 123-145 (1994)
2. Lions, JL, Stampacchia, G: Variational inequalities. Commun. Pure Appl. Math. **20**, 493-517 (1967)
3. Diaz, JB, Metcalf, FT: On the structure of the set of subsequential limit points of successive approximations. Bull. Am. Math. Soc. **73**, 516-519 (1967)
4. Su, Y, Wang, D, Shang, M: Strong convergence of monotone hybrid algorithm for quasi-nonexpansive mappings. Int. J. Math. Anal. **1**, 1235-1241 (2007)
5. Tian, M, Jin, X: Strong convergent result for quasi-nonexpansive mappings in Hilbert spaces. Fixed Point Theory Appl. **2011**, 88 (2011)

6. Dong, QL, Guo, YN, Su, F: Hybrid iterative scheme by a relaxed extragradient method for equilibrium problems, a general system of variational inequalities and fixed point problems of a countable family of nonexpansive mappings. *J. Appl. Math.* **2012**, Article ID 572326 (2012)
7. Ceng, LC, Wang, CY, Yao, JC: Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. *Math. Methods Oper. Res.* **67**, 375-390 (2008)
8. Kangtunyakarn, A: An iterative algorithm to approximate a common element of the set of common fixed points for a finite family of strict pseudo-contractions and of the set of solutions for a modified system of variational inequalities. *Fixed Point Theory Appl.* **2013**, 143 (2013)
9. Opial, Z: Weak convergence of the sequence of successive approximation of nonexpansive mappings. *Bull. Am. Math. Soc.* **73**, 591-597 (1967)
10. Osilike, MO, Igboke, DI: Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations. *Comput. Math. Appl.* **40**, 559-567 (2000)
11. Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. *J. Nonlinear Convex Anal.* **6**(1), 117-136 (2005)
12. Xu, HK: An iterative approach to quadric optimization. *J. Optim. Theory Appl.* **116**, 659-678 (2003)
13. Takahashi, W: *Nonlinear Functional Analysis*. Yokohama Publishers, Yokohama (2000)
14. Cheawchan, K, Kangtunyakarn, A: Approximation method for fixed points of nonlinear mapping and variational inequalities with application. *Thai J. Math.* (in press)
15. Takahashi, S, Takahashi, T: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.* **331**, 506-515 (2007)
16. Kohsaka, F, Takahashi, W: Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces. *Arch. Math.* **91**, 166-177 (2008)
17. Iemoto, S, Takahashi, W: Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space. *Nonlinear Anal.* **71**, 2082-2089 (2009)
18. Kangtunyakarn, A, Suantai, S: Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings. *Nonlinear Anal. Hybrid Syst.* **3**, 296-309 (2009)
19. Kangtunyakarn, A, Suantai, S: A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings. *Nonlinear Anal.* **71**, 4448-4460 (2009)
20. Kangtunyakarn, A: Strong convergence of the hybrid method for a finite family of nonspreading mappings and variational inequality problem. *Fixed Point Theory Appl.* **2012**, 188 (2012)

1.2 A new iterative process for a hybrid pair of generalized asymptotically nonexpansive single-valued and generalized nonexpansive multi-valued mappings in Banach spaces

In this paper, we construct an iterative process involving a hybrid pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of generalized nonexpansive multi-valued mappings and prove weak and strong convergence theorems of the proposed iterative process in Banach spaces. Our main results extend and generalize many results in the literature.

1 Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. Let X be a Banach space and let D be a nonempty subset of X . Let $CB(D)$ and $KC(D)$ denote the families of nonempty, closed, and bounded subsets and nonempty, compact, and convex subsets of D , respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\} \quad \text{for } A, B \in CB(D),$$

where $\text{dist}(x, D) = \inf\{\|x - y\| : y \in D\}$ is the distance from a point x to a subset D . Let t be a single-valued mapping of D into D and T be a multi-valued mapping of D into

$CB(D)$. The set of fixed points of t and T will be denoted by $F(t) = \{x \in D : x = tx\}$ and $F(T) = \{x \in D : x \in Tx\}$, respectively. A point x is called a *common fixed point* of t and T if

$$x = tx \in Tx.$$

Definition 1.1 A single-valued mapping $t : D \rightarrow D$ is said to be *generalized asymptotically nonexpansive* if there exist sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$, $\lim_{n \rightarrow \infty} s_n = 0$ such that

$$\|t^n x - t^n y\| \leq k_n \|x - y\| + s_n,$$

for all $x, y \in D$ and $n \in \mathbb{N}$.

In the case of $s_n = 0$, for all $n \in \mathbb{N}$, a single-valued mapping t is called an *asymptotically nonexpansive mapping*. In particular, if $k_n = 1$ and $s_n = 0$, for all $n \in \mathbb{N}$, a single-valued mapping t reduce to a *nonexpansive mapping*. The fixed point property for generalized asymptotically nonexpansive single-valued mappings can be found in [1]. The following example shows that the fixed point set of a generalized asymptotically nonexpansive mapping is not necessarily closed; see also [2].

Example 1.2 ([1]) Define a single-valued mapping $t : [-\frac{2}{3}, \frac{2}{3}] \rightarrow [-\frac{2}{3}, \frac{2}{3}]$ by

$$tx = \begin{cases} x, & \text{if } x \in [-\frac{2}{3}, 0), \\ \frac{16}{81}, & \text{if } x = 0, \\ x^4, & \text{if } x \in (0, \frac{2}{3}]. \end{cases}$$

Then t is generalized asymptotically nonexpansive and $F(t) = [-\frac{2}{3}, 0)$ which is not closed.

Definition 1.3 A multi-valued mapping $T : D \rightarrow CB(D)$ is said to be

- (i) *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in D$;
- (ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$, for all $x \in D$ and $p \in F(T)$.

The study of fixed points for nonexpansive multi-valued mappings using the Hausdorff metric was initiated by Markin [3]. Different iterative processes have been used to approximate fixed points of nonexpansive and quasi-nonexpansive multi-valued mappings; in particular, Sastry and Babu [4] considered Mann and Ishikawa iterates for a multi-valued mapping T with a fixed point p and proved that these iterates converge to a fixed point q of T under certain conditions. Moreover, they illustrated that the fixed point q may be different from p . Later in 2007, Panyanak [5] generalized results of Sastry and Babu [4] to uniformly convex Banach spaces and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain. In 2009, Shahzad and Zegeye [6] proved strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multi-valued mappings. They constructed an iterative process which removes the restriction of T , namely *end-point condition*, i.e., $Tp = \{p\}$ for any $p \in F(T)$; see also [7, 8].

In 2011, Garcia-Falset *et al.* [9] introduced a new condition on single-valued mappings, called *condition (E)*, which is weaker than nonexpansiveness. Later, Abkar and Eslamian [10] used a modified condition for multi-valued mappings as follows.

Definition 1.4 A multi-valued mapping $T : D \rightarrow CB(D)$ is said to satisfy *condition (E)_μ* where $\mu \geq 0$ if for each $x, y \in D$,

$$\text{dist}(x, Ty) \leq \mu \text{dist}(x, Tx) + \|x - y\|.$$

We say that T satisfies *condition (E)* whenever T satisfies (E_μ) for some $\mu \geq 1$.

Remark 1.5 From the above definitions, it is clear that if T is nonexpansive, then T satisfies the condition (E_1) .

In 2011, Sokhuma and Kaewkhao [11] introduced the following iterative process for approximating a common fixed point of a pair of a nonexpansive single-valued mapping t and a nonexpansive multi-valued mapping T :

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n z_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n t y_n, \end{cases} \quad n \in \mathbb{N}, \quad (1.1)$$

where $x_1 \in D$, $z_n \in T x_n$, and $0 < a \leq \alpha_n, \beta_n \leq b < 1$. They also proved a strong convergence theorem for the iterative process (1.1) in uniformly convex Banach spaces.

In 2013, Eslamian [12] extended the results of [11, 13] in uniformly convex Banach spaces. He used the following iterative process for a pair of a finite family of asymptotically nonexpansive single-valued mappings $\{t_i\}_{i=1}^N$ and a finite family of quasi-nonexpansive multi-valued mapping $\{T_i\}_{i=1}^N$:

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \end{cases} \quad n \in \mathbb{N}, \quad (1.2)$$

where $x_1 \in D$, $z_n^{(i)} \in T_i x_n$, and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, \dots, N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = \sum_{i=0}^N \beta_n^{(i)} = 1$.

In this paper, motivated by the above results, we propose an iterative process for approximating a common fixed point of a pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of quasi-nonexpansive multi-valued mappings and prove weak and strong convergence theorems of the proposed iterative process in Banach spaces.

2 Preliminaries

A Banach space X is called *uniformly convex* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. The following result was proved by Xu [14].

Proposition 2.1 *Let X be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r = \{z \in X : \|z\| \leq r\}$ and $\lambda \in [0, 1]$.

A Banach space X is said to satisfy the *Opial property* (see [15]) if it is given that whenever $\{x_n\}$ converges weakly to $x \in X$,

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for each $y \in X$ with $y \neq x$. The examples of Banach spaces which satisfy the Opial property are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy the Opial property.

The following results are needed for proving our results.

Definition 2.2 (see [2]) Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X . We say that $\{x_n\}$ is of *monotone type (I) with respect to F* if there exist sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, and $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| + \varepsilon_n$ for all $n \in \mathbb{N}$ and $p \in F$.

Proposition 2.3 (see [2]) Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X . If $\{x_n\}$ is of monotone type (I) with respect to F and $\liminf_{n \rightarrow \infty} \text{dist}(x_n, F) = 0$, then $\lim_{n \rightarrow \infty} x_n = p$ for some $p \in F$ satisfying $\text{dist}(p, F) = 0$. In particular, if F is closed, then $p \in F$.

Lemma 2.4 (see [16]) Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfy

$$a_{n+1} \leq (1 + c_n)a_n + b_n, \quad \text{for all } n \in \mathbb{N},$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then:

- (i) $\lim_{n \rightarrow \infty} a_n$ exists.
- (ii) If $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.5 (see [17]) Let X be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < 1$, for all $n \in \mathbb{N}$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X satisfying, for some $r \geq 0$,

- (i) $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$,
- (ii) $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and
- (iii) $\lim_{n \rightarrow \infty} \|\lambda_n x_n + (1 - \lambda_n)y_n\| = r$.

Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 2.6 (see [18]) Let X be a Banach space which satisfies the Opial property and $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be such that $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v , respectively, then $u = v$.

3 Main results

In this section, we prove weak and strong convergence theorems of the proposed iterative process in Banach spaces. We first note that if $\{t_i\}_{i=1}^N$ is a finite family of generalized asymptotically nonexpansive single-valued mappings of D into itself, where D is a nonempty convex subset of a Banach space X . Then we have $\|t_i^n x - t_i^n y\| \leq k_n^{(i)} \|x - y\| + s_n^{(i)}$, for all $x, y \in D$ and all $i = 1, 2, \dots, N$, where $\{k_n^{(i)}\} \subset [1, \infty)$ and $\{s_n^{(i)}\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} k_n^{(i)} = 1$ and $\lim_{n \rightarrow \infty} s_n^{(i)} = 0$. Put $k_n = \max_{1 \leq i \leq N} \{k_n^{(i)}\}$ and $s_n = \max_{1 \leq i \leq N} \{s_n^{(i)}\}$. It is clear that $\lim_{n \rightarrow \infty} k_n = 1$ and $\lim_{n \rightarrow \infty} s_n = 0$ and

$$\|t_i^n x - t_i^n y\| \leq k_n \|x - y\| + s_n$$

for all $x, y \in D$, $i = 1, 2, \dots, N$, and all $n \in \mathbb{N}$.

In order to prove our main results, the following lemma is needed.

Lemma 3.1 *Let D be a nonempty, closed, and convex subset of a Banach space X . Let $\{t_i\}_{i=1}^N$ be a finite family of generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \dots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, \dots, N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$.

Proof Let $p \in \mathcal{F}$, for $i = 1, 2, \dots, N$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} \|t_i^n y_n - p\| \\ &\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} (k_n \|y_n - p\| + s_n) \\ &= \alpha_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \|y_n - p\| + s_n \sum_{i=1}^N \alpha_n^{(i)} \\ &\leq \alpha_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \|y_n - p\| + s_n \\ &\leq \alpha_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \left(\beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \beta_n^{(i)} \|z_n^{(i)} - p\| \right) + s_n \\ &= \left(\alpha_n^{(0)} + k_n \beta_n^{(0)} \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \sum_{i=1}^N \beta_n^{(i)} \|z_n^{(i)} - p\| + s_n \\ &= \left(\alpha_n^{(0)} + k_n \beta_n^{(0)} \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \sum_{i=1}^N \beta_n^{(i)} \text{dist}(z_n^{(i)}, T_i p) + s_n \\ &\leq \left(\alpha_n^{(0)} + k_n \beta_n^{(0)} \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \sum_{i=1}^N \beta_n^{(i)} H(T_i x_n, T_i p) + s_n \\ &\leq \left(\alpha_n^{(0)} + k_n \beta_n^{(0)} \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \sum_{i=1}^N \beta_n^{(i)} \|x_n - p\| + s_n \\ &= \left(\alpha_n^{(0)} + k_n \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + s_n \\ &\leq k_n \|x_n - p\| + s_n \\ &= (1 + (k_n - 1)) \|x_n - p\| + s_n. \end{aligned}$$

By Lemma 2.4, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, we conclude that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$. $\square$

Theorem 3.2 *Let D be a nonempty, closed, and convex subset of a Banach space X . Let $\{t_i\}_{i=1}^N$ be a finite family of generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \dots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, \dots, N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$.

Proof The necessity is obvious and thus we prove only the sufficiency. Suppose that $\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$. In the proof of Lemma 3.1, we see that the sequence $\{x_n\}$ is of monotone type (I) with respect to $\mathcal{F}$. It follows by Proposition 2.3 that $\{x_n\}$ converges to a point in $\mathcal{F}$. $\square$

The closedness of $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ can be dropped if t_i is asymptotically nonexpansive for all $i = 1, 2, \dots, N$. Then the following corollary is obtained directly from Theorem 3.2.

Corollary 3.3 *Let D be a nonempty, closed, and convex subset of a Banach space X . Let $\{t_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive single-valued mappings of D into itself with a sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \dots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, \dots, N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$.

Recall that a mapping $t : D \rightarrow D$ is called *uniformly L -Lipschitzian* if there exists a constant $L > 0$ such that $\|t^n x - t^n y\| \leq L \|x - y\|$ for all $x, y \in D$ and $n \in \mathbb{N}$. Next, we prove a strong convergence theorem in a uniformly convex Banach space.

Lemma 3.4 *Let D be a nonempty, closed, and convex subset of a uniformly convex Banach space X . Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L -Lipschitzian and generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that*

$\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \dots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, \dots, N$ such that $0 < a \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$, and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then we have the following:

- (i) $\lim_{n \rightarrow \infty} \|x_n - z_n^{(i)}\| = 0$ for all $i = 1, 2, \dots, N$;
- (ii) $\lim_{n \rightarrow \infty} \|x_n - t_i x_n\| = 0$ for all $i = 1, 2, \dots, N$.

Proof (i) By Lemma 3.1, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Put $\lim_{n \rightarrow \infty} \|x_n - p\| = c$. By the definition of $\{x_n\}$, we have

$$\begin{aligned} \|t_i^n y_n - p\| &\leq k_n \|y_n - p\| + s_n \\ &\leq k_n \left(\beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \beta_n^{(i)} \|z_n^{(i)} - p\| \right) + s_n \\ &= k_n \beta_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \beta_n^{(i)} \|z_n^{(i)} - p\| + s_n \\ &= k_n \beta_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \beta_n^{(i)} \text{dist}(z_n^{(i)}, T_i p) + s_n \\ &\leq k_n \beta_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \beta_n^{(i)} H(T_i x_n, T_i p) + s_n \\ &\leq k_n \beta_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \beta_n^{(i)} \|x_n - p\| + s_n \\ &= k_n \left(\beta_n^{(0)} + \sum_{i=1}^N \beta_n^{(i)} \right) \|x_n - p\| + s_n \\ &= k_n \|x_n - p\| + s_n. \end{aligned}$$

Then we have

$$\limsup_{n \rightarrow \infty} \|t_i^n y_n - p\| \leq \limsup_{n \rightarrow \infty} (k_n \|y_n - p\| + s_n) \leq \limsup_{n \rightarrow \infty} (k_n \|x_n - p\| + s_n).$$

By $\lim_{n \rightarrow \infty} k_n = 1$ and $\lim_{n \rightarrow \infty} s_n = 0$, we have

$$\limsup_{n \rightarrow \infty} \|t_i^n y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = c. \quad (3.1)$$

Since $c = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \lim_{n \rightarrow \infty} \|\alpha_n^{(0)}(x_n - p) + \sum_{i=1}^N \alpha_n^{(i)}(t_i^n y_n - p)\|$, it follows by Lemma 2.5 that

$$\lim_{n \rightarrow \infty} \|x_n - t_i^n y_n\| = 0 \quad \text{for all } i = 1, 2, \dots, N. \quad (3.2)$$

Consider

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} \|t_i^n y_n - p\| \\
&= \left(1 - \sum_{i=1}^N \alpha_n^{(i)}\right) \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} \|t_i^n y_n - p\| \\
&\leq \left(1 - \sum_{i=1}^N \alpha_n^{(i)}\right) \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} (k_n \|y_n - p\| + s_n).
\end{aligned}$$

This implies that

$$\|x_{n+1} - p\| - \|x_n - p\| \leq \sum_{i=1}^N \alpha_n^{(i)} (k_n \|y_n - p\| - \|x_n - p\| + s_n).$$

Therefore,

$$\begin{aligned}
\frac{\|x_{n+1} - p\| - \|x_n - p\|}{bN} + \|x_n - p\| &\leq \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\sum_{i=1}^N \alpha_n^{(i)}} + \|x_n - p\| \\
&\leq k_n \|y_n - p\| + s_n.
\end{aligned}$$

By (3.1), we obtain

$$\begin{aligned}
c &= \liminf_{n \rightarrow \infty} \left(\frac{\|x_{n+1} - p\| - \|x_n - p\|}{bN} + \|x_n - p\| \right) \\
&\leq \liminf_{n \rightarrow \infty} (k_n \|y_n - p\| + s_n) \\
&= \liminf_{n \rightarrow \infty} \|y_n - p\| \\
&\leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c.
\end{aligned}$$

Thus,

$$c = \lim_{n \rightarrow \infty} \|y_n - p\| = \lim_{n \rightarrow \infty} \left\| \beta_n^{(0)}(x_n - p) + \sum_{i=1}^N \beta_n^{(i)}(z_n^{(i)} - p) \right\|.$$

Since

$$\|z_n^{(i)} - p\| = \text{dist}(z_n^{(i)}, T_i p) \leq H(T_i x_n, T_i p) \leq \|x_n - p\|,$$

it implies that

$$\limsup_{n \rightarrow \infty} \|z_n^{(i)} - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = c.$$

Hence, by Lemma 2.5, we have

$$\lim_{n \rightarrow \infty} \|x_n - z_n^{(i)}\| = 0 \quad \text{for all } i = 1, 2, \dots, N.$$

(ii) Since t_i is generalized asymptotically nonexpansive, for all $i = 1, 2, \dots, N$, we get

$$\|t_i^n x_n - x_n\| \leq \|t_i^n x_n - t_i^n y_n\| + \|t_i^n y_n - x_n\| \leq k_n \|x_n - y_n\| + s_n + \|t_i^n y_n - x_n\|.$$

By the definition of $\{x_n\}$, we have $y_n - x_n = \sum_{i=1}^N \beta_n^{(i)} (z_n^{(i)} - x_n)$. This implies that

$$\begin{aligned} \|t_i^n x_n - x_n\| &\leq k_n \sum_{i=1}^N \beta_n^{(i)} \|z_n^{(i)} - x_n\| + \|t_i^n y_n - x_n\| + s_n \\ &\leq k_n \|z_n^{(i)} - x_n\| + \|t_i^n y_n - x_n\| + s_n. \end{aligned}$$

Then, by (i) and (3.2), we get

$$\lim_{n \rightarrow \infty} \|x_n - t_i^n x_n\| = 0 \quad \text{for all } i = 1, 2, \dots, N. \quad (3.3)$$

For $i = 1, 2, \dots, N$, we have

$$\begin{aligned} \|x_n - t_i x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + \|t_i^{n+1} x_{n+1} - t_i^{n+1} x_n\| + \|t_i^{n+1} x_n - t_i x_n\| \\ &\leq (1 + L) \|x_n - x_{n+1}\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + L \|t_i^n x_n - x_n\| \\ &\leq (1 + L) \sum_{i=1}^N \alpha_n^{(i)} \|x_n - t_i^n y_n\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + L \|t_i^n x_n - x_n\|. \end{aligned}$$

By (3.2) and (3.3), we conclude that $\lim_{n \rightarrow \infty} \|x_n - t_i x_n\| = 0$ for all $i = 1, 2, \dots, N$. $\square$

Theorem 3.5 *Let D be a nonempty, compact, and convex subset of a uniformly convex Banach space X . Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L -Lipschitzian and generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \dots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, \dots, N$ such that $0 < a \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$, and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.

Proof By Lemma 3.1, we have $\{x_n\}$ is bounded. Since D is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging strongly to $p \in D$. By condition (E), there exists $\mu \geq 1$ such that for $i = 1, 2, \dots, N$,

$$\begin{aligned} \text{dist}(p, T_i p) &\leq \|p - x_{n_j}\| + \text{dist}(x_{n_j}, T_i p) \\ &\leq \|x_{n_j} - p\| + \mu \text{dist}(x_{n_j}, T_i x_{n_j}) + \|x_{n_j} - p\| \end{aligned}$$

$$\begin{aligned}
&= 2\|x_{n_j} - p\| + \mu \operatorname{dist}(x_{n_j}, T_i x_{n_j}) \\
&\leq 2\|x_{n_j} - p\| + \mu \|x_{n_j} - z_{n_j}^{(i)}\|.
\end{aligned}$$

Then, by Lemma 3.4(i), we have $p \in T_i p$ for all $i = 1, 2, \dots, N$. So $p \in \bigcap_{i=1}^N F(T_i)$.

Since t_i is uniformly L -Lipschitzian, for all $i = 1, 2, \dots, N$, we have

$$\begin{aligned}
\|t_i p - p\| &\leq \|t_i p - t_i x_{n_j}\| + \|t_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\
&\leq (L+1)\|x_{n_j} - p\| + \|t_i x_{n_j} - x_{n_j}\|.
\end{aligned}$$

By Lemma 3.4(ii), it implies that $t_i p = p$ for all $i = 1, 2, \dots, N$. Thus, $p \in \bigcap_{i=1}^N F(t_i)$. Therefore, $p \in \mathcal{F}$. Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, we get $\lim_{n \rightarrow \infty} \|x_n - p\| = \lim_{j \rightarrow \infty} \|x_{n_j} - p\| = 0$. This shows that $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. $\square$

Next, we give a numerical example to support Theorem 3.5.

Example 3.6 Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $D = [0, 3]$. Define two single-valued mappings t_1 and t_2 on D as follows:

$$t_1 x = \sin x, \quad t_2 x = x.$$

Also we define two multi-valued mappings T_1 and T_2 on D as follows:

$$T_1 x = \begin{cases} [0, \frac{x}{3}], & x \neq 3; \\ \{1\}, & x = 3; \end{cases} \quad T_2 x = \left[\frac{x}{4}, \frac{x}{2} \right].$$

Let $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^2 \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^2 \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases} \quad (3.4)$$

where $\alpha_n^{(0)} = \frac{3n+4}{10n}$, $\alpha_n^{(1)} = \frac{2n-1}{5n}$, $\alpha_n^{(2)} = \frac{3n-2}{10n}$, $\beta_n^{(0)} = \frac{15n+7}{60n}$, $\beta_n^{(1)} = \frac{5n-1}{20n}$, $\beta_n^{(2)} = \frac{15n-2}{30n}$, for all $n \in \mathbb{N}$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 0, where $\{0\} = \bigcap_{i=1}^2 F(t_i) \cap \bigcap_{i=1}^2 F(T_i)$.

Solution It is shown in [19] that both t_1 and t_2 are generalized asymptotically nonexpansive single-valued mappings. Moreover, they are uniformly L -Lipschitzian mappings and $\bigcap_{i=1}^2 F(t_i) = \{0\}$. It is easy to see that both T_1 and T_2 are quasi-nonexpansive multi-valued mappings satisfying condition (E) and $\bigcap_{i=1}^2 F(T_i) = \{0\}$. Thus, $\bigcap_{i=1}^2 F(t_i) \cap \bigcap_{i=1}^2 F(T_i) = \{0\}$. For every $n \in \mathbb{N}$, $\alpha_n^{(0)} = \frac{3n+4}{10n}$, $\alpha_n^{(1)} = \frac{2n-1}{5n}$, $\alpha_n^{(2)} = \frac{3n-2}{10n}$, $\beta_n^{(0)} = \frac{15n+7}{60n}$, $\beta_n^{(1)} = \frac{5n-1}{20n}$, $\beta_n^{(2)} = \frac{15n-2}{30n}$. Then the sequences $\{\alpha_n^{(0)}\}$, $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\beta_n^{(0)}\}$, $\{\beta_n^{(1)}\}$, and $\{\beta_n^{(2)}\}$ satisfy all the conditions of Theorem 3.5. Put $z_n^{(1)} = \frac{x_n}{2}$ and $z_n^{(2)} = \frac{x_n}{3}$ for all $n \in \mathbb{N}$. Then the algorithm (3.4) becomes

$$\begin{cases} y_n = (\frac{13}{24} + \frac{5}{72n})x_n, \\ x_{n+1} = (\frac{37}{80} + \frac{5}{16n} - \frac{1}{72n^2})x_n + (\frac{2n-1}{5n})t_1^n y_n, & n \in \mathbb{N}. \end{cases} \quad (3.5)$$

Using the algorithm (3.5) with the initial point $x_1 = 2.5$, we have numerical results in Table 1.

Table 1 The values of the sequences $\{x_n\}$ and $\{y_n\}$ in Example 3.6

n	x_n	y_n
1	2.5000000	1.5277778
2	2.1025927	1.2119111
3	1.5352877	0.8671533
4	1.0799923	0.6037457
5	0.7544605	0.4191447
$\vdots$	$\vdots$	$\vdots$
21	0.0023377	0.0012740
$\vdots$	$\vdots$	$\vdots$
38	0.0000040	0.0000022
39	0.0000027	0.0000015
40	0.0000019	0.0000010
41	0.0000013	0.0000007
42	0.0000009	0.0000005

Finally, we prove a weak convergence theorem in uniformly convex Banach spaces.

Theorem 3.7 *Let D be a nonempty, closed, and convex subset of a uniformly convex Banach space X with the Opial property. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L -Lipschitzian and generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $KC(D)$ satisfying the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \dots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, \dots, N$ such that $0 < a \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$, and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges weakly to a point in $\mathcal{F}$.

Proof By Lemma 3.1, $\{x_n\}$ is bounded. Since X is uniformly convex, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to $p \in D$. By Lemma 3.4, we have $\lim_{j \rightarrow \infty} \|x_{n_j} - z_{n_j}^{(i)}\| = 0$ and $\lim_{j \rightarrow \infty} \|x_{n_j} - t_i x_{n_j}\| = 0$ for all $i = 1, 2, \dots, N$. We will show that $p \in \mathcal{F}$. Since $T_1 p$ is compact, for all $j \in \mathbb{N}$, we can choose $w_{n_j} \in T_1 p$ such that $\|x_{n_j} - w_{n_j}\| = \text{dist}(x_{n_j}, T_1 p)$ and the sequence $\{w_{n_j}\}$ has a convergent subsequence $\{w_{n_k}\}$ with $\lim_{k \rightarrow \infty} w_{n_k} = w \in T_1 p$. By condition (E), we have

$$\text{dist}(x_{n_k}, T_1 p) \leq \mu \text{dist}(x_{n_k}, T_1 x_{n_k}) + \|x_{n_k} - p\|.$$

Then we have

$$\begin{aligned} \|x_{n_k} - w\| &\leq \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - w\| \\ &= \text{dist}(x_{n_k}, T_1 p) + \|w_{n_k} - w\| \\ &\leq \mu \text{dist}(x_{n_k}, T_1 x_{n_k}) + \|x_{n_k} - p\| + \|w_{n_k} - w\| \\ &\leq \mu \|x_{n_k} - z_{n_k}^{(i)}\| + \|x_{n_k} - p\| + \|w_{n_k} - w\|. \end{aligned}$$

This implies that

$$\limsup_{k \rightarrow \infty} \|x_{n_k} - w\| \leq \limsup_{k \rightarrow \infty} \|x_{n_k} - p\|.$$

From the Opial property, we have $p = w \in T_1 p$. Similarly, it can be shown that $p \in T_i p$ for all $i = 2, \dots, N$. Thus, $p \in \bigcap_{i=1}^N F(T_i)$.

Next, by mathematical induction, we can prove that, for $i = 1, 2, \dots, N$,

$$\lim_{j \rightarrow \infty} \|x_{n_j} - t_i^m x_{n_j}\| = 0 \quad \text{for each } m \in \mathbb{N}. \quad (3.6)$$

Indeed, it is obvious that the conclusion is true for $m = 1$. Suppose the conclusion holds for $m \geq 1$. Since t_i is uniformly L -Lipschitzian, we have

$$\begin{aligned} \|x_{n_j} - t_i^{m+1} x_{n_j}\| &\leq \|x_{n_j} - t_i^m x_{n_j}\| + \|t_i^m x_{n_j} - t_i^{m+1} x_{n_j}\| \\ &\leq \|x_{n_j} - t_i^m x_{n_j}\| + L \|x_{n_j} - t_i x_{n_j}\|. \end{aligned}$$

This shows that $\lim_{j \rightarrow \infty} \|x_{n_j} - t_i^{m+1} x_{n_j}\| = 0$ for all $i = 1, 2, \dots, N$. Hence, (3.6) holds.

From (3.6), we have for each $x \in D$, $m \in \mathbb{N}$ and $i = 1, 2, \dots, N$,

$$\limsup_{j \rightarrow \infty} \|x_{n_j} - x\| = \limsup_{j \rightarrow \infty} \|t_i^m x_{n_j} - x\|. \quad (3.7)$$

Since t_i is generalized asymptotically nonexpansive, we get

$$\limsup_{j \rightarrow \infty} \|t_i^m x_{n_j} - t_i^m p\| \leq \limsup_{j \rightarrow \infty} (k_m \|x_{n_j} - p\| + s_m).$$

Then we have

$$\limsup_{m \rightarrow \infty} \left(\limsup_{j \rightarrow \infty} \|t_i^m x_{n_j} - t_i^m p\| \right) \leq \limsup_{j \rightarrow \infty} \|x_{n_j} - p\|. \quad (3.8)$$

By Proposition 2.1, we have

$$\begin{aligned} \left\| x_{n_j} - \frac{p + t_i^m p}{2} \right\|^2 &= \left\| \frac{1}{2}(x_{n_j} - p) + \frac{1}{2}(x_{n_j} - t_i^m p) \right\|^2 \\ &\leq \frac{1}{2} \|x_{n_j} - p\|^2 + \frac{1}{2} \|x_{n_j} - t_i^m p\|^2 - \frac{1}{4} g(\|p - t_i^m p\|). \end{aligned}$$

It implies that

$$\begin{aligned} \limsup_{j \rightarrow \infty} \left\| x_{n_j} - \frac{p + t_i^m p}{2} \right\|^2 &\leq \frac{1}{2} \limsup_{j \rightarrow \infty} \|x_{n_j} - p\|^2 + \frac{1}{2} \limsup_{j \rightarrow \infty} \|x_{n_j} - t_i^m p\|^2 \\ &\quad - \frac{1}{4} g(\|p - t_i^m p\|). \end{aligned} \quad (3.9)$$

By the Opial property and $\{x_{n_j}\}$ converging weakly to p , we obtain

$$\limsup_{j \rightarrow \infty} \|x_{n_j} - p\|^2 \leq \limsup_{j \rightarrow \infty} \left\| x_{n_j} - \frac{p + t_i^m p}{2} \right\|^2.$$

Then, by (3.9), we have

$$g(\|p - t^m p\|) \leq 2 \limsup_{j \rightarrow \infty} \|x_{n_j} - t_i^m p\|^2 - 2 \limsup_{j \rightarrow \infty} \|x_{n_j} - p\|^2. \quad (3.10)$$

It implies by (3.7), (3.8), and (3.10) that

$$\begin{aligned} \limsup_{m \rightarrow \infty} g(\|p - t_i^m p\|) &\leq 2 \limsup_{m \rightarrow \infty} \left(\limsup_{j \rightarrow \infty} \|x_{n_j} - t_i^m p\|^2 \right) - 2 \limsup_{j \rightarrow \infty} \|x_{n_j} - p\|^2 \\ &\leq 0. \end{aligned}$$

This shows that $\lim_{m \rightarrow \infty} g(\|p - t_i^m p\|) = 0$ for all $i = 1, 2, \dots, N$. Then the properties of g yield $\lim_{m \rightarrow \infty} \|p - t_i^m p\| = 0$ for all $i = 1, 2, \dots, N$. So we have

$$\begin{aligned} \|t_i p - p\| &\leq \|t_i p - t_i^{m+1} p\| + \|t_i^{m+1} p - p\| \\ &\leq L \|p - t_i^m p\| + \|t_i^{m+1} p - p\| \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$

This implies that $t_i p = p$ for all $i = 1, 2, \dots, N$. Thus, $p \in \bigcap_{i=1}^N F(t_i)$.

Hence, we obtain $p \in \mathcal{F}$.

Finally, we show that $\{x_n\}$ converges weakly to p . To show this, suppose not. Then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\{x_{n_l}\}$ converges weakly to $q \in D$ and $q \neq p$. By the same method as given above, we can prove that $q \in \mathcal{F}$. By Lemma 3.1, $\lim_{n \rightarrow \infty} \|x_n - p\|$ and $\lim_{n \rightarrow \infty} \|x_n - q\|$ exist. It follows by Lemma 2.6 that $q = p$. Thus, $\{x_n\}$ converges weakly to a point in $\mathcal{F}$. $\square$

Remark 3.8 Theorem 3.5 extends and generalizes the results of Sokhuma and Kaewkhao [11] to a pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of quasi-nonexpansive multi-valued mappings satisfying condition (E). Theorems 3.5 and 3.7 extend and generalize the results of Eslamian [12] and Eslamian and Abkar [13] to a pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of quasi-nonexpansive multi-valued mappings satisfying condition (E).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand. ²Department of Mathematics, Faculty of Science and Technology, Nakhon Pathom Rajabhat University, Nakhon Pathom, 73000, Thailand.

³Research Center for Pure and Applied Mathematics, Research and Development Institute, Nakhon Pathom Rajabhat University, Nakhon Pathom, 73000, Thailand.

Acknowledgements

This paper was supported by the Thailand Research Fund under the project RTA5780007.

Received: 21 January 2015 Accepted: 1 April 2015 Published online: 22 April 2015

References

1. Phuengrattana, W, Suantai, S: Existence theorems for generalized asymptotically nonexpansive mappings in uniformly convex metric spaces. *J. Convex Anal.* **20**(3), 753-761 (2013)

2. Saejung, S, Suantai, S, Yotkaew, P: A note on "Common Fixed Point of Multistep Noor Iteration with Errors for a Finite Family of Generalized Asymptotically Quasi-Nonexpansive Mapping". *Abstr. Appl. Anal.* **2009**, Article ID 283461 (2009)
3. Markin, JT: Continuous dependence of fixed point sets. *Proc. Am. Math. Soc.* **38**, 545-547 (1973)
4. Sastry, KPR, Babu, GVR: Convergence of Ishikawa iterates for a multivalued mapping with a fixed point. *Czechoslov. Math. J.* **55**, 817-826 (2005)
5. Panyanak, B: Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces. *Comput. Math. Appl.* **54**, 872-877 (2007)
6. Shahzad, N, Zegeye, H: On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces. *Nonlinear Anal.* **71**, 838-844 (2009)
7. Khan, SH, Yildirim, I: Fixed points of multivalued nonexpansive mappings in Banach spaces. *Fixed Point Theory Appl.* **2012**, Article ID 73 (2012)
8. Khan, SH, Abbas, M, Ali, S: Fixed points of multivalued quasi-nonexpansive mappings using a faster iterative process. *Acta Math. Sin. Engl. Ser.* **30**(7), 1231-1241 (2014)
9. García-Falset, J, Llorens-Fuster, E, Suzuki, T: Fixed point theory for a class of generalized nonexpansive mappings. *J. Math. Anal. Appl.* **375**, 185-195 (2011)
10. Abkar, A, Eslamian, M: Common fixed point results in CAT(0) spaces. *Nonlinear Anal.* **74**, 1835-1840 (2011)
11. Sokhuma, K, Kaewkhao, A: Ishikawa iterative process for a pair of single-valued and multivalued nonexpansive mappings in Banach spaces. *Fixed Point Theory Appl.* **2011**, Article ID 618767 (2011)
12. Eslamian, M: Weak and strong convergence theorems of iterative process for two finite families of mappings. *Sci. Bull. "Politeh." Univ. Buchar, Ser. A, Appl. Math. Phys.* **75**(4), 81-90 (2013)
13. Eslamian, M, Abkar, A: One-step iterative process for a finite family of multivalued mappings. *Math. Comput. Model.* **54**, 105-111 (2011)
14. Xu, HK: Inequalities in Banach spaces with applications. *Nonlinear Anal.* **16**, 1127-1138 (1991)
15. Opial, Z: Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **73**, 595-597 (1967)
16. Xu, HK: Another control condition in an iterative method for nonexpansive mappings. *Bull. Aust. Math. Soc.* **65**, 109-113 (2002)
17. Schu, J: Weak and strong convergence of fixed points of asymptotically nonexpansive mappings. *Bull. Aust. Math. Soc.* **43**, 153-159 (1991)
18. Suantai, S: Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings. *J. Math. Anal. Appl.* **311**, 506-517 (2005)
19. Kiziltunc, H, Yolacan, E: Strong convergence theorems for two total asymptotically nonexpansive nonself mappings in Banach spaces. *Fixed Point Theory Appl.* **2013**, Article ID 90 (2013)

1.3 A HYBRID SHRINKING PROJECTION METHOD FOR COMMONFIXED POINTS OF A FINITE FAMILY DEMICONTRACTIVE MAPPINGS WITH VARIATIONAL INEQUALITY PROBLEMS

In this section, we prove some properties of a demicontractive mapping defined on a nonempty closed convex subset of a Hilbert space. By using these properties, we obtain strong convergence theorems of a hybrid shrinking projection method for finding a common element of the set of common fixed points of a finite family of demicontractive mappings and the set of common solutions of a finite family of variational inequality problems in a Hilbert space. A numerical example is presented to illustrate the proposed method and convergence result. Our results improve and extend the corresponding results existing in the literature.

1. INTRODUCTION

Throughout this article, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, that C is a nonempty closed convex subset of H , and that P_C is the metric projection of H onto C . Let $T : C \rightarrow C$ be a mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and that it is called *quasi-nonexpansive* if $\overline{F(T)} \neq \emptyset$ and $\|Tx - z\| \leq \|x - z\|$ for all $x \in C$ and $z \in F(T)$. Clearly, every

nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive. It is well known that the fixed point set of a nonexpansive (quasi-nonexpansive) mapping in a Hilbert space is always closed and convex.

In 2011, Osilike and Isiogugu [10] introduced a class of nonlinear mappings which contains the class of nonexpansive mappings in a Hilbert space, namely, *strictly pseudo-nonspreading*, and obtained weak and strong convergence theorems of Halpern's type for this mapping in Hilbert spaces. Recall that a mapping T is called *strictly pseudo-nonspreading* if there exists $k \in [0, 1)$ such that, for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty \rangle,$$

where I denotes the identity mapping. Note that if $k = 0$, a mapping T is called *nonspreading* (see [9]). Iterative methods for strictly pseudo-nonspreading mapping have been extensively investigated (see, e.g., [2], [16]).

As a generalization of the class of strictly pseudo-nonspreading mappings, the class of demicontractive mappings was introduced by Hicks and Kubicek [4] in 1977 and has been studied by several authors (see, e.g., [3]). Recall that a mapping $T : C \rightarrow C$ is said to be *demicontractive* if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that, for all $x \in C$ and for all $z \in F(T)$,

$$\|Tx - z\|^2 \leq \|x - z\|^2 + k\|x - Tx\|^2.$$

We call k the *contraction coefficient*.

Remark 1.1. From the above definitions, we have the following facts.

- (i) Every nonspreading mapping with a nonempty fixed point set is quasi-nonexpansive.
- (ii) Every strictly pseudo-nonspreading mapping with a nonempty fixed point set is demicontractive.
- (iii) The class of demicontractive mappings includes the class of quasi-nonexpansive mappings.

We now give three examples for the class of demicontractive mappings.

Example 1.2. Let H be the real line, and let $C = [-1, 1]$. Define a mapping $T : C \rightarrow C$ by

$$Tx = \begin{cases} \frac{4}{7}x \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Obviously, $F(T) = \{0\}$. Also, for all $x \in C$, we have $|Tx - T0|^2 = |Tx|^2 = |\frac{4}{7}x \sin(\frac{1}{x})|^2 \leq |\frac{4x}{7}|^2 \leq |x|^2 \leq |x - 0|^2 + k|x - Tx|^2$ for all $k \in [0, 1)$. Therefore, T is demicontractive.

Example 1.3. Let H be the real line, and let $C = [-1, 1]$. Define a mapping $T : C \rightarrow C$ by

$$Tx = \begin{cases} \frac{9-x}{10}, & x \in [-1, 0), \\ \frac{x+9}{10}, & x \in [0, 1]. \end{cases}$$

Obviously, $F(T) = \{1\}$. We will show that there exists $k \in [0, 1)$ such that

$$|Tx - 1|^2 \leq |x - 1|^2 + k|x - Tx|^2 \quad (1.1)$$

for all $x \in [-1, 1]$. Then, it suffices to consider the following two cases.

Case 1: $x \in [-1, 0)$. Then we have $Tx = \frac{9-x}{10}$. From the definition of T , we have $|Tx - 1|^2 = |\frac{9-x}{10} - 1|^2 = \frac{1}{10}|x + 1|^2$ and $|x - Tx|^2 = |x - \frac{9-x}{10}|^2 = \frac{1}{100}|11x - 9|^2 \geq 0$. So, there exists $k \in [0, 1)$ such that

$$\begin{aligned} |x - 1|^2 + k|x - Tx|^2 &\geq |x - 1|^2 = x^2 - 2x + 1 = x^2 + 2x + 1 - 4x \\ &= |x + 1|^2 - 4x. \end{aligned}$$

Since $x \in [-1, 0)$, we have $-4x \geq 0$. Then we get $|x - 1|^2 + k|x - Tx|^2 \geq |x + 1|^2 \geq \frac{1}{10}|x + 1|^2 = |Tx - 1|^2$.

Case 2: $x \in [0, 1]$. Then we have $Tx = \frac{x+9}{10}$. From the definition of T , we have $|Tx - 1|^2 = |\frac{x+9}{10} - 1|^2 = \frac{1}{10}|x - 1|^2$ and $|x - Tx|^2 = |x - \frac{x+9}{10}|^2 = \frac{81}{100}|x - 1|^2 \geq 0$. Then, there exists $k \in [0, 1)$ such that

$$|x - 1|^2 + k|x - Tx|^2 \geq |x - 1|^2 \geq \frac{1}{10}|x - 1|^2 = |Tx - 1|^2.$$

Therefore, inequality (1.1) holds. Hence, T is demicontractive.

Example 1.4 ([3, p. 862]). Let H be the real line, and let $C = [-2, 1]$. Define a mapping $T : C \rightarrow C$ by $Tx = -x^2 - x$ for all $x \in C$. This mapping is demicontractive but not quasi-nonexpansive.

Let $B : C \rightarrow H$ be a mapping. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.2)$$

The set of solutions of (1.2) is denoted by $\text{VI}(C, B)$.

A mapping $B : C \rightarrow H$ is called *ϕ -inverse strongly monotone* (see [5]) if there exists a positive real number ϕ such that

$$\langle x - y, Bx - By \rangle \geq \phi \|Bx - By\|^2, \quad \forall x, y \in C.$$

The variational inequality theory, which was first introduced by Stampacchia [17] in 1964, has emerged as an interesting and fascinating branch of applied mathematics with a wide range of applications in economics, network analysis, optimizations, pure and applied sciences, and so on. In recent years, much attention has been given to developing efficient iterative methods for solving variational inequality problems (see, e.g., [1], [11]).

In 2003, Takahashi and Toyoda [20] introduced the following iterative scheme for finding a solution of the variational inequality problem in a Hilbert space. For an initial point $x_1 \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(x_n - \eta_n Bx_n), \quad n \geq 1, \quad (1.3)$$

where $T : C \rightarrow C$ is a nonexpansive mapping, $B : C \rightarrow H$ is a ϕ -inverse strongly monotone mapping, and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Under some restrictions on parameters, they proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to a point $u \in F(T) \cap \text{VI}(C, B)$, where $u = \lim_{n \rightarrow \infty} P_{F(T) \cap \text{VI}(C, B)} x_n$.

In 2008, Takahashi, Takeuchi, and Kubota [19] introduced the following iterative scheme, known as the *shrinking projection method*, for finding a fixed point of a nonexpansive mapping $T : C \rightarrow C$ in a Hilbert space. For an initial point $x_1 \in C$, $C_1 = C$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\| \leq \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1, \end{cases} \quad (1.4)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a point $u = P_{F(T)}x_1$ under some suitable conditions.

In 2012, Kangtunyakarn [6] introduced the following iterative scheme based on the shrinking projection method for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of nonspreading mappings $\{T_i\}_{i=1}^N$ in a Hilbert space. For an initial point $x_1 \in C$, $C_1 = C$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$\begin{cases} y_n = \alpha_n x_n + \beta_n Sx_n + \gamma_n \sum_{i=1}^N \delta_n^i P_C(I - \eta B_i)x_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\| \leq \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1, \end{cases} \quad (1.5)$$

where $S : C \rightarrow C$ is an S -mapping generated by $T_1, T_2, \dots, T_N$ and $\rho_1, \rho_2, \dots, \rho_N$, and where $\{B_i\}_{i=1}^N$ is a finite family of ϕ -inverse strongly monotone mappings of C into H . He proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a point $u = P_{\mathcal{F}}x_1$ under some suitable conditions, where $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N \text{VI}(C, B_i)$.

The shrinking projection method is a popular method that plays an important role in studying the strong convergence for finding fixed points of nonlinear mappings. Many researchers developed the shrinking projection method for solving variational inequality problems and fixed point problems in Hilbert and Banach spaces (see, e.g., [8], [7]).

Motivated and inspired by the above results and related literature, we introduce a new hybrid shrinking projection method for finding a common element of the set of common fixed points of a finite family of demicontractive mappings and of the set of common solutions of a finite family of variational inequalities for ϕ -inverse strongly monotone mappings in a Hilbert space. Then we prove some strong convergence theorems which extend and improve the corresponding results of Takahashi and Toyoda [20], Takahashi, Takeuchi, and Kubota [19], Kangtunyakarn [6], and many others. Finally, we provide numerical examples in support of our main results.

2. PRELIMINARIES

In this section, we give some useful lemmas to prove our main results. Let C be a nonempty closed convex subset of a real Hilbert space H . It is also known that a Hilbert space H satisfies *Opial's condition* (see [15, Lemma 1]); that is, for

any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Let P_C be the metric projection of H onto C ; that is, for $x \in H$, $P_C x$ satisfies the property $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. It is well known that P_C is a nonexpansive mapping of H onto C .

Lemma 2.1 ([18, Lemma 3.1.3]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H , and let B be a mapping of C into H . Let $u \in C$. Then, for $\eta > 0$, $u = P_C(I - \eta B)u$ if and only if $u \in \text{VI}(C, B)$.*

Lemma 2.2 ([12, Lemma 1.3]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $P_C : H \rightarrow C$ be the metric projection. Given $x \in H$ and $z \in C$, we have $z = P_C x$ if and only if the following holds:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.3 ([14, p. 375]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $P_C : H \rightarrow C$ be the metric projection. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, y \in C.$$

Lemma 2.4 ([21, Lemma 1.1]). *Let H be a Hilbert space. Let $x_1, x_2, \dots, x_N \in H$ and*

$\alpha_1, \alpha_2, \dots, \alpha_N$ be real numbers in $[0, 1]$ such that $\sum_{i=1}^N \alpha_i = 1$. Then,

$$\left\| \sum_{i=1}^N \alpha_i x_i \right\|^2 = \sum_{i=1}^N \alpha_i \|x_i\|^2 - \sum_{1 \leq i, j \leq N} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.5 ([13, Lemma 1.3]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Given $x, y, z \in H$ and α a real number, the set $\{u \in C : \|y - u\|^2 \leq \|x - u\|^2 + \langle z, u \rangle + \alpha\}$ is closed and convex.*

3. MAIN RESULTS

In this section, we prove the strong convergence theorems for finding a common element of the set of common fixed points of a finite family of demicontractive mappings and the set of common solutions of a finite family of variational inequality problems in a Hilbert space, and we give a numerical example in support our main results.

We now present the following useful properties of a demicontractive mapping in a Hilbert space.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H . Let $T : C \rightarrow C$ be a demicontractive mapping with contraction coefficient k . Then the following hold:*

- (i) $F(T) = \text{VI}(C, I - T)$,
- (ii) $F(T) = F(P_C(I - \lambda(I - T)))$ for all $\lambda > 0$,
- (iii) $F(T)$ is closed and convex,

(iv) $P_C(I - \lambda(I - T))$ is quasi-nonexpansive, for all $\lambda \in (0, 1 - k]$.

Proof. (i) It is easy to see that $F(T) \subseteq \text{VI}(C, I - T)$. Assume that $z \in \text{VI}(C, I - T)$ and $q \in F(T)$. Since $z \in \text{VI}(C, I - T)$, we have $\langle y - z, (I - T)z \rangle \geq 0, \forall y \in C$. Consider

$$\begin{aligned} \|Tz - Tq\|^2 &= \|(I - (I - T))z - (I - (I - T))q\|^2 \\ &= \|(z - q) - ((I - T)z - (I - T)q)\|^2 \\ &= \|z - q\|^2 + \|(I - T)z - (I - T)q\|^2 \\ &\quad - 2\langle z - q, (I - T)z - (I - T)q \rangle \\ &= \|z - q\|^2 + \|(I - T)z\|^2 - 2\langle z - q, (I - T)z \rangle. \end{aligned} \quad (3.1)$$

Since T is demicontractive, we have

$$\|Tz - Tq\|^2 \leq \|z - q\|^2 + k\|(I - T)z\|^2. \quad (3.2)$$

By (3.1) and (3.2), we have

$$(1 - k)\|(I - T)z\|^2 \leq 2\langle z - q, (I - T)z \rangle = -2\langle q - z, (I - T)z \rangle \leq 0.$$

Then we have $z \in F(T)$, and so $\text{VI}(C, I - T) \subseteq F(T)$. Hence, $F(T) = \text{VI}(C, I - T)$.

(ii) From Lemmas 2.1 and 3.1, we have $F(T) = F(P_C(I - \lambda(I - T)))$, $\forall \lambda > 0$.

(iii) To show that $F(T)$ is closed, we let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \rightarrow x$. Since T is demicontractive, we have $\|Tx - Tx_n\|^2 \leq \|x - x_n\|^2 + k\|x - Tx\|^2$. It follows that

$$\begin{aligned} \|x - Tx\| &\leq \|x - Tx_n\| + \|Tx_n - Tx\| \\ &\leq \|x - x_n\| + \sqrt{\|x - x_n\|^2 + k\|x - Tx\|^2}. \end{aligned}$$

By letting $n \rightarrow \infty$ in the above inequality, we get $(1 - \sqrt{k})\|x - Tx\| \leq 0$. This implies by $k \in [0, 1)$ that $\|x - Tx\| = 0$. Hence, $x = Tx$ so that $F(T)$ is closed.

Finally, we show that $F(T)$ is convex. Let $z = \alpha x + (1 - \alpha)y$, where $x, y \in F(T)$ and $\alpha \in (0, 1)$. We need to show that $z \in F(T)$. By the definition of T , we obtain

$$\begin{aligned} \|z - Tz\|^2 &= \|\alpha(x - Tz) + (1 - \alpha)(y - Tz)\|^2 \\ &= \alpha\|Tx - x\|^2 + (1 - \alpha)\|Ty - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha(\|z - x\|^2 + k\|z - Tx\|^2) + (1 - \alpha)(\|z - y\|^2 + k\|z - Ty\|^2) \\ &\quad - \alpha(1 - \alpha)\|x - y\|^2 \\ &= k\|z - Tz\|^2. \end{aligned}$$

Thus, $(1 - k)\|z - Tz\|^2 = 0$. This implies that $z = Tz$ and hence $F(T)$ is convex.

(iv) Let $x \in C$ and $z \in F(T)$. Then, by (ii), we have $z \in F(P_C(I - \lambda(I - T)))$. This implies that

$$\begin{aligned} \|P_C(I - \lambda(I - T))x - z\|^2 &\leq \|(I - \lambda(I - T))x - z\|^2 \\ &= \|(x - z) - \lambda(I - T)x\|^2 \\ &= \|x - z\|^2 - 2\lambda\langle x - z, (I - T)x \rangle \end{aligned}$$

$$+ \lambda^2 \|(I - T)x\|^2. \quad (3.3)$$

By the fact that T is demicontractive, we have

$$\|Tx - Tz\|^2 \leq \|x - z\|^2 + k\|(I - T)x\|^2. \quad (3.4)$$

Since

$$\begin{aligned} \|Tx - Tz\|^2 &= \|(I - (I - T))x - (I - (I - T))z\|^2 \\ &= \|(x - z) - ((I - T)x - (I - T)z)\|^2 \\ &= \|x - z\|^2 - 2\langle x - z, (I - T)x \rangle + \|(I - T)x\|^2, \end{aligned}$$

it follows by (3.4) that $(1 - k)\|(I - T)x\|^2 \leq 2\langle x - z, (I - T)x \rangle$. Therefore, by (3.3), we have

$$\begin{aligned} \|P_C(I - \lambda(I - T))x - z\|^2 &\leq \|x - z\|^2 - (1 - k)\lambda\|(I - T)x\|^2 + \lambda^2\|(I - T)x\|^2 \\ &= \|x - z\|^2 - \lambda(1 - k - \lambda)\|(I - T)x\|^2. \end{aligned}$$

This implies by $\lambda \in (0, 1 - k]$ that $\|P_C(I - \lambda(I - T))x - z\| \leq \|x - z\|$. Thus, $P_C(I - \lambda(I - T))$ is a quasi-nonexpansive mapping. $\square$

We now prove the following strong convergence theorems.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_i\}_{i=1}^N$ be a finite family of continuous and demicontractive mappings of C into itself with contraction coefficient k , and let $\{B_i\}_{i=1}^N$ be a finite family of ϕ -inverse strongly monotone mappings from C into H . Assume that $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N \text{VI}(C, B_i) \neq \emptyset$. For an initial point $x_1 \in C$ with $C_1 = C$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by*

$$\begin{cases} w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)x_n, \\ z_n = \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i))x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\| \leq \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1, \end{cases} \quad (3.5)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$, and suppose that the following conditions hold:

- (i) $\sigma_i, \xi_i \in (0, 1]$ and $\sum_{i=1}^N \sigma_i = \sum_{i=1}^N \xi_i = 1$;
- (ii) $\eta \in [d, e]$ for some $d, e \in (0, 2\phi)$ and $\lambda \in (0, 1 - k]$;
- (iii) $0 < a \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $u = P_{\mathcal{F}}x_1$.

Proof. Let $x, y \in C$. Since B_i is ϕ -inverse strongly monotone, for all $i = 1, 2, \dots, N$, we have

$$\begin{aligned} &\|P_C(I - \eta B_i)x - P_C(I - \eta B_i)y\|^2 \\ &\leq \|(I - \eta B_i)x - (I - \eta B_i)y\|^2 \\ &= \|x - y\|^2 - 2\eta\langle x - y, B_i x - B_i y \rangle + \eta^2\|B_i x - B_i y\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|x - y\|^2 - 2\eta\phi\|B_ix - B_iy\|^2 + \eta^2\|B_ix - B_iy\|^2 \\
&= \|x - y\|^2 - \eta(2\phi - \eta)\|B_ix - B_iy\|^2 \\
&\leq \|x - y\|^2 - d(2\phi - e)\|B_ix - B_iy\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

This shows that $P_C(I - \eta B_i)$ is nonexpansive for all $i = 1, 2, \dots, N$. It follows by Lemma 2.1 and the closedness and convexity of $F(P_C(I - \eta B_i))$ that $\text{VI}(C, B_i) = F(P_C(I - \eta B_i))$ is closed and convex for all $i = 1, 2, \dots, N$. Thus, $\bigcap_{i=1}^N \text{VI}(C, B_i)$ is closed and convex. Then, by Lemma 3.1(iii), we have that $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N \text{VI}(C, B_i)$ is also closed and convex; consequently, $P_{\mathcal{F}}x_1$ is well defined for every $x_1 \in C$.

Next, we divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is well defined for every $n \in \mathbb{N}$.

Let $p \in \mathcal{F}$. Then, by Lemmas 2.1 and 3.1(ii), we have $P_C(I - \eta B_i)p = p$ and $P_C(I - \lambda(I - T_i))p = p$ for all $i = 1, 2, \dots, N$. By Lemma 3.1(iv), we have

$$\begin{aligned}
\|z_n - p\| &\leq \sum_{i=1}^N \xi_i \|P_C(I - \lambda(I - T_i))x_n - p\| \leq \sum_{i=1}^N \xi_i \|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.6}$$

By the nonexpansiveness of $P_C(I - \eta B_i)$, we have

$$\|w_n - p\| \leq \sum_{i=1}^N \sigma_i \|P_C(I - \eta B_i)x_n - p\| \leq \sum_{i=1}^N \sigma_i \|x_n - p\| = \|x_n - p\|. \tag{3.7}$$

From the above, we get that $\|y_n - p\| \leq \alpha_n \|x_n - p\| + \beta_n \|z_n - p\| + \gamma_n \|w_n - p\| \leq \|x_n - p\|$. This shows that $p \in C_{n+1}$ and hence that $\mathcal{F} \subset C_{n+1} \subset C_n$. By Lemma 2.5, we observe that C_n is closed and convex. Hence, $P_{C_{n+1}}x_1$ is well defined for every $x_1 \in C$. Therefore, $\{x_n\}$ is well defined.

Step 2. We show that $\lim_{n \rightarrow \infty} x_n = q$ for some $q \in C$.

Since $\mathcal{F}$ is a nonempty closed convex subset of H , there exists a unique $\omega \in \mathcal{F}$ such that $\omega = P_{\mathcal{F}}x_1$. From $x_n = P_{C_n}x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$, for all $n \geq 1$, we have $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$ for all $n \geq 1$. On the other hand, by $\mathcal{F} \subset C_n$, we obtain that $\|x_n - x_1\| \leq \|\omega - x_1\|$ for all $n \geq 1$. Hence, $\{\|x_n - x_1\|\}$ is bounded; so are $\{w_n\}$, $\{z_n\}$, and $\{y_n\}$. Therefore, $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists. By the construction of the set C_n , we know that $x_m = P_{C_m}x_1 \in C_m \subset C_n$ for $m > n \geq 1$. This implies by Lemma 2.3 that

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2 \rightarrow 0, \quad \text{as } m, n \rightarrow \infty. \tag{3.8}$$

Since $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists, it implies that $\{x_n\}$ is a Cauchy sequence. By the completeness of H and the closedness of C , we get that there exists an element $q \in C$ such that $\lim_{n \rightarrow \infty} x_n = q$.

Step 3. We show that $q \in \bigcap_{i=1}^N F(T_i)$.

From (3.8), we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Since $x_{n+1} \in C_{n+1}$, we get that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\|.$$

This implies by (3.9) that

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (3.10)$$

Thus, $\lim_{n \rightarrow \infty} y_n = q$.

For $p \in \mathcal{F}$, by Lemma 2.4, (3.6), and (3.7), we have

$$\begin{aligned} \|y_n - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + \beta_n \|z_n - p\|^2 + \gamma_n \|w_n - p\|^2 - \alpha_n \beta_n \|z_n - x_n\|^2 \\ &\quad - \alpha_n \gamma_n \|w_n - x_n\|^2 - \beta_n \gamma_n \|z_n - w_n\|^2 \\ &\leq \|x_n - p\|^2 - \alpha_n \beta_n \|z_n - x_n\|^2 - \alpha_n \gamma_n \|w_n - x_n\|^2 - \beta_n \gamma_n \|z_n - w_n\|^2. \end{aligned}$$

This implies by condition (iii) that

$$\begin{aligned} &a^2 \|z_n - x_n\|^2 + a^2 \|w_n - x_n\|^2 + a^2 \|z_n - w_n\|^2 \\ &\leq \alpha_n \beta_n \|z_n - x_n\|^2 + \alpha_n \gamma_n \|w_n - x_n\|^2 + \beta_n \gamma_n \|z_n - w_n\|^2 \\ &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq \|x_n - y_n\| (\|x_n - p\| + \|y_n - p\|). \end{aligned}$$

Then, by (3.10) and $a > 0$, we have

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0, \quad \lim_{n \rightarrow \infty} \|w_n - x_n\| = 0, \quad \lim_{n \rightarrow \infty} \|z_n - w_n\| = 0. \quad (3.11)$$

Since T_i is continuous for all $i = 1, 2, \dots, N$, we have that $P_C(I - \lambda(I - T_i))$ is continuous for all $i = 1, 2, \dots, N$. So, $\sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i))$ is continuous. By $\lim_{n \rightarrow \infty} x_n = q$, this implies that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i)) x_n = \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i)) q. \quad (3.12)$$

Since

$$\begin{aligned} &\left\| \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i)) q - q \right\| \\ &\leq \left\| \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i)) q - \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i)) x_n \right\| \\ &\quad + \left\| \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i)) x_n - x_n \right\| + \|x_n - q\| \\ &= \left\| \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i)) q - \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i)) x_n \right\| + \|z_n - x_n\| \\ &\quad + \|x_n - q\|, \end{aligned}$$

this implies by $\lim_{n \rightarrow \infty} x_n = q$, (3.11), and (3.12) that

$$\sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i))q = q. \quad (3.13)$$

Next, we will show that $q \in \bigcap_{i=1}^N F(P_C(I - \lambda(I - T_i)))$. To show this, we let $z \in \bigcap_{i=1}^N F(P_C(I - \lambda(I - T_i)))$. Then, by (3.13) and Lemma 3.1(iv), we have

$$\begin{aligned} \|q - z\| &= \left\| \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i))q - z \right\| \leq \sum_{i=1}^N \xi_i \|P_C(I - \lambda(I - T_i))q - z\| \\ &\leq \|q - z\|. \end{aligned}$$

This shows that $\|q - z\| = \sum_{i=1}^N \xi_i \|P_C(I - \lambda(I - T_i))q - z\|$. Thus,

$$\|q - z\| = \|P_C(I - \lambda(I - T_i))q - z\|, \quad \forall i = 1, 2, \dots, N. \quad (3.14)$$

By Lemma 2.4, we obtain

$$\begin{aligned} &\left\| \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i))q - z \right\|^2 \\ &= \sum_{i=1}^N \xi_i \|P_C(I - \lambda(I - T_i))q - z\|^2 \\ &\quad - \sum_{1 \leq i, j \leq N} \xi_i \xi_j \|P_C(I - \lambda(I - T_i))q - P_C(I - \lambda(I - T_j))q\|^2. \end{aligned}$$

By (3.14), we have

$$\|q - z\|^2 = \|q - z\|^2 - \sum_{1 \leq i, j \leq N} \xi_i \xi_j \|P_C(I - \lambda(I - T_i))q - P_C(I - \lambda(I - T_j))q\|^2,$$

and so $\sum_{1 \leq i, j \leq N} \xi_i \xi_j \|P_C(I - \lambda(I - T_i))q - P_C(I - \lambda(I - T_j))q\|^2 = 0$. Since $\xi_i \in (0, 1]$, we get $\|P_C(I - \lambda(I - T_i))q - P_C(I - \lambda(I - T_j))q\|^2 = 0$ for all $i, j \in \{1, 2, \dots, N\}$. That is,

$$P_C(I - \lambda(I - T_i))q = P_C(I - \lambda(I - T_j))q, \quad \forall i, j \in \{1, 2, \dots, N\}.$$

Then, for all $j = 1, 2, \dots, N$, we have

$$q = \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_i))q = \sum_{i=1}^N \xi_i P_C(I - \lambda(I - T_j))q = P_C(I - \lambda(I - T_j))q.$$

Thus, $q \in \bigcap_{i=1}^N F(P_C(I - \lambda(I - T_i)))$. By Lemma 3.1(ii), we obtain that $F(P_C(I - \lambda(I - T_i))) = F(T_i)$ for all $i = 1, 2, \dots, N$. Hence, $q \in \bigcap_{i=1}^N F(T_i)$.

Step 4. We show that $q \in \bigcap_{i=1}^N \text{VI}(C, B_i)$.

In fact, since $P_C(I - \eta B_i)$ is nonexpansive, we have

$$\begin{aligned}
\left\| \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)q - w_n \right\| &= \left\| \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)q - \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)x_n \right\| \\
&= \left\| \sum_{i=1}^N \sigma_i (P_C(I - \eta B_i)q - P_C(I - \eta B_i)x_n) \right\| \\
&\leq \sum_{i=1}^N \sigma_i \|P_C(I - \eta B_i)q - P_C(I - \eta B_i)x_n\| \\
&\leq \|q - x_n\|.
\end{aligned} \tag{3.15}$$

From $x_n \rightarrow q \in C$ as $n \rightarrow \infty$, it follows by (3.11) and (3.15) that

$$\begin{aligned}
\left\| \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)q - q \right\| &\leq \left\| \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)q - w_n \right\| + \|w_n - x_n\| \\
&\quad + \|x_n - q\| \\
&\leq 2\|x_n - q\| + \|w_n - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

This shows that

$$\sum_{i=1}^N \sigma_i P_C(I - \eta B_i)q = q. \tag{3.16}$$

Next, we will show that $q \in \bigcap_{i=1}^N F(P_C(I - \eta B_i))$. To show this, we let $z \in \bigcap_{i=1}^N F(P_C(I - \eta B_i))$. Then, by (3.16), we have

$$\|q - z\| = \left\| \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)q - z \right\| \leq \sum_{i=1}^N \sigma_i \|P_C(I - \eta B_i)q - z\| \leq \|q - z\|.$$

This shows that $\|q - z\| = \sum_{i=1}^N \sigma_i \|P_C(I - \eta B_i)q - z\|$. Thus,

$$\|q - z\| = \|P_C(I - \eta B_i)q - z\|, \quad \forall i = 1, 2, \dots, N. \tag{3.17}$$

By Lemma 2.4, we obtain

$$\begin{aligned}
\left\| \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)q - z \right\|^2 &= \sum_{i=1}^N \sigma_i \|P_C(I - \eta B_i)q - z\|^2 \\
&\quad - \sum_{1 \leq i, j \leq N} \sigma_i \sigma_j \|P_C(I - \eta B_i)q - P_C(I - \eta B_j)q\|^2.
\end{aligned}$$

By (3.17), we have $\|q - z\|^2 = \|q - z\|^2 - \sum_{1 \leq i, j \leq N} \sigma_i \sigma_j \|P_C(I - \eta B_i)q - P_C(I - \eta B_j)q\|^2$, and so $\sum_{1 \leq i, j \leq N} \sigma_i \sigma_j \|P_C(I - \eta B_i)q - P_C(I - \eta B_j)q\|^2 = 0$. Since $\sigma_i \in (0, 1]$, we get $\|P_C(I - \eta B_i)q - P_C(I - \eta B_j)q\|^2 = 0$ for all $i, j \in \{1, 2, \dots, N\}$. That is,

$$P_C(I - \eta B_i)q = P_C(I - \eta B_j)q, \quad \forall i, j \in \{1, 2, \dots, N\}.$$

Then, for all $j = 1, 2, \dots, N$, we have

$$q = \sum_{i=1}^N \sigma_i P_C(I - \eta B_i)q = \sum_{i=1}^N \sigma_i P_C(I - \eta B_j)q = P_C(I - \eta B_j)q.$$

Thus, $q \in \bigcap_{i=1}^N F(P_C(I - \eta B_i))$. By Lemma 2.1, we have $F(P_C(I - \eta B_i)) = \text{VI}(C, B_i)$ for all $i = 1, 2, \dots, N$. Then we have $q \in \bigcap_{i=1}^N \text{VI}(C, B_i)$.

Step 5. Finally, we show that $q = u = P_{\mathcal{F}}x_1$. Since $x_n = P_{C_n}x_1$ and $\mathcal{F} \subset C_n$, we obtain $\langle x_1 - x_n, x_n - p \rangle \geq 0$ for all $p \in \mathcal{F}$. Thus, we get $\langle x_1 - q, q - p \rangle \geq 0$ for all $p \in \mathcal{F}$. This shows that $q = P_{\mathcal{F}}x_1 = u$.

By Steps 1–5, we can conclude that $\{x_n\}$ and $\{y_n\}$ converge strongly to $u = P_{\mathcal{F}}x_1$. This completes the proof. $\square$

As a direct consequence of Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a continuous and demicontractive mapping with contraction coefficient k , and let $B : C \rightarrow H$ be a ϕ -inverse strongly monotone mapping. Assume that $\mathcal{F} := F(T) \cap \text{VI}(C, B) \neq \emptyset$. For an initial point $x_1 \in C$ with $C_1 = C$, define sequences $\{x_n\}$ and $\{y_n\}$ recursively by*

$$\begin{cases} y_n = \alpha_n x_n + \beta_n P_C(I - \lambda(I - T))x_n + \gamma_n P_C(I - \eta B)x_n, \\ C_{n+1} = \{p \in C_n : \|y_n - p\| \leq \|x_n - p\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1, \end{cases} \quad (3.18)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$, and suppose that the following conditions hold:

- (i) $\eta \in [d, e]$ for some $d, e \in (0, 2\phi)$ and $\lambda \in (0, 1 - k]$;
- (ii) $0 < a \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $u = P_{\mathcal{F}}x_1$.

Proof. In Theorem 3.2, we put $N = 1$, $T_1 = T$, $B_1 = B$, $\sigma_1 = 1$, and $\xi_1 = 1$. Hence, we obtain the desired result from Theorem 3.2. $\square$

Remark 3.4. It is known that the class of demicontractive mappings contains the classes of nonexpansive mappings, nonspreading mappings, quasi-nonexpansive mappings, and strictly pseudo-nonspreading mappings. Thus, Theorem 3.2 and Corollary 3.3 can be applied to these classes of mappings.

Finally, we give a numerical example supporting Theorem 3.2. All codes were written in Scilab.

Example 3.5. Let H be a real line with the Euclidean norm, and let $C = [0, 15]$. For all $x \in C$, we define mappings T_1, T_2, B_1, B_2 on C as follows:

$$T_1x = \begin{cases} \frac{4}{7}x \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0, \end{cases} \quad T_2x = -\frac{5x}{7}, \quad B_1x = \frac{x}{15}, \quad B_2x = \frac{x}{5}.$$

Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by (3.5), where $\xi_1 = \xi_2 = \frac{1}{2}$, $\lambda = \frac{1}{6}$, $\sigma_1 = \frac{1}{4}$, $\sigma_2 = \frac{3}{4}$, $\eta = 4$, $\alpha_n = \frac{19}{27} + \frac{4}{135n}$, $\beta_n = \frac{10n-1}{135n}$, $\gamma_n = \frac{10n-1}{45n}$, for all $n \geq 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to zero.

Solution. Obviously, T_1 and T_2 are demicontractive mappings with contraction coefficient $\frac{2}{3}$, and B_1 and B_2 are 5-inverse strongly monotone mappings. It can be observed that all the assumptions of Theorem 3.2 are satisfied. It is also easy to see that $\bigcap_{i=1}^2 F(T_i) \cap \bigcap_{i=1}^2 \text{VI}(C, B_i) = \{0\}$. For any arbitrary $x_1 \in C = C_1 = [0, 15]$, it follows by (3.5) that $0 \leq y_1 \leq x_1 \leq 15$. Then we have $C_2 = \{p \in C_1 : |y_1 - p| \leq |x_1 - p|\} = [0, \frac{x_1+y_1}{2}]$. Since $\frac{x_1+y_1}{2} \leq x_1$, we get $x_2 = P_{C_2}x_1 = \frac{x_1+y_1}{2}$. By continuing this process, we obtain that $C_{n+1} = \{p \in C_n : |y_n - p| \leq |x_n - p|\} = [0, \frac{x_n+y_n}{2}]$, and hence that $x_{n+1} = P_{C_{n+1}}x_1 = \frac{x_n+y_n}{2}$.

Now, we rewrite the algorithm (3.5) as follows:

$$\begin{aligned} x_1 &\in [0, 15], \\ y_n &= \begin{cases} (\frac{9470n+187}{11340n})x_n + (\frac{10n-1}{2835n})x_n \sin(\frac{1}{x_n}), & x_n \neq 0, \\ 0, & x_n = 0, \end{cases} \\ x_{n+1} &= \frac{x_n + y_n}{2}, \quad n \geq 1. \end{aligned}$$

The values of the sequences $\{x_n\}$ and $\{y_n\}$ with different n 's are reported in Table 1.

Remark 3.6. Table 1 and Figure 1 show that the sequences $\{x_n\}$ and $\{y_n\}$ converge to zero, where $\{0\} = \bigcap_{i=1}^2 F(T_i) \cap \bigcap_{i=1}^2 \text{VI}(C, B_i)$.

TABLE 1. The values of the sequences $\{x_n\}$ and $\{y_n\}$ in Example 3.5.

	$x_1 = 4$		$x_1 = 11.5$	
n	x_n	y_n	x_n	y_n
1	4.0000000	3.4094908	11.5000000	9.7964246
2	3.7047454	3.1276784	10.6482123	8.9834323
3	3.4162119	2.8750077	9.8158223	8.2545229
4	3.1456098	2.6432389	9.0351726	7.5859259
5	2.8944243	2.4300595	8.3105492	6.9709719
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	0.0008595	0.0007206	0.0024875	0.0020767
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
176	0.0000012	0.0000010	0.0000035	0.0000030
177	0.0000011	0.0000010	0.0000032	0.0000027
178	0.0000010	0.0000009	0.0000030	0.0000025
179	0.0000010	0.0000008	0.0000027	0.0000023
180	0.0000009	0.0000007	0.0000025	0.0000021

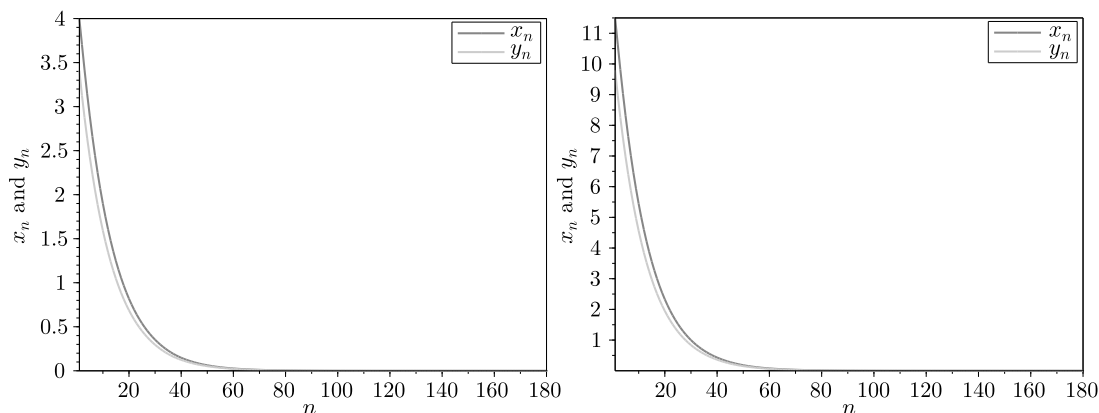


FIGURE 1. The convergence of the sequences $\{x_n\}$ and $\{y_n\}$ with initial values 4 and 11.5 in Example 3.5.

REFERENCES

1. Q. H. Ansari and J. C. Yao, *Systems of generalized variational inequalities and their applications*, Appl. Anal. **76** (2000), no. 3–4, 203–217. [Zbl 1018.49005](#). [MR1812468](#). [DOI 10.1080/00036810008840877](#). [663](#)
2. K. Aoyama and F. Kohsaka, *Fixed point and mean convergence theorems for a family of λ -hybrid mappings*, J. Nonlinear Anal. Optim. **2** (2011), no. 1, 87–95. [MR2981792](#). [662](#)
3. C. E. Chidume and S. Mărușter, *Iterative methods for the computation of fixed points of demicontractive mappings*, J. Comp. Appl. Math. **234** (2010), no. 3, 861–882. [Zbl 1191.65064](#). [MR2606672](#). [DOI 10.1016/j.cam.2010.01.050](#). [662](#), [663](#)
4. T. L. Hicks and J. D. Kubicek, *On the Mann iteration process in a Hilbert space*, J. Math. Anal. Appl. **59** (1977), no. 3, 498–504. [Zbl 0361.65057](#). [MR0513062](#). [DOI 10.1016/0022-247X\(77\)90076-2](#). [662](#)
5. H. Iiduka and W. Takahashi, *Weak convergence theorems by Cesáro means for nonexpansive mappings and inverse-strongly-monotone mappings*, J. Nonlinear Convex Anal. **7** (2006), no. 1, 105–113. [Zbl 1104.47059](#). [MR2218893](#). [663](#)
6. A. Kangtunyakarn, *Strong convergence of the hybrid method for a finite family of nonspreading mappings and variational inequality problems*, Fixed Point Theory Appl. **2012**, no. 188. [Zbl 06235292](#). [MR3017198](#). [664](#)
7. Y. Kimura, *Convergence of a sequence of sets in a Hadamard space and the shrinking projection method for a real Hilbert ball*, Abstr. Appl. Anal. **2010**, no. 582475. [Zbl 1206.54049](#). [MR2754196](#). [DOI 10.1155/2010/582475](#). [664](#)
8. Y. Kimura, K. Nakajo, and W. Takahashi, *Strongly convergent iterative schemes for a sequence of nonlinear mappings*, J. Nonlinear Convex Anal. **9** (2008), no. 3, 407–416. [Zbl 1162.47049](#). [MR2478973](#). [664](#)
9. F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), no. 2, 166–177. [Zbl 1149.47045](#). [MR2430800](#). [DOI 10.1007/s00013-008-2545-8](#). [662](#)
10. M. O. Osilike and F. O. Isiogugu, *Weak and strong convergence theorems for nonspreading mappings in Hilbert spaces*, Nonlinear Anal. **74** (2011), no. 5, 1814–1822. [Zbl 1281.47050](#). [MR2764382](#). [DOI 10.1016/j.na.2010.04.060](#). [662](#)

11. A. Latif, L. C. Ceng, and Q. H. Ansari, *Multi-step hybrid viscosity method for systems of variational inequalities defined over sets of solutions of an equilibrium problem and fixed point problems*, Fixed Point Theory Appl. **2012**, no. 186. [Zbl 1277.49010](#). [MR3017197](#). [663](#)
12. G. Marino and H. K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), no. 1, 336–346. [Zbl 1116.47053](#). [MR2306805](#). [665](#)
13. C. Martinez-Yanesa and H. K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal. **64** (2006), no. 11, 2400–2411. [Zbl 1105.47060](#). [MR2215815](#). [DOI 10.1016/j.na.2005.08.018](#). [665](#)
14. K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), no. 2, 372–379. [Zbl 1035.47048](#). [MR1974031](#). [DOI 10.1016/S0022-247X\(02\)00458-4](#). [665](#)
15. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. (N.S.) **73** (1967), 591–597. [Zbl 0179.19902](#). [MR0211301](#). [DOI 10.1090/S0002-9904-1967-11761-0](#). [664](#)
16. N. Petrot and R. Wangkeeree, *A general iterative scheme for strict pseudononspreading mapping related to optimization problem in Hilbert spaces*, J. Nonlinear Anal. Optim. **2** (2011), no. 2, 329–336. [MR2912086](#). [662](#)
17. G. Stampacchia, *Formes bilinéaires coercitives sur les ensembles convexes*, C. R. Math. Acad. Sci. Paris **258** (1964), 4413–4416. [Zbl 0124.06401](#). [MR0166591](#). [663](#)
18. W. Takahashi, *Nonlinear Function Analysis: Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, 2000. [Zbl 0997.47002](#). [MR1864294](#). [665](#)
19. W. Takahashi, Y. Takeuchi, and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), no. 1, 276–286. [Zbl 1134.47052](#). [MR2394082](#). [DOI 10.1016/j.jmaa.2007.09.062](#). [664](#)
20. W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), no. 2, 417–428. [Zbl 1055.47052](#). [MR2006529](#). [DOI 10.1023/A:1025407607560](#). [663](#), [664](#)
21. H. Zegeye and N. Shahzad, *Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings*, Comput. Math. Appl. **62** (2011), no. 11, 4007–4014. [Zbl 06026774](#). [MR2859956](#). [DOI 10.1016/j.camwa.2011.09.018](#). [665](#)

1.4 Viscosity iteration method in CAT(0) spaces without the nice projection property

In this section, we first define a concept of concerning some nice property of CAT(0) spaces. A complete CAT(0) space X is said to have the nice projection property (property $\mathcal{N}$ for short) if its metric projection onto a geodesic segment preserves points on each geodesic segment, that is, for any geodesic segment L in X and $x, y \in X, m \in [x, y]$ implies $P_L(m) \in [P_L(x), P_L(y)]$, where P_L denotes the metric projection from X onto L . In this paper, we prove a strong convergence theorem of a two-step viscosity iteration method for nonexpansive mappings in CAT(0) spaces without the condition on the property $\mathcal{N}$. Our result gives an affirmative answer to a problem raised by Piatek

1 Introduction

A mapping T on a metric space (X, ρ) is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$\rho(T(x), T(y)) \leq k\rho(x, y) \quad \text{for all } x, y \in X. \quad (1)$$

If (1) is valid when $k = 1$, then T is called *nonexpansive*. A point $x \in X$ is called a *fixed point* of T if $x = T(x)$. We shall denote by $\text{Fix}(T)$ the set of all fixed points of T .

One of the powerful iteration methods for finding fixed points of nonexpansive mappings was given by Moudafi [1]. More precisely, let C be a nonempty, closed, and convex subset of a Hilbert space H and $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$, the following scheme is known as the *viscosity iteration method*:

$$\begin{aligned} x_1 &= u \in C \text{ arbitrarily chosen,} \\ x_{n+1} &= \frac{\alpha_n}{1 + \alpha_n} f(x_n) + \frac{1}{1 + \alpha_n} T(x_n), \end{aligned} \quad (2)$$

where $f : C \rightarrow C$ is a contraction and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (iii) $\lim_{n \rightarrow \infty} (1/\alpha_n - 1/\alpha_{n+1}) = 0$. In [1], the author proved that the sequence $\{x_n\}$ defined by (2) converges strongly to a fixed point z of T . The point z also satisfies the following *variational inequality*:

$$\langle f(z) - z, z - x \rangle \geq 0, \quad x \in \text{Fix}(T).$$

The first extension of Moudafi's result to the so-called CAT(0) space was proved by Shi and Chen [2]. They assumed that the space (X, ρ) must satisfy the property $\mathcal{P}$, i.e., for $x, u, y_1, y_2 \in X$, one has

$$\rho(x, m_1)\rho(x, y_1) \leq \rho(x, m_2)\rho(x, y_2) + \rho(x, u)\rho(y_1, y_2),$$

where m_1 and m_2 are the unique nearest points of u on the segments $[x, y_1]$ and $[x, y_2]$, respectively. By using the concept of quasi-linearization introduced by Berg and Nikolaev [3], Wangkeeree and Preechasilp [4] could omit the property $\mathcal{P}$ from Shi and Chen's result as the following theorem.

Theorem A ([4], Theorem 3.4) *Let C be a nonempty, closed, and convex subset of a complete CAT(0) space X , $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$, and $f : C \rightarrow C$ be a contraction with $k \in [0, 1)$. For $x_1 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (iii) either $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} (\alpha_{n+1}/\alpha_n) = 1$. Then $\{x_n\}$ converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$ which is equivalent to the variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \text{Fix}(T).$$

Among other things, by using the geometric properties of CAT(0) spaces, Piatek [5] proved the strong convergence of a two-step viscosity iteration method as the following result.

Theorem B ([5], Theorem 4.3) *Let X be a complete CAT(0) space with the property $\mathcal{N}$. Let $T : X \rightarrow X$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$ and $f : X \rightarrow X$ be a contraction with $k \in [0, \frac{1}{2})$. Then there is a unique point $q \in \text{Fix}(T)$ such that $q = P_{\text{Fix}(T)}(f(q))$. Moreover, for each $u \in X$ and for each couple of sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ satisfying (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (iii) $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$, the viscosity iterative sequence defined by $x_1 = u$,*

$$y_n = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n),$$

$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)y_n, \quad \forall n \geq 1,$$

converges to q .

In [5], the author provided an example of a CAT(0) space lacking property $\mathcal{N}$ and also raised the following open problem.

Problem Can we omit the property $\mathcal{N}$ in Theorem B?

In this paper, by combining the ideas of [4] and [5] intensively, we can omit the property $\mathcal{N}$ from Theorem B. This gives a complete solution to the problem mentioned above.

2 Preliminaries

Let $[0, l]$ be a closed interval in $\mathbb{R}$ and x, y be two points in a metric space (X, ρ) . A *geodesic* joining x to y is a map $\xi : [0, l] \rightarrow X$ such that $\xi(0) = x$, $\xi(l) = y$, and $\rho(\xi(s), \xi(t)) = |s - t|$ for all $s, t \in [0, l]$. The image of ξ is called a *geodesic segment* joining x and y which when unique is denoted by $[x, y]$. The space (X, ρ) is said to be a *geodesic space* if every two points in X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset C of X is said to be *convex* if every pair of points $x, y \in C$ can be joined by a geodesic in X and the image of every such geodesic is contained in C .

A *geodesic triangle* $\Delta(p, q, r)$ in a geodesic space (X, ρ) consists of three points p, q, r in X and a choice of three geodesic segments $[p, q]$, $[q, r]$, $[r, p]$ joining them. A *comparison triangle* for the geodesic triangle $\Delta(p, q, r)$ in X is a triangle $\bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{p}, \bar{q}) = \rho(p, q)$, $d_{\mathbb{R}^2}(\bar{q}, \bar{r}) = \rho(q, r)$, and $d_{\mathbb{R}^2}(\bar{r}, \bar{p}) = \rho(r, p)$. A point $\bar{u} \in [\bar{p}, \bar{q}]$ is called a *comparison point* for $u \in [p, q]$ if $\rho(p, u) = d_{\mathbb{R}^2}(\bar{p}, \bar{u})$. Comparison points on $[\bar{q}, \bar{r}]$ and $[\bar{r}, \bar{p}]$ are defined in the same way.

Definition 2.1 A geodesic triangle $\Delta(p, q, r)$ in (X, ρ) is said to satisfy the *CAT(0) inequality* if for any $u, v \in \Delta(p, q, r)$ and for their comparison points $\bar{u}, \bar{v} \in \bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$, one has

$$\rho(u, v) \leq d_{\mathbb{R}^2}(\bar{u}, \bar{v}).$$

A geodesic space X is said to be a *CAT(0) space* if all of its geodesic triangles satisfy the CAT(0) inequality. For other equivalent definitions and basic properties of CAT(0) spaces, we refer the reader to standard texts, such as [6, 7]. It is well known that every CAT(0) space is uniquely geodesic. Notice also that pre-Hilbert spaces, $\mathbb{R}$ -trees, Euclidean buildings are examples of CAT(0) spaces (see [6, 8]). Let C be a nonempty, closed, and convex subset of a complete CAT(0) space (X, ρ) . It follows from Proposition 2.4 of [6] that for each $x \in X$, there exists a unique point $x_0 \in C$ such that

$$\rho(x, x_0) = \inf\{\rho(x, y) : y \in C\}.$$

In this case, x_0 is called the *unique nearest point* of x in C . The *metric projection* of X onto C is the mapping $P_C : X \rightarrow C$ defined by

$$P_C(x) := \text{the unique nearest point of } x \text{ in } C.$$

Definition 2.2 A complete CAT(0) space X is said to have the *nice projection property* [9] if for any geodesic segment L in X , it is the case that $P_L(m) \in [P_L(x), P_L(y)]$ for any $x, y \in X$ and $m \in [x, y]$.

Let (X, ρ) be a CAT(0) space. For each $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$\rho(x, z) = (1 - t)\rho(x, y) \quad \text{and} \quad \rho(y, z) = t\rho(x, y). \quad (3)$$

We shall denote by $tx \oplus (1 - t)y$ the unique point z satisfying (3). Now, we collect some elementary facts about CAT(0) spaces which will be used in the proof of our main theorem.

Lemma 2.3 ([10], Lemma 2.4) *Let (X, ρ) be a CAT(0) space. Then*

$$\rho(tx \oplus (1-t)y, z) \leq t\rho(x, z) + (1-t)\rho(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 2.4 ([10], Lemma 2.5) *Let (X, ρ) be a CAT(0) space. Then*

$$\rho^2(tx \oplus (1-t)y, z) \leq t\rho^2(x, z) + (1-t)\rho^2(y, z) - t(1-t)\rho^2(x, y)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 2.5 ([11], Lemma 3) *Let (X, ρ) be a CAT(0) space. Then*

$$\rho(tx \oplus (1-t)z, ty \oplus (1-t)z) \leq t\rho(x, y)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 2.6 (cf. [12, 13]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a CAT(0) space (X, ρ) and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)y_n$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n \rightarrow \infty} (\rho(y_{n+1}, y_n) - \rho(x_{n+1}, x_n)) \leq 0.$$

Then $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = 0$.

Lemma 2.7 ([14], Lemma 2.1) *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \beta_n \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$.

Then $\{s_n\}$ converges to zero as $n \rightarrow \infty$.

We finish this section by recalling an important concept of quasi-linearization introduced by Berg and Nikolaev [3]. Let us denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a *vector*. The *quasi-linearization* is a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} (\rho^2(a, d) + \rho^2(b, c) - \rho^2(a, c) - \rho^2(b, d)) \quad \text{for all } a, b, c, d \in X.$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that (X, ρ) satisfies the *Cauchy-Schwarz inequality* if

$$|\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle| \leq \rho(a, b)\rho(c, d) \quad \text{for all } a, b, c, d \in X.$$

It is known from [3], Corollary 3, that a geodesic space X is a $CAT(0)$ space if and only if X satisfies the Cauchy-Schwarz inequality. Some other properties of quasi-linearization are included as follows.

Lemma 2.8 ([4], Lemma 2.9) *Let X be a $CAT(0)$ space. Then*

$$\rho^2(x, u) \leq \rho^2(y, u) + 2\langle \vec{xy}, \vec{xu} \rangle$$

for all $u, x, y \in X$.

Lemma 2.9 ([4], Lemma 2.10) *Let u and v be two points in a $CAT(0)$ space X . For each $t \in [0, 1]$, we set $u_t = tu \oplus (1-t)v$. Then, for each $x, y \in X$, we have*

- (i) $\langle \vec{u_t x}, \vec{u_t y} \rangle \leq t\langle \vec{ux}, \vec{uy} \rangle + (1-t)\langle \vec{vx}, \vec{vy} \rangle$;
- (ii) $\langle \vec{u_t x}, \vec{u_t y} \rangle \leq t\langle \vec{ux}, \vec{uy} \rangle + (1-t)\langle \vec{vx}, \vec{uy} \rangle$ and $\langle \vec{u_t x}, \vec{v_t y} \rangle \leq t\langle \vec{ux}, \vec{vy} \rangle + (1-t)\langle \vec{vx}, \vec{vy} \rangle$.

The following fact, which can be found in [15], is an immediate consequence of Lemma 2.4.

Lemma 2.10 *Let X be a $CAT(0)$ space. Then*

$$\rho^2(tx \oplus (1-t)y, z) \leq t^2\rho^2(x, z) + (1-t)^2\rho^2(y, z) + 2t(1-t)\langle \vec{xz}, \vec{yz} \rangle$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

3 Main theorem

Before proving our main theorem, we need one more lemma, which is proved by Wang-keeree and Preechasilp (see [4], Theorem 3.1).

Lemma 3.1 *Let C be a nonempty, closed, and convex subset of a complete $CAT(0)$ space X , $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$, and $f : C \rightarrow C$ be a contraction with $k \in [0, 1)$. For each $t \in (0, 1)$, let $\{z_t\}$ be given by*

$$z_t = tf(z_t) \oplus (1-t)T(z_t).$$

Then $\{z_t\}$ converges strongly to $\tilde{x}$ as $t \rightarrow 0$. Moreover, $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$ and $\tilde{x}$ also satisfies the following variational inequality:

$$\langle \vec{\tilde{x}f(\tilde{x})}, \vec{\tilde{x}\tilde{x}} \rangle \geq 0, \quad x \in \text{Fix}(T). \quad (4)$$

Now, we are ready to prove our main theorem.

Theorem 3.2 *Let C be a nonempty, closed, and convex subset of a complete $CAT(0)$ space X , $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$, and $f : C \rightarrow C$ be a contraction with $k \in [0, \frac{1}{2})$. For the arbitrary initial point $u \in C$, let $\{x_n\}$ be generated by*

$$x_1 = u,$$

$$y_n = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n),$$

$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)y_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (iii) $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$ and $\tilde{x}$ also satisfies

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \text{Fix}(T).$$

Proof We divide the proof into three steps.

Step 1. We show that $\{x_n\}$, $\{y_n\}$, $\{T(x_n)\}$, and $\{f(x_n)\}$ are bounded sequences. Let $p \in \text{Fix}(T)$. By Lemma 2.3, we have

$$\begin{aligned} \rho(x_{n+1}, p) &\leq \beta_n \rho(x_n, p) + (1 - \beta_n) \rho(y_n, p) \\ &\leq \beta_n \rho(x_n, p) + (1 - \beta_n) [\alpha_n \rho(f(x_n), p) + (1 - \alpha_n) \rho(T(x_n), p)] \\ &\leq [\beta_n + (1 - \beta_n)(1 - \alpha_n)] \rho(x_n, p) + (1 - \beta_n) \alpha_n \rho(f(x_n), f(p)) \\ &\quad + (1 - \beta_n) \alpha_n \rho(f(p), p) \\ &\leq [1 - (1 - k)\alpha_n + (1 - k)\alpha_n \beta_n] \rho(x_n, p) + (1 - \beta_n) \alpha_n \rho(f(p), p) \\ &\leq \max \left\{ \rho(x_n, p), \frac{\rho(f(p), p)}{1 - k} \right\}. \end{aligned}$$

By induction, we also have

$$\rho(x_n, p) \leq \max \left\{ \rho(x_1, p), \frac{\rho(f(p), p)}{1 - k} \right\}.$$

Hence, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{f(x_n)\}$, and $\{T(x_n)\}$.

Step 2. We show that $\lim_{n \rightarrow \infty} \rho(x_n, T(x_n)) = 0$. By applying Lemma 2.5 twice for geodesic triangles $\triangle(f(x_n), T(x_n), T(x_{n+1}))$ and $\triangle(f(x_n), f(x_{n+1}), T(x_{n+1}))$, respectively, we obtain

$$\begin{aligned} \rho(y_n, y_{n+1}) &\leq (1 - \alpha_n) \rho(T(x_n), T(x_{n+1})) + |\alpha_n - \alpha_{n+1}| \rho(f(x_n), T(x_{n+1})) \\ &\quad + \alpha_{n+1} \rho(f(x_n), f(x_{n+1})) \\ &\leq (1 - \alpha_n) \rho(x_n, x_{n+1}) + |\alpha_n - \alpha_{n+1}| \rho(f(x_n), T(x_{n+1})) \\ &\quad + \alpha_{n+1} k \rho(x_n, x_{n+1}), \end{aligned}$$

which implies

$$\rho(y_n, y_{n+1}) - \rho(x_n, x_{n+1}) \leq (\alpha_{n+1} k - \alpha_n) \rho(x_n, x_{n+1}) + |\alpha_n - \alpha_{n+1}| \rho(f(x_n), T(x_{n+1})).$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\limsup_{n \rightarrow \infty} (\rho(y_{n+1}, y_n) - \rho(x_{n+1}, x_n)) \leq 0$. By Lemma 2.6 we have $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = 0$. Thus,

$$\begin{aligned} \rho(x_n, T(x_n)) &\leq \rho(x_n, y_n) + \rho(y_n, T(x_n)) \\ &= \rho(x_n, y_n) + \alpha_n \rho(f(x_n), T(x_n)) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Step 3. We show that $\{x_n\}$ converges to $\tilde{x}$, which satisfies $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$ and

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \text{Fix}(T).$$

Let $\{z_m\}$ be a sequence in C defined by

$$z_m = \alpha_m f(z_m) \oplus (1 - \alpha_m)T(z_m), \quad \forall m \in \mathbb{N}.$$

By Lemma 3.1, $\{z_m\}$ converges strongly as $m \rightarrow \infty$ to $\tilde{x}$ which satisfies (4) and $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$. We claim that

$$\limsup_{n \rightarrow \infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq 0.$$

It follows from Lemma 2.9(i) that

$$\begin{aligned} \rho^2(z_m, x_n) &= \langle \overrightarrow{z_m x_n}, \overrightarrow{z_m x_n} \rangle \\ &\leq \alpha_m \langle \overrightarrow{f(z_m)x_n}, \overrightarrow{z_m x_n} \rangle + (1 - \alpha_m) \langle \overrightarrow{T(z_m)x_n}, \overrightarrow{z_m x_n} \rangle \\ &= \alpha_m \langle \overrightarrow{f(z_m)f(\tilde{x})}, \overrightarrow{z_m x_n} \rangle + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_n} \rangle + \alpha_m \langle \overrightarrow{\tilde{x}z_m}, \overrightarrow{z_m x_n} \rangle + \alpha_m \langle \overrightarrow{z_m x_n}, \overrightarrow{z_m x_n} \rangle \\ &\quad + (1 - \alpha_m) \langle \overrightarrow{T(z_m)T(x_n)}, \overrightarrow{z_m x_n} \rangle + (1 - \alpha_m) \langle \overrightarrow{T(x_n)x_n}, \overrightarrow{z_m x_n} \rangle \\ &\leq \alpha_m k \rho(z_m, \tilde{x}) \rho(z_m, x_n) + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_n} \rangle + \alpha_m \rho(\tilde{x}, z_m) \rho(z_m, x_n) \\ &\quad + \alpha_m \rho^2(z_m, x_n) + (1 - \alpha_m) \rho^2(z_m, x_n) + (1 - \alpha_m) \rho(T(x_n), x_n) \rho(z_m, x_n) \\ &\leq \alpha_m (k + 1) \rho(z_m, \tilde{x}) M + \rho(T(x_n), x_n) M + \rho^2(z_m, x_n) + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_n} \rangle, \end{aligned}$$

for some $M > 0$. This implies

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n z_m} \rangle \leq (k + 1) \rho(z_m, \tilde{x}) M + \frac{\rho(x_n, T(x_n))}{\alpha_m} M. \quad (5)$$

Taking the upper limit as $n \rightarrow \infty$ first and then $m \rightarrow \infty$, the inequality (5) yields

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n z_m} \rangle \leq 0. \quad (6)$$

Notice also that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n \tilde{x}} \rangle = \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n z_m} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m \tilde{x}} \rangle \leq \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n z_m} \rangle + \rho(f(\tilde{x}), \tilde{x}) \rho(z_m, \tilde{x}).$$

This, together with (6), implies that

$$\limsup_{n \rightarrow \infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n \tilde{x}} \rangle \leq 0.$$

Finally, we show that $x_n \rightarrow \tilde{x}$ as $n \rightarrow \infty$. It follows from Lemmas 2.4, 2.8, 2.9, and 2.10 that

$$\begin{aligned} \rho^2(x_{n+1}, \tilde{x}) &\leq \beta_n \rho^2(x_n, \tilde{x}) + (1 - \beta_n) \rho^2(y_n, \tilde{x}) \\ &\leq \beta_n \rho^2(x_n, \tilde{x}) + (1 - \beta_n) [\alpha_n^2 \rho^2(f(x_n), \tilde{x}) + (1 - \alpha_n)^2 \rho^2(T(x_n), \tilde{x})] \end{aligned}$$

$$\begin{aligned}
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{T(x_n)\tilde{x}} \rangle \\
\leq & \beta_n\rho^2(x_n, \tilde{x}) + (1-\beta_n)(1-\alpha_n)^2\rho^2(x_n, \tilde{x}) \\
& + \alpha_n^2(1-\beta_n)[\rho^2(x_{n+1}, f(x_n)) + 2\langle \overrightarrow{\tilde{x}x_{n+1}}, \overrightarrow{\tilde{x}f(x_n)} \rangle] \\
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)[\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{T(x_n)x_n} \rangle + \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle] \\
\leq & [\beta_n + (1-\beta_n)(1-\alpha_n)]\rho^2(x_n, \tilde{x}) + \alpha_n^2(1-\beta_n)\rho^2(x_{n+1}, f(x_n)) \\
& + 2\alpha_n^2(1-\beta_n)[\langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle] \\
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{T(x_n)x_n} \rangle \\
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)[\langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_n\tilde{x}} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle] \\
\leq & [\beta_n + (1-\beta_n)(1-\alpha_n)]\rho^2(x_n, \tilde{x}) + \alpha_n^2(1-\beta_n)\rho^2(x_{n+1}, f(x_n)) \\
& + 2\alpha_n^2(1-\beta_n)\rho(f(x_n), f(\tilde{x}))\rho(x_{n+1}, \tilde{x}) + 2\alpha_n^2(1-\beta_n)\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)\rho(f(x_n), \tilde{x})\rho(T(x_n), x_n) \\
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)\rho(f(x_n), f(\tilde{x}))\rho(x_n, \tilde{x}) \\
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\
\leq & [\beta_n + (1-\beta_n)(1-\alpha_n)]\rho^2(x_n, \tilde{x}) + \alpha_n^2(1-\beta_n)\rho^2(x_{n+1}, f(x_n)) \\
& + 2k\alpha_n^2(1-\beta_n)\rho(x_n, \tilde{x})\rho(x_{n+1}, \tilde{x}) + 2\alpha_n^2(1-\beta_n)\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)\rho(f(x_n), \tilde{x})\rho(x_n, T(x_n)) \\
& + 2k\alpha_n(1-\alpha_n)(1-\beta_n)\rho^2(x_n, \tilde{x}) + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\
\leq & [\beta_n + (1-\beta_n)(1-\alpha_n)]\rho^2(x_n, \tilde{x}) + \alpha_n^2(1-\beta_n)\rho^2(x_{n+1}, f(x_n)) \\
& + k\alpha_n^2(1-\beta_n)[\rho^2(x_n, \tilde{x}) + \rho^2(x_{n+1}, \tilde{x})] + 2\alpha_n^2(1-\beta_n)\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\
& + 2\alpha_n(1-\alpha_n)(1-\beta_n)\rho(f(x_n), \tilde{x})\rho(x_n, T(x_n)) \\
& + 2k\alpha_n(1-\alpha_n)(1-\beta_n)\rho^2(x_n, \tilde{x}) + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\rho^2(x_{n+1}, \tilde{x}) \leq & \left[\frac{\beta_n + (1-\beta_n)(1-\alpha_n) + 2k\alpha_n(1-\alpha_n)(1-\beta_n)}{1 - k\alpha_n^2(1-\beta_n)} \right] \rho^2(x_n, \tilde{x}) \\
& + \frac{k\alpha_n^2(1-\beta_n)}{1 - k\alpha_n^2(1-\beta_n)} \rho^2(x_n, \tilde{x}) + \frac{\alpha_n^2(1-\beta_n)}{1 - k\alpha_n^2(1-\beta_n)} \rho^2(x_{n+1}, f(x_n)) \\
& + \frac{2\alpha_n(1-\alpha_n)(1-\beta_n)}{1 - k\alpha_n^2(1-\beta_n)} \rho(f(x_n), \tilde{x})\rho(x_n, T(x_n)) \\
& + \frac{2\alpha_n^2(1-\beta_n)}{1 - k\alpha_n^2(1-\beta_n)} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \frac{2\alpha_n(1-\alpha_n)(1-\beta_n)}{1 - k\alpha_n^2(1-\beta_n)} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle.
\end{aligned}$$

Thus,

$$\rho^2(x_{n+1}, \tilde{x}) \leq (1 - \alpha'_n)\rho^2(x_n, \tilde{x}) + \alpha'_n\beta'_n, \quad (7)$$

where $\alpha'_n = \frac{\alpha_n(1-\beta_n)(1-k(2-\alpha_n))}{1-k\alpha_n(1-\beta_n)}$ and

$$\begin{aligned}\beta'_n &= \frac{k\alpha_n}{1-k(2-\alpha_n)}\rho^2(x_n, \tilde{x}) + \frac{\alpha_n}{1-k(2-\alpha_n)}\rho^2(x_{n+1}, f(x_n)) \\ &\quad + \frac{2(1-\alpha_n)}{1-k(2-\alpha_n)}\rho(f(x_n), \tilde{x})\rho(x_n, T(x_n)) \\ &\quad + \frac{2\alpha_n}{1-k(2-\alpha_n)}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \frac{2(1-\alpha_n)}{1-k(2-\alpha_n)}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle.\end{aligned}$$

Since $k \in [0, \frac{1}{2})$, $\alpha'_n \in (0, 1)$. Applying Lemma 2.7 to the inequality (7), we can conclude that $x_n \rightarrow \tilde{x}$ as $n \rightarrow \infty$. This completes the proof. $\square$

4 Concluding remarks and open problems

- (1) Our main theorem can be applied to $\text{CAT}(\kappa)$ spaces with $\kappa \leq 0$ since any $\text{CAT}(\kappa)$ space is a $\text{CAT}(\kappa')$ space for $\kappa' \geq \kappa$ (see [6]). However, the result for $\kappa > 0$ is still unknown (see [5], p.1264).
- (2) Our main theorem can be viewed as an extension of Corollary 8 in [16] for a contraction f with $k \in [0, \frac{1}{2})$. It remains an open problem whether Theorem 3.2 holds for $k \in [\frac{1}{2}, 1)$.

References

1. Moudafi, A: Viscosity approximation methods for fixed-points problems. *J. Math. Anal. Appl.* **241**, 46-55 (2000)
2. Shi, LY, Chen, RD: Strong convergence of viscosity approximation methods for nonexpansive mappings in $\text{CAT}(0)$ spaces. *J. Appl. Math.* **2012**, Article ID 421050 (2012)
3. Berg, ID, Nikolaev, IG: Quasilinearization and curvature of Alexandrov spaces. *Geom. Dedic.* **133**, 195-218 (2008)
4. Wangkeeree, R, Preechasilp, P: Viscosity approximation methods for nonexpansive mappings in $\text{CAT}(0)$ spaces. *J. Inequal. Appl.* **2013**, Article ID 93 (2013)
5. Piatek, B: Viscosity iteration in $\text{CAT}(\kappa)$ spaces. *Numer. Funct. Anal. Optim.* **34**, 1245-1264 (2013)
6. Bridson, M, Haefliger, A: *Metric Spaces of Non-Positive Curvature*. Springer, Berlin (1999)
7. Burago, D, Burago, Y, Ivanov, S: *A Course in Metric Geometry*. Graduate Studies in Math., vol. 33. Am. Math. Soc., Providence (2001)
8. Brown, KS (ed.): *Buildings*. Springer, New York (1989)
9. Espinola, R, Fernandez-Leon, A: $\text{CAT}(k)$ -Spaces, weak convergence and fixed points. *J. Math. Anal. Appl.* **353**, 410-427 (2009)
10. Dhompongsa, S, Panyanak, B: On Δ -convergence theorems in $\text{CAT}(0)$ spaces. *Comput. Math. Appl.* **56**, 2572-2579 (2008)
11. Kirk, WA: Geodesic geometry and fixed point theory II. In: *International Conference on Fixed Point Theory and Applications*, pp. 113-142. Yokohama Publishers, Yokohama (2004)
12. Laowang, W, Panyanak, B: Strong and Δ convergence theorems for multivalued mappings in $\text{CAT}(0)$ spaces. *J. Inequal. Appl.* **2009**, Article ID 730132 (2009)
13. Suzuki, T: Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces. *Fixed Point Theory Appl.* **2005**, 103-123 (2005)
14. Xu, HK: An iterative approach to quadratic optimization. *J. Optim. Theory Appl.* **116**, 659-678 (2003)
15. Dehghan, H, Rooin, J: Metric projection and convergence theorems for nonexpansive mappings in Hadamard spaces. *J. Nonlinear Convex Anal.* (to appear)
16. Nilsrakoo, W, Saejung, S: Equilibrium problems and Moudafi's viscosity approximation methods in Hilbert spaces. *Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal.* **17**, 195-213 (2010)

Chapter II

Some Fixed Point Iterative Methods for Some Multi-valued Mappings in Hilbert Spaces and Geodesic Metric Spaces

In this chapter, we study fixed point iteration methods for multi-valued nonexpansive mappings, quasi-nonexpansive mappings and lambda-hybrid mappings in Hilbert and Geodesic metric spaces. We divide in this chapter into 5 sections as follows. In Section 2.1, we study a new proximal point algorithm for finding a common element of the set of fixed points of nonexpansive single-valued mappings, the set of fixed points of nonexpansive multi-valued mappings, and the set of minimizers of convex and lower semi-continuous functions. We obtain Δ -convergence and strong convergence of the proposed algorithm to a common element of the three sets in CAT(0) spaces. Furthermore, we apply our convergence results to obtain in a special space of CAT(0) spaces, so-called R-tree, under the gate condition. A numerical example to support our main results is also given. In Section 2.2, we prove a condition of the existence for generalized multi-valued mappings satisfying some inequalities in metric spaces. These results are improved versions of results of Boško Damjanović and Dragan Dorić (Filomat 25:125-131, 2011). In Section 2.3, we introduce a new class of set-valued mappings in a Hilbert space, called *Lambda-hybrid*, and prove some fixed point theorems for mappings in this class. Finally, we establish weak convergence of modified SP-iteration to a fixed point of *Lambda-hybrid* set-valued mappings in Hilbert spaces. In section 2.4, we propose a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in a CAT(0) space. We also prove Δ and strong convergence theorems of the proposed iterative process under some control conditions. In the last section, Section 2.4, we prove strong convergence of the viscosity approximation method for multivalued nonexpansive mappings in CAT(0) spaces. Our results generalize the results of Dhompongsa *et al.* (2012), Wangkeeree and Preechasilp (2013) and many others. Some related results in R-trees are also given.

2.1 Proximal Point Algorithms for a Hybrid Pair of Nonexpansive Single-Valued and Multi-Valued Mappings in Geodesic Metric Spaces

In this section, we propose a new proximal point algorithm for finding a common element of the set of fixed points of nonexpansive single-valued mappings, the set of fixed points of nonexpansive multi-valued mappings, and the set of minimizers of convex and lower semi-continuous functions. We obtain Δ -convergence and strong convergence of the proposed algorithm to a common element of the three sets in $\text{CAT}(0)$ spaces. Furthermore, we apply our convergence results to obtain in a special space of $\text{CAT}(0)$ spaces, so-called $\mathbb{R}$ -tree, under the gate condition. A numerical example to support our main results is also given.

1. Introduction

Let X be a geodesic metric space. We consider the problem of finding a point $x \in X$ such that

$$h(x) = \min_{u \in X} h(u), \tag{1.1}$$

where $h : X \rightarrow (-\infty, \infty]$ is a proper and convex function. The problem of finding a solution of (1.1) has interesting interpretations in various fields such as convex minimization problems, saddle point problems, variational inequalities, and complementarity problems; see, e.g., [7, 28, 31]. Until now, a variety of algorithms for solving (1.1) has been proposed and investigated. Among them, the proximal point algorithm, which was first proposed by Martinet [26] in 1970, is known for its theoretically nice convergence properties. Many of the algorithms for solving (1.1) have been generalized from classical linear spaces to the setting of manifolds. In particular, the proximal point algorithm

in the context of Riemannian manifold (of nonpositive sectional curvature) was studied in [8, 17, 24].

In 2013, Bačák [6] introduced the proximal point algorithm into geodesic metric spaces of nonpositive curvature, so-called CAT(0) spaces, as follows. For any initial point x_1 in a CAT(0) space X , a sequence $\{x_n\}$ generated by

$$x_{n+1} = \operatorname{argmin}_{u \in X} \left[h(u) + \frac{1}{2\lambda_n} d(u, x_n)^2 \right],$$

where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was shown that if h has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ -converges to its minimizer.

Later in 2015, Choleamjiak et al. [11] modified the proximal point algorithm for solving the minimization problem and the fixed point problem of two nonexpansive single-valued mappings in a CAT(0) space X as follows. For any initial point $x_1 \in X$, a sequence $\{x_n\}$ generated by

$$\begin{cases} z_n = \operatorname{argmin}_{u \in X} \left[h(u) + \frac{1}{2\lambda_n} d(u, x_n)^2 \right], \\ y_n = \beta_n x_n \oplus (1 - \beta_n) T_1 z_n, \\ x_{n+1} = \alpha_n T_1 x_n \oplus (1 - \alpha_n) T_2 y_n, \end{cases} \quad (1.2)$$

where $\lambda_n > 0$ for all $n \in \mathbb{N}$ and T_1, T_2 are nonexpansive single-valued mappings. They proved Δ -convergence and strong convergence of the sequence to minimizers of a convex and lower semi-continuous function and common fixed points of two nonexpansive single-valued mappings in complete CAT(0) spaces under suitable conditions.

In the recent years, the problem of finding a common element of the set of fixed points of single-valued mappings and the set of minimizers of convex and lower semi-continuous functions in the framework of CAT(0) spaces has been intensively studied by many researchers. However, up to now, no researchers have studied the problem of finding a common element of three sets, i.e., the set of fixed points of multi-valued mappings, the set of fixed points of single-valued mappings, and the set of minimizers of convex and lower semi-continuous functions.

In this paper, motivated by the research described above, we propose a new proximal point algorithm which is a modification of (1.2) for finding a common element of the set of fixed points of nonexpansive single-valued mappings, the set of fixed points of nonexpansive multi-valued mappings, and the set of minimizers of convex and lower semi-continuous functions. We prove Δ -convergence and strong convergence of the proposed algorithm to a common element of the set of fixed points of nonexpansive single-valued mappings, the set of fixed points of nonexpansive multi-valued mappings, and the set of minimizers of convex and lower semi-continuous functions in complete CAT(0) spaces. Furthermore, we apply our convergence results to obtain in the framework of $\mathbb{R}$ -trees under the gate condition. A numerical example to support our main results is also given.

2. Preliminaries and Lemmas

A *geodesic metric space* is a metric space X such that every two points of X are joined by a geodesic; see for more details in [9]. Let D be a nonempty subset of a geodesic metric space X and let $T : D \rightarrow D$ be a single-valued mapping. An element $x \in D$ is called a *fixed point* of T if $x = Tx$. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in D : x = Tx\}$.

Definition 2.1. A single-valued mapping $T : D \rightarrow D$ is said to be

- (i) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$;
- (ii) *semi-compact* if for any sequence $\{x_n\}$ in D such that

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0,$$

there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in D$.

Let $CB(D)$, $CC(D)$ and $KC(D)$ denote the families of nonempty closed bounded subsets, closed convex subsets and compact convex subsets of D , respectively. The *Pompeiu–Hausdorff distance* [33] on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\} \quad \text{for } A, B \in CB(D),$$

where $\text{dist}(x, D) = \inf\{d(x, y) : y \in D\}$ is the distance from a point x to a subset D . Let S be a multi-valued mapping of D into $CB(D)$. An element $x \in D$ is called a *fixed point* of S if $x \in Sx$. The set of all fixed points of S is denoted by $F(S)$, that is, $F(S) = \{x \in D : x \in Sx\}$.

A nonempty subset D of X is said to be *gated* if for any point $x \notin D$ there is a unique point y_x such that for any $z \in D$,

$$d(x, z) = d(x, y_x) + d(y_x, z).$$

Clearly, gate sets in a complete geodesic metric space are always closed and convex. The point y_x is called the *gate* of x in D . It is easy to see that y_x is also the unique nearest point of x in D . If $S : D \rightarrow CC(D)$ is a multi-valued mapping with $F(S) \neq \emptyset$, then we say that a point $u \in D$ is a *key* of S if, for each $x \in F(S)$, x is the gate of u in Sx . We say that S satisfies the *gate condition* if S has a key in D .

Definition 2.2. A multi-valued mapping $S : D \rightarrow CB(D)$ is said to be

- (i) *nonexpansive* if $H(Sx, Sy) \leq d(x, y)$ for all $x, y \in D$;
- (ii) *hemi-compact* if for any sequence $\{x_n\}$ in D such that

$$\lim_{n \rightarrow \infty} \text{dist}(x_n, Sx_n) = 0,$$

there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in D$.

One of the special spaces of geodesic metric spaces is a CAT(0) space (for further details on the subject, the reader is referred to [9]). The useful inequality of CAT(0) space is *(CN) inequality* [10], that is, if z, x, y are points

in a CAT(0) space and if $\frac{x \oplus y}{2}$ is the midpoint of the geodesic segment $[x, y]$, then the CAT(0) inequality implies

$$d\left(z, \frac{x \oplus y}{2}\right)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2, \quad (\text{CN})$$

which is equivalent to

$$d(z, \lambda x \oplus (1 - \lambda)y)^2 \leq \lambda d(z, x)^2 + (1 - \lambda)d(z, y)^2 - \lambda(1 - \lambda)d(x, y)^2, \quad (\text{CN}^*)$$

for any $\lambda \in [0, 1]$, where $\lambda x \oplus (1 - \lambda)y$ denotes the unique point in $[x, y]$. The (CN*) inequality has appeared in [15]. Moreover, if X is a CAT(0) space and $x, y \in X$, then for any $\lambda \in [0, 1]$, there exists a unique point $\lambda x \oplus (1 - \lambda)y \in [x, y]$ such that

$$d(z, \lambda x \oplus (1 - \lambda)y) \leq \lambda d(z, x) + (1 - \lambda)d(z, y),$$

for any $z \in X$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X . For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known by [14] that in a complete CAT(0) space, the asymptotic center $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ and call x the Δ -limit of $\{x_n\}$. Notice that given $\{x_n\} \subseteq X$ such that $\{x_n\}$ Δ -converge to x and given $y \in X$ with $x \neq y$,

$$\limsup_{n \rightarrow \infty} d(x_n, x) < \limsup_{n \rightarrow \infty} d(x_n, y).$$

Thus, every CAT(0) space X satisfies a condition which is known in Banach space theory as the Opial property.

We next collect some basic properties of the Δ -convergence which will be used in the sequel.

Lemma 2.3 [23]. *Every bounded sequence in a CAT(0) space has a Δ -convergent subsequence.*

Lemma 2.4 [13]. *Let D be a nonempty closed convex subset of a CAT(0) space X . If $\{x_n\}$ is a bounded sequence in D , then the asymptotic center of $\{x_n\}$ is in D .*

Lemma 2.5 [15]. *Let $\{x_n\}$ be a sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.6 [15]. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and $T : D \rightarrow D$ be a nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in D such that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ then $x = Tx$.*

Convex and lower semi-continuous functions on CAT(0) spaces are our principal object of interest in this paper. Recall that a function $h : D \rightarrow (-\infty, \infty]$ defined on a nonempty convex subset D of a CAT(0) space is *convex* if, for any geodesic $\gamma : [0, 1] \rightarrow D$, the composite function $h \circ \gamma$ is convex. We say that a function h defined on D is *lower semi-continuous* at a point $x \in D$ if

$$h(x) \leq \liminf_{n \rightarrow \infty} h(x_n)$$

for each sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = x$. A function h is said to be *lower semi-continuous* on D if it is lower semi-continuous at any point in D . For any $\lambda > 0$, define the *Moreau–Yosida resolvent* of h in CAT(0) spaces as follows:

$$J_\lambda x = \operatorname{argmin}_{u \in D} \left[h(u) + \frac{1}{2\lambda} d(u, x)^2 \right]$$

and put $J_0 x = x$, for all $x \in D$. This definition in metric spaces with no linear structure first appeared in [18]. The mapping J_λ is well defined for all $\lambda \geq 0$, see [18]. If h is a proper, convex and lower semi-continuous function, then the set of fixed points of the resolvent associated with h coincides with the set of minimizers of h ; see [5]. In addition, the resolvent J_λ of h is nonexpansive for all $\lambda > 0$; see [20].

Lemma 2.7 [20, 27]. *Assume that X is a complete CAT(0) space and $h : X \rightarrow (-\infty, \infty]$ is a proper convex and lower semi-continuous function. For each $x \in X$ and $\lambda > \mu > 0$, the following identity holds:*

$$J_\lambda x = J_\mu \left(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x \right),$$

where J_λ is a Moreau–Yosida resolvent of h .

Lemma 2.8 [3]. *Assume that X is a complete CAT(0) space and $h : X \rightarrow (-\infty, \infty]$ is a proper convex and lower semi-continuous function. For each $x, y \in X$ and $\lambda > 0$, the following inequality holds:*

$$\frac{1}{2\lambda} d(J_\lambda x, y)^2 - \frac{1}{2\lambda} d(x, y)^2 + \frac{1}{2\lambda} d(J_\lambda x, x)^2 \leq h(y) - h(J_\lambda x),$$

where J_λ is a Moreau–Yosida resolvent of h .

3. Convergence Results in CAT(0) Spaces

In this section, we prove that the modified proximal point algorithm converges to a common element of the set of fixed points of nonexpansive single-valued mappings, the set of fixed points of nonexpansive multi-valued mappings, and the set of minimizers of convex and lower semi-continuous functions, on a nonempty subset of complete CAT(0) spaces.

Theorem 3.1. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X . Let $T : D \rightarrow D$ be a nonexpansive single-valued mapping, $S : D \rightarrow CB(D)$ be a nonexpansive multi-valued mapping, and $h : D \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap F(S) \cap \operatorname{argmin}_{u \in D} h(u)$ is nonempty and $Sq = \{q\}$ for all $q \in \mathcal{F}$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$\begin{cases} y_n = \operatorname{argmin}_{u \in D} \left[h(u) + \frac{1}{2\lambda_n} d(u, x_n)^2 \right], \\ z_n = \beta_n y_n \oplus (1 - \beta_n) w_n, \quad w_n \in Sy_n, \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) Tz_n, \quad \forall n \in \mathbb{N}, \end{cases} \quad (3.1)$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ . Then, we have the following:

- (i) $\lim_{n \rightarrow \infty} d(x_n, q)$ exists for all $q \in \mathcal{F}$;
- (ii) $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$;
- (iii) $\lim_{n \rightarrow \infty} \operatorname{dist}(x_n, Sx_n) = 0$;
- (iv) $\lim_{n \rightarrow \infty} d(x_n, J_\lambda x_n) = 0$.

Proof. (i) Let $q \in \mathcal{F}$. Then we have $q = Tq \in Sq$ and $h(q) \leq h(u)$ for all $u \in D$. It follows that

$$h(q) + \frac{1}{2\lambda_n} d(q, q)^2 \leq h(u) + \frac{1}{2\lambda_n} d(u, q)^2, \quad \forall u \in D,$$

and hence $q = J_{\lambda_n} q$ for all $n \in \mathbb{N}$. Since $y_n = J_{\lambda_n} x_n$, it follows by nonexpansiveness of J_{λ_n} that

$$d(y_n, q) = d(J_{\lambda_n} x_n, J_{\lambda_n} q) \leq d(x_n, q). \quad (3.2)$$

For $q \in \mathcal{F}$, by $Sq = \{q\}$, we obtain that

$$\begin{aligned} d(z_n, q) &\leq \beta_n d(y_n, q) + (1 - \beta_n) d(w_n, q) \\ &\leq \beta_n d(y_n, q) + (1 - \beta_n) \operatorname{dist}(q, Sy_n) \\ &\leq \beta_n d(y_n, q) + (1 - \beta_n) H(Sq, Sy_n) \\ &\leq d(y_n, q). \end{aligned} \quad (3.3)$$

By (3.2) and (3.3), we have

$$\begin{aligned} d(x_{n+1}, q) &\leq \alpha_n d(x_n, q) + (1 - \alpha_n) d(Tz_n, q) \\ &= \alpha_n d(x_n, q) + (1 - \alpha_n) d(Tz_n, Tq) \\ &\leq \alpha_n d(x_n, q) + (1 - \alpha_n) d(z_n, q) \\ &\leq \alpha_n d(x_n, q) + (1 - \alpha_n) d(y_n, q) \\ &\leq d(x_n, q). \end{aligned} \quad (3.4)$$

This implies that $\{d(x_n, q)\}$ is decreasing and bounded below, thus $\lim_{n \rightarrow \infty} d(x_n, q)$ exists for all $q \in \mathcal{F}$.

(ii) By (i), we put

$$\lim_{n \rightarrow \infty} d(x_n, q) = l \quad \text{for some } l. \quad (3.5)$$

From (3.2), we have

$$\limsup_{n \rightarrow \infty} d(y_n, q) \leq \limsup_{n \rightarrow \infty} d(x_n, q) = l. \quad (3.6)$$

It follows by (3.4) that

$$\begin{aligned} d(x_n, q) &\leq \frac{1}{1-\alpha_n}(d(x_n, q) - d(x_{n+1}, q)) + d(y_n, q) \\ &\leq \frac{1}{1-b}(d(x_n, q) - d(x_{n+1}, q)) + d(y_n, q). \end{aligned}$$

This implies that

$$l = \liminf_{n \rightarrow \infty} d(x_n, q) \leq \liminf_{n \rightarrow \infty} d(y_n, q).$$

Therefore, by (3.6), we have

$$\lim_{n \rightarrow \infty} d(y_n, q) = l. \quad (3.7)$$

By Lemma 2.8, we get that

$$\frac{1}{2\lambda_n}d(y_n, q)^2 - \frac{1}{2\lambda_n}d(x_n, q)^2 + \frac{1}{2\lambda_n}d(y_n, x_n)^2 \leq h(q) - h(y_n).$$

Since $h(q) \leq h(y_n)$ for all $n \in \mathbb{N}$, we have

$$d(x_n, y_n)^2 \leq d(x_n, q)^2 - d(y_n, q)^2.$$

It implies by (3.5) and (3.7) that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0. \quad (3.8)$$

By (CN*) inequality, $Sq = \{q\}$, and (3.2), we get that

$$\begin{aligned} d(z_n, q)^2 &\leq \beta_n d(y_n, q)^2 + (1 - \beta_n) d(w_n, q)^2 - \beta_n(1 - \beta_n) d(y_n, w_n)^2 \\ &\leq \beta_n d(x_n, q)^2 + (1 - \beta_n) \text{dist}(q, Sy_n)^2 - \beta_n(1 - \beta_n) d(y_n, w_n)^2 \\ &\leq \beta_n d(x_n, q)^2 + (1 - \beta_n) H(Sq, Sy_n)^2 - \beta_n(1 - \beta_n) d(y_n, w_n)^2 \\ &\leq \beta_n d(x_n, q)^2 + (1 - \beta_n) d(q, y_n)^2 - \beta_n(1 - \beta_n) d(y_n, w_n)^2 \\ &\leq d(x_n, q)^2 - \beta_n(1 - \beta_n) d(y_n, w_n)^2, \end{aligned}$$

and so

$$\begin{aligned} d(x_{n+1}, q)^2 &\leq \alpha_n d(x_n, q)^2 + (1 - \alpha_n) d(Tz_n, q)^2 - \alpha_n(1 - \alpha_n) d(x_n, Tz_n)^2 \\ &\leq \alpha_n d(x_n, q)^2 + (1 - \alpha_n) d(z_n, q)^2 - \alpha_n(1 - \alpha_n) d(x_n, Tz_n)^2 \\ &\leq \alpha_n d(x_n, q)^2 + (1 - \alpha_n) (d(x_n, q)^2 - \beta_n(1 - \beta_n) d(y_n, w_n)^2) \\ &\quad - \alpha_n(1 - \alpha_n) d(x_n, Tz_n)^2 \\ &\leq d(x_n, q)^2 - \beta_n(1 - \beta_n)(1 - \alpha_n) d(y_n, w_n)^2 \\ &\quad - \alpha_n(1 - \alpha_n) d(x_n, Tz_n)^2. \end{aligned}$$

This implies that

$$\begin{aligned} \beta_n(1 - \beta_n)(1 - \alpha_n) d(y_n, w_n)^2 + \alpha_n(1 - \alpha_n) d(x_n, Tz_n)^2 &\leq d(x_n, q)^2 \\ &\quad - d(x_{n+1}, q)^2. \end{aligned}$$

Thus, by (3.5) and the condition $0 < a \leq \alpha_n, \beta_n \leq b < 1$, we have

$$\lim_{n \rightarrow \infty} d(x_n, Tz_n) = 0, \quad (3.9)$$

and

$$\lim_{n \rightarrow \infty} d(y_n, w_n) = 0. \quad (3.10)$$

Since $z_n = \beta_n y_n \oplus (1 - \beta_n)w_n$, we have

$$d(z_n, y_n) = (1 - \beta)d(y_n, w_n).$$

Then, by nonexpansiveness of T , we get that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, Tz_n) + d(Tz_n, Ty_n) + d(Ty_n, Tx_n) \\ &\leq d(x_n, Tz_n) + d(z_n, y_n) + d(y_n, x_n) \\ &= d(x_n, Tz_n) + (1 - \beta)d(y_n, w_n) + d(y_n, x_n). \end{aligned}$$

It follows by (3.8), (3.9), and (3.10) that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$.

(iii) By nonexpansiveness of S , we have

$$\begin{aligned} \text{dist}(x_n, Sx_n) &\leq d(x_n, y_n) + \text{dist}(y_n, Sy_n) + H(Sy_n, Sx_n) \\ &\leq 2d(x_n, y_n) + \text{dist}(y_n, Sy_n) \\ &\leq 2d(x_n, y_n) + d(y_n, w_n). \end{aligned}$$

This implies by (3.8) and (3.10) that $\lim_{n \rightarrow \infty} \text{dist}(x_n, Sx_n) = 0$.

(iv) Since $\lambda_n \geq \lambda > 0$, by Lemma 2.7 and nonexpansiveness of J_λ , we have

$$\begin{aligned} d(x_n, J_\lambda x_n) &\leq d(x_n, y_n) + d(y_n, J_\lambda x_n) \\ &= d(x_n, y_n) + d(J_{\lambda_n} x_n, J_\lambda x_n) \\ &= d(x_n, y_n) + d\left(J_\lambda \left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right), J_\lambda x_n\right) \\ &\leq d(x_n, y_n) + d\left(\left(\frac{\lambda_n - \lambda}{\lambda_n}\right) J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n, x_n\right) \\ &= d(x_n, y_n) + \left(1 - \frac{\lambda}{\lambda_n}\right) d(J_{\lambda_n} x_n, x_n) \\ &= d(x_n, y_n) + \left(1 - \frac{\lambda}{\lambda_n}\right) d(y_n, x_n) \\ &= \left(2 - \frac{\lambda}{\lambda_n}\right) d(y_n, x_n). \end{aligned}$$

Thus, by (3.8), we obtain that $\lim_{n \rightarrow \infty} d(x_n, J_\lambda x_n) = 0$. $\square$

We now get the Δ -convergence theorem in CAT(0) spaces.

Theorem 3.2. *Let D be a nonempty closed convex subset of a complete CAT(0) space X . Let $T : D \rightarrow D$ be a nonexpansive single-valued mapping, $S : D \rightarrow KC(D)$ be a nonexpansive multi-valued mapping, and $h : D \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap F(S) \cap \text{argmin}_{u \in D} h(u)$ is nonempty and $Sq = \{q\}$ for all $q \in \mathcal{F}$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ . Then the sequence $\{x_n\}$ Δ -converges to a point in $\mathcal{F}$.*

Proof. We put $\omega_\Delta(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Let $p \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{p\}$. By Lemmas 2.3 and 2.4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that

$$\Delta - \lim_{n \rightarrow \infty} v_n = v \in D. \quad (3.11)$$

From Theorem 3.1, we have $\lim_{n \rightarrow \infty} d(v_n, Tv_n) = 0$ and $\lim_{n \rightarrow \infty} d(v_n, J_\lambda v_n) = 0$. Then, by nonexpansiveness of T and J_λ , it implies by Lemma 2.6 that $v = Tv = J_\lambda v$. So, we get

$$v \in F(T) \cap F(J_\lambda) = F(T) \cap \operatorname{argmin}_{u \in D} h(u). \quad (3.12)$$

Since S is compact valued, for each $n \in \mathbb{N}$, there exist $r_n \in Sv_n$ and $\gamma_n \in Sv$ such that $d(v_n, r_n) = \operatorname{dist}(v_n, Sv_n)$ and $d(r_n, \gamma_n) = \operatorname{dist}(r_n, Sv)$. By Theorem 3.1(iii), it follows that

$$\lim_{n \rightarrow \infty} d(v_n, r_n) = 0.$$

By the compactness of Sv , there exists a subsequence $\{\gamma_{n_i}\}$ of $\{\gamma_n\}$ such that $\lim_{i \rightarrow \infty} \gamma_{n_i} = \gamma \in Sv$. Then we have

$$\begin{aligned} \limsup_{i \rightarrow \infty} d(v_{n_i}, \gamma) &\leq \limsup_{i \rightarrow \infty} (d(v_{n_i}, r_{n_i}) + d(r_{n_i}, \gamma_{n_i}) + d(\gamma_{n_i}, \gamma)) \\ &\leq \limsup_{i \rightarrow \infty} (d(v_{n_i}, r_{n_i}) + \operatorname{dist}(r_{n_i}, Sv) + d(\gamma_{n_i}, \gamma)) \\ &\leq \limsup_{i \rightarrow \infty} (d(v_{n_i}, r_{n_i}) + H(Sv_{n_i}, Sv) + d(\gamma_{n_i}, \gamma)) \\ &\leq \limsup_{i \rightarrow \infty} (d(v_{n_i}, r_{n_i}) + d(v_{n_i}, v) + d(\gamma_{n_i}, \gamma)) \\ &= \limsup_{i \rightarrow \infty} d(v_{n_i}, v). \end{aligned}$$

By (3.11) and the uniqueness of asymptotic centers, we obtain $v = \gamma \in Sv$. Thus, by (3.12), we have

$$v \in F(T) \cap F(S) \cap \operatorname{argmin}_{u \in D} h(u) = \mathcal{F}.$$

It follows by Lemma 2.5 and Theorem 3.1(i) that $p = v$, and hence $\omega_\Delta(x_n) \subseteq \mathcal{F}$.

To show that $\{x_n\}$ Δ -converges to a point in $\mathcal{F}$, it suffices to show that $\omega_\Delta(x_n)$ consists of exactly one point. Suppose that $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u^*\}$ and $A(\{x_n\}) = \{x\}$. Since $u^* \in \omega_\Delta(x_n) \subseteq \mathcal{F}$ and $\{d(x_n, u^*)\}$ converges, it implies by Lemma 2.5 that $x = u^*$. This completes the proof. $\square$

The following result is a strong convergence theorem for a modified proximal point algorithm in $\operatorname{CAT}(0)$ spaces.

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete $\operatorname{CAT}(0)$ space X . Let $T : D \rightarrow D$ be a nonexpansive single-valued mapping, $S : D \rightarrow CB(D)$ be a nonexpansive multi-valued mapping, and $h : D \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap F(S) \cap \operatorname{argmin}_{u \in D} h(u)$ is nonempty and $Sq = \{q\}$ for all $q \in \mathcal{F}$.*

For $x_1 \in D$, the sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ . If J_λ is semi-compact or T is semi-compact or S is hemi-compact, then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.

Proof. Without loss of generality, we assume S is hemi-compact. By Theorem 3.1(iii), $\lim_{n \rightarrow \infty} \text{dist}(x_n, Sx_n) = 0$. Then, by hemi-compactness of S , there exists a subsequence $\{u_n\}$ of $\{x_n\}$ which converges strongly to q in D . Again, by Theorem 3.1(ii)–(iv), we obtain that $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$, $\lim_{n \rightarrow \infty} \text{dist}(u_n, Su_n) = 0$, and $\lim_{n \rightarrow \infty} d(u_n, J_\lambda u_n) = 0$. It follows by nonexpansiveness of T and J_λ that $q = Tq = J_\lambda q$. So, we get

$$q \in F(T) \cap F(J_\lambda) = F(T) \cap \underset{u \in D}{\text{argmin}} h(u). \quad (3.13)$$

By nonexpansiveness of S , we have

$$\begin{aligned} \text{dist}(q, Sq) &\leq d(q, u_n) + \text{dist}(u_n, Su_n) + H(Su_n, Sq) \\ &\leq 2d(q, u_n) + \text{dist}(u_n, Su_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This shows that $\text{dist}(q, Sq) = 0$, that is, $q \in Sq$. Thus, $q \in F(S)$. Therefore, by (3.13), we have

$$q \in F(T) \cap F(S) \cap \underset{u \in D}{\text{argmin}} h(u) = \mathcal{F}.$$

By double extract subsequence principle, we can conclude that the sequence $\{x_n\}$ converges strongly to a point q in $\mathcal{F}$. $\square$

Since every multi-valued mapping $S : D \rightarrow CB(D)$ is hemi-compact if D is a compact subset of X . Then the following result can be obtained from Theorem 3.3 immediately.

Theorem 3.4. *Let D be a nonempty compact convex subset of a complete $CAT(0)$ space X . Let $T : D \rightarrow D$ be a nonexpansive single-valued mapping, $S : D \rightarrow CB(D)$ be a nonexpansive multi-valued mapping, and $h : D \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap F(S) \cap \underset{u \in D}{\text{argmin}} h(u)$ is nonempty and $Sq = \{q\}$ for all $q \in \mathcal{F}$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ . Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.*

Remark 3.5. (i) Since any $CAT(\kappa)$ space is a $CAT(\kappa')$ space for $\kappa' \geq \kappa$ (see [9]), all our results immediately apply to any $CAT(\kappa)$ space with $\kappa \leq 0$.

(ii) Theorem 3.2 extends and improves the results of Chulamjiak et al. [11], Khan and Abbas [21], Sokhuma and Kaewkhao [30]. In fact, we present a new proximal point algorithm for solving the constrained convex minimization problem as well as the fixed point problem of a pair of nonexpansive single-valued and multi-valued mappings in $CAT(0)$ spaces.

(iii) Theorems 3.3 and 3.4 generalize the results of Khan and Abbas [21] from one nonexpansive mapping to a pair of nonexpansive single-valued and multi-valued mappings involving the convex and lower semi-continuous function in $CAT(0)$ spaces.

We next give a numerical example supporting Theorems 3.1–3.4.

Example 3.6. Let $X = \mathbb{R}^2$ with the Euclidean norm and $D = \{\mathbf{x} = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2 : 0 \leq x^{(1)}, x^{(2)} \leq 1\}$. For each $\mathbf{x} = (x^{(1)}, x^{(2)}) \in D$, we define mappings T and S on D as follows:

$$T\mathbf{x} = \left(\frac{2x^{(1)} + 1}{4}, \frac{x^{(2)} + 7}{8} \right) \text{ and } S\mathbf{x} = \{x^{(1)}\} \times \left[\frac{x^{(2)} + 1}{2}, 1 \right].$$

For each $\mathbf{x} \in D$, we define $h : D \rightarrow (-\infty, \infty]$ by

$$(3.14) \quad h(\mathbf{x}) = \|\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{x}\|_2^2 + (-1.5, -2) \cdot \mathbf{x} + 7.$$

It is easy to check that T, S are nonexpansive and h is proper convex and lower semi-continuous. Using the proximity operator [12] and the soft thresholding operator [19], we have

$$\begin{aligned} J_1 \mathbf{x} &= \operatorname{argmin}_{\mathbf{u} \in D} \left[h(\mathbf{u}) + \frac{1}{2}\|\mathbf{u} - \mathbf{x}\|^2 \right] = \operatorname{prox}_h \mathbf{x} \\ &= \operatorname{prox}_{\frac{\|\cdot\|_1}{2}} \left(\frac{\mathbf{x} - (-1.5, -2)}{2} \right) \\ &= \left(\max \left\{ \frac{|x^{(1)} + 1.5| - 1}{2}, 0 \right\} \operatorname{sgn}(x^{(1)} + 1.5), \max \left\{ \frac{|x^{(2)} + 2| - 1}{2}, 0 \right\} \operatorname{sgn}(x^{(2)} + 2) \right), \end{aligned}$$

where $\operatorname{sgn}(\cdot)$ is the signum function defined by

$$\operatorname{sgn}(\xi) = \begin{cases} 1, & \xi > 0 \\ 0, & \xi = 0 \\ -1, & \xi < 0. \end{cases}$$

Then the algorithm (3.1) becomes:

$$\begin{cases} \mathbf{y}_n = J_1 \mathbf{x}_n, \\ \mathbf{z}_n = \beta_n \mathbf{y}_n + (1 - \beta_n) \mathbf{w}_n, & \mathbf{w}_n \in S\mathbf{y}_n, \\ \mathbf{x}_{n+1} = \alpha_n \mathbf{x}_n + (1 - \alpha_n) T\mathbf{z}_n, & \forall n \in \mathbb{N}, \end{cases} \quad (3.15)$$

where $\mathbf{x}_n = (x_n^{(1)}, x_n^{(2)})$, $\mathbf{y}_n = (y_n^{(1)}, y_n^{(2)})$, and $\mathbf{z}_n = (z_n^{(1)}, z_n^{(2)})$ are points in $\mathbb{R}^2$.

We choose $\mathbf{w}_n = (y_n^{(1)}, \frac{y_n^{(2)} + 1}{2}) \in S\mathbf{y}_n$, $\alpha_n = \frac{250n-1}{500n}$ and $\beta_n = \frac{80n-1}{100n}$. Using the algorithm (3.15) with the initial point $\mathbf{x}_1 = (0.1, 0.2)$, we have numerical results in Table 1.

From Table 1, we see that the sequence $\{\mathbf{x}_n\}$ converges to a point $(0.5, 1)$ which is a solution of the fixed point problems of a pair of a nonexpansive single-valued mapping T and a nonexpansive multi-valued mapping S and also a solution of the following constrained convex minimization problems:

$$\min_{\mathbf{x} \in D \subseteq \mathbb{R}^2} \|\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{x}\|_2^2 + (-1.5, -2) \cdot \mathbf{x} + 7.$$

Table 1. Numerical results of Example 3.6 for the algorithm (3.15)

n	$\mathbf{x}_n = (x_n^{(1)}, x_n^{(2)})$	$\ \mathbf{x}_n - \mathbf{x}_{n-1}\ _2$	$h(\mathbf{x}_n)$
1	(0.1000000, 0.2000000)	—	6.7750000
2	(0.3502000, 0.7992515)	6.4939e-01	6.4063700
3	(0.4250500, 0.8991883)	1.2486e-01	6.3828902
4	(0.4625167, 0.9495212)	6.2747e-02	6.3769766
5	(0.4812562, 0.9747423)	3.1421e-02	6.3754946
6	(0.4906275, 0.9873657)	1.5722e-02	6.3751237
7	(0.4953135, 0.9936810)	7.8640e-03	6.3750309
8	(0.4976567, 0.9968399)	3.9330e-03	6.3750077
9	(0.4988283, 0.9984197)	1.9669e-03	6.3750019
10	(0.4994141, 0.9992097)	9.8356e-04	6.3750005
11	(0.4997071, 0.9996048)	4.9183e-04	6.3750001
12	(0.4998535, 0.9998024)	2.4594e-04	6.3750000
13	(0.4999268, 0.9999012)	1.2298e-04	6.3750000
14	(0.4999634, 0.9999506)	6.1491e-05	6.3750000
15	(0.4999817, 0.9999753)	3.0747e-05	6.3750000
16	(0.4999908, 0.9999876)	1.5374e-05	6.3750000
17	(0.4999954, 0.9999938)	7.6873e-06	6.3750000
18	(0.4999977, 0.9999969)	3.8437e-06	6.3750000
19	(0.4999989, 0.9999985)	1.9219e-06	6.3750000
20	(0.4999994, 0.9999992)	9.6098e-07	6.3750000

4. Convergence Results in $\mathbb{R}$ -Trees

In this section, we apply our results to obtain in the framework of $\mathbb{R}$ -trees under the gate condition. We first introduce some definitions and useful lemmas of a special space of CAT(0) spaces, called $\mathbb{R}$ -tree.

In 1977, Tits [32] introduced an $\mathbb{R}$ -tree as follows.

Definition 4.1. A geodesic metric space X is called an $\mathbb{R}$ -tree if there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in X$, and if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Remark 4.2. It follows by Definition 4.1 that if u, v, w are points in $\mathbb{R}$ -tree X , then $[u, v] \cap [u, w] = [u, z]$ for some $z \in X$.

We now collect some basic properties of $\mathbb{R}$ -trees.

Lemma 4.3. Let X be a complete $\mathbb{R}$ -tree. Then the following statements hold:

- (i) [16] The gate subsets of X are precisely its closed and convex subsets.
- (ii) [25] If A and B are bounded closed convex subsets of X , then

$$d(P_A(\eta), P_B(\eta)) \leq H(A, B)$$

for any $\eta \in X$, where $P_A(\eta), P_B(\eta)$ are, respectively, the unique nearest points of η in A and B .

Fixed point theory for nonexpansive mappings in $\mathbb{R}$ -trees was first studied by Kirk [22]. Since then fixed point theorems for various types of single-valued and multi-valued mappings in $\mathbb{R}$ -trees have been rapidly developed and many of papers have appeared (e.g., see [1, 2, 4, 16, 22, 25]).

The aim of this section is to avoid the restriction of a multi-valued mapping S , that is, $Sq = \{q\}$ for $q \in F(S)$. We use the following iteration process defined in $\mathbb{R}$ -trees. Let η be a key of S . For $x_1 \in D$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = \operatorname{argmin}_{u \in D} \left[h(u) + \frac{1}{2\lambda_n} d(u, x_n)^2 \right], \\ z_n = \beta_n y_n \oplus (1 - \beta_n) w_n, \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T z_n, \quad \forall n \in \mathbb{N}, \end{cases} \quad (4.1)$$

where w_n is the gate of η in Sy_n ; $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Now, we get the convergence theorems in complete $\mathbb{R}$ -trees.

Theorem 4.4. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$ -tree X . Let $T : D \rightarrow D$ be a nonexpansive single-valued mapping, $S : D \rightarrow KC(D)$ be a nonexpansive multi-valued mapping, and $h : D \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap F(S) \cap \operatorname{argmin}_{u \in D} h(u)$ is nonempty and S satisfying the gate condition. Let η be a key of S . For $x_1 \in D$, the sequence $\{x_n\}$ generated by (4.1) where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ . Then, we have the following:*

- (i) $\lim_{n \rightarrow \infty} d(x_n, q)$ exists for all $q \in \mathcal{F}$;
- (ii) $\lim_{n \rightarrow \infty} d(x_n, T x_n) = 0$;
- (iii) $\lim_{n \rightarrow \infty} \operatorname{dist}(x_n, S x_n) = 0$;
- (iv) $\lim_{n \rightarrow \infty} d(x_n, J_\lambda x_n) = 0$.

Proof. Let $q \in \mathcal{F}$. By the gate condition, it implies by Lemma 4.3(ii) that

$$d(w_n, q) \leq d(P_{Sy_n}(\eta), P_{Sq}(\eta)) \leq H(Sq, Sy_n). \quad (4.2)$$

Using (4.2), we can prove this theorem by following the steps of the proof of Theorem 3.1. $\square$

Theorem 4.5. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$ -tree X . Let $T : D \rightarrow D$ be a nonexpansive single-valued mapping, $S : D \rightarrow KC(D)$ be a nonexpansive multi-valued mapping, and $h : D \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap F(S) \cap \operatorname{argmin}_{u \in D} h(u)$ is nonempty and S satisfying the gate condition. Let η be a key of S . For $x_1 \in D$, the sequence $\{x_n\}$ generated by (4.1) where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ . Then the sequence $\{x_n\}$ Δ -converges to a point in $\mathcal{F}$.*

Proof. We can prove this by following the steps of the proof of Theorem 3.2. $\square$

Theorem 4.6. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$ -tree X . Let $T : D \rightarrow D$ be a nonexpansive single-valued mapping, $S : D \rightarrow KC(D)$ be a nonexpansive multi-valued mapping, and $h : D \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap F(S) \cap \operatorname{argmin}_{u \in D} h(u)$ is nonempty and S satisfying the gate condition. Let η be a key of S . For $x_1 \in D$, the sequence $\{x_n\}$ generated by (4.1) where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ . If J_λ is semi-compact or T is semi-compact or S is hemi-compact, then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.*

Proof. Without loss of generality, we assume S is hemi-compact. By Theorem 4.4(iii), $\lim_{n \rightarrow \infty} \operatorname{dist}(x_n, Sx_n) = 0$. Then, by hemi-compactness of S , there exists a subsequence $\{u_n\}$ of $\{x_n\}$ converges strongly to q in D . Again, by Theorem 4.4(ii)–(iv), we obtain that $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$, $\lim_{n \rightarrow \infty} \operatorname{dist}(u_n, Su_n) = 0$, and $\lim_{n \rightarrow \infty} d(u_n, J_\lambda u_n) = 0$. Then, by the same arguments as in the proof of Theorem 3.3, we can conclude that the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. $\square$

Finally, we get the following consequence of Theorem 4.6.

Theorem 4.7. *Let D be a nonempty compact convex subset of a complete $\mathbb{R}$ -tree X . Let $T : D \rightarrow D$ be a nonexpansive single-valued mapping, $S : D \rightarrow KC(D)$ be a nonexpansive multi-valued mapping, and $h : D \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\mathcal{F} = F(T) \cap F(S) \cap \operatorname{argmin}_{u \in D} h(u)$ is nonempty and S satisfying the gate condition. Let η be a key of S . For $x_1 \in D$, the sequence $\{x_n\}$ generated by (4.1) where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ . Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.*

- Remark 4.8.* (i) Theorem 3.2 improves the results of Chulamjiak et al. [11], Khan and Abbas [21] to a pair of nonexpansive single-valued and multi-valued mappings involving the convex and lower semi-continuous function in $\mathbb{R}$ -trees.
- (iii) Theorems 3.3 and 3.4 extend and improve the results of Khan and Abbas [21] from one nonexpansive mapping to a pair of nonexpansive single-valued and multi-valued mappings involving the convex and lower semi-continuous function in $\mathbb{R}$ -trees.

References

- [1] Aksoy, A.G., Khamsi, M.A.: A selection theorem in metric trees. *Proc. Am. Math. Soc.* **134**, 2957–2966 (2006)
- [2] Aleomraninejad, S.M.A., Rezapour, S., Shahzad, N.: Some fixed point results on a metric space with a graph. *Topol. Appl.* **159**, 659–663 (2012)
- [3] Ambrosio, L., Gigli, N., Savare, G.: *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Lectures in Mathematics ETH Zurich, 2nd edn. Birkhuser, Basel (2008)
- [4] Amini-Harandi, A., Farajzadeh, A.P.: Best approximation, coincidence and fixed point theorems for set-valued maps in $\mathbb{R}$ -trees. *Nonlinear Anal.* **71**, 1649–1653 (2009)
- [5] Ariza-Ruiz, D., Leustean, L., Lopez, G.: Firmly nonexpansive mappings in classes of geodesic spaces. *Trans. Am. Math. Soc.* **366**, 4299–4322 (2014)
- [6] Bačák, M.: The proximal point algorithm in metric spaces. *Isr. J. Math.* **194**, 689–701 (2013)
- [7] Barbu, V., Precupanu, T.: *Convexity and Optimization in Banach Spaces*, 2nd edn. D. Reidel, Dordrecht (1986)
- [8] Bento, G., Ferreira, O., Oliveira, P.: Local convergence of the proximal point method for a special class of nonconvex functions on Hadamard manifolds. *Nonlinear Anal.* **73**, 564–572 (2010)
- [9] Bridson, M., Haefliger, A.: *Metric Spaces of Non-Positive Curvature*. Springer, Berlin (1999)
- [10] Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.* **41**, 5–251 (1972)
- [11] Cholakjia, P., Abdou, A.A., Cho, Y.J.: Proximal point algorithms involving fixed points of nonexpansive mappings in $CAT(0)$ spaces. *Fixed Point Theory Appl.* **227**, 1–13 (2015)
- [12] Combettes, P.L., Pesquet, J.C.: Proximal splitting methods in signal processing. In: Bauschke, H.H., Burachik, R., Combettes, P.L., Elser, V., Luke, D.R., Wolkowicz, H. (eds.) *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, pp. 185–212. Springer, New York (2011)
- [13] Dhompongsa, S., Kirk, W.A., Panyanak, B.: Nonexpansive set-valued mappings in metric and Banach spaces. *J. Nonlinear Convex Anal.* **8**, 35–45 (2007)
- [14] Dhompongsa, S., Kirk, W.A., Sims, B.: Fixed points of uniformly Lipschitzian mappings. *Nonlinear Anal.* **65**, 762–772 (2006)
- [15] Dhompongsa, S., Panyanak, B.: On Δ -convergence theorems in $CAT(0)$ spaces. *Comput. Math. Appl.* **56**, 2572–2579 (2008)
- [16] Espínola, R., Kirk, W.A.: Fixed point theorems in $\mathbb{R}$ -trees with applications to graph theory. *Topol. Appl.* **153**, 1046–1055 (2006)
- [17] Ferreira, O., Oliveira, P.: Proximal point algorithm on Riemannian manifolds. *Optimization* **51**, 257–270 (2002)
- [18] Guler, O.: On the convergence of the proximal point algorithm for convex minimization. *SIAM J. Control Optim.* **29**, 403–419 (1991)
- [19] E. T. Hale, W. Yin and Y. Zhang, A fixed-point continuation method for l_1 -regularized minimization with applications to compressed sensing. Tech. rep., CAAM TR07-07 (2007)

- [20] Jost, J.: Convex functionals and generalized harmonic maps into spaces of nonpositive curvature. *Comment. Math. Helv.* **70**, 659–673 (1995)
- [21] Khan, S.H., Abbas, M.: Strong and Δ -convergence of some iterative schemes in CAT(0) spaces. *Comput. Math. Appl.* **61**, 109–116 (2011)
- [22] Kirk, W.A.: Fixed point theorems in CAT(0) spaces and $\mathbb{R}$ -trees. *Fixed Point Theory Appl.* **2004**, 309–316 (2004)
- [23] Kirk, W.A., Panyanak, B.: A concept of convergence in geodesic spaces. *Nonlinear Anal.* **68**, 3689–3696 (2008)
- [24] Li, C., Lopez, G., Martyn-Marquez, V.: Monotone vector fields and the proximal point algorithm on Hadamard manifolds. *J. Lond. Math. Soc.* **79**, 663–683 (2009)
- [25] Markin, J.T.: Fixed points, selections and best approximation for multivalued mappings in $\mathbb{R}$ -trees. *Nonlinear Anal.* **67**, 2712–2716 (2007)
- [26] Martinet, B.: Regularisation d'inquations variationnelles par approximations successives. *Rev. Fr. Inform. Rech. Oper.* **4**, 154–158 (1970)
- [27] Mayer, U.F.: Gradient flows on nonpositively curved metric spaces and harmonic maps. *Commun. Anal. Geom.* **6**, 199–253 (1998)
- [28] Rockafellar, R.T., Wets, R.J.-B.: *Variational Analysis*. Springer, Berlin (1998)
- [29] Sokhuma, K.: Δ -convergence theorems for a pair of single-valued and multivalued nonexpansive mappings in CAT(0) spaces. *J. Math. Anal.* **4**(2), 23–31 (2013)
- [30] Sokhuma, K., Kaewkhao, A.: Ishikawa iterative process for a pair of single-valued and multivalued nonexpansive mappings in Banach spaces. *Fixed Point Theory Appl.* 2011 (2011) (Article ID 618767)
- [31] Takahashi, W.: *Nonlinear Functional Analysis. Fixed Points Theory and Its Applications*. Yokohama Publishers, Yokohama (2000)
- [32] Tits, J.: A Theorem of Lie-Kolchin for Trees. *Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin*. Academic Press, New York (1977)
- [33] Tyrrell Rockafellar, R., Wets, R.J.-B.: *Variational Analysis*. Springer, Berlin (2005)

2.2 Generalized multi-valued mappings satisfy some inequalities conditions on metric spaces

In this **section**, we prove a condition of the existence for generalized multi-valued mappings satisfying some inequalities in metric spaces. These results are improved versions of results of Boško Damjanović and Dragan Dorić (Filomat 25:125-131, 2011).

1 Introduction and preliminaries

Let (X, d) be a metric space. We denote by $CB(X)$ the family of all non-empty closed bounded subsets of X . Let $H(\cdot, \cdot)$ be the Hausdorff metric, *i.e.*,

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

for $A, B \in CB(X)$, where

$$d(x, B) = \inf_{y \in B} d(x, y).$$

- (i) Let T be a self-mapping on X . Then T is called a Banach contraction mapping if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$.

- (ii) T is called a Kannan mapping if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$$

for all $x, y \in X$.

(iii) If T is a mapping such that

$$d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty)\},$$

such that $r \in [0, 1)$ and all $x, y \in X$, then T is called a generalized Kannan mapping.

In 1973, Hardy and Rogers [1] introduced a condition as follows:

(iv) Let $x, y \in X$. Then there exists $a_i \geq 0$ such that

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx),$$

where $\sum_{i=1}^5 a_i < 1$.

(v) Ciric [2] defined the following condition which generalizes the Banach contraction and Kannan mapping, that is,

$$d(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

such that $r \in [0, 1)$ and all $x, y \in X$.

If X is complete and at least one of (i), (ii), (iii), (iv), and (v) holds, then T has a unique fixed point (see [1–5]).

In 2008, Suzuki [6] introduced the condition C as follows. T is said to satisfy condition C if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in C$.

In the same year, Kikkawa and Suzuki [7] generalized the Kannan mapping resulting in the following theorem.

Theorem 1.1 (Kikkawa and Suzuki [7]) *Let T be a mapping on complete metric space (X, d) and let φ be a non-increasing function from $[0, 1)$ into $(\frac{1}{2}, 1]$ defined by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Let $\alpha \in [0, \frac{1}{2})$ and put $r = \frac{\alpha}{1-\alpha} \in [0, 1)$. Suppose that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \quad (1.1)$$

for all $x, y \in X$. Then T has a unique fixed point z and $\lim_{n \rightarrow \infty} T^n x = z$ holds for every $x \in X$.

Theorem 1.2 (Kikkawa and Suzuki [7]) *Let T be a mapping on a complete metric space (X, d) and let θ be a non-increasing function from $[0, 1)$ into $(\frac{1}{2}, 1]$ defined by*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{1}{2}(\sqrt{5} - 1), \\ \frac{1-r}{r^2}, & \text{if } \frac{1}{2}(\sqrt{5} - 1) \leq r < \frac{1}{\sqrt{2}}, \\ \frac{1}{r+1}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Suppose that $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty)\} \quad (1.2)$$

for all $x, y \in X$. Then T has a unique fixed point z and $\lim_{n \rightarrow \infty} T^n x = z$ holds for every $x \in X$.

In 2011, Karapinar and Tas [8] stated some new conditions which are modifications of Suzuki's condition C , as follows. T is said to satisfy condition SCC if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq M(x, y)$$

for all $x, y \in K$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}.$$

In 1969, Nadler [9] proved a multi-valued extension of the Banach contraction theorem as follows.

Theorem 1.3 (Nadler [10]) *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that*

$$H(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Next, the result of Kikkawa and Suzuki [9] is a generalization of Nadler.

Theorem 1.4 (Kikkawa and Suzuki [9]) *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Define a strictly decreasing function η from $[0, 1)$ onto $(\frac{1}{2}, 1]$ by*

$$\eta(r) = \frac{r}{1+r}$$

and assume that there exists $r \in [0, 1)$ such that

$$\eta(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

In 2011, Damjanović and Dorić [11] generalized the result of Kannan (iii) and Nadler.

Theorem 1.5 (Damjanović and Dorić [11]) *Define a non-increasing function φ from $[0, 1)$ into $(0, 1]$ by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{\sqrt{5}-1}{2}, \\ 1-r, & \text{if } \frac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty)\} \quad (1.3)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Corollary 1.6 (Damjanović and Dorić [11]) *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Let $\alpha \in [0, \frac{1}{2})$ and put $r = 2\alpha$. Suppose that*

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \quad (1.4)$$

for all $x, y \in X$, where the function φ is defined as in Theorem 1.5. Then there exists $z \in X$ such that $z \in Tz$.

In this paper, we prove a condition of the existence for generalized multi-valued mappings under SCC conditions in metric spaces. These results are improved versions of results of Boško Damjanović and Dragan Dorić [11].

2 Main results

Theorem 2.1 *Define a non-increasing function φ from $[0, \frac{1}{2})$ into $(0, 1]$ by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{\sqrt{5}-1}{\sqrt{5}+1}, \\ \frac{1-2r}{1-r}, & \text{if } \frac{\sqrt{5}-1}{\sqrt{5}+1} \leq r < \frac{1}{2}. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq rM(x, y) \quad (2.1)$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < \frac{1}{2}$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, we have $d(u_2, Tu_2) \leq H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \leq d(u_1, Tu_1) \leq d(u_1, u_2).$$

Thus from the assumption (2.1),

$$d(u_2, Tu_2) \leq H(Tu_1, Tu_2) \leq rM(u_1, u_2)$$

where $M(u_1, u_2) = \max\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\}$. Consider

$$\begin{aligned} d(u_2, Tu_2) &\leq r \max\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\} \\ &= r \max\{d(u_1, u_2), d(u_1, Tu_2)\}. \end{aligned}$$

If $\max\{d(u_1, u_2), d(u_1, Tu_2)\} = d(u_1, Tu_2)$, then

$$\begin{aligned} d(u_2, Tu_2) &\leq rd(u_1, Tu_2) \\ &\leq rd(u_1, u_2) + rd(u_2, Tu_2) \end{aligned}$$

and then

$$d(u_2, Tu_2) \leq \left(\frac{r}{1-r}\right)d(u_1, u_2).$$

If $\max\{d(u_1, u_2), d(u_1, Tu_2)\} = d(u_1, u_2)$, then

$$d(u_2, Tu_2) \leq rd(u_1, u_2) \leq \left(\frac{r}{1-r}\right)d(u_1, u_2).$$

So

$$d(u_2, Tu_2) \leq \left(\frac{r}{1-r}\right)d(u_1, u_2).$$

So there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \leq \left(\frac{r_1}{1-r_1}\right)d(u_1, u_2)$. Thus, we can construct a sequence $\{x_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq \left(\frac{r_1}{1-r_1}\right)d(u_n, u_{n+1}).$$

Hence, by induction,

$$d(u_n, u_{n+1}) \leq \left(\frac{r_1}{1-r_1}\right)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} \left(\frac{r_1}{1-r_1}\right)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n \rightarrow \infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq rd(x, Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$. Since $u_n \rightarrow z$, there exists $n_0 \in \mathbb{N}$ such that $d(z, u_n) \leq \left(\frac{1}{3}\right)d(z, x)$ for all $n \geq n_0$. Then we have

$$\begin{aligned} \varphi(r)d(u_n, Tu_n) &\leq d(u_n, Tu_n) \\ &\leq d(u_n, u_{n+1}) \\ &\leq d(u_n, z) + d(z, u_{n+1}) \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{2}{3}\right)d(z, x) \\
&= d(z, x) - \frac{1}{3}d(z, x) \\
&\leq d(z, x) - d(z, u_n) \\
&\leq d(x, u_n).
\end{aligned} \tag{2.2}$$

Then from (2.1) we have

$$H(Tu_n, Tx) \leq r \max\{d(u_n, x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n)\}.$$

Since $u_{n+1} \in Tu_n$, $d(u_{n+1}, Tx) \leq H(Tu_n, Tx)$, so that

$$d(u_{n+1}, Tx) \leq r \max\{d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1})\}$$

for all $n \geq n_0$. Letting $n \rightarrow \infty$, we obtain

$$d(z, Tx) \leq r \max\{d(z, x), d(x, Tx), d(z, Tx)\}.$$

It follows that

$$d(z, Tx) \leq \left(\frac{r}{1-r}\right)d(x, Tx) \quad \text{for all } x \in X \setminus \{z\}. \tag{2.3}$$

Next, we show that $z \in Tz$. Suppose that z is not an element in Tz .

Case (i): $0 \leq r < \frac{\sqrt{5}-1}{\sqrt{5}+1}$. Let $a \in Tz$. Then $a \neq z$ and so by (2.3), we have

$$d(z, Ta) \leq \left(\frac{r}{1-r}\right)d(a, Ta).$$

On the other hand, since $\varphi(r)d(z, Tz) = d(z, Tz) \leq d(z, a)$, from (2.1) we have

$$H(Tz, Ta) \leq r \max\{d(z, a), d(z, Tz), d(a, Ta), d(z, Ta), d(a, Tz)\}.$$

So

$$d(a, Ta) \leq H(Tz, Ta) \leq r \max\{d(z, a), d(z, Tz), d(z, Ta)\}. \tag{2.4}$$

It implies that

$$d(a, Ta) \leq r \max\{d(z, a), d(z, Tz), d(z, Ta)\}.$$

Since $d(z, a) \leq d(z, Tz) + d(Tz, a) = d(z, Tz)$, we have

$$d(a, Ta) \leq \left(\frac{r}{1-r}\right)d(z, Tz). \tag{2.5}$$

Using (2.3), (2.4), and (2.5), we have

$$\begin{aligned}
d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\
&\leq \left(\frac{r}{1-r}\right)d(a, Ta) + r \max\{d(z, a), d(z, Tz), d(z, Ta)\} \\
&\leq \left(\frac{r}{1-r}\right)d(a, Ta) + r \max\left\{d(z, a), d(z, Tz), \left(\frac{r}{1-r}\right)d(a, Ta)\right\} \\
&\leq \left(\frac{r}{1-r}\right)d(a, Ta) + r \max\{d(z, a), d(z, Tz)\} \\
&\leq \left(\frac{r}{1-r}\right)d(a, Ta) + rd(z, Tz) \\
&\leq \left(\frac{r}{1-r}\right)^2 d(z, Tz) + rd(z, Tz) \\
&\leq \left(\frac{r}{1-r}\right)^2 d(z, Tz) + \left(\frac{r}{1-r}\right)d(z, Tz) \\
&\leq \left[\left(\frac{r}{1-r}\right)^2 + \left(\frac{r}{1-r}\right)\right]d(z, Tz) \\
&\leq [k^2 + k]d(z, Tz),
\end{aligned}$$

where $k = \frac{r}{1-r}$. Since $r < \frac{\sqrt{5}-1}{\sqrt{5}+1}$, we have $k^2 + k < 1$ and so $d(z, Tz) < d(z, Tz)$, a contradiction. Thus $z \in Tz$.

Case (ii): $\frac{\sqrt{5}-1}{\sqrt{5}+1} \leq r < \frac{1}{2}$. Let $x \in X$. If $x = z$, then $H(Tx, Tz) \leq r \max\{d(x, z), d(x, Tx), d(z, Tz), d(x, Tz), d(z, Tx)\}$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \leq d(z, Tx) + \left(\frac{1}{n}\right)d(x, z).$$

We consider

$$\begin{aligned}
d(x, Tx) &\leq d(x, y_n) \\
&\leq d(x, z) + d(z, y_n) \\
&\leq d(x, z) + d(z, Tx) + \left(\frac{1}{n}\right)d(x, z) \\
&\leq d(x, z) + \left(\frac{r}{1-r}\right)d(x, Tx) + \left(\frac{1}{n}\right)d(x, z).
\end{aligned}$$

Thus, $\left(\frac{1-2r}{1-r}\right)d(x, Tx) \leq \left(1 + \frac{1}{n}\right)d(x, z)$. Take $n \rightarrow \infty$, we obtain

$$\left(\frac{1-2r}{1-r}\right)d(x, Tx) \leq d(x, z),$$

by using (2.1), implies $H(Tx, Tz) \leq r \max\{d(x, z), d(x, Tx), d(z, Tz), d(x, Tz), d(z, Tx)\}$. Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$

$$\begin{aligned}
d(z, Tz) &= \lim_{n \rightarrow \infty} d(u_{n+1}, Tz) \\
&\leq H(Tu_n, Tz)
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} r \max \{d(u_n, z), d(u_n, Tu_n), d(z, Tz), d(u_n, Tz), d(z, Tu_n)\} \\
&\leq \lim_{n \rightarrow \infty} r \max \{d(u_n, z), d(u_n, u_{n+1}), d(z, Tz), d(u_n, Tz), d(z, u_{n+1})\} \\
&\leq rd(z, Tz).
\end{aligned}$$

Therefore, $(1-r)d(z, Tz) \leq 0$, which implies $d(z, Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. $\square$

Example 2.2 Let $X = [0, \infty)$ be endowed with the usual metric d . Define $T : X \rightarrow \text{CB}(X)$ by

$$T(x) = \begin{cases} [0, x^2], & 0 \leq x \leq \frac{1}{2}, \\ [0, \frac{x}{3}], & \frac{1}{2} < x < 1, \\ [0, \log(x)], & 1 \leq x. \end{cases} \quad (2.6)$$

Proof We show that T satisfies (2.1). Let $x, y \in X$. We prove by cases.

Case (i): Suppose that $x, y \in [0, \frac{1}{2}]$. Thus, if $x^2 \leq y$, then

$$\varphi\left(\frac{1}{4}\right)d(x, Tx) = |x - x^2| \geq |x - y| = d(x, y).$$

But if $x^2 > y$, then

$$\varphi\left(\frac{1}{4}\right)d(x, Tx) = |x - x^2| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned}
H(Tx, Ty) &= |x^2 - y^2| \\
&\leq \frac{1}{4} |(2x)^2 - (2y)^2| \\
&\leq \frac{1}{4} |x - 2y^2| \\
&\leq \frac{1}{4} |x - y^2| \\
&= \frac{1}{4} \max \{ |x - y|, |x - x^2|, |y - y^2|, |x - y^2|, |y - x^2| \} \\
&= \frac{1}{4} \max \{ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \} \\
&= rM(x, y),
\end{aligned} \quad (2.7)$$

where $r = \frac{1}{4}$. Hence T satisfies (2.1).

Case (ii): Suppose that $x, y \in (\frac{1}{2}, 1)$. Thus, if $\frac{x}{3} \leq y$, then

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = \left|x - \frac{x}{3}\right| \geq |x - y| = d(x, y).$$

But if $\frac{x}{3} > y$, then

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = \left|x - \frac{x}{3}\right| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned} H(Tx, Ty) &= \frac{1}{3}|x - y| \\ &\leq \frac{1}{3}\left|x - \frac{y}{3}\right| \\ &= \frac{1}{3} \max\left\{|x - y|, \left|x - \frac{x}{3}\right|, \left|y - \frac{y}{3}\right|, \left|x - \frac{y}{3}\right|, \left|y - \frac{x}{3}\right|\right\} \\ &= \frac{1}{3} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= rM(x, y), \end{aligned} \tag{2.8}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (iii): Suppose that $x, y \in [1, \infty]$. Thus, if $\log(x) \leq y$, then

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = |x - \log(x)| \geq |x - y| = d(x, y).$$

But if $\log(x) > y$, then

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = |x - \log(x)| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned} H(Tx, Ty) &= |\log(x) - \log(y)| \\ &= \frac{1}{3}(3\log(x) - 3\log(y)) \\ &\leq \frac{1}{3}|x - \log(y)| \\ &= \frac{1}{3} \max\{|x - y|, |x - \log(x)|, |y - \log(y)|, |x - \log(y)|, |y - \log(x)|\} \\ &= \frac{1}{3} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= rM(x, y), \end{aligned} \tag{2.9}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (iv): Suppose that $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1)$. Then $x^2 < x < y$. Thus, $\varphi(\frac{1}{3})d(x, Tx) = |x - x^2| \geq |x - y| = d(x, y)$. Hence T satisfies (2.1).

Case (v): Suppose that $x \in (\frac{1}{2}, 1)$ and $y \in [0, \frac{1}{2}]$. So $x > y$. Thus, if $\frac{x}{3} \leq y$, then

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = \left|x - \frac{x}{3}\right| \geq |x - y| = d(x, y).$$

But if $\frac{x}{3} > y$, then

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = \left|x - \frac{x}{3}\right| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned} H(Tx, Ty) &= \left|\frac{x}{3} - y^2\right| \\ &\leq \frac{1}{3}|x - 3y^2| \\ &\leq \frac{1}{3}|x - y^2| \\ &= \frac{1}{3} \max\left\{|x - y|, \left|x - \frac{x}{3}\right|, |y - y^2|, |x - y^2|, \left|y - \frac{x}{3}\right|\right\} \\ &= \frac{1}{3} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= rM(x, y), \end{aligned} \tag{2.10}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (vi): Suppose that $x \in [0, \frac{1}{2}]$ and $y \in [1, \infty]$.

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = |x - x^2| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned} H(Tx, Ty) &= |x^2 - \log(y)| \\ &= \frac{1}{3}|3x^2 - 3\log(y)| = \frac{1}{3}|3\log(y) - 3x^2| \\ &\leq \frac{1}{3} \max\{|y - \log(y)|, |y - x^2|\} \\ &= \frac{1}{3} \max\{|x - y|, |x - x^2|, |y - \log(y)|, |x - \log(y)|, |y - x^2|\} \\ &= \frac{1}{3} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= rM(x, y), \end{aligned} \tag{2.11}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (vii): Suppose that $x \in [1, \infty]$ and $y \in [0, \frac{1}{2}]$. Thus, if $\log(x) \leq y$, then

$$\varphi\left(\frac{1}{4}\right)d(x, Tx) = |x - \log(x)| \geq |x - y| = d(x, y).$$

But if $\log(x) > y$, then

$$\varphi\left(\frac{1}{4}\right)d(x, Tx) = |x - \log(x)| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned}
H(Tx, Ty) &= |\log(x) - y^2| \\
&= \frac{1}{4} |4\log(x) - 4y^2| \\
&\leq \frac{1}{4} |x - y^2| \\
&= \frac{1}{4} \max\{|x - y|, |x - \log(x)|, |y - y^2|, |x - y^2|, |y - \log(x)|\} \\
&= \frac{1}{4} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\
&= rM(x, y),
\end{aligned} \tag{2.12}$$

where $r = \frac{1}{4}$. Hence T satisfies (2.1).

Case (viii): Suppose that $x \in (\frac{1}{2}, 1)$ and $y \in [1, \infty]$.

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = \left|x - \frac{x}{3}\right| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned}
H(Tx, Ty) &= \left|\frac{x}{3} - \log(y)\right| \\
&= \frac{1}{3} |x - 3\log(y)| = \frac{1}{3} |3\log(y) - x| \\
&\leq \frac{1}{3} \max\left\{|y - \log(y)|, \left|y - \frac{x}{3}\right|\right\} \\
&= \frac{1}{3} \max\left\{|x - y|, \left|x - \frac{x}{3}\right|, |y - \log(y)|, |x - \log(y)|, \left|y - \frac{x}{3}\right|\right\} \\
&= \frac{1}{3} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\
&= rM(x, y),
\end{aligned} \tag{2.13}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Case (ix): Suppose that $x \in [1, \infty]$ and $y \in (\frac{1}{2}, 1)$. Thus, if $\log(x) \leq y$, then

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = |x - \log(x)| \geq |x - y| = d(x, y).$$

But if $\log(x) > y$, then

$$\varphi\left(\frac{1}{3}\right)d(x, Tx) = |x - \log(x)| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned}
H(Tx, Ty) &= \left|\log(x) - \frac{y}{3}\right| \\
&= \frac{1}{3} |3\log(x) - y|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{3}|x-y| \\
&= \frac{1}{3} \max \left\{ |x-y|, |x-\log(x)|, \left|y-\frac{y}{3}\right|, \left|x-\frac{y}{3}\right|, |y-\log(x)| \right\} \\
&= \frac{1}{3} \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\
&= rM(x, y),
\end{aligned} \tag{2.14}$$

where $r = \frac{1}{3}$. Hence T satisfies (2.1).

Thus we see that T satisfies condition (2.1) and satisfies Theorem 2.1. So there exists $z \in X$ such that $z \in Tz$. Moreover, $0 \in T(0)$. $\square$

Theorem 2.3 Define a non-increasing function φ from $[0, \frac{1}{5})$ into $(0, 1]$ by

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}, \\ \frac{1-5r}{1-2r}, & \text{if } \frac{\sqrt{5}-1}{4+2\sqrt{5}} \leq r < \frac{1}{5}. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $\text{CB}(X)$. Assume that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq S(x, y) \tag{2.15}$$

where $S(x, y) = rd(x, y) + rd(x, Tx) + rd(y, Ty) + rd(x, Ty) + rd(y, Tx)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < 1$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, we have $d(u_2, Tu_2) \leq H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \leq d(u_1, Tu_1) \leq d(u_1, u_2).$$

Thus, from the assumption (2.15),

$$d(u_2, Tu_2) \leq H(Tu_1, Tu_2) \leq S(u_1, u_2)$$

where $S(u_1, u_2) = rd(u_1, u_2) + rd(u_1, Tu_1) + rd(u_2, Tu_2) + rd(u_1, Tu_2) + rd(u_2, Tu_1)$. Consider

$$\begin{aligned}
d(u_2, Tu_2) &\leq rd(u_1, u_2) + rd(u_1, Tu_1) + rd(u_2, Tu_2) + rd(u_1, Tu_2) + rd(u_2, Tu_1) \\
&\leq 3rd(u_1, u_2) + 2rd(u_2, Tu_2).
\end{aligned}$$

So

$$d(u_2, Tu_2) \leq \left(\frac{3r}{1-2r} \right) d(u_1, u_2).$$

So there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \leq \left(\frac{3r_1}{1-2r_1} \right) d(u_1, u_2)$. Thus, we can construct a sequence $\{x_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq \left(\frac{3r_1}{1-2r_1} \right) d(u_n, u_{n+1}).$$

Hence, by induction,

$$d(u_n, u_{n+1}) \leq \left(\frac{3r_1}{1-2r_1} \right)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} \left(\frac{3r_1}{1-2r_1} \right)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n \rightarrow \infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq \left(\frac{3r}{1-2r} \right) d(x, Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$. Since $u_n \rightarrow z$, there exists $n_0 \in \mathbb{N}$ such that $d(z, u_n) \leq \left(\frac{1}{3} \right) d(z, x)$ for all $n \geq n_0$. By using (2.2), we get

$$\varphi(r) d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (2.15) we have

$$H(Tu_n, Tx) \leq r[d(u_n, x) + d(u_n, Tu_n) + d(x, Tx) + d(u_n, Tx) + d(x, Tu_n)].$$

Since $u_{n+1} \in Tu_n$, $d(u_{n+1}, Tx) \leq H(Tu_n, Tx)$, so that

$$d(u_{n+1}, Tx) \leq r[d(u_n, x) + d(u_n, u_{n+1}) + d(x, Tx) + d(u_n, Tx) + d(x, u_{n+1})]$$

for all $n \geq n_0$. Letting $n \rightarrow \infty$, we obtain

$$\begin{aligned} d(z, Tx) &\leq r[2d(z, x) + d(x, Tx) + d(z, Tx)] \\ &\leq r3d(z, x) + r2d(z, Tx). \end{aligned}$$

It follows that

$$d(z, Tx) \leq \left(\frac{3r}{1-2r} \right) d(x, Tx) \quad \text{for all } x \in X \setminus \{z\}. \quad (2.16)$$

Next, we show that $z \in Tz$. Suppose that z is not an element in Tz .

Case (i): $0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}$. Let $a \in Tz$. Then $a \neq z$ and so by (2.16), we have

$$d(z, Ta) \leq \left(\frac{3r}{1-2r} \right) d(a, Ta).$$

On the other hand, since $\varphi(r)d(z, Tz) = d(z, Tz) \leq d(z, a)$, from (2.15) we have

$$H(Tz, Ta) \leq r[d(z, a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)].$$

So

$$\begin{aligned} d(a, Ta) &\leq H(Tz, Ta) \leq r[2d(z, a) + d(a, Ta) + d(z, Ta)] \\ &\leq r[3d(z, a) + 2d(a, Ta)]. \end{aligned} \quad (2.17)$$

Since $d(z, a) \leq d(z, Tz) + d(Tz, a) = d(z, Tz)$, we have

$$d(a, Ta) \leq \left(\frac{3r}{1-2r} \right) d(z, Tz).$$

Using (2.15), (2.16), and (2.17), we have

$$\begin{aligned} d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\ &\leq \left(\frac{3r}{1-2r} \right) d(a, Ta) + S(a, z) \\ &\leq \left(\frac{3r}{1-2r} \right) d(a, Ta) + r[d(z, a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)] \\ &\leq \left(\frac{3r}{1-2r} \right) d(a, Ta) + 3rd(z, a) \\ &\leq \left(\frac{3r}{1-2r} \right)^2 d(z, Tz) + \left(\frac{3r}{1-2r} \right) d(z, Tz) \\ &\leq (k^2 + k)d(z, Tz), \end{aligned}$$

where $k = \frac{3r}{1-2r}$.

Since $0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}$, we have $0 \leq k^2 + k < 1$ and so, $d(z, Tz) < d(z, Tz)$, a contradiction. Thus $z \in Tz$.

Case (ii): $\frac{\sqrt{5}-1}{4+2\sqrt{5}} \leq r < \frac{1}{5}$. Let $x \in X$.

If $x = z$, then $H(Tx, Tz) \leq r[d(x, z) + d(x, Tx) + d(z, Tz) + d(x, Tz) + d(z, Tx)]$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \leq d(z, Tx) + \left(\frac{1}{n} \right) d(x, z).$$

We consider

$$\begin{aligned} d(x, Tx) &\leq d(x, y_n) \\ &\leq d(x, z) + d(z, y_n) \\ &\leq d(x, z) + d(z, Tx) + \left(\frac{1}{n} \right) d(x, z) \\ &\leq d(x, z) + \left(\frac{3r}{1-2r} \right) d(x, Tx) + \left(\frac{1}{n} \right) d(x, z). \end{aligned}$$

Thus, $(\frac{1-5r}{1-2r})d(x, Tx) \leq (1 + \frac{1}{n})d(x, z)$. Take $n \rightarrow \infty$, we obtain

$$\left(\frac{1-5r}{1-2r} \right) d(x, Tx) \leq d(x, z),$$

by using (2.15), implies $H(Tx, Tz) \leq S(x, z)$, where $S(x, z) = r[d(x, z) + d(x, Tx) + d(z, Tz) + d(x, Tz) + d(z, Tx)]$.

Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$

$$\begin{aligned}
 d(z, Tz) &= \lim_{n \rightarrow \infty} d(u_{n+1}, Tz) \\
 &\leq H(Tu_n, Tz) \\
 &\leq \lim_{n \rightarrow \infty} r[d(u_n, z) + d(u_n, Tu_n) + d(z, Tz) + d(u_n, Tz) + d(z, Tu_n)] \\
 &\leq \lim_{n \rightarrow \infty} [rd(u_n, z) + rd(u_n, u_{n+1}) + rd(z, Tz) + rd(u_n, Tz) + rd(z, u_{n+1})] \\
 &\leq (2r)d(z, Tz).
 \end{aligned} \tag{2.18}$$

Using (2.18), we have $(1 - 2r)d(z, Tz) \leq 0$, which implies $d(z, Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. $\square$

Example 2.4 Let $X = [0, \frac{1}{2}]$ with the metric $d(x, y) = \frac{|x-y|}{|x-y|+1}$ for all $x, y \in X$. Define $T : X \rightarrow CB(X)$ by

$$T(x) = [0, x^2].$$

Proof We show that T satisfies (2.15). Let $x, y \in X$. Thus, if $x^2 \leq y$, then

$$\varphi\left(\frac{1}{6}\right)d(x, Tx) = \frac{|x - x^2|}{|x - x^2| + 1} \geq \frac{|x - y|}{|x - y| + 1} = d(x, y).$$

But if $x^2 > y$, then

$$\varphi\left(\frac{1}{6}\right)d(x, Tx) = \frac{|x - x^2|}{|x - x^2| + 1} \leq \frac{|x - y|}{|x - y| + 1} = d(x, y)$$

and

$$\begin{aligned}
 H(Tx, Ty) &= \frac{|x^2 - y^2|}{|x^2 - y^2| + 1} \\
 &= \frac{1}{6} \frac{6|x^2 - y^2|}{|x^2 - y^2| + 1} \\
 &= \frac{1}{6} \left\{ \frac{|x^2 - y^2|}{|x^2 - y^2| + 1} + \frac{|x^2 - y^2|}{|x^2 - y^2| + 1} + \frac{|x^2 - y^2|}{|x^2 - y^2| + 1} + \frac{2|x^2 - y^2|}{|x^2 - y^2| + 1} + \frac{|x^2 - y^2|}{|x^2 - y^2| + 1} \right\} \\
 &< \frac{1}{6} \left\{ \frac{|x - y|}{|x - y| + 1} + \frac{|x - x^2|}{|x - x^2| + 1} + \frac{|y - y^2|}{|y - y^2| + 1} + \frac{|x - y^2|}{|x - y^2| + 1} + \frac{|y - x^2|}{|y - x^2| + 1} \right\} \\
 &= \frac{1}{6} \{d(x, y) + d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)\} \\
 &= \frac{1}{6} S(x, y),
 \end{aligned} \tag{2.19}$$

where $r = \frac{1}{6}$.

Thus we see that T satisfies condition (2.15) and satisfies Theorem 2.3. So there exists $z \in X$ such that $z \in Tz$. Moreover, $0 \in T(0)$. $\square$

Theorem 2.5 Define a non-increasing function φ from $[0, 1)$ into $(0, 1]$ by

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{\sqrt{5}-1}{2}, \\ 1-r, & \text{if } \frac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

Let $\alpha \in [0, \frac{1}{2})$ and $r = \frac{\alpha}{1-\alpha}$, and let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$.

Assume that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq \alpha M(x, y) \quad (2.20)$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < \frac{1}{2}$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, we have $d(u_2, Tu_2) \leq H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \leq d(u_1, Tu_1) \leq d(u_1, u_2).$$

Thus, from the assumption (2.20),

$$d(u_2, Tu_2) \leq H(Tu_1, Tu_2) \leq \alpha M(u_1, u_2)$$

where $M(u_1, u_2) = \max\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\}$.

Consider

$$\begin{aligned} d(u_2, Tu_2) &\leq \alpha \max\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\} \\ &= \alpha \max\{d(u_1, u_2), d(u_1, Tu_2)\}. \end{aligned}$$

If $\max\{d(u_1, u_2), d(u_1, Tu_2)\} = d(u_1, Tu_2)$, then

$$\begin{aligned} d(u_2, Tu_2) &\leq \alpha d(u_1, Tu_2) \\ &\leq \alpha d(u_1, u_2) + \alpha d(u_2, Tu_2) \end{aligned}$$

and then

$$d(u_2, Tu_2) \leq \left(\frac{\alpha}{1-\alpha} \right) d(u_1, u_2) = rd(u_1, u_2),$$

where $r = \frac{\alpha}{1-\alpha}$.

So there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \leq r_1 d(u_1, u_2)$. Thus, we can construct a sequence $\{x_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq r_1 d(u_n, u_{n+1}).$$

Hence, by induction

$$d(u_n, u_{n+1}) \leq (r_1)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} (r_1)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n \rightarrow \infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq rd(x, Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$. Since $u_n \rightarrow z$, there exists $n_0 \in \mathbb{N}$ such that $d(z, u_n) \leq (\frac{1}{3})d(z, x)$ for all $n \geq n_0$. By using (2.2), we get

$$\varphi(r)d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (2.20), we have

$$H(Tu_n, Tx) \leq \alpha \max\{d(u_n, x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n)\}.$$

Since $u_{n+1} \in Tu_n$, we have $d(u_{n+1}, Tx) \leq H(Tu_n, Tx)$, so that

$$d(u_{n+1}, Tx) \leq \alpha \max\{d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1})\}$$

for all $n \geq n_0$. Letting $n \rightarrow \infty$, we obtain

$$d(z, Tx) \leq \alpha \max\{d(z, x), d(x, Tx), d(z, Tx)\}.$$

We obtain

$$d(z, Tx) \leq \left(\frac{\alpha}{1-\alpha}\right)d(x, Tx) = rd(x, Tx) \quad \text{for all } x \in X \setminus \{z\}. \quad (2.21)$$

Next, we show that $z \in Tz$. Suppose that z is not an element in Tz .

Case (i): $0 \leq r < \frac{\sqrt{5}-1}{2}$. Let $a \in Tz$. Then $a \neq z$ and so by (2.21), we have

$$d(z, Ta) \leq rd(a, Ta).$$

On the other hand, since $\varphi(r)d(z, Tz) = d(z, Tz) \leq d(z, a)$, from (2.20) we have

$$H(Tz, Ta) \leq \alpha \max\{d(z, a), d(z, Tz), d(a, Ta), d(z, Ta), d(a, Tz)\}.$$

So

$$d(a, Ta) \leq H(Tz, Ta) \leq \alpha \max\{d(z, a), d(z, Tz), d(z, Ta)\}. \quad (2.22)$$

It implies that

$$d(a, Ta) \leq \alpha \max\{d(z, a), d(z, Tz), d(z, Ta)\}.$$

Since $d(z, a) \leq d(z, Tz) + d(Tz, a) = d(z, Tz)$, we have

$$d(a, Ta) \leq rd(z, Tz). \quad (2.23)$$

Using (2.20), (2.21), (2.22), and (2.23), we have

$$\begin{aligned} d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\ &\leq rd(a, Ta) + \alpha \max\{d(z, a), d(z, Tz), d(z, Ta)\} \\ &\leq rd(a, Ta) + \alpha \max\{d(z, a), d(z, Tz), rd(a, Ta)\} \\ &\leq rd(a, Ta) + \alpha \max\{d(z, a), d(z, Tz)\} \\ &\leq rd(a, Ta) + \alpha d(z, Tz) \\ &\leq (r)^2 d(z, Tz) + rd(z, Tz) \\ &\leq (r^2 + r)d(z, Tz), \end{aligned}$$

where $r = \frac{\alpha}{1-\alpha}$.

Since $r < \frac{\sqrt{5}-1}{2}$, we have $r^2 + r < 1$ and so $d(z, Tz) < d(z, Tz)$, a contradiction. Thus $z \in Tz$.

Case (ii) $\frac{\sqrt{5}-1}{2} \leq r < 1$. Let $x \in X$. If $x = z$, then $H(Tx, Tz) \leq \alpha \max\{d(x, z), d(x, Tx), d(z, Tz), d(x, Tz), d(z, Tx)\}$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \leq d(z, Tx) + \left(\frac{1}{n}\right)d(x, z).$$

We consider

$$\begin{aligned} d(x, Tx) &\leq d(x, y_n) \\ &\leq d(x, z) + d(z, y_n) \\ &\leq d(x, z) + d(z, Tx) + \left(\frac{1}{n}\right)d(x, z) \\ &\leq d(x, z) + rd(x, Tx) + \left(\frac{1}{n}\right)d(x, z). \end{aligned}$$

Thus, $(1 - r)d(x, Tx) \leq (1 + \frac{1}{n})d(x, z)$. Take $n \rightarrow \infty$, we obtain

$$(1 - r)d(x, Tx) \leq d(x, z),$$

by using (2.20), this implies $H(Tx, Tz) \leq \alpha \max\{d(x, z), d(x, Tx), d(z, Tz), d(x, Tz), d(z, Tx)\}$. Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$

$$\begin{aligned} d(z, Tz) &= \lim_{n \rightarrow \infty} d(u_{n+1}, Tz) \\ &\leq H(Tu_n, Tz) \\ &\leq \lim_{n \rightarrow \infty} \alpha \max\{d(u_n, z), d(u_n, Tu_n), d(z, Tz), d(u_n, Tz), d(z, Tu_n)\} \\ &\leq \lim_{n \rightarrow \infty} \alpha \max\{d(u_n, z), d(u_n, u_{n+1}), d(z, Tz), d(u_n, Tz), d(z, u_{n+1})\} \\ &\leq \alpha d(z, Tz). \end{aligned}$$

Therefore, $(1 - \alpha)d(z, Tz) \leq 0$, which implies $d(z, Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. $\square$

Corollary 2.6 *Let be (X, d) a complete metric space and let T be a mapping from X into $CB(X)$. Let $\alpha \in [0, \frac{1}{5})$ and $r = 5\alpha$. Assume that*

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq S(x, y)$$

where $S(x, y) = \alpha d(x, y) + \alpha d(x, Tx) + \alpha d(y, Ty) + \alpha d(x, Ty) + \alpha d(y, Tx)$ for all $x, y \in X$, where the function φ is defined as Theorem 2.5. Then there exists $z \in X$ such that $z \in Tz$.

Remark 2.7 We see that Theorem 2.5 is a multi-valued mapping generalization of Theorem 2.3 of Kikkawa and Suzuki [7] and therefore the Kannan fixed point theorem [6] for generalized Kannan mappings.

Theorem 2.8 *Define a non-increasing function φ from $[0, 1)$ into $(0, 1]$ by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{1}{2}, \\ 1 - r, & \text{if } \frac{1}{2} \leq r < 1. \end{cases}$$

Let $\alpha \in [0, \frac{1}{5})$ and $r = \frac{3\alpha}{1-2\alpha}$, and let be (X, d) a complete metric space and let T be a mapping from X into $CB(X)$.

Assume that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq S(x, y) \quad (2.24)$$

where $S(x, y) = \alpha d(x, y) + \alpha d(x, Tx) + \alpha d(y, Ty) + \alpha d(x, Ty) + \alpha d(y, Tx)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Proof Let r_1 be a real number such that $0 \leq r < r_1 < 1$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, we have $d(u_2, Tu_2) \leq H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \leq d(u_1, Tu_1) \leq d(u_1, u_2).$$

Thus, from the assumption (2.24),

$$d(u_2, Tu_2) \leq H(Tu_1, Tu_2) \leq S(u_1, u_2)$$

where $S(u_1, u_2) = \alpha d(u_1, u_2) + \alpha d(u_1, Tu_1) + \alpha d(u_2, Tu_2) + \alpha d(u_1, Tu_2) + \alpha d(u_2, Tu_1)$. Consider

$$\begin{aligned} d(u_2, Tu_2) &\leq \alpha d(u_1, u_2) + \alpha d(u_1, Tu_1) + \alpha d(u_2, Tu_2) + \alpha d(u_1, Tu_2) + \alpha d(u_2, Tu_1) \\ &\leq 3\alpha d(u_1, u_2) + 2\alpha d(u_2, Tu_2). \end{aligned}$$

Then

$$d(u_2, Tu_2) \leq \left(\frac{3\alpha}{1-2\alpha} \right) d(u_1, u_2) = rd(u_1, u_2),$$

where $r = \frac{3\alpha}{1-2\alpha}$.

So there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \leq r_1 d(u_1, u_2)$. Thus, we can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq r_1 d(u_n, u_{n+1}).$$

Hence, by induction

$$d(u_n, u_{n+1}) \leq (r_1)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} (r_1)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n \rightarrow \infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq rd(x, Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$. Since $u_n \rightarrow z$, there exists $n_0 \in \mathbb{N}$ such that $d(z, u_n) \leq (\frac{1}{3})d(z, x)$ for all $n \geq n_0$. By using (2.2), we get

$$\varphi(r)d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (2.1), we have

$$H(Tu_n, Tx) \leq \alpha [d(u_n, x) + d(u_n, Tu_n) + d(x, Tx) + d(u_n, Tx) + d(x, Tu_n)].$$

Since $u_{n+1} \in Tu_n$, $d(u_{n+1}, Tx) \leq H(Tu_n, Tx)$, so that

$$d(u_{n+1}, Tx) \leq \alpha [d(u_n, x) + d(u_n, u_{n+1}) + d(x, Tx) + d(u_n, Tx) + d(x, u_{n+1})]$$

for all $n \geq n_0$. Letting $n \rightarrow \infty$, we obtain

$$\begin{aligned} d(z, Tx) &\leq \alpha[2d(z, x) + d(x, Tx) + d(z, Tx)] \\ &\leq \alpha 3d(z, x) + \alpha 2d(z, Tx). \end{aligned}$$

It follows that

$$d(z, Tx) \leq \left(\frac{3\alpha}{1-2\alpha} \right) d(x, Tx) = rd(x, Tx) \quad \text{for all } x \in X \setminus \{z\}. \quad (2.25)$$

Next, we show that $z \in Tz$. Suppose that z is not element in Tz .

Case (i): $0 \leq r < \frac{1}{2}$. Let $a \in Tz$. Then $a \neq z$ and so by (2.25), we have

$$d(z, Ta) \leq rd(a, Ta).$$

On the other hand, since $\varphi(r)d(z, Tz) = d(z, Tz) \leq d(z, a)$, from (2.24) we have

$$H(Tz, Ta) \leq \alpha[d(z, a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)].$$

So

$$\begin{aligned} d(a, Ta) &\leq H(Tz, Ta) \leq \alpha[2d(z, a) + d(a, Ta) + d(z, Ta)] \\ &\leq \alpha[3d(z, a) + 2d(a, Ta)]. \end{aligned} \quad (2.26)$$

Since $d(z, a) \leq d(z, Tz) + d(Tz, a) = d(z, Tz)$, we have

$$d(a, Ta) \leq \left(\frac{3\alpha}{1-2\alpha} \right) d(z, Tz) = rd(z, Tz). \quad (2.27)$$

Using (2.24), (2.26), and (2.27), we have

$$\begin{aligned} d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\ &\leq rd(a, Ta) + S(a, z) \\ &\leq rd(a, Ta) + \alpha[d(z, a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)] \\ &\leq (r + 2\alpha)d(a, Ta) + 3\alpha d(z, a) \\ &\leq (r + 2\alpha)rd(z, Tz) + 3\alpha d(z, Tz) \\ &\leq (r + r)rd(z, Tz) + rd(z, Tz) \\ &\leq (2r^2 + r)d(z, Tz). \end{aligned}$$

Since $0 \leq r < \frac{1}{2}$, we have $0 \leq 2r^2 + r < 1$ and so, $d(z, Tz) < d(z, Tz)$, a contradiction. Thus $z \in Tz$.

Case (ii): $\frac{1}{2} \leq r < 1$. Let $x \in X$. If $x = z$, then $H(Tx, Tz) \leq \alpha[d(x, z) + d(x, Tx) + d(z, Tz) + d(x, Tz) + d(z, Tx)]$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \leq d(z, Tx) + \left(\frac{1}{n} \right) d(x, z).$$

We consider

$$\begin{aligned}
 d(x, Tx) &\leq d(x, y_n) \\
 &\leq d(x, z) + d(z, y_n) \\
 &\leq d(x, z) + d(z, Tx) + \left(\frac{1}{n}\right)d(x, z) \\
 &\leq d(x, z) + rd(x, Tx) + \left(\frac{1}{n}\right)d(x, z).
 \end{aligned}$$

Thus, $(1 - r)d(x, Tx) \leq (1 + \frac{1}{n})d(x, z)$. Take $n \rightarrow \infty$, we obtain

$$(1 - r)d(x, Tx) \leq d(x, z),$$

by using (2.24), this implies $H(Tx, Tz) \leq S(x, z)$, where $S(x, z) = \alpha[d(x, z) + d(x, Tx) + d(z, Tz) + d(x, Tz) + d(z, Tx)]$. Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$

$$\begin{aligned}
 d(z, Tz) &= \lim_{n \rightarrow \infty} d(u_{n+1}, Tz) \\
 &\leq H(Tu_n, Tz) \\
 &\leq \lim_{n \rightarrow \infty} \alpha[d(u_n, z) + d(u_n, Tu_n) + d(z, Tz) + d(u_n, Tz) + d(z, Tu_n)] \\
 &\leq \lim_{n \rightarrow \infty} [\alpha d(u_n, z) + \alpha d(u_n, u_{n+1}) + \alpha d(z, Tz) + \alpha d(u_n, Tz) + \alpha d(z, u_{n+1})] \\
 &\leq (2\alpha)d(z, Tz).
 \end{aligned} \tag{2.28}$$

Therefore, $(1 - 2\alpha)d(z, Tz) \leq 0$, which implies $d(z, Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. $\square$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand. ²Research Center for Academic Excellence in Mathematics, Naresuan University, Phitsanulok, 65000, Thailand. ³Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand.

Acknowledgements

The authors would like to thank the Thailand Research Fund under the project RTA5780007 and Science Achievement Scholarship of Thailand, which provides funding for our research.

Received: 28 April 2015 Accepted: 13 October 2015 Published online: 29 October 2015

References

- Hardy, GE, Rogers, TD: A generalization of a fixed point theorem of Reich. *Can. Math. Bull.* **16**, 201-206 (1973)
- Ciric, LB: A generalization of Banach's contraction principle. *Proc. Am. Math. Soc.* **45**, 267-273 (1974)
- Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **3**, 133-181 (1922)
- Kannan, R: Some results on fixed points-II. *Am. Math. Mon.* **76**, 405-408 (1969)
- Bianchini, RMT: Su un problema di S. Reich riguardante la teoria dei punti fissi. *Boll. Unione Mat. Ital.* **5**, 103-108 (1972)
- Suzuki, T: Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.* **340**, 1088-1095 (2008)
- Kikkawa, M, Suzuki, T: Some similarity between contractions and Kannan mappings. *Fixed Point Theory Appl.* **2008**, Article ID 649749 (2008)

8. Karapinar, E, Tas, K: Generalized (C)-conditions and related fixed point theorems. *Comput. Math. Appl.* **61**, 3370-3380 (2011)
9. Kikkawa, M, Suzuki, T: Three fixed point theorems for generalized contractions with constants in complete metric spaces. *Nonlinear Anal.* **69**, 2942-2949 (2008)
10. Nadler, SB Jr.: Multi-valued contraction mappings. *Pac. J. Math.* **30**, 475-488 (1969)
11. Damjanović, B, Dorić, D: Multivalued generalizations of the Kannan fixed point theorem. *Filomat* **25**, 125-131 (2011)

2.3 EXISTENCE AND CONVERGENCE THEOREMS FOR λ -HYBRID SET-VALUE MAPPINGS IN HILBERT SPACES

In this section, we define a new class of set-valued mappings in a Hilbert space, called λ -*hybrid*. We prove some fixed point theorems for mappings in this class. Finally, we establish weak convergence of modified SP-iteration to a fixed point of λ -hybrid set-valued mappings in Hilbert spaces. Our results extend many known recent results in the literature.

1 Introduction

Let H be a real Hilbert space and D be a nonempty subset of H . Let $CB(D)$ and $K(D)$ denote the families of nonempty closed bounded subsets and nonempty compact subsets of D , respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\} \quad \text{for } A, B \in CB(D),$$

where $\text{dist}(x, D) = \inf \{ \|x - y\| : y \in D \}$. A single-valued mapping $T : D \rightarrow D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A set-valued mapping $T : D \rightarrow CB(D)$ is called *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $z \in D$ is called a *fixed point* of $T : D \rightarrow D$ (respectively, $T : D \rightarrow CB(D)$) if $z = Tz$ (respectively, $z \in Tz$). The set of fixed points of T is denoted by $F(T)$. A set-valued mapping $T : D \rightarrow CB(D)$ is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tz) \leq \|x - z\|$ for all

$x \in D$ and all $z \in F(T)$. It is clear that every nonexpansive set-valued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. It is known that if T is a quasi-nonexpansive set-valued mapping, then $F(T)$ is closed.

The following lemma shows that the fixed point set of a quasi-nonexpansive set-valued mapping T is convex under the assumption that $Tp = \{p\}$ for all $p \in F(T)$. The proof of this fact is very easy, for convenience of the readers, we include the details.

Lemma 1.1 *Let D be a nonempty closed convex subset of a real Hilbert space H . Assume that $T : D \rightarrow CB(D)$ is a quasi-nonexpansive set-valued mapping with $F(T) \neq \emptyset$. If $Tp = \{p\}$ for all $p \in F(T)$, then $F(T)$ is convex.*

Proof. Let $z = \alpha x + (1 - \alpha)y$ where $x, y \in F(T)$ and $\alpha \in (0, 1)$. We need to show that $z \in F(T)$. Let $w \in Tz$. By the definition of T , we obtain

$$\begin{aligned} \|z - w\|^2 &= \|\alpha(x - w) + (1 - \alpha)(y - w)\|^2 \\ &= \alpha\|x - w\|^2 + (1 - \alpha)\|y - w\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha \operatorname{dist}(w, Tx)^2 + (1 - \alpha) \operatorname{dist}(w, Ty)^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha H(Tz, Tx)^2 + (1 - \alpha) H(Tz, Ty)^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha\|z - x\|^2 + (1 - \alpha)\|z - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= 0. \end{aligned}$$

This implies that $z = w \in Tz$. Therefore, $z \in F(T)$ and hence $F(T)$ is convex. $\square$

An important example of nonexpansive single-valued mappings in a Hilbert space is a firmly nonexpansive mapping. Recall that a single-valued mapping $T : D \rightarrow D$ is called *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$$

for all $x, y \in D$; see [4]. We also know that a firmly nonexpansive mapping can be deduced from an equilibrium problem in a Hilbert space (see *e.g.*, [3, 5]). In 2008, Kohsaka and Takahashi [12] introduced a new class of nonlinear single-valued mappings which contains the class of firmly nonexpansive mappings in a Hilbert space as follows. A single-valued mapping $T : D \rightarrow D$ is called *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|y - Tx\|^2 + \|x - Ty\|^2$$

for all $x, y \in D$; see also [9]. Further, Takahashi [22] introduced the class of hybrid single-valued mappings in a Hilbert space, which also contains the class of firmly nonexpansive mappings in a Hilbert space as follows. A single-valued mapping $T : D \rightarrow D$ is called *hybrid* if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2$$

for all $x, y \in D$. In 2010, Aoyama *et al.* [1] introduced the class of λ -hybrid single-valued mappings in a Hilbert space containing the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings as follows. A single-valued mapping $T : D \rightarrow D$ is called λ -hybrid if there exists $\lambda \in \mathbb{R}$ such that

$$(1 + \lambda)\|Tx - Ty\|^2 \leq (1 - \lambda)\|x - y\|^2 + \lambda\|y - Tx\|^2 + \lambda\|x - Ty\|^2$$

for all $x, y \in D$; see also [23]. They obtained a fixed point theorem and an ergodic theorem for these mappings in Hilbert spaces. We note that 0-hybrid is nonexpansive, 1-hybrid is nonspreading, and $\frac{1}{2}$ -hybrid is hybrid.

Fixed point theory for set-valued mappings has more complicated and difficult than the corresponding theory of single-valued mappings. However, those fixed point theorems have many useful applications in various fields, *i.e.*, game theory and mathematical economics. Thus, it is natural to extend the known fixed point results for single-valued mappings to the setting of set-valued mappings. Some classical fixed point theorems for single-valued mappings have already been extended to set-valued mappings, for example, Banach contraction principle was extended to a set-valued case in 1969 by Nadler [16]. Since then many authors have studied fixed points for set-valued mappings; see for example [2, 6, 7, 8, 11, 13, 15, 20, 25] and the references therein.

The aim of this paper is to extend the class of λ -hybrid single-valued mappings to the set-valued case in Hilbert spaces and to prove some fixed point theorems for set-valued mappings in this setting.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. In a real Hilbert space H , it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$$

for all $x, y \in H$ and $\alpha \in [0, 1]$; see [21]. Furthermore, in a real Hilbert space, we have that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$$

for all $x, y, z, w \in H$. It is also known that a Hilbert space H satisfies *Opial's condition* [18], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

The following two lemmas are useful for our main results.

Lemma 2.1 *Let H be a real Hilbert space. Then the following well known results hold:*

$$(i) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \text{ for all } x, y \in H.$$

$$(ii) \quad \|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 \text{ for all } x, y \in H.$$

Let μ be a continuous linear functional on l^∞ , the Banach space of bounded real sequences, and $(a_1, a_2, \dots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_1, a_2, \dots))$. We call μ a *Banach limit* if μ satisfies $\|\mu\| = \mu(1, 1, \dots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for each $(a_1, a_2, \dots) \in l^\infty$. For a Banach limit μ , we know that

$$\liminf_{n \rightarrow \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \rightarrow \infty} a_n$$

for all $(a_1, a_2, \dots) \in l^\infty$. So, if $(a_1, a_2, \dots) \in l^\infty$ with $\lim_{n \rightarrow \infty} a_n = c$, then $\mu_n(a_n) = c$; see also [21] for more details.

Lemma 2.2 ([24]) *Let D be a nonempty closed convex subset of a real Hilbert space H , let $\{x_n\}$ be a bounded sequence in H , and let μ be a Banach limit. If $f : D \rightarrow \mathbb{R}$ is defined by*

$$f(z) = \mu_n \|x_n - z\|^2 \text{ for all } z \in D,$$

then there exists a unique $z^* \in D$ such that

$$f(z^*) = \min\{f(z) : z \in D\}.$$

3 Main results

In this section, we first define a new set-valued mapping in a real Hilbert space H as follows. Let D be a nonempty closed convex subset of H . A set-valued mapping $T : D \rightarrow CB(D)$ is called λ -*hybrid* if there exists $\lambda \in \mathbb{R}$ such that

$$(1 + \lambda)H(Tx, Ty)^2 \leq (1 - \lambda)\|x - y\|^2 + \lambda \text{dist}(y, Tx)^2 + \lambda \text{dist}(x, Ty)^2$$

for all $x, y \in D$. Clearly every nonexpansive set-valued mapping is λ -hybrid for $\lambda = 0$. It can be shown that if T is a λ -hybrid set-valued mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Indeed, if $z \in F(T)$ and T is a λ -hybrid set-valued mapping, we obtain for all $x \in D$ that

$$\begin{aligned} (1 + \lambda)H(Tx, Tz)^2 &\leq (1 - \lambda)\|x - z\|^2 + \lambda \text{dist}(z, Tx)^2 + \lambda \text{dist}(x, Tz)^2 \\ &\leq (1 - \lambda)\|x - z\|^2 + \lambda H(Tz, Tx)^2 + \lambda \|x - z\|^2. \end{aligned}$$

This implies that $H(Tx, Tz) \leq \|x - z\|$.

We now give an example for the class of λ -hybrid set-valued mappings.

Example 3.1 Let $D = [1, 7]$ with the usual metric and $T : D \rightarrow CB(D)$ be the set-valued mapping defined by

$$Tx = \begin{cases} \{1\}, & x \in [1, 4]; \\ \left[1, \frac{2x^2 + 1}{x^2 + 1}\right], & x \in (4, 7]. \end{cases}$$

It is easy to see that T is not nonexpansive since $H(T(4), T(4.2)) = H(\{1\}, [1, \frac{907}{466}]) = \frac{441}{466} > \frac{1}{5} = |4 - 4.2|$. Next, we will show that T is λ -hybrid for all $\lambda \in [\frac{1}{2}, 1]$, that is,

$$(1 + \lambda)H(Tx, Ty)^2 \leq (1 - \lambda)|x - y|^2 + \lambda \text{dist}(y, Tx)^2 + \lambda \text{dist}(x, Ty)^2 \quad (3.1)$$

for all $x, y \in D$. Obviously, (3.1) holds if $x, y \in [1, 4]$. Then, it suffices to consider the following 2 cases.

Case 1: $x, y \in (4, 7]$. We have $Tx = \left[1, \frac{2x^2 + 1}{x^2 + 1}\right]$ and $Ty = \left[1, \frac{2y^2 + 1}{y^2 + 1}\right]$. This implies that $|x - y| \geq 0$, $\text{dist}(y, Tx) > |4 - \frac{2x^2 + 1}{x^2 + 1}| = |2 + \frac{1}{x^2 + 1}| > 2$, and $\text{dist}(x, Ty) > |4 - \frac{2y^2 + 1}{y^2 + 1}| = |2 + \frac{1}{y^2 + 1}| > 2$. By $\lambda \in [\frac{1}{2}, 1]$, we obtain that

$$\begin{aligned} (1 + \lambda)H(Tx, Ty)^2 &= (1 + \lambda) \left(\frac{2x^2 + 1}{x^2 + 1} - \frac{2y^2 + 1}{y^2 + 1} \right)^2 \\ &= (1 + \lambda) \left(\frac{1}{x^2 + 1} - \frac{1}{y^2 + 1} \right)^2 \\ &< 2 \\ &< (1 - \lambda)|x - y|^2 + \lambda \text{dist}(y, Tx)^2 + \lambda \text{dist}(x, Ty)^2. \end{aligned}$$

Case 2: $x \in [1, 4]$ and $y \in (4, 7]$. We have $Tx = \{1\}$ and $Ty = \left[1, \frac{2y^2 + 1}{y^2 + 1}\right]$. This implies that $|x - y| > 0$, $\text{dist}(y, Tx) > 3$, and $\text{dist}(x, Ty) > 0$. By $\lambda \in [\frac{1}{2}, 1]$, we obtain that

$$\begin{aligned} (1 + \lambda)H(Tx, Ty)^2 &= (1 + \lambda) \left(\frac{2x^2 + 1}{x^2 + 1} - 1 \right)^2 \\ &= (1 + \lambda) \left(\frac{x^2}{x^2 + 1} \right)^2 \\ &< 2 \\ &< (1 - \lambda)|x - y|^2 + \lambda \text{dist}(y, Tx)^2 + \lambda \text{dist}(x, Ty)^2. \end{aligned}$$

Therefore, inequality (3.1) holds.

We define $T(D) = \bigcup_{x \in D} Tx$ and $T^n x = T^{n-1}(Tx)$ for all $x \in D$, $n \in \mathbb{N}$. The following lemma is very useful for the proof of our main theorems.

Lemma 3.2 *Let D be a nonempty closed convex subset of a real Hilbert space H and $T : D \rightarrow K(D)$ be a λ -hybrid set-valued mapping. If $x, y \in D$ and $a \in Tx$, then there exists $b \in Ty$ such that*

$$\|a - b\|^2 \leq H(Tx, Ty)^2 \leq \|x - y\|^2 + 2\lambda\langle x - a, y - b \rangle.$$

Proof. Let $x, y \in D$ and $a \in Tx$. From the compactness of $T(D)$, there exists $b \in Ty$ such that $\|a - b\| \leq H(Tx, Ty)$. It follows that

$$\begin{aligned} (1 + \lambda)H(Tx, Ty)^2 &\leq (1 - \lambda)\|x - y\|^2 + \lambda \text{dist}(y, Tx)^2 + \lambda \text{dist}(x, Ty)^2 \\ &\leq (1 - \lambda)\|x - y\|^2 + \lambda\|y - a\|^2 + \lambda\|x - b\|^2 \\ &= (1 - \lambda)\|x - y\|^2 + \lambda(\|y - x\|^2 + 2\langle y - x, x - a \rangle \\ &\quad + \|x - a\|^2) + \lambda(\|x - a\|^2 + 2\langle x - a, a - b \rangle + \|a - b\|^2) \\ &= (1 - \lambda)\|x - y\|^2 + \lambda(\|y - x\|^2 + 2\langle x - a, a - b - x + y \rangle \\ &\quad + 2\|x - a\|^2 + \|a - b\|^2) \\ &= \|x - y\|^2 + 2\lambda\langle x - a, a - x + y - b \rangle + 2\lambda\|x - a\|^2 \\ &\quad + \lambda\|a - b\|^2 \\ &\leq \|x - y\|^2 + 2\lambda\langle x - a, y - b \rangle + \lambda H(Tx, Ty)^2. \end{aligned}$$

This implies that $H(Tx, Ty)^2 \leq \|x - y\|^2 + 2\lambda\langle x - a, y - b \rangle$. $\square$

We next prove an existence theorem for λ -hybrid set-valued mappings in Hilbert spaces.

Theorem 3.3 *Let D be a nonempty closed convex subset of a real Hilbert space H and $T : D \rightarrow K(D)$ be a λ -hybrid set-valued mapping. Then, $F(T) \neq \emptyset$ if and only if there exists $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded.*

Proof. The necessity is obvious. Conversely, assume that there exists $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded. For a Banach limit μ on l^∞ , we define a function $f : D \rightarrow \mathbb{R}$ by

$$f(z) = \mu_n \|z_n - z\|^2 \text{ for all } z \in D.$$

By Lemma 2.2, there exists a unique $z^* \in D$ such that

$$f(z^*) = \min\{f(z) : z \in D\}.$$

Let $z_{n+1} \in Tz_n$. It implies by Lemma 3.2 that there exists $a \in Tz^*$ such that

$$\|z_{n+1} - a\|^2 \leq \|z_n - z^*\|^2 + 2\lambda\langle z_n - z_{n+1}, z^* - a \rangle.$$

Then we have

$$\begin{aligned} \|z_{n+1} - a\|^2 &\leq \|z_n - z^*\|^2 + \lambda(\|z_n - a\|^2 + \|z_{n+1} - z^*\|^2 - \|z_n - z^*\|^2 \\ &\quad - \|z_{n+1} - a\|^2) \\ &= (1 - \lambda)\|z_n - z^*\|^2 + \lambda\|z_n - a\|^2 + \lambda\|z_{n+1} - z^*\|^2 \\ &\quad - \lambda\|z_{n+1} - a\|^2. \end{aligned}$$

This implies that

$$(1 + \lambda)\|z_{n+1} - a\|^2 - \lambda\|z_n - a\|^2 \leq (1 - \lambda)\|z_n - z^*\|^2 + \lambda\|z_{n+1} - z^*\|^2.$$

For any $n \in \mathbb{N}$, we have

$$(1 + \lambda)\mu_n\|z_{n+1} - a\|^2 - \lambda\mu_n\|z_n - a\|^2 \leq (1 - \lambda)\mu_n\|z_n - z^*\|^2 + \lambda\mu_n\|z_{n+1} - z^*\|^2.$$

Then, we get

$$f(a) = \mu_n\|z_n - a\|^2 \leq \mu_n\|z_n - z^*\|^2 = f(z^*).$$

Since $z^* \in D$ is a unique element such that $f(z^*) = \min\{f(z) : z \in D\}$ and $a \in D$, we have $z^* = a \in Tz^*$. Hence, z^* is a fixed point of T . $\square$

As a direct consequence of Theorem 3.3, we obtain the following results.

Corollary 3.4 *Let D be a nonempty closed convex subset of a real Hilbert space H . Let $T : D \rightarrow K(D)$ be a nonspreading set-valued mapping, i.e.,*

$$2H(Tx, Ty)^2 \leq \text{dist}(y, Tx)^2 + \text{dist}(x, Ty)^2 \text{ for all } x, y \in D.$$

Then, $F(T) \neq \emptyset$ if and only if there exists $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded.

Corollary 3.5 *Let D be a nonempty closed convex subset of a real Hilbert space H . Let $T : D \rightarrow K(D)$ be a hybrid set-valued mapping, i.e.,*

$$3H(Tx, Ty)^2 \leq \|x - y\|^2 + \text{dist}(y, Tx)^2 + \text{dist}(x, Ty)^2 \text{ for all } x, y \in D.$$

Then, $F(T) \neq \emptyset$ if and only if there exists $x \in D$ and $z_n \in T^n x$ such that $\{z_n\}$ is bounded.

Next, we study a weak convergence theorem for λ -hybrid set-valued mappings in Hilbert spaces. Before proving, we need the following lemma.

Lemma 3.6 *Let D be a nonempty closed convex subset of a real Hilbert space H and $T : D \rightarrow K(D)$ be a λ -hybrid set-valued mapping. If $\{x_n\}$ is a sequence in D such that $x_n \rightharpoonup x$ and $y_n \in Tx_n$ with $x_n - y_n \rightarrow 0$, then $x \in Tx$.*

Proof. Let $\{x_n\}$ be a sequence in D such that $x_n \rightharpoonup x$ and $y_n \in Tx_n$ with $x_n - y_n \rightarrow 0$. We will show that $x \in Tx$. By Lemma 3.2, there exists $z_n \in Tx$ such that

$$\|y_n - z_n\|^2 \leq \|x_n - x\|^2 + 2\lambda\langle x_n - y_n, x - z_n \rangle. \quad (3.2)$$

Since Tx is compact and $z_n \in Tx$, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightarrow z \in Tx$.

Suppose that $z \neq x$. It follows by the Opial's condition that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \|x_{n_i} - x\| &< \limsup_{i \rightarrow \infty} \|x_{n_i} - z\| \\ &= \limsup_{i \rightarrow \infty} \|x_{n_i} - y_{n_i} + y_{n_i} - z\| \\ &\leq \limsup_{i \rightarrow \infty} \|y_{n_i} - z\|. \end{aligned}$$

This implies by (3.2) that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \|x_{n_i} - x\|^2 &< \limsup_{i \rightarrow \infty} \|y_{n_i} - z\|^2 \\ &= \limsup_{i \rightarrow \infty} \|y_{n_i} - z_{n_i} + z_{n_i} - z\|^2 \\ &\leq \limsup_{i \rightarrow \infty} \|y_{n_i} - z_{n_i}\|^2 \\ &\leq \limsup_{i \rightarrow \infty} (\|x_{n_i} - x\|^2 + 2\lambda \langle x_{n_i} - y_{n_i}, x - z_{n_i} \rangle) \\ &= \limsup_{i \rightarrow \infty} \|x_{n_i} - x\|^2. \end{aligned}$$

This is a contradiction. Hence, $x = z \in Tx$. $\square$

Fixed point iteration methods are very useful for finding a fixed point of various nonlinear mappings such as Mann iteration [14], Ishikawa iteration [10], Noor iteration [17], SP-iteration [19] and so on. Among these iteration methods, Phuengrattana and Suantai [19] showed that SP-iteration converges faster than the others. So, we are interested to modify the SP-iteration for a single-valued mapping to a set-valued mapping as follows. For $x_1 \in D$, the sequence $\{x_n\}$ of the *modified SP-iteration* is defined by

$$\begin{aligned} z_n &= \alpha_n x_n + (1 - \alpha_n) w_n^{(1)}, \quad w_n^{(1)} \in Tx_n \\ y_n &= \beta_n z_n + (1 - \beta_n) w_n^{(2)}, \quad w_n^{(2)} \in Tz_n \\ x_{n+1} &= \gamma_n y_n + (1 - \gamma_n) w_n^{(3)}, \quad w_n^{(3)} \in Ty_n, \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{3.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$.

We now prove a weak convergence theorem for modified SP-iteration of λ -hybrid set-valued mappings in Hilbert spaces.

Theorem 3.7 *Let D be a nonempty closed convex subset of a real Hilbert space H and $T : D \rightarrow K(D)$ be a λ -hybrid set-valued mapping with $F(T) \neq \emptyset$. For $x_1 \in D$, let $\{x_n\}$ be the sequence of the modified SP-iteration defined by (3.3). Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences in $[0, 1]$ satisfying $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$. If $Tp = \{p\}$ for each $p \in F(T)$, then $\{x_n\}$ converges weakly to a fixed point of T .*

Proof. Since T is λ -hybrid and $F(T) \neq \emptyset$, we get T is quasi-nonexpansive. Then, for each $p \in F(T)$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \gamma_n \|y_n - p\| + (1 - \gamma_n) \|w_n^{(3)} - p\| \\
&= \gamma_n \|y_n - p\| + (1 - \gamma_n) \text{dist}(w_n^{(3)}, Tp) \\
&\leq \gamma_n \|y_n - p\| + (1 - \gamma_n) H(Ty_n, Tp) \\
&\leq \|y_n - p\| \\
&\leq \beta_n \|z_n - p\| + (1 - \beta_n) \|w_n^{(2)} - p\| \\
&= \beta_n \|z_n - p\| + (1 - \beta_n) \text{dist}(w_n^{(2)}, Tp) \\
&\leq \beta_n \|z_n - p\| + (1 - \beta_n) H(Tz_n, Tp) \\
&\leq \|z_n - p\| \\
&\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|w_n^{(1)} - p\| \\
&= \alpha_n \|x_n - p\| + (1 - \alpha_n) \text{dist}(w_n^{(1)}, Tp) \\
&\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) H(Tx_n, Tp) \\
&\leq \|x_n - p\|.
\end{aligned} \tag{3.4}$$

Therefore, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and hence $\{x_n\}$ is bounded.

For each $p \in F(T)$, we obtain by (3.4) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|z_n - p\|^2 \\
&= \|\alpha_n(x_n - p) + (1 - \alpha_n)(w_n^{(1)} - p)\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|w_n^{(1)} - p\|^2 \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - w_n^{(1)}\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \text{dist}(w_n^{(1)}, Tp)^2 \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - w_n^{(1)}\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) H(Tx_n, Tp)^2 \\
&\quad - \alpha_n(1 - \alpha_n) \|x_n - w_n^{(1)}\|^2 \\
&\leq \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_n - w_n^{(1)}\|^2.
\end{aligned}$$

This implies that

$$\alpha_n(1 - \alpha_n) \|x_n - w_n^{(1)}\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

By $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$, we have

$$\lim_{n \rightarrow \infty} \|x_n - w_n^{(1)}\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z_1 \in D$. By Lemma 3.6, it implies that $z_1 \in F(T)$.

Next, we will show that $x_n \rightharpoonup z_1$. To show this, we suppose not. So, there must exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z_2 \in D$ and $z_2 \neq z_1$. By Lemma 3.6, we get $z_2 \in F(T)$. Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T)$, it follows by the Opial's condition that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - z_1\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - z_1\| \\ &< \lim_{i \rightarrow \infty} \|x_{n_i} - z_2\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_2\| \\ &= \lim_{k \rightarrow \infty} \|x_{n_k} - z_2\| \\ &< \lim_{k \rightarrow \infty} \|x_{n_k} - z_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_1\|. \end{aligned}$$

This is a contradiction. Therefore, $\{x_n\}$ converges weakly to $z_1 \in F(T)$. $\square$

As a direct consequence of Theorem 3.7, we have the following two corollaries.

Corollary 3.8 *Let D be a nonempty closed convex subset of a real Hilbert space H and $T : D \rightarrow K(D)$ be a λ -hybrid set-valued mapping with $F(T) \neq \emptyset$. For $x_1 \in D$, let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) w_n^{(1)}, \quad w_n^{(1)} \in T x_n \\ x_{n+1} &= \beta_n y_n + (1 - \beta_n) w_n^{(2)}, \quad w_n^{(2)} \in T y_n, \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$. If $Tp = \{p\}$ for each $p \in F(T)$, then $\{x_n\}$ converges weakly to a fixed point of T .

Corollary 3.9 *Let D be a nonempty closed convex subset of a real Hilbert space H and $T : D \rightarrow K(D)$ be a λ -hybrid set-valued mapping with $F(T) \neq \emptyset$. For $x_1 \in D$, let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) w_n^{(1)}, \quad w_n^{(1)} \in T x_n, \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$. If $Tp = \{p\}$ for each $p \in F(T)$, then $\{x_n\}$ converges weakly to a fixed point of T .

4 Acknowledgements

The authors would like to thank the Thailand Research Fund under the project RTA5780007 for the financial support, and the referees for valuable suggestions on the research paper.

5 References

- [1] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, Fixed point and ergodic theorems for λ -hybrid mappings in Hilbert spaces, *J. Nonlinear Convex Anal.*, **11**, (2010) 335-343.
- [2] N.A. Assad and W.A. Kirk, Fixed point theorems for set-valued mappings of contractive type, *Pacific J. Math.*, **43**, (1972) 553-562.
- [3] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63**, (1994) 123-145.
- [4] F.E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Math. Z.*, **100**, (1967) 201-225.
- [5] P.L. Combettes and A. Hirstoaga, Equilibrium problems in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6**, (2005) 117-136.
- [6] T. Dominguez Benavides and B. Gavira, A fixed point property for multivalued nonexpansive mappings, *J. Math. Anal. Appl.*, **328**, (2007) 1471-1483.
- [7] D. Downing and W.A. Kirk, Fixed point theorems for set-valued mappings in metric and Banach spaces, *Math. Japonica*, **22**, (1977) 99-112.
- [8] K. Geobel, On a fixed point theorem for multivalued nonexpansive mappings, *Ann. Univ. M. Curie-Skłodowska*, **29**, (1975) 70-72.
- [9] S. Iemoto and W. Takahashi, Approximating fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, *Nonlinear Anal.*, **71**, (2009) 2082-2089.
- [10] S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44**, (1974) 1471-150.
- [11] W.A. Kirk and S. Massa, Remark on asymptotic and Chebyshev centers, *Houston J. Math.*, **16**, (1990) 357-364.
- [12] F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, *Arch. Math. (Basel)*, **91**, (2008) 166-177.
- [13] T.C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space, *Bull. Amer. Math. Soc.*, **80**, (1974) 1123-1126.
- [14] W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4**, (1953) 506-510.
- [15] J.T. Markin, A fixed point theorem for set valued mappings, *Bull. Amer. Math. Soc.*, **74**, (1968) 639-640.
- [16] S.B. Nadler Jr., Multi-valued contraction mappings, *Pacific J. Math.*, **30**, (1969) 475-488.
- [17] M.A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.*, **251**, (2000) 2172-229.
- [18] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73**, (1967) 591-597.
- [19] W. Phuengrattana and S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, *J. Comput. Appl. Math.*, **235**, (2011) 3006-3014.
- [20] S. Reich, Approximate selections, best approximations, fixed points, and invariant sets, *J. Math. Anal. Appl.*, **62**, (1978) 104-113.
- [21] W. Takahashi, Introduction to Nonlinear and Convex Analysis, Yokohama Publishers, Yokohama, 2009.

- [22] W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space, *J. Nonlinear Convex Anal.*, **11**, (2010) 79-88.
- [23] W. Takahashi, Nonlinear mappings in equilibrium problems and an open problem in fixed point theory, *Proc. of the 9th International Conference on Fixed Point Theory and Its Applications*, Yokohama Publishers, 2010, pp. 177-197.
- [24] W. Takahashi and J.C. Yao, Fixed point theorems and ergodic theorems for non-linear mappings in a Hilbert space, *Taiwanese J. Math.*, **15**, (2011) 457-472.
- [25] H.K. Xu, Multivalued nonexpansive mappings in Banach spaces, *Nonlinear Anal.*, **43**, (2001) 693-706.

2.4 A NEW ONE-STEP ITERATIVE PROCESS FOR APPROXIMATING COMMON FIXED POINTS OF A COUNTABLE FAMILY OF QUASI-NONEXPANSIVE MULTI-VALUED MAPPINGS IN CAT(0) SPACES

In this section, we propose a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in a CAT(0) space. We also prove Δ and strong convergence theorems of the proposed iterative process under some control conditions. Our main results extend and generalize many results in the literature.

1. Introduction

Let (X, d) be a metric space. A *geodesic* joining x to y (where $x, y \in X$) is a map γ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(l) = y$ and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. Thus γ is an isometry and $d(x, y) = l$. The image of γ is called a *geodesic* (or *metric*) *segment* joining x and y . When it is unique, this geodesic is denoted by $[x, y]$. We write $\alpha x \oplus (1 - \alpha)y$ for the unique point z in the geodesic segment joining from x to y such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(y, z) = \alpha d(x, y)$ for $\alpha \in [0, 1]$. The space X is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset D of X is said to be *convex* if D includes every geodesic segment joining any two of its points.

Following [3], a metric space X is said to be a *CAT(0) space* if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane $\mathbb{E}^2$. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces [3], R-trees [18], the complex Hilbert ball with a hyperbolic metric [15], and many others. It follows from [3] that CAT(0) spaces are uniquely geodesic metric spaces. The fixed point theory in CAT(0) spaces was first studied by Kirk [16, 17]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0)

space always has a fixed point. Since then, there have been many researches concerning the existence and the convergence of fixed points for single-valued and multi-valued mappings in such spaces (e.g., see [2, 5, 7, 8, 18, 19, 20]).

The study of fixed points for nonexpansive multi-valued mappings using the Pompeiu-Hausdorff metric was initiated by Markin [21]. Different iterative processes have been used to approximate fixed points of nonexpansive and quasi-nonexpansive multi-valued mappings; in particular, Sastry and Babu [23] considered Mann and Ishikawa iterative processes for a multi-valued mapping T with a fixed point p and proved that these iterative processes converge to a fixed point q of T under certain conditions in Hilbert spaces. Moreover, they illustrated that fixed point q may be different from p . Later, in 2007, Panyanak [22] generalized the results of Sastry and Babu [23] to uniformly convex Banach spaces and proved a convergence theorem of Mann iterative processes for a mapping defined on a noncompact domain. Since then, the strong convergence of the Mann and Ishikawa iterative processes for multi-valued mappings has been rapidly developed, and many papers have appeared (e.g., see [6, 12, 24, 28]). Among other things, Shahzad and Zegeye [27] defined two types of Ishikawa iterative processes and proved strong convergence theorems for such iterative processes involving quasi-nonexpansive multi-valued mappings in uniformly convex Banach spaces. Recently, Abkar and Eslamian [1] established Δ and strong convergence theorems for the multi-step iterative process for a finite family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

In this paper, motivated by the above results, we propose a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces and prove Δ and strong convergence theorems for the proposed iterative process in CAT(0) spaces. We finally provide an example to support our main result.

2. Preliminaries

For a nonempty set X , we let $\mathcal{P}(X)$ be the power set of X and $2^X = \mathcal{P}(X) - \{\emptyset\}$. For a metric space (X, d) , $x \in X$, and $A, B \in 2^X$, let $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$, $\text{dist}(x, B) = \inf\{d(x, y) : y \in B\}$, and $h(A, B) = \sup\{\text{dist}(x, B) : x \in A\}$.

We now recall some definitions of continuity for multi-valued mappings (see [4, 14] for more details). Let (X, d) and (Y, d) be metric spaces. A multi-valued mapping $T : X \rightarrow 2^Y$ is said to be

- *Hausdorff upper semi-continuous* at x if for each $\varepsilon > 0$, there is $\delta > 0$ such that $h(Ty, Tx) < \varepsilon$ for each $y \in B(x, \delta)$;
- *Hausdorff lower semi-continuous* at x if for each $\varepsilon > 0$, there is $\delta > 0$ such that $h(Tx, Ty) < \varepsilon$ for each $y \in B(x, \delta)$;
- *continuous* at x if T is Hausdorff upper and lower semi-continuous at x .

We say that the multi-valued mapping T is continuous if it is continuous at each point in X .

Let D be a nonempty subset of a metric space X . Let $CB(D)$ and $KC(D)$ denote the families of nonempty closed bounded subsets and nonempty compact convex subsets of D , respectively. The *Pompeiu-Hausdorff distance* [25] on $CB(D)$ is defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\} \quad \text{for } A, B \in CB(D),$$

where $\text{dist}(x, D) = \inf\{d(x, y) : y \in D\}$ is the distance from a point x to a subset D .

Note that a continuous multi-valued mapping behaves like a continuous single-valued mapping [14], that is, if a multi-valued mapping $T : D \rightarrow CB(D)$ is continuous then for every sequence $\{x_n\}$ in D such that $\lim_{n \rightarrow \infty} x_n = x$, we have $\lim_{n \rightarrow \infty} H(Tx_n, Tx) = 0$.

The set of fixed points of a multi-valued mapping $T : D \rightarrow CB(D)$ will be denoted by $F(T) = \{x \in D : x \in Tx\}$.

Definition 2.1. A multi-valued mapping $T : D \rightarrow CB(D)$ is said to be

- (i) *nonexpansive* [21] if $H(Tx, Ty) \leq d(x, y)$, for all $x, y \in D$.
- (ii) *quasi-nonexpansive* [23] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq d(x, p)$, for all $x \in D$ and $p \in F(T)$.
- (iii) *hemicompact* if for any sequence $\{x_n\}$ in D such that $\lim_{n \rightarrow \infty} \text{dist}(x_n, Tx_n) = 0$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i \rightarrow \infty} x_{n_i} = p \in D$.

Definition 2.2. A multi-valued mapping $T : D \rightarrow CB(D)$ is said to satisfy *condition* (E_μ) provided that

$$\text{dist}(x, Ty) \leq \mu \text{dist}(x, Tx) + d(x, y)$$

for each $x, y \in D$. We say that T satisfies *condition* (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Remark 2.3. From the above definitions, it is clear that:

- (i) if T is nonexpansive, then T satisfies the condition (E_1) ;
- (ii) if D is compact, then T is hemicompact.

Although the condition (E) implies the quasi-nonexpansiveness for single-valued mappings ([13]), but it is not true for multi-valued mappings as the following example.

Example 2.4 ([26, Example 1]). Let $D = [0, \infty)$ and $T : D \rightarrow CB(D)$ be defined by

$$Tx = [x, 2x] \quad \text{for all } x \in D.$$

Then T satisfies condition (E) and is not quasi-nonexpansive.

Notice also that the classes of (multi-valued) quasi-nonexpansive mappings, continuous mappings and mappings satisfying condition (E) are different (see Examples 2.5-2.7).

Example 2.5 ([13, Example 2]). Let $D = [-1, 1]$ and $T : D \rightarrow CB(D)$ be defined by

$$Tx = \begin{cases} \left\{ \frac{x}{1+|x|} \sin\left(\frac{1}{x}\right) \right\} & \text{if } x \neq 0; \\ \{0\} & \text{if } x = 0. \end{cases}$$

Then T is quasi-nonexpansive and does not satisfy condition (E).

Example 2.6 ([5, p. 984]). Let $D = [0, 1]$ and $T : D \rightarrow CB(D)$ be defined by

$$Tx = \begin{cases} \{x^2\} & \text{if } 0 \leq x < 1; \\ \{0\} & \text{if } x = 1. \end{cases}$$

Then T is quasi-nonexpansive and is not continuous. Notice also that the mapping $Tx = \{x^2\}$ on $[0, 1]$ is continuous but is not quasi-nonexpansive nor satisfies condition (E).

Example 2.7 ([13, Example 3]). Let $D = [-2, 1]$ and $T : D \rightarrow CB(D)$ be defined by

$$Tx = \begin{cases} \left\{ \frac{|x|}{2} \right\} & \text{if } -2 \leq x < 1; \\ \left\{ -\frac{1}{2} \right\} & \text{if } x = 1. \end{cases}$$

Then T satisfies condition (E) and is not continuous.

The notion of the asymptotic center can be introduced in the general setting of a CAT(0) space X as follows: Let $\{x_n\}$ be a bounded sequence in X . For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known by [10] that in a CAT(0) space, the asymptotic center $A(\{x_n\})$ consists of exactly one point.

We now give the definition and collect some basic properties of the Δ -convergence which will be used in the sequel.

Definition 2.8 ([19]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ and call x the Δ -limit of $\{x_n\}$.

Lemma 2.9 ([19]). *Every bounded sequence in a $CAT(0)$ space has a Δ -convergent subsequence.*

Lemma 2.10 ([9]). *If D is a nonempty closed convex subset of a $CAT(0)$ space X and if $\{x_n\}$ is a bounded sequence in D , then the asymptotic center of $\{x_n\}$ is in D .*

Lemma 2.11 ([11]). *Let $\{x_n\}$ be a sequence in a $CAT(0)$ space X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.12 ([3]). *Let X be a geodesic metric space. The following are equivalent:*

- (i) X is a $CAT(0)$ space.
- (ii) X satisfies the (CN) inequality: If $x, y \in X$ and $\frac{x \oplus y}{2}$ is the midpoint of x and y , then

$$d\left(z, \frac{x \oplus y}{2}\right)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2, \text{ for all } z \in X.$$

The following lemma is a generalization of the (CN) inequality which can be found in [11].

Lemma 2.13. *Let X be a $CAT(0)$ space. Then*

$$d(z, \lambda x \oplus (1 - \lambda)y)^2 \leq \lambda d(z, x)^2 + (1 - \lambda)d(z, y)^2 - \lambda(1 - \lambda)d(x, y)^2,$$

for any $\lambda \in [0, 1]$ and $x, y, z \in X$.

In 2012, Dhompongsa et al. [8] introduced the following notation in $CAT(0)$ spaces: Let $x_1, \dots, x_n$ be points in a $CAT(0)$ space X and $\lambda_1, \dots, \lambda_n \in (0, 1)$ with $\sum_{i=1}^n \lambda_i = 1$, we write

$$(2.1) \quad \bigoplus_{i=1}^n \lambda_i x_i := (1 - \lambda_n) \left(\frac{\lambda_1}{1 - \lambda_n} x_1 \oplus \frac{\lambda_2}{1 - \lambda_n} x_2 \oplus \dots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} x_{n-1} \right) \oplus \lambda_n x_n.$$

The definition of $\bigoplus$ is an ordered one in the sense that it depends on the order of points $x_1, \dots, x_n$. Under (2.1) we obtain that

$$d\left(\bigoplus_{i=1}^n \lambda_i x_i, y\right) \leq \sum_{i=1}^n \lambda_i d(x_i, y) \text{ for each } y \in X.$$

3. Main results

In this section, we first introduce a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in $CAT(0)$ spaces. Let D be a nonempty closed convex subset of a $CAT(0)$ space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings

of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by

$$(3.1) \quad x_{n+1} = \bigoplus_{i=0}^n \lambda_n^{(i)} y_n^{(i)}, \text{ for all } n \in \mathbb{N},$$

where $y_n^{(0)} = x_n$, $y_n^{(i)} \in T_i x_n$ and the sequences $\{\lambda_n^{(i)}\} \subset (0, 1)$ satisfying $\sum_{i=0}^n \lambda_n^{(i)} = 1$.

Note that, if we put

$$W_n^{(m)} = \bigoplus_{i=0}^m \delta_n^{(i,m)} y_n^{(i)},$$

where $\delta_n^{(i,m)} = \frac{\lambda_n^{(i)}}{\sum_{j=0}^m \lambda_n^{(j)}}$ for $i = 0, 1, \dots, m$. Then, we get

$$\begin{aligned} W_n^{(m)} &= \left(1 - \delta_n^{(m,m)}\right) \left(\frac{\delta_n^{(0,m)}}{1 - \delta_n^{(m,m)}} x_n \oplus \frac{\delta_n^{(1,m)}}{1 - \delta_n^{(m,m)}} y_n^{(1)} \oplus \dots \oplus \frac{\delta_n^{(m-1,m)}}{1 - \delta_n^{(m,m)}} y_n^{(m-1)} \right) \oplus \delta_n^{(m,m)} y_n^{(m)} \\ &= \left(1 - \delta_n^{(m,m)}\right) \left(\delta_n^{(0,m-1)} x_n \oplus \delta_n^{(1,m-1)} y_n^{(1)} \oplus \dots \oplus \delta_n^{(m-1,m-1)} y_n^{(m-1)} \right) \oplus \delta_n^{(m,m)} y_n^{(m)} \\ &= \left(1 - \delta_n^{(m,m)}\right) \left(\frac{\lambda_n^{(0)}}{\sum_{j=0}^{m-1} \lambda_n^{(j)}} x_n \oplus \frac{\lambda_n^{(1)}}{\sum_{j=0}^{m-1} \lambda_n^{(j)}} y_n^{(1)} \oplus \dots \oplus \frac{\lambda_n^{(m-1)}}{\sum_{j=0}^{m-1} \lambda_n^{(j)}} y_n^{(m-1)} \right) \oplus \delta_n^{(m,m)} y_n^{(m)} \\ &= \left(1 - \delta_n^{(m,m)}\right) W_n^{(m-1)} \oplus \delta_n^{(m,m)} y_n^{(m)} \\ &= \frac{\sum_{j=0}^{m-1} \lambda_n^{(j)}}{\sum_{j=0}^m \lambda_n^{(j)}} W_n^{(m-1)} \oplus \frac{\lambda_n^{(m)}}{\sum_{j=0}^m \lambda_n^{(j)}} y_n^{(m)}. \end{aligned}$$

Therefore, the following result holds:

$$(3.2) \quad W_n^{(m)} = \frac{\sum_{j=0}^{m-1} \lambda_n^{(j)}}{\sum_{j=0}^m \lambda_n^{(j)}} W_n^{(m-1)} \oplus \frac{\lambda_n^{(m)}}{\sum_{j=0}^m \lambda_n^{(j)}} y_n^{(m)}.$$

The following two lemmas are useful and crucial for our main theorems.

Lemma 3.1. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by (3.1). Then, $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$.*

Proof. For $p \in \cap_{i=1}^{\infty} F(T_i)$, we have by (3.1) that

$$d(x_{n+1}, p) = d\left(\bigoplus_{i=0}^n \lambda_n^{(i)} y_n^{(i)}, p\right)$$

$$\begin{aligned}
&\leq \sum_{i=0}^n \lambda_n^{(i)} d(y_n^{(i)}, p) \\
&= \sum_{i=0}^n \lambda_n^{(i)} \text{dist}(y_n^{(i)}, T_i p) \\
&\leq \sum_{i=0}^n \lambda_n^{(i)} H(T_i x_n, T_i p) \\
&\leq \sum_{i=0}^n \lambda_n^{(i)} d(x_n, p) \\
&= d(x_n, p).
\end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$. $\square$

Lemma 3.2. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by (3.1). If $\lim_{n \rightarrow \infty} \lambda_n^{(i)}$ exist for all $i \in \mathbb{N} \cup \{0\}$ and lie in $(0, 1)$, then $\lim_{n \rightarrow \infty} \text{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$.*

Proof. For each $p \in \cap_{i=1}^{\infty} F(T_i)$, we obtain by (3.1) that

$$d(x_{n+1}, p) = d\left(\bigoplus_{i=0}^n \lambda_n^{(i)} y_n^{(i)}, p\right) = d\left(\bigoplus_{i=0}^n \frac{\lambda_n^{(i)}}{\sum_{j=0}^n \lambda_n^{(j)}} y_n^{(i)}, p\right) = d(W_n^{(n)}, p).$$

It follows by Lemma 2.13 and (3.2) that

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d\left(\bigoplus_{j=0}^{n-1} \frac{\lambda_n^{(j)}}{\sum_{j=0}^n \lambda_n^{(j)}} W_n^{(n-1)} \oplus \frac{\lambda_n^{(n)}}{\sum_{j=0}^n \lambda_n^{(j)}} y_n^{(n)}, p\right)^2 \\
&\leq \frac{\sum_{j=0}^{n-1} \lambda_n^{(j)}}{\sum_{j=0}^n \lambda_n^{(j)}} d(W_n^{(n-1)}, p)^2 + \frac{\lambda_n^{(n)}}{\sum_{j=0}^n \lambda_n^{(j)}} d(y_n^{(n)}, p)^2 \\
&\quad - \frac{\lambda_n^{(n)}}{\sum_{j=0}^n \lambda_n^{(j)}} \frac{\sum_{j=0}^{n-1} \lambda_n^{(j)}}{\sum_{j=0}^n \lambda_n^{(j)}} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&= \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, p)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{n-1} \lambda_n^{(j)} d \left(\frac{\sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} W_n^{(n-2)} \oplus \frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} y_n^{(n-1)}, p \right)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
&\quad - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&\leq \sum_{j=0}^{n-1} \lambda_n^{(j)} \left(\frac{\sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, p)^2 + \frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(y_n^{(n-1)}, p)^2 \right. \\
&\quad \left. - \frac{\sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} \frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, y_n^{(n-1)})^2 \right) + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
&\quad - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&= \sum_{j=0}^{n-2} \lambda_n^{(j)} d(W_n^{(n-2)}, p)^2 + \lambda_n^{(n-1)} d(y_n^{(n-1)}, p)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
&\quad - \frac{\lambda_n^{(n-1)} \sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, y_n^{(n-1)})^2 - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&\leq \sum_{j=0}^{n-3} \lambda_n^{(j)} d(W_n^{(n-3)}, p)^2 + \lambda_n^{(n-2)} d(y_n^{(n-2)}, p)^2 + \lambda_n^{(n-1)} d(y_n^{(n-1)}, p)^2 \\
&\quad + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 - \frac{\lambda_n^{(n-2)} \sum_{j=0}^{n-3} \lambda_n^{(j)}}{\sum_{j=0}^{n-2} \lambda_n^{(j)}} d(W_n^{(n-3)}, y_n^{(n-2)})^2 \\
&\quad - \frac{\lambda_n^{(n-1)} \sum_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, y_n^{(n-1)})^2 - \lambda_n^{(N)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&\quad \vdots \\
&\leq \lambda_n^{(0)} d(W_n^{(0)}, p)^2 + \sum_{k=1}^n \lambda_n^{(k)} d(y_n^{(k)}, p)^2 - \sum_{k=1}^n \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^k \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2 \\
&\leq \sum_{k=0}^n \lambda_n^{(k)} d(x_n, p)^2 - \sum_{k=1}^n \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^k \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2
\end{aligned}$$

$$= d(x_n, p)^2 - \sum_{k=1}^n \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^k \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2.$$

This implies that

$$(3.3) \quad \sum_{k=1}^n \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^k \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

Since $0 < \lambda_n^{(0)} \leq \sum_{j=0}^k \lambda_n^{(j)} \leq 1$ for all $k = 1, 2, \dots, n$, we have $0 < \lambda_n^{(0)} \lambda_n^{(k)} \leq \lambda_n^{(k)} \sum_{j=0}^k \lambda_n^{(j)}$. So, $0 < \lambda_n^{(0)} \lambda_n^{(k)} \leq \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^k \lambda_n^{(j)}}$ for all $k = 1, 2, \dots, n$. Then (3.3) becomes

$$(3.4) \quad \sum_{k=1}^n \lambda_n^{(0)} \lambda_n^{(k)} d(W_n^{(k-1)}, y_n^{(k)})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

By Lemma 3.1 and the condition $\lim_{n \rightarrow \infty} \lambda_n^{(i)}$ exist for all $i \in \mathbb{N} \cup \{0\}$ and lie in $(0, 1)$, we get that

$$(3.5) \quad \lim_{n \rightarrow \infty} d(x_n, y_n^{(1)}) = 0 \text{ and } \lim_{n \rightarrow \infty} d(W_n^{(k-1)}, y_n^{(k)}) = 0 \text{ for all } k \geq 2.$$

Then, for $k \geq 2$, we have

$$\begin{aligned} d(x_n, y_n^{(k)}) &\leq d(x_n, W_n^{(k-1)}) + d(W_n^{(k-1)}, y_n^{(k)}) \\ &= d\left(x_n, \bigoplus_{i=0}^{k-1} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{k-1} \lambda_n^{(j)}} y_n^{(i)}\right) + d(W_n^{(k-1)}, y_n^{(k)}) \\ &\leq \sum_{i=0}^{k-1} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{k-1} \lambda_n^{(j)}} d(x_n, y_n^{(i)}) + d(W_n^{(k-1)}, y_n^{(k)}) \\ &= \sum_{i=1}^{k-1} \frac{\lambda_n^{(i)}}{\sum_{j=0}^{k-1} \lambda_n^{(j)}} d(x_n, y_n^{(i)}) + d(W_n^{(k-1)}, y_n^{(k)}). \end{aligned}$$

This implies by (3.5) that $\lim_{n \rightarrow \infty} d(x_n, y_n^{(k)}) = 0$ for all $k \geq 1$. Since $\text{dist}(x_n, T_i x_n) \leq d(x_n, y_n^{(i)})$ for all $i \in \mathbb{N}$, it follows that $\lim_{n \rightarrow \infty} \text{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. $\square$

We now get a Δ -convergence theorem for a countable family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $KC(D)$ satisfying the condition (E). Assume that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. Suppose that $\lim_{n \rightarrow \infty} \lambda_n^{(i)}$ exist for all $i \in \mathbb{N} \cup \{0\}$ and lie in $(0, 1)$. Then, the sequence $\{x_n\}$ generated by (3.1) Δ -converges to a common fixed point of $\{T_i\}$.*

Proof. By Lemmas 3.1 and 3.2, we have $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$ and $\lim_{n \rightarrow \infty} \text{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. Then, the sequence $\{x_n\}$ is bounded. We put $\omega_{\Delta}(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Let $u \in \omega_{\Delta}(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.9, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{j \rightarrow \infty} u_{n_j} = z \in D$. We will show that $z \in T_1 z$. Since $T_1 z$ is compact, for all $j \in \mathbb{N}$, we can choose $y_{n_j} \in T_1 z$ such that $d(u_{n_j}, y_{n_j}) = \text{dist}(u_{n_j}, T_1 z)$ and $\{y_{n_j}\}$ has a convergent subsequence $\{y_{n_k}\}$ with $\lim_{k \rightarrow \infty} y_{n_k} = q \in T_1 z$. By condition (E), we have

$$\text{dist}(u_{n_k}, T_1 z) \leq \mu \text{dist}(u_{n_k}, T_1 u_{n_k}) + d(u_{n_k}, z).$$

Then we have

$$\begin{aligned} d(u_{n_k}, q) &\leq d(u_{n_k}, y_{n_k}) + d(y_{n_k}, q) \\ &= \text{dist}(u_{n_k}, T_1 z) + d(y_{n_k}, q) \\ &\leq \mu \text{dist}(u_{n_k}, T_1 u_{n_k}) + d(u_{n_k}, z) + d(y_{n_k}, q). \end{aligned}$$

This implies that

$$\limsup_{k \rightarrow \infty} d(u_{n_k}, q) \leq \limsup_{k \rightarrow \infty} d(u_{n_k}, z).$$

By the uniqueness of asymptotic centers, we have $z = q \in T_1 z$. Similarly, it can be shown that $z \in T_i z$ for all $i = 2, \dots, N$. Then, $z \in \cap_{i=1}^{\infty} F(T_i)$ and so $\lim_{n \rightarrow \infty} d(x_n, z)$ exists. Suppose that $u \neq z$. By the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{j \rightarrow \infty} d(u_{n_j}, z) &< \limsup_{j \rightarrow \infty} d(u_{n_j}, u) \\ &\leq \limsup_{n \rightarrow \infty} d(u_n, u) \\ &< \limsup_{n \rightarrow \infty} d(u_n, z) \\ &= \limsup_{n \rightarrow \infty} d(x_n, z) \\ &= \limsup_{j \rightarrow \infty} d(u_{n_j}, z). \end{aligned}$$

This is a contradiction, hence $u = z \in \cap_{i=1}^{\infty} F(T_i)$. This shows that $\omega_{\Delta}(x_n) \subset \cap_{i=1}^{\infty} F(T_i)$.

Next, we show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{p\}$ and let $A(\{x_n\}) = \{q\}$. Since $p \in \omega_\Delta(x_n) \subset \cap_{i=1}^\infty F(T_i)$, it follows that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. By Lemma 2.11, we obtain that $p = q$. Hence, the sequence $\{x_n\}$ Δ -converges to a common fixed point of $\{T_i\}$. $\square$

The following result is a strong convergence theorem for a countable family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

Theorem 3.4. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of continuous and quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^\infty F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^\infty F(T_i)$. Let the sequence $\{x_n\}$ generated by (3.1) with $\lim_{n \rightarrow \infty} \lambda_n^{(i)}$ exist for all $i \in \mathbb{N} \cup \{0\}$ and lie in $(0, 1)$. Assume that one member of the family $\{T_i\}$ is hemicompact. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.*

Proof. By Lemma 3.2, $\lim_{n \rightarrow \infty} \text{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. Without loss of generality, we assume that T_1 is hemicompact. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j \rightarrow \infty} x_{n_j} = p \in D$. By continuity of T_i , we have $\lim_{j \rightarrow \infty} \text{dist}(x_{n_j}, T_i x_{n_j}) = \text{dist}(p, T_i p)$ for all $i \in \mathbb{N}$. This implies that $\text{dist}(p, T_i p) = 0$ for all $i \in \mathbb{N}$ and hence $p \in \cap_{i=1}^\infty F(T_i)$. It follows by Lemma 3.1 that $\{x_n\}$ converges strongly to p . $\square$

Remark 3.5. Since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$ (see [3]), all our results immediately apply to any CAT(κ) space with $\kappa \leq 0$.

Finally, we give a numerical example supporting Theorems 3.3 and 3.4.

Example 3.6. Let X be a real line with the Euclidean norm and $D = [0, 1]$. For $x \in D$, $i = 1, 2, \dots$, we define mappings T_i on D as follows:

$$T_i x = \left[0, \frac{x}{i}\right] \text{ for all } i \in \mathbb{N}.$$

Let the sequence $\{x_n\}$ be generated by

$$(3.6) \quad x_{n+1} = \bigoplus_{i=0}^n \lambda_n^{(i)} y_n^{(i)}, \text{ for all } n \in \mathbb{N},$$

where $y_n^{(0)} = x_n$, $y_n^{(i)} \in T_i x_n$ and the sequences $\{\lambda_n^{(i)}\}$ defined by

$$\lambda_n^{(i)} = \begin{cases} \frac{1}{2^{i+1}} \left(\frac{n}{n+1} \right), & n \geq i+1 \\ 1 - \frac{n}{n+1} \left(\sum_{k=1}^n \frac{1}{2^k} \right), & n = i \\ 0, & n < i. \end{cases}$$

Obviously, T_i is quasi-nonexpansive and satisfies condition (E) for all $i \in \mathbb{N}$ and $T_i(0) = \{0\}$ such that $\cap_{i=1}^\infty F(T_i) = \{0\}$. It can be observed that all the assumptions of Theorems 3.3 and 3.4 are satisfied.

For any arbitrary $x_1 \in D = [0, 1]$, we put $y_n^{(i)} = \frac{x_n}{5^i}$ for all $i \in \mathbb{N}$. Then, we rewrite the algorithm (3.6) as follows:

$$x_{n+1} = \lambda_n^{(0)} x_n + \frac{\lambda_n^{(1)} x_n}{5} + \frac{\lambda_n^{(2)} x_n}{10} + \cdots + \frac{\lambda_n^{(n)} x_n}{5n}, \text{ for all } n \in \mathbb{N},$$

where

$$\left(\lambda_n^{(i)} \right) = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{3}{8} & \frac{3}{16} & \frac{3}{32} & \frac{11}{32} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ \frac{n}{2(n+1)} & \frac{n}{4(n+1)} & \frac{n}{8(n+1)} & \frac{n}{16(n+1)} & \frac{n}{32(n+1)} & \frac{n}{64(n+1)} & \cdots & \frac{n}{2^i(n+1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

The values of the sequence $\{x_n\}$ with different n are reported in Table 1.

TABLE 1. The values of the sequence $\{x_n\}$ in Example 3.6.

	$x_1 = 0.11$	$x_1 = 0.95$
n	x_n	x_n
1	0.1100000	0.9500000
2	0.0440000	0.3800000
3	0.0183333	0.1583333
4	0.0081545	0.0704253
5	0.0037986	0.0328065
6	0.0018280	0.0157875
7	0.0009008	0.0077801
8	0.0004520	0.0039036
9	0.0002300	0.0019863
10	0.0001184	0.0010222
$\vdots$	$\vdots$	$\vdots$
17	0.0000014	0.0000118
18	0.0000007	0.0000064
19	0.0000004	0.0000034
20	0.0000002	0.0000019

From Table 1, it is clear that $\{x_n\}$ converges to 0, where $\{0\} = \cap_{i=1}^{\infty} F(T_i)$.

Acknowledgments

This paper was supported by the Thailand Research Fund under the project RTA5780007. The second author was also supported by the Research Administration Center, Chiang Mai University, Chiang Mai, Thailand.

REFERENCES

- [1] A. Abkar and M. Eslamian, Convergence theorems for a finite family of generalized nonexpansive multivalued mappings in CAT(0) spaces, *Nonlinear Anal.* **75** (2012), 1895–1903.
- [2] A. Abkar and M. Eslamian, Geodesic metric spaces and generalized nonexpansive multivalued mappings, *Bull. Iranian Math. Soc.* **39** (2013), 993–1008.
- [3] M. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer, Berlin, 1999.
- [4] P. Chanthorn and P. Chaocha, Fixed point sets of set-valued mappings, *Fixed Point Theory and Applications* (2015) 2015:56.
- [5] P. Chaocha and A. Phon-on, A note on fixed point sets in CAT(0) spaces, *J. Math. Anal. Appl.* **320** (2006), 983–987.
- [6] W. Cholanjiak and S. Suantai, Approximation of common fixed points of two quasi-nonexpansive multi-valued maps in Banach spaces, *Comput. Math. Appl.* **61** (2011), 941–949.
- [7] S. Dhompongsa, A. Kaewkhao and B. Panyanak, Lim’s theorems for multivalued mappings in CAT(0) spaces, *J. Math. Anal. Appl.* **312** (2005), 478–487.
- [8] S. Dhompongsa, A. Kaewkhao and B. Panyanak, On Kirk’s strong convergence theorem for multivalued nonexpansive mappings on CAT(0) spaces, *Nonlinear Anal.* **75** (2012), 459–468.
- [9] S. Dhompongsa, W.A. Kirk and B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, *J. Nonlinear Convex Anal.* **8** (2007), 35–45.
- [10] S. Dhompongsa, W.A. Kirk and B. Sims, Fixed points of uniformly Lipschitzian mappings, *Nonlinear Anal.* **65** (2006), 762–772.
- [11] S. Dhompongsa and B. Panyanak, On Δ -convergence theorems in CAT(0) spaces, *Comput. Math. Appl.* **56** (2008), 2572–2579.
- [12] M. Eslamian and A. Abkar, One-step iterative process for a finite family of multivalued mappings, *Math. Comp. Model.* **54** (2011), 105–111.
- [13] J. Garcia-Falset, E. Lorens-Fuster and T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, *J. Math. Anal. Appl.* **375** (2011), 185–195.
- [14] A. Geletu, Introduction to Topological Spaces and Set-Valued Maps, Lecture Notes in Math. Institute of Mathematics, Department of Operations Research & Stochastics, Ilmenau University of Technology (2006).
- [15] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, vol. 83 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, NY, USA, 1984.
- [16] W.A. Kirk, Geodesic geometry and fixed point theory, in: Seminar of Mathematical Analysis, Malaga/Seville, 2002/2003, in: Colecc. Abierta, Vol. 64, Univ. Sevilla Secr. Publ., Seville, 2003, pp. 195–225.
- [17] W.A. Kirk, Geodesic geometry and fixed point theory II, in: International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, 2004, pp. 113–142.
- [18] W.A. Kirk, Fixed point theorems in CAT(0) spaces and R-trees, *Fixed Point Theory Appl.* **2004** (2004), 309–316.
- [19] W.A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear Anal.* **68** (2008), 3689–3696.
- [20] L. Leuştean, A quadratic rate of asymptotic regularity for CAT(0)-spaces, *J. Math. Anal. Appl.* **325** (2007), 386–399.
- [21] J.T. Markin, Continuous dependence of fixed point sets, *Proc. Amer. Math. Soc.* **38** (1973), 545–547.
- [22] B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, *Comput. Math. Appl.* **54** (2007), 872–877.

- [23] K.P.R. Sastry and G.V.R. Babu, Convergence of Ishikawa iterates for a multivalued mapping with a fixed point, *Czechoslovak Math. J.* **55** (2005), 817–826.
- [24] Y. Song and H. Wang, Erratum to Mann and Ishikawa iterative processes for multi-valued mappings in Banach spaces, *Comput. Math. Appl.* **52** (2008), 2999–3002.
- [25] R. Tyrrell Rockafellar and Roger J.-B. Wets, *Variational Analysis*, Springer-Verlag, 2005.
- [26] H.K. Xu, On weakly nonexpansive and $*$ –nonexpansive multivalued mappings, *Math. Japonica.* **36** (1991), 441–445.
- [27] N. Shahzad and H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, *Nonlinear Anal.* **71** (2009), 838–844.
- [28] H. Zegeye and N. Shahzad, Viscosity approximation methods for nonexpansive multimapings in Banach spaces, *Acta Math. Sin. Engl. Ser.* **26** (2010), 1165–1176.

2.5 Viscosity approximation methods for multivalued nonexpansive mappings in geodesic spaces

In this section, we prove strong convergence of the viscosity approximation method for multivalued nonexpansive mappings in CAT(0) spaces. Our results generalize the results of Dhompongsa *et al.* (2012), Wangkeeree and Preechasilp (2013) and many others. Some related results in $\mathbb{R}$ -trees are also given.

1 Introduction

One of the successful approximation methods for finding fixed points of nonexpansive mappings was given by Moudafi [1]. Let E be a nonempty closed convex subset of a Hilbert space H and $t : E \rightarrow E$ be a nonexpansive mapping with a nonempty fixed point set $\text{Fix}(t)$. The following scheme is known as the *viscosity approximation method* or *Moudafi's viscosity approximation method*:

$$\begin{aligned} x_1 &\in E \quad \text{arbitrarily chosen,} \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)t(x_n), \quad n \in \mathbb{N}, \end{aligned} \tag{1}$$

where $f : E \rightarrow E$ is a contraction and $\{\alpha_n\}$ is a sequence in $(0, 1)$. In [1], under some suitable assumptions, the author proved that the sequence $\{x_n\}$ defined by (1) converges strongly to a point z in $\text{Fix}(t)$ which satisfies the following *variational inequality*:

$$\langle f(z) - z, z - x \rangle \geq 0, \quad x \in \text{Fix}(t).$$

We note that the Halpern approximation method [2],

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)t(x_n), \quad n \in \mathbb{N},$$

where u is a fixed element in E , is a special case of the Moudafi one. Notice also that the Moudafi viscosity approximation method can be applied to convex optimization, linear programming, monotone inclusions, and elliptic differential equations.

The first extension of Moudafi's result to the so-called CAT(0) space was proved by Shi and Chen [3]. However, they assumed that the space (X, ρ) must satisfy the property $\mathcal{P}$, i.e., for $x, u, y_1, y_2 \in X$, one has

$$\rho(x, m_1)\rho(x, y_1) \leq \rho(x, m_2)\rho(x, y_2) + \rho(x, u)\rho(y_1, y_2),$$

where m_1 and m_2 are the unique nearest points of u on the segments $[x, y_1]$ and $[x, y_2]$, respectively. By using the concept of quasi-linearization introduced by Berg and Nikolaev [4], Wangkeeree and Preechasilp [5] could omit the property $\mathcal{P}$ from Shi and Chen's result. Precisely, they obtained the following theorems.

Theorem 1.1 (Theorem 3.1 of [5]) *Let E be a nonempty closed convex subset of a complete CAT(0) space X , $t : E \rightarrow E$ be a nonexpansive mapping with $\text{Fix}(t) \neq \emptyset$, and $f : E \rightarrow E$ be a contraction with constant $k \in [0, 1)$. For each $s \in (0, 1)$, let x_s be given by*

$$x_s = sf(x_s) \oplus (1-s)t(x_s).$$

Then $\{x_s\}$ converges strongly as $s \rightarrow 0$ to $\tilde{x}$ such that $\tilde{x} = P_{\text{Fix}(t)}(f(\tilde{x}))$, which is equivalent to the variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \text{Fix}(t).$$

Theorem 1.2 (Theorem 3.4 of [5]) *Let E be a nonempty closed convex subset of a complete CAT(0) space X , $t : E \rightarrow E$ be a nonexpansive mapping with $\text{Fix}(t) \neq \emptyset$, and $f : E \rightarrow E$ be a contraction with constant $k \in [0, 1)$. Suppose that $x_1 \in E$ is arbitrarily chosen and $\{x_n\}$ is iteratively generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)t(x_n), \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying:

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n \rightarrow \infty} (\frac{\alpha_n}{\alpha_{n+1}}) = 1$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$, where $\tilde{x} = P_{\text{Fix}(t)}(f(\tilde{x}))$.

However, on p.14 of [5], in order to conclude that $\alpha'_n = \frac{2(1-k)\alpha_n}{1-k\alpha_n} \in (0, 1)$, the sequence $\{\alpha_n\}$ must be contained in $(0, \frac{1}{2-k})$. Therefore, Theorem 1.2 should be rewritten as follows.

Theorem 1.3 *Let E be a nonempty closed convex subset of a complete CAT(0) space X , $t : E \rightarrow E$ be a nonexpansive mapping with $\text{Fix}(t) \neq \emptyset$, and $f : E \rightarrow E$ be a contraction with constant $k \in [0, 1)$. Suppose that $x_1 \in E$ is arbitrarily chosen and $\{x_n\}$ is iteratively generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)t(x_n), \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, \frac{1}{2-k})$ satisfying (C1), (C2), and (C3). Then $\{x_n\}$ converges strongly to $\tilde{x} = P_{\text{Fix}(t)}(f(\tilde{x}))$.

Fixed point theory for multivalued mappings has many useful applications in applied sciences, in particular, in game theory and optimization theory. Thus, it is natural to study the extension of the known fixed point results for single-valued mappings to the setting of multivalued mappings.

Let E be a closed convex subset of a complete CAT(0) space, $f : E \rightarrow E$ be a contraction, and T be a nonexpansive mapping on E whose values are nonempty bounded closed subsets of E . For each $s \in (0, 1)$, we can define a multivalued contraction G_s on E by

$$G_s(x) = (1 - s)f(x) \oplus sT(x), \quad x \in E.$$

Applying Nadler's theorem [6], G_s has a (not necessarily unique) fixed point $x_s \in E$, that is,

$$x_s \in sf(x_s) \oplus (1 - s)T(x_s). \quad (2)$$

Recently, Bo and Yi [7] extended Theorems 1.1 and 1.2 to multivalued nonexpansive mappings. We observe that there are many gaps in the proof of [7]. In fact, Theorem 3.1 of [7] is absolutely wrong since there is a closed convex subset E of the Euclidean plane $\mathbb{R}^2$ and a nonexpansive mapping on E such that the net $\{x_s\}$ defined by (2) does not converge (see [8], p.697).

The purpose of this paper is to extend Theorems 1.1 and 1.2 in the right way. Of course, a condition like the endpoint condition must be added. Our main discoveries are Theorems 3.1 and 3.3.

2 Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers. Let $[0, l]$ be a closed interval in $\mathbb{R}$ and x, y be two points in a metric space (X, ρ) . A *geodesic* joining x to y is a map $\xi : [0, l] \rightarrow X$ such that $\xi(0) = x$, $\xi(l) = y$, and $\rho(\xi(s), \xi(t)) = |s - t|$ for all $s, t \in [0, l]$. The image of ξ is called a *geodesic segment* joining x and y , which when unique is denoted by $[x, y]$. The space (X, ρ) is said to be a *geodesic space* if every two points in X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset E of X is said to be *convex* if every pair of points $x, y \in E$ can be joined by a geodesic in X and the image of every such geodesic is contained in E .

A *geodesic triangle* $\Delta(p, q, r)$ in a geodesic space (X, ρ) consists of three points p, q, r in X and a choice of three geodesic segments $[p, q]$, $[q, r]$, $[r, p]$ joining them. A *comparison triangle* for geodesic triangle $\Delta(p, q, r)$ in X is a triangle $\bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{p}, \bar{q}) = \rho(p, q)$, $d_{\mathbb{R}^2}(\bar{q}, \bar{r}) = \rho(q, r)$, and $d_{\mathbb{R}^2}(\bar{r}, \bar{p}) = \rho(r, p)$. A point $\bar{u} \in [\bar{p}, \bar{q}]$ is called a *comparison point* of $u \in [p, q]$ if $\rho(p, u) = d_{\mathbb{R}^2}(\bar{p}, \bar{u})$. Comparison points on $[\bar{q}, \bar{r}]$ and $[\bar{r}, \bar{p}]$ are defined in the same way.

Definition 2.1 A geodesic triangle $\Delta(p, q, r)$ in (X, ρ) is said to satisfy the CAT(0) *inequality* if for any $u, v \in \Delta(p, q, r)$ and for their comparison points $\bar{u}, \bar{v} \in \bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$, one has

$$\rho(u, v) \leq d_{\mathbb{R}^2}(\bar{u}, \bar{v}).$$

A geodesic space X is said to be a CAT(0) *space* if all of its geodesic triangles satisfy the CAT(0) inequality. For other equivalent definitions and basic properties of CAT(0) spaces, we refer the reader to standard texts, such as [9, 10]. It is well known that every CAT(0) space is uniquely geodesic. Notice also that Pre-Hilbert spaces, $\mathbb{R}$ -trees, Euclidean buildings are examples of CAT(0) spaces (see [9, 11]). Let E be a nonempty closed convex subset of a complete CAT(0) space (X, ρ) . It follows from Proposition 2.4 of [9] that for each $x \in X$, there exists a unique point $x_0 \in E$ such that

$$\rho(x, x_0) = \inf\{\rho(x, y) : y \in E\}.$$

In this case, x_0 is called the *unique nearest point* of x in E . The *metric projection* of X onto E is the mapping $P_E : X \rightarrow E$ defined by

$$P_E(x) := \text{the unique nearest point of } x \text{ in } E.$$

By Lemma 2.1 of [12], for each $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$\rho(x, z) = (1 - t)\rho(x, y) \quad \text{and} \quad \rho(y, z) = t\rho(x, y). \quad (3)$$

We shall denote by $tx \oplus (1 - t)y$ the unique point z satisfying (3). Now, we collect some elementary facts about CAT(0) spaces.

Lemma 2.2 (Lemma 2.4 of [12]) *Let (X, ρ) be a CAT(0) space. Then*

$$\rho(tx \oplus (1 - t)y, z) \leq t\rho(x, z) + (1 - t)\rho(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 2.3 (Lemma 2.5 of [12]) *Let (X, ρ) be a CAT(0) space. Then*

$$\rho^2(tx \oplus (1 - t)y, z) \leq t\rho^2(x, z) + (1 - t)\rho^2(y, z) - t(1 - t)\rho^2(x, y)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 2.4 (Lemma 3 of [13]) *Let (X, ρ) be a CAT(0) space. Then*

$$\rho(tx \oplus (1 - t)z, ty \oplus (1 - t)z) \leq t\rho(x, y)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Let $\{x_n\}$ be a bounded sequence in X . For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} \rho(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is well known from Proposition 7 of [14] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ and call x the Δ -limit of $\{x_n\}$.

Lemma 2.5 ([15], p.3690) *Every bounded sequence in a complete CAT(0) space always has a Δ -convergent subsequence.*

Lemma 2.6 (Proposition 2.1 of [16]) *If E is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in E , then the asymptotic center of $\{x_n\}$ is in E .*

The concept of quasi-linearization was introduced by Berg and Nikolaev [4]. Let (X, ρ) be a metric space. We denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a *vector*. The *quasi-linearization* is a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} (\rho^2(a, d) + \rho^2(b, c) - \rho^2(a, c) - \rho^2(b, d)) \quad \text{for all } a, b, c, d \in X.$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that (X, ρ) satisfies the *Cauchy-Schwarz inequality* if

$$|\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle| \leq \rho(a, b)\rho(c, d) \quad \text{for all } a, b, c, d \in X.$$

It is well known from Corollary 3 of [4] that a geodesic space X is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality. Some other properties of quasi-linearization are included as follows.

Lemma 2.7 (Theorem 3.1 of [17]) *Let E be a nonempty closed convex subset of a complete CAT(0) space X , $u \in X$ and $v \in E$. Then*

$$v = P_E(u) \quad \text{if and only if} \quad \langle \overrightarrow{vu}, \overrightarrow{wv} \rangle \geq 0 \quad \text{for all } w \in E.$$

Lemma 2.8 (Lemma 2.9 of [5]) *Let X be a CAT(0) space. Then*

$$\rho^2(x, z) \leq \rho^2(y, z) + 2\langle \overrightarrow{xy}, \overrightarrow{xz} \rangle \quad \text{for all } x, y, z \in X.$$

Lemma 2.9 (Lemma 2.10 of [5]) *Let u and v be two points in a CAT(0) space X . For each $t \in [0, 1]$, we set $u_t = tu \oplus (1 - t)v$. Then, for each $x, y \in X$, we have*

- (i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1 - t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$;
- (ii) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1 - t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ and $\langle \overrightarrow{u_t x}, \overrightarrow{v y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{v y} \rangle + (1 - t)\langle \overrightarrow{vx}, \overrightarrow{v y} \rangle$.

Lemma 2.10 (Theorem 2.6 of [18]) *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X , and $x \in X$. Then $\{x_n\}$ Δ -converges to x if and only if $\limsup_{n \rightarrow \infty} \langle \overrightarrow{x_n x}, \overrightarrow{y x} \rangle \leq 0$ for all $y \in X$.*

Recall that a continuous linear functional μ on ℓ_∞ , the Banach space of bounded real sequences, is called a *Banach limit* if $\|\mu\| = \mu(1, 1, \dots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in \ell_\infty$.

Lemma 2.11 (Proposition 2 of [19]) *Let α be a real number and let $(a_1, a_2, \dots) \in \ell_\infty$ be such that $\mu_n(a_n) \leq \alpha$ for all Banach limits μ and $\limsup_n (a_{n+1} - a_n) \leq 0$. Then $\limsup_n a_n \leq \alpha$.*

Lemma 2.12 (Lemma 2.1 of [20]) *Let $\{c_n\}$ be a sequence of non-negative real numbers satisfying*

$$c_{n+1} \leq (1 - \gamma_n)c_n + \gamma_n \eta_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\eta_n\} \subset \mathbb{R}$ such that

- (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (ii) $\sum_{n=1}^{\infty} |\gamma_n \eta_n| < \infty$ or $\limsup_{n \rightarrow \infty} \eta_n \leq 0$.

Then $\{c_n\}$ converges to zero as $n \rightarrow \infty$.

Let E be a nonempty subset of a CAT(0) space (X, ρ) . We shall denote the family of nonempty bounded closed subsets of E by $BC(E)$, the family of nonempty bounded closed convex subsets of E by $BCC(E)$, and the family of nonempty compact subsets of E by $K(E)$. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $BC(X)$, i.e.,

$$H(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}, \quad A, B \in BC(X),$$

where $\text{dist}(a, B) := \inf\{\rho(a, b) : b \in B\}$ is the distance from the point a to the set B .

Definition 2.13 A multivalued mapping $T : E \rightarrow BC(X)$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$H(T(x), T(y)) \leq k\rho(x, y), \quad x, y \in E. \quad (4)$$

If (4) is valid when $k = 1$, then T is called *nonexpansive*. A point $x \in E$ is called a fixed point of T if $x \in T(x)$. We shall denote by $\text{Fix}(T)$ the set of all fixed points of T . A multivalued mapping T is said to satisfy the *endpoint condition* [21] if $\text{Fix}(T) \neq \emptyset$ and $T(x) = \{x\}$ for all $x \in \text{Fix}(T)$.

The following fact is a consequence of Lemma 3.2 in [22]. Notice also that it is an extension of Proposition 3.7 in [15].

Lemma 2.14 *If E is a closed convex subset of a complete CAT(0) space X and $T : E \rightarrow K(X)$ is a nonexpansive mapping, then the conditions $\{x_n\}$ Δ -converges to x and $\text{dist}(x_n, T(x_n)) \rightarrow 0$ imply $x \in \text{Fix}(T)$.*

The following fact is also needed.

Lemma 2.15 (Lemma 3.1 of [22]) *Let E be a closed convex subset of a complete CAT(0) space X and $T : E \rightarrow BC(X)$ be a nonexpansive mapping. If T satisfies the endpoint condition, then $\text{Fix}(T)$ is closed and convex.*

3 Main results

Theorem 3.1 *Let E be a nonempty closed convex subset of a complete CAT(0) space X , $T : E \rightarrow K(E)$ be a nonexpansive mapping satisfying the endpoint condition, and $f : E \rightarrow E$ be a contraction with $k \in [0, 1)$. Then the following statements hold:*

- (i) $\{x_s\}$ defined by (2) converges strongly to $\tilde{x}$ as $s \rightarrow 0$, where $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$.
- (ii) If $\{x_n\}$ is a bounded sequence in E such that $\lim_{n \rightarrow \infty} \text{dist}(x_n, T(x_n)) = 0$, then

$$\rho^2(f(\tilde{x}), \tilde{x}) \leq \mu_n \rho^2(f(\tilde{x}), x_n)$$

for all Banach limits μ .

Proof (i) We first show that $\{x_s\}$ is bounded. From (2), for each x_s , there exists $y_s \in T(x_s)$ such that $x_s = sf(x_s) \oplus (1-s)y_s$. By the endpoint condition, for each $p \in \text{Fix}(T)$, we have

$$\begin{aligned} \rho(x_s, p) &\leq s\rho(f(x_s), p) + (1-s)\rho(y_s, p) \\ &= s\rho(f(x_s), p) + (1-s)\text{dist}(y_s, T(p)) \\ &\leq s\rho(f(x_s), p) + (1-s)H(T(x_s), T(p)) \\ &\leq s\rho(f(x_s), p) + (1-s)\rho(x_s, p), \end{aligned}$$

which implies

$$\begin{aligned} \rho(x_s, p) &\leq \rho(f(x_s), p) \leq \rho(f(x_s), f(p)) + \rho(f(p), p) \\ &\leq k\rho(x_s, p) + \rho(f(p), p). \end{aligned}$$

Thus $\rho(x_s, p) \leq \frac{1}{1-k} \rho(f(p), p)$. Hence, $\{x_s\}$ is bounded and so are $\{f(x_s)\}$ and $\{y_s\}$. We note that

$$\text{dist}(x_s, T(x_s)) \leq \rho(x_s, y_s) \leq s\rho(f(x_s), y_s) \rightarrow 0 \quad \text{as } s \rightarrow 0. \quad (5)$$

Next, we show that $\{x_s\}$ converges strongly to $\tilde{x}$ where $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$. Let a sequence $\{s_n\}$ in $(0, 1)$ converging to 0 and put $x_n := x_{s_n}$. It suffices to show that there exists a subsequence of $\{x_n\}$ converging to $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$. By Lemmas 2.5 and 2.14, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $\tilde{x}$ in $\text{Fix}(T)$ such that $\Delta\text{-}\lim_{k \rightarrow \infty} x_{n_k} = \tilde{x}$. It follows from the endpoint condition and Lemma 2.9(i) that

$$\begin{aligned} \rho^2(x_{n_k}, \tilde{x}) &= \langle \overrightarrow{x_{n_k} \tilde{x}}, \overrightarrow{x_{n_k} \tilde{x}} \rangle \\ &\leq s_{n_k} \langle \overrightarrow{f(x_{n_k}) \tilde{x}}, \overrightarrow{x_{n_k} \tilde{x}} \rangle + (1-s_{n_k}) \langle \overrightarrow{y_{n_k} \tilde{x}}, \overrightarrow{x_{n_k} \tilde{x}} \rangle \\ &\leq s_{n_k} \langle \overrightarrow{f(x_{n_k}) \tilde{x}}, \overrightarrow{x_{n_k} \tilde{x}} \rangle + (1-s_{n_k}) \rho(y_{n_k}, \tilde{x}) \rho(x_{n_k}, \tilde{x}) \\ &= s_{n_k} \langle \overrightarrow{f(x_{n_k}) \tilde{x}}, \overrightarrow{x_{n_k} \tilde{x}} \rangle + (1-s_{n_k}) \text{dist}(y_{n_k}, T(\tilde{x})) \rho(x_{n_k}, \tilde{x}) \\ &\leq s_{n_k} \langle \overrightarrow{f(x_{n_k}) \tilde{x}}, \overrightarrow{x_{n_k} \tilde{x}} \rangle + (1-s_{n_k}) H(T(x_{n_k}), T(\tilde{x})) \rho(x_{n_k}, \tilde{x}) \\ &\leq s_{n_k} \langle \overrightarrow{f(x_{n_k}) \tilde{x}}, \overrightarrow{x_{n_k} \tilde{x}} \rangle + (1-s_{n_k}) \rho^2(x_{n_k}, \tilde{x}), \end{aligned}$$

which implies

$$\begin{aligned}
\rho^2(x_{n_k}, \tilde{x}) &\leq \langle \overrightarrow{f(x_{n_k})\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle \\
&= \langle \overrightarrow{f(x_{n_k})f(\tilde{x})}, \overrightarrow{x_{n_k}\tilde{x}} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle \\
&\leq \rho(f(x_{n_k}), f(\tilde{x}))\rho(x_{n_k}, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle \\
&\leq k\rho^2(x_{n_k}, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle.
\end{aligned}$$

Thus

$$\rho^2(x_{n_k}, \tilde{x}) \leq \frac{1}{1-k} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle. \quad (6)$$

Since $\Delta\text{-}\lim_{k \rightarrow \infty} x_{n_k} = \tilde{x}$, by Lemma 2.10 we have

$$\limsup_{k \rightarrow \infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_k}\tilde{x}} \rangle \leq 0.$$

This, together with (6), implies that $\{x_{n_k}\}$ converges strongly to $\tilde{x}$.

Next, we show that $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$. Since T satisfies the endpoint condition, we have

$$\begin{aligned}
\text{dist}(f(x_{n_k}), T(x_{n_k})) &\leq \rho(f(x_{n_k}), f(\tilde{x})) + \rho(f(\tilde{x}), \tilde{x}) + \text{dist}(\tilde{x}, T(x_{n_k})) \\
&\leq \rho(x_{n_k}, \tilde{x}) + \rho(f(\tilde{x}), \tilde{x}) + H(T(\tilde{x}), T(x_{n_k})) \\
&\leq \rho(f(\tilde{x}), \tilde{x}) + 2\rho(x_{n_k}, \tilde{x})
\end{aligned}$$

and

$$\begin{aligned}
\rho(f(\tilde{x}), \tilde{x}) &= \text{dist}(f(\tilde{x}), T(\tilde{x})) \\
&\leq \rho(f(\tilde{x}), f(x_{n_k})) + \text{dist}(f(x_{n_k}), T(x_{n_k})) + H(T(x_{n_k}), T(\tilde{x})) \\
&\leq \text{dist}(f(x_{n_k}), T(x_{n_k})) + 2\rho(x_{n_k}, \tilde{x}).
\end{aligned}$$

Thus

$$|\text{dist}(f(x_{n_k}), T(x_{n_k})) - \rho(f(\tilde{x}), \tilde{x})| \leq 2\rho(x_{n_k}, \tilde{x}). \quad (7)$$

Applying Lemma 2.3, for any $q \in \text{Fix}(T)$, we have

$$\begin{aligned}
\rho^2(x_{n_k}, q) &= \rho^2(s_{n_k}f(x_{n_k}) \oplus (1-s_{n_k})y_{n_k}, q) \\
&\leq s_{n_k}\rho^2(f(x_{n_k}), q) + (1-s_{n_k})\rho^2(y_{n_k}, q) - s_{n_k}(1-s_{n_k})\rho^2(f(x_{n_k}), y_{n_k}) \\
&\leq s_{n_k}\rho^2(f(x_{n_k}), q) + (1-s_{n_k})H^2(T(x_{n_k}), T(q)) - s_{n_k}(1-s_{n_k})\rho^2(f(x_{n_k}), y_{n_k}) \\
&\leq s_{n_k}\rho^2(f(x_{n_k}), q) + (1-s_{n_k})\rho^2(x_{n_k}, q) - s_{n_k}(1-s_{n_k})\rho^2(f(x_{n_k}), y_{n_k}).
\end{aligned}$$

This implies that

$$\begin{aligned}
\rho^2(x_{n_k}, q) &\leq \rho^2(f(x_{n_k}), q) - (1-s_{n_k})\rho^2(f(x_{n_k}), y_{n_k}) \\
&\leq \rho^2(f(x_{n_k}), q) - (1-s_{n_k})[\text{dist}(f(x_{n_k}), T(x_{n_k}))]^2.
\end{aligned}$$

Taking $k \rightarrow \infty$, together with (7), we get

$$\rho^2(\tilde{x}, q) \leq \rho^2(f(\tilde{x}), q) - \rho^2(f(\tilde{x}), \tilde{x}).$$

Hence

$$0 \leq \frac{1}{2} [\rho^2(\tilde{x}, \tilde{x}) + \rho^2(f(\tilde{x}), q) - \rho^2(\tilde{x}, q) - \rho^2(f(\tilde{x}), \tilde{x})] = \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}} \rangle \quad \text{for all } q \in \text{Fix}(T).$$

By Lemma 2.7, $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$ and this completes the proof.

(ii) Let $\{x_n\}$ be a bounded sequence in E such that $\lim_{n \rightarrow \infty} \text{dist}(x_n, T(x_n)) = 0$ and let μ be a Banach limit. Suppose that $\mu_n \rho^2(f(\tilde{x}), x_n) < \eta < \gamma < \rho^2(f(\tilde{x}), \tilde{x})$ for some $\eta, \gamma \in \mathbb{R}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\rho^2(f(\tilde{x}), x_{n_k}) < \gamma \quad \text{for all } k \in \mathbb{N}. \quad (8)$$

Otherwise $\rho^2(f(\tilde{x}), x_n) \geq \gamma$ for all large n which implies that $\mu_n \rho^2(f(\tilde{x}), x_n) \geq \gamma > \eta$, a contradiction, and therefore (8) holds. By Lemmas 2.5 and 2.14, we can assume that $\{x_{n_k}\}$ Δ -converges to a point p in $\text{Fix}(T)$. Then by (8) and Lemma 2.6, p is contained in the closed ball centered at $f(\tilde{x})$ of radius $\sqrt{\gamma}$. This contradicts the fact that $\tilde{x}$ is the unique nearest point of $f(\tilde{x})$ in $\text{Fix}(T)$ and hence the proof is complete. $\square$

As an immediate consequence of Theorem 3.1, we obtain the following.

Corollary 3.2 (Theorem 3.4 of [22]) *Let E be a nonempty closed convex subset of a complete CAT(0) space X , and $T : E \rightarrow K(E)$ be a nonexpansive mapping satisfying the endpoint condition. Fix $u \in E$, and for each $s \in (0, 1)$ let x_s be a fixed point of $G_s : E \rightarrow K(E)$ defined by $G_s(x) = su \oplus (1-s)T(x)$, that is, $x_s \in E$ and $x_s \in su \oplus (1-s)T(x_s)$. Then the following statements hold:*

- (i) $\{x_s\}$ converges strongly to $\tilde{x}$ as $s \rightarrow 0$, where $\tilde{x} = P_{\text{Fix}(T)}(u)$.
- (ii) If $\{x_n\}$ is a bounded sequence in E such that $\lim_{n \rightarrow \infty} \text{dist}(x_n, T(x_n)) = 0$, then

$$\rho^2(u, \tilde{x}) \leq \mu_n \rho^2(u, x_n)$$

for all Banach limits μ .

Now, we define an explicit approximation method for multivalued nonexpansive mappings. Let $T : E \rightarrow K(E)$ be a nonexpansive mapping, $f : E \rightarrow E$ be a contraction, and $\{\alpha_n\}$ be a sequence in $(0, 1)$. Fix $x_1 \in E$ and $y_1 \in T(x_1)$. Let

$$x_2 = \alpha_1 f(x_1) \oplus (1 - \alpha_1) y_1.$$

By the definition of Hausdorff distance and the nonexpansiveness of T , we can choose $y_2 \in T(x_2)$ such that

$$\rho(y_1, y_2) \leq \rho(x_1, x_2).$$

Inductively, we have

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) y_n, \quad y_n \in T(x_n), \quad (9)$$

and $\rho(y_n, y_{n+1}) \leq \rho(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Theorem 3.3 *Let E be a nonempty closed convex subset of a complete CAT(0) space X , $T : E \rightarrow K(E)$ be a nonexpansive mapping satisfying the endpoint condition. Let $f : E \rightarrow E$ be a contraction with $k \in [0, \frac{1}{2})$ and $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2-k})$ satisfying:*

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n \rightarrow \infty} (\frac{\alpha_n}{\alpha_{n+1}}) = 1$.

Then the sequence $\{x_n\}$ defined by (9) converges strongly to $\tilde{x}$, where $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$.

Proof We divide the proof into three steps.

Step 1. We show that $\{x_n\}$, $\{y_n\}$, and $\{f(x_n)\}$ are bounded sequences. Let $p \in \text{Fix}(T)$. By Lemma 2.2, we have

$$\begin{aligned} \rho(x_{n+1}, p) &\leq \alpha_n \rho(f(x_n), p) + (1 - \alpha_n) \rho(y_n, p) \\ &\leq \alpha_n [\rho(f(x_n), f(p)) + \rho(f(p), p)] + (1 - \alpha_n) H(T(x_n), T(p)) \\ &\leq \max \left\{ \rho(x_n, p), \frac{\rho(f(p), p)}{1 - k} \right\}. \end{aligned}$$

By induction, we also have

$$\rho(x_n, p) \leq \max \left\{ \rho(x_1, p), \frac{\rho(f(p), p)}{1 - k} \right\} \quad \text{for all } n \in \mathbb{N}.$$

Hence, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{f(x_n)\}$.

Step 2. We show that $\lim_{n \rightarrow \infty} \rho(x_{n+1}, x_n) = 0$. Observe that

$$\begin{aligned} \rho(x_{n+1}, x_n) &\leq \rho(\alpha_n f(x_n) \oplus (1 - \alpha_n) y_n, \alpha_{n-1} f(x_{n-1}) \oplus (1 - \alpha_{n-1}) y_{n-1}) \\ &\leq \rho(\alpha_n f(x_n) \oplus (1 - \alpha_n) y_n, \alpha_n f(x_n) \oplus (1 - \alpha_n) y_{n-1}) \\ &\quad + \rho(\alpha_n f(x_n) \oplus (1 - \alpha_n) y_{n-1}, \alpha_n f(x_{n-1}) \oplus (1 - \alpha_n) y_{n-1}) \\ &\quad + \rho(\alpha_n f(x_{n-1}) \oplus (1 - \alpha_n) y_{n-1}, \alpha_{n-1} f(x_{n-1}) \oplus (1 - \alpha_{n-1}) y_{n-1}) \\ &\leq (1 - \alpha_n) \rho(y_n, y_{n-1}) + \alpha_n \rho(f(x_n), f(x_{n-1})) \\ &\quad + |\alpha_n - \alpha_{n-1}| \rho(f(x_{n-1}), y_{n-1}) \\ &\leq (1 - \alpha_n(1 - k)) \rho(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| \rho(f(x_{n-1}), y_{n-1}). \end{aligned}$$

Putting, in Lemma 2.12, $c_n = \rho(x_n, x_{n-1})$, $\gamma_n = (1 - k)\alpha_n$ and $\eta_n = \frac{1}{1-k} |1 - \frac{\alpha_{n-1}}{\alpha_n}| \rho(f(x_{n-1}), y_{n-1})$, we get by (C2) and (C3) that $\lim_{n \rightarrow \infty} \rho(x_{n+1}, x_n) = 0$.

Step 3. We show that $\{x_n\}$ converges strongly to a point $\tilde{x} \in \text{Fix}(T)$ with $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$. For each $s \in (0, 1)$, let x_s be defined by (2). By Theorem 3.1, $\{x_s\}$ converges strongly to a

point $\tilde{x} \in \text{Fix}(T)$ and $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$. We note that

$$\begin{aligned} \text{dist}(x_n, T(x_n)) &\leq \rho(x_n, y_n) \\ &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, y_n) \\ &\leq \rho(x_n, x_{n+1}) + \alpha_n \rho(f(x_n), y_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Again by Theorem 3.1, we have $\mu_n(\rho^2(f(\tilde{x}), \tilde{x}) - \rho^2(f(\tilde{x}), x_n)) \leq 0$ for all Banach limits μ . Moreover, since $\lim_{n \rightarrow \infty} \rho(x_{n+1}, x_n) = 0$,

$$\limsup_{n \rightarrow \infty} [\rho^2(f(\tilde{x}), \tilde{x}) - \rho^2(f(\tilde{x}), x_{n+1})] - (\rho^2(f(\tilde{x}), \tilde{x}) - \rho^2(f(\tilde{x}), x_n)) = 0.$$

It follows from Lemma 2.11 that

$$\limsup_{n \rightarrow \infty} (\rho^2(f(\tilde{x}), \tilde{x}) - \rho^2(f(\tilde{x}), x_n)) \leq 0. \quad (10)$$

For each $n \in \mathbb{N}$, we set $z_n = \alpha_n \tilde{x} \oplus (1 - \alpha_n) y_n$. It follows from Lemmas 2.8 and 2.9 that

$$\begin{aligned} \rho^2(x_{n+1}, \tilde{x}) &\leq \rho^2(z_n, \tilde{x}) + 2 \langle \overrightarrow{x_{n+1} z_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle \\ &\leq (1 - \alpha_n)^2 \rho^2(y_n, \tilde{x}) + 2 [\alpha_n \langle \overrightarrow{f(x_n) z_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + (1 - \alpha_n) \langle \overrightarrow{y_n z_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle] \\ &\leq (1 - \alpha_n)^2 H^2(T(x_n), T(\tilde{x})) + 2 [\alpha_n^2 \langle \overrightarrow{f(x_n) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle \\ &\quad + \alpha_n (1 - \alpha_n) \langle \overrightarrow{f(x_n) y_n}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + \alpha_n (1 - \alpha_n) \langle \overrightarrow{y_n \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle] \\ &\leq (1 - \alpha_n)^2 \rho^2(x_n, \tilde{x}) + 2 [\alpha_n^2 \langle \overrightarrow{f(x_n) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + \alpha_n (1 - \alpha_n) \langle \overrightarrow{f(x_n) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle] \\ &= (1 - \alpha_n)^2 \rho^2(x_n, \tilde{x}) + 2 \alpha_n \langle \overrightarrow{f(x_n) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle \\ &= (1 - \alpha_n)^2 \rho^2(x_n, \tilde{x}) + 2 \alpha_n \langle \overrightarrow{f(x_n) f(\tilde{x})}, \overrightarrow{x_{n+1} \tilde{x}} \rangle + 2 \alpha_n \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle \\ &\leq (1 - \alpha_n)^2 \rho^2(x_n, \tilde{x}) + 2 k \alpha_n \rho(x_n, \tilde{x}) \rho(x_{n+1}, \tilde{x}) + 2 \alpha_n \langle \overrightarrow{f(\tilde{x}) \tilde{x}}, \overrightarrow{x_{n+1} \tilde{x}} \rangle \\ &\leq (1 - \alpha_n)^2 \rho^2(x_n, \tilde{x}) + k \alpha_n [\rho^2(x_n, \tilde{x}) + \rho^2(x_{n+1}, \tilde{x})] \\ &\quad + \alpha_n [\rho^2(f(\tilde{x}), \tilde{x}) + \rho^2(x_{n+1}, \tilde{x}) - \rho^2(f(\tilde{x}), x_{n+1})], \end{aligned}$$

yielding

$$\begin{aligned} \rho^2(x_{n+1}, \tilde{x}) &\leq \frac{1 - (2 - k)\alpha_n + \alpha_n^2}{1 - (1 + k)\alpha_n} \rho^2(x_n, \tilde{x}) + \frac{\alpha_n}{1 - (1 + k)\alpha_n} [\rho^2(f(\tilde{x}), \tilde{x}) - \rho^2(f(\tilde{x}), x_{n+1})] \\ &\leq \frac{1 - (2 - k)\alpha_n}{1 - (1 + k)\alpha_n} \rho^2(x_n, \tilde{x}) + \frac{\alpha_n^2}{1 - (1 + k)\alpha_n} M \\ &\quad + \frac{\alpha_n}{1 - (1 + k)\alpha_n} [\rho^2(f(\tilde{x}), \tilde{x}) - \rho^2(f(\tilde{x}), x_{n+1})], \end{aligned}$$

where $M \geq \sup_{n \in \mathbb{N}} \{\rho(x_n, \tilde{x})\}$. It follows that

$$\rho^2(x_{n+1}, \tilde{x}) \leq (1 - \gamma_n) \rho^2(x_n, \tilde{x}) + \gamma_n \eta_n, \quad (11)$$

where

$$\gamma_n = \frac{(1-2k)\alpha_n}{1-(1+k)\alpha_n} \quad \text{and} \quad \eta_n = \frac{\alpha_n}{1-2k}M + \frac{1}{1-2k}[\rho^2(f(\tilde{x}), \tilde{x}) - \rho^2(f(\tilde{x}), x_{n+1})].$$

Since $\alpha_n \in (0, \frac{1}{2-k})$ and $k \in [0, \frac{1}{2})$, we have $\gamma_n \in (0, 1)$. By (C1) and (10), $\limsup_n \eta_n \leq 0$. Applying Lemma 2.12 to the inequality (11), we can conclude that $x_n \rightarrow \tilde{x}$ as $n \rightarrow \infty$. Therefore, the proof is complete. $\square$

Corollary 3.4 *Let E be a nonempty closed convex subset of a complete CAT(0) space X , and $T : E \rightarrow K(E)$ be a nonexpansive mapping satisfying the endpoint condition. Suppose that $u, x_1 \in E$ are arbitrarily chosen and $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)y_n,$$

where $y_n \in T(x_n)$ such that $\rho(y_n, y_{n+1}) \leq \rho(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $\{\alpha_n\}$ is a sequence in $(0, \frac{1}{2})$ satisfying (C1), (C2), and (C3). Then $\{x_n\}$ converges strongly to the unique nearest point of u in $\text{Fix}(T)$.

Proof We define $f : E \rightarrow E$ by $f(x) = u$ for all $x \in E$. Then f is a contraction with $k = 0$. The conclusion follows immediately from Theorem 3.3. $\square$

Remark 3.5 There are some results in Banach spaces related to our work (see, e.g., [23–26]). Notice that our approach is quite different from that of [23–26].

As we have observed, Theorem 3.3 can be viewed as an extension of Theorem 1.3 for a contraction f with $k \in [0, \frac{1}{2})$. It remains an open question whether Theorem 3.3 holds for $k \in [\frac{1}{2}, 1)$.

Question 3.6 Let E be a nonempty closed convex subset of a complete CAT(0) space X , $T : E \rightarrow K(E)$ be a nonexpansive mapping satisfying the endpoint condition. Let $f : E \rightarrow E$ be a contraction with $k \in [0, 1)$ and $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying (C1), (C2), and (C3) and $\{x_n\}$ be a sequence defined by (9). Does $\{x_n\}$ converge to $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$?

4 $\mathbb{R}$ -Trees

Definition 4.1 An $\mathbb{R}$ -tree is a geodesic space X such that:

- (i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in X$;
- (ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

By (i) and (ii) we have

- (iii) if $u, v, w \in X$, then $[u, v] \cap [u, w] = [u, z]$ for some $z \in X$.

It is well known that every $\mathbb{R}$ -tree is a CAT(0) space which does not contain the Euclidean plane. To avoid the endpoint condition, we prefer to work on $\mathbb{R}$ -trees. Although an $\mathbb{R}$ -tree is not strong enough to make all nonexpansive mappings having the endpoint condition (see Example 5.3 in [27]), but it is strong enough to make our theorems hold without this condition.

Let E be a closed convex subset of a complete $\mathbb{R}$ -tree (X, ρ) and $T : E \rightarrow BCC(E)$ a multivalued mapping. Then, by Theorem 4.1 of [28], there exists a single-valued mapping

$t : E \rightarrow E$ such that $t(x) \in T(x)$ and

$$\rho(t(x), t(y)) \leq H(T(x), T(y)) \quad \text{for all } x, y \in E. \quad (12)$$

In this case, we call t a *nonexpansive selection* of T .

Let $f : E \rightarrow E$ be a contraction and fix $x_1 \in E$. We define a sequence $\{x_n\}$ in E by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) y_n, \quad (13)$$

where $y_n = t(x_n) \in T(x_n)$ for all $n \in \mathbb{N}$.

Theorem 4.2 *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$ -tree X , and $T : E \rightarrow BCC(E)$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Let $f : E \rightarrow E$ be a contraction with $k \in [0, 1)$ and $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2-k})$ satisfying:*

$$(C1) \quad \lim_{n \rightarrow \infty} \alpha_n = 0;$$

$$(C2) \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(C3) \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty \text{ or } \lim_{n \rightarrow \infty} \left(\frac{\alpha_n}{\alpha_{n+1}} \right) = 1.$$

Then the sequence $\{x_n\}$ defined by (13) converges strongly to $\tilde{x} = P_{\text{Fix}(T)}(f(\tilde{x}))$.

Proof By Theorem 4.2 of [28] (see also Theorem 2 of [29]), $\text{Fix}(t) = \text{Fix}(T)$. The set is closed and convex by Proposition 1 of [30] and t is nonexpansive by (12). The conclusion follows from Theorem 1.3. $\square$

Corollary 4.3 *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$ -tree X , and $T : E \rightarrow BCC(E)$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, \frac{1}{2})$ satisfying (C1), (C2), and (C3). Fix $x_1 \in E$ and let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) t(x_n), \quad n \in \mathbb{N},$$

where $t : E \rightarrow E$ is a nonexpansive selection of T with $\text{Fix}(t) = \text{Fix}(T)$. Then $\{x_n\}$ converges strongly to the unique nearest point of u in $\text{Fix}(T)$.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

The authors read and approved the final manuscript.

Acknowledgements

This research was supported by Chiang Mai University and Thailand Research Fund under Grant RTA5780007.

Received: 2 February 2015 Accepted: 18 June 2015 Published online: 14 July 2015

References

1. Moudafi, A: Viscosity approximation methods for fixed-points problems. *J. Math. Anal. Appl.* **241**, 46-55 (2000)
2. Halpern, B: Fixed points of nonexpanding maps. *Bull. Am. Math. Soc.* **73**, 957-961 (1967)
3. Shi, LY, Chen, RD: Strong convergence of viscosity approximation methods for nonexpansive mappings in $\text{CAT}(0)$ spaces. *J. Appl. Math.* **2012**, Article ID 421050 (2012)
4. Berg, ID, Nikolaev, IG: Quasilinearization and curvature of Alexandrov spaces. *Geom. Dedic.* **133**, 195-218 (2008)
5. Wangkeeree, R, Preechasilp, P: Viscosity approximation methods for nonexpansive mappings in $\text{CAT}(0)$ spaces. *J. Inequal. Appl.* **2013**, 93 (2013)
6. Nadler, SB: Multi-valued contraction mappings. *Pac. J. Math.* **30**, 475-487 (1969)
7. Bo, LH, Yi, L: Viscosity approximation methods for a nonexpansive multi-valued mapping in $\text{CAT}(0)$ spaces and variational inequality. *Theor. Math. Appl.* **4**, 45-63 (2014)

8. Pietramala, P: Convergence of approximating fixed point sets for multivalued nonexpansive mappings. *Comment. Math. Univ. Carol.* **32**, 697-701 (1991)
9. Bridson, M, Haefliger, A: *Metric Spaces of Non-Positive Curvature*. Springer, Berlin (1999)
10. Burago, D, Burago, Y, Ivanov, S: *A Course in Metric Geometry*. Graduate Studies in Math., vol. 33. Am. Math. Soc., Providence (2001)
11. Brown, KS: *Buildings*. Springer, New York (1989)
12. Dhompongsa, S, Panyanak, B: On Δ -convergence theorems in CAT(0) spaces. *Comput. Math. Appl.* **56**, 2572-2579 (2008)
13. Kirk, WA: Geodesic geometry and fixed point theory II. In: *International Conference on Fixed Point Theory and Applications*, pp. 113-142. Yokohama Publ., Yokohama (2004)
14. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. *Nonlinear Anal.* **65**, 762-772 (2006)
15. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. *Nonlinear Anal.* **68**, 3689-3696 (2008)
16. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. *J. Nonlinear Convex Anal.* **8**, 35-45 (2007)
17. Dehghan, H, Rooin, J: Metric projection and convergence theorems for nonexpansive mappings in Hadamard spaces. *J. Nonlinear Convex Anal.* (to appear)
18. Kakavandi, BA: Weak topologies in complete CAT(0) metric spaces. *Proc. Am. Math. Soc.* **141**, 1029-1039 (2013)
19. Shioji, N, Takahashi, W: Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces. *Proc. Am. Math. Soc.* **125**, 3641-3645 (1997)
20. Xu, HK: An iterative approach to quadratic optimization. *J. Optim. Theory Appl.* **116**, 659-678 (2003)
21. Dhompongsa, S, Kaewkhao, A, Panyanak, B: Browder's convergence theorem for multivalued mappings without endpoint condition. *Topol. Appl.* **159**, 2757-2763 (2012)
22. Dhompongsa, S, Kaewkhao, A, Panyanak, B: On Kirk's strong convergence theorem for multivalued nonexpansive mappings on CAT(0) spaces. *Nonlinear Anal.* **75**, 459-468 (2012)
23. Jung, JS: Convergence of approximating fixed points for multivalued nonself-mappings in Banach spaces. *Korean J. Math.* **16**, 215-231 (2008)
24. Zegeye, H, Shahzad, N: Viscosity approximation methods for nonexpansive multimaps in Banach spaces. *Acta Math. Sin. Engl. Ser.* **26**, 1165-1176 (2010)
25. Cui, Y, Hudzik, H: Viscosity approximation methods for multivalued mappings in Banach spaces. *Numer. Funct. Anal. Optim.* **33**, 1288-1303 (2012)
26. Jung, JS: Convergence of a viscosity iterative method for multivalued nonself-mappings in Banach spaces. *Abstr. Appl. Anal.* **2013**, Article ID 369412 (2013)
27. Samanmit, K, Panyanak, B: On multivalued nonexpansive mappings in $\mathbb{R}$ -trees. *J. Appl. Math.* **2012**, Article ID 629149 (2012)
28. Aksoy, AG, Khamsi, MA: A selection theorem in metric trees. *Proc. Am. Math. Soc.* **134**, 2957-2966 (2006)
29. Khamsi, MA, Kirk, WA, Yanez, CM: Fixed point and selection theorems in hyperconvex spaces. *Proc. Am. Math. Soc.* **128**, 3275-3283 (2000)
30. Markin, JT: Fixed points for generalized nonexpansive mappings in $\mathbb{R}$ -trees. *Comput. Math. Appl.* **62**, 4614-4618 (2011)

Chapter III

Fixed Point Theorems for Some Contractive Mappings and G-Nonexpansive Mappings with Directed Graphs

In this chapter, we study fixed point theorems and iteration methods for some contractive mappings and G-nonexpansive mappings in metric spaces and Banach spaces with directed graph. In section 3.1, we introduce the concepts of weak g -graph-preserving for multivalued mappings and weak G -contractions in a metric space endowed with a directed graph. We establish the coincidence point theorems for this type of mappings in a complete metric space endowed with a directed graph. Examples illustrating our main results are also presented. In section 3.2, we present some existence and uniqueness results for coupled coincidence point and common fixed point of θ - ψ -contraction mappings in complete metric spaces endowed with a directed graph. Our results generalize the results obtained by Kadelburg *et al.* (Fixed Point Theory Appl. 2015:27, 2015, doi:10.1007/s11590-013-0708-4). We also have an application to some integral system to support the results. In section 3.3, we prove some fixed point theorems for multivalued nonself G -almost contractions in Banach spaces with a directed graph and give some examples to illustrate our main results. In section 3.4, we prove some fixed point theorems for non-self G -almost contractive mappings in Banach spaces with a directed graph and give some examples to illustrate our main results. In section 3.5, we prove Browder's convergence theorem for G -nonexpansive mappings in a Hilbert space with a directed graph. Moreover, we also prove strong convergence of the Halpern iteration process to a fixed point of G -nonexpansive mappings in a Hilbert space endowed with a directed graph. In section 3.6, we prove the weak and strong convergence of a sequence generated by a modified S-iteration process for finding a common fixed point of two G -nonexpansive mappings in a uniformly convex Banach space with a directed graph. We also give some numerical examples for supporting our main theorem and compare convergence rate between the studied iteration and the Ishikawa iteration. In section 3.7, we prove a strong convergence theorem for two different hybrid methods by using CQ method for a finite family of G -nonexpansive mappings in a Hilbert space. We give an example and numerical results for supporting our main results and compare the rate of convergence of the two iterative methods.

3.1 Coincidence point theorems for some multi-valued mappings in complete metric spaces endowed with a graph

In this **chapter**, we introduce the concepts of weak g -graph-preserving for multi-valued mappings and weak G -contractions in a metric space endowed with a directed graph. We establish the coincidence point theorems for this type of mappings in a complete metric space endowed with a directed graph. Examples illustrating our main results are also presented. Our results extend and generalize various known results in the literature.

1 Introduction

Fixed point theorems for contraction mappings and their generalizations play an important role in the study of theory of equations. The Banach contraction principle [1] is a fundamental result which can be applied widely for solving the existence of solutions of various equations. Over the years, it has been widely extended and generalized in different directions by many authors (see [1–10]).

Let (X, d) be a complete metric space. A mapping $T : X \rightarrow X$ is called a *contraction* if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X. \quad (1.1)$$

A mapping $T : X \rightarrow X$ is called a *Kannan mapping* if there exists a real number $\alpha \in [0, \frac{1}{2})$ such that for every $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty). \quad (1.2)$$

In 1968, Kannan [6] established the fixed point theorem for the Kannan mapping T .

One year later, Chatterjea [5] introduced the new concept of contraction mapping. Based on the Kannan mapping, he changed some variables in (1.2) and obtained new condition, that is,

$$d(Tx, Ty) \leq \alpha d(x, Ty) + \alpha d(y, Tx). \quad (1.3)$$

It is easily seen that conditions (1.1) and (1.2) are independent from each other, (1.1) and (1.3) as well.

Zamfirescu [10] proved some fixed point theorems by combining (1.1), (1.2) and (1.3), which is stated as follows.

Theorem 1.1 [10] *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a map for which there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 < b, c < \frac{1}{2}$ such that for each pair $x, y \in X$, at least one of the following is true:*

- (i) $d(Tx, Ty) \leq ad(x, y)$;
- (ii) $d(Tx, Ty) \leq bd(x, Tx) + bd(y, Ty)$;
- (iii) $d(Tx, Ty) \leq cd(x, Ty) + cd(y, Tx)$.

Then T is a Picard operator, that is, T has a unique fixed point $x_0 \in X$ and for each $x \in X$, $T^n x \rightarrow x_0$.

In [2] Berinde introduced and studied a weak contraction mapping on a complete metric space which is weaker than the Zamfirescu operators.

Definition 1.2 [2] *Let (X, d) be a metric space. A map $T : X \rightarrow X$ is called *weak contraction* if there exist a constant $\alpha \in (0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + Ld(x, Ty) \quad \text{for all } x, y \in X. \quad (1.4)$$

The following results can be found in [2].

Proposition 1.3 [2]

- (1) *Any Kannan mapping is a weak contraction.*
- (2) *Any mapping T satisfying the contractive condition (1.3) is a weak contraction.*
- (3) *Any Zamfirescu mapping, i.e., any mapping satisfying the assumptions in Theorem 1.1, is a weak contraction.*

A partial ordering is a binary relation $\preceq$ over the set X which satisfies the following conditions:

- (1) $x \preceq x$ (reflexivity);
- (2) $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetry);
- (3) $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity)

for all $x, y, z \in X$. A set with a partial ordering $\preceq$ is called a *partially ordered set*. We write $x < y$ if $x \preceq y$ and $x \neq y$.

Fixed point theorems for monotone single-valued mappings have been investigated and studied in partially ordered metric spaces by many mathematicians (see [7–9, 11, 12]). Nieto and Rodriguez-Lopez [7, 8] were the first who studied some fixed point theorems for monotone nondecreasing mappings in partially ordered metric spaces and applied the obtained results to study an existence problem of ordinary differential equations.

The study of fixed point for multi-valued contraction mappings using the Pompeiu-Hausdorff metric was first performed by Nadler [13].

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $D(x, A) = \inf\{d(x, y) : y \in A\}$. The class of all nonempty bounded and closed subsets of X is denoted by $CB(X)$. Let H be

a Pompeiu-Hausdorff metric induced by the metric d on X , that is,

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\}$$

for every $A, B \in CB(X)$.

We let $\text{Comp}(X)$ be the set of all nonempty compact subsets of X . It is clear that $\text{Comp}(X)$ is included in $CB(X)$.

Let $T : X \rightarrow 2^X$ (collection of all nonempty subsets of X) and $g : X \rightarrow X$. An element $x \in X$ is called

- (i) *fixed point* of T if $x \in T(x)$,
- (ii) *coincidence point* of a hybrid pair $\{T, g\}$ if $g(x) \in T(x)$.

In 1969, Nadler [13] introduced the concept of Banach contraction principle for a multi-valued mapping and proved the existence of fixed point for multi-valued version of the Banach contraction principle. The following theorem is the first well-known theorem of multi-valued contractions studied by Nadler.

Theorem 1.4 [13] *Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ be a multi-valued mapping. If there exists $k \in [0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X,$$

then T has a fixed point in X .

In 2007, Berinde and Berinde [14] provided the new type of contraction which is a generalization of the contraction principle considered by Nadler.

Definition 1.5 [14] *Let (X, d) be a metric space and $T : X \rightarrow CB(X)$ be a multi-valued mapping. T is said to be a *multi-valued weak contraction* or a *multi-valued (θ, L) -weak contraction* if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that*

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X.$$

We now recall some notions concerning a directed graph. Let (X, d) be a metric space and Δ denote the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges is a subset of $X \times X$. We assume that the graph G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$. We denote by G^{-1} the conversion of a graph G , i.e.,

$$G^{-1} = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

The next definition, G -contraction, was introduced by Jachymski [15] in 2008.

Definition 1.6 [15] *Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, i.e., $\Delta \subseteq E(G)$. We say that a mapping $T : X \rightarrow X$ is a G -contraction if T preserves edges of G , i.e., for every $x, y \in X$,*

$$(x, y) \in E(G) \quad \Rightarrow \quad (Tx, Ty) \in E(G) \tag{1.5}$$

and there exists $\alpha \in (0, 1)$ such that $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow d(Tx, Ty) \leq \alpha d(x, y).$$

The mapping $T : X \rightarrow X$ satisfying condition (1.5) is called a graph-preserving mapping. Under some additional properties of a metric space X endowed with a directed graph, Jachymski showed that a G -contraction $T : X \rightarrow X$ has a fixed point if and only if there exists $x \in X$ such that $(x, T(x)) \in E(G)$.

Subsequently, Beg and Butt [16] tried to introduce the concept of G -contraction for multi-valued mappings, but their extension was not carried correctly (see [17, 18]).

In 2011, Nicolae *et al.* [19] extended the notion of multivalued contraction on a metric space with a graph.

Recently, Dinevari and Frigon [20] introduced a new concept of G -contraction multi-valued mappings.

Definition 1.7 [20] Let $T : X \rightarrow 2^X$ be a map with nonempty values. We say that T is a G -contraction (in the sense of Dinevari and Frigon) if there exists $\alpha \in (0, 1)$ such that for all $(x, y) \in E(G)$ and $u \in Tx$, there exists $v \in Ty$ such that $(u, v) \in E(G)$ and $d(u, v) \leq \alpha d(x, y)$.

They showed that under some properties, weaker than Property (A), a multi-valued G -contraction with the closed value has a fixed point. Recently, Tiammee and Suantai [21] introduced the concept of graph-preserving for multi-valued mappings and proved their fixed point theorem in a complete metric space endowed with a graph.

Definition 1.8 [21] Let X be a nonempty set, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and $T : X \rightarrow CB(X)$. Then T is said to be *graph-preserving* if $(x, y) \in E(G) \Rightarrow (u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$.

Definition 1.9 [21] Let X be a nonempty set, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, $g : X \rightarrow X$, and $T : X \rightarrow CB(X)$. Then T is said to be *g-graph-preserving* if for any $x, y \in X$ such that

$$(g(x), g(y)) \in E(G) \Rightarrow (u, v) \in E(G)$$

for all $u \in Tx$ and $v \in Ty$.

Recently, Phon-on *et al.* [22] introduced a new type of weak G -contraction which is weaker than that of Tiammee and Suantai [21], and they proved some fixed point theorems for this type of mappings with compact values which is a generalization of several known results in a complete metric space endowed with a graph.

Definition 1.10 [22] Let X be a nonempty set and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and $T : X \rightarrow \text{Comp}(X)$. Then T is said to be *weak graph-preserving* if it satisfies the following: for each $x, y \in X$, if $(x, y) \in E(G)$, then for each $u \in Tx$ there is $v \in P_{Ty}(u)$ such that $(u, v) \in E(G)$, where $P_{Ty}(u) = \{a \in Ty \mid d(u, a) = D(u, Ty)\}$.

Motivated and inspired by all of those works mentioned above, we aim to introduce a new type of multi-valued contractions which is more general than that of Berinde [2].

These classes of mappings are defined for multi-valued mappings in complete metric spaces endowed with directed graphs. Fixed point theorems for this type of mappings are established. We also apply our main results for proving the existence theorem of a fixed point for a multi-valued mapping defined on a partially ordered metric space. Moreover, we also apply the obtained results to prove the existence theorem of a coupled fixed point. Some examples are given to illustrate our results.

2 Preliminaries

Let (X, d) be a metric space and let $CB(X)$ be the class of all nonempty bounded and closed subsets of X . For $x \in X$ and $A \subseteq X$, we denote

$$D(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

Let H be a Pompeiu-Hausdorff metric induced by the metric d on X , that is,

$$H(A, B) = \max\left\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\right\}$$

for every $A, B \in CB(X)$. An element $y_0 \in A$ is said to be a *best approximation* to x if

$$d(x, y_0) = D(x, A).$$

A nonempty subset A of X is a *proximal set* if for each $x \in X$, there exists a point $y \in A$ such that $d(x, y) = D(x, A)$. Let $PB(X)$ be the family of all bounded proximal subsets of X . The set of all best approximations from x to A is denoted by

$$P_A(x) = \{y \in A \mid d(x, y) = D(x, A)\}.$$

This defines a mapping $P_A : X \rightarrow 2^A$ and is called *the metric projection into A* . It is clear that $PB(X)$ is included in $CB(X)$.

The following lemma is useful for our main results.

Lemma 2.1 [13] *Let $A, B \in CB(X)$ and $a \in A$. Then, for $\varepsilon > 0$, there exists an element $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.*

Definition 2.2 Let X be a nonempty set, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and let $T : X \rightarrow PB(X)$ and $g : X \rightarrow X$. Then T is said to be *weak g -graph-preserving* if for any $x, y \in X$ such that $(g(x), g(y)) \in E(G)$, then for each $u \in Tx$ there exists $v \in P_{Ty}(u)$ such that $(u, v) \in E(G)$.

Example 2.3 Let $X = \{0, 1\} \cup \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$ and let d be a usual metric on X . Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = X$ and

$$\begin{aligned} E(G) = & \left\{(1, 0), \left(1, \frac{1}{2}\right), (1, 1)\right\} \cup \left\{\left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) \mid n \in \mathbb{N}\right\} \\ & \cup \left\{\left(\frac{1}{2^n}, 0\right), \left(\frac{1}{2^n}, \frac{1}{2^n}\right) \mid n \in \mathbb{N}\right\}. \end{aligned}$$

Let $T : X \rightarrow PB(X)$ and $g : X \rightarrow X$ be defined by

$$Tx = \begin{cases} \{0, 1\} & \text{if } x = 0; \\ \{\frac{1}{2^{n+1}}, 1\} & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N}; \\ \{\frac{1}{2}, 1\} & \text{if } x = 1, \end{cases}$$

and $g(x) = x^2$.

Now we will show that T is weak g -graph-preserving. Let $(g(x), g(y)) \in E(G)$.

If $(g(x), g(y)) = (1, 0)$, then $(x, y) = (1, 0)$, $Tx = \{\frac{1}{2}, 1\}$, and $Ty = \{0, 1\}$, we have $P_{Ty}(\frac{1}{2}) = \{0, 1\}$, $P_{Ty}(1) = \{1\}$ and $(\frac{1}{2}, 0), (1, 1) \in E(G)$.

If $(g(x), g(y)) = (\frac{1}{2^{2n}}, \frac{1}{2^{2n}})$, then $(x, y) = (\frac{1}{2^n}, \frac{1}{2^n})$ and $Tx = \{\frac{1}{2^{n+1}}, 1\} = Ty$, we have $P_{Ty}(1) = \{1\}$, $P_{Ty}(\frac{1}{2^{n+1}}) = \{\frac{1}{2^{n+1}}\}$ and $(1, 1), (\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}) \in E(G)$.

If $(g(x), g(y)) = (\frac{1}{2^{2n}}, 0)$, then $(x, y) = (\frac{1}{2^n}, 0)$, $Tx = \{\frac{1}{2^{n+1}}, 1\}$, and $Ty = \{0, 1\}$, we have $P_{Ty}(\frac{1}{2^{n+1}}) = \{0\}$, $P_{Ty}(1) = \{1\}$ and $(\frac{1}{2^{n+1}}, 0), (1, 1) \in E(G)$.

If $(g(x), g(y)) = (1, 1)$, then $(x, y) = (1, 1)$ and $Tx = \{\frac{1}{2}, 1\} = Ty$, we have $P_{Ty}(\frac{1}{2}) = \{\frac{1}{2}\}$, $P_{Ty}(1) = \{1\}$ and $(\frac{1}{2}, \frac{1}{2}), (1, 1) \in E(G)$.

Hence T is weak g -graph-preserving.

Definition 2.4 Let X be a nonempty set, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and let $T : X \rightarrow X$ and $g : X \rightarrow X$ be self-mappings on X . Then T is said to be G -edge-preserving with respect to g if for any $x, y \in X$,

$$(g(x), g(y)) \in E(G) \quad \text{implies} \quad (Tx, Ty) \in E(G).$$

3 Main results

We start with introducing a new type of weak G -contraction with respect to g .

Definition 3.1 Let (X, d) be a metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, $g : X \rightarrow X$ and $T : X \rightarrow PB(X)$. T is said to be a (k, L, r) weak G -contraction with respect to g if there exist $L \geq 0$ and $k, r \in [0, 1)$ with $0 < k + r < 1$ such that

$$H(Tx, Ty) \leq kd(g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty)$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$.

Example 3.2 Let $X = \{0, 1\} \cup \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$ and let d be a usual metric on X . Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = X$ and

$$E(G) = \left\{ (0, 0), (1, 0), \left(1, \frac{1}{2}\right) \right\} \cup \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) \mid n \in \mathbb{N} \right\} \\ \cup \left\{ \left(\frac{1}{2^n}, 0\right), \left(\frac{1}{2^n}, \frac{1}{2^n}\right) \mid n \in \mathbb{N} \right\}.$$

Let $T : X \rightarrow PB(X)$ and $g : X \rightarrow X$ be defined by

$$Tx = \begin{cases} \{0\} & \text{if } x = 1; \\ \{\frac{1}{2^{n+1}}, 0\} & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N}; \\ \{\frac{1}{2}\} & \text{if } x = 0, \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{if } x = 0, 1, \frac{1}{2}; \\ \frac{1}{2^{n-1}} & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Now we prove that T is a (k, L, r) weak G -contraction with respect to g , where $k = \frac{1}{2}$, $\frac{2}{7} \leq r < \frac{1}{2}$, and $L \geq 0$. Let $x, y \in X$ such that $(x, y) \in E(G)$.

If $(g(x), g(y)) = (\frac{1}{2^{n-1}}, \frac{1}{2^{n-1}})$, $n \neq 1$. Then $H(Tx, Ty) = 0$, so

$$H(Tx, Ty) \leq kd(g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

If $(g(x), g(y)) = (1, \frac{1}{2})$, then $(x, y) = (1, \frac{1}{4}), (0, \frac{1}{4}), (\frac{1}{2}, \frac{1}{4})$.

Case $(x, y) = (1, \frac{1}{4})$, we have $Tx = \{0\}$ and $Ty = \{\frac{1}{8}, 0\}$. Thus

$$H(Tx, Ty) = \frac{1}{8} \leq \frac{1}{2} (g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

Case $(x, y) = (0, \frac{1}{4})$, we have $Tx = \{\frac{1}{2}\}$ and $Ty = \{\frac{1}{8}, 0\}$. Thus

$$H(Tx, Ty) = \frac{1}{2} \leq \frac{1}{2} (g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

Case $(x, y) = (\frac{1}{2}, \frac{1}{4})$, we have $Tx = \{\frac{1}{4}, 0\}$ and $Ty = \{\frac{1}{8}, 0\}$. Thus

$$H(Tx, Ty) = \frac{1}{8} \leq \frac{1}{2} (g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

If $(g(x), g(y)) = (\frac{1}{2^{n-1}}, \frac{1}{2^n})$, $n \neq 1$, then $(x, y) = (\frac{1}{2^n}, \frac{1}{2^{n+1}})$. We have $Tx = \{\frac{1}{2^{n+1}}, 0\}$ and $Ty = \{\frac{1}{2^{n+2}}, 0\}$. Thus

$$H(Tx, Ty) = \frac{1}{2^{n+2}} = \frac{1}{2} \left(\frac{1}{2^{n+1}} \right) \leq \frac{1}{2} (g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

Hence T is a (k, L, r) weak G -contraction with respect to g .

The following property is useful for our main results.

Property (A) [15] For any sequence $(x_n)_{n \in \mathbb{N}}$ in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

We first prove our main result.

Theorem 3.3 Let (X, d) be a complete metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and let $g : X \rightarrow X$ be a surjective mapping. If $T : X \rightarrow PB(X)$ is a multi-valued mapping satisfying the following properties:

- (i) T is a weak g -graph-preserving mapping;
- (ii) there exists $x_0 \in X$ such that $(g(x_0), y) \in E(G)$ for some $y \in Tx_0$;
- (iii) X has Property (A);
- (iv) T is a (k, L, r) weak G -contraction,

then there exists $u \in X$ such that $g(u) \in Tu$.

Proof Let $x_0 \in X$ be such that $(g(x_0), y) \in E(G)$ for some $y \in Tx_0$. Since g is surjective, there exists $x_1 \in X$ such that $y = g(x_1) \in Tx_0$. By (ii), we have $(g(x_0), g(x_1)) \in E(G)$. Since T is weak g -graph-preserving and g is surjective, there exists $x_2 \in X$ such that $g(x_2) \in P_{Tx_1}(g(x_1))$ and $(g(x_1), g(x_2)) \in E(G)$. By Lemma 2.1, there exists $y_1 \in Tx_1$ such that

$$d(g(x_1), y_1) \leq H(Tx_0, Tx_1) + \frac{1}{2}.$$

It follows that

$$\begin{aligned} d(g(x_1), g(x_2)) &= D(g(x_1), Tx_1) \\ &\leq d(g(x_1), y_1) \\ &\leq H(Tx_0, Tx_1) + \frac{1}{2}. \end{aligned} \tag{3.1}$$

Since $(g(x_0), g(x_1)) \in E(G)$ and T is a (k, L, r) weak G -contraction, we have

$$\begin{aligned} H(Tx_0, Tx_1) &\leq kd(x_0, g(x_1)) + LD(g(x_1), Tx_0) + rD(g(x_0), Tx_1) \\ &= rD(g(x_0), Tx_1) + kd(g(x_0), g(x_1)) \\ &\leq rd(g(x_0), g(x_2)) + kd(g(x_0), g(x_1)) \\ &\leq r[d(g(x_0), g(x_1)) + d(g(x_1), g(x_2))] + kd(g(x_0), g(x_1)) \\ &= (k + r)d(g(x_0), g(x_1)) + rd(g(x_1), g(x_2)) \\ &= (k + r)d(g(x_0), g(x_1)) + rD(g(x_1), Tx_1) \\ &\leq (k + r)d(g(x_0), g(x_1)) + rH(Tx_0, Tx_1), \end{aligned}$$

which implies

$$H(Tx_0, Tx_1) \leq \frac{k + r}{1 - r} d(g(x_0), g(x_1)).$$

From this together with (3.1), we obtain

$$d(g(x_1), g(x_2)) \leq \frac{k + r}{1 - r} d(g(x_0), g(x_1)) + \frac{1}{2}. \tag{3.2}$$

Since T is weak g -graph-preserving and g is surjective, there exists $x_3 \in X$ such that $g(x_3) \in P_{Tx_2}(g(x_2))$ and $(g(x_2), g(x_3)) \in E(G)$. By Lemma 2.1, there exists $y_2 \in Tx_2$ such that

$$d(g(x_2), y_2) \leq H(Tx_1, Tx_2) + \frac{1}{2^2}.$$

It follows that

$$\begin{aligned} d(g(x_2), g(x_3)) &= D(g(x_2), Tx_2) \\ &\leq d(g(x_2), y_2) \\ &\leq H(Tx_1, Tx_2) + \frac{1}{2^2}. \end{aligned} \tag{3.3}$$

Since T is a (k, L, r) weak G -contraction, by (3.3), we get

$$\begin{aligned}
H(Tx_1, Tx_2) &\leq kd(g(x_1), g(x_2)) + LD(g(x_2), Tx_1) + rD(g(x_1), Tx_2) \\
&= rD(g(x_1), Tx_2) + kd(g(x_1), g(x_2)) \\
&\leq rd(g(x_1), g(x_3)) + kd(g(x_1), g(x_2)) \\
&\leq r[d(g(x_1), g(x_2)) + d(g(x_2), g(x_3))] + kd(g(x_1), g(x_2)) \\
&= (r + k)d(g(x_1), g(x_2)) + rd(g(x_2), g(x_3)) \\
&\leq (k + r)d(g(x_1), g(x_2)) + rD(g(x_2), Tx_2) \\
&\leq (k + r)d(g(x_1), g(x_2)) + rH(Tx_1, Tx_2).
\end{aligned}$$

This implies

$$H(Tx_1, Tx_2) \leq \frac{k+r}{1-r} d(g(x_1), g(x_2)).$$

From this together with (3.3), we obtain

$$\begin{aligned}
d(g(x_2), g(x_3)) &\leq \left(\frac{k+r}{1-r}\right) d(g(x_1), g(x_2)) + \frac{1}{2^2} \\
&= \left(\frac{k+r}{1-r}\right) D(g(x_1), T(x_1)) + \frac{1}{2^2} \\
&\leq \left(\frac{k+r}{1-r}\right) H(T(x_0), T(x_1)) + \frac{1}{2^2} \\
&\leq \left(\frac{k+r}{1-r}\right)^2 d(g(x_0), g(x_1)) + \frac{1}{2^2}.
\end{aligned}$$

By induction, we obtain a sequence $\{g(x_n)\}$ in X such that for each $n \in \mathbb{N}$, $g(x_{n+1}) \in Tx_n$, $(g(x_n), g(x_{n+1})) \in E(G)$ and

$$d(g(x_n), g(x_{n+1})) \leq \left(\frac{k+r}{1-r}\right)^n d(g(x_0), g(x_1)) + \frac{1}{2^n}. \quad (3.4)$$

Put $q = \frac{k+r}{1-r}$. For any $n, m \in \mathbb{N}$ with $m > n$, by using (3.4), we have

$$\begin{aligned}
d(g(x_n), g(x_m)) &\leq d(g(x_n), g(x_{n+1})) + d(g(x_{n+1}), g(x_{n+2})) + \cdots + d(g(x_{m-1}), g(x_m)) \\
&\leq q^n d(g(x_0), g(x_1)) + \frac{1}{2^n} + q^{n+1} d(g(x_0), g(x_1)) + \frac{1}{2^{n+1}} + \cdots \\
&\quad + q^{m-1} d(g(x_0), g(x_1)) + \frac{1}{2^{m-1}} \\
&\leq d(g(x_0), g(x_1)) \sum_{i=n}^{m-1} q^i + \sum_{i=n}^{m-1} \frac{1}{2^i} \\
&\leq \frac{q^n}{1-q} d(g(x_0), g(x_1)) + \frac{2}{2^n},
\end{aligned}$$

which, in view of $0 < q < 1$, implies that $\{g(x_n)\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $\lim_{n \rightarrow \infty} g(x_n) = g(u)$. By assumption (iii), there is a subsequence $\{g(x_{n_k})\}$ of $\{g(x_n)\}$ such that $(g(x_{n_k}), g(u)) \in E(G)$ for all $k \in \mathbb{N}$. Since T is a weak- G -contraction, we have

$$\begin{aligned} D(g(u), Tu) &\leq d(g(u), g(x_{n_k+1})) + D(g(x_{n_k+1}), Tu) \\ &\leq d(g(u), g(x_{n_k+1})) + H(Tx_{n_k}, Tu) \\ &\leq d(g(u), g(x_{n_k+1})) + kd(g(x_{n_k}), g(u)) + LD(g(u), Tx_{n_k}) + rD(g(x_{n_k}), Tu) \\ &\leq d(g(u), g(x_{n_k+1})) + kd(g(x_{n_k}), g(u)) + Ld(g(u), g(x_{n_k+1})) + rD(g(x_{n_k}), Tu). \end{aligned}$$

Let $y \in Tu$. From the above inequality, we get

$$\begin{aligned} D(g(u), Tu) &\leq d(g(u), g(x_{n_k+1})) + kd(g(x_{n_k}), g(u)) \\ &\quad + Ld(g(u), g(x_{n_k+1})) + rd(g(x_{n_k}), y). \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain

$$D(g(u), Tu) \leq rd(g(u), y).$$

Since y is arbitrary, it follows that $D(g(u), Tu) \leq rD(g(u), Tu)$, which implies $D(g(u), Tu) = 0$. As Tu is closed, $g(u) \in Tu$. This completes the proof. $\square$

The following result is obtained directly from Theorem 3.3 in case that T is a single-valued mapping.

Theorem 3.4 *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and let $g : X \rightarrow X$ be a surjective mapping. Let $T : X \rightarrow X$ be a single-valued mapping satisfying the following properties:*

- (i) *T is a G -edge-preserving mapping with respect to g ;*
- (ii) *there exists $x_0 \in X$ such that $(g(x_0), Tx_0) \in E(G)$;*
- (iii) *X has Property (A);*
- (iv) *there exist $L \geq 0$ and $k, r \in [0, 1]$ with $0 < k + r < 1$ such that*

$$d(Tx, Ty) \leq kd(g(x), g(y)) + Ld(g(y), Tx) + rd(g(x), Ty)$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$, then there exists $u \in X$ such that $g(u) \in Tu$.

Example 3.5 Let $X = \{0, 1\} \cup \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$ and let d be a usual metric on X . Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = X$ and

$$\begin{aligned} E(G) &= \{(0, 0), (1, 0)\} \cup \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{n+1}} \right) \mid n \in \mathbb{N} \right\} \\ &\quad \cup \left\{ \left(\frac{1}{2^n}, 0 \right), \left(\frac{1}{2^n}, \frac{1}{2^n} \right), \left(1, \frac{1}{2^n} \right) \mid n \in \mathbb{N} \right\}. \end{aligned}$$

Let $T : X \rightarrow PB(X)$ and $g : X \rightarrow X$ be defined by

$$Tx = \begin{cases} \{1\} & \text{if } x = 0; \\ \{\frac{1}{2^{n+1}}, 0\} & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N}; \\ \{0\} & \text{if } x = 1, \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{if } x = 0; \\ \frac{1}{2^n} & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N}; \\ 0 & \text{if } x = 1. \end{cases}$$

Now we will show that T is weak g -graph-preserving. Let $(g(x), g(y)) \in E(G)$.

If $(g(x), g(y)) = (0, 0)$, then $(x, y) = (1, 1)$ and so $Tx = \{0\} = Ty$. We have $P_{Ty}(0) = \{0\}$ and $(0, 0) \in E(G)$.

If $(g(x), g(y)) = (1, 0)$, then $(x, y) = (0, 1)$ and so $Tx = \{1\}$, $Ty = \{0\}$. We have $P_{Ty}(1) = \{0\}$ and $(1, 0) \in E(G)$.

If $(g(x), g(y)) = (1, \frac{1}{2^n})$, then $(x, y) = (0, \frac{1}{2^n})$ and so $Tx = \{1\}$, $Ty = \{\frac{1}{2^{n+1}}, 0\}$. We have $P_{Ty}(1) = \{\frac{1}{2^{n+1}}\}$ and $(1, \frac{1}{2^{n+1}}) \in E(G)$.

If $(g(x), g(y)) = (\frac{1}{2^n}, \frac{1}{2^n})$, then $(x, y) = (\frac{1}{2^n}, \frac{1}{2^n})$ and so $Tx = \{\frac{1}{2^{n+1}}, 0\} = Ty$. We have $P_{Ty}(0) = \{0\}$, $P_{Ty}(\frac{1}{2^{n+1}}) = \{\frac{1}{2^{n+1}}\}$ and $(0, 0), (\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}) \in E(G)$.

If $(g(x), g(y)) = (\frac{1}{2^n}, 0)$, then $(x, y) = (\frac{1}{2^n}, 1)$ and so $Tx = \{\frac{1}{2^{n+1}}, 0\}$, $Ty = \{0\}$. We have $P_{Ty}(0) = \{0\} = P_{Ty}(\frac{1}{2^{n+1}})$ and $(0, 0), (\frac{1}{2^{n+1}}, 0) \in E(G)$.

If $(g(x), g(y)) = (\frac{1}{2^n}, \frac{1}{2^{n+1}})$, then $(x, y) = (\frac{1}{2^n}, \frac{1}{2^{n+1}})$ and so $Tx = \{\frac{1}{2^{n+1}}, 0\}$, $Ty = \{\frac{1}{2^{n+2}}, 0\}$. We have $P_{Ty}(0) = \{0\}$, $P_{Ty}(\frac{1}{2^{n+1}}) = \{\frac{1}{2^{n+1}}\}$ and $(0, 0), (\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}) \in E(G)$.

Hence T is weak g -graph-preserving.

Next we prove that T is a (k, L, r) weak G -contraction with respect to g , where $k = \frac{1}{2}$, $r = \frac{1}{4}$, and $L \geq 1$. Let $x, y \in X$ such that $(g(x), g(y)) \in E(G)$.

If $(g(x), g(y)) = (0, 0)$ and $(g(x), g(y)) = (\frac{1}{2^n}, \frac{1}{2^n})$, then $H(Tx, Ty) = 0$, so

$$H(Tx, Ty) \leq kd(g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

If $(g(x), g(y)) = (1, 0)$, then $(x, y) = (0, 1)$ and so $Tx = \{1\}$, $Ty = \{0\}$. Thus

$$H(Tx, Ty) = 1 \leq \frac{1}{2}d(g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

If $(g(x), g(y)) = (1, \frac{1}{2^n})$, then $(x, y) = (0, \frac{1}{2^n})$ and so $Tx = \{1\}$, $Ty = \{\frac{1}{2^{n+1}}, 0\}$. Thus

$$H(Tx, Ty) = 1 - \frac{1}{2^{n+1}} \leq \frac{1}{2}d(g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

If $(g(x), g(y)) = (\frac{1}{2^n}, 0)$, then $(x, y) = (\frac{1}{2^n}, 1)$ and so $Tx = \{\frac{1}{2^{n+1}}, 0\}$, $Ty = \{0\}$. Thus

$$H(Tx, Ty) = \frac{1}{2^{n+1}} = \frac{1}{2} \left(\frac{1}{2^n} \right) \leq \frac{1}{2}d(g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

If $(g(x), g(y)) = (\frac{1}{2^n}, \frac{1}{2^{n+1}})$, then $(x, y) = (\frac{1}{2^n}, \frac{1}{2^{n+1}})$ and so $Tx = \{\frac{1}{2^{n+1}}, 0\}$, $Ty = \{\frac{1}{2^{n+2}}, 0\}$. Thus

$$H(Tx, Ty) = \frac{1}{2^{n+2}} \leq \frac{1}{2} d(g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty).$$

Hence T is a (k, L, r) weak G -contraction with respect to g . So conditions (i) and (iv) of Theorem 3.3 are satisfied. Also, it is easy to check that condition (iii) of Theorem 3.3 is satisfied. Then all the hypotheses in Theorem 3.3 are satisfied, and we see that 0 and 1 are coincidence points of T and g .

We next apply Theorem 3.3 to obtain fixed point theorems for some contraction mappings in partially ordered metric spaces.

Definition 3.6 Let (X, d) be a metric space endowed with a partial ordering $\preceq$. For each $A, B \in PB(X)$, we write $A \prec_1 B$ if for each $a \in A$, there exists $b \in P_B(a)$ such that $a \prec b$.

Definition 3.7 Let (X, d) be a metric space endowed with a partial ordering $\preceq$, $g : X \rightarrow X$ be a surjective map and $T : X \rightarrow PB(X)$. Then T is said to be g -increasing if for any $x, y \in X$,

$$g(x) \prec g(y) \Rightarrow Tx \prec_1 Ty.$$

If g is the identity map and $T : X \rightarrow PB(X)$ is g -increasing, we simply say that T is increasing.

Example 3.8 Let $X = [1, \infty)$ and let $\leq$ be a usual partial ordering on X , that is, $x \leq y$ if and only if $y - x \geq 0$. Suppose that d is a usual metric on X . Let $T : X \rightarrow PB(X)$ and $g : X \rightarrow X$ be defined by

$$Tx = \begin{cases} [1, 4] & \text{if } x = 1; \\ \{2 + \frac{5}{2}x\} & \text{otherwise,} \end{cases}$$

and $g(x) = x^2$. It can be shown easily that T is g -increasing.

Moreover, we take $g(x) = x$, then it is easy to see that T is increasing.

Corollary 3.9 Let (X, d) be a complete metric space endowed with a partial ordering $\preceq$, $g : X \rightarrow X$ be a surjective and $T : X \rightarrow PB(X)$ be a multi-valued mapping. Suppose that

- (1) T is g -increasing;
- (2) there exist $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) \prec u$;
- (3) for each sequence $\{x_n\}$ such that $g(x_n) \prec g(x_{n+1})$ for all $n \in \mathbb{N}$ and $g(x_n)$ converges to $g(x)$, for some $x \in X$, then $g(x_n) \prec g(x)$ for all $n \in \mathbb{N}$;
- (4) there exist $L \geq 0$ and $k, r \in [0, 1)$ with $0 < k + r < 1$ such that

$$H(Tx, Ty) \leq kd(g(x), g(y)) + LD(g(y), Tx) + rD(g(x), Ty)$$

for all $x, y \in X$ with $g(x) \prec g(y)$.

Then there exists $u \in X$ such that $g(u) \in Tu$.

Proof Define $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, y) : x < y\}$. Let $x, y \in X$ be such that $(g(x), g(y)) \in E(G)$. Then $g(x) < g(y)$. Since T is g -increasing, we have $Tx <_1 Ty$. Then, for any $u \in Tx$, there exists $v \in P_{Ty}(u)$ such that $u < v$ and so $(u, v) \in E(G)$. Thus T is weak g -graph-preserving. By (2), there exist $x_0 \in X$ and $u \in Tx_0$ such that $g(x_0) < u$, and so $(g(x_0), u) \in E(G)$. So assumption (ii) of Theorem 3.3 is satisfied. It is easy to see that (iii) and (iv) of Theorem 3.3 are also satisfied. Therefore, this corollary is obtained directly by Theorem 3.3. $\square$

If g is the identity map, then the following result is directly obtained by Corollary 3.9.

Corollary 3.10 *Let (X, d) be a complete metric space endowed with a partial ordering $\preceq$. Let $T : X \rightarrow PB(X)$ be a multi-valued mapping. Suppose that*

- (1) *T is increasing;*
- (2) *there exist $x_0 \in X$ and $u \in Tx_0$ such that $x_0 < u$;*
- (3) *for each sequence $\{x_n\}$ such that $x_n < x_{n+1}$ for all $n \in \mathbb{N}$ and x_n converges to x , for some $x \in X$, then $x_n < x$ for all $n \in \mathbb{N}$;*
- (4) *there exist $L \geq 0$ and $k, r \in [0, \infty)$ with $0 < k + r < 1$ such that*

$$H(Tx, Ty) \leq kd(x, y) + LD(y, Tx) + rD(x, Ty)$$

for all $x, y \in X$ with $x < y$.

Then there exists $u \in X$ such that $u \in Tu$.

4 Applications

In this section, we prove the existence of a coupled fixed point for a single-valued mapping in a complete metric space endowed with a directed graph.

Let X be a nonempty set and $F : X \times X \rightarrow X$ be a single-valued mapping. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of F if $x = F(x, y)$ and $y = F(y, x)$. We denote by $C\text{Fix}(F)$ the set of all coupled fixed points of the mapping F , i.e., $C\text{Fix}(F) = \{(x, y) \in X \times X \mid F(x, y) = x \text{ and } F(y, x) = y\}$.

Coupled fixed point theorems and their application were investigated by many authors (see [12, 19, 23–27] for examples).

Definition 4.1 [12] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \rightarrow X$ be a given mapping. The mapping F is said to have the mixed monotone property if it is monotone nondecreasing in x and monotone nonincreasing in y , that is,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \preceq x_2 &\Rightarrow F(x_1, y) \preceq F(x_2, y), \\ y_1, y_2 \in X, \quad y_1 \preceq y_2 &\Rightarrow F(x, y_1) \succeq F(x, y_2). \end{aligned}$$

Recently, Chifu and Petrusel [28] introduced the concept of edge-preserving as follows.

Definition 4.2 [28] We say that $F : X \times X \rightarrow X$ is *edge-preserving* if $(x, u) \in E(G)$, $(y, v) \in E(G^{-1})$ implies $(F(x, y), F(u, v)) \in E(G)$ and $(F(y, x), F(v, u)) \in E(G^{-1})$.

Let (X, d) be a metric space and let $Y = X \times X$. Then the mapping $\eta : Y \times Y \rightarrow [0, \infty)$ defined by

$$\eta((x, y), (u, v)) = d(x, u) + d(y, v) \quad \text{for all } (x, y), (u, v) \in Y$$

is a metric on Y , and it is easy to see that (X, d) is complete if and only if (Y, η) is complete.

For a mapping $F : X \times X \rightarrow X$, we define the mapping $T_F : Y \rightarrow Y$ by

$$T_F(x, y) = (F(x, y), F(y, x)) \quad \text{for all } (x, y) \in Y.$$

Note that an element $(x, y) \in X \times X$ is a coupled fixed point of F if and only if (x, y) is a fixed point of T_F .

Theorem 4.3 *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Let the mapping $F : X \times X \rightarrow X$ be edge-preserving and suppose that the following properties hold:*

- (i) *there exist $x_0, y_0 \in X$ such that $(x_0, F(x_0, y_0)) \in E(G)$ and $(y_0, F(y_0, x_0)) \in E(G^{-1})$;*
- (ii) *X has the following property:*
 - (a) *if any sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;*
 - (b) *if any sequence $\{y_n\}$ in X such that $y_n \rightarrow y$ and $(y_n, y_{n+1}) \in E(G^{-1})$ for $n \in \mathbb{N}$, then $(y_n, y) \in E(G^{-1})$ for all $n \in \mathbb{N}$;*
- (iii) *there exist $L \geq 0$ and $k, r \in [0, 1)$ with $0 < k + r < 1$ such that*

$$\begin{aligned} & d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ & \leq k[d(x, u) + d(y, v)] + L[d(u, F(x, y)) + d(v, F(y, x))] \\ & \quad + r[d(x, F(u, v)) + d(y, F(v, u))] \end{aligned}$$

for all $x, y, u, v \in X$ with $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$.

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof Let $Y = X \times X$ and η be a metric on Y given by

$$\eta((x, y), (u, v)) = d(x, u) + d(y, v) \quad \text{for all } (x, y), (u, v) \in Y.$$

Then (Y, η) is a complete metric space. Let G_Y be a directed graph defined by $G_Y = (V(G_Y), E(G_Y))$, where $V(G_Y) = Y$ and

$$E(G_Y) = \{((x, y), (u, v)) \mid (x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1})\}.$$

We shall show that T_F satisfies all conditions of Theorem 3.4. Let $(x, y), (u, v) \in Y$ such that $((x, y), (u, v)) \in E(G_Y)$. Then $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$. Since F is edge-preserving, we have $(F(x, y), F(u, v)) \in E(G)$ and $(F(y, x), F(v, u)) \in E(G^{-1})$. Hence $((F(x, y), F(y, x)), (F(u, v), F(v, u))) \in E(G_Y)$. Thus $(T_F(x, y), T_F(u, v)) \in E(G_Y)$. So T_F satisfies assumption (i) of Theorem 3.4. By (2), there exists $(x_0, y_0) \in Y$ such that $(x_0, F(x_0, y_0)) \in E(G)$ and $(y_0, F(y_0, x_0)) \in E(G^{-1})$.

$E(G^{-1})$, so $((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \in E(G_Y)$. Thus $((x_0, y_0), T_F(x_0, y_0)) \in E(G_Y)$, so assumption (ii) of Theorem 3.4 is satisfied by T_F . Next, we will show that T_F satisfies assumption (iv) of Theorem 3.4. By condition (iii), we have

$$\begin{aligned}
 \eta(T_F(x, y), T_F(u, v)) &= \eta((F(x, y), F(y, x)), (F(u, v), F(v, u))) \\
 &= d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\
 &\leq k[d(x, u) + d(y, v)] + L[d(u, F(x, y)) + d(v, F(y, x))] \\
 &\quad + r[d(x, F(u, v)) + d(y, F(v, u))] \\
 &= k\eta((x, y), (u, v)) + L\eta((u, v), (F(x, y), F(y, x))) \\
 &\quad + r\eta((x, y), (F(u, v), F(v, u))) \\
 &= k\eta((x, y), (u, v)) + L\eta((u, v), T_F(x, y)) \\
 &\quad + r\eta((x, y), T_F(u, v))
 \end{aligned}$$

for all $(x, y) \in E(G)$ and $(y, v) \in E(G^{-1})$. So assumption (iv) of Theorem 3.4 is satisfied. Finally, we will show that Y has Property (A). For any sequence (x_n, y_n) in Y , it is easy to see that if $(x_n, y_n) \rightarrow (x, y)$ and $((x_n, y_n), (x_{n+1}, y_{n+1})) \in E(G_Y)$ for $n \in \mathbb{N}$, then $x_n \rightarrow x$, $y_n \rightarrow y$, $(x_n, x_{n+1}) \in E(G)$, and $(y_n, y_{n+1}) \in E(G^{-1})$. By (ii), we have $(x_n, x) \in E(G)$ and $(y_n, y) \in E(G^{-1})$. So $((x_n, y_n), (x, y)) \in E(G_Y)$. Thus all conditions of Theorem 3.4 are satisfied. Hence there exists $(x, y) \in Y$ such that $(x, y) = T_F(x, y)$, which implies that (x, y) is a coupled fixed point of F . $\square$

The following corollary is a consequence of Theorem 4.3.

Corollary 4.4 *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X and there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Assume that there exist $L \geq 0$ and $k, r \in [0, 1)$ with $0 < k + r < 1$ such that*

$$\begin{aligned}
 &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\
 &\leq k[d(x, u) + d(y, v)] + L[d(u, F(x, y)) + d(v, F(y, x))] \\
 &\quad + r[d(x, F(u, v)) + d(y, F(v, u))]
 \end{aligned}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$. Suppose that X has the following property:

- (a) *if a nondecreasing sequence $\{x_n\} \rightarrow x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*
- (b) *if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.*

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in $X \times X$.

Proof Let $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, y) \mid x \preceq y\}$. We can directly check that all conditions of Theorem 4.3 are satisfied. Therefore F has a coupled fixed point. $\square$

References

1. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **3**, 133-181 (1922)
2. Berinde, V: Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Anal. Forum* **9**(1), 43-53 (2004)
3. Bojor, F: Fixed point theorems for Reich type contraction on metric spaces with a graph. *Nonlinear Anal.* **75**, 3895-3901 (2012)
4. Branciari, A: A fixed point theorem for mappings satisfying a general contractive condition of integral type. *J. Math. Anal. Appl.* **29**, 531-536 (2002)
5. Chatterjea, SK: Fixed point theorems. *C. R. Acad. Bulgare Sci.* **25**, 727-730 (1972)
6. Kannan, R: Some results on fixed points. *Bull. Calcutta Math. Soc.* **10**, 71-76 (1968)
7. Nieto, JJ, Rodríguez-Lopez, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation. *Order* **22**, 223-239 (2005)
8. Nieto, JJ, Rodríguez-Lopez, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation. *Acta Math. Sin. Engl. Ser.* **23**(12), 2205-2212 (2007)
9. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **132**, 1435-1443 (2004)
10. Zamfirescu, T: Fixed point theorems in metric spaces. *Arch. Math. (Basel)* **23**, 292-298 (1972)
11. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. *Appl. Anal.* **87**, 1-8 (2008)
12. Gnana Bhaskar, T, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal. TMA* **65**, 1379-1393 (2006)
13. Nadler, S: Multi-valued contraction mappings. *Pac. J. Math.* **20**(2), 475-488 (1969)
14. Berinde, M, Berinde, V: On a general class multi-valued weakly Picard mappings. *J. Math. Anal. Appl.* **326**, 772-782 (2007)
15. Jachymski, J: The contraction principle for mappings on a metric with a graph. *Proc. Am. Math. Soc.* **139**, 1359-1373 (2008)
16. Beg, I, Butt, AR: Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces. *Nonlinear Anal.* **71**, 3699-3704 (2009)
17. Alfuraidan, MR: Fixed points of multivalued mappings in modular function spaces with a graph. *Fixed Point Theory Appl.* **2015**, Article ID 42 (2015). doi:10.1186/s13663-015-0292-7
18. Alfuraidan, MR: Remarks on monotone multivalued mappings on a metric space with a graph. *J. Inequal. Appl.* (2015). doi:10.1186/s13660-015-0712-6
19. Nicolae, A, O'Regan, D, Petrusel, A: Fixed point theorems for single-valued and multi-valued generalized contractions in metric spaces endowed with a graph. *Georgian Math. J.* **18**, 307-327 (2011)
20. Dinevari, T, Frigon, T: Fixed point results for multi-valued contractions on a metric space with a graph. *J. Math. Anal. Appl.* **405**, 507-517 (2013)
21. Tiammee, J, Suantai, S: Coincidence point theorems for graph-preserving multi-valued mappings. *Fixed Point Theory Appl.* (2014). doi:10.1186/1687-1812-2014-70
22. Phon-on, A, Sama-Ae, A, Makaje, N, Riyapan, P, Busaman, S: Coincidence point theorems for weak graph preserving multi-valued mapping. *Fixed Point Theory Appl.* (2014). doi:10.1186/1687-1812-2014-248
23. Agarwal, RP, Karapinar, E: Remarks on some coupled fixed point theorems in G -metric spaces. *Fixed Point Theory Appl.* **2013**, Article ID 2 (2013)
24. Mursaleen, M, Mohiuddine, SA, Agarwal, RP: Coupled fixed point theorems for α - ψ -contractive type mappings in partially ordered metric spaces. *Fixed Point Theory Appl.* **2012**, Article ID 228 (2012)
25. Roldán, A, Martínez-Moreno, J, Roldán, C, Karapinar, E: Some remarks on multidimensional fixed point theorems. *Fixed Point Theory* **15**(2), 545-558 (2014)
26. Urs, C: Coupled fixed point theorems and applications to periodic boundary value problems. *Miskolc Math. Notes* **14**(1), 323-333 (2013)
27. Urs, C: Ulam-Hyers stability for coupled fixed points of contractive type operators. *J. Nonlinear Sci. Appl.* **6**, 124-136 (2013)
28. Chifu, C, Petrusel, G: New results on coupled fixed point theory in metric spaces endowed with a directed graph. *Fixed Point Theory Appl.* **2014**, Article ID 151 (2014)

3.2 Common coupled fixed point theorems for θ - ψ -contraction mappings endowed with a directed graph

In this **section**, we present some existence and uniqueness results for coupled coincidence point and common fixed point of θ - ψ -contraction mappings in complete metric spaces endowed with a directed graph. Our results generalize the results obtained by Kadelburg *et al.* (Fixed Point Theory Appl. 2015:27, 2015, doi:10.1007/s11590-013-0708-4). We also have an application to some integral system to support the results.

1 Introduction and preliminaries

For $F : X \times X \rightarrow X$ and $g : X \rightarrow X$, a concept of coupled coincidence point $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$ was first introduced by Lakshmikantham and Ćirić [2]. Their results extended the result in [4, 3]. Also, the existence and uniqueness of a coupled coincidence point for such a mapping that satisfies the mixed monotone property in a partially ordered metric space were studied. Consequently, a number of coupled fixed point and coupled coincidence point results have been shown recently. For example, see [71–5].

Choudhury and Kundu [7] give a notion of compatibility.

Definition 1.1 ([7]) Let (X, d) be a metric space, and let $g : X \rightarrow X$ and $F : X \times X \rightarrow X$. The mappings g and F are said to be *compatible* if

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} gx_n$ and $\lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} gy_n$.

Let Θ denote the class of all functions $\theta : [0, \infty) \times [0, \infty) \rightarrow [0, 1)$ that satisfy the following conditions:

$$(\theta_1) \quad \theta(s, t) = \theta(t, s) \text{ for all } s, t \in [0, \infty);$$

(θ_2) for any two sequences $\{s_n\}$ and $\{t_n\}$ of nonnegative real numbers,

$$\theta(s_n, t_n) \rightarrow 1 \quad \Rightarrow \quad s_n, t_n \rightarrow 0.$$

In 2015, Kadelburg *et al.* [1] used the monotone and g -monotone properties to obtained common coupled fixed point theorems for Geraghty-type contraction with compatibility of F and g .

Let (X, d) be a metric space, Δ be a diagonal of $X \times X$, and G be a directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. That is, G is determined by $(V(G), E(G))$. We will use this notation of G throughout this work.

The fixed point theorem using the context of metric spaces endowed with a graph was first studied by Jachymski [18]. The result generalized the Banach contraction principle to mappings on metric spaces with a graph. Since then, many authors studied the problem of existence of fixed points for single-valued mappings and multivalued mappings in several spaces with graphs; see [19–23].

Recently, Chifu and Petrusel [24] give the concept of G -continuity for a mapping $F : X^2 \rightarrow X$ and the property A as follows.

Definition 1.2 Let (X, d) be a complete metric space, G be a directed graph, and $F : X^2 \rightarrow X$ be a mapping. Then

- (i) F is called G -continuous if for all $(x^*, y^*) \in X^2$ and for any sequence $(n_i)_i \in \mathbb{N}$ of positive integers such that $F(x_{n_i}, y_{n_i}) \rightarrow x^*$, $F(y_{n_i}, x_{n_i}) \rightarrow y^*$ as $i \rightarrow \infty$ and $(F(x_{n_i}, y_{n_i}), F(x_{n_i+1}, y_{n_i+1})), (F(y_{n_i}, x_{n_i}), F(y_{n_i+1}, x_{n_i+1})) \in E(G)$, we have that

$$\begin{aligned} F(F(x_{n_i}, y_{n_i}), F(y_{n_i}, x_{n_i})) &\rightarrow F(x^*, y^*) \quad \text{and} \\ F(F(y_{n_i}, x_{n_i}), F(x_{n_i}, y_{n_i})) &\rightarrow F(y^*, x^*) \quad \text{as } i \rightarrow \infty; \end{aligned}$$

- (ii) (X, d, G) has property A if for any sequence $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \rightarrow x$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$.

Their results generalized the result in [4] by using the context of metric spaces endowed with a directed graph.

The aim of this work is to prove some existence and uniqueness results for a coupled coincidence point and a common fixed point of θ - ψ contraction mappings in complete metric spaces endowed with a directed graph. The results generalize the results obtained by Kadelburg *et al.* [1]. An application to some integral system is provided to support the results.

2 Common coupled fixed point

We define the set $\text{CcFix}(F)$ of all coupled coincidence points of mappings $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ and the set $(X^2)_g^F$ as follows:

$$\text{CcFix}(F) = \{(x, y) \in X^2 : F(x, y) = gx \text{ and } F(y, x) = gy\}$$

and

$$(X^2)_g^F = \{(x, y) \in X^2 : (gx, F(x, y)), (gy, F(y, x)) \in E(G)\}.$$

Now, we give some definitions that are useful for our main results.

Definition 2.1 We say that $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ are G -edge preserving if

$$[(gx, gu), (gy, gv) \in E(G)] \Rightarrow [(F(x, y), F(u, v)), (F(y, x), F(v, u)) \in E(G)].$$

Definition 2.2 Let (X, d) be a complete metric space, and $E(G)$ be the set of the edges of the graph. We say that $E(G)$ satisfies the transitivity property if and only if, for all $x, y, a \in X$,

$$(x, a), (a, y) \in E(G) \rightarrow (x, y) \in E(G).$$

Let Ψ denote the class of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ that satisfy the following conditions:

- (ψ_1) ψ is nondecreasing;
- (ψ_2) $\psi(s + t) \leq \psi(s) + \psi(t)$;
- (ψ_3) ψ is continuous;
- (ψ_4) $\psi(t) = 0 \Leftrightarrow t = 0$.

Definition 2.3 Let (X, d) be a complete metric space endowed with a directed graph G . The mappings $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ are called a θ - ψ -contraction if:

- (1) F and g is G -edge preserving;
- (2) there exist $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y, u, v \in X$ satisfying $(gx, gu), (gy, gv) \in E(G)$,

$$\psi(d(F(x, y), F(u, v))) \leq \theta(d(gx, gu), d(gy, gv))\psi(M(gx, gu, gy, gv)), \quad (1)$$

where $M(gx, gu, gy, gv) = \max\{d(gx, gu), d(gy, gv)\}$.

Lemma 2.4 Let (X, d) be a complete metric space endowed with a directed graph G , and let $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ be a θ - ψ -contraction. Assume that there exist $x_0, y_0, a_0, b_0 \in X$ and $F(X \times X) \subset g(X)$. Then:

- (i) There exists sequences $\{x_n\}, \{y_n\}, \{a_n\}, \{b_n\}$ in X for which

$$\begin{aligned} gx_n &= F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}), \\ ga_n &= F(a_{n-1}, b_{n-1}) \quad \text{and} \quad gb_n = F(b_{n-1}, a_{n-1}) \quad \text{for } n = 1, 2, \dots \end{aligned} \quad (2)$$

- (ii) If (gx_n, ga_n) and $(gy_n, gb_n) \in E(G)$ for all $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} M(gx_n, ga_n, gy_n, gb_n) = 0.$$

Proof (i) Let $x_0, y_0, a_0, b_0 \in X$. By the assumption that $F(X \times X) \subset g(X)$ and $F(x_0, y_0), F(y_0, x_0), F(a_0, b_0), F(b_0, a_0) \in X$, it easy to construct sequences $\{x_n\}, \{y_n\}, \{a_n\}$, and $\{b_n\}$ in X for which

$$\begin{aligned} gx_n &= F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}), \\ ga_n &= F(a_{n-1}, b_{n-1}) \quad \text{and} \quad gb_n = F(b_{n-1}, a_{n-1}) \quad \text{for } n = 1, 2, \dots \end{aligned}$$

(ii) Let (gx_n, ga_n) and $(gy_n, gb_n) \in E(G)$ for all $n \in \mathbb{N}$. Using the θ - ψ -contraction (1) and (2), we obtain that

$$\begin{aligned}\psi(d(gx_{n+1}, ga_{n+1})) &= \psi(d(F(x_n, y_n), F(a_n, b_n))) \\ &\leq \theta(d(gx_n, ga_n), d(gy_n, gb_n))\psi(M(gx_n, ga_n, gy_n, gb_n))\end{aligned}\quad (3)$$

and

$$\begin{aligned}\psi(d(gy_{n+1}, gb_{n+1})) &= \psi(d(F(y_n, x_n), F(b_n, a_n))) \\ &\leq \theta(d(gy_n, gb_n), d(gx_n, ga_n))\psi(M(gy_n, gb_n, gx_n, ga_n)) \\ &= \theta(d(gx_n, ga_n), d(gy_n, gb_n))\psi(M(gx_n, ga_n, gy_n, gb_n))\end{aligned}\quad (4)$$

for all $n \in \mathbb{N}$. From (3) and (4) we get

$$\begin{aligned}\psi(M(gx_{n+1}, ga_{n+1}, gy_{n+1}, gb_{n+1})) &= \psi(\max\{d(gx_{n+1}, ga_{n+1}), d(gy_{n+1}, gb_{n+1})\}) \\ &\leq \theta(d(gx_n, ga_n), d(gy_n, gb_n))\psi(M(gx_n, ga_n, gy_n, gb_n)) \\ &< \psi(M(gx_n, ga_n, gy_n, gb_n))\end{aligned}\quad (5)$$

for all $n \in \mathbb{N}$, that is,

$$\psi(M(gx_{n+1}, ga_{n+1}, gy_{n+1}, gb_{n+1})) < \psi(M(gx_n, ga_n, gy_n, gb_n)).$$

Regarding the properties of ψ , we conclude that

$$M(gx_{n+1}, ga_{n+1}, gy_{n+1}, gb_{n+1}) < M(gx_n, ga_n, gy_n, gb_n).$$

It follows that $d_n := M(gx_n, ga_n, gy_n, gb_n)$ is decreasing. Then $d_n \rightarrow d$ as $n \rightarrow \infty$ for some $d \geq 0$. We claim that $d = 0$. Suppose not. Using (5), we have

$$\frac{\psi(M(gx_{n+1}, ga_{n+1}, gy_{n+1}, gb_{n+1}))}{\psi(M(gx_n, ga_n, gy_n, gb_n))} \leq \theta(d(gx_n, ga_n), d(gy_n, gb_n)) < 1.$$

Taking the limit as $n \rightarrow \infty$, we have

$$\theta(d(gx_n, ga_n), d(gy_n, gb_n)) \rightarrow 1.$$

Since $\theta \in \Theta$,

$$d(gx_n, ga_n) \rightarrow 0 \quad \text{and} \quad d(gy_n, gb_n) \rightarrow 0$$

as $n \rightarrow \infty$. Therefore,

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} M(gx_n, ga_n, gy_n, gb_n) = 0,$$

which is a contradiction. Hence,

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} M(gx_n, ga_n, gy_n, gb_n) = 0$$

□

Next, we will prove the main result.

Theorem 2.5 *Let (X, d) be a complete metric space endowed with a directed graph G , and let $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ be a θ - ψ -contraction. Suppose that:*

- (i) *g is continuous, and $g(X)$ is closed;*
- (ii) *$F(X \times X) \subset g(X)$, and g and F are compatible;*
- (iii) *F is G -continuous, or the tripled (X, d, G) has property A ;*
- (iv) *$E(G)$ satisfies the transitivity property.*

Under these conditions, $\text{CcFix}(F) \neq \emptyset$ if and only if $(X^2)_g^F \neq \emptyset$.

Proof Let $\text{CcFix}(F) \neq \emptyset$. Then there exists $(u, v) \in \text{CcFix}(F)$ such that $(gu, F(u, v)) = (gu, gu)$ and $(gv, F(v, u)) = (gv, gv) \in \Delta \subset E(G)$. Thus, $(gu, F(u, v))$ and $(gv, F(v, u)) \in E(G)$. It follows that $(u, v) \in (X^2)_g^F$, so that $(X^2)_g^F \neq \emptyset$.

Now, suppose that $(X^2)_g^F \neq \emptyset$. Let $x_0, y_0 \in X$ be such that $(x_0, y_0) \in (X^2)_g^F$. Then $(gx_0, F(x_0, y_0))$ and $(gy_0, F(y_0, x_0)) \in E(G)$. From Lemma 2.4(i) we have sequences $\{x_n\}$ and $\{y_n\}$ in X for which

$$gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}) \quad \text{for } n = 1, 2, \dots$$

Since $(gx_0, F(x_0, y_0)) = (gx_0, gx_1)$ and $(gy_0, F(y_0, x_0)) = (gy_0, gy_1) \in E(G)$ and F and g are G -edge preserving, we have $(F(x_0, y_0), F(x_1, y_1)) = (gx_1, gx_2)$ and $(F(y_0, x_0), F(y_1, x_1)) = (gy_1, gy_2) \in E(G)$. By induction we shall obtain (gx_{n-1}, gx_n) and $(gy_{n-1}, gy_n) \in E(G)$ for each $n \in \mathbb{N}$. By Lemma 2.4(ii) we have

$$d_n := M(gx_{n-1}, gx_n, gy_{n-1}, gy_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (6)$$

Now, we shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Applying a similar argument as in the proof of Theorem 3.1 in [1] and using (6), condition (iv), and property of ψ , it follows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. By condition (i) there exist $u, v \in g(X)$ such that

$$\lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} F(x_n, y_n) = u \quad \text{and} \quad \lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} F(y_n, x_n) = v.$$

By the compatibility of g and F we have that

$$\lim_{n \rightarrow \infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0. \quad (7)$$

Now, suppose that (a) F is G -continuous. It is easy to see that

$$d(gu, F(gx_n, gy_n)) \leq d(gu, gF(x_n, y_n)) + d(gF(x_n, y_n), F(gx_n, gy_n)).$$

Taking the limit as $n \rightarrow \infty$ and using (7), the continuity of g , and G -continuity of F , we have that $d(gu, F(u, v)) = 0$, that is, $gu = F(u, v)$. Using a similar idea, we also have that $gv = F(v, u)$. Therefore, $\text{CcFix}(F) \neq \emptyset$.

Suppose now that (b) the tripled (X, d, G) with property A. Let $gx = u$ and $gy = v$ for some $x, y \in X$. Then we have (gx_n, gx) and $(gy_n, gy) \in E(G)$ for each $n \in \mathbb{N}$. By (1) we have

$$\begin{aligned}
& \psi(d(gx, F(x, y)) + d(gy, F(y, x))) \\
& \leq \psi(d(gx, gx_{n+1}) + d(gx_{n+1}, F(x, y)) + d(gy, gy_{n+1}) + d(gy_{n+1}, F(y, x))) \\
& \leq \psi(d(F(x_n, y_n), F(x, y))) + \psi(d(F(y_n, x_n), F(y, x))) \\
& \quad + \psi(d(gx, gx_{n+1})) + \psi(d(gy, gy_{n+1})) \\
& \leq 2\theta(d(gx_n, gx), d(gy_n, gy))\psi(M(gx_n, gx, gy_n, gy)) \\
& \quad + \psi(d(gx, gx_{n+1})) + \psi(d(gy, gy_{n+1})).
\end{aligned}$$

Letting $n \rightarrow \infty$, we have $\psi(d(gx, F(x, y)) + d(gy, F(y, x))) = 0$. By properties of ψ , we can see that $d(gx, F(x, y)) + d(gy, F(y, x)) = 0$. Finally, $gx = F(x, y)$ and $gy = F(y, x)$. $\square$

We denote by $\text{CmFix}(F)$ the set of all common fixed points of mappings $F : X^2 \rightarrow X$ and $g : X \rightarrow X$, that is,

$$\text{CmFix}(F) = \{(x, y) \in X^2 : F(x, y) = gx = x \text{ and } F(y, x) = gy = y\}.$$

Theorem 2.6 *In addition to hypotheses of Theorem 2.5, assume that*

- (vi) *for any two elements $(x, y), (u, v) \in X \times X$, there exists $(a, b) \in X \times X$ such that $(gx, ga), (gu, ga), (gy, gb), (gv, gb) \in E(G)$.*

Then, $\text{CmFix}(F) \neq \emptyset$ if and only if $(X^2)_g^F \neq \emptyset$.

Proof Theorem 2.5 implies that there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$. Suppose that there exists another $(u, v) \in X \times X$ such that $gu = F(u, v)$ and $gv = F(v, u)$. We will show that $gx = gu$ and $gy = gv$.

By condition (vi) there exists $(a, b) \in X \times X$ such that $(gx, ga), (gu, ga), (gy, gb), (gv, gb) \in E(G)$. Set $a_0 = a, b_0 = b, x_0 = x, y_0 = y, u_0 = u$, and $v_0 = v$. By Lemma 2.4(i) we have sequences $\{a_n\}, \{b_n\}, \{x_n\}, \{y_n\}, \{u_n\}$, and $\{v_n\}$ in X for which

$$\begin{aligned}
ga_n &= F(a_{n-1}, b_{n-1}) \quad \text{and} \quad gb_n = F(b_{n-1}, a_{n-1}), \\
gx_n &= F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}), \\
gu_n &= F(u_{n-1}, v_{n-1}) \quad \text{and} \quad gv_n = F(v_{n-1}, u_{n-1})
\end{aligned}$$

for $n \in \mathbb{N}$. By the properties of coincidence points, $x_n = x, y_n = y$ and $u_n = u, v_n = v$, that is,

$$gx_n = F(x, y), \quad gy_n = F(y, x) \quad \text{and} \quad gu_n = F(u, v), \quad gv_n = F(v, u) \quad \text{for all } n \in \mathbb{N}.$$

Since $(gx, ga), (gy, gb) \in E(G)$, we have (gx, ga_0) and $(gy, gb_0) \in E(G)$. Because F and g are G -edge preserving, we have $(F(x, y), F(a_0, b_0)) = (gx, ga_1)$ and $(F(y, x), F(b_0, a_0)) = (gy, gb_1) \in E(G)$. Similarly, (gx, ga_n) and $(gy, gb_n) \in E(G)$. By Lemma 2.4(ii) we obtain

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} M(gx, ga_n, gy, gb_n) = 0,$$

and then

$$\lim_{n \rightarrow \infty} d(gx, ga_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(gy, gb_n) = 0.$$

Similarly, from $(gu, ga), (gv, gb) \in E(G)$ we have

$$\lim_{n \rightarrow \infty} d(gu, ga_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(gv, gb_n) = 0.$$

By the triangle inequality we have

$$d(gx, gu) \leq d(gx, ga_n) + d(ga_n, gu) \quad \text{and} \quad d(gy, gv) \leq d(gy, gb_n) + d(gb_n, gv)$$

for all $n \in \mathbb{N}$. Letting $n \rightarrow \infty$ in these two inequalities, we get that $d(gx, gu) = 0$ and $d(gy, gv) = 0$. Therefore, we have $gx = gu$ and $gy = gv$.

The proof of the existence and uniqueness of a common fixed point can be derived using a similar argument as in Theorem 3.7 in [1]. $\square$

Remark 2.1 In the case where $(X, d, \preceq)$ is a partially ordered complete metric space, letting $E(G) = \{(x, y) \in X \times X : x \preceq y\}$ and $\psi(t) = t$, we obtain Theorem 3.1 and Theorem 3.7 in [1].

3 Applications

In this section, we apply our theorem to the existence theorem for a solution of the following integral system:

$$\begin{aligned} x(t) &= \int_0^T f(t, s, x(s), y(s)) ds + h(t), \\ y(t) &= \int_0^T f(t, s, y(s), x(s)) ds + h(t), \end{aligned} \tag{8}$$

where $t \in [0, T]$ with $T > 0$.

Let $X := C([0, T], \mathbb{R}^n)$ with $\|x\| = \max_{t \in [0, T]} |x(t)|$, for $x \in C([0, T], \mathbb{R}^n)$.

We define the graph G with partial order relation by

$$x, y \in X, \quad x \leq y \quad \Leftrightarrow \quad x(t) \leq y(t) \quad \text{for any } t \in [0, T].$$

Thus, $(X, \|\cdot\|)$ is a complete metric space endowed with a directed graph G .

Let $E(G) = \{(x, y) \in X \times X : x \leq y\}$. Then $E(G)$ satisfies the transitivity property, and $(X, \|\cdot\|, G)$ has property A.

Theorem 3.1 Consider system (8). Suppose that

- (i) $f : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $h : [0, T] \rightarrow \mathbb{R}^n$ are continuous;
- (ii) for all $x, y, u, v \in \mathbb{R}^n$ with $x \leq u, y \leq v$, we have $f(t, s, x, y) \leq f(t, s, u, v)$ for all $t, s \in [0, T]$;
- (iii) there exist $0 \leq k < 1$ and $T > 0$ such that

$$|f(t, s, x, y) - f(t, s, u, v)| \leq \frac{k}{T} (|x - u| + |y - v|)$$

for all $t, s \in [0, T], x, y, u, v \in \mathbb{R}^n, x \leq u, y \leq v$;

(iv) there exists $(x_0, y_0) \in X \times X$ such that

$$\begin{aligned} x_0(t) &\leq \int_0^T f(t, s, x_0(s), y_0(s)) ds + h(t) \quad \text{and} \\ y_0(t) &\leq \int_0^T f(t, s, y_0(s), x_0(s)) ds + h(t), \end{aligned}$$

where $t \in [0, T]$.

Then there exists at least one solution of the integral system (8).

Proof Let $F : X \times X \rightarrow X$, $(x, y) \mapsto F(x, y)$, where

$$F(x, y)(t) = \int_0^T f(t, s, x(s), y(s)) ds + h(t), \quad t \in [0, T],$$

and define $g : X \rightarrow X$ by $gx(t) = \frac{x(t)}{2}$.

System (8) can be written as

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Let $x, y, u, v \in X$ be such that $gx \leq gu$ and $gy \leq gv$. We have $x \leq u, y \leq v$ and

$$\begin{aligned} F(x, y)(t) &= \int_0^T f(t, s, x(s), y(s)) ds + h(t) \\ &\leq \int_0^T f(t, s, u(s), v(s)) ds + h(t) = F(u, v)(t) \quad \text{for all } t \in [0, T] \end{aligned}$$

and

$$\begin{aligned} F(y, x)(t) &= \int_0^T f(t, s, y(s), x(s)) ds + h(t) \\ &\leq \int_0^T f(t, s, v(s), u(s)) ds + h(t) = F(v, u)(t) \quad \text{for all } t \in [0, T]. \end{aligned}$$

Thus, F and g are G -edge preserving.

By condition (iv) it follows that $(X^2)_g^F = \{(x, y) \in X \times X : gx \leq F(x, y) \text{ and } gy \leq F(y, x)\} \neq \emptyset$.

On the other hand,

$$\begin{aligned} &|F(x, y)(t) - F(u, v)(t)| \\ &\leq \int_0^T |f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))| ds \\ &= \int_0^T |f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))| ds \\ &\leq \frac{k}{T} \int_0^T (|x(s) - u(s)| + |y(s) - v(s)|) ds \\ &\leq k \left(\frac{\|gx - gu\| + \|gy - gv\|}{2} \right) \\ &\leq kM(gx, gu, gy, gv) \quad \text{for all } t \in [0, T]. \end{aligned}$$

Then, there exist $\psi(t) = t$ and $\theta \in \Theta$, where $\theta(s, t) = k$ for $s, t \in [0, \infty)$ with $k \in [0, 1)$, such that

$$\psi(\|F(x, y) - F(u, v)\|) \leq \theta(\|gx - gu\|, \|gy - gv\|)\psi(M(gx, gu, gy, gv)),$$

where $M(gx, gu, gy, gv) = \max\{\|gx - gu\|, \|gy - gv\|\}$. Hence, F and g are a θ - ψ -contraction.

Thus, there exists a coupled common fixed point $(x^*, y^*) \in X \times X$ of the mapping F and g , which is the solution of the integral system (8). $\square$

Theorem 3.2 *Consider system (8). Suppose that*

- (i) $f : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $h : [0, T] \rightarrow \mathbb{R}^n$ are continuous;
- (ii) for all $x, y, u, v \in \mathbb{R}^n$ with $x \leq u, y \leq v$, we have $f(t, s, x, y) \leq f(t, s, u, v)$ for all $t, s \in [0, T]$;
- (iii) for all $t, s \in [0, T], x, y, u, v \in \mathbb{R}^n, x \leq u, y \leq v$,

$$|f(t, s, x, y) - f(t, s, u, v)| \leq \frac{1}{T} \ln(1 + \max\{|x - u|, |y - v|\});$$

- (iv) there exists $(x_0, y_0) \in X \times X$ such that

$$\begin{aligned} x_0(t) &\leq \int_0^T f(t, s, x_0(s), y_0(s)) ds + h(t), \\ y_0(t) &\leq \int_0^T f(t, s, y_0(s), x_0(s)) ds + h(t), \end{aligned}$$

where $t \in [0, T]$.

Then there exists at least one solution of the integral system (8).

Proof Let $F : X \times X \rightarrow X, (x, y) \mapsto F(x, y)$, where

$$F(x, y)(t) = \int_0^T f(t, s, x(s), y(s)) ds + h(t), \quad t \in [0, T],$$

and define $g : X \rightarrow X$ by $gx(t) = x(t)$. As in Theorem 3.1, we have that F and g are G -edge preserving.

By condition (iv) it follows that $(X^2)_g^F = \{(x, y) \in X \times X : gx \leq F(x, y) \text{ and } gy \leq F(y, x)\} \neq \emptyset$.

On the other hand,

$$\begin{aligned} &|F(x, y)(t) - F(u, v)(t)| \\ &\leq \int_0^T |f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))| ds \\ &= \int_0^T |f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))| ds \\ &\leq \frac{1}{T} \int_0^T \ln(1 + \max\{|x(s) - u(s)|, |y(s) - v(s)|\}) ds \\ &\leq \ln\left(1 + \max\left\{\max_{t \in [0, T]} |x(t) - u(t)|, \max_{t \in [0, T]} |y(t) - v(t)|\right\}\right) \end{aligned}$$

$$\begin{aligned}
&\leq \ln(1 + \max\{\|x - u\|, \|y - v\|\}) \\
&= \ln(1 + M(gx, gu, gy, gv)) \quad \text{for all } t \in [0, T],
\end{aligned}$$

where $M(gx, gu, gy, gv) = \max\{\|gx - gu\|, \|gy - gv\|\}$, which yields

$$\begin{aligned}
&\ln(|F(x, y)(t) - F(u, v)(t)| + 1) \\
&\leq \ln(\ln(1 + M(gx, gu, gy, gv)) + 1) \\
&= \frac{\ln(\ln(1 + M(gx, gu, gy, gv)) + 1)}{\ln(1 + M(gx, gu, gy, gv))} \ln(1 + M(gx, gu, gy, gv)).
\end{aligned}$$

Hence, there exist $\psi(x) = \ln(x + 1)$ and $\theta \in \Theta$ defined by

$$\theta(s, t) = \begin{cases} \frac{\ln(\ln(1 + \max\{s, t\}))}{\ln(1 + \max\{s, t\})}, & s > 0 \text{ or } t > 0, \\ r \in [0, 1), & s = 0, t = 0, \end{cases}$$

such that

$$\begin{aligned}
\psi(d(F(x, y), F(u, v))) &= \psi(\|F(x, y) - F(u, v)\|) \\
&\leq \theta(d(gx, gu), d(gy, gv)) \psi(M(gx, gu, gy, gv)).
\end{aligned}$$

Hence, we see that F and g are a θ - ψ -contraction. Thus, there exists a coupled common fixed point $(x^*, y^*) \in X \times X$ of the mapping F and g , which is a solution for the integral system (8). $\square$

References

1. Kadelburg, Z, Kumam, P, Radenović, S, Sintunavarat, W: Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property. *Fixed Point Theory Appl.* **2015**, 27 (2015). doi:10.1186/s13663-015-0278-5
2. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal. TMA* **70**, 4341-4349 (2009)
3. Guo, D, Lakshmikantham, V: Coupled fixed points of nonlinear operators with applications. *Nonlinear Anal., Theory Methods Appl.* **11**, 623-632 (1987)
4. Gnana Bhaskar, T, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal. TMA* **65**, 1379-1393 (2006)
5. Berinde, V: Coupled fixed point theorems for φ -contractive mixed monotone mappings in partially ordered metric spaces. *Nonlinear Anal., Theory Methods Appl.* **75**(6), 3218-3228 (2012)
6. Berinde, V: Coupled coincidence point theorems for mixed monotone nonlinear operators. *Comput. Math. Appl.* **64**(6), 1770-1777 (2012)
7. Choudhury, BS, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonlinear Anal. TMA* **73**, 2524-2531 (2010)

8. Sintunavarat, W, Radenović, S, Golubović, Z, Kuman, P: Coupled fixed point theorems for F -invariant set. *Appl. Math. Inf. Sci.* **7**(1), 247-255 (2013)
9. Karapinar, E, Luong, NV, Thuan, NX, Hai, TT: Coupled coincidence points for mixed monotone operators in partially ordered metric spaces. *Arab. J. Math.* **1**, 329-339 (2012)
10. Borcut, M: Tripled fixed point theorems for monotone mappings in partially ordered metric spaces. *Carpath. J. Math.* **28**(2), 207-214 (2012)
11. Borcut, M: Tripled coincidence theorems for monotone mappings in partially ordered metric spaces. *Creative Math. Inform.* **21**(2), 135-142 (2012)
12. Radenović, S: Coupled fixed point theorems for monotone mappings in partially ordered metric spaces. *Kragujev. J. Math.* **38**(2), 249-257 (2014)
13. Radenović, S: Some coupled coincidence points results of monotone mappings in partially ordered metric spaces. *Int. J. Anal. Appl.* **5**(2), 174-184 (2014)
14. Radenović, S: Bhaskar-Lakshmikantham type results for monotone mappings in partially ordered metric spaces. *Int. J. Nonlinear Anal. Appl.* **5**(2), 37-49 (2014)
15. Radenović, S: A note on tripled coincidence and tripled common fixed point theorems in partially ordered metric spaces. *Appl. Math. Comput.* **236**, 367-372 (2014)
16. Fadail, ZM, Rad, GS, Ozturk, V, Radenović, S: Some remarks on coupled, tripled and n -tupled fixed points theorems in ordered abstract metric spaces. *Far East J. Math. Sci.* **97**, 809-839 (2015)
17. Dorić, D, Kadelburg, Z, Radenović, S: Coupled fixed point results for mappings without mixed monotone property. *Appl. Math. Lett.* (2012). doi:10.1016/j.aml.2012.02.022
18. Jachymski, J: The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **136**, 1359-1373 (2008)
19. Beg, I, Butt, AR, Radojević, S: The contraction principle for set valued mappings on a metric space with a graph. *Comput. Math. Appl.* **60**, 1214-1219 (2010)
20. Bojor, F: Fixed point theorems for Reich type contractions on metric spaces with a graph. *Nonlinear Anal.* **75**(9), 3895-3901 (2012)
21. Alfuraidan, MR: The contraction principle for multivalued mappings on a modular metric space with a graph. *Can. Math. Bull.* **2015**, 29 (2015). doi:10.4153/CMB-2015-029-x
22. Alfuraidan, MR: Remarks on monotone multivalued mappings on a metric space with a graph. *J. Inequal. Appl.* **2015**, 202 (2015). doi:10.1186/s13660-015-0712-6
23. Alfuraidan, MR, Khamsi, MA: Caristi fixed point theorem in metric spaces with a graph. *Abstr. Appl. Anal.* **2014**, 303484 (2014). doi:10.1155/2014/303484
24. Chifu, C, Petrusel, G: New results on coupled fixed point theory in metric spaces endowed with a directed graph. *Fixed Point Theory Appl.* **2014**, 151 (2014). doi:10.1186/1687-1812-2014-151

3.3 Fixed Point Theorems for Multivalued Nonself G -Almost Contractions in Banach Spaces Endowed with Graphs

In this section, we prove some fixed point theorems for multivalued nonself G -almost contractions in Banach spaces with a directed graph and give some examples to illustrate our main results. The main results in this paper extend and generalize many known results in the literature therein.

1. Introduction

Fixed point theory plays a very important role in nonlinear analysis, economics, and applications. The important tool for solving existence problems in many branches of mathematics and applied sciences is Banach contraction principle [1]. It has been generalized in many directions by many authors ([2–15]).

On the other hand, the fixed point theory for nonself mappings has been studied by many authors ([3–5]). In 2013, Berinde and Păcurar [5] introduced the new type of contraction for nonself single valued mappings.

Definition 1. Let K be a nonempty subset of a normed space X . A mapping $T : K \rightarrow X$ is said to be almost contractive if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|y - Tx\| \quad (1)$$

for all $x, y \in K$

By the above definition, they proved some fixed point theorem for almost contractive mappings in Banach spaces.

Later, Alghamdi et al. [4] extended the almost contractive mappings to a multivalued nonself almost contractions.

Definition 2. Let (X, d) be a complete convex metric space and K a nonempty subset of X . A mapping $T : K \rightarrow CB(X)$

is said to be a *multivalued almost contraction* if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \quad (2)$$

for all $x, y \in K$, where $H(\cdot, \cdot)$ is the Pompeiu-Hausdorff metric induced by d .

They proved a fixed point theorem for multivalued nonself almost contraction on convex metric spaces.

Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ is a set of vertices of graph and $E(G)$ is a set of its edges. Assume that G has no parallel edges. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges: that is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}. \quad (3)$$

Let x and y be two vertices in G ; a *path* in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^n$ of $n+1$ vertices such

that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for each $i = 1, 2, \dots, n$. A graph G is said to be *connected* if there exists a (directed) path between any two vertices of G . We denote

$$[x]_G^N = \{y \in X : \text{there is a path in } G \text{ of length } N \text{ from } x \text{ to } y\}. \quad (4)$$

On the other hand, in 2008, Jachymski [6] extended Banach contraction principle by introducing the notion of G -contraction and proved a fixed point theorem for G -contractions in a metric space endowed with a directed graph. These results have been generalized by some authors in several ways (see [7–14]).

Motivated by the previous results, we introduced the concept of multivalued nonself G -almost contractions in normed spaces and establish some fixed point theorems for this type of mappings in normed spaces endowed with a directed graph. We give some examples to illustrate our main results.

2. Preliminaries

In this section, we give some basic and useful definitions and known results that are useful for the main results in this paper.

Let X be a normed space. We denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X . For any $A, B \in CB(X)$, define the function $H : CB(X) \times CB(X) \rightarrow \mathbb{R}^+$ by

$$H(A, B) = \max \{d(A, B), d(B, A)\}, \quad (5)$$

where

$$\begin{aligned} d(A, B) &= \sup \{d(a, B) : a \in A\}, \\ d(B, A) &= \sup \{d(b, A) : b \in B\}, \\ d(a, C) &= \inf \{\|a - x\| : x \in C\}. \end{aligned} \quad (6)$$

Note that H is called the *Pompeiu-Hausdorff metric* induced by metric d .

The following lemma is very useful for our main results.

Lemma 3 (see [2]). *Let (X, d) be a metric space. If $A, B \in CB(X)$ and $a \in A$, then, for each $\epsilon > 0$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.*

Let X be a normed space, K a nonempty closed subset of X , and $T : K \rightarrow CB(X)$ a nonself mapping. If $x \in K$ with $Tx \subset K$, then we can always choose an $y \in \partial K$ such that $y = (1 - \lambda)x + \lambda z$, where $0 < \lambda < 1$ and $z \in Tx \setminus K$, which expresses the fact that

$$\|x - z\| = \|x - y\| + \|y - z\|. \quad (7)$$

We denote

$$\begin{aligned} P_T(x) &= \{y \in \partial K : \|x - z\| = \|x - y\| \\ &\quad + \|y - z\| \text{ where } z \in Tx\}. \end{aligned} \quad (8)$$

The following property was introduced by Jachymski [6]. It is very useful for our main results.

Property A (see [7]). With any sequence (x_n) in X , if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for any $n \in \mathbb{N}$.

In order to obtain our main results, we need the following definition of domination in graphs [16, 17].

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if, for all $v \in V(G) \setminus X$, there exists $x \in X$ such that $(x, v) \in E(G)$ and we say that x dominates v or v is dominated by x . For any $v \in V$, a set $X \subseteq V$ is *dominated* by v if $(v, x) \in E(G)$ for any $x \in X$ and we say that X *dominates* v if $(x, v) \in E(G)$ for all $x \in X$.

3. Main Results

In this section, we prove some fixed point theorems for multivalued nonself G -almost contractions in a Banach space endowed with a directed graph. First, we recall some definitions which are useful for our main results.

Definition 4. Let K be a nonempty subset of a normed space X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$. A mapping $T : K \rightarrow CB(X)$ is said to be G -almost contraction

- (1) if there exist $\delta \in (0, 1)$ and $L \geq 0$ with $\delta(1 + L) < 1$ such that, for any $x, y \in K$,

$$H(Tx, Ty) \leq \delta \|x - y\| + L \|y - Tx\| \quad (9)$$

whenever $(x, y) \in E(G)$,

- (2) if $u \in Tx$ and $v \in Ty$ are such that $d(u, v) \leq d(x, y)$; then $(u, v) \in E(G)$.

First, we prove a fixed point theorem for a G -almost contraction in a Banach space endowed with a directed graph.

Theorem 5. *Let K be a nonempty closed subset of a Banach space X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$. Let $T : K \rightarrow CB(X)$ be a G -almost contraction. Suppose that*

- (1) *T satisfies Rothe's boundary condition: that is, $x \in \partial K \Rightarrow Tx \subset K$,*
- (2) *K has Property A,*
- (3) *there is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in Tx_0$*
- (4) *$P_T(x)$ is dominated by x for all $x \in C$ with $Tx \not\subset K$.*

Then T has a fixed point in K .

Proof. Let $x_0 \in K$ be such that $(x_0, y_1) \in E(G)$, where $y_1 \in Tx_0$. If $y_1 \in K$, we denote $x_1 = y_1$; that is, $(x_0, x_1) \in E(G)$. If $y_1 \notin K$, there exists $x_1 \in \partial K$ such that

$$\|x_0 - x_1\| + \|x_1 + y_1\| = \|x_0 - y_1\|. \quad (10)$$

Since $P_T(x_0)$ is dominated by x_0 , we obtain $(x_0, x_1) \in E(G)$. If $x_0 = x_1$ then $x_0 \in Tx_0$. Suppose $x_0 \neq x_1$; we can choose $k_0 \in \mathbb{N}$ such that

$$\delta^{k_0} < (1 - \delta) \|x_0 - x_1\|. \quad (11)$$

By Lemma 3, we can choose $y_2 \in Tx_1$ such that

$$\begin{aligned} \|y_1 - y_2\| &\leq H(Tx_0, Tx_1) + \delta^{k_0} \\ &\leq \delta \|x_0 - x_1\| + Ld(x_1, Tx_0) + \delta^{k_0} \\ &= \delta \|x_0 - x_1\| + \delta^{k_0} < \|x_0 - x_1\|. \end{aligned} \quad (12)$$

Hence, $(y_1, y_2) \in E(G)$. If $y_2 \in K$, we denote $x_2 = y_2$. If $y_2 \notin K$, then there exists $x_2 \in \partial K$ such that

$$\|x_1 - x_2\| + \|x_2 - y_2\| = \|x_1 - y_2\|. \quad (13)$$

If $x_1 = x_2$ then $x_1 \in Tx_1$. Suppose $x_1 \neq x_2$; we can choose $k_1 \in \mathbb{N}$ with $k_1 > k_0$ such that

$$\delta^{k_1} < (1 - \delta) \|x_1 - x_2\|. \quad (14)$$

By Lemma 3, we can choose $y_3 \in Tx_2$ such that

$$\begin{aligned} \|y_2 - y_3\| &\leq H(Tx_1, Tx_2) + \delta^{k_1} \\ &\leq \delta \|x_1 - x_2\| + Ld(x_2, Tx_1) + \delta^{k_1} \\ &= \delta \|x_1 - x_2\| + \delta^{k_1} < \|x_1 - x_2\|. \end{aligned} \quad (15)$$

Hence, $(y_1, y_2) \in E(G)$. By induction, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

- (i) $y_{n+1} \in Tx_n$;
- (ii) $\|y_n - y_{n+1}\| \leq H(Tx_{n-1}, Tx_n) + \delta^{k_n}$, where
 - (a) $y_n = x_n$ if $y_n \in K$,
 - (b) $y_n \neq x_n$ whenever $y_n \notin K$, and then $x_n \in \partial K$ is such that

$$\|x_{n-1} - x_n\| + \|x_n - y_n\| = \|x_{n-1} - y_n\|, \quad (16)$$
 - (c) $\{k_n\}$ is a sequence of positive integers with $\delta^{k_n} < (1 - \delta)\|x_n - x_{n+1}\|$.

Suppose that

$$\begin{aligned} P &= \{x_k \in \{x_n\} : x_k = y_k\}, \\ Q &= \{x_k \in \{x_n\} : x_k \neq y_k\}. \end{aligned} \quad (17)$$

Note that $\{y_n\} \subset K$ and $(y_n, y_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Moreover, if $x_k \in Q$, then both x_{k-1} and x_k belong to the set P . Since x_{k-1} and x_k are not in the set Q for all k , we cannot have two consecutive terms of $\{x_k\}$ in the set Q .

Now, we claim that $\{x_n\}$ is a Cauchy sequence. We conclude that there are three possibilities.

Case 1. If $x_n, x_{n+1} \in P$, then $x_n \in Tx_{n-1}$ and $x_{n+1} \in Tx_n$. Since T is a G -almost contraction and $(x_n, x_{n+1}) \in E(G)$, we have

$$\begin{aligned} \|x_n - x_{n+1}\| &= H(Tx_{n-1}, Tx_n) + \delta^{k_{n-1}} \\ &\leq \delta \|x_{n-1} - x_n\| + Ld(x_n, Tx_{n-1}) + \delta^{k_{n-1}} \\ &= \delta \|x_{n-1} - x_n\| + \delta^{k_{n-1}}. \end{aligned} \quad (18)$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then $x_n \in Tx_{n-1}$ and $y_{n+1} \neq x_{n+1}$; that is,

$$\|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| = \|x_n - Tx_n\|. \quad (19)$$

Since T is a G -almost contraction and $(x_n, x_{n+1}) \in E(G)$, we have

$$\begin{aligned} \|x_n - x_{n+1}\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_{n+1}\| \\ &= \|x_n - y_{n+1}\| = \|y_n - y_{n+1}\| \\ &\leq H(Tx_{n-1}, Tx_n) + \delta^{k_{n-1}} \\ &\leq \delta \|x_{n-1} - x_n\| + Ld(x_n, Tx_{n-1}) + \delta^{k_{n-1}} \\ &= \delta \|x_{n-1} - x_n\| + \delta^{k_{n-1}}. \end{aligned} \quad (20)$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$, then $x_{n-1} \in P$ and $y_n \neq x_n$ such that

$$\|x_{n-1} - x_n\| + \|x_n - Tx_{n-1}\| = \|x_{n-1} - Tx_{n-1}\|. \quad (21)$$

Moreover, $y_{n-1} = x_{n-1}$ and $y_n \in Tx_{n-1}$. Since T is a G -almost contraction, we have

$$\begin{aligned} \|x_n - x_{n+1}\| &= \|x_n - x_{n+1}\| \leq \|x_n - y_n\| + \|y_n, y_{n+1}\| \\ &\leq \|x_n - y_n\| + H(Tx_{n-1}, Tx_n) + \delta^{k_{n-1}} \\ &\leq \|x_n - y_n\| + \delta \|x_{n-1} - x_n\| \\ &\quad + Ld(x_n, Tx_{n-1}) + \delta^{k_{n-1}} \\ &\leq \|x_n - y_n\| + \|x_{n-1} - x_n\| \\ &\quad + Ld(x_n, Tx_{n-1}) + \delta^{k_{n-1}} \\ &= \|x_{n-1} - y_n\| + Ld(x_n, Tx_{n-1}) + \delta^{k_{n-1}} \\ &\leq \|x_{n-1} - y_n\| + L \|x_n, y_n\| + \delta^{k_{n-1}} \\ &= \|x_{n-1} - y_n\| + L \|x_{n-1}, y_n\| \\ &\quad - L \|x_{n-1}, x_n\| + \delta^{k_{n-1}} \\ &= (1 + L) \|x_{n-1} - y_n\| - L \|x_{n-1}, x_n\| \\ &\quad + \delta^{k_{n-1}} \leq (1 + L) \|y_{n-1} - y_n\| + \delta^{k_{n-1}} \\ &\leq (1 + L) H(x_{n-2}, x_{n-1}) + (1 + L) \delta^{k_{n-2}} \\ &\quad + \delta^{k_{n-1}} \\ &\leq (1 + L) \delta \|x_{n-2} - x_{n-1}\| \\ &\quad + (1 + L) Ld(x_{n-1}, Tx_{n-2}) \\ &\quad + (1 + L) \delta^{k_{n-2}} + \delta^{k_{n-1}} \\ &= (1 + L) \delta \|x_{n-2} - x_{n-1}\| + (1 + L) \delta^{k_{n-2}} \\ &\quad + \delta^{k_{n-1}}. \end{aligned} \quad (22)$$

We denote $h = (1 + L)\delta < 1$, so

$$\|x_n - x_{n+1}\| \leq h \|y_{n-2} - y_{n-1}\| + h\delta^{k_{n-3}} + \delta^{k_{n-1}}. \quad (23)$$

Therefore, combining Cases 1, 2, and 3, we obtain

$$\begin{aligned} \|x_n - x_{n+1}\| &\leq h \max \{\|x_{n-1} - x_n\|, \|x_{n-2} - x_{n-1}\|\} \\ &\quad + h\delta^{k_{n-3}} + \delta^{k_{n-1}} \\ &\leq h \max \{\|x_{n-1} - x_n\|, \|x_{n-2} - x_{n-1}\|\} \\ &\quad + h^{k_{n-3}+1} + h^{k_{n-1}} \\ &\leq h \max \{\|x_{n-1} - x_n\|, \|x_{n-2} - x_{n-1}\|\} \\ &\quad + h^{n-3} + h^{n-1} \end{aligned} \quad (24)$$

for all $n \geq 2$. By inductivity, it follows that, for any $n > 1$,

$$\|x_n - x_{n+1}\| \leq h^{n/2} \max \{\|x_0 - x_1\|, \|x_1 - x_2\|\} + h^{(n-1)/2} n. \quad (25)$$

Now, for any $n > m$, we obtain

$$\begin{aligned} \|x_m - x_n\| &\leq \|x_m - x_{m+1}\| + \|x_{m+1} - x_{m+2}\| + \cdots \\ &\quad + \|x_{n-1} - x_n\| \leq (h^{(m-1)/2} + h^{m/2} + \cdots + h^{(n-2)/2}) \\ &\quad \cdot \max \{\|x_0 - x_1\|, \|x_1 - x_2\|\} + h^{(m-1)/2} m \\ &\quad + h^{m/2} (m+1) + \cdots + h^{(n-1)/2} n. \end{aligned} \quad (26)$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since C is closed and $\{x_n\} \subset C$, there exists $x \in C$ such that $\lim_{n \rightarrow \infty} x_n = x$. Let $\{x_{n_k}\} \subset P$ be an infinite subsequence of $\{x_n\}$. Since X has Property A, $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x, Tx) &\leq \|x - x_{n_k+1}\| + d(x_{n_k+1}, Tx) \\ &= \|x - x_{n_k+1}\| + H(Tx_{n_k}, Tx) \\ &\leq \|x - x_{n_k+1}\| + \delta \|x_{n_k} - x\| + Ld(x, Tx_{n_k}) \\ &= (1 + L) \|x - x_{n_k+1}\| + \delta \|x_{n_k} - x\|. \end{aligned} \quad (27)$$

Letting $k \rightarrow \infty$, we obtain

$$d(x, Tx) = 0 \quad (28)$$

and hence $x \in Tx$: that is, x is a fixed point of T . This completes the proof. $\square$

As a consequence of Theorem 5, by putting $E(G) = C \times C$, we obtain the following corollary.

Corollary 6. *Let K be a nonempty closed subset of a Banach space X and $T : K \rightarrow CB(X)$ be an almost contractive mapping with Rothe's boundary condition. Then T has a fixed point.*

Remark 7. (i) In Theorem 5, if we take $E(G) = X \times X$, we obtain Theorem 9 of Alghmdy et al. [4] in Banach spaces.

(ii) In Theorem 5, if we take $E(G) = X \times X$ and $T : K \rightarrow X$, we obtain the existence result which is similar to Berinde and Păcurar's result [5].

(iii) In Theorem 5, if we take $T : X \rightarrow X$ and $L = 0$, we obtain the existence result which is similar to Jachymski's result [5] in Banach spaces.

(iv) In Theorem 5, if we take $E(G) = X \times X$ and $L = 0$, we obtain the existence result which is similar to Nadler's theorem [5] in Banach spaces.

Next, we give an example to illustrate Theorem 5.

Example 8. Let $X = \mathbb{R}$ and $K = [0, \infty)$ with the usual norm $\|x - y\| = |x - y|$ and $G = (V(G), E(G))$ be such that $V(G) = K$,

$$\begin{aligned} E(G) &= \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{2n+1}} \right), \left(\frac{1}{2^n}, 0 \right), \left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n+2}} \right) : n \geq 2 \right\} \\ &\quad \cup \left\{ \left(\frac{1}{2}, 0 \right), (0, 0) \right\}. \end{aligned} \quad (29)$$

Notice that K has Property A. Define a mapping $T : K \rightarrow CB(X)$ by

$$Tx = \begin{cases} \{0\} & \text{if } x = 0, \\ \left\{ -\frac{1}{8} \right\} & \text{if } x = \frac{1}{2}, \\ \left\{ \frac{1}{2^{2k}}, \frac{1}{2^{2k+1}} \right\} & \text{if } x = \frac{1}{2^{2k+1}}, k \in \mathbb{N} \cup \{0\} \\ \left\{ \frac{1}{2^{2k+2}}, \frac{1}{2^{2k+3}} \right\} & \text{if } x = \frac{1}{2^{2k}}, k \in \mathbb{N} \cup \{0\} \setminus \left\{ \frac{1}{2} \right\}. \end{cases} \quad (30)$$

We see that the only $x \in K$ with $Tx \notin K$ is $x = 1/2$, and we see that $P_T(1/2) = \{0\}$ and $(1/2, 0) \in E(G)$; hence, $P_T(x)$ is dominated by x . By direct computation, we can show that T is G -almost contraction with $\delta = 1/4$ and $L = 2$. Moreover, we see that

$$\begin{aligned} T(\partial K) &= T(\{0, 1\}) = \{0, 1\} \\ &\implies T0 \text{ and } T1 \text{ are subset of } K, \end{aligned} \quad (31)$$

which implies T satisfies Rothe's boundary condition. Choose $x_0 = 0 \in \partial C$, we see that $(x_0, Tx_0) = (0, 0) \in E(G)$. Therefore, all conditions of Theorem 5 are satisfied and we see that $F(T) = \{0\}$.

References

- [1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.
- [2] J. Nadler, "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.
- [3] N. A. Assad, "A fixed point theorem for some non-self-mappings," *Tamkang Journal of Mathematics*, vol. 21, no. 4, pp. 387–393, 1990.
- [4] M. A. Alghamdi, V. Berinde, and N. Shahzad, "Fixed points of non-self almost contractions," *Carpathian Journal of Mathematics*, vol. 30, no. 1, pp. 7–14, 2014.
- [5] V. Berinde and M. Păcurar, "Fixed point theorems for nonself single-valued almost contractions," *Fixed Point Theory*, vol. 14, no. 2, pp. 301–312, 2013.
- [6] J. Jachymski, "The contraction principle for mappings on a metric space with a graph," *Proceedings of the American Mathematical Society*, vol. 136, no. 4, pp. 1359–1373, 2008.
- [7] I. Beg and A. R. Butt, "Fixed point of set-valued graph contractive mappings," *Journal of Inequalities and Applications*, vol. 2013, article 252, 2013.
- [8] F. Bojor, "Fixed point theorems for Reich type contractions on metric spaces with a graph," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 75, no. 9, pp. 3895–3901, 2012.
- [9] M. R. Alfuraidan, "Fixed points of monotone nonexpansive mappings with a graph," *Fixed Point Theory and Applications*, vol. 2015, article 49, 2015.
- [10] J. Tiammee and S. Suantai, "Coincidence point theorems for graph-preserving multi-valued mappings," *Fixed Point Theory and Applications*, vol. 2014, article 70, 2014.
- [11] J. Tiammee, A. Kaewkhao, and S. Suantai, "On Browder's convergence theorem and Halpern iteration process for G -nonexpansive mappings in Hilbert spaces endowed with graphs," *Fixed Point Theory and Applications*, vol. 2015, article no. 187, 2015.
- [12] N. Hussain, J. Ahmad, and A. Azam, "Generalized fixed point theorems for multi-valued α - ϕ contractive mappings," *Journal of Inequalities and Applications*, vol. 2014, p. 384, 2014.
- [13] N. Hussain, J. Ahmad, and M. A. Kutbi, "Fixed point theorems for generalized Mizoguchi-Takahashi graphic contractions," *Journal of Function Spaces*, vol. 2016, Article ID 6514920, 7 pages, 2016.
- [14] M. R. Alfuraidana, M. Bacharb, and M. A. Khamsi, "Almost monotone contractions on weighted graphs," *The Journal of Nonlinear Science and Applications*, vol. 9, pp. 5189–5195, 2016.
- [15] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, Springer, New York, NY, USA, 2009.
- [16] C. Pang, R. Zhang, Q. Zhang, and J. Wang, "Dominating sets in directed graphs," *Information Sciences*, vol. 180, no. 19, pp. 3647–3652, 2010.
- [17] J. Bang-Jensen and G. Gutin, *Digraphs Theory, Algorithms and Applications*, Springer Monographs in Mathematics, Springer, London, UK, 2007.

3.4 Fixed point theorems for nonself G -almost contractive mappings in Banach spaces endowed with graphs

In this section, we prove some fixed point theorems for non-self G -almost contractive mappings in Banach spaces with a directed graph and give some examples to illustrate our main results. The main results in this paper extend and generalize many known results in the literature therein.

1. INTRODUCTION

Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be *contractive* if there exists $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. Forward, $F(T)$ stands for the set of all fixed point of T , i.e., $F(T) = \{x \in X : x = Tx\}$.

Banach's contraction principle [6] (or Banach's fixed point theorem) is a very important tool for solving existence and uniqueness problems in many branches of mathematics and applied sciences. In the effort to extend its area of theoretical and practical applications, Banach's contraction principle has been generalized in many directions by many authors, see [1]-[5],[8]-[16],[18, 19], for a limited list of papers strictly related to the topic of the present paper.

In 2013, as a natural counterpart of the selfmap mapping case, first treated in [10], Berinde and Păcurar [13] introduced a new type of contraction condition for *non-self mappings*.

Definition 1.1 ([13]). Let C be a nonempty subset of a normed space X . A mapping $T : C \rightarrow X$ is said to be *almost contractive* if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|y - Tx\|$$

for all $x, y \in C$

Using this concept, in [13], the authors proved some fixed point theorem for non-self almost contractive mappings on Banach spaces. This type of mappings was extended to the case of multi-valued non-self almost contractions by Alghamdi et al. [4].

Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ is a set of vertices of a graph and $E(G)$ is the set of its edges. Assume that G has no parallel edges. We denote by G^{-1}

the *directed graph* obtained from G by reversing the direction of edges, that is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

If x and y are vertices in G , then a *path* in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^n$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for each $i = 1, 2, \dots, n$. A closed directed path of length $N > 1$ from x to y , that is, $x = y$, is called a *directed cycle*. A *cycle digraph* is a directed graph that has no directed cycle. A directed graph G is called an *oriented graph* if, whenever $(x, y) \in E(G)$, then $(y, x) \notin E(G)$.

In 2008, Jachymski [16] combined the concept of fixed point theory and graph theory to study fixed point theorems in a metric space endowed with a directed graph. He introduced the concept of G -contraction and generalized Banach's contraction principle in a metric space endowed with a directed graph.

Definition 1.2 ([16]). Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, i.e., $\Delta = \{(x, x) : x \in X\} \subseteq E(G)$. A mapping $f : X \rightarrow X$ is said to be G -contractive if f edge-preserving, i.e.,

$$x, y \in X, (x, y) \in E(G) \implies (f(x), f(y)) \in E(G)$$

and there exists $\alpha \in [0, 1)$ such that, for any $x, y \in X$,

$$(x, y) \in E(G) \implies d(f(x), f(y)) \leq \alpha d(x, y).$$

Using this concept, in [16], he proved the following theorem:

Theorem 1.1 ([16]). Let (X, d) be complete metric space. Suppose that a triple (X, d, G) have the property, for any sequence $(x_n)_{n \in \mathbb{N}}$, if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$ and there is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with $(x_{n_k}, x) \in E(G)$ for each $n \in \mathbb{N}$. Let $f : X \rightarrow X$ be a G -contractive mapping and $X_f = \{x \in X : (x, f(x)) \in E(G)\}$. Then $F(T) \neq \emptyset$ if and only if $X_f \neq \emptyset$.

The above theorem has been generalized in many directions. (see [2, 8, 15, 19])

In 2015, Berinde and Păcurar [14] proved a fixed point theorem for non-self G -contraction mappings in a Banach space endowed with a graph.

In this paper, we prove some fixed point theorems for a non-self G -almost contractive mappings in a Banach space endowed with a directed graph and also prove some fixed point theorems for a non-self G -contractive mapping in a Banach space. Further, we give some examples to illustrate our main results.

2. PRELIMINARIES

In this section, we give some basic and useful definitions and known results that are useful for the main results in this paper.

Let X be a Banach space, C a nonempty closed subset of X and $T : C \rightarrow X$ a non-self mapping. If $x \in C$ with $Tx \notin C$, then we can always choose an $y \in \partial C$ such that $y = (1 - \lambda)x + \lambda Tx$, where $0 < \lambda < 1$, which expresses the fact that

$$\|x - Tx\| = \|x - y\| + \|y - Ty\|,$$

we denote

$$P_T(x) = \{y \in \partial C : \|x - Tx\| = \|x - y\| + \|y - Ty\|\}.$$

Property A. ([8]) For any sequence $(x_n)_{n \in \mathbb{N}}$, if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for any $n \in \mathbb{N}$.

In order to obtain our main results, we need the following definition of domination in graphs ([7, 17]):

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a *dominating set* if, for all $v \in V(G) \setminus X$, there exists $x \in X$ such that $(x, v) \in E(G)$ and we call that x dominates v or v is dominated by x . For any $v \in V$, a set $X \subseteq V$ is *dominated* by v if $(v, x) \in E(G)$ for any $x \in X$ and we say that X *dominates* v if $(x, v) \in E(G)$ for all $x \in X$.

3. MAIN RESULTS

In this section, we prove some fixed point theorems for non-self G -almost contractive mappings in a Banach space endowed with a directed graph. First, we recall some definitions which are useful for our main results.

Definition 3.3. Let C be a nonempty subset of a normed space X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$.

(1) A mapping $T : C \rightarrow X$ is said to be G -almost contractive if there exist $\delta \in (0, 1)$ and $L \geq 0$ with $\delta(1 + L) < 1$ are such that, for any $x, y \in C$,

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|y - Tx\|$$

whenever $(x, y) \in E(G)$,

(2) A mapping $T : C \rightarrow X$ is said to be G -contractive if there exists $k \in (0, 1)$ such that, for any $x, y \in C$,

$$\|Tx - Ty\| \leq k\|x - y\|$$

whenever $(x, y) \in E(G)$.

First, we prove a fixed point theorem for a G -almost contractive mapping in a Banach space endowed with a directed graph.

Theorem 3.2. Let C be a nonempty closed subset of a Banach space X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Suppose that C has Property A. Let $T : C \rightarrow X$ a G -almost contractive mapping, edge-preserving and satisfies Rothe's boundary condition

$$(3.1) \quad T(\partial C) \subset C.$$

Suppose that $P_T(x)$ is dominated by x for all $x \in C$ with $Tx \notin C$. If there exists $x_0 \in \partial C$ such that $(x_0, Tx_0) \in E(G)$. Then T has a fixed point in C .

Proof. If $T(C) \subset C$, then T is a self mapping on the closed set C and the conclusion follows by Theorem 3.3 in [19] for $X = C$. If $T(C) \not\subset C$. Let $x_0 \in \partial C$ be such that $(x_0, Tx_0) \in E(G)$. By (1), we get $x_1 = Tx_0 \in C$. Since T is edge-preserving, we have $(x_1, x_2) = (Tx_0, Tx_1) \in E(G)$. If $Tx_1 \in C$, we denote $x_2 = Tx_1$. If $Tx_1 \notin C$, there exists $x_2 \in P_T(x_1)$ such that

$$\|x_0 - x_1\| = \|x_0 - x_2\| + \|x_2 - x_1\|.$$

Since $P_T(x_1)$ is dominated by x_1 , we obtain $(x_1, x_2) \in E(G)$.

Continuing arguments, we can obtain a sequence $\{x_n\}$ satisfying one of the following properties:

- (a) $x_n = Tx_{n-1}$, if $Tx_{n-1} \in C$;
- (b) $x_n \in P_T(x_{n-1})$ if $Tx_{n-1} \notin C$.

Suppose that

$$P = \{x_k \in x_n : x_k = Tx_{k-1}\}$$

and

$$Q = \{x_k \in \{x_n\} : x_k \neq Tx_{k-1}\}.$$

Note that $\{x_n\} \subseteq C$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Moreover, if $x_k \in C$, then both $\{x_{k-1}\}$ and $\{x_{k+1}\}$ belong to the set P . By (1), we cannot have two consecutive terms of $\{x_n\}$ in the set Q .

Now, we claim that $\{x_n\}$ is a Cauchy sequence. We conclude that there are three possibilities.

Case 1. If $x_n, x_{n+1} \in P$, then $x_n = Tx_{n-1}$ and $x_{n+1} = Tx_n$. Since T is a G -almost contractive mapping and $(x_n, x_{n+1}) \in E(G)$, we have

$$\|x_n - x_{n+1}\| = \|Tx_{n-1} - Tx_n\| \leq \delta\|x_{n-1} - x_n\| + L\|x_n - Tx_{n-1}\|,$$

which implies

$$\|x_n - x_{n+1}\| \leq \delta\|x_{n-1} - x_n\|.$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then $x_n = Tx_{n-1}$ and $x_{n+1} \in P_T(x_n)$ i.e.,

$$\|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| = \|x_n - Tx_n\|.$$

Since T is a G -almost contractive mapping and $(x_n, x_{n+1}) \in E(G)$, we have

$$\begin{aligned} \|x_n - x_{n+1}\| &= \|x_n - Tx_n\| - \|x_{n+1} - Tx_n\| \leq \|Tx_{n-1} - Tx_n\| \\ &\leq \delta\|x_{n-1} - x_n\| + L\|x_n - Tx_{n-1}\| \\ &= \delta\|x_{n-1} - x_n\|, \end{aligned}$$

which implies

$$\|x_n, x_{n+1}\| \leq \delta\|x_{n-1} - x_n\|.$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$, then $x_{n-1} \in P$ and $x_n \in P_T(x_{n-1})$ such that

$$\|x_{n-1} - x_n\| + \|x_n - Tx_{n-1}\| = \|x_{n-1} - Tx_{n-1}\|.$$

Since T is a G -almost contractive mapping, we have

$$\begin{aligned} \|x_n - x_{n+1}\| &\leq \|x_n - Tx_n\| \leq \|x_n - Tx_{n-1}\| + \|Tx_{n-1} - Tx_n\| \\ &\leq \|x_n - Tx_{n-1}\| + \delta\|x_{n-1}, x_n\| + L\|x_n - Tx_{n-1}\| \\ &\leq \|x_n - Tx_{n-1}\| + \|x_{n-1} - x_n\| + L\|x_n - Tx_{n-1}\| \\ &= \|x_{n-1} - Tx_{n-1}\| + L\|x_n - Tx_{n-1}\| \\ &= \|Tx_{n-2} - Tx_{n-1}\| + L\|x_n - Tx_{n-1}\| \\ &\leq \delta\|x_{n-2} - x_{n-1}\| + L\|x_{n-1} - Tx_{n-2}\| + L\|x_n - Tx_{n-1}\| \\ &= \delta\|x_{n-2} - x_{n-1}\| + L(\|x_n - Tx_{n-1}\|) \\ &= \delta\|x_{n-2} - x_{n-1}\| + L\|x_{n-1} - Tx_{n-1}\| - L\|x_{n-1} - x_n\| \\ &\leq \delta\|x_{n-2} - x_{n-1}\| + L\delta\|x_{n-2}, Tx_{n-1}\| + L^2\|x_{n-1} - Tx_{n-2}\| \\ &= \delta(1 + L)\|x_{n-2} - x_{n-1}\|. \end{aligned}$$

We denote $h = \delta(1 + L) < 1$ and hence

$$\|x_n - x_{n+1}\| \leq h\|x_{n-2} - x_{n-1}\|.$$

Thus, combining Cases 1, 2 and 3, we have

$$\|x_n - x_{n+1}\| \leq h \max\{\|x_{n-2} - x_{n-1}\|, \|x_{n-1} - x_n\|\},$$

for all $n \geq 2$. Now, by induction, it follows that, for any $n > 1$,

$$\|x_n - x_{n+1}\| \leq h^{\lfloor \frac{n}{2} \rfloor} \max\{\|x_0 - x_1\|, \|x_1 - x_2\|\},$$

where $\left[\frac{n}{2}\right]$ denote the greatest integer not exceeding $\frac{n}{2}$. Now, for any $m > n > N$, we obtain

$$\|x_n - x_m\| \leq \sum_{i=N}^{\infty} \|x_i - x_{i-1}\| \leq 2 \frac{\delta^{[\frac{n}{2}]}}{2} \max\{\|x_0 - x_1\|, \|x_1 - x_2\|\}.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since C is closed and $\{x_n\} \subset C$, there exists $x \in C$ such that $\lim_{n \rightarrow \infty} x_n = x$. Let $\{x_{n_k}\} \subset P$ be an infinite subsequence of $\{x_n\}$. Since X has Property A, $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x, Tx) &\leq \|x - x_{n_{k+1}}\| + \|x_{n_{k+1}} - Tx\| \\ &= \|x - x_{n_{k+1}}\| + \|Tx_{n_k} - Tx\| \\ &\leq \|x - x_{n_{k+1}}\| + \delta \|x_{n_k} - x\| + L \|x - Tx_{n_k}\| \\ &= (1 + L) \|x - x_{n_{k+1}}\| + \delta \|x_{n_k} - x\|. \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain

$$\|x - Tx\| = 0$$

and hence x is a fixed point of T . This completes the proof. $\square$

As a consequence of Theorem 3.2, by putting $E(G) = C \times C$, we obtain the following:

Corollary 3.1. *Let C be a nonempty closed subset of a Banach space X and $T : C \rightarrow X$ be an almost contractive mapping with Rothe's boundary condition. Then T has a fixed point.*

Next, we give an example to illustrate Theorem 3.2.

Example 3.1. Let $X = \mathbb{R}$ and $C = [0, \infty)$ with the usual norm $\|x - y\| = |x - y|$ and $G = (V(G), E(G))$ be such that $V(G) = C$,

$$E(G) = \{(1/2^n, 1/2^{n+1}), (1/2^n, 0) : n \geq 2\} \cup \{(1/2, 0), (0, 0)\}.$$

Notice that C has Property A. Define a mapping $T : C \rightarrow X$ by

$$Tx = \begin{cases} 1 & \text{if } x = 1, \\ -\frac{1}{8} & \text{if } x = \frac{1}{2}, \\ \frac{1}{4} & \text{if } x = \frac{3}{8}, \\ \frac{x}{4} & \text{otherwise.} \end{cases}$$

We see that the only $x \in C$ with $Tx \notin C$ is $x = 1/2$, and we see that $P_T(1/2) = \{0\}$ and $(1/2, 0) \in E(G)$, hence $P_T(x)$ is dominated by x . By direct computation, we can show that T is edge-preserving and G -almost contractive mapping with $\delta = \frac{1}{4}$ and $L = 2$. Moreover, we see that

$$T(\partial C) = T(\{0, 1\}) = \{0, 1\} \subset C,$$

which implies T satisfies Rothe's boundary condition. Choose $x_0 = 0 \in \partial C$, we see that $(x_0, Tx_0) = (0, 0) \in E(G)$. Therefore, all conditions of Theorem 3.2 are satisfied and we see that $F(T) = \{0, 1\}$

Noted that T is not almost contractive since, for all $0 \leq k < 1$ and $L \geq 0$,

$$\left\|T\left(\frac{3}{8}\right) - T\left(\frac{1}{4}\right)\right\| = \frac{3}{16} > k \left\|\frac{3}{8} - \frac{1}{4}\right\| + L \left\|\frac{1}{4} - T\left(\frac{3}{8}\right)\right\|.$$

Now, if we apply Theorem 3.2, we can prove the following fixed point theorem for a non-self G -contractive mapping in Banach spaces.

Theorem 3.3. *Let C be a nonempty closed subset of a Banach space X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Suppose that C has Property A. Let $T : C \rightarrow X$ be a G -contractive mapping, edge-preserving and satisfies Rothe's boundary condition*

$$(3.2) \quad T(\partial C) \subset C.$$

Suppose that $P_T(x)$ is dominated by x for all $x \in C$ with $Tx \notin C$. If there exists $x_0 \in \partial C$ such that $(x_0, Tx_0) \in E(G)$. Then T has a fixed point in C . Moreover, if $F(T) \times F(T) \subset E(G)$, then T has a unique fixed point.

Proof. Let $x, y \in C$ with $(x, y) \in E(G)$, by the assumption, we have

$$\|Tx - Ty\| \leq k\|x - y\| = k\|x - y\| + 0\|y - Tx\|.$$

This implies that T is a G -almost contractive mapping, where $\delta = k$ and $L = 0$. Therefore, all the conditions of Theorem 3.2 are satisfied. Then T has a fixed point.

Next, assume that $F(T) \times F(T) \subseteq E(G)$ and let $x, y \in F(T)$. Then $(x, y) \in E(G)$ and

$$\|x - y\| = \|Tx - Ty\| = k\|x - y\|,$$

which implies $\|x - y\| = 0$, that is, $x = y$. Therefore, T has a unique fixed point. This completes the proof. $\square$

As a consequence of Theorem 3.3, by putting $E(G) = C \times C$, we obtain the following:

Corollary 3.2. *Let C be a nonempty closed subset of a Banach space X and $T : C \rightarrow X$ be a contractive mapping with Rothe's boundary condition. Then T has a fixed point.*

Next, we give an example to illustrate Theorem 3.3.

Example 3.2. Let $X = \mathbb{R}$ and $C = [0, 1]$ with the usual norm $\|x - y\| = |x - y|$ and $G = (V(G), E(G))$ be such that $V(G) = C$,

$$E(G) = \{(1/2^n, 1/2^{n+1}), (1/2^n, 0) : n \geq 2\} \cup \{(1/2, 0), (0, 0)\}.$$

Note that C has Property A. Now define the mapping $T : C \rightarrow X$ by

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \neq \frac{1}{2}, \\ -\frac{1}{4}, & \text{if } x = \frac{1}{2}. \end{cases}$$

We see that the only $x \in C$ with $Tx \notin C$ is $x = 1/2$, $P_T(1/2) = \{0\}$ and $(1/2, 0) \in E(G)$ and hence $P_T(x)$ is dominated by x . By direct computation, we can show that T is edge-preserving and G -contractive mapping with $k = \frac{1}{2}$. Moreover,

$$T(\partial C) = T(\{0, 1\}) = \left\{0, \frac{1}{2}\right\} \subset C,$$

which implies that T satisfies Rothe's boundary condition. Choose $x_0 = 0 \in \partial C$, so $(x_0, Tx_0) = (0, 0) \in E(G)$. Therefore, all the conditions of Theorem 3.3 are satisfied and we see that $F(T) = \{0\}$.

Note that T is not a contraction since, for all $0 \leq k < 1$,

$$\left\|T\left(\frac{1}{2}\right) - T\left(\frac{9}{16}\right)\right\| = \left\|-\frac{1}{4} - \frac{9}{32}\right\| = \frac{17}{32} > k\frac{1}{16} = k\left\|\frac{1}{2} - \frac{9}{16}\right\|.$$

REFERENCES

- [1] Agarwal, R. P., O'Regan, D. and Sahu, D. R., *Fixed Point Theory for Lipschitzian Type-Mappings with Applications*, Springer, New York, NY, USA, 2009
- [2] Alfuraidan, M. R., *Fixed points of monotone nonexpansive mappings with a graph*, Fixed Point Theory Appl., 2015, 2015:49, 6 pp.
- [3] Alghamdi, M. A., Berinde, V. and Shahzad, N., *Fixed points of non-self almost contractions*, Carpathian J. Math., **30** (2014), No. 1, 7–14
- [4] Alghamdi, M. A., Berinde, V. and Shahzad, N., *Fixed points of multivalued nonself almost contractions*, J. Appl. Math., 2013, Art. ID 621614, 6 pp.
- [5] Assad, N. A., *A fixed point theorem for some non-self-mappings*, Tamkang J. Math., **21** (1990), 387–393
- [6] Banach, S., *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181
- [7] Bang-Jensen, J. and Gutin, G., *Digraphs Theory, Algorithms and Applications*, Springer Monographs in Mathematics, Springer-Verlag London Ltd., London, 2007
- [8] Beg, I. and Butt, A. R., *Fixed point of set-valued graph contractive mappings*, J. Inequal. Appl., 2013, 2013:252
- [9] Berinde, V. and Păcurar, M., *Fixed point theorems for nonself single-valued almost contractions*, Fixed Point Theory, **14** (2013), No. 2, 301–312
- [10] Berinde, V., *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, **9** (2004), No. 1, 43–53
- [11] Berinde, V., *Approximating fixed points of weak ϕ -contractions using the Picard iteration*, Fixed Point Theory, **4** (2003), No. 2, 131–142
- [12] Berinde, V., *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math., **19** (2003), No. 1, 7–22
- [13] Berinde, V. and Păcurar, M., *Fixed point theorems for nonself single-valued almost contractions*, Fixed Point Theory, **14** (2013), No. 2, 301–311
- [14] Berinde, V. and Păcurar, M., *The contraction principle for nonself mappings on Banach spaces endowed with a graph*, J. Nonlinear Convex Anal., **16** (2015), No. 9, 1925–1936
- [15] Bojor, F., *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal., **75** (2012), 3895–3901
- [16] Jachymski, J., *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373
- [17] Pang, C., Zhang R., Zhang, Q. and Wang, J., *Dominating sets in directed graphs*, Inform. Sci., **180** (2010) 3647–3652
- [18] Takahashi, W., *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000
- [19] Tiammee, J. and Suantai, S., *Coincidence point theorems for graph-preserving multi-valued mappings*, Fixed Point Theory Appl., 2014, 2014:70, 11 pp.

¹SCHOOL OF MATHEMATICS AND STATISTICS
 FACULTY OF SCIENCE AND TECHNOLOGY
 CHIANG MAI RAJABHAT UNIVERSITY
 CHIANG MAI 50300, THAILAND
 E-mail address: jukrapong.benz@gmail.com

²DEPARTMENT OF EDUCATION AND THE RINS
 GYEONGSANG NATIONAL UNIVERSITY
 JINJU 660-701, KOREA
 E-mail address: yjcho@gnu.ac.kr

3.5 On Browder's convergence theorem and Halpern iteration process for G -nonexpansive mappings in Hilbert spaces endowed with graphs

In this paper, we prove Browder's convergence theorem for G -nonexpansive mappings in a Hilbert space with a directed graph. Moreover, we also prove strong convergence of the Halpern iteration process to a fixed point of G -nonexpansive mappings in a Hilbert space endowed with a directed graph. The main results obtained in this paper extend and generalize many well-known results in the literature.

1 Introduction

Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be contraction if there is $0 < k < 1$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. A mapping T is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. We use the notation $F(T)$ to stand for the set of all fixed points of T , i.e., $x \in F(T)$ if and only if $x = Tx$.

The study of contractive-type mappings is a famous topic in a metric fixed point theory. Banach [1] proved a classical theorem, known as the Banach contraction principle, which is a very important tool for solving existence problems in many branches of mathematics and physics.

Theorem 1.1 ([1]) *Let (X, d) be a complete metric space and $T : X \rightarrow X$ a contraction mapping. Then T has a unique fixed point.*

There are many generalizations of the Banach contraction principle in the literature (see [2–4]).

Let $G = (V(G), E(G))$ be a directed graph where $V(G)$ is a set of vertices of graph and $E(G)$ be a set of its edges. Assume that G has no parallel edges. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges. That is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

If x and y are vertices in G , then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^n$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \dots, n$. A graph G is connected if there is a (directed) path between any two vertices of G .

In 2008, Jachymski [5] combined the concept of fixed point theory and graph theory to study fixed point theory in a metric space endowed with a directed graph. He introduced a concept of G -contraction and generalized the Banach contraction principle in a metric space endowed with a directed graph.

Definition 1.2 ([5]) Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, i.e., $\Delta = \{(x, x) : x \in X\} \subseteq E(G)$.

We say that a mapping $f : X \rightarrow X$ is a G -contraction if f preserves edges of G , i.e.,

$$x, y \in X, \quad (x, y) \in E(G) \quad \Rightarrow \quad (f(x), f(y)) \in E(G) \quad (1.1)$$

and there exists $\alpha \in (0, 1)$ such that for any $x, y \in X$,

$$(x, y) \in E(G) \quad \Rightarrow \quad d(f(x), f(y)) \leq \alpha d(x, y).$$

Using this concept, he proved in [5] the following theorem.

Theorem 1.3 ([5]) Let (X, d) be complete, and let a triple (X, d, G) have the following property:

for any $(x_n)_{n \in \mathbb{N}}$ if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$
and there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let f be a G -contraction, and $X_f = \{x \in X : (x, f(x)) \in E(G)\}$. Then $F(T) \neq \emptyset$ if and only if $X_f \neq \emptyset$.

The above theorem has been improved and extended in many ways, see [6–8] for examples.

Let C be a nonempty convex subset of a Banach space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $T : C \rightarrow C$. Then T is said to be G -nonexpansive if the following conditions hold:

- (1) T is edge-preserving, i.e., for any $x, y \in C$ such that $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$;
- (2) $\|Tx - Ty\| \leq \|x - y\|$, whenever $(x, y) \in E(G)$ for any $x, y \in C$.

Example 1.4 Let c be the Banach space of convergent sequences and $k > 1$. Let $G = (X, E(G))$, where $X = c$ and

$$E(G) = \left\{ ((x_n), (y_n)) \mid \text{for all } n \in \mathbb{N}, x_n, y_n \in \mathbb{Z} \text{ and } y_n = x_n + 1, n \geq 2 \right\}.$$

Define a mapping $T : X \rightarrow X$ by

$$T(x_1, x_2, \dots, x_i, \dots) = \begin{cases} (0, x_2, x_3, x_4, \dots) & \text{if } x_n \in \mathbb{Z} \text{ for all } n \in \mathbb{N}, \\ (kx_1, kx_2, kx_3, \dots) & \text{if } x_n \notin \mathbb{Z} \text{ for some } n \in \mathbb{N}. \end{cases}$$

Note that T is G -nonexpansive, but it is not nonexpansive.

We note that $E(G)$ in the above example is not convex in $C \times C$, while $E(G)$ in the following example is convex.

Example 1.5 Let c be a closed unit ball of the space l_1 with the norm $\|\{x_k\}\| = \sum_k |x_k|$. Let $G = (C, E(G))$ be the graph on C defined by

$$E(G) = \left\{ (\{x_k\}, \{y_k\}) : |x_k| + |y_k| \leq 1 \text{ and } \|\{x_k\} - \{y_k\}\| \leq \frac{3}{8} \right\}.$$

It is easy to show that $E(G)$ is convex. Now let $T : C \rightarrow C$ be defined by

$$T(\{x_k\}) = \{x_k^2\}, \quad \{x_k\} \in C.$$

We can easily show that T is G -nonexpansive. However, it is not nonexpansive because $\|Tx - Ty\| > \|x - y\|$ where $\{x\} = \{\frac{1}{2}, 0, 0, \dots\}$ and $\{y\} = \{1, 0, 0, \dots\}$.

The study of fixed point theorems for nonexpansive mappings and the structure of their fixed point sets on both Hilbert and Banach spaces were widely investigated by many authors (see [9–18]). In 1967, Browder [9] proved a strong convergence theorem to a fixed point of a nonexpansive mapping in a Hilbert space by using the Banach contraction principle.

Very recently, in 2015, Alfuraidan [10] proved a fixed point theorem for a G -nonexpansive mapping $T : C \rightarrow C$ in a Banach space X which satisfies the τ -Opial condition and C is a bounded convex τ -compact subset of X .

In this paper, we prove Browder's convergence theorem for a G -nonexpansive mapping in a Hilbert space endowed with a directed graph and we also prove a strong convergence theorem of the Halpern iteration process for this type of mappings.

2 Preliminaries

In this section, we give some basic and useful definitions and well-known results that will be used in the other sections.

Proposition 2.1 ([11]) *Let X be a Hilbert space. For any $x, y \in X$. If $\|x + y\| = \|x\| + \|y\|$, then there exists $t \geq 0$ such that $y = tx$ or $x = ty$.*

Definition 2.2 A sequence $\{x_n\}$ in a Hilbert space X is said to converge weakly to $x \in X$ if $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ for all $y \in X$. In this case, we write $x_n \rightharpoonup x$.

The following useful result is due to [11].

Theorem 2.3 ([11]) *Let X be a Banach space. Then X is reflexive if and only if every closed convex bounded subset C of X is weakly compact, i.e., every bounded sequence in C has a weakly convergent subsequence.*

Let C be a nonempty closed convex subset of a real Hilbert space X . For every point $x \in X$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of X onto C .

The following lemma shows some useful properties of P_C on a Hilbert space.

Lemma 2.4 ([12], Lemma 3.1.2) *Let C be a convex subset of a Hilbert space H and let $x \in H$ and $y \in C$. Then the following are equivalent:*

- (1) $\|x - y\| = d(x, C)$;
- (2) $\langle x - y, y - z \rangle \geq 0$ for every $z \in C$.

Theorem 2.5 ([12]) *Let X be a Hilbert space. Let $\{x_n\}$ be a sequence of X with $x_n \rightharpoonup x$. If $x \neq y$, then*

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|.$$

The following property is useful for our main results.

Property G Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$, where $V(G) = C$, be a directed graph. Then C is said to have Property G if every sequence $\{x_n\}$ in C converging weakly to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Definition 2.6 Let C be a nonempty closed convex subset of a Hilbert space H and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then T is said to be G -monotone if $\langle Tx - Ty, x - y \rangle \geq 0$ whenever $(x, y) \in E(G)$ for any $x, y \in C$.

In order to obtain our main result, we need some basic definitions of domination in graphs [19, 20].

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if every $v \in V(G) \setminus X$ there exists $x \in X$ such that $(x, v) \in E(G)$ and we say that x dominates v or v is dominated by x . Let $v \in V$, a set $X \subseteq V$ is dominated by v if $(v, x) \in E(G)$ for any $x \in X$ and we say that X dominates v if $(x, v) \in E(G)$ for all $x \in X$. In this paper, we always assume that $E(G)$ contains all loops.

3 Main result

In this section, we prove a fixed point theorem for G -nonexpansive mapping in a Hilbert space endowed with a directed graph. First, we begin with the property of G -nonexpansive mapping and the structure of its fixed point set.

Lemma 3.1 *Let X be a normed space and $G = (V(G), E(G))$ a directed graph with $V(G) = X$. Suppose $T : X \rightarrow X$ is a G -nonexpansive mapping. If X has a Property G , then T is continuous.*

Proof Let $\{x_n\}$ be a sequence in X such that $x_n \rightarrow x$. We will show that $Tx_n \rightarrow Tx$. To show this, let $\{Tx_{n_k}\}$ be a subsequence of $\{Tx_n\}$. Since $x_{n_k} \rightarrow x$, by Property G , there is a subsequence (x_{m_k}) such that $(x_{m_k}, x) \in E(G)$ for each $k \in \mathbb{N}$. Since T is G -nonexpansive and $(x_{m_k}, x) \in E(G)$, we obtain

$$\|Tx_{m_k} - Tx\| \leq \|x_{m_k} - x\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence $Tx_{m_k} \rightarrow Tx$. By the double extract subsequence principle, we conclude that $Tx_n \rightarrow Tx$. Therefore T is continuous. $\square$

We now discuss the structure of the fixed point set of G -nonexpansive mappings.

Theorem 3.2 *Let X be a normed space and C be a subset of X having Property G . Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose $T : C \rightarrow C$ is a G -nonexpansive mapping and $F(T) \times F(T) \subseteq E(G)$. Then $F(T)$ is closed and convex.*

Proof Suppose $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \rightarrow x$. Since C has Property G , there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Since T is G -nonexpansive, we obtain

$$\begin{aligned} \|x - Tx\| &\leq \|x - x_{n_k}\| + \|x_{n_k} - Tx\| \\ &= \|x - x_{n_k}\| + \|Tx_{n_k} - Tx\| \\ &= \|x - x_{n_k}\| + \|x_{n_k} - x\| \rightarrow 0. \end{aligned}$$

Therefore $x = Tx$, i.e., $x \in F(T)$. This shows that $F(T)$ is closed.

Next, we will show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\lambda \in [0, 1]$. Then $(x, x), (x, y) \in E(G)$. Denote $z = \lambda x + (1 - \lambda)y$. Since $E(G)$ is convex, we obtain

$$(x, z) = (\lambda x + (1 - \lambda)x, \lambda x + (1 - \lambda)y) \in E(G).$$

Similarly, we also have $(y, z) \in E(G)$. Since T is G -nonexpansive, we obtain

$$\|x - Tz\| = \|Tx - Tz\| \leq \|x - z\| = (1 - \lambda)\|x - y\| \quad (3.1)$$

and

$$\|y - Tz\| = \|Ty - Tz\| \leq \|y - z\| = \lambda\|x - y\|. \quad (3.2)$$

Hence

$$\begin{aligned} \|x - y\| &= \|(x - Tz) + (Tz - y)\| \\ &\leq \|x - Tz\| + \|Tz - y\| \\ &\leq \|x - z\| + \|y - z\| = \|x - y\|. \end{aligned}$$

This implies that $\|x - y\| = \|x - Tz\| + \|Tz - y\| = \|x - z\| + \|y - z\|$ and

$$\|(x - Tz) + (Tz - y)\| = \|x - Tz\| + \|Tz - y\|.$$

By (3.1) and (3.2), we can conclude that

$$\|x - Tz\| = \|x - z\| \quad \text{and} \quad \|y - Tz\| = \|y - z\|.$$

By Proposition 2.1, there exists $t \geq 0$ such that $x - Tz = t(Tz - y)$, so

$$Tz = \beta x + (1 - \beta)y, \quad \text{where } \beta = \frac{1}{1+t}.$$

Hence $x - Tz = (1 - \beta)(x - y) = \frac{1-\beta}{1-\beta}(x - z)$, which implies that $x - Tz = x - z$. Therefore $z = Tz$, i.e., $z \in F(T)$. Thus $F(T)$ is convex. $\square$

Proposition 3.3 *Let C be a nonempty closed convex subset of a Hilbert space H and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. If T is G -nonexpansive, then $I - T$ is G -monotone, where I is the identity mapping on C .*

Proof Let $x, y \in C$ be such that $(x, y) \in E(G)$. By the Cauchy-Schwarz inequality and G -nonexpansiveness of T , we have

$$\begin{aligned} 0 &\leq \|Tx - Ty\| \|x - y\| - \langle Tx - Ty, x - y \rangle \\ &\leq \|x - y\|^2 - \langle Tx - Ty, x - y \rangle \\ &= \langle x - y, x - y \rangle - \langle Tx - Ty, x - y \rangle \\ &= \langle (x - y) - (Tx - Ty), x - y \rangle \\ &= \langle (I - T)x - (I - T)y, x - y \rangle. \end{aligned}$$

Hence $I - T$ is G -monotone. $\square$

Next, we prove a Browder's fixed point theorem for a G -nonexpansive mapping.

Theorem 3.4 *Let C be a bounded closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose C has Property G. Let $T : C \rightarrow C$ be a G -nonexpansive. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Define $T_n : C \rightarrow C$ by*

$$T_n x = (1 - \alpha_n)Tx + \alpha_n x_0$$

for each $x \in C$ and $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \rightarrow 0$. Then the following hold:

- (i) T_n has a fixed point $u_n \in C$;
- (ii) $F(T) \neq \emptyset$;
- (iii) if $F(T) \times F(T) \subseteq E(G)$ and Px_0 is dominated by $\{u_n\}$, then the sequence $\{u_n\}$ converges strongly to $w_0 = Px_0$ where P is the metric projection onto $F(T)$.

Proof (i) Let x_0 be such that $(x_0, Tx_0) \in E(G)$. We first show that T_n is G -contraction for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $x, y \in C$ such that $(x, y) \in E(G)$. Since T is G -nonexpansive, we obtain

$$\|T_n x - T_n y\| = (1 - \alpha_n)\|Tx - Ty\| \leq (1 - \alpha_n)\|x - y\|.$$

Since T is edge-preserving, $(Tx, Ty) \in E(G)$. By convexity of $E(G)$, we have

$$(T_n x, T_n y) = ((1 - \alpha_n)Tx + \alpha_n x_0, (1 - \alpha_n)Ty + \alpha_n x_0) \in E(G).$$

Therefore T_n is G -contraction. For any sequence $\{x_n\}$ in C such that $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$, by Property G of C , there is a subsequence (x_{n_k}) such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$. Since $E(G)$ is convex and $(x_0, x_0) \in E(G)$, we have

$$(x_0, T_n x_0) = ((1 - \alpha_n)x_0 + \alpha_n x_0, (1 - \alpha_n)Tx_0 + \alpha_n x_0) \in E(G).$$

Therefore all conditions of Theorem 1.3 are satisfied, so T_n has a fixed point, *i.e.*, $u_n = T_n u_n$.

(ii) We will show that $F(T) \neq \emptyset$. Since $\{u_n\}$ is bounded, by Theorem 2.3, there is a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup v$ for some $v \in C$. Suppose $Tv \neq v$. By Property G , without loss of generality, we may assume that $(u_{n_i}, v) \in E(G)$ for all $i \in \mathbb{N}$. Since $u_{n_i} - Tu_{n_i} \rightarrow 0$ as $i \rightarrow \infty$, by Theorem 2.5, we have

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|u_{n_i} - v\| &< \liminf_{i \rightarrow \infty} \|u_{n_i} - Tv\| \\ &= \liminf_{i \rightarrow \infty} \|u_{n_i} - Tu_{n_i} + Tu_{n_i} - Tv\| \\ &= \liminf_{i \rightarrow \infty} \|Tu_{n_i} - Tv\| \\ &\leq \liminf_{i \rightarrow \infty} \|u_{n_i} - v\|, \end{aligned}$$

which is a contradiction. Hence $Tv = v$.

(iii) Next, assume that $F(T) \times F(T) \subseteq E(G)$ and $\{Px_0\}$ is dominated by $\{u_n\}$. We will show that $u_n \rightarrow w_0 = Px_0$. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$, we denote $v_i = u_{n_i}$. For each i , v_i is a fixed point of T_{n_i} . Hence we have

$$\alpha_{n_i} v_i + (1 - \alpha_{n_i})(v_i - Tv_i) = \alpha_{n_i} x_0.$$

Since w_0 is a fixed point of T , we have

$$\alpha_{n_i} w_0 + (1 - \alpha_{n_i})(w_0 - Tw_0) = \alpha_{n_i} w_0.$$

If we subtract these two equations and take the inner product of the difference with $v_i - w_0$, we obtain

$$\alpha_{n_i} \langle v_i - w_0, v_i - w_0 \rangle + (1 - \alpha_{n_i}) \langle Uv_i - Uw_0, v_i - w_0 \rangle = \alpha_{n_i} \langle x_0 - w_0, v_i - w_0 \rangle, \quad (3.3)$$

where $U = I - T$ and I is the identity map. Since Px_0 is dominated by $\{u_n\}$, we obtain $(v_i, w_0) \in E(G)$ for all $i \in \mathbb{N}$. By Proposition 3.3, U is G -monotone, so $\langle Uv_i - Uw_0, v_i - w_0 \rangle \geq 0$ for all $i \in \mathbb{N}$. This together with (3.1) shows

$$\alpha_{n_i} \|v_i - w_0\|^2 \leq \alpha_{n_i} \langle x_0 - w_0, v_i - w_0 \rangle.$$

Hence

$$\begin{aligned} \|v_i - w_0\|^2 &\leq \langle x_0 - w_0, v_i - w_0 \rangle \\ &= \langle x_0 - w_0, v - w_0 \rangle + \langle x_0 - w_0, v_i - v \rangle. \end{aligned}$$

By Lemma 2.4, we know that $\langle x_0 - w_0, v - w_0 \rangle \leq 0$, so we get

$$\|v_i - w_0\|^2 \leq \langle x_0 - w_0, v_i - v \rangle \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

because $v_i \rightarrow v$. Hence $v_i \rightarrow w_0 = Px_0$. By the double extract subsequence principle, we can conclude that $u_n \rightarrow w_0 = Px_0$. $\square$

Next, we give an example which supports Theorem 3.4.

Example 3.5 Let $H = \mathbb{R}$ and $C = [0, \frac{1}{2}]$ with the usual norm $\|x - y\| = |x - y|$ and let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(x, y) : x, y \in [0, \frac{3}{8}] \text{ such that } |x - y| \leq \frac{1}{8}\}$. Define $T : C \rightarrow C$ by

$$Tx = \begin{cases} \frac{8}{6}x^2 & \text{if } x \in [0, \frac{1}{2}), \\ \frac{25}{64} & \text{if } x = \frac{1}{2}. \end{cases}$$

Proof We see that $F(T) = \{0\}$. Choose $x_0 = \frac{1}{8}$, so $(x_0, Tx_0) \in E(G)$. It is easy to see that $E(G)$ is convex. Let $(x, y) \in E(G)$. Then $x, y \in [0, \frac{3}{8}]$ and $|x - y| \leq \frac{1}{8}$. So, we have $|Tx - Ty| = \frac{8}{6}|x^2 - y^2| \leq \frac{8}{6}|x + y||x - y| \leq |x - y| \leq \frac{1}{8}$, which implies that $(Tx, Ty) \in E(G)$ and $\|Tx - Ty\| \leq \|x - y\|$. Thus T is G -nonexpansive. Next, for each $n \in \mathbb{N}$, define $T_n : C \rightarrow C$ by

$$T_n x = \frac{1}{8(n+5)} + \left(1 - \frac{1}{n+5}\right)Tx.$$

Then the unique fixed point of T_n is $u_n = \frac{3n+15-\sqrt{3\sqrt{3n^2+28n+67}}}{8(n+4)}$. By using elementary calculus, we can show that $u_n \leq \frac{1}{8}$ for all $n \in \mathbb{N}$. Thus $(u_n, Px_0) = (u_n, 0) \in E(G)$, i.e., Px_0 is dominated by $\{u_n\}$ and $u_n \rightarrow 0 = Px_0$ as $n \rightarrow \infty$. $\square$

It is noted that T is not nonexpansive because

$$\left\|T\left(\frac{1}{2}\right) - T\left(\frac{3}{8}\right)\right\| = \left\|\frac{25}{64} - \frac{3}{16}\right\| = \frac{13}{64} > \frac{1}{8} = \left\|\frac{1}{2} - \frac{3}{8}\right\|.$$

Open question It is noted that the set C in the above example has no Property G but we still have the Browder convergence theorem for a G -nonexpansive mapping T . Is it possible to obtain Theorem 3.4 with a property which is weaker than the Property G or without the Property G ?

As a consequence of Theorem 3.4, by putting $E(G) = C \times C$, we obtain the Browder convergence theorem.

Corollary 3.6 ([9]) *Let C be a bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Let x_0 be an arbitrary point of C and define $T_n : C \rightarrow C$ by*

$$T_n = \left(1 - \frac{1}{n}\right)Tx + \frac{1}{n}x_0$$

for each $x \in C$ and $n \in \mathbb{N}$. Then the following hold:

- T_n has a unique fixed point u_n in C ;
- the sequence $\{u_n\}$ converges strongly to $Px_0 \in F(T)$, where P is the metric projection onto $F(T)$.

4 Convergence of Halpern iteration process

In this section, we prove strong convergence of Halpern iteration process for G -nonexpansive mappings in a Hilbert space endowed with a graph.

Definition 4.1 ([13]) Let C be a nonempty convex subset of a linear space and $T : C \rightarrow C$ a mapping. Let $u \in C$ and $\{\alpha_n\}$ be a sequence in $[0, 1]$. Then a sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \end{cases} \quad (4.1)$$

is called the Halpern iteration.

In 1992, Wittmann [14] proved the strong convergence of the Halpern iteration for a nonexpansive mapping in a Hilbert space and $\{\alpha_n\}$ satisfies

$$\alpha_n \in [0, 1], \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \quad (4.2)$$

The following is also useful for proving our main result.

Lemma 4.2 ([16]) Let (s_n) be a sequence of non-negative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad n \geq 0,$$

where (α_n) , (β_n) , and (γ_n) satisfy the conditions:

1. $(\alpha_n) \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, or equivalently, $\prod_{n=1}^{\infty} (1 - \alpha_n) = 0$;
2. $\limsup_{n \rightarrow \infty} \beta_n \leq 0$;
3. $\gamma_n \geq 0$ for all $n \geq 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Definition 4.3 Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, then $(x, z) \in E(G)$.

The following result is needed for proving strong convergence of Halpern iteration process for G -nonexpansive mapping in Hilbert spaces endowed with a directed graph.

Proposition 4.4 Let C be a convex subset of a vector space X and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let G be transitive and $T : C \rightarrow C$ be edge-preserving. Let $\{x_n\}$ be a sequence defined by (4.1), where $u = x_0$ and $(x_0, Tx_0) \in E(G)$. If $\{x_n\}$ dominates x_0 , then (x_n, x_{n+1}) , (x_0, x_n) , and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$.

Proof We prove by induction. Since $E(G)$ is convex, (x_0, x_0) and (x_0, Tx_0) are in $E(G)$, we have $(x_0, x_1) \in E(G)$. Then $(Tx_0, Tx_1) \in E(G)$, since T is edge-preserving. Because G

is transitive, we have $(x_0, Tx_1) \in E(G)$. By convexity of $E(G)$ and $(x_0, Tx_1), (Tx_0, Tx_1) \in E(G)$, we get $(x_1, Tx_1) \in E(G)$. By assumption, $(x_1, x_0) \in E(G)$. So, by convexity of $E(G)$, we get $(x_1, x_2) \in E(G)$. Next, assume that $(x_k, x_{k+1}), (x_0, Tx_k)$, and (x_k, Tx_k) are in $E(G)$. Then $(Tx_k, Tx_{k+1}) \in E(G)$, since T is edge-preserving. By transitivity of G , we have $(x_0, Tx_{k+1}) \in E(G)$. By convexity of $E(G)$ and $(x_0, Tx_{k+1}), (Tx_k, Tx_{k+1}) \in E(G)$, we get $(x_{k+1}, Tx_{k+1}) \in E(G)$. Since $\{x_0\}$ is dominated by $\{x_n\}$, we have $(x_{k+1}, x_0) \in E(G)$. By convexity of $E(G)$, we get $(x_{k+1}, x_{k+2}) \in E(G)$. So, by induction, we can conclude that $(x_n, x_{n+1}), (x_0, x_n)$, and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$. $\square$

We now ready to prove the strong convergence theorem.

Theorem 4.5 *Let C be a nonempty closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has Property G . Let $T : C \rightarrow C$ be a G -nonexpansive mapping. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{\alpha_n\}$ be a sequence satisfying (4.2). Let $\{x_n\}$ be a sequence defined by Halpern iteration, where $u = x_0$. If $\{x_n\}$ is dominated by Px_0 and $\{x_n\}$ dominates x_0 , then $\{x_n\}$ converges strongly to Px_0 , where P is the metric projection on $F(T)$.*

Proof Let $z_0 = Px_0$. From Proposition 4.4, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. First we will show that $\{x_n\}$ is bounded. Since $z_0 \in F(T)$ and $z_0 = Px_0$ is dominated by $\{x_n\}$, we have $(x_n, z_0) \in E(G)$, we get

$$\begin{aligned} \|x_{n+1} - z_0\| &\leq (1 - \alpha_n)\|Tx_n - z_0\| + \alpha_n\|x_0 - z_0\| \\ &= (1 - \alpha_n)\|Tx_n - Tx_0\| + \alpha_n\|x_0 - z_0\| \\ &\leq (1 - \alpha_n)\|x_n - z_0\| + \alpha_n\|x_0 - z_0\| \\ &\leq \max\{\|x_n - z_0\|, \|x_0 - z_0\|\} \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded. Moreover, $\{Tx_n\}$ is bounded. By (4.1) and $(x_n, x_{n+1}) \in E(G)$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}|(\|x_0\| + \|Tx_{n-1}\|) + (1 - \alpha_n)\|x_n - x_{n-1}\| \\ &\leq \|\alpha_n - \alpha_{n-1}\|K + (1 - \alpha_n)\|x_n - x_{n-1}\|, \end{aligned} \tag{4.3}$$

where $K = \sup\{\|x_0\| + \|Tx_n\| : n \in \mathbb{N}\}$. By using (4.3), for $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_{n+m+1} - x_{n+m}\| &\leq \left(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k| \right) K + \left(\prod_{k=m}^{n+m-1} |1 - \alpha_{k+1}| \right) \|x_{m+1} - x_m\| \\ &\leq \left(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k| \right) K + \exp\left(- \sum_{k=m}^{n+m-1} \alpha_{k+1} \right) \|x_{m+1} - x_m\|. \end{aligned}$$

Since $\{x_n\}$ is bounded and $\sum_{k=0}^{\infty} \alpha_k = \infty$, we obtain

$$\limsup_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \limsup_{n \rightarrow \infty} \|x_{n+m+1} - x_{n+m}\| \leq \left(\sum_{k=m}^{\infty} |\alpha_{k+1} - \alpha_k| \right) K$$

for all $m \in \mathbb{N}$. Hence, by $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, we get

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (4.4)$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n \|x_0 - Tx_n\|. \end{aligned}$$

Because $\{Tx_n\}$ is bounded with (4.4), we obtain

$$\|x_n - Tx_n\| \rightarrow 0 \quad (4.5)$$

as $n \rightarrow \infty$. We next show that

$$\limsup_{n \rightarrow \infty} \langle x_n - z_0, x_0 - z_0 \rangle \leq 0.$$

Indeed, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle x_n - z_0, x_0 - z_0 \rangle = \lim_{k \rightarrow \infty} \langle x_{n_k} - z_0, x_0 - z_0 \rangle.$$

Because all the x_{n_k} lie in the weakly compact set C and C has Property G , we may assume without loss of generality that $x_{n_k} \rightharpoonup y$ for some $y \in C$ and $(x_{n_k}, y) \in E(G)$. Suppose $y \neq Ty$. By Theorem 2.5, (4.5), and G -nonexpansiveness of T , we get

$$\begin{aligned} \liminf_{k \rightarrow \infty} \|x_{n_k} - y\| &< \liminf_{k \rightarrow \infty} \|x_{n_k} - Ty\| \\ &\leq \liminf_{k \rightarrow \infty} (\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Ty\|) \\ &= \liminf_{k \rightarrow \infty} \|Tx_{n_k} - Ty\| \\ &\leq \liminf_{k \rightarrow \infty} \|x_{n_k} - y\|, \end{aligned}$$

which is a contradiction. So $y = Ty$. Hence, by Lemma 2.4, we get

$$\lim_{k \rightarrow \infty} \langle x_{n_k} - z_0, x_0 - z_0 \rangle = \langle y - z_0, x_0 - z_0 \rangle \leq 0. \quad (4.6)$$

Therefore $\limsup_{n \rightarrow \infty} \langle x_n - z_0, x_0 - z_0 \rangle \leq 0$.

Since $(1 - \alpha_n)(Tx_n - z_0) = (x_{n+1} - z_0) - \alpha_n(x_0 - z_0)$, we have

$$\begin{aligned} \|(1 - \alpha_n)(Tx_n - z_0)\|^2 &= \|x_{n+1} - z_0\|^2 + \alpha_n^2 \|x_0 - z_0\|^2 - 2\alpha_n \langle x_{n+1} - z_0, x_0 - z_0 \rangle \\ &\geq \|x_{n+1} - z_0\|^2 - 2\alpha_n \langle x_{n+1} - z_0, x_0 - z_0 \rangle. \end{aligned}$$

This implies, by G -nonexpansiveness of T and $(z_0, x_n) \in E(G)$, that

$$\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle x_{n+1} - z_0, x_0 - z_0 \rangle$$

for each $n \in \mathbb{N}$. By Lemma 4.2, we can conclude that

$$\lim_{n \rightarrow \infty} \|x_n - z_0\|^2 = 0.$$

Therefore $\{x_n\}$ converges strongly to $z_0 = Px_0$. □

References

1. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **3**, 133-181 (1922)
2. Nadler, SB Jr.: Multi-valued contraction mappings. *Pac. J. Math.* **30**, 475-488 (1969)
3. Kirk, WA: A fixed point theorem for mappings which do not increase distances. *Am. Math. Mon.* **72**, 1004-1006 (1965)
4. Zamfirescu, T: Fixed point theorems in metric spaces. *Arch. Math.* **23**(1), 292-298 (1972)
5. Jachymski, J: The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **136**(4), 1359-1373 (2008)
6. Bojor, F: Fixed point theorems for Reich type contractions on metric spaces with a graph. *Nonlinear Anal.* **75**, 3895-3901 (2012)
7. Dinevari, T, Frigon, M: Fixed point results for multivalued contractions on a metric space with a graph. *J. Math. Anal. Appl.* **405**, 507-517 (2013)
8. Sultana, A, Vetrivel, V: Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications. *J. Math. Anal. Appl.* **417**, 336-344 (2014)
9. Browder, FE: Convergence of approximants to fixed points of non-expansive maps in Banach spaces. *Arch. Ration. Mech. Anal.* **24**, 82-90 (1967)
10. Alfuraidan, MR: Fixed points of monotone nonexpansive mappings with a graph. *Fixed Point Theory Appl.* **2015**, 49 (2015). doi:10.1186/s13663-015-0299-0
11. Agarwal, RP, O'Regan, D, Sahu, DR: *Fixed Point Theory for Lipschitzian Type-Mappings with Applications*. Springer, New York (2009)
12. Takahashi, W: *Nonlinear Functional Analysis*. Yokohama Publishers, Yokohama (2000)
13. Halpern, B: Fixed points of nonexpansive maps. *Bull. Am. Math. Soc.* **73**, 957-961 (1967)
14. Wittmann, R: Approximation of fixed points of nonexpansive mappings. *Arch. Math.* **58**, 486-491 (1992)
15. Lorentz, GG: A contribution to the theory of divergent series. *Acta Math.* **80**, 167-190 (1948)
16. Xu, HK: Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66**, 240-256 (2002)
17. Yao, Z, Zhu, LJ, Liou, YC: Strong convergence of a Halpern-type iteration algorithm for fixed point problems in Banach spaces. *J. Nonlinear Sci. Appl.* **8**(5), 489-495 (2015)
18. Li, P, Kang, SM, Zhu, LJ: Visco-resolvent algorithms for monotone operators and nonexpansive mappings. *J. Nonlinear Sci. Appl.* **7**(5), 325-344 (2014)
19. Pang, C, Zhang, R, Zhang, Q, Wang, J: Dominating sets in directed graphs. *Inf. Sci.* **180**, 3647-3652 (2010)
20. Bang-Jensen, J, Gutin, G: *Digraphs Theory, Algorithms and Applications*. Springer Monographs in Mathematics. Springer, London (2007)

3.6 A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs

In this section, we prove the weak and strong convergence of a sequence generated by a modified S-iteration process for finding a common fixed point of two G-nonexpansive mappings in a uniformly convex Banach space with a directed graph. We also give some numerical examples for supporting our main theorem and compare convergence rate between the studied iteration and the Ishikawa iteration.

1 Introduction

In 1922, Banach [1] established the famous fixed point result of a contractive mapping in complete metric spaces, known as Banach's contraction principle, which is an important tool for solving the existence problems of nonlinear mappings. Since then,

many generalizations of this principle have been studied in several directions. In the last 40 years, many authors studied various iteration schemes for several classes of contractive and nonexpansive mappings. In 2007, Agarwal et al. [2] introduced and studied the S-iteration process for a class of nearly asymptotically nonexpansive mappings in Banach spaces and this scheme has a better convergence rate than Ishikawa iteration for a class of contractions in metric spaces. In 2008, by combination of the concepts in fixed point theory and graph theory, Jachymski [3] generalized the Banach's contraction principle in a complete metric space endowed with a directed graph. In 2012, Aleomraninejad et al. [4] presented some iterative scheme for G-contraction and G-nonexpansive mappings in a Banach space with a graph. Recently, Alfuraidan [5] studied the existence of fixed

of monotone nonexpansive mappings on a Banach space endowed with a directed graph. After that, Alfuraidan and Khamsi [6] defined the concept of G -monotone nonexpansive multivalued mappings defined on a hyperbolic metric space with a graph. In 2015, Tiammee et al. [7] proved Browder's convergence theorem for G -nonexpansive mapping in a Hilbert space with a directed graph. They also proved the strong convergence of the Halpern iteration for a G -nonexpansive mapping. Recently, Tripak [8] proved weak and strong convergence of the Ishikawa iteration for G -nonexpansive mappings in a Banach space with a directed graph.

Motivated by [2] and [8], we study the following modified S -iteration process: $x_0 \in C$ and

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n S_1 x_n, \\ x_{n+1} &= (1 - \alpha_n)S_1 x_n + \alpha_n S_2 y_n, \quad n \geq 0, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $S_1, S_2 : C \rightarrow C$ are G -nonexpansive mappings. We prove, under some certain conditions, weak and strong convergence theorems of a modified S -iteration process for approximating common fixed points of two G -nonexpansive mappings in a uniformly convex Banach space X endowed with a graph.

2 Preliminaries and lemmas

In this section, we provide and recall some definitions and lemmas which will be used in the later sections.

Let C be a nonempty subset of a real Banach space X . Let Δ denote the diagonal of the cartesian product $C \times C$, i.e., $\Delta = \{(x, x) : x \in C\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C , and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edge. So we can identify the graph G with the pair $(V(G), E(G))$. A mapping $S : C \rightarrow C$ is said to be

1. *G*-contraction if S satisfies the conditions:

- (i) S preserves edges of G (or S is edge-preserving), i.e.,

$$(x, y) \in E(G) \Rightarrow (Sx, Sy) \in E(G),$$

- (ii) S decreases weights of edges of G in the following way: there exists $\alpha \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow \|Sx - Sy\| \leq \alpha \|x - y\|,$$

2. G -nonexpansive if S satisfies the conditions:

- (i) S preserves edges of G , i.e.,

$$(x, y) \in E(G) \Rightarrow (Sx, Sy) \in E(G),$$

- (ii) S non-increases weights of edges of G in the following way:

$$(x, y) \in E(G) \Rightarrow \|Sx - Sy\| \leq \|x - y\|.$$

Definition 1 The conversion of a graph G is the graph obtained from G by reversing the direction of edges denoted by G^{-1} , and

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Definition 2 Let x and y be vertices of a graph G . A path in G from x to y of length N ($N \in \mathbb{N} \cup \{0\}$) is a sequence $\{x_i\}_{i=0}^N$ of $N + 1$ vertices for which $x_0 = x$, $x_N = y$, and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \dots, N - 1$.

Definition 3 A graph G is said to be connected if there is a path between any two vertices of the graph G .

Definition 4 A directed graph $G = (V(G), E(G))$ is said to be transitive if, for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, we have $(x, z) \in E(G)$.

Definition 5 Let $x_0 \in V(G)$ and A a subset of $V(G)$. We say that

- (i) A is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in A$.
(ii) A dominates x_0 if for each $x \in A$, $(x, x_0) \in E(G)$.

Definition 6 A Banach space X is said to satisfy Opial's property if whenever a sequence $\{x_n\}$ in X converges weakly to $x \in X$, then

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

for all $y \in X$, $y \neq x$.

Definition 7 Let C be a nonempty subset of a Banach space X and let $S : C \rightarrow X$ be a mapping. Then, S is said to be G -demiclosed at $y \in X$ if, for any sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to $x \in C$, $\{Sx_n\}$ converges strongly to y and $(x_n, x_{n+1}) \in E(G)$ imply $Sx = y$.

The following property is useful for our main results.

Property WG (SG) Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then, C is said to have

Property WG (SG) if for each sequence $\{x_n\}$ in C converging weakly (strongly) to $x \in C$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Definition 8 [9] Let C be a nonempty closed convex subset of a real uniformly convex Banach space X . The mappings S_i ($i = 1, 2$) on C are said to satisfy Condition B if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that, for all $x \in C$,

$$\max\{\|x - S_1x\|, \|x - S_2x\|\} \geq f(d(x, F)),$$

where $F = F(S_1) \cap F(S_2)$ and $F(S_i)$ ($i = 1, 2$) are the sets of fixed points of S_i .

Definition 9 [9] Let C be a subset of a metric space (X, d) . A mapping $S : C \rightarrow C$ is semicompact if for a sequence $\{x_n\}$ in C with $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow p \in C$.

Lemma 1 [10] Let X be a Banach space, and $R > 1$ be a fixed number. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_R(0) = \{x \in X : \|x\| \leq R\}$ and $\lambda \in [0, 1]$.

Lemma 2 [11] Let X be a uniformly convex Banach space, and $\{\alpha_n\}$ a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq c$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq c$ and

$\limsup_{n \rightarrow \infty} \|\alpha_n x_n + (1 - \alpha_n)y_n\| = c$ for some $c \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 3 [12] Let X be a Banach space that satisfies Opial's property, and let $\{x_n\}$ be a sequence in X . Let x, y in X be such that $\lim_{n \rightarrow \infty} \|x_n - x\|$ and $\lim_{n \rightarrow \infty} \|x_n - y\|$ exist. If $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$ that converge weakly to x and y , respectively, then $x = y$.

3 Main results

Unless otherwise stated, we shall suppose throughout this section that C is a nonempty closed convex subset of a Banach space X with a directed graph $G = (V(G), E(G))$ such that $V(G) = C$ and $E(G)$ is convex. We also suppose that the graph G is transitive. The mappings S_i ($i = 1, 2$) : $C \rightarrow C$ are G -nonexpansive mappings with $F = F(S_1) \cap F(S_2) \neq \emptyset$. For arbitrary $x_0 \in C$, let $\{x_n\}$ be the sequence generated by (1).

Proposition 1 *Let $z_0 \in F$ be such that $(x_0, z_0), (z_0, x_0)$ are in $E(G)$. Then $(x_n, z_0), (y_n, z_0), (z_0, x_n), (z_0, y_n), (x_n, y_n)$, and (x_n, x_{n+1}) are in $E(G)$.*

Proof We proceed by induction. Since S_1 is edge-preserving and $(x_0, z_0) \in E(G)$, we have $(S_1x_0, z_0) \in E(G)$ and so $(y_0, z_0) \in E(G)$, by $E(G)$ is convex. Then, since S_2 is edge-preserving and $(y_0, z_0) \in E(G)$, we get $(S_2y_0, z_0) \in E(G)$. By the convexity of $E(G)$ and $(S_1x_0, z_0), (S_2y_0, z_0) \in E(G)$, we get $(x_1, z_0) \in E(G)$. Thus, by edge-preserving of S_1 , $(S_1x_1, z_0) \in E(G)$. Again, by the convexity of $E(G)$ and $(S_1x_1, z_0), (x_1, z_0) \in E(G)$, we have $(y_1, z_0) \in E(G)$ and hence, $(S_2y_1, z_0) \in E(G)$. Next, we assume that $(x_k, z_0) \in E(G)$. Since S_1 is edge-preserving, we get $(S_1x_k, z_0) \in E(G)$ and hence, $(y_k, z_0) \in E(G)$, since $E(G)$ is convex. Since S_2 is edge-preserving, we have $(S_2y_k, z_0) \in E(G)$. By the convexity of $E(G)$, we get $(x_{k+1}, z_0) \in E(G)$. Hence, by edge-preserving of S_1 , we obtain $(S_1x_{k+1}, z_0) \in E(G)$, and so $(y_{k+1}, z_0) \in E(G)$, since $E(G)$ is convex. Therefore, $(x_n, z_0), (y_n, z_0) \in E(G)$ for all $n \geq 1$. Using a similar argument, we can show that $(z_0, x_n), (z_0, y_n) \in E(G)$ under the assumption that $(z_0, x_0) \in E(G)$. By the transitivity of G , we get $(x_n, y_n), (x_n, x_{n+1}) \in E(G)$. This completes the proof. $\square$

Lemma 4 *If X is a uniformly convex Banach space, $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, and $(x_0, z_0), (z_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $z_0 \in F$, then*

- (i) $\|x_{n+1} - z_0\| \leq \|x_n - z_0\|$ for all $n \in \mathbb{N}$, and hence $\lim_{n \rightarrow \infty} \|x_n - z_0\|$ exists;
(ii) $\lim_{n \rightarrow \infty} \|x_n - S_1x_n\| = 0 = \lim_{n \rightarrow \infty} \|x_n - S_2x_n\|$.

Proof Let $z_0 \in F$. By Proposition 1, $(x_n, z_0), (y_n, z_0) \in E(G)$. Then, by G -nonexpansiveness of $S_i (i = 1, 2)$, we have

$$\begin{aligned} \|x_{n+1} - z_0\| &= \|(1 - \alpha_n)S_1x_n + \alpha_nS_2y_n - z_0\| \\ &\leq (1 - \alpha_n)\|S_1x_n - z_0\| + \alpha_n\|S_2y_n - z_0\| \\ &\leq (1 - \alpha_n)\|x_n - z_0\| + \alpha_n\|y_n - z_0\| \\ &= (1 - \alpha_n)\|x_n - z_0\| + \alpha_n\|(1 - \beta_n)x_n + \beta_nS_1x_n - z_0\| \\ &\leq (1 - \alpha_n)\|x_n - z_0\| + \alpha_n\beta_n\|x_n - z_0\| + \alpha_n(1 - \beta_n)\|x_n - z_0\| \\ &= (1 - \alpha_n)\|x_n - z_0\| + \alpha_n\|x_n - z_0\| \\ &= \|x_n - z_0\|. \end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \|x_n - z_0\|$ exists and thus, $\{x_n\}$ is bounded and this proves part (i). We next prove part (ii). Since $\{x_n\}$ and $\{y_n\}$ are bounded, there exists $R > 0$ such that $x_n - z_0, y_n - z_0 \in B_R(0)$ for all $n \geq 1$. Set $c = \lim_{n \rightarrow \infty} \|x_n - z_0\|$. If $c = 0$, then, by G -nonexpansiveness of $S_i (i = 1, 2)$, we get

$$\|x_n - S_ix_n\| \leq \|x_n - z_0\| + \|z_0 - S_ix_n\| \leq \|x_n - z_0\| + \|z_0 - x_n\|.$$

Hence, the result follows. Suppose that $c > 0$. By Lemma 1 together with G-nonexpansiveness of S_1 , we have

$$\begin{aligned}
 \|y_n - z_0\|^2 &= \|(1 - \beta_n)x_n + \beta_n S_1 x_n - z_0\|^2 \\
 &= \|\beta_n(S_1 x_n - z_0) + (1 - \beta_n)(x_n - z_0)\|^2 \\
 &\leq \beta_n \|S_1 x_n - z_0\|^2 + (1 - \beta_n) \|x_n - z_0\|^2 \\
 &\quad - \beta_n(1 - \beta_n)g(\|S_1 x_n - x_n\|) \\
 &\leq \beta_n \|x_n - z_0\|^2 + (1 - \beta_n) \|x_n - z_0\|^2 \\
 &= \|x_n - z_0\|^2,
 \end{aligned} \tag{2}$$

so

$$\limsup_{n \rightarrow \infty} \|y_n - z_0\| \leq \limsup_{n \rightarrow \infty} \|x_n - z_0\| = c. \tag{3}$$

Also by Lemma 1 together with G-nonexpansiveness of $S_i (i = 1, 2)$, we have

$$\begin{aligned}
 \|x_{n+1} - z_0\|^2 &= \|(1 - \alpha_n)S_1 x_n + \alpha_n S_2 y_n - z_0\|^2 \\
 &\leq \alpha_n \|y_n - z_0\|^2 + (1 - \alpha_n) \|x_n - z_0\|^2 \\
 &\quad - \alpha_n(1 - \alpha_n)g(\|S_2 y_n - S_1 x_n\|) \\
 &\leq \alpha_n \|y_n - z_0\|^2 + (1 - \alpha_n) \|x_n - z_0\|^2,
 \end{aligned} \tag{4}$$

which implies

$$\alpha_n \|x_n - z_0\|^2 \leq \alpha_n \|y_n - z_0\|^2 + \|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2.$$

It follows that

$$\begin{aligned}
 \|x_n - z_0\|^2 &\leq \|y_n - z_0\|^2 + \frac{1}{\alpha_n} \left[\|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2 \right] \\
 &\leq \|y_n - z_0\|^2 + \frac{1}{\delta} \left[\|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2 \right].
 \end{aligned}$$

Take $\liminf$ yields

$$c \leq \liminf_{n \rightarrow \infty} \|y_n - z_0\|.$$

From (3), we get $\lim_{n \rightarrow \infty} \|y_n - z_0\| = c$. Now using (2), $\limsup_{n \rightarrow \infty} \|S_1 x_n - z_0\| \leq c$ and by Lemma 2, we get

$$\lim_{n \rightarrow \infty} \|S_1 x_n - x_n\| = 0. \tag{5}$$

Since

$$\begin{aligned}
 \|y_n - x_n\| &= \|(1 - \beta_n)x_n + \beta_n S_1 x_n - x_n\| \\
 &= \beta_n \|S_1 x_n - x_n\|.
 \end{aligned}$$

This implies by (5),

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{6}$$

Next, from (4), we get

$$\begin{aligned}
 \|x_{n+1} - z_0\|^2 &\leq \|x_n - z_0\|^2 - \alpha_n(1 - \alpha_n)g(\|S_2 y_n - S_1 x_n\|) \\
 &\leq \|x_n - z_0\|^2 - \delta^2 g(\|S_2 y_n - S_1 x_n\|).
 \end{aligned}$$

Thus, $\delta^2 g(\|S_2 y_n - S_1 x_n\|) \leq \|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2$. It implies that $\lim_{n \rightarrow \infty} g(\|S_2 y_n - S_1 x_n\|) = 0$, and since g is strictly increasing and continuous at 0,

$$\lim_{n \rightarrow \infty} \|S_2 y_n - S_1 x_n\| = 0. \quad (7)$$

Finally,

$$\begin{aligned} \|S_2 x_n - x_n\| &\leq \|S_2 x_n - S_2 y_n\| + \|S_2 y_n - S_1 x_n\| + \|S_1 x_n - x_n\| \\ &\leq \|x_n - y_n\| + \|S_2 y_n - S_1 x_n\| + \|S_1 x_n - x_n\|, \end{aligned}$$

which by (5), (6), and (7), we can conclude that $\lim_{n \rightarrow \infty} \|S_2 x_n - x_n\| = 0$. This completes the proof. $\square$

Proposition 2 Suppose that X is a Banach space having Opial's property, C has Property WG and let $S : C \rightarrow C$ be a G -nonexpansive mapping. Then, $I - S$ is G -demiclosed at 0.

Proof Suppose that $\{x_n\}$ is a sequence in C that converges weakly to q with $(x_n, x_{n+1}) \in E(G)$ and $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$. From Property WG, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, q) \in E(G)$ for all $k \in \mathbb{N}$. Suppose for contradiction that $q \neq Sq$. By Opial's property, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|x_{n_k} - q\| &< \limsup_{n \rightarrow \infty} \|x_{n_k} - Sq\| \\ &\leq \limsup_{n \rightarrow \infty} (\|x_{n_k} - Sx_{n_k}\| + \|Sx_{n_k} - Sq\|) \\ &\leq \limsup_{n \rightarrow \infty} \|x_{n_k} - q\|, \end{aligned}$$

a contradiction. Thus, $Sq = q$. This completes the proof. $\square$

We now prove the weak convergence theorem.

Theorem 1 Suppose that X is a uniformly convex Banach space having Opial's property, C has Property WG and $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$. If $(x_0, z_0), (z_0, x_0) \in E(G)$ for some $z_0 \in F$, then $\{x_n\}$ converges weakly to a common fixed point of S_i .

Proof Let $z_0 \in F$ be such that $(x_0, z_0), (z_0, x_0) \in E(G)$. From Lemma 4 (i), $\lim_{n \rightarrow \infty} \|x_n - z_0\|$ exists, so $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ and $\{x_{n_j}\}$ be subsequences of the sequence $\{x_n\}$ with two weak limits q_1 and q_2 , respectively. By Lemma 4 (ii), we obtain that $\|x_{n_k} - S_i x_{n_k}\| \rightarrow 0$ and $\|x_{n_j} - S_i x_{n_j}\| \rightarrow 0$ as $k, j \rightarrow \infty$. By Proposition 2, we can conclude that $S_i q_1 = q_1$ and $S_i q_2 = q_2$ and so $q_1, q_2 \in F$. By Lemma 3, we have $q_1 = q_2$. Therefore, $\{x_n\}$ converges weakly to a common fixed point in F . $\square$

We next prove the strong convergence theorem.

Theorem 2 Suppose that X is a uniformly convex Banach space, $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, $S_i (i = 1, 2)$ satisfy Condition B, F is dominated by x_0 and F dominates x_0 , then $\{x_n\}$ converges strongly to a common fixed point of S_i .

Proof Let $z \in F$. Then, by Lemma 4 (i), we have $d(x_{n+1}, F) \leq d(x_n, F)$. So $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. Since each $S_i (i = 1, 2)$ satisfies Condition B and by Lemma 4 (ii), it follows that $\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0$ and thus, $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Using the same proof as in Theorem 3.5 [9], we can deduce that $x_n \rightarrow q \in F$. $\square$

Theorem 3 Suppose that X is a uniformly convex Banach space, C has Property SG, $\{\alpha_n\}, \{\beta_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, one of $S_i (i = 1, 2)$ is semicompact, F is dominated by x_0 and F dominates x_0 , then $\{x_n\}$ converges strongly to a common fixed point of S_i .

Proof From Lemma 4, $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - S_i x_n\| = 0$. By the semicompactness of $S_i (i = 1, 2)$, there exists $q \in C$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow q$ as $j \rightarrow \infty$. Since C has Property SG and transitivity of graph G , we obtain $(x_{n_j}, q) \in E(G)$. Then

$$\begin{aligned} \|q - S_i q\| &\leq \|q - x_{n_j}\| + \|x_{n_j} - S_i x_{n_j}\| + \|S_i x_{n_j} - S_i q\| \\ &\leq \|q - x_{n_j}\| + \|x_{n_j} - S_i x_{n_j}\| + \|x_{n_j} - q\| \\ &\rightarrow 0 \text{ as } j \rightarrow \infty, \end{aligned}$$

hence $q \in F$. As in the proof of Theorem 2, $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. We note that $d(x_{n_j}, F) \leq d(x_{n_j}, q) \rightarrow 0$ as $j \rightarrow \infty$, hence $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. It follows, as in the proof of Theorem 2, that $\{x_n\}$ converges strongly to a common fixed point of $S_i (i = 1, 2)$. This completes the proof. $\square$

Remark 1 In our main theorems (Theorems 2 and 3), the domination assumptions are crucial for proving strong convergence of the studied method. We note that the initial point x_0 of the iteration and the common fixed point set of the mappings, F , should satisfy the domination assumptions this is because of G-nonexpansiveness of the mappings S_i . However, in Theorem 1, we can relax the domination assumptions of Theorems 2 and 3. In this theorem, we assume only x_0 dominates z_0 and z_0 dominates x_0 for some $z_0 \in F$, but we obtain only weak convergence. On the other hand, in order to obtain strong convergence in Theorems 2 and 3, we need a condition B and property SG, respectively.

4 Numerical examples

In this section, we give an example and its numerical experiments for supporting our main theorem. In 1976, Rhoades [14] gave the idea how to compare the rate of convergence between two iterative methods as follows:

Definition 10 [14] Let C be a nonempty closed convex subset of a Banach space X and $S : C \rightarrow C$ be a mapping. Suppose $\{x_n\}$ and $\{z_n\}$ are two iterations which converge to a fixed point q of S . Then $\{x_n\}$ is said to converge faster than $\{z_n\}$ if

$$\|x_n - q\| \leq \|z_n - q\|$$

for all $n \geq 1$.

In 2011, Phuengrattana and Suantai [15] showed that the Ishikawa iteration converges faster than the Mann iteration for a class of continuous functions on the closed interval in a real line.

In order to study, the order of convergence of a real sequence $\{a_n\}$ converging to a , we usually use the well-known terminology in numerical analysis, see [16], for example.

Definition 11 [16] Suppose $\{a_n\}$ is a sequence that converges to a , with $a_n \neq a$ for all n . If positive constants λ and α exist with

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1} - a|}{|a_n - a|^\alpha} = \lambda,$$

then $\{a_n\}$ converges to a of order α , with asymptotic error constant λ . If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent and if $\alpha = 2$, the sequence is quadratically convergent.

In 2002, Berinde [17] employed above concept for comparing the rate of convergence between the two iterative methods as follows:

Definition 12 [17] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converge to a, b , respectively. Assume there exists

$$\lim_{n \rightarrow \infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

- (i) If $l = 0$, then it is said that the sequence $\{a_n\}$ converges to a faster than the sequence $\{b_n\}$ to b .
- (ii) If $0 < l < \infty$, then we say that the sequence $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Table 1 Numerical experiments of Example 1

n	Ishikawa iteration	Modified S-iteration process	Rate of Convergence		
	z_n	x_n	$ z_n - 1 $	$ x_n - 1 $	$ x_n - 1 / z_n - 1 $
1	1.27895	1.10845	0.27895	0.10845	0.38878
2	1.24538	1.02129	0.24538	0.02129	0.08676
3	1.22001	1.00290	0.22001	0.00290	0.01318
4	1.19884	1.00036	0.19884	0.00036	0.00181
5	1.18047	1.00004	0.18047	0.00004	0.00022
6	1.16423	1.00001	0.16423	0.00001	0.00006

Table 2 Numerical errors of Ishikawa iteration and modified S-iteration process

n	Ishikawa iteration		Modified S-iteration process	
	z_n	$ z_n - z_{n-1} $	x_n	$ x_n - x_{n-1} $
1	1.27895	0.07105	1.10845	0.24155
2	1.24538	0.03358	1.02129	0.08716
3	1.22001	0.02536	1.00290	0.01838
4	1.19884	0.02117	1.00036	0.00255
5	1.18047	0.01837	1.00004	0.00031
6	1.16423	0.01624	1.00001	0.00004

Definition 13 [17, 18] Let C be a nonempty closed convex subset of a Banach space X and $S : C \rightarrow C$ be a mapping. Suppose $\{x_n\}$ and $\{z_n\}$ are two iterations which converge to a fixed point q of S . We say that $\{x_n\}$ converges faster than $\{z_n\}$ to q if

$$\lim_{n \rightarrow \infty} \frac{\|x_n - q\|}{\|z_n - q\|} = 0.$$

We now give an example which shows numerical experiment for supporting our main results and comparing the rate of convergence of the studied method and Ishikawa iteration.

Example 1 Let $X = \mathbb{R}$ and $C = [0, 2]$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$ and $(x, y) \in E(G)$ if and only if $0.80 \leq x, y \leq 1.70$ and $x, y \in \mathbb{Q}$. Define a mapping $S_1, S_2 : C \rightarrow C$ by

$$S_1x = \begin{cases} x^{\ln x}, & \text{if } x \neq \sqrt{2}, \\ 2, & \text{if } x = \sqrt{2} \end{cases}$$

and

$$S_2x = \begin{cases} \frac{2}{3} \arcsin(x - 1) + 1, & \text{if } x \neq \sqrt{3}, \\ 0, & \text{if } x = \sqrt{3} \end{cases}$$

for any $x \in C$. It is easy to show that S_1, S_2 are G-nonexpansive but S_1, S_2 are not nonexpansive because $|S_1x - S_1y| > 0.50 > |x - y|$ and $|S_2u - S_2v| > 0.80 > |u - v|$ when $x = \sqrt{2}$, $u = \sqrt{3}$ and $y = 1 = v$. Let $\alpha_n = \frac{n+1}{5n+3}$ and $\beta_n = \frac{n+4}{10n+7}$. Choose $z_0 = x_0 = 1.35$. Let $\{x_n\}$ be a sequence generated by (1) and $\{z_n\}$ be a sequence generated by Ishikawa iteration. We obtain the following numerical experiments for common fixed point of S_1 and S_2 and rate of convergence of $\{x_n\}$ and $\{z_n\}$.

We note that $x = 1$ is a common fixed point of S_1 and S_2 .

From Tables 1 and 2, we see that both $\{z_n\}$ and $\{x_n\}$ converge to $1 \in F$ and observe that $|x_n - 1| \leq |z_n - 1|$ and $\lim_{n \rightarrow \infty} \frac{|x_n - 1|}{|z_n - 1|} = 0$, so the sequence $\{x_n\}$ converges faster than $\{z_n\}$ generated by Ishikawa iteration (Fig. 1).

As far as we know, there is no any paper which can compare rate of convergence for nonexpansive mappings in general, so it is interesting to ask the following question:

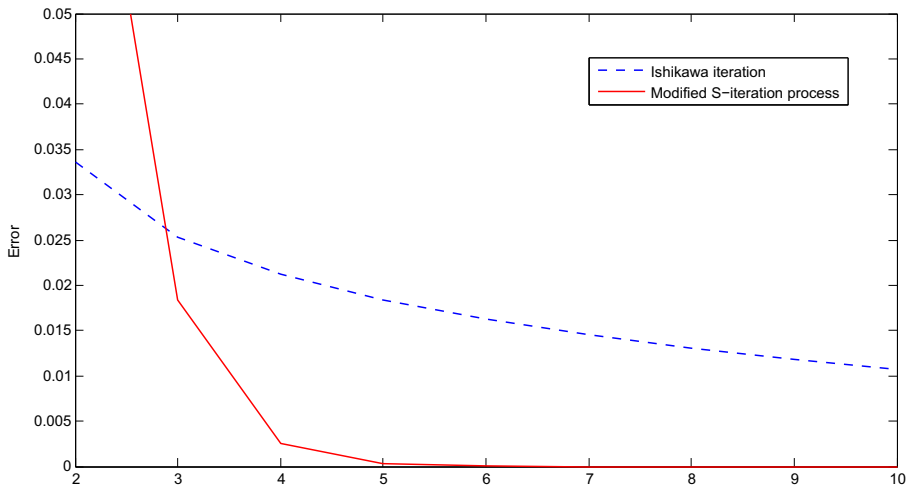


Fig. 1 Comparison of errors of the Ishikawa iteration and a modified S-iteration process.

Which one converges faster than the other between Ishikawa iteration and modified S-iteration process for the class of nonexpansive mappings ?

Acknowledgements The authors would like to thank the referees for valuable comments and suggestions for improving this work. R. Suparatulatorn and S. Suantai would like to thank the Thailand Research Fund under the project RTA5780007, the Royal Golden Jubilee (RGJ) Ph.D. Programme (PHD/0021/2559) and Chiang Mai University. W. Chulamjiak would like to thank University of Phayao.

References

1. Banach, S.: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math* **3**, 133–181 (1922)
2. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. *J. Nonlinear Convex. Anal.* **8**, 61–79 (2007)
3. Jachymski, J.: The contraction principle for mappings on a metric space with a graph. *Proc. Amer. Math. Soc* **136**(4), 1359–1373 (2008)
4. Aleomraninejad, S.M.A., Rezapour, S., Shahzad, N.: Some fixed point result on a metric space with a graph. *Topol. Appl* **159**, 659–663 (2012)
5. Alfuraidan, M.R.: Fixed points of monotone nonexpansive mappings with a graph. *Fixed Point Theory Appl.* doi:[10.1186/s13663-015-0299-0](https://doi.org/10.1186/s13663-015-0299-0) (2015)
6. Alfuraidan, M.R., Khamsi, M.A.: Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph. *Fixed Point Theory Appl.* doi:[10.1186/s13663-015-0294-5](https://doi.org/10.1186/s13663-015-0294-5) (2015)
7. Tiammee, J., Kaewkhao, A., Suantai, S.: On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs. *Fixed Point Theory Appl.* doi:[10.1186/s13663-015-0436-9](https://doi.org/10.1186/s13663-015-0436-9) (2015)
8. Tripak, O.: Common fixed points of G-nonexpansive mappings on Banach spaces with a graph. *Fixed Point Theory Appl.* doi:[10.1186/s13663-016-0578-4](https://doi.org/10.1186/s13663-016-0578-4) (2016)
9. Shahzad, S., Al-Dubiban, R.: Approximating common fixed points of nonexpansive mappings in Banach spaces. *Georgian Math. J.* **13**(3), 529–537 (2006)
10. Xu, H.K.: Inequalities in Banach spaces with applications. *Nonlinear Anal.* **16**(12), 1127–1138 (1991)

11. Schu, J.: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bull. Aust. Math. Soc.* **43**(1), 153–159 (1991)
12. Suantai, S.: Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings. *J. Math. Anal. Appl.* **331**, 506–517 (2005)
13. Agarwal, R.P., O'Regan, D., Sahu, D.R.: *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*. Springer, New York (2009)
14. Rhoades, B.E.: Comments on two fixed point iteration method. *J. Math. Anal. Appl.* **56**(2), 741–750 (1976)
15. Phuengrattana, W., Suantai, S.: On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval. *J. Comput. Appl. Math.* **235**, 3006–3014 (2011)
16. Burden, R.L., Faires, J.D. *Numerical Analysis*, 9th edn. Brooks/Cole Cengage Learning, Boston (2010)
17. Berinde, V.: *Iterative Approximation of Fixed Points*. Editura Efemeride, Baia Mare (2002)
18. Phuengrattana, W., Suantai, S.: Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach spaces. *Thai J. Math.* **11**, 217–226 (2013)

3.7 Hybrid methods for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with graphs

In this paper, we prove a strong convergence theorem for two different hybrid methods by using CQ method for a finite family of G-nonexpansive mappings in a Hilbert space. We give an example and numerical results for supporting our main results and compare the rate of convergence of the two iterative methods.

Introduction

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and C be a nonempty subset of H .

A nonlinear mapping $T : C \rightarrow C$ is said to be

1. *contraction* if there exists $\alpha \in (0, 1)$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$;
2. *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

Since 1922, fixed point theorems and the existence of fixed points of a single-valued nonlinear mapping have been intensively studied and considered by many authors (see, for examples [1–7]).

In 1953, Mann [8] introduced the famous iteration procedure as follows:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\mathbb{N}$ the set of all positive integers. Many researchers have used Mann's iteration for obtaining weak convergence theorem (see for example [9–11]).

In 2003, Nakajo and Takahashi [12] introduced a modification of the Mann iteration and called it CQ method. They generated the sequence $\{x_n\}$ by

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 1, \end{cases} \quad (1.1)$$

for $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. They showed that $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

In 2015, Khan et al. [13] used the following definition defined by Berinde [14] to compare the convergence rate:

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers with limits a and b , respectively. Assume that there exists $\lim_{n \rightarrow \infty} \frac{|a_n - a|}{|b_n - b|} = \ell$. If $\ell = 0$, then we say that $\{a_n\}$ converges faster to a than $\{b_n\}$ to b .

Let C be a nonempty subset of a real Banach space X . Let Δ denote the diagonal of the cartesian product $C \times C$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C , and the set $E(G)$ of its edges with $\Delta \subseteq E(G)$. We assume G has no parallel edge. So we can identify the graph G with the pair $(V(G), E(G))$. A mapping $T : C \rightarrow C$ is said to be

1. *G-contraction* if T satisfies the conditions:

(i) T preserves edges of G , i.e.,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G), \quad \forall (x, y) \in E(G);$$

(ii) T decreases weights of edges of G in the following way: there exists $\alpha \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow \|Tx - Ty\| \leq \alpha \|x - y\|, \quad \forall (x, y) \in E(G);$$

2. *G-nonexpansive* if T satisfies the conditions:

(i) T preserves edges of G , i.e.,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G), \quad \forall (x, y) \in E(G);$$

(ii) T non-increases weights of edges of G in the following way:

$$(x, y) \in E(G) \Rightarrow \|Tx - Ty\| \leq \|x - y\|, \quad \forall (x, y) \in E(G).$$

In 2008, Jachymski [15] proved some generalizations of the Banach's contraction principle in complete metric spaces endowed with a graph. To be more precise, Jachymski proved the following result.

Theorem 1.1 ([15]). *Let (X, d) be a complete metric space, and a triple (X, d, G) has the following property:*

for any sequence $\{x_n\}$ if $x_n \rightarrow x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ and there is the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $(x_{n_k}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let $T : X \rightarrow X$ be a G -contraction, and $X_T = \{x \in X : (x, Tx) \in E(G)\}$. Then $F(T) \neq \emptyset$ if and only if $X_T \neq \emptyset$.

When $\{x_n\}$ is a sequence in X , we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \rightarrow x$ and the weak convergence by $x_n \rightharpoonup x$.

In 2015, Tiammee et al. [16] and Alfuraidan [17] employed the above theorem to establish the existence and the convergence results for G -nonexpansive mappings with graphs.

Motivated by Nakajo and Takahashi [12] and Tiammee et al. [16], we introduce the modified CQ method for proving a strong convergence theorem for G -nonexpansive mappings in a Hilbert space endowed with a directed graph. Moreover, we provide some numerical examples to support our main theorem.

2. Preliminaries and lemmas

Let C be a nonempty, closed and convex subset of a Hilbert space H . The nearest point projection of H onto C is denoted by P_C , that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the *metric projection* of H onto C . We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x \rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [18].

A mapping $A : C \rightarrow H$ is called α -inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

We know that if $T : C \rightarrow C$ is nonexpansive, then $A = I - T$ is $\frac{1}{2}$ -inverse strongly monotone; see [19] for more details. We consider the following *variational inequality* problem: find a $\hat{x} \in C$ such that

$$\langle A\hat{x}, \hat{x} - y \rangle \geq 0, \quad \forall y \in C.$$

The set of solutions of the variational inequality (2.1) is denoted by $VI(C, A)$.

We need the following known results.

Lemma 2.1 ([18]). *Let H be a real Hilbert space. Then*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2,$$

for all $t \in [0, 1]$ and $x, y \in H$.

Lemma 2.2 ([20]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H . Given that $x, y, z \in H$ and $a \in \mathbb{R}$, the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\}$$

is convex and closed.

Lemma 2.3 ([12]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $P_C : H \rightarrow C$ be the metric projection from H onto C . Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \forall y \in C.$$

Lemma 2.4 ([21]). *Let H be a real Hilbert space and let $\{x_i\}_{i=1}^m \subseteq H$. For $\alpha_i \in (0, 1)$, $i = 1, 2, \dots, m$ such that $\sum_{i=1}^m \alpha_i = 1$, the following identity holds:*

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1 \leq i < j \leq m} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.5. *Let C be a nonempty subset of a Hilbert space H and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. Let $T : C \rightarrow C$ be a G -nonexpansive mapping. Then, for any $\varepsilon > 0$, there exists a positive number $\xi(\varepsilon) > 0$ such that $\|x - Tx\| < \varepsilon$ for all $x \in \text{co}(\{x_0, x_1\})$, whenever $x_0, x_1 \in C$ with $(x_0, x), (x_1, x) \in E(G)$, $\|x_0 - Tx_0\| \leq \xi(\varepsilon)$ and $\|x_1 - Tx_1\| \leq \xi(\varepsilon)$.*

Proof. Let $x = (1 - \lambda)x_0 + \lambda x_1$, for some $\lambda \in [0, 1]$ and $\varepsilon > 0$.

We consider the following two cases.

Case I. If $\|x_0 - x_1\| < \frac{\varepsilon}{3}$, then

$$\|x - x_0\| = \lambda \|x_0 - x_1\| < \frac{\varepsilon}{3}.$$

If $\xi(\varepsilon) < \frac{\varepsilon}{3}$, then we have

$$\begin{aligned} \|Tx - x\| &\leq \|Tx - Tx_0\| + \|Tx_0 - x_0\| + \|x_0 - x\| \\ &\leq 2\|x - x_0\| + \|Tx_0 - x_0\| \\ &< 2\left(\frac{\varepsilon}{3}\right) + \xi(\varepsilon) \\ &< \varepsilon. \end{aligned}$$

Case II. If $\|x_0 - x_1\| \geq \frac{\varepsilon}{3}$, then for any nonnegative number $\lambda < \frac{\varepsilon}{3\|x_0 - x_1\|}$, we have

$$\|x - x_0\| = \lambda \|x_0 - x_1\| < \frac{\varepsilon}{3}.$$

If $\xi(\varepsilon) < \frac{\varepsilon}{3}$ and $\lambda < \frac{\varepsilon}{3\|x_0 - x_1\|}$, then we have

$$\begin{aligned}\|Tx - x\| &\leq \|Tx - Tx_0\| + \|Tx_0 - x_0\| + \|x_0 - x\| \\ &\leq 2\|x - x_0\| + \|Tx_0 - x_0\| \\ &< 2\left(\frac{\varepsilon}{3}\right) + \xi(\varepsilon) \\ &< \varepsilon.\end{aligned}$$

We may assume that $\lambda \in [\frac{\varepsilon}{3\|x_0 - x_1\|}, 1]$ and $\|x_0 - x_1\| \geq \frac{\varepsilon}{3}$. Then we obtain

$$\begin{aligned}\|Tx - x_0\| &\leq \|Tx - Tx_0\| + \|Tx_0 - x_0\| \\ &\leq \|x - x_0\| + \xi(\varepsilon) \\ &= \lambda\|x_1 - x_0\| + \xi(\varepsilon)\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\|Tx - x_1\| &\leq \|Tx - Tx_1\| + \|Tx_1 - x_1\| \\ &\leq \|x - x_1\| + \xi(\varepsilon) \\ &= (1 - \lambda)\|x_1 - x_0\| + \xi(\varepsilon).\end{aligned}\tag{2.2}$$

From (2.1), (2.2) and $\lambda \in [\frac{\varepsilon}{3\|x_0 - x_1\|}, 1]$, we get that

$$\begin{aligned}\|Tx - x\| &\leq (1 - \lambda)\|Tx - x_0\| + \lambda\|Tx - x_1\| \\ &\leq 2(1 - \lambda)\lambda\|x_1 - x_0\| + \xi(\varepsilon) \\ &< \varepsilon. \quad \square\end{aligned}$$

We next prove the demiclosedness principle of a G-nonexpansive mapping.

Lemma 2.6. *Let C be a nonempty, closed and convex subset of a Hilbert space H and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. Let $T : C \rightarrow C$ be a G-nonexpansive mapping and $\{x_n\}$ be a sequence in C such that $x_n \rightarrow x$ for some $x \in C$. If, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$ and $\{x_n - Tx_n\} \rightarrow y$ for some $y \in H$, then $(I - T)x = y$.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightarrow x$ and $\lim_{n \rightarrow \infty} \|x_n - Tx_n - y\| = 0$ for some $y \in H$. Set $T_y x = Tx + y$, $x \in C$. Then $(I - T_y)x_n = (I - T)x_n - y$. We may assume, without loss of generality, that $y = 0$. By the assumption, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$. We set $\varepsilon_{n_k} = \|x_{n_k} - Tx_{n_k}\|$. Let $\varepsilon > 0$. Since $\varepsilon_{n_k} \rightarrow 0$ as $k \rightarrow \infty$, there exists $N \in \mathbb{N}$ such that

$$\varepsilon_{n_k} < \varepsilon, \quad \forall k \geq N.$$

Lemma 2.5 gives that, for each $z \in \overline{co}(\{x_{n_k} : k \geq N\})$, $\|z - Tz\| < \varepsilon$. By the weak compactness of $\overline{co}(\{x_{n_k} : k \geq N\})$, it contains the weak limit x of $\{x_{n_k}\}$. This shows that $\|x - Tx\| < \varepsilon$. Hence $\|x - Tx\| = 0$, that is, $x = Tx$, since ε is arbitrary. $\square$

3. Main results

In this section, by using the CQ method, we obtain two different strong convergence theorems for finding the same common fixed point of a finite family of G-nonexpansive mappings in real Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $\{T_i : i = 1, 2, \dots, N\}$ be a finite family of G-nonexpansive mappings of C into itself. Assume that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$, F is closed and $F(T_i) \times F(T_i) \subseteq E(G)$*

for all $i \in \{1, 2, \dots, N\}$. For an initial point $x_1 \in H$ with $Q_1 = C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_{i,n} = \alpha_{i,n}x_n + (1 - \alpha_{i,n})T_i x_n, \\ C_{i,n} = \{z \in C : \|y_{i,n} - z\| \leq \|x_n - z\|\}, \\ C_n = \bigcap_{i=1}^N C_{i,n}, \\ Q_n = \{z \in Q_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \quad n \geq 2, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \quad (3.1)$$

where $\{\alpha_{i,n}\} \subset [0, 1)$ for all $i \in \{1, 2, \dots, N\}$ such that $\limsup_{n \rightarrow \infty} \alpha_{i,n} < 1$ for all $i \in \{1, 2, \dots, N\}$. Assume that $\{x_n\}$ dominates z for all $z \in F$ and if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$, then $(x_{n_k}, w) \in E(G)$. Then the sequence $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. We split the proof into six steps.

Step 1. Show that $P_F x_1$ is well-defined for each $x_1 \in H$.

As shown in Theorem 3.2 of Tiammee et al. [16], $F(T_i)$ is convex for all $i = 1, 2, \dots, N$. It follows now from the assumption that $F = \bigcap_{i=1}^N F(T_i)$ is closed and convex. Hence, $P_F x_1$ is well-defined.

Step 2. Show that $P_{C_n \cap Q_n} x_1$ is well-defined for each $x_1 \in H$.

From the definition of C_n and Q_n , it is obvious that Q_n is closed and convex for all $n \in \mathbb{N}$. It follows from Lemma 2.2 that for each $i \in \{1, 2, \dots, N\}$, $C_{i,n}$ is closed and convex for all $n \in \mathbb{N}$. This implies that C_n is also closed and convex for all $n \in \mathbb{N}$. Let $p \in F$ and $i \in \{1, 2, \dots, N\}$. Since $\{x_n\}$ dominates p , we have

$$\begin{aligned} \|y_{i,n} - p\| &\leq \alpha_{i,n}\|x_n - p\| + (1 - \alpha_{i,n})\|T_i x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \quad (3.2)$$

So, we have $p \in C_{i,n}$ for all $i = 1, 2, \dots, N$ and $n \in \mathbb{N}$. We can conclude that $p \in \bigcap_{i=1}^N C_{i,n} = C_n$ for all $n \in \mathbb{N}$. Thus $F \subset C_n$ for all $n \in \mathbb{N}$. Using induction, it can be shown as in [22] that $F \subset Q_n$ for all $i = 1, 2, \dots, N$. Hence $F \subset C_n \cap Q_n$. This implies that $P_{C_n \cap Q_n} x_1$ is well-defined.

Step 3. Show that $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists.

Since F is a nonempty, closed and convex subset of H , there exists a unique $v \in F$ such that $v = P_F x_1$. From $x_n = P_{Q_n} x_1$ and $x_{n+1} \in Q_n$, $\forall n \in \mathbb{N}$, we get

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \quad \forall n \in \mathbb{N}. \quad (3.3)$$

On the other hand, as $F \subset Q_n$, we obtain

$$\|x_n - x_1\| \leq \|v - x_1\|, \quad \forall n \in \mathbb{N}. \quad (3.4)$$

It follows from (3.3) and (3.4) that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists.

Step 4. Show that $x_n \rightarrow w \in C$ as $n \rightarrow \infty$.

For $m > n$, by the definition of Q_n , we see that $Q_m \subset Q_n$. Noting that $x_m = P_{Q_m} x_1$ and $x_n = P_{Q_n} x_1$, by Lemma 2.3, we get

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2.$$

From Step 3, we obtain that $\{x_n\}$ is a Cauchy sequence. Hence, there exists $w \in C$ such that $x_n \rightarrow w$ as $n \rightarrow \infty$. In particular, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.5)$$

Step 5. Show that $w \in F$.

Since $x_{n+1} \in Q_n$, it follows from (3.5) that

$$\|y_{i,n} - x_n\| \leq \|y_{i,n} - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \rightarrow 0 \quad (3.6)$$

as $n \rightarrow \infty$ for all $i = 1, 2, \dots, N$. Since $y_{i,n} = \alpha_{i,n}x_n + (1 - \alpha_{i,n})T_i x_n$, we have

$$\|x_n - T_i x_n\| = \frac{1}{(1 - \alpha_{i,n})} \|y_{i,n} - x_n\|.$$

It follows from $\limsup_{n \rightarrow \infty} \alpha_{i,n} < 1$ for all $i = 1, 2, \dots, N$ and (3.6) that

$$\|x_n - T_i x_n\| \rightarrow 0 \quad (3.7)$$

as $n \rightarrow \infty$ for all $i \in \{1, 2, \dots, N\}$. From $(x_n, w) \in E(G)$ and Lemma 2.6, we have $w \in F$.

Step 6. Show that $w = v = P_F x_1$.

Since $x_n = P_{Q_n} x_1$, we have

$$\langle x_1 - x_n, x_n - p \rangle \geq 0, \quad \forall p \in Q_n. \quad (3.8)$$

By taking the limit in (3.8), we obtain

$$\langle x_1 - w, w - p \rangle \geq 0, \quad \forall p \in Q_n. \quad (3.9)$$

Since $F \subset Q_n$, so $w = v = P_F x_1$. This completes the proof. $\square$

Theorem 3.2. Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let $\{T_i : i = 1, 2, \dots, N\}$ be a finite family of G -nonexpansive mappings of C into itself. Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, F is closed and $F(T_i) \times F(T_i) \subseteq E(G)$ for all $i \in \{1, 2, \dots, N\}$. For an initial point $x_1 \in H$ with $Q_1 = C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = \alpha_{0,n} x_n + \sum_{i=1}^N \alpha_{i,n} T_i x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in Q_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \quad n \geq 2, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \quad (3.10)$$

where $\{\alpha_{i,n}\} \subset (0, 1]$ for all $i \in \{0, 1, 2, \dots, N\}$ such that $\sum_{i=0}^N \alpha_{i,n} = 1$ and $\liminf_{n \rightarrow \infty} \alpha_{0,n} > 0$ for all $i \in \{1, 2, \dots, N\}$. Assume that $\{x_n\}$ dominates z for all $z \in F$ and if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$, then $(x_{n_k}, w) \in E(G)$. Then the sequence $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. From Step 1 in the proof of Theorem 3.1, we have that $P_F x_1$ is well-defined for each $x_1 \in H$. We know from Step 2 in the proof of Theorem 3.1 that $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. Let $p \in F$. Since $\{x_n\}$ dominates p , we have

$$\begin{aligned} \|y_n - p\| &= \left\| \alpha_{0,n} x_n + \sum_{i=1}^N \alpha_{i,n} T_i x_n \right\| \\ &\leq \alpha_{0,n} \|x_n - p\| + \sum_{i=1}^N \alpha_{i,n} \|T_i x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

So, we have $p \in C_n$ for all $n \in \mathbb{N}$. Thus $F \subset C_n$ for all $n \in \mathbb{N}$. It is clear that $F \subset Q_n$ for all $n \in \mathbb{N}$ (see Marino and Xu [22]). Hence, $F \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $P_{C_n \cap Q_n} x_1$ is well-defined. As in Steps 3–4 of the proof of Theorem 3.1, we have that the sequence $\{x_n\}$ is Cauchy. By the completeness of the space H , $x_n \rightarrow w \in C$ as $n \rightarrow \infty$. We next show that $w \in F$. Since $x_{n+1} \in Q_n$ and $\{x_n\}$ is Cauchy, it follows that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \rightarrow 0 \quad (3.11)$$

as $n \rightarrow \infty$. For $p \in F$, it follows from [Lemma 2.4](#) and $\{x_n\}$ dominates p that

$$\begin{aligned} \|y_n - p\|^2 &= \left\| \alpha_{0,n}x_n + \sum_{i=1}^N \alpha_{i,n}T_i x_n \right\|^2 \\ &= \alpha_{0,n}\|x_n - p\|^2 + \sum_{i=1}^N \alpha_{i,n}\|T_i x_n - p\|^2 - \sum_{i=1}^N \alpha_{0,n}\alpha_{i,n}\|T_i x_n - x_n\|^2 \\ &\leq \|x_n - p\|^2 - \sum_{i=1}^N \alpha_{0,n}\alpha_{i,n}\|T_i x_n - x_n\|^2. \end{aligned}$$

This implies that

$$\sum_{i=1}^N \alpha_{0,n}\alpha_{i,n}\|T_i x_n - x_n\|^2 \leq \|x_n - p\|^2 - \|y_n - p\|^2.$$

It follows from the assumption and [\(3.11\)](#), we obtain

$$\lim_{n \rightarrow \infty} \|T_i x_n - x_n\| = 0$$

for all $i = 1, 2, \dots, N$. From $(x_n, w) \in E(G)$ and [Lemma 2.6](#), we have $w \in F$. As in Step 6 of the proof of [Theorem 3.1](#), we obtain $w = P_F x_1$. This completes the proof. $\square$

Remark 3.3. The sequences $\{x_n\}$ generated in [Theorems 3.1](#) and [3.2](#) converge to $P_F x_1$ under the different conditions on the sequence $\alpha_{i,n}$.

We know that every G-nonexpansive mapping is nonexpansive [\[16\]](#), then we obtain the following results.

Corollary 3.4. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $\{T_i : i = 1, 2, \dots, N\}$ be a finite family of nonexpansive mappings of C into itself. Assume that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. For an initial point $x_1 \in H$ with $Q_1 = C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_{i,n} = \alpha_{i,n}x_n + (1 - \alpha_{i,n})T_i x_n, \\ C_{i,n} = \{z \in C : \|y_{i,n} - z\| \leq \|x_n - z\|\}, \\ C_n = \bigcap_{i=1}^N C_{i,n}, \\ Q_n = \{z \in Q_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \quad n \geq 2, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \quad (3.12)$$

where $\{\alpha_{i,n}\} \subset [0, 1]$ for all $i \in \{1, 2, \dots, N\}$. Assume that $\limsup_{n \rightarrow \infty} \alpha_{i,n} < 1$ for all $i \in \{1, 2, \dots, N\}$. Then the sequence $\{x_n\}$ converges strongly to $P_F x_1$.

Corollary 3.5. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $\{T_i : i = 1, 2, \dots, N\}$ be a finite family of nonexpansive mappings of C into itself. Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. For an initial point $x_1 \in H$ with $Q_1 = C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = \alpha_{0,n}x_n + \sum_{i=1}^N \alpha_{i,n}T_i x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in Q_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \quad n \geq 2, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \quad (3.13)$$

where $\{\alpha_{i,n}\} \subset (0, 1]$ for all $i \in \{0, 1, 2, \dots, N\}$ such that $\sum_{i=0}^N \alpha_{i,n} = 1$. Assume that $\liminf_{n \rightarrow \infty} \alpha_{0,n}\alpha_{i,n} > 0$ for all $i \in \{1, 2, \dots, N\}$. Then the sequence $\{x_n\}$ converges strongly to $P_F x_1$.

Putting $T_i = I$ ($i = 2, 3, \dots, N$) where I is an identity mapping in [Theorem 3.1–3.2](#), we obtain the following result.

Corollary 3.6. Let C be a nonempty closed and convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let T be a G -nonexpansive mapping of C into itself. Assume that $F(T) \neq \emptyset$, $F(T)$ is closed and $F(T) \times F(T) \subseteq E(G)$. For an initial point $x_1 \in H$ with $Q_1 = C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in Q_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \quad n \geq 2, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \quad (3.14)$$

where $\{\alpha_n\} \subset (0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Assume that $\{x_n\}$ dominates z for all $z \in F(T)$ and if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$, then $(x_{n_k}, w) \in E(G)$. Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_1$.

Remark 3.7. The iterative scheme (3.14) extends the CQ method (1.1) from a nonexpansive mapping to a G -nonexpansive mapping.

If $\lambda \in [0, 2\alpha]$, we know that $I - \lambda A$ is nonexpansive; see [19]. Then, we obtain the following results.

Corollary 3.8. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $\{A_i : C \rightarrow H, i = 1, 2, \dots, N\}$ be a finite family of α -inverse strongly monotone with $\bigcap_{i=1}^N V(C, A_i) \neq \emptyset$. For an initial point $x_1 \in H$ with $Q_1 = C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_{i,n} = \alpha_{i,n} x_n + (1 - \alpha_{i,n})P_C(x_n - \lambda_n A_i x_n), \\ C_{i,n} = \{z \in C : \|y_{i,n} - z\| \leq \|x_n - z\|\}, \\ C_n = \bigcap_{i=1}^N C_{i,n}, \\ Q_n = \{z \in Q_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \quad n \geq 2, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \quad (3.15)$$

where $\{\alpha_{i,n}\} \subset [0, 1]$ for all $i \in \{1, 2, \dots, N\}$ and $\{\lambda_n\} \subset [0, 2\alpha]$. Assume that $\limsup_{n \rightarrow \infty} \alpha_{i,n} < 1$ for all $i \in \{1, 2, \dots, N\}$. Then the sequence $\{x_n\}$ converges strongly to $P_{\bigcap_{i=1}^N V(C, A_i)} x_1$.

Corollary 3.9. Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $\{A_i : C \rightarrow H, i = 1, 2, \dots, N\}$ be a finite family of α -inverse strongly monotone with $\bigcap_{i=1}^N V(C, A_i) \neq \emptyset$. For an initial point $x_1 \in H$ with $Q_1 = C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = \alpha_{0,n} x_n + \sum_{i=1}^N \alpha_{i,n} P_C(x_n - \lambda_n A_i x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in Q_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \quad n \geq 2, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \quad (3.16)$$

where $\{\alpha_{i,n}\} \subset (0, 1]$ for all $i \in \{0, 1, 2, \dots, N\}$ such that $\sum_{i=0}^N \alpha_{i,n} = 1$ and $\{\lambda_n\} \subset [0, 2\alpha]$. Assume that $\liminf_{n \rightarrow \infty} \alpha_{0,n} > 0$ for all $i \in \{1, 2, \dots, N\}$. Then the sequence $\{x_n\}$ converges strongly to $P_{\bigcap_{i=1}^N V(C, A_i)} x_1$.

4. Examples and numerical results

In this section, we give examples and numerical results for supporting our main theorem.

Example 4.1. Let $H = \mathbb{R}$ and $C = [0, 3]$. Assume that $(x, y) \in E(G)$ if and only if $0 \leq x, y \leq 2$ or $x = y$. Define two mappings $T_1, T_2 : C \rightarrow C$ by

$$T_1 x = \frac{1}{4} \tan(x - 1) + 1 \quad \text{and} \quad T_2 x = 2^{\frac{3}{4}(x-1)},$$

Table 1
Numerical results of iteration (3.1).

n	$y_{1,n}$	$y_{2,n}$	$C_n \cap Q_n$	x_n
1	0.48852	0.50966	[0.25483, 3.00000]	0.00000
2	0.64072	0.60174	[0.44777, 3.00000]	0.25483
3	0.73736	0.68992	[0.59257, 3.00000]	0.44777
4	0.80652	0.76353	[0.69955, 3.00000]	0.59257
5	0.85695	0.82151	[0.77825, 3.00000]	0.69955
6	0.89402	0.86604	[0.83613, 3.00000]	0.77825
7	0.92137	0.89978	[0.87875, 3.00000]	0.83613
8	0.94160	0.92516	[0.91018, 3.00000]	0.87875
9	0.95660	0.94418	[0.93339, 3.00000]	0.91018
10	0.96772	0.95839	[0.95055, 3.00000]	0.93339
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
30	0.99991	0.99988	[0.99986, 3.00000]	0.99981

Table 2
Numerical results of iteration (3.10).

n	y_n	$C_n \cap Q_n$	x_n
1	0.20707	[0.10354, 3.00000]	0.00000
2	0.34473	[0.22413, 3.00000]	0.10354
3	0.45763	[0.34088, 3.00000]	0.22413
4	0.55155	[0.44621, 3.00000]	0.34088
5	0.62963	[0.53792, 3.00000]	0.44621
6	0.69442	[0.61617, 3.00000]	0.53792
7	0.74809	[0.68213, 3.00000]	0.61617
8	0.79247	[0.73730, 3.00000]	0.68213
9	0.82913	[0.78321, 3.00000]	0.73730
10	0.85938	[0.82130, 3.00000]	0.78321
$\vdots$	$\vdots$	$\vdots$	$\vdots$
30	0.99731	[0.99655, 3.00000]	0.99579

for any $x \in C$. Let $\alpha_{1,n} = \frac{n}{3n+2}$ and $\alpha_{2,n} = \frac{n}{4n+3}$. It is easy to check that T_1, T_2 are G-nonexpansive. On the other hand, T_1, T_2 are not nonexpansive for $x = \frac{14}{5}$ and $y = 2$. This implies that $|T_1x - T_1y| > \frac{4}{5} = |x - y|$ and $|T_2x - T_2y| > \frac{4}{5} = |x - y|$.

For $\{x_n\}$ generated by (3.1), we divide the process of our iteration into 3 Steps as follows:

Step 1. Find $C_n = \bigcap_{i=1}^2 C_{i,n}$. Since $C_{1,n} = \{z \in C : \|y_{1,n} - z\| \leq \|x_n - z\|\}$ and $C_{2,n} = \{z \in C : \|y_{2,n} - z\| \leq \|x_n - z\|\}$, we obtain that $(2z - (y_{1,n} + x_n))(x_n - y_{1,n}) \leq 0$ and $(2z - (y_{2,n} + x_n))(x_n - y_{2,n}) \leq 0$. We observe the following cases:

Case 1 : If $x_n - y_{1,n} = 0$, then $C_{1,n} = C, \forall n \geq 1$.

Case 2 : If $x_n - y_{1,n} > 0$, then $z \leq \frac{y_{1,n} + x_n}{2}$. Thus, $C_{1,n} = [0, \frac{y_{1,n} + x_n}{2}], \forall n \geq 1$.

Case 3 : If $x_n - y_{1,n} < 0$, then $z \geq \frac{y_{1,n} + x_n}{2}$. Thus, $C_{1,n} = [\frac{y_{1,n} + x_n}{2}, 3], \forall n \geq 1$.

Similarly, we have $C_{2,n} = C$; where $x_n - y_{2,n} = 0$, $C_{2,n} = [0, \frac{y_{2,n} + x_n}{2}]$; where $x_n - y_{2,n} > 0$ and $C_{2,n} = [\frac{y_{2,n} + x_n}{2}, 3]$; where $x_n - y_{2,n} < 0, \forall n \geq 1$.

Step 2. Find $Q_n = \{z \in Q_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\} = \{z \in Q_{n-1} : (x_1 - x_n)(z - x_n) \leq 0\}$. We observe the following cases:

Case 1 : If $x_1 - x_n = 0$, then $Q_n = C, \forall n \geq 2$.

Case 2 : If $x_1 - x_n > 0$, then $z \leq x_n$. Thus, $Q_n = Q_{n-1} \cap [0, x_n], \forall n \geq 2$.

Case 3 : If $x_1 - x_n < 0$, then $z \geq x_n$. Thus, $Q_n = Q_{n-1} \cap [x_n, 3], \forall n \geq 2$.

Step 3. Compute the numerical results of $x_{n+1} = P_{C_n \cap Q_n} x_1$. Choose $x_1 = 0$, we have From Table 1, we see that 1 is the solution of iteration (3.1).

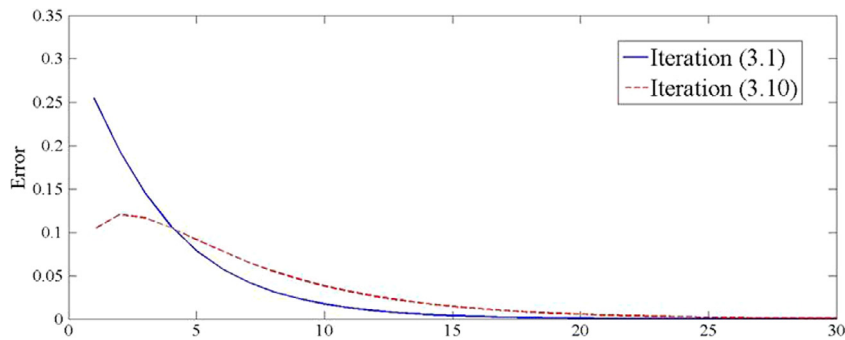


Fig. 1. Error plots for sequences $\{x_n\}$ in Tables 1 and 2.

We next show that our iteration generated by (3.10) also converges to 1. As the same above process, choose $x_1 = 0$. (See Table 2.)

From Table 2, we see that the iteration (3.10) converges to 1. (See Fig. 1.)

In the same way of Khan ([13], Example 3), we can conclude the following remark.

Remark 4.2. We see that the iteration (3.1) converges faster than the iteration (3.10) under the same conditions.

Acknowledgments

The authors would like to thank the referees for valuable comments and suggestions for improving this work. R. Suparatulatorn and S. Suantai would like to thank the Thailand Research Fund under the project RTA5780007 and the Graduate School of Chiang Mai University. W. Cholamjiak would like to thank University of Phayao and the Thailand Research Fund.

References

- [1] R.P. Agarwal, D. O'Regan, D.R. Sahu, *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*, Springer, 2009.
- [2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* 3 (1922) 133–181.
- [3] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci. USA* 54 (1965) 1041–1044.
- [4] F.E. Browder, Convergence of approximants to fixed points of non-expansive maps in Banach spaces, *Arch. Ration. Mech. Anal.* 24 (1967) 82–90.
- [5] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.
- [6] K. Goebel, S. Reich, in: *Uniform Convexity Hyperbolic Geometry, and Nonexpansive Mappings*, vol. 73, Marcel Dekker, New York, NY, USA, 1984.
- [7] W. Takahashi, J.C. Yao, Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces, *Taiwanese J. Math.* 15 (2011) 457–472.
- [8] W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* 4 (1953) 506–510.
- [9] A. Genat, J. Lindenstrass, An example concerning fixed points, *Israel J. Math.* 22 (1975) 81–86.
- [10] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* 67 (1979) 274–276.
- [11] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mapping, *J. Math. Anal. Appl.* 158 (1991) 407–413.
- [12] K. Nakajo, W. Takahashi, Strongly convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.* 279 (2003) 372–379.
- [13] A.R. Khan, F. Gürsoy, V. Karakaya, Jungck–Khan iterative scheme and higher convergence rate, *Int. J. Comput. Math.* (2015). <http://dx.doi.org/10.1080/00207160.2015.1085030>.
- [14] V. Berinde, Picard iteration converges faster than the Mann iteration in the class of quasi-contractive operators, *Fixed Point Theory Appl.* 2 (2004) 97–105.
- [15] J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.* 136 (4) (2008) 1359–1373.
- [16] J. Tiammee, A. Kaewkhao, S. Suantai, On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, *Fixed Point Theory Appl.* 2015 (2015) 187. <http://dx.doi.org/10.1186/s13663-015-0436-9>.
- [17] M.R. Alfuraidan, Fixed points of monotone nonexpansive mappings with a graph, *Fixed Point Theory Appl.* 2015 (2015) 49. <http://dx.doi.org/10.1186/s13663-015-0299-0>.
- [18] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

- [19] S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, *Nonlinear Anal.* 69 (2008) 1025–1033.
- [20] T.H. Kim, H.K. Xu, Strongly convergence of modified Mann iterations for with asymptotically nonexpansive mappings and semigroups, *Nonlinear Anal.* 64 (2006) 1140–1152.
- [21] C.E. Chidume, J.N. Ezeora, Krasnoselkii-type algorithm for family of multi-valued strictly pseudo-contractive mappings, *Fixed Point Theory Appl.* 2014 (2014) 111.
- [22] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.* 329 (2007) 336–346.

Chapter IV

Split Common Fixed point and Null point Problems for Nonlinear Mappings in Hilbert spaces

4.1 Split common fixed point and null point problems for demicontractive operators in Hilbert spaces

In this section, we consider a split common fixed point and null point problem which includes the split common fixed point problem, the split common null problem and other problems related to the fixed point problem and the null point problem. We introduce an algorithm for studying the split common fixed point and null problem for demicontractive operators and maximal monotone operators in real Hilbert spaces. We establish a strong convergence result under some suitable conditions and reduce our main result to above-mentioned problems. Moreover, we also apply our main results to the split equilibrium problem. Finally, we give numerical results to demonstrate the convergence of our algorithms.

1. Introduction

The split common fixed point problem (SCFP) is a generalization of the split feasibility problem (SFP). This problem was studied by many authors [7,9,17–19,23,26] due to its applications are desirable and can be used in real-world applications, for example, in image processing, in signal processing, in modelling inverse problems, in image reconstruction, in computerized tomography, in the intensity-modulated radiation therapy, etc [4,8,10,11]. Let us recall the SFP and the SCFP:

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 , respectively, and let $A : H_1 \rightarrow H_2$ be a bounded linear operator. The SFP is formulated as finding a point

$$x \in C \quad \text{such that} \quad Ax \in Q. \quad (1)$$

Censor and Elfving [8] were the first who introduced the SFP for modelling inverse problems in finite-dimensional Hilbert spaces. Recently, many authors introduced various algorithms for solving the SFP [3,21,27,29]. It is known that x solves the SFP (1) if and only if x solves the

fixed point equation:

$$P_C(I - \gamma A^*(I - P_Q)A)x = x, \quad (2)$$

where A^* is the adjoint operator of A and $\gamma > 0$. Let $S : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ be two operators with nonempty fixed point sets $F(S) := \{x \in H_1 : x = Sx\}$ and $F(T)$, respectively. The SCFP for two operators is to find a point

$$x \in F(S) \quad \text{such that} \quad Ax \in F(T). \quad (3)$$

We see that the SCFP (3) is immediately reduced to the SFP (1) when we set $S := P_C$ and $T := P_Q$.

In 2009, Censor and Segal [9] introduced the following algorithm:

$$x_{n+1} = S(x_n + \gamma A^t(T - I)Ax_n), \quad n \in \mathbb{N}, \quad (4)$$

for solving the SCFP (3) in finite-dimensional Hilbert spaces where S and T are directed operators and $\mathbb{N} := \{1, 2, 3, \dots\}$.

After that in 2011, by modification of Mann's iteration, Moudafi [19] introduced the following algorithm:

$$\begin{aligned} y_n &= x_n + \gamma \beta A^*(T - I)Ax_n, \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n Sy_n, \quad n \in \mathbb{N}, \end{aligned} \quad (5)$$

for solving the SCFP (3) in the infinite-dimensional real Hilbert spaces and proved a weak convergence result of this algorithm where S and T are quasi-nonexpansive operators such that $I - S$ and $I - T$ are demiclosed at zero.

In 2014, Saejung and Kraikaew [17] proposed the following algorithm for solving the SCFP (3) where S is a strongly quasi-nonexpansive operator and T is a quasi-nonexpansive operator:

$$\begin{aligned} x_0 &\in H_1, \\ x_{n+1} &= \alpha_n x_0 + (1 - \alpha_n)S(x_n + \gamma A^*(T - I)Ax_n), \quad n \in \mathbb{N}, \end{aligned} \quad (6)$$

They also proved a strong convergence result when $\gamma \in (0, 1/\|A\|^2)$, $\alpha_n \in (0, 1)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The SCFP (3) for demicontractive operators was first studied by Moudafi [18]. He proved weak convergence of the algorithm (5) for this problem under some control conditions. Recently, in 2016, Shehu and Choleamjiak [23] modified the algorithm (5) to obtain a strong convergence result for demicontractive operators.

Let us recall the split common null point problem (SCNP). Byrne *et al.* [5] introduced and studied the SCNP for set-valued operators in Hilbert spaces: Given set-valued operators $S_i : H_1 \rightarrow 2^{H_1}$, $1 \leq i \leq p$, and $T_j : H_2 \rightarrow 2^{H_2}$, $1 \leq j \leq r$, respectively, and bounded linear operators $A_j : H_1 \rightarrow H_2$, $1 \leq j \leq r$, the problem is to find a point

$$z \in H_1 \quad \text{such that} \quad 0 \in \bigcap_{i=1}^p S_i(z) \quad \text{and} \quad 0 \in \bigcap_{j=1}^r T_j(A_j z). \quad (7)$$

In the case $p = r = 1$, above problem is reduced to find a point

$$z \in H_1 \quad \text{such that} \quad 0 \in B_1(z) \quad \text{and} \quad 0 \in B_2(Az), \quad (8)$$

where $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ are set-valued operators and $A : H_1 \rightarrow H_2$ is a bounded linear operator.

Byrne *et al.* [5] proposed algorithms and proved convergence theorems for finding such a solution of the SCNP (8) when B_1 and B_2 are maximal monotone operators.

Recently, in 2015, Takahashi *et al.* [26] considered the following problem: Find a point

$$z \in H_1 \quad \text{such that} \quad z \in F(S) \cap B^{-1}0 \quad \text{and} \quad Az \in F(T), \quad (9)$$

where $S : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ are two single-valued operators and $B : H_1 \rightarrow 2^{H_1}$ is a set-valued operator. Note that $B^{-1}0 := \{x \in H : 0 \in Bx\}$ is a null point set of B . They introduced an algorithm for solving above problem as shown below.

THEOREM 1.1 [26] *Let C be a nonempty closed convex subset of H_1 . Let $B : H_1 \rightarrow 2^{H_1}$ be a maximal monotone operator such that the domain of B is included in C and let J_λ^B be the resolvent of B for $\lambda > 0$. Let $S : C \rightarrow C$ be a generalized hybrid operator and $T : H_2 \rightarrow H_2$ a nonexpansive operator. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Suppose that $\Gamma := F(S) \cap B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. For any $x_1 \in C$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S(J_{\lambda_n}^B (I - \lambda_n A^* (I - T) A) x_n), \quad n \in \mathbb{N}, \quad (10)$$

where $0 < c \leq \beta_n \leq d < 1$ and $0 < a \leq \lambda_n \leq b < 1/\|A\|^2$. Then the sequence $\{x_n\}$ converges weakly to a point $z_0 \in \Gamma$, where $z_0 = \lim_{n \rightarrow \infty} P_\Gamma(x_n)$.

In this article, inspired and motivated by these works, we are interested to study a split common fixed point and null point problem which is more general than the problem (9). Our problem can be reduced to the SCFP, the split common null problem and other problems which are connected with the fixed point problem and the null point problem. We introduce an algorithm for studying the split common fixed point and null point problem for demicontractive operators and maximal monotone operators, and prove a strong convergence theorem of the proposed algorithm under some suitable conditions. Moreover, we reduce our main result to above-mentioned problems. As applications, we also apply our main results to the split equilibrium problem. In the last section, we give numerical examples to show the convergence of our algorithms.

2. Preliminaries

Throughout this paper, let $\mathbb{R}$ be the set of real numbers. We shall assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, and let I be the identity operator on H . We denote the strong and weak convergence of a sequence $\{x_n\}$ in H to an element $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. Let C be a nonempty closed convex subset of H . Recall that the (metric) projection from H onto C , denoted by P_C is defined for each $x \in H$, $P_C x$ is the unique element in C such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is known that $P_C x \in C$ is characterized by the following property:

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \quad \text{for all } y \in C.$$

For an operator $T : H \rightarrow H$ and $\delta \in \mathbb{R}$, define the operator $T_\delta : H \rightarrow H$ by

$$T_\delta := I + \delta(T - I).$$

It is clear that $(T_\delta)_\lambda = T_{\delta\lambda}$ for all $\delta, \lambda \in \mathbb{R}$. Moreover, $F(T) = F(T_\delta)$ for $\delta \neq 0$.

Now let us recall the definitions of some operators concerned in our study.

Definition 2.1 An operator $T : H \rightarrow H$ is said to be

(i) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in H,$$

(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \quad \text{for all } x \in H, \quad p \in F(T),$$

(iii) *strictly quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| < \|x - p\| \quad \text{for all } x \notin F(T), \quad p \in F(T),$$

(iv) *strongly quasi-nonexpansive* if T is quasi-nonexpansive and

$$x_n - Tx_n \rightarrow 0,$$

whenever $\{x_n\}$ is a bounded sequence in H and $\|x_n - p\| - \|Tx_n - p\| \rightarrow 0$ for some $p \in F(T)$,

(v) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2 \quad \text{for all } x, y \in H,$$

(vi) *firmly quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2 \quad \text{for all } x \in H, \quad p \in F(T),$$

(vii) *strictly pseudocontractive* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2 \quad \text{for all } x, y \in H,$$

(viii) *demicontractive* if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2 \quad \text{for all } x \in H, \quad p \in F(T).$$

Remark 2.2 We easily observe that the class of demicontractive operators includes the class of (i)–(vii) operators in Definition 2.1.

Example 2.3 [15] Let H be the real line and $C = [-1, 1]$. Define T on C by

$$Tx = \begin{cases} \frac{2}{3}x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This operator is demicontractive which is not strictly pseudocontractive.

Example 2.4 [12] Let H be the real line and $C = [-2, 1]$. Define T on C by $Tx = -x^2 - x$. This operator is demicontractive which is not quasi-nonexpansive.

We need a class of operators called by Yamada and Ogura [28] as α -attracting quasi-nonexpansive.

Definition 2.5 An operator $T : H \rightarrow H$ is said to be α -attracting quasi-nonexpansive where $\alpha \geq 0$ if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \alpha\|x - Tx\|^2 \quad \text{for all } x \in H, p \in F(T).$$

Note that a firmly quasi-nonexpansive operator T is 1-attracting quasi-nonexpansive.

PROPOSITION 2.6 A α -attracting quasi-nonexpansive operator $T : H \rightarrow H$ where $\alpha > 0$ is strongly quasi-nonexpansive.

Proof Let T be a α -attracting quasi-nonexpansive operator, where $\alpha > 0$. Then

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|x - Tx\|^2, \quad (11)$$

for all $x \in H, z \in F(T)$. By (11), we have $\|Tx - z\| \leq \|x - z\|$ for all $x \in H, z \in F(T)$ and hence T is quasi-nonexpansive. Let $\{x_n\}$ be a bounded sequence in H and $\|x_n - p\| - \|Tx_n - p\| \rightarrow 0$ for some $p \in F(T)$. Since a sequence $\{\|x_n - p\| + \|Tx_n - p\|\}$ is bounded, in view of (11), we have

$$\begin{aligned} \alpha\|x_n - Tx_n\|^2 &\leq \|x_n - p\|^2 - \|Tx_n - p\|^2 \\ &\leq (\|x_n - p\| - \|Tx_n - p\|)(\|x_n - p\| + \|Tx_n - p\|) \rightarrow 0. \end{aligned}$$

This implies that

$$\|x_n - Tx_n\| \rightarrow 0.$$

Hence T is strongly quasi-nonexpansive. ■

Next, we give some important tools for proving our results.

LEMMA 2.7 [6] Assume that $T : H \rightarrow H$ has a fixed point and let $\delta \in (0, 2]$. Then T is firmly quasi-nonexpansive if and only if the operator T_δ is $((2 - \delta)/\delta)$ -attracting quasi-nonexpansive.

Remark 2.8 In the case $\delta > 2$, by using the same proof as in [6, Theorem 2.1.39], we obtain that T is firmly quasi-nonexpansive if and only if T_δ is $((\delta - 2)/\delta)$ -demiccontractive.

LEMMA 2.9 [6] Let $T : H \rightarrow H$ be a strictly quasi-nonexpansive operator and $S : H \rightarrow H$ a quasi-nonexpansive operator. Suppose that $F(T) \cap F(S) \neq \emptyset$. Then

$$F(TS) = F(ST) = F(T) \cap F(S).$$

LEMMA 2.10 [7] Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and $T : H_2 \rightarrow H_2$ be a quasi-nonexpansive operator such that the equation $(I - T)Ax = 0$ has a solution. Let $V := I + (1/\|A\|^2)A^*(T - I)A$, then the following hold:

- (i) $Ax \in F(T)$ if and only if $x \in F(V)$;
- (ii) If $I - T$ is demiclosed at zero, then $I - V$ is also demiclosed at zero.
- (iii) V is quasi-nonexpansive.

Remark 2.11 Using the same proof as in [7, Lemma 4.1], we obtain that Lemma 2.10 is still true when $V = I + \gamma A^*(T - I)A$, where $\gamma \in (0, 1/\|A\|^2)$.

Definition 2.12 An operator $T : H \rightarrow H$ is called *demiclosed at zero* if for any sequence $\{x_n\}$ which converges weakly to x , and if the sequence $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$.

LEMMA 2.13 [20] *If $T : H \rightarrow H$ is a nonexpansive operator, then the operator $I - T$ is demiclosed at zero.*

Recall that $f : C \rightarrow C$ is called a θ -contraction with respect to a nonempty subset F of C , where $\theta \in [0, 1)$ if $\|f(x) - f(z)\| \leq \theta\|x - z\|$ for all $x \in C$ and $z \in F$.

THEOREM 2.14 [1] *Let C be a nonempty closed convex subset of H and let $T : C \rightarrow C$ be a strongly quasi-nonexpansive operator such that $I - T$ is demiclosed at zero. Let $f : C \rightarrow C$ be a θ -contraction with respect to $F := F(T)$, where $0 \leq \theta < 1$. Suppose that $\{x_n\}$ is a sequence generated iteratively by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}, \quad (12)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z_0 \in F$, where z_0 is the unique fixed point of a contraction $P_F f$.

Let us recall the maximal monotone operator. Let B be a set-valued operator of H into 2^H . The effective domain of B is denoted by $D(B)$, that is,

$$D(B) := \{x \in H : Bx \neq \emptyset\}.$$

Then B is called a maximal monotone operator if B is monotone, i.e.

$$\langle x - y, u - v \rangle \geq 0 \quad \text{for all } x, y \in D(B), \quad u \in Bx, \quad v \in By,$$

and the graph $G(B)$ of B ,

$$G(B) := \{(x, u) \in H \times H : u \in Bx\},$$

is not properly contained in the graph of any other monotone operator. For a maximal monotone operator $B : H \rightarrow 2^H$ and $\lambda > 0$, the resolvent of B with parameter λ is denoted and defined by

$$J_\lambda^B := (I + \lambda B)^{-1} : H \rightarrow D(B).$$

It is known [2] that if $B : H \rightarrow 2^H$ is a maximal monotone operator and $\lambda > 0$, then J_λ^B is single-valued, firmly nonexpansive and $F(J_\lambda^B) = B^{-1}0 = \{x \in H : 0 \in Bx\}$.

Let $h : H \rightarrow (-\infty, \infty]$ be a convex function, then a subdifferential ∂h of h at $x \in H$ is defined by

$$\partial h(x) := \{z \in H : h(x) + \langle z, y - x \rangle \leq h(y), \quad \forall y \in H\}.$$

In [22], it was shown that ∂h is maximal monotone where h is a proper, lower semicontinuous and convex function. In particular, if h is a constant function, it is not difficult to show that $\partial h = 0$, where 0 is a zero operator. Then 0 is maximal monotone and $J_\lambda^0 = I$ for $\lambda > 0$.

3. Some properties of quasi-nonexpansive operators

In this section, we present some properties of attracting quasi-nonexpansive operators related to quasi-nonexpansive, strongly quasi-nonexpansive and demicontractive operators. We first give

some relationship between an operator $T : H_2 \rightarrow H_2$ and the operator $I + \gamma A^*(T - I)A$, where $A : H_1 \rightarrow H_2$ is a bounded linear operator.

LEMMA 3.1 *Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and $T : H_2 \rightarrow H_2$ be a quasi-nonexpansive operator such that the equation $(I - T)Ax = 0$ has a solution. For a positive real number γ , define the operator $W : H_1 \rightarrow H_1$ by*

$$W := I + \gamma A^*(T - I)A.$$

If $\gamma \in (0, 1/\|A\|^2)$, then W is $((1 - \gamma\|A\|^2)/\gamma\|A\|^2)$ -attracting quasi-nonexpansive.

Proof Let $\lambda := \gamma\|A\|^2 \in (0, 1)$. Then we can write the operator W in the form $W = V_\lambda = (V_{1/2})_{2\lambda}$ where $V = I + (1/\|A\|^2)A^*(T - I)A$. By Lemma 2.10 (iii), we have V is quasi-nonexpansive. By applying Lemma 2.7, we obtain $V_{1/2}$ is firmly quasi-nonexpansive and W is $((1 - \lambda)/\lambda)$ -attracting quasi-nonexpansive. ■

Remark 3.2 In Lemma 3.1, if T is a strongly quasi-nonexpansive operator, then W is also strongly quasi-nonexpansive.

The following theorem gives some properties of the composition of attracting quasi-nonexpansive operators and strongly quasi-nonexpansive operators which is a generalization of [7, Theorem 4.2], in the case of two operators.

THEOREM 3.3 *Let $T : H \rightarrow H$ be a α -attracting quasi-nonexpansive operator where $\alpha > 0$ and $S : H \rightarrow H$ a strongly quasi-nonexpansive operator. Suppose that $F(T) \cap F(S) \neq \emptyset$. Then*

- (i) *Both TS and ST are strongly quasi-nonexpansive;*
- (ii) *If $I - T$ and $I - S$ are demiclosed at zero, then $I - TS$ and $I - ST$ are also demiclosed at zero.*

Proof By definition of T and S , we see that T is strictly quasi-nonexpansive and S is quasi-nonexpansive. From Lemma 2.9, $F(TS) = F(ST) = F(T) \cap F(S)$. By the quasi-nonexpansivity of T and S , we have

$$\|TSx - z\| \leq \|Sx - z\| \leq \|x - z\|, \quad (13)$$

and

$$\|STx - z\| \leq \|Tx - z\| \leq \|x - z\| \quad (14)$$

for all $x \in H, z \in F(T) \cap F(S)$. Hence TS and ST are quasi-nonexpansive.

(i) We show that TS is strongly quasi-nonexpansive. Let $\{x_n\}$ be a bounded sequence in H and $\|x_n - p\| - \|TSx_n - p\| \rightarrow 0$ for some $p \in F(T) \cap F(S)$. By (13), we have

$$0 \leq \|x_n - p\| - \|Sx_n - p\| \leq \|x_n - p\| - \|TSx_n - p\| \rightarrow 0,$$

Since S is strongly quasi-nonexpansive, we get

$$x_n - Sx_n \rightarrow 0. \quad (15)$$

By (13) again, we also have

$$0 \leq \|Sx_n - p\| - \|TSx_n - p\| \leq \|x_n - p\| - \|TSx_n - p\| \rightarrow 0.$$

By the boundedness of a sequence $\{\|Sx_n - p\| + \|TSx_n - p\|\}$, we have

$$\|Sx_n - p\|^2 - \|TSx_n - p\|^2 \rightarrow 0.$$

Since T is α -attracting quasi-nonexpansive where $\alpha > 0$, we have

$$\alpha \|TSx_n - Sx_n\|^2 \leq \|Sx_n - p\|^2 - \|TSx_n - p\|^2 \rightarrow 0.$$

It follows that

$$\|TSx_n - Sx_n\| \rightarrow 0. \quad (16)$$

From (15) and (16), we obtain

$$\|TSx_n - x_n\| \leq \|TSx_n - Sx_n\| + \|Sx_n - x_n\| \rightarrow 0.$$

Therefore, TS is strongly quasi-nonexpansive. Similarly, by using inequality (14), we can prove that ST is strongly quasi-nonexpansive.

(ii) Suppose that $I - T$ and $I - S$ are demiclosed at zero. We show that $I - TS$ is demiclosed at zero. Let $\{y_n\}$ be a sequence such that $y_n \rightharpoonup y$ for some $y \in H$ and $y_n - TSy_n \rightarrow 0$. Choose an arbitrary $z \in F(T) \cap F(S)$. By (13), we obtain that

$$0 \leq \|y_n - z\| - \|Sy_n - z\| \leq \|z - y_n\| - \|TSy_n - z\| \leq \|TSy_n - y_n\| \rightarrow 0.$$

We see that $\{y_n\}$ is bounded because $y_n \rightharpoonup y$. Since S is strongly quasi-nonexpansive, we obtain that

$$y_n - Sy_n \rightarrow 0. \quad (17)$$

By the demiclosedness of $I - S$ at zero, $y \in F(S)$. From (13) again, we have

$$0 \leq \|Sy_n - z\| - \|TSy_n - z\| \leq \|z - y_n\| - \|TSy_n - z\| \leq \|TSy_n - y_n\| \rightarrow 0.$$

By the boundedness of a sequence $\{\|Sy_n - z\| + \|TSy_n - z\|\}$, we get

$$\|Sy_n - z\|^2 - \|TSy_n - z\|^2 \rightarrow 0.$$

Since T is α -attracting quasi-nonexpansive where $\alpha > 0$, we have

$$\alpha \|TSy_n - Sy_n\|^2 \leq \|Sy_n - z\|^2 - \|TSy_n - z\|^2 \rightarrow 0,$$

it follows that

$$\|TSy_n - Sy_n\| \rightarrow 0.$$

From (17) and $y_n \rightharpoonup y$, we get $Sy_n \rightharpoonup y$. By the demiclosedness of $I - T$ at zero, we obtain that $y \in F(T)$ and hence $y \in F(T) \cap F(S) = F(TS)$. Therefore, $I - TS$ is demiclosed at zero. Similarly, by applying inequality (14), we can show that $I - ST$ is also demiclosed at zero. ■

The following lemma gives some relationship between demicontractive operators and attracting quasi-nonexpansive operators.

LEMMA 3.4 *Assume that $T : H \rightarrow H$ has a fixed point, and let $k \in [0, 1)$ and $\delta \in (0, 1 - k)$. Then T is k -demicontractive if and only if the operator T_δ is $((1 - k - \delta)/\delta)$ -attracting quasi-nonexpansive.*

Proof Since $2/(1 - k) > 2$ and by Remark 2.8, we get $T_{(1-k)/2}$ is firmly quasi-nonexpansive if and only if T is k -demicontractive. Now we consider the operator $T_\delta = (T_{(1-k)/2})^{2\delta/(1-k)}$. Since $2\delta/(1 - k) \in (0, 2)$ and by using Lemma 2.7, we have $T_{(1-k)/2}$ is firmly quasi-nonexpansive if and only if T_δ is $((1 - k - \delta)/\delta)$ -attracting quasi-nonexpansive. This completes the proof. ■

4. Split common fixed point and null point problems

We first present an algorithm for studying the split common fixed point and null problem for demicontractive operators and maximal monotone operators and prove a strong convergence theorem.

THEOREM 4.1 *Let $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ be maximal monotone operators and let $J_{\lambda_1}^{B_1}$ and $J_{\lambda_2}^{B_2}$ be resolvents of B_1 and B_2 , respectively for $\lambda_1, \lambda_2 > 0$. Let $S : H_1 \rightarrow H_1$ be a k_1 -demicontractive operator and $T : H_2 \rightarrow H_2$ a k_2 -demicontractive operator such that both $I - S$ and $I - T$ are demiclosed at zero. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $\Omega := \{x \in F(S) \cap B_1^{-1}0 : Ax \in F(T) \cap B_2^{-1}0\} \neq \emptyset$. Let $f : H_1 \rightarrow H_1$ be a θ -contraction with respect to Ω , where $0 \leq \theta < 1$. Suppose that $\{x_n\} \subset H_1$ is a sequence generated by $x_1 \in H_1$ and*

$$\begin{aligned} y_n &= x_n + \gamma A^*(T_{\delta_2} J_{\lambda_2}^{B_2} - I)Ax_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)S_{\delta_1}(J_{\lambda_1}^{B_1} y_n), \quad n \in \mathbb{N}, \end{aligned} \quad (18)$$

where the parameters $\gamma, \delta_1, \delta_2$ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

- (C1) $\gamma \in (0, 1/\|A\|^2)$, $\delta_1 \in (0, 1 - k_1)$, and $\delta_2 \in (0, 1 - k_2)$;
 (C2) $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where z_0 is the unique fixed point of a contraction $P_{\Omega}f$.

Proof Set $U := S_{\delta_1} J_{\lambda_1}^{B_1}$ and $V := T_{\delta_2} J_{\lambda_2}^{B_2}$. By Lemma 3.4, we obtain S_{δ_1} is $((1 - k_1 - \delta_1)/\delta_1)$ -attracting quasi-nonexpansive and T_{δ_2} is $((1 - k_2 - \delta_2)/\delta_2)$ -attracting quasi-nonexpansive. Since $J_{\lambda_1}^{B_1}$ and $J_{\lambda_2}^{B_2}$ are firmly nonexpansive, by Proposition 2.6 we have both $J_{\lambda_1}^{B_1}$ and $J_{\lambda_2}^{B_2}$ are strongly quasi-nonexpansive. It follows from Theorem 3.3 (i) that U and V are strongly quasi-nonexpansive. Clearly, $I - S_{\delta_1}$ and $I - T_{\delta_2}$ are demiclosed at zero. By Lemma 2.13, $I - J_{\lambda_1}^{B_1}$ and $I - J_{\lambda_2}^{B_2}$ are demiclosed at zero. By Theorem 3.3 (ii), $I - U$ and $I - V$ are also demiclosed at zero. Next, let $W := I + \gamma A^*(V - I)A$. By Lemma 3.1 and Remark 2.11, we obtain W is $((1 - \gamma\|A\|^2)/\gamma\|A\|^2)$ -attracting quasi-nonexpansive and $I - W$ is demiclosed at zero. From Lemma 2.9 and Remark 2.11, we get

$$\begin{aligned} \emptyset \neq \Omega &= F(S) \cap B_1^{-1}0 \cap A^{-1}(F(T) \cap B_2^{-1}0) \\ &= F(S_{\delta_1}) \cap F(J_{\lambda_1}^{B_1}) \cap A^{-1}(F(T_{\delta_2}) \cap F(J_{\lambda_2}^{B_2})) \\ &= F(U) \cap A^{-1}F(V) \\ &= F(U) \cap F(W) = F(UW). \end{aligned}$$

By using Theorem 3.3 (i) and (ii) again, we have UW is strongly quasi-nonexpansive and $I - UW$ is demiclosed at zero. We also have f is a θ -contraction with respect to $F(UW)$, where $0 \leq \theta < 1$. By Theorem 2.14, we can conclude that $\{x_n\}$ converges to $z_0 \in \Omega$, where $z_0 = P_{\Omega}f(z_0)$. This completes the proof. $\blacksquare$

When we take $B_2 = 0$ in Theorem 4.1, we obtain the following corollary.

COROLLARY 4.2 *Let $B : H_1 \rightarrow 2^{H_1}$ be a maximal monotone operators and let J_{λ}^B be the resolvent of B for $\lambda > 0$. Let $S : H_1 \rightarrow H_1$ be a k_1 -demicontractive operator and $T : H_2 \rightarrow H_2$ a k_2 -demicontractive operator such that both $I - S$ and $I - T$ are demiclosed at zero. Let $A : H_1 \rightarrow$*

H_2 be a bounded linear operator. Assume that $\Gamma := \{x \in F(S) \cap B^{-1}0 : Ax \in F(T)\} \neq \emptyset$. Let $f : H_1 \rightarrow H_1$ be a θ -contraction with respect to Γ , where $0 \leq \theta < 1$. Suppose that $\{x_n\} \subset H_1$ is a sequence generated by $x_1 \in H_1$ and

$$\begin{aligned} y_n &= x_n + \gamma A^*(T - I)Ax_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)S_\delta(J_\lambda^B y_n), \quad n \in \mathbb{N}, \end{aligned} \quad (19)$$

where the parameters γ, δ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

- (C1) $\gamma \in (0, (1 - k_2)/\|A\|^2)$, and $\delta \in (0, 1 - k_1)$;
(C2) $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Gamma$, where z_0 is the unique fixed point of a contraction $P_\Gamma f$.

When we take $S = I$ in Corollary 4.2, we obtain the following result.

COROLLARY 4.3 Let $B : H_1 \rightarrow 2^{H_1}$ be a maximal monotone operators and let J_λ^B be the resolvent of B for $\lambda > 0$. Let $T : H_2 \rightarrow H_2$ be a k -demicontractive operator such that $I - T$ is demiclosed at zero. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $\Gamma := \{x \in B^{-1}0 : Ax \in F(T)\} \neq \emptyset$. Let $f : H_1 \rightarrow H_1$ be a θ -contraction with respect to Γ , where $0 \leq \theta < 1$. Suppose that $\{x_n\} \subset H_1$ is a sequence generated by $x_1 \in H_1$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J_\lambda^B(x_n + \gamma A^*(T - I)Ax_n), \quad n \in \mathbb{N}, \quad (20)$$

where the parameters γ, δ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

- (C1) $\gamma \in (0, (1 - k)/\|A\|^2)$;
(C2) $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Gamma$, where z_0 is the unique fixed point of a contraction $P_\Gamma f$.

When we take $B = 0$ in Corollary 4.2, we obtain the algorithm which converges strongly to the SCFP (3) for demicontractive operators.

THEOREM 4.4 Let $S : H_1 \rightarrow H_1$ be a k_1 -demicontractive operator and $T : H_2 \rightarrow H_2$ a k_2 -demicontractive operator such that both $I - S$ and $I - T$ are demiclosed at zero. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $\Gamma := \{x \in F(S) : Ax \in F(T)\} \neq \emptyset$. Let $f : H_1 \rightarrow H_1$ be a θ -contraction with respect to Γ , where $0 \leq \theta < 1$. Suppose that $\{x_n\} \subset H_1$ is a sequence generated by $x_1 \in H_1$ and

$$\begin{aligned} y_n &= x_n + \gamma A^*(T - I)Ax_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)S_\delta y_n, \quad n \in \mathbb{N}, \end{aligned} \quad (21)$$

where the parameters γ, δ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

- (C1) $\gamma \in (0, (1 - k_2)/\|A\|^2)$, and $\delta \in (0, 1 - k_1)$;
(C2) $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Gamma$, where z_0 is the unique fixed point of a contraction $P_\Gamma f$.

When we take $S = T = I$ in Theorem 4.1, we obtain the algorithm converging strongly to the SCNP (8).

COROLLARY 4.5 *Given two set-valued maximal monotone operators $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ and let $J_{\lambda_1}^{B_1}$ and $J_{\lambda_2}^{B_2}$ be resolvents of B_1 and B_2 , respectively for $\lambda_1, \lambda_2 > 0$. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $\Gamma := \{x \in B_1^{-1}0 : Ax \in B_2^{-1}0\} \neq \emptyset$. Let $f : H_1 \rightarrow H_1$ be a θ -contraction with respect to Γ , where $0 \leq \theta < 1$. Suppose that $\{x_n\} \subset H_1$ is a sequence generated by $x_1 \in H_1$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{\lambda_1}^{B_1}(x_n + \gamma A^*(J_{\lambda_2}^{B_2} - I)Ax_n), \quad n \in \mathbb{N}, \quad (22)$$

where the parameter γ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

- (C1) $\gamma \in (0, 1/\|A\|^2)$;
- (C2) $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Gamma$, where z_0 is the unique fixed point of a contraction $P_{\Gamma}f$.

Note that Corollary 4.5 is a result of Byrne *et al.* [5] in case that f is a constant function and the interval $(0, 1/\|A\|^2)$ of the parameter γ can be extended to $(0, 2/\|A\|^2)$ which is explained in [17].

5. Applications

Let C be a nonempty closed convex subset of H , and let $g : C \times C \rightarrow \mathbb{R}$ be a bifunction. Recall that the equilibrium problem is to find a point $z \in C$ such that

$$g(z, y) \geq 0 \quad \text{for all } y \in C. \quad (23)$$

The solution set of (23) is denoted by $EP(g)$. To study the equilibrium problem, we assume that the bifunction $g : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions:

- (A1) $g(x, x) = 0$ for all $x \in C$;
- (A2) g is monotone, i.e. $g(x, y) + g(y, x) \leq 0$ for all $x, y \in C$;
- (A3) For each $x, y, z \in C$, $\limsup_{t \rightarrow 0} g(tz + (1-t)x, y) \leq g(x, y)$;
- (A4) For each $x \in C$, the function $y \mapsto g(x, y)$ is convex and lower semicontinuous.

LEMMA 5.1 [13] *Let C be a nonempty closed convex subset of H , and let g be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For any $r > 0$ and $x \in H$, define the resolvent $T_r^g : H \rightarrow C$ as follows:*

$$T_r^g x = \left\{ z \in C : g(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

- (i) T_r^g is nonempty and single-valued;
- (ii) T_r^g is firmly nonexpansive;
- (iii) $F(T_r^g) = EP(g)$;
- (iv) $EP(g)$ is closed and convex.

THEOREM 5.2 [25] *Let C be a nonempty closed convex subset of H , and let g be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Define A_g as follows:*

$$A_g(x) = \begin{cases} \{z \in H : g(x, y) \geq \langle y - x, z \rangle, \forall y \in C\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases}$$

Then the following hold:

- (i) A_g is maximal monotone;
- (ii) $EP(g) = A_g^{-1}0$;
- (iii) $T_r^g = (I + rA_g)^{-1}$, for $r > 0$.

Recently, Kazmi and Rizvi [16] studied the split equilibrium problem (SEP): Let C and Q be nonempty closed convex subsets of H_1 and H_2 , respectively. Let $g_1 : C \times C \rightarrow \mathbb{R}$ and $g_2 : Q \times Q \rightarrow \mathbb{R}$ be two bifunctions and $A : H_1 \rightarrow H_2$ a bounded linear operator, then the SEP is to find a point $x^* \in C$ such that

$$g_1(x^*, x) \geq 0 \quad \text{for all } x \in C, \quad (24)$$

and

$$y^* := Ax^* \in Q \text{ solves } g_2(y^*, y) \geq 0 \quad \text{for all } y \in Q. \quad (25)$$

Then above problem is to find a point $x^* \in C$ such that

$$x^* \in EP(g_1) \quad \text{and} \quad Ax^* \in EP(g_2). \quad (26)$$

The following result is obtained directly from Theorem 4.1.

THEOREM 5.3 *Let C and Q be nonempty closed convex subsets of H_1 and H_2 , respectively. Let $g_1 : C \times C \rightarrow \mathbb{R}$ and $g_2 : Q \times Q \rightarrow \mathbb{R}$ be bifunctions satisfying (A1)–(A4) and let $T_{r_1}^{g_1}$ and $T_{r_2}^{g_2}$ be resolvents of A_{g_1} and A_{g_2} in Theorem 5.2, respectively for $r_1, r_2 > 0$. Let $S : H_1 \rightarrow H_1$ be a k_1 -demicontractive operator and $T : H_2 \rightarrow H_2$ a k_2 -demicontractive operator such that both $I - S$ and $I - T$ are demiclosed at zero. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $\Omega := \{x \in F(S) \cap EP(g_1) : Ax \in F(T) \cap EP(g_2)\} \neq \emptyset$. Let $f : H_1 \rightarrow H_1$ be a θ -contraction with respect to Ω , where $0 \leq \theta < 1$. Suppose that $\{x_n\} \subset H_1$ is a sequence generated by $x_1 \in H_1$ and*

$$\begin{aligned} y_n &= x_n + \gamma A^*((I + \delta_2(T - I))T_{r_2}^{g_2} - I)Ax_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)(I + \delta_1(S - I))T_{r_1}^{g_1}y_n, \quad n \in \mathbb{N}, \end{aligned} \quad (27)$$

where the parameters $\gamma, \delta_1, \delta_2$ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

- (C1) $\gamma \in (0, 1/\|A\|^2)$, $\delta_1 \in (0, 1 - k_1)$, and $\delta_2 \in (0, 1 - k_2)$;
- (C2) $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $z_0 \in \Omega$, where z_0 is the unique fixed point of a contraction $P_{\Omega}f$.

Proof We set $B_1 := A_{g_1}$ and $B_2 := A_{g_2}$. By Theorem 5.2, we know that B_1 and B_2 are maximal monotone, $EP(g_1) = B_1^{-1}0$, $EP(g_2) = B_2^{-1}0$, $T_{r_1}^{g_1} = J_{r_1}^{B_1}$ and $T_{r_2}^{g_2} = J_{r_2}^{B_2}$, so the result is obtained directly by Theorem 4.1. ■

The following result is obtained directly from Corollary 4.2.

THEOREM 5.4 Let C be a nonempty closed convex subset of H_1 . Let $g : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and let T_r^g be the resolvent of A_g in Theorem 5.2 for $r > 0$. Let $S : H_1 \rightarrow H_1$ be a k_1 -demicontractive operator and $T : H_2 \rightarrow H_2$ a k_2 -demicontractive operator such that both $I - S$ and $I - T$ are demiclosed at zero. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Assume that $\Gamma := \{x \in F(S) \cap EP(g) : Ax \in F(T)\} \neq \emptyset$. Let $f : H_1 \rightarrow H_1$ be a θ -contraction with respect to Γ , where $0 \leq \theta < 1$. Suppose that $\{x_n\} \subset H_1$ is a sequence generated by $x_1 \in H_1$ and

$$\begin{aligned} y_n &= x_n + \gamma A^*(T - I)Ax_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)(I + \delta(S - I))T_r^g y_n, \quad n \in \mathbb{N}, \end{aligned} \quad (28)$$

where the parameters γ , δ and the sequence $\{\alpha_n\}$ satisfy the following conditions:

- (C1) $\gamma \in (0, (1 - k_2)/\|A\|^2)$, and $\delta \in (0, 1 - k_1)$;
(C2) $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $z_0 \in \Gamma$, where z_0 is the unique fixed point of a contraction $P_\Gamma f$.

Proof We set $B := A_g$. By Theorem 5.2, we know that B is maximal monotone, $EP(g) = B^{-1}0$ and $T_r^g = J_r^B$, so the result is obtained directly by Corollary 4.2. ■

Remark 5.5 Let C and Q be nonempty closed convex subsets of H_1 and H_2 , respectively. Let $g_1 : C \times C \rightarrow \mathbb{R}$ and $g_2 : Q \times Q \rightarrow \mathbb{R}$ be bifunctions satisfying (A1)–(A4). Let $T : H_2 \rightarrow H_2$ be a k -demicontractive operator such that $I - T$ is demiclosed at zero and $A : H_1 \rightarrow H_2$ a bounded linear operator.

- (i) If we set $B := A_{g_1}$ in Corollary 4.3, we obtain an algorithm converging strongly to a point of the set $\{x \in EP(g_1) : Ax \in F(T)\}$.
(ii) By setting $B_1 := A_{g_1}$ and $B_2 := A_{g_2}$ in Corollary 4.5, we also obtain an algorithm converging strongly to a point of the set $\{x \in EP(g_1) : Ax \in EP(g_2)\}$.

6. Numerical examples

Example 6.1 Let $H_1 = \mathbb{R} = H_2$. Let $S : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$Sx = \begin{cases} \sqrt{x} & \text{if } x \geq 1, \\ x & \text{otherwise,} \end{cases}$$

and let $T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $Tx = \frac{2}{3}(x + 1)$. We see that both T and S are 0-demicontractive operators. Define a function $h : \mathbb{R} \rightarrow (-\infty, \infty]$ by

$$h(x) = \begin{cases} -\ln x + x & \text{if } x > 0, \\ \infty & \text{otherwise.} \end{cases}$$

Hence h is a proper, lower semicontinuous and convex function and $B := \partial h$ is maximal monotone. By [14], we obtain the explicit resolvent of B in the form:

$$J_1^B x = (I + B)^{-1}x = \frac{1}{2}(x - 1 + \sqrt{(x - 1)^2 + 4}).$$

We also define a bounded linear operator $A : \mathbb{R} \rightarrow \mathbb{R}$ by $Ax = 2x$. It is clear that $A^t x = 2x$ and $\|A\| = 2$. Let $\Gamma := \{x \in F(S) \cap B^{-1}0 : Ax \in F(T)\}$. We see that $1 \in \Gamma$. Put $\gamma = \frac{1}{8}$, $\delta = \frac{1}{2}$ and

$\alpha_n = 10^{-5}/n$ for all $n \in \mathbb{N}$, and let a contraction $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = 2$, then an algorithm (19) becomes

$$x_{n+1} = \begin{cases} \frac{1}{50000n} + \frac{1}{2} \left(1 - \frac{1}{100000n} \right) \left\{ \sqrt{\frac{1}{12} (5x_n - 5 + \sqrt{25(x_n - 1)^2 + 144})} \right. \\ \quad \left. + \frac{1}{12} (5x_n - 5 + \sqrt{25(x_n - 1)^2 + 144}) \right\} & \text{if } x_n \geq 1, \\ \frac{1}{50000n} + \frac{1}{12} \left(1 - \frac{1}{100000n} \right) (5x_n - 5 + \sqrt{25(x_n - 1)^2 + 144}) & \text{otherwise.} \end{cases} \quad (29)$$

We first start with the initial point $x_1 = 5$. The stopping criterion for our testing method is taken as: $|x_{n+1} - x_n| < 10^{-7}$. Now, a convergence of the algorithm (29) is shown in Table 1 and it converges to $1 \in \Gamma$.

Example 6.2 Let $S : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $Sx = \sin x$ and let $T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$Tx = \begin{cases} \sqrt{x} & \text{if } x \geq 1, \\ x & \text{otherwise.} \end{cases}$$

Let $C = [-9, 3]$, we define a bifunction $g : C \times C \rightarrow \mathbb{R}$ by

$$g(x, y) = y^2 + xy - 2x^2.$$

By [24], we can write the explicit resolvent of A_g in the form: $T_r^g x = x/(3r + 1)$ for $r > 0$. Define a bounded linear operator $A : \mathbb{R} \rightarrow \mathbb{R}$ by $Ax = 2x$. We consider the set $\Gamma := \{x \in F(S) \cap EP(g) : Ax \in F(T)\}$. It is clear that $0 \in \Gamma$. Let $\alpha_n = 1/40000n$ for all $n \in \mathbb{N}$, $\gamma = \frac{1}{10}$,

Table 1. Numerical experiment of the algorithm (29).

n	x_n	$ x_{n+1} - x_n $
2	2.755193	1.0682596
3	1.686933	0.4478996
4	1.239034	0.1612722
5	1.077761	0.0531321
$\vdots$	$\vdots$	$\vdots$
17	1.0000010	0.0000001
18	1.0000009	0.00000007

Table 2. Numerical experiment of the algorithm (30).

n	x_n	$ x_{n+1} - x_n $
2	1.355956	1.0721562
3	0.283799	0.2128757
4	0.070924	0.0531902
5	0.017734	0.0132976
$\vdots$	$\vdots$	$\vdots$
14	0.0000014	0.0000001
15	0.0000012	0.00000009

$\delta = \frac{1}{2}$, $r = 1$, and let a contraction $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = \frac{1}{2}$. Then an algorithm (28) becomes

$$x_{n+1} = \begin{cases} \frac{1}{80000n} + \frac{1}{2} \left(1 - \frac{1}{40000n} \right) \left\{ \sin \left(\frac{3x_n + \sqrt{2x_n}}{20} \right) + \frac{3x_n + \sqrt{2x_n}}{20} \right\} & \text{if } x_n \geq \frac{1}{2}, \\ \frac{1}{80000n} + \frac{1}{2} \left(1 - \frac{1}{40000n} \right) \left\{ \sin \left(\frac{x_n}{4} \right) + \frac{x_n}{4} \right\} & \text{otherwise.} \end{cases} \quad (30)$$

We start with the initial point $x_1 = 10$. The stopping criterion for our testing method is taken as: $|x_{n+1} - x_n| < 10^{-7}$. Now, a convergence of the algorithm (30) is shown in Table 2 and it converges to $0 \in \Gamma$.

References

- [1] K. Aoyama and F. Kohsaka, *Viscosity approximation process for a sequence of quasi-nonexpansive mappings*, Fixed Point Theory App. 2014 (2014), p. 17. 11 pages.
- [2] H.H Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.
- [3] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inv. Probl. 18 (2002), pp. 441–453.
- [4] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inv. Probl. 20 (2004), pp. 103–120.
- [5] C. Byrne, Y. Censor, A. Gibali, and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. 13 (2012), pp. 759–775.
- [6] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Lecture Notes in Mathematics, vol. 2057, Springer, Heidelberg, 2012.
- [7] A. Cegielski, *General method for solving the split common fixed point problem*, J. Optim. Theory Appl. 165 (2015), pp. 385–404.
- [8] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms 8 (1994), pp. 221–239.
- [9] Y. Censor and A. Segal, *The split common fixed point problem for directed operators*, J. Convex Anal. 16 (2009), pp. 587–600.
- [10] Y. Censor, T. Elfving, N. Kopf, and T. Bortfeld, *The multiple-sets split feasibility problem and its applications for inverse problems*, Inv. Probl. 21 (2005), pp. 2071–2084.
- [11] Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol. 51 (2006), pp. 2353–2365.
- [12] C.E. Chidume and S. Maruster, *Iterative methods for the computation of fixed points of demicontractive mappings*, J. Comput. Appl. Math. 234 (2010), pp. 861–882.
- [13] P.L. Combettes and A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. 6 (2005), pp. 117–136.
- [14] P.L. Combettes and J.C. Pesquet, *Proximal splitting methods in signal processing*, Fixed-Point Algor. Inv. Probl. Sci. Eng. 49 (2011), pp. 185–212.

- [15] T.L. Hicks and J.D. Kubicek, *On the Mann iteration process in a Hilbert space*, J. Math. Anal. Appl. 59 (1977), pp. 498–504.
- [16] K.R. Kazmi and S.H. Rizvi, *Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem*, J. Egypt. Math. Soc. 21 (2013), pp. 44–51.
- [17] R. Kraikaew and S. Saejung, *On split common fixed point problems*, J. Math. Anal. Appl. 415 (2014), pp. 513–524.
- [18] A. Moudafi, *The split common fixed-point problem for demicontractive mappings*, Inv. Probl. 26 (2010), pp. 587–600.
- [19] A. Moudafi, *A note on the split common fixed-point problem for quasi-nonexpansive operators*, Nonlinear Anal. 74 (2011), pp. 4083–4087.
- [20] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Am. Math. Soc. 73 (1967), pp. 591–598.
- [21] B. Qu and N. Xiu, *A note on the CQ algorithm for the split feasibility problem*, Inv. Probl. 21 (2005), pp. 1655–1665.
- [22] R.T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pac. J. Math. 33 (1970), pp. 209–216.
- [23] Y. Shehu and P. Chulamjiak, *Another look at the split common fixed point problem for demicontractive operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 110 (2016), pp. 201–218.
- [24] U. Singthong and S. Suantai, *Equilibrium problems and fixed point problems for nonspreading-type mappings in Hilbert space*, Int. J. Nonlinear Anal. Appl. 2 (2011), pp. 51–61.
- [25] S. Takahashi, W. Takahashi, and M.T. Toyoda, *Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces*, J. Optim. Theory Appl. 147 (2010), pp. 27–41.
- [26] W. Takahashi, H.-K. Xu, and J.C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal. 23 (2015), pp. 205–221.
- [27] H.K. Xu, *A variable Krasnosel'skiĭ-Mann algorithm and the multiple-set split feasibility problem*, Inv. Probl. 22 (2006), pp. 2021–2034.
- [28] I. Yanada and N. Ogura, *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, Numer. Funct. Anal. Optim. 25 (2004), pp. 619–655.
- [29] Q. Yang, *The relaxed CQ algorithm for solving the split feasibility problem*, Inv. Probl. 20 (2004), pp. 1261–1266.

4.2 On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces

In this section, we introduce and study iterative schemes for solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces and prove that the modified Mann iteration converges weakly to a common solution of the considered problems. Moreover, we present some examples and numerical results for the main results.

1 Introduction

In the following, let H_1 and H_2 be real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty subset of H_1 . The *equilibrium problem* is to find a point $\hat{x} \in C$ such that

$$F_1(\hat{x}, y) \geq 0 \quad (1.1)$$

for all $y \in C$. Since its inception by Blum and Oettli [1] in 1994, the equilibrium problem (1.1) has received much attention due to its applications in a large variety of problems arising in numerous problems in physics, optimizations, and economics. Some methods have been rapidly established for solving this problem (see [2–5]).

Very recently, Kazmi and Rizvi [6] introduced and studied the following split equilibrium problem:

Let $C \subseteq H_1$ and $Q \subseteq H_2$. Let $F_1 : C \times C \rightarrow \mathbb{R}$ and $F_2 : Q \times Q \rightarrow \mathbb{R}$ be two bifunctions. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. The *split equilibrium problem* is to find $\hat{x} \in C$ such that

$$F_1(\hat{x}, x) \geq 0 \quad \text{for all } x \in C \quad (1.2)$$

and such that

$$\hat{y} = A\hat{x} \in Q \text{ solves } F_2(\hat{y}, y) \geq 0 \quad \text{for all } y \in Q. \quad (1.3)$$

Note that the problem (1.2) is the classical equilibrium problem and we denote its solution set by $EP(F_1)$. The inequalities (1.2) and (1.3) constitute a pair of equilibrium problems which have to find the image $\hat{y} = A\hat{x}$, under a given bounded linear operator A , of the solution $\hat{x}$ of (1.2) in H_1 is the solution of (1.3) in H_2 . We denote the solution set of (1.3) by $EP(F_2)$. The solution set of the split equilibrium problem (1.2) and (1.3) is denoted by $\Omega = \{z \in EP(F_1) : Az \in EP(F_2)\}$.

A subset $C \subset H_1$ is said to be *proximal* if, for each $x \in H_1$,

$$\|x - y\| = d(x, C) = \inf\{\|x - z\| : z \in C\}.$$

Let $CB(C)$, $K(C)$, and $P(C)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximal bounded subset of C , respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}$$

for all $A, B \in CB(C)$ where $d(x, B) = \inf_{b \in B} \|x - b\|$. An element $p \in C$ is called a *fixed point* of $T : C \rightarrow CB(C)$ if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$. We say that $T : C \rightarrow CB(C)$ is:

(1) *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in C$;

(2) *quasi-nonexpansive* if

$$H(Tx, Tp) \leq \|x - p\|$$

for all $x \in C$ and $p \in F(T)$.

Recently, the existence of fixed points and the convergence theorems of multi-valued mappings have been studied by many authors (see [7–12]).

Hussain and Khan [13] presented the fixed point theorems of a $*$ -nonexpansive multi-valued mapping and the strong convergence of its iterates to a fixed point defined on a closed and convex subset of a Hilbert space by using the best approximation operator $P_T x$, which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$. The convergence theorems and its applications in this direction have been established by many authors (for instance, see [10, 14, 15]).

In 2011, Song and Cho [16] gave the example of a multi-valued mapping T which is not necessary nonexpansive, but P_T is nonexpansive. This is an important tool for studying the fixed point theory for multi-valued mappings.

Kohsaka and Takahashi [17] introduced a class of mappings which is called *nonspreading mapping*. Let C be a subset of Hilbert spaces H_1 . A mapping $T : C \rightarrow C$ is said to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. Subsequently, Iemoto and Takahashi [18] showed that $T : C \rightarrow C$ is non-spreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle$$

for all $x, y \in C$.

Very recently, Liu [19] introduced the following class of multi-valued mappings: a multi-valued mapping $T : C \rightarrow CB(C)$ is said to be *nonspreading* if

$$2\|u_x - u_y\|^2 \leq \|u_x - y\|^2 + \|u_y - x\|^2$$

for some $u_x \in Tx$ and $u_y \in Ty$ for all $x, y \in C$. He proved a weak convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points.

In this paper, we introduce, by using Hausdorff metric, the class of nonspreading multi-valued mappings. We say that a mapping $T : C \rightarrow CB(C)$ is a *k-nonspreading multi-valued mapping* if there exists $k > 0$ such that

$$H(Tx, Ty)^2 \leq k(d(Tx, y)^2 + d(x, Ty)^2) \quad (1.4)$$

for all $x, y \in C$.

It is easy to see that, if T is $\frac{1}{2}$ -nonspreading, then T is nonspreading in the case of single-valued mappings (see [17, 20]). Moreover, if T is a $\frac{1}{2}$ -nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Indeed, for all $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned} 2H(Tx, Tp)^2 &\leq d(Tx, p)^2 + d(x, Tp)^2 \\ &\leq H(Tx, Tp)^2 + \|x - p\|^2. \end{aligned}$$

It follows that

$$H(Tx, Tp) \leq \|x - p\|. \quad (1.5)$$

We now give an example of a $\frac{1}{2}$ -nonspreading multi-valued mapping which is not non-expansive.

Example 1.1 Consider $C = [-3, 0]$ with the usual norm. Define $T : C \rightarrow CB(C)$ by

$$Tx = \begin{cases} \{0\}, & x \in [-2, 0]; \\ [-\frac{|x|}{|x|+1}, 0], & x \in [-3, -2). \end{cases}$$

Now, we show that T is $\frac{1}{2}$ -nonspreading. In fact, we have the following cases:

Case 1: If $x, y \in [-2, 0]$, then $H(Tx, Ty) = 0$.

Case 2: If $x \in [-2, 0]$ and $y \in [-3, -2)$, then $Tx = \{0\}$ and $Ty = [-\frac{|y|}{|y|+1}, 0]$. This implies that

$$2H(Tx, Ty)^2 = 2\left(\frac{|y|}{|y|+1}\right)^2 < 2 < y^2 \leq d(Tx, y)^2 + d(x, Ty)^2.$$

Case 3: If $x, y \in [-3, -2)$, then $Tx = [-\frac{|x|}{|x|+1}, 0]$ and $Ty = [-\frac{|y|}{|y|+1}, 0]$. This implies that

$$2H(Tx, Ty)^2 = 2\left(\frac{|x|}{|x|+1} - \frac{|y|}{|y|+1}\right)^2 < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

On the other hand, T is not nonexpansive since for $x = -2$ and $y = -\frac{5}{2}$, we have $Tx = \{0\}$ and $Ty = [-\frac{5}{7}, 0]$. This shows that $H(Tx, Ty) = \frac{5}{7} > \frac{1}{2} = |-2 - (-\frac{5}{2})| = \|x - y\|$.

In 1953, Mann [21] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H :

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \quad (1.6)$$

for each $n \in \mathbb{N}$, where the initial point x_1 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Motivated by the previous results, in this paper, we introduce and study the Mann-type iteration to approximate a common solution of the split equilibrium problem and the fixed point problem for a $\frac{1}{2}$ -nonspreading multi-valued mapping and prove some weak convergence theorems in Hilbert spaces. Finally, we give some examples and numerical results to illustrate our main results.

2 Preliminaries

We now provide some results for the main results. In a Hilbert space H_1 , let C be a nonempty closed convex subset of H_1 . For every point $x \in H_1$, there exists a unique nearest point of C , denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H_1 onto C . We know that P_C is a firmly nonexpansive mapping from H_1 onto C , i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H_1.$$

Further, for any $x \in H_1$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

A mapping $A : C \rightarrow H_1$ is called α -inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Lemma 2.1 *Let H_1 be a real Hilbert space. Then the following equations hold:*

- (1) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$ for all $x, y \in H_1$;
- (2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$ for all $x, y \in H_1$;
- (3) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$ for all $t \in [0, 1]$ and $x, y \in H_1$;
- (4) If $\{x_n\}_{n=1}^\infty$ is a sequence in H_1 which converges weakly to $z \in H_1$, then

$$\limsup_{n \rightarrow \infty} \|x_n - y\|^2 = \limsup_{n \rightarrow \infty} \|x_n - z\|^2 + \|z - y\|^2$$

for all $y \in H_1$.

A space X is said to satisfy *Opial's condition* if, for any sequence x_n with $x_n \rightharpoonup x$, then

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known that every Hilbert space satisfies Opial's condition.

Lemma 2.2 [22] *Let X be a Banach space which satisfies Opial's condition and $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be such that $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v , respectively, then $u = v$.*

Assumption 2.3 [1] Let $F_1 : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying the following assumptions:

- (1) $F_1(x, x) = 0$ for all $x \in C$;
- (2) F_1 is monotone, i.e., $F_1(x, y) + F_1(y, x) \leq 0$ for all $x \in C$;
- (3) for each $x, y, z \in C$, $\limsup_{t \rightarrow 0^+} F_1(tz + (1-t)x, y) \leq F_1(x, y)$;
- (4) for each $x \in C$, $y \rightarrow F_1(x, y)$ is convex and lower semi-continuous.

Lemma 2.4 [3] *Assume that $F_1 : C \times C \rightarrow \mathbb{R}$ satisfies Assumption 2.3. For any $r > 0$ and $x \in H_1$, define a mapping $T_r^{F_1} : H_1 \rightarrow C$ as follows:*

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:

- (1) $T_r^{F_1}$ is nonempty and single-valued;
- (2) $T_r^{F_1}$ is firmly nonexpansive, i.e., for any $x, y \in H_1$,

$$\|T_r^{F_1}x - T_r^{F_1}y\|^2 \leq \langle T_r^{F_1}x - T_r^{F_1}y, x - y \rangle;$$

- (3) $F(T_r^{F_1}) = EP(F_1)$;
- (4) $EP(F_1)$ is closed and convex.

Further, assume that $F_2 : Q \times Q \rightarrow \mathbb{R}$ satisfying Assumption 2.3. For each $s > 0$ and $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \rightarrow Q$ as follows:

$$T_s^{F_2}(w) = \left\{ d \in Q : F_2(d, e) + \frac{1}{s} \langle e - d, d - w \rangle \geq 0, \forall e \in Q \right\}.$$

Then we have the following:

- (5) $T_s^{F_2}$ is nonempty and single-valued;
- (6) $T_s^{F_2}$ is firmly nonexpansive;
- (7) $F(T_s^{F_2}) = EP(F_2, Q)$;
- (8) $EP(F_2, Q)$ is closed and convex.

Condition (A) Let H_1 be a Hilbert space and C be a subset of H_1 . A multi-valued mapping $T : C \rightarrow CB(C)$ is said to satisfy *Condition (A)* if $\|x - p\| = d(x, Tp)$ for all $x \in H_1$ and $p \in F(T)$.

Remark 2.5 We see that T satisfies Condition (A) if and only if $Tp = \{p\}$ for all $p \in F(T)$. It is well known that the best approximation operator P_T , which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$, also satisfies Condition (A).

3 Main results

Now, we are ready to prove some weak convergence theorem for $\frac{1}{2}$ -nonspreading multi-valued mappings in Hilbert spaces. To this end, we need the following crucial results.

Lemma 3.1 *Let C be a closed and convex subset of a real Hilbert space H_1 and $T : C \rightarrow K(C)$ be a k -nonspreading multi-valued mapping such that $k \in (0, \frac{1}{2}]$. If $x, y \in C$ and $a \in Tx$, then there exists $b \in Ty$ such that*

$$\|a - b\|^2 \leq H(Tx, Ty)^2 \leq \frac{k}{1-k} (\|x - y\|^2 + 2\langle x - a, y - b \rangle).$$

Proof Let $x, y \in C$ and $a \in Tx$. By Nadler's theorem (see [8]), there exists $b \in Ty$ such that

$$\|a - b\|^2 \leq H(Tx, Ty)^2.$$

It follows that

$$\begin{aligned} & \frac{1}{k} H(Tx, Ty)^2 \\ & \leq d(Tx, y)^2 + d(x, Ty)^2 \\ & \leq \|a - y\|^2 + \|x - b\|^2 \\ & \leq \|a - x\|^2 + 2\langle a - x, x - y \rangle + \|x - y\|^2 + \|x - a\|^2 + 2\langle x - a, a - b \rangle + \|a - b\|^2 \\ & = 2\|a - x\|^2 + \|x - y\|^2 + \|a - b\|^2 + 2\langle a - x, x - a - (y - b) \rangle \\ & \leq 2\|a - x\|^2 + \|x - y\|^2 + H(Tx, Ty)^2 + 2\langle a - x, x - a - (y - b) \rangle. \end{aligned}$$

This implies that

$$H(Tx, Ty)^2 \leq \frac{k}{1-k} (\|x - y\|^2 + 2\langle x - a, y - b \rangle).$$

This completes the proof. $\square$

Lemma 3.2 *Let C be a closed and convex subset of a real Hilbert space H_1 and $T : C \rightarrow K(C)$ be a k -nonspreading multi-valued mapping such that $k \in (0, \frac{1}{2}]$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

Proof Let $\{x_n\}$ be a sequence in C which converges weakly to p and let $y_n \in Tx_n$ be such that $\|x_n - y_n\| \rightarrow 0$.

Now, we show that $p \in F(T)$. By Lemma 3.1, there exists $z_n \in Tp$ such that

$$\|y_n - z_n\|^2 \leq \frac{k}{1-k} (\|x_n - p\|^2 + 2\langle x_n - y_n, p - z_n \rangle).$$

Since Tp is compact and $z_n \in Tp$, there exists $\{z_{n_i}\} \subset \{z_n\}$ such that $z_{n_i} \rightarrow z \in Tp$. Since $\{x_n\}$ converges weakly, it is bounded. For each $x \in H_1$, define a function $f : H_1 \rightarrow [0, \infty)$ by

$$f(x) := \limsup_{i \rightarrow \infty} \frac{k}{1-k} \|x_{n_i} - x\|^2.$$

Then, by Lemma 2.1(4), we obtain

$$f(x) = \limsup_{i \rightarrow \infty} \frac{k}{1-k} (\|x_{n_i} - p\|^2 + \|p - x\|^2)$$

for all $x \in H_1$. Thus $f(x) = f(p) + \frac{k}{1-k} \|p - x\|^2$ for all $x \in H_1$. It follows that

$$f(z) = f(p) + \frac{k}{1-k} \|p - z\|^2. \quad (3.1)$$

We observe that

$$\begin{aligned} f(z) &= \limsup_{i \rightarrow \infty} \frac{k}{1-k} \|x_{n_i} - z\|^2 = \limsup_{i \rightarrow \infty} \frac{k}{1-k} \|x_{n_i} - y_{n_i} + y_{n_i} - z\|^2 \\ &\leq \limsup_{i \rightarrow \infty} \frac{k}{1-k} \|y_{n_i} - z\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} f(z) &\leq \limsup_{i \rightarrow \infty} \frac{k}{1-k} \|y_{n_i} - z\|^2 \\ &= \limsup_{i \rightarrow \infty} \frac{k}{1-k} (\|y_{n_i} - z_{n_i} + z_{n_i} - z\|)^2 \\ &\leq \limsup_{i \rightarrow \infty} \frac{k}{1-k} (\|x_{n_i} - p\|^2 + 2\langle x_{n_i} - y_{n_i}, p - z_{n_i} \rangle) \\ &\leq \limsup_{i \rightarrow \infty} \frac{k}{1-k} \|x_{n_i} - p\|^2 \\ &= f(p). \end{aligned} \quad (3.2)$$

Hence it follows from (3.1) and (3.2) that $\|p - z\| = 0$. This completes the proof. $\square$

Theorem 3.3 *Let H_1, H_2 be two real Hilbert space and $C \subset H_1, Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2 , respectively. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and $T : C \rightarrow K(C)$ a $\frac{1}{2}$ -nonspreading multi-valued mapping. Let $F_1 : C \times C \rightarrow \mathbb{R}, F_2 : Q \times Q \rightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.3 and F_2 is upper semi-continuous in the first argument. Assume that T satisfies Condition (A) and $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily,} \\ u_n = T_{r_n}^{F_1} (I - \gamma A^* (I - T_{r_n}^{F_2}) A) x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n) T u_n, \end{cases} \quad (3.3)$$

for all $n \geq 1$, where $\{\alpha_n\} \subset (0, 1), r_n \subset (0, \infty),$ and $\gamma \in (0, 1/L)$ such that L is the spectral radius of $A^* A$ and A^* is the adjoint of A . Assume that the following conditions hold:

- (1) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
 (2) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then the sequence $\{x_n\}$ defined by (3.3) converges weakly to $p \in \Theta$.

Proof We first show that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$ -inverse strongly monotone mapping. Since $T_{r_n}^{F_2}$ is firmly nonexpansive and $I - T_{r_n}^{F_2}$ is 1-inverse strongly monotone, we see that

$$\begin{aligned}
 \|A^*(I - T_{r_n}^{F_2})Ax - A^*(I - T_{r_n}^{F_2})Ay\|^2 &= \langle A^*(I - T_{r_n}^{F_2})(Ax - Ay), A^*(I - T_{r_n}^{F_2})(Ax - Ay) \rangle \\
 &= \langle (I - T_{r_n}^{F_2})(Ax - Ay), AA^*(I - T_{r_n}^{F_2})(Ax - Ay) \rangle \\
 &\leq L \langle (I - T_{r_n}^{F_2})(Ax - Ay), (I - T_{r_n}^{F_2})(Ax - Ay) \rangle \\
 &= L \|(I - T_{r_n}^{F_2})(Ax - Ay)\|^2 \\
 &\leq L \langle Ax - Ay, (I - T_{r_n}^{F_2})(Ax - Ay) \rangle \\
 &= L \langle x - y, A^*(I - T_{r_n}^{F_2})Ax - A^*(I - T_{r_n}^{F_2})Ay \rangle
 \end{aligned}$$

for all $x, y \in H_1$. This implies that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$ -inverse strongly monotone mapping. Since $\gamma \in (0, \frac{1}{L})$, it follows that $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is nonexpansive.

Now, we divide the proof into six steps as follows:

Step 1. Show that $\{x_n\}$ is bounded.

Let $p \in \Theta$. Then $p = T_{r_n}^{F_1}p$ and $(I - \gamma A^*(I - T_{r_n}^{F_2})A)p = p$. Thus we have

$$\begin{aligned}
 \|u_n - p\| &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)p\| \\
 &\leq \|(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - (I - \gamma A^*(I - T_{r_n}^{F_2})A)p\| \\
 &\leq \|x_n - p\|.
 \end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|z_n - p\| \quad \text{for some } z_n \in Tu_n \\
 &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) d(z_n, Tp) \\
 &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) H(Tu_n, Tp) \\
 &\leq \|x_n - p\|.
 \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Step 2. Show that $\|z_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$ for all $z_n \in Tu_n$. From Lemma 2.1 and T satisfying Condition (A), we have

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 - \alpha_n (1 - \alpha_n) \|x_n - z_n\|^2 \\
 &= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) d(z_n, Tp)^2 - \alpha_n (1 - \alpha_n) \|x_n - z_n\|^2 \\
 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) H(Tu_n, Tp)^2 - \alpha_n (1 - \alpha_n) \|x_n - z_n\|^2 \\
 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|u_n - p\|^2 - \alpha_n (1 - \alpha_n) \|x_n - z_n\|^2 \\
 &\leq \|x_n - p\|^2 - \alpha_n (1 - \alpha_n) \|x_n - z_n\|^2.
 \end{aligned}$$

This implies that

$$\alpha_n(1 - \alpha_n)\|x_n - z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

From Condition (1) and the existence of $\lim_{n \rightarrow \infty} \|x_n - p\|$, we have

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (3.5)$$

Step 3. Show that $\|u_n - z_n\| \rightarrow 0$ as $n \rightarrow \infty$ for all $z_n \in Tu_n$. For any $p \in \Theta$, we estimate

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - p\|^2 \\ &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}p\|^2 \\ &\leq \|x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \gamma^2 \|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 + 2\gamma \langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n \rangle. \end{aligned}$$

Thus we have

$$\begin{aligned} \|u_n - p\|^2 &\leq \|x_n - p\|^2 + \gamma^2 \langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*(I - T_{r_n}^{F_2})Ax_n \rangle \\ &\quad + 2\gamma \langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n \rangle. \end{aligned} \quad (3.6)$$

On the other hand, we have

$$\begin{aligned} \gamma^2 \langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*(I - T_{r_n}^{F_2})Ax_n \rangle &\leq L\gamma^2 \langle Ax_n - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n \rangle \\ &= L\gamma^2 \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} &2\gamma \langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n \rangle \\ &= 2\gamma \langle A(p - x_n), Ax_n - T_{r_n}^{F_2}Ax_n \rangle \\ &= 2\gamma \langle A(p - x_n) + (Ax_n - T_{r_n}^{F_2}Ax_n) \\ &\quad - (Ax_n - T_{r_n}^{F_2}Ax_n), Ax_n - T_{r_n}^{F_2}Ax_n \rangle \\ &= 2\gamma \{ \langle Ap - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n \rangle - \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 \} \\ &\leq 2\gamma \left\{ \frac{1}{2} \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 - \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 \right\} \\ &= -\gamma \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2. \end{aligned} \quad (3.8)$$

Using (3.6), (3.7), and (3.8), we have

$$\begin{aligned} \|u_n - p\|^2 &\leq \|x_n - p\|^2 + L\gamma^2 \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 - \gamma \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma(L\gamma - 1) \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2. \end{aligned} \quad (3.9)$$

It follows that, for all $z_n \in Tu_n$,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)z_n - p\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) d(z_n, Tp)^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) H(Tu_n, Tp)^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|u_n - p\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) (\|x_n - p\|^2 + \gamma(L\gamma - 1) \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2) \\
&\leq \|x_n - p\|^2 + \gamma(L\gamma - 1) \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2.
\end{aligned}$$

Therefore, we have

$$-\gamma(L\gamma - 1) \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $\gamma(L\gamma - 1) < 0$ and $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, by (3.8), we obtain

$$\lim_{n \rightarrow \infty} \|Ax_n - T_{r_n}^{F_2} Ax_n\| = 0. \quad (3.10)$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive and $I - \gamma A^*(T_{r_n}^{F_2} - I)A$ is nonexpansive, it follows that

$$\begin{aligned}
&\|u_n - p\|^2 \\
&= \|T_{r_n}^{F_1}(x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n) - T_{r_n}^{F_1}p\|^2 \\
&\leq \langle T_{r_n}^{F_1}(x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n) - T_{r_n}^{F_1}p, x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p \rangle \\
&= \langle u_n - p, x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p \rangle \\
&= \frac{1}{2} \{ \|u_n - p\|^2 + \|x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\|^2 - \|u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \} \\
&\leq \frac{1}{2} \{ \|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \} \\
&= \frac{1}{2} \{ \|u_n - p\|^2 + \|x_n - p\|^2 - (\|u_n - x_n\|^2 + \gamma^2 \|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \\
&\quad - 2\gamma \langle u_n - x_n, A^*(I - T_{r_n}^{F_2} - I)Ax_n \rangle) \},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma \langle u_n - x_n, A^*(I - T_{r_n}^{F_2})Ax_n \rangle \\
&\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma \|u_n - x_n\| \|A^*(I - T_{r_n}^{F_2})Ax_n\|.
\end{aligned} \quad (3.11)$$

It follows from (3.4) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2 \quad \text{for all } z_n \in Tu_n
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) d(z_n, Tp)^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) H(Tu_n, Tp)^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|u_n - p\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) (\|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma \|u_n - x_n\| \|A^*(I - T_{r_n}^{F_2})Ax_n\|).
\end{aligned}$$

Therefore, we have

$$(1 - \alpha_n) \|u_n - x_n\|^2 \leq 2\gamma \|u_n - x_n\| \|A^*(I - T_{r_n}^{F_2})Ax_n\| + \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

From Condition (1) and (3.10), we have

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (3.12)$$

From (3.5) and (3.12), we have

$$\|u_n - z_n\| \leq \|u_n - x_n\| + \|x_n - z_n\| \rightarrow 0 \quad (3.13)$$

as $n \rightarrow \infty$.

Step 4. Show that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. From (3.12) and (3.13), we have

$$\begin{aligned}
\|x_{n+1} - u_n\| &= \|\alpha_n x_n + (1 - \alpha_n) z_n - u_n\| \\
&\leq \alpha_n \|x_n - u_n\| + (1 - \alpha_n) \|z_n - u_n\| \rightarrow 0
\end{aligned} \quad (3.14)$$

as $n \rightarrow \infty$. From (3.12) and (3.14), we also have

$$\|u_n - u_{n+1}\| \leq \|u_n - x_{n+1}\| + \|x_{n+1} - u_{n+1}\| \rightarrow 0 \quad (3.15)$$

as $n \rightarrow \infty$. It follows from (3.12) and (3.14) that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|u_n - x_n\| \rightarrow 0 \quad (3.16)$$

as $n \rightarrow \infty$.

Step 5. Show that $\omega_w(x_n) \subset \Theta$, where $\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x, \{x_{n_i}\} \subset \{x_n\}\}$. Since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Let $q \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ converging weakly to q . From (3.11), it follows that $u_{n_i} \rightharpoonup q$ as $i \rightarrow \infty$. By Lemma 3.2 and (3.13), we obtain $q \in F(T)$.

Next, we show that $q \in EP(F_1)$. From $u_n = T_{r_n}^{F_1}(I + \gamma A^*(I - T_{r_n}^{F_2})A)x_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n \rangle \geq 0$$

for all $y \in C$, which implies that

$$F_1(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle - \frac{1}{r_n} \langle y - u_n, \gamma A^*(I - T_{r_n}^{F_2})Ax_n \rangle \geq 0$$

for all $y \in C$. By Assumption 2.3(2), we have

$$\frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle - \frac{1}{r_{n_i}} \langle y - u_{n_i}, \gamma A^* (I - T_{r_{n_i}}^{F_1}) A x_{n_i} \rangle \geq F_1(y, u_{n_i})$$

for all $y \in C$. From $\liminf_{n \rightarrow \infty} r_n > 0$, from (3.10), (3.12), and Assumption 2.3(4), we obtain

$$F_1(y, q) \leq 0$$

for all $y \in C$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, $y_t \in C$, and hence $F_1(y_t, q) \leq 0$. So, by Assumption 2.3(1) and (4), we have

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, q) \leq tF_1(y_t, y)$$

and hence $F_1(y_t, y) \geq 0$. So $F_1(q, y) \geq 0$ for all $y \in C$ by (3.15) and hence $q \in EP(F_1)$. Since A is a bounded linear operator, $Ax_{n_i} \rightharpoonup Aq$. Then it follows from (3.10) that

$$T_{r_{n_i}}^{F_2} Ax_{n_i} \rightharpoonup Aq \quad (3.17)$$

as $i \rightarrow \infty$. By the definition of $T_{r_{n_i}}^{F_2} Ax_{n_i}$, we have

$$F_2(T_{r_{n_i}}^{F_2} Ax_{n_i}, y) + \frac{1}{r_{n_i}} \langle y - T_{r_{n_i}}^{F_2} Ax_{n_i}, T_{r_{n_i}}^{F_2} Ax_{n_i} - Ax_{n_i} \rangle \geq 0$$

for all $y \in C$. Since F_2 is upper semi-continuous in the first argument and (3.17), it follows that

$$F_2(Aq, y) \geq 0$$

for all $y \in C$. This shows that $Aq \in EP(F_2)$. Hence $q \in \Omega$.

Step 6. Show that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of Θ . It is sufficient to show that $\omega_w(x_n)$ is single point set. Let $p, q \in \omega_w(x_n)$ and $\{x_{n_k}\}, \{x_{n_m}\} \subset \{x_n\}$ be such that $x_{n_k} \rightharpoonup p$ and $x_{n_m} \rightharpoonup q$. From (3.12), we also have $u_{n_k} \rightharpoonup p$ and $u_{n_m} \rightharpoonup q$. By Lemma 3.2 and (3.13), it follows that $p, q \in F(T)$. Applying Lemma 2.2, we obtain $p = q$. This completes the proof. $\square$

If $Tp = \{p\}$ for all $p \in F(T)$, then T satisfies Condition (A) and so we can obtain the following result.

Theorem 3.4 *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1, Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2 , respectively. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and $T : C \rightarrow K(C)$ a $\frac{1}{2}$ -nonspreading multi-valued mapping. Let $F_1 : C \times C \rightarrow \mathbb{R}, F_2 : Q \times Q \rightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.3 and F_2 is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily,} \\ u_n = T_{r_n}^{F_1} (I - \gamma A^* (I - T_{r_n}^{F_2}) A) x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n) T u_n, \end{cases} \quad (3.18)$$

for all $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$, $r_n \subset (0, \infty)$, and $\gamma \in (0, 1/L)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A . Assume that the following conditions hold:

- (1) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (2) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then the sequence $\{x_n\}$ defined by (3.18) converges weakly to $p \in \Theta$.

Since P_T satisfies Condition (A), we also obtain the following results.

Theorem 3.5 Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2 , respectively. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and $T : C \rightarrow P(C)$ a multi-valued mapping. Let $F_1 : C \times C \rightarrow \mathbb{R}$, $F_2 : Q \times Q \rightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.3 and F_2 is upper semi-continuous in the first argument. Assume that P_T is $\frac{1}{2}$ -nonspreading multi-valued mapping and $I - T$ is demiclosed at 0 with $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 \in C \text{ arbitrarily,} \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)P_T u_n, \end{cases} \quad (3.19)$$

for all $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$, $r_n \subset (0, \infty)$, and $\gamma \in (0, 1/L)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A . Assume that the following conditions hold:

- (1) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (2) $\liminf_{n \rightarrow \infty} r_n > 0$.

Then the sequence $\{x_n\}$ defined by (3.19) converges weakly to $p \in \Theta$.

Proof By the same proof as in Theorem 3.3, we have $u_n \rightarrow z_n \in P_T u_n$. This implies that

$$d(u_n, Tu_n) \leq d(u_n, P_T u_n) \leq \|u_n - z_n\| \rightarrow 0 \quad (3.20)$$

as $n \rightarrow \infty$. Since $I - T$ is demiclosed at 0, we obtain this result. $\square$

4 Examples and numerical results

In this section, we give examples and numerical results for supporting our main theorem.

Example 4.1 Let $H_1 = H_2 = \mathbb{R}$, $C = [-3, 0]$, and $Q = (-\infty, 0]$. Let $F_1(u, v) = (u - 1)(v - u)$ for all $u, v \in C$ and $F_2(x, y) = (x + 15)(y - x)$ for all $x, y \in Q$. Define two mappings $A : \mathbb{R} \rightarrow \mathbb{R}$ and $T : C \rightarrow K(C)$ by $Ax = 3x$ for all $x \in \mathbb{R}$ and

$$Tx = \begin{cases} \{0\}, & x \in [-2, 0]; \\ [-\frac{|x|}{|x|+1}, 0], & x \in [-3, -2). \end{cases}$$

Choose $\alpha_n = \frac{n}{2n+1}$, $r_n = \frac{n}{n+1}$, and $\gamma = \frac{1}{10}$. It is easy to check that F_1 and F_2 satisfy all conditions in Theorem 3.3 and T satisfies Condition (A) such that $F(T) = \{0\}$. For each $r > 0$ and $x \in C$, we divide the process of our iteration into five steps as follows:

Table 1 Numerical results of Example 1.1 being randomized in the first time

n	z_n	x_n	$ x_{n+1} - x_n $
1	-5.74683E-01	-3.00000E+00	1.61688E+00
2	0	-1.38312E+00	8.29873E-01
3	0	-5.53249E-01	3.16142E-01
4	0	-2.37107E-01	1.31726E-01
5	0	-1.05381E-01	5.74804E-02
6	0	-4.79003E-02	2.57925E-02
7	0	-2.21078E-02	1.17909E-02
8	0	-1.03170E-02	5.46194E-03
9	0	-4.85506E-03	2.55529E-03
10	0	-2.29976E-03	1.20464E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	0	-9.26099E-16	4.67634E-16

Step 1. Find $z \in Q$ such that $F_2(z, y) + \frac{1}{r} \langle y - z, z - Ax \rangle \geq 0$ for all $y \in Q$. Noting that $Ax = 3x$, we have

$$\begin{aligned}
 F_2(z, y) + \frac{1}{r} \langle y - z, z - Ax \rangle \geq 0 &\iff (z + 15)(y - z) + \frac{1}{r} \langle y - z, z - 3x \rangle \geq 0 \\
 &\iff r(z + 15)(y - z) + (y - z)(z - 3x) \geq 0 \\
 &\iff (y - z)((1 + r)z - (3x - 15r)) \geq 0.
 \end{aligned}$$

By Lemma 2.4, we know that $T_r^{F_2} Ax$ is single-valued. Hence $z = \frac{3x - 15r}{1 + r}$.

Step 2. Find $s \in C$ such that $s = x - \gamma A^*(I - T_r^{F_2})Ax$. From Step 1, we have

$$\begin{aligned}
 s &= x - \gamma A^*(I - T_r^{F_2})Ax = x - \gamma A^*(Ax - T_r^{F_2} Ax) \\
 &= x - \gamma \left(9x - \frac{3(3x - 15r)}{1 + r} \right) \\
 &= (1 - 9\gamma)x + \frac{3\gamma}{1 + r}(3x - 15r).
 \end{aligned}$$

Step 3. Find $u \in C$ such that $F_1(u, v) + \frac{1}{r} \langle v - u, u - s \rangle \geq 0$ for all $v \in C$. From Step 2, we have

$$\begin{aligned}
 F_1(u, v) + \frac{1}{r} \langle v - u, u - s \rangle \geq 0 &\iff (u - 1)(v - u) + \frac{1}{r} \langle v - u, u - s \rangle \geq 0 \\
 &\iff r(u - 1)(v - u) + (v - u)(u - s) \geq 0 \\
 &\iff (v - u)((1 + r)u - (s + r)) \geq 0.
 \end{aligned}$$

Similarly, by Lemma 2.4, we obtain $u = \frac{s + r}{1 + r} = \frac{(1 - 9\gamma)x + r}{1 + r} + \frac{3\gamma(3x - 15r)}{(1 + r)^2}$.

Step 4. Find $x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)Tu_n$, where $u_n = \frac{(1 - 9\gamma)x_n + r_n}{1 + r_n} + \frac{3\gamma(3x_n - 15r_n)}{(1 + r_n)^2}$. From

$$Tx = \begin{cases} \{0\}, & x \in [-2, 0]; \\ [-\frac{|x|}{|x|+1}, 0], & x \in [-3, -2), \end{cases}$$

and $\alpha_n = \frac{n}{2n+1}$, $r_n = \frac{n}{n+1}$, and $\gamma = \frac{1}{10}$, we have

$$x_{n+1} = \left(\frac{n}{2n+1} \right) x_n + \left(1 - \frac{n}{2n+1} \right) z_n, \tag{4.1}$$

Table 2 Numerical results of Example 1.1 being randomized in the second time

n	z_n	x_n	$ x_{n+1} - x_n $
1	-1.26779E-00	-3.00000E+00	1.91548E+00
2	0	-1.08452E+00	6.50711E-01
3	0	-4.33808E-01	2.47890E-01
4	0	-1.85918E-01	1.03288E-01
5	0	-8.26300E-02	4.50709E-02
6	0	-3.75591E-02	2.02241E-02
7	0	-1.73350E-02	9.24532E-03
8	0	-8.08965E-03	4.28276E-03
9	0	-3.80690E-03	2.00363E-03
10	0	-1.80327E-03	9.44568E-04
⋮	⋮	⋮	⋮
50	0	-7.26163E-16	3.66677E-16

where

$$z_n \in \begin{cases} \{0\}, & u_n \in [-2, 0]; \\ [-\frac{|u_n|}{|u_n|+1}, 0], & u_n \in [-3, -2). \end{cases}$$

Step 5. Compute the numerical results. Choosing $x_1 = -3$ and taking randomly z_n in the above interval, we obtain Tables 1 and 2.

From Table 1 and Table 2, we see that 0 is the solution in Example 1.1.

References

- Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123-145 (1994)
- Ceng, LC, Yao, JC: A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. J. Comput. Appl. Math. **214**, 186-201 (2008)
- Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117-136 (2005)
- Peng, JW, Liou, YC, Yao, JC: An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions. Fixed Point Theory Appl. **2009**, Article ID 794178 (2009)
- Tada, A, Takahashi, W: Strong convergence theorem for an equilibrium problem and a nonexpansive mapping. In: Takahashi, W, Tanaka, T (eds.) Nonlinear Analysis and Convex Analysis. Yokohama Publishers, Yokohama (2005)
- Kazmi, KR, Rizvi, SH: Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem. J. Egypt. Math. Soc. **21**, 44-51 (2013)
- Assad, NA, Kirk, WA: Fixed point theorems for set-valued mappings of contractive type. Pac. J. Math. **43**, 553-562 (1972)
- Nadler, SB Jr.: Multi-valued contraction mappings. Pac. J. Math. **30**, 475-488 (1969)
- Pietramala, P: Convergence of approximating fixed points sets for multivalued nonexpansive mappings. Comment. Math. Univ. Carol. **32**, 697-701 (1991)
- Shahzad, N, Zegeye, H: Strong convergence results for nonself multimaps in Banach spaces. Proc. Am. Math. Soc. **136**, 539-548 (2008)
- Shahzad, N, Zegeye, H: On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces. Nonlinear Anal. **71**, 838-844 (2009)
- Song, Y, Wang, H: Convergence of iterative algorithms for multivalued mappings in Banach spaces. Nonlinear Anal. **70**, 1547-1556 (2009)
- Hussain, N, Khan, AR: Applications of the best approximation operator to *-nonexpansive maps in Hilbert spaces. Numer. Funct. Anal. Optim. **24**, 327-338 (2003)
- Jung, JS: Convergence of approximating fixed points for multivalued nonself-mappings in Banach spaces. Korean J. Math. **16**, 215-231 (2008)
- Zegeye, H, Shahzad, N: Viscosity approximation methods for nonexpansive multimaps in Banach space. Acta Math. Sin. **26**, 1165-1176 (2010)
- Song, Y, Cho, YJ: Some note on Ishikawa iteration for multi-valued mappings. Bull. Korean Math. Soc. **48**, 575-584 (2011)
- Kohsaka, F, Takahashi, W: Fixed point theorems for a class of nonlinear mappings relate to maximal monotone operators in Banach spaces. Arch. Math. **91**, 166-177 (2008)
- Iemoto, S, Takahashi, W: Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space. Nonlinear Anal. **71**, 2082-2089 (2009)
- Liu, HB: Convergence theorems for a finite family of nonspreading and nonexpansive multivalued mappings and nonexpansive multivalued mappings and equilibrium problems with application. Theor. Math. Appl. **3**, 49-61 (2013)
- Kohsaka, F, Takahashi, W: Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces. SIAM J. Optim. **19**, 824-835 (2008)
- Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)
- Suantai, S: Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings. J. Math. Anal. Appl. **311**, 506-517 (2005)

4.3 On solving split mixed equilibrium problems and fixed point problems of hybrid-type multivalued mappings in Hilbert spaces

In this section, we introduce and study iterative algorithms for solving split mixed equilibrium problems and fixed point problems of λ -hybrid multivalued mappings in real Hilbert spaces and prove that the proposed iterative algorithm converges weakly to a common solution of the considered problems. We also provide an example to illustrate the convergence behavior of the proposed iteration process.

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H , $\varphi : C \rightarrow \mathbb{R}$ be a function, and $F : C \times C \rightarrow \mathbb{R}$ be a bifunction. The *mixed equilibrium problem* is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \quad (1.1)$$

The solution set of mixed equilibrium problem is denoted by $MEP(F, \varphi)$. In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem, which is to find $x \in C$ such that $F(x, y) \geq 0, \forall y \in C$. The solution set of equilibrium problem is denoted by $EP(F)$.

The mixed equilibrium problem is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, minimization problems, fixed point problems, Nash equilibrium problems in noncooperative games, and others; see, e.g., [1–4].

In 1994, Censor and Elfving [5] firstly introduced the following split feasibility problem in finite-dimensional Hilbert spaces: Let H_1, H_2 be two Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2 , respectively, and let $A : H_1 \rightarrow H_2$ be a bounded linear operator. The split feasibility problem is formulated as finding a point x^*

with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q.$$

The split feasibility problem can extensively be applied in fields such as intensity-modulated radiation therapy, signal processing and image reconstruction, then the split feasibility problem has received so much attention by so many scholars; see [6–9].

In 2013, Kazmi and Rizvi [10] introduced and studied the following split equilibrium problem: let $C \subseteq H_1$ and $Q \subseteq H_2$. Let $F_1 : C \times C \rightarrow \mathbb{R}$ and $F_2 : Q \times Q \rightarrow \mathbb{R}$ be nonlinear bi-functions and let $A : H_1 \rightarrow H_2$ be a bounded linear operator. The *split equilibrium problem* is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C \text{ and such that } y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \forall y \in Q. \quad (1.2)$$

The solution set of the split equilibrium problem is denoted by

$$SEP(F_1, F_2) := \{x^* \in C : x^* \in EP(F_1) \text{ and } Ax^* \in EP(F_2)\}.$$

The authors gave an iterative algorithm to find the common element of sets of solution of the split equilibrium problem and hierarchical fixed point problem; for more details refer to [11, 12].

In 2016, Suantai *et al.* [13] proposed the iterative algorithm to solve the problems for finding a common elements the set of solution of the split equilibrium problem and the fixed point of a nonspreading multivalued mapping in Hilbert space, given sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in C \quad \text{arbitrarily,} \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n)Su_n, \quad \forall n \in \mathbb{N}, \end{cases} \quad (1.3)$$

where $\{\alpha_n\} \subset (0, 1)$, $r_n \subset (0, \infty)$ and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A , $C \subset H_1$, $Q \subset H_2$, $S : C \rightarrow K(C)$ is a $\frac{1}{2}$ -nonspreading multivalued mapping, $F_1 : C \times C \rightarrow \mathbb{R}$ and $F_2 : Q \times Q \rightarrow \mathbb{R}$ are two bifunctions. The authors showed that under certain conditions, the sequence $\{x_n\}$ converges weakly to an element of $F(S) \cap SEP(F_1, F_2)$.

Several iterative algorithms have been developed for solving split feasibility problems and related split equilibrium problems; see, *e.g.*, [14–16].

Motivated and inspired by the above results and related literature, we propose an iterative algorithm for finding a common element of the set of solutions of split mixed equilibrium problems and the set of fixed points of λ -hybrid multivalued mappings in real Hilbert spaces. Then we prove some weak convergence theorems which extend and improve the corresponding results of Kazmi and Rizvi [10] and Suantai *et al.* [13] and many others. We finally provide numerical examples for supporting our main result.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H . We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. It is also well known [17] that a Hilbert space H satisfies *Opial's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

The following two lemmas are useful for our main results.

Lemma 2.1 *In a real Hilbert space H , the following inequalities hold:*

- (1) $\|x - y\|^2 \leq \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \forall x, y \in H$;
- (2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H$;
- (3) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1], \forall x, y \in H$;
- (4) *If $\{x_n\}$ is a sequence in H which converges weakly to $z \in H$, then*

$$\limsup_{n \rightarrow \infty} \|x_n - y\|^2 = \limsup_{n \rightarrow \infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.$$

Lemma 2.2 ([18]) *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H . Let $u, v \in H$ be such that $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v , respectively, then $u = v$.*

A single-valued mapping $T : C \longrightarrow H$ is called δ -inverse strongly monotone [19] if there exists a positive real number δ such that

$$\langle x - y, Tx - Ty \rangle \geq \delta \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

For each $\gamma \in (0, 2\delta]$, we see that $I - \gamma T$ is a nonexpansive single-valued mapping, that is,

$$\|(I - \gamma T)x - (I - \gamma T)y\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $CB(C)$ and $K(C)$ the collection of all nonempty closed bounded subsets and nonempty compact subsets of C , respectively. The *Hausdorff metric* $\mathcal{H}$ on $CB(C)$ is defined by

$$\mathcal{H}(A, B) := \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\}, \quad \forall A, B \in CB(C),$$

where $\text{dist}(x, B) = \inf\{d(x, y) : y \in B\}$ is the distance from a point x to a subset B . Let $S : C \rightarrow CB(C)$ be a multivalued mapping. An element $x \in C$ is called a *fixed point* of S if $x \in Sx$. The set of all fixed points of S is denoted by $F(S)$, that is, $F(S) = \{x \in C : x \in Sx\}$. Recall that a multivalued mapping $S : C \rightarrow CB(C)$ is called

- (i) *nonexpansive* if

$$\mathcal{H}(Sx, Sy) \leq \|x - y\|, \quad x, y \in C;$$

(ii) *quasi-nonexpansive* if $F(S) \neq \emptyset$ and

$$\mathcal{H}(Sx, Sp) \leq \|x - p\|, \quad \forall x \in C, \forall p \in F(S);$$

(iii) *nonspreading* [13] if

$$2\mathcal{H}(Sx, Sy)^2 \leq \text{dist}(y, Sx)^2 + \text{dist}(x, Sy)^2, \quad \forall x, y \in C;$$

(iv) λ -*hybrid* [20] if there exists $\lambda \in \mathbb{R}$ such that

$$(1 + \lambda)\mathcal{H}(Sx, Sp)^2 \leq (1 - \lambda)\|x - y\|^2 + \lambda \text{dist}(y, Sx)^2 + \lambda \text{dist}(x, Sy)^2, \quad \forall x, y \in C.$$

We note that 0-hybrid is nonexpansive, 1-hybrid is nonspreading, and if S is λ -hybrid with $F(S) \neq \emptyset$, then S is quasi-nonexpansive. It is well known [20] that if S is λ -hybrid, then $F(S)$ is closed. In addition, if S satisfies the condition: $Sp = \{p\}$ for all $p \in F(S)$, then $F(S)$ is also convex.

The following result is a demiclosedness principle for λ -hybrid multivalued mapping in a real Hilbert space.

Lemma 2.3 ([20]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \rightarrow K(C)$ be a λ -hybrid multivalued mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $y_n \in Sx_n$ with $x_n - y_n \rightarrow 0$, then $x \in Sx$.*

For solving the mixed equilibrium problem, we assume that the bifunction $F_1 : C \times C \rightarrow \mathbb{R}$ satisfies the following assumption:

Assumption 2.4 Let C be a nonempty closed and convex subset of a Hilbert space H_1 . Let $F_1 : C \times C \rightarrow \mathbb{R}$ be the bifunction, $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous satisfies the following conditions:

- (A1) $F_1(x, x) = 0$ for all $x \in C$;
- (A2) F_1 is monotone, i.e., $F_1(x, y) + F_1(y, x) \leq 0, \forall x, y \in C$;
- (A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F_1(tz + (1-t)x, y) \leq F_1(x, y)$;
- (A4) for each $x \in C$, $y \mapsto F_1(x, y)$ is convex and lower semicontinuous;
- (B1) for each $x \in H_1$ and fixed $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$F_1(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0;$$

- (B2) C is a bounded set.

Lemma 2.5 ([21]) *Let C be a nonempty closed and convex subset of a Hilbert space H_1 . Let $F_1 : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfies Assumption 2.4 and let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap \text{dom } \varphi \neq \emptyset$. For $r > 0$ and $x \in H_1$. Define a mapping $T_r^{F_1} : H_1 \rightarrow C$ as follows:*

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\},$$

for all $x \in H_1$. Assume that either (B1) or (B2) holds. Then the following conclusions hold:

- (1) for each $x \in H_1$, $T_r^{F_1} \neq \emptyset$;
- (2) $T_r^{F_1}$ is single-valued;
- (3) $T_r^{F_1}$ is firmly nonexpansive, i.e., for any $x, y \in H_1$,

$$\|T_r^{F_1}x - T_r^{F_1}y\|^2 \leq \langle T_r^{F_1}x - T_r^{F_1}y, x - y \rangle;$$

- (4) $F(T_r^{F_1}) = \text{MEP}(F_1, \varphi)$;
- (5) $\text{MEP}(F_1, \varphi)$ is closed and convex.

Further, assume that $F_2 : Q \times Q \rightarrow \mathbb{R}$ satisfying Assumption 2.4 and $\phi : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function such that $Q \cap \text{dom } \phi \neq \emptyset$, where Q is a nonempty closed and convex subset of a Hilbert space H_2 . For each $s > 0$ and $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \rightarrow Q$ as follows:

$$T_s^{F_2}(v) = \left\{ w \in Q : F_2(w, d) + \phi(d) - \phi(w) + \frac{1}{r} \langle d - w, w - v \rangle \geq 0, \forall d \in Q \right\}.$$

Then we have the following:

- (6) for each $v \in H_2$, $T_s^{F_2} \neq \emptyset$;
- (7) $T_s^{F_2}$ is single-valued;
- (8) $T_s^{F_2}$ is firmly nonexpansive;
- (9) $F(T_s^{F_2}) = \text{MEP}(F_2, \phi)$;
- (10) $\text{MEP}(F_2, \phi)$ is closed and convex.

3 Main results

In this section, we prove the weak convergence theorems for finding a common element of the set of solutions of split mixed equilibrium problems and the set of fixed points of λ -hybrid multivalued mappings in real Hilbert spaces and give a numerical example to support our main result.

We introduce the definition of split mixed equilibrium problems in real Hilbert spaces as follows.

Definition 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2 . Let $F_1 : C \times C \rightarrow \mathbb{R}$ and $F_2 : Q \times Q \rightarrow \mathbb{R}$ be nonlinear bifunctions, let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions such that $C \cap \text{dom } \varphi \neq \emptyset$ and $Q \cap \text{dom } \phi \neq \emptyset$, and let $A : H_1 \rightarrow H_2$ be a bounded linear operator. The *split mixed equilibrium problem* is to find $x^* \in C$ such that

$$F_1(x^*, x) + \varphi(x) - \varphi(x^*) \geq 0, \quad \forall x \in C, \quad (3.1)$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) + \phi(y) - \phi(y^*) \geq 0, \quad \forall y \in Q. \quad (3.2)$$

The solution set of the split mixed equilibrium problem (3.1) and (3.2) is denoted by

$$\text{SMEP}(F_1, \varphi, F_2, \phi) := \{x^* \in C : x^* \in \text{MEP}(F_1, \varphi) \text{ and } Ax^* \in \text{MEP}(F_2, \phi)\}.$$

We now get our main result.

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2 . Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and $S : C \rightarrow K(C)$ a λ -hybrid multivalued mapping. Let $F_1 : C \times C \rightarrow \mathbb{R}$, $F_2 : Q \times Q \rightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.4, let $\varphi : C \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex functions such that $C \cap \text{dom } \varphi \neq \emptyset$ and $Q \cap \text{dom } \phi \neq \emptyset$, respectively, and F_2 is upper semicontinuous in the first argument. Assume that $\Theta = F(S) \cap \text{SMEP}(F_1, \varphi, F_2, \phi) \neq \emptyset$ and $Sp = \{p\}$ for all $p \in F(S)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)w_n, \quad w_n \in Su_n, \\ x_{n+1} = \beta_n w_n + (1 - \beta_n)z_n, \quad z_n \in Sy_n, \forall n \in \mathbb{N}, \end{cases} \quad (3.3)$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$, and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A . Assume that the following conditions hold:

$$(C1) \quad 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1;$$

$$(C2) \quad 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1;$$

$$(C3) \quad 0 < \liminf_{n \rightarrow \infty} r_n.$$

Then the sequence $\{x_n\}$ generated by (3.3) converges weakly to $p \in \Theta$.

Proof First, we show that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$ -inverse strongly monotone mapping. Since $T_{r_n}^{F_2}$ is firmly nonexpansive and $I - T_{r_n}^{F_2}$ is 1-inverse strongly monotone, we see that

$$\begin{aligned} \|A^*(I - T_{r_n}^{F_2})Ax - A^*(I - T_{r_n}^{F_2})Ay\|^2 &= \langle A^*(I - T_{r_n}^{F_2})(Ax - Ay), A^*(I - T_{r_n}^{F_2})(Ax - Ay) \rangle \\ &= \langle (I - T_{r_n}^{F_2})(Ax - Ay), AA^*(I - T_{r_n}^{F_2})(Ax - Ay) \rangle \\ &\leq L \langle (I - T_{r_n}^{F_2})(Ax - Ay), (I - T_{r_n}^{F_2})(Ax - Ay) \rangle \\ &= L \|(I - T_{r_n}^{F_2})(Ax - Ay)\|^2 \\ &\leq L \langle Ax - Ay, (I - T_{r_n}^{F_2})(Ax - Ay) \rangle \\ &= L \langle x - y, A^*(I - T_{r_n}^{F_2})Ax - A^*(I - T_{r_n}^{F_2})Ay \rangle \end{aligned}$$

for all $x, y \in H_1$. This implies that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$ -inverse strongly monotone mapping. Since $\gamma \in (0, \frac{1}{L})$, it follows that $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is a nonexpansive mapping.

Now, we divide the proof into five steps as follows:

Step 1. Show that $\{x_n\}$ is bounded.

Let $q \in \Theta$. Then we have $q = T_{r_n}^{F_1}q$ and $q = (I - \gamma A^*(I - T_{r_n}^{F_2})A)q$. By nonexpansiveness of $I - \gamma A^*(I - T_{r_n}^{F_2})A$, it implies that

$$\begin{aligned} \|u_n - q\| &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)q\| \\ &\leq \|(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - (I - \gamma A^*(I - T_{r_n}^{F_2})A)q\| \\ &\leq \|x_n - q\|. \end{aligned} \quad (3.4)$$

This implies that

$$\|w_n - q\| = \text{dist}(w_n, Sq) \leq H(Su_n, Sq) \leq \|u_n - q\| \leq \|x_n - q\|, \quad (3.5)$$

and so

$$\begin{aligned} \|y_n - q\| &= \|\alpha_n x_n + (1 - \alpha_n)w_n - q\| \\ &\leq \alpha_n \|x_n - q\| + (1 - \alpha_n)\|w_n - q\| \\ &= \|x_n - q\|. \end{aligned} \quad (3.6)$$

It follows that

$$\|z_n - q\| = \text{dist}(z_n, Sq) \leq H(Sy_n, Sq) \leq \|y_n - q\| \leq \|x_n - q\|. \quad (3.7)$$

By (3.5) and (3.7), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\beta_n w_n + (1 - \beta_n)z_n - q\| \\ &\leq \beta_n \|w_n - q\| + (1 - \beta_n)\|z_n - q\| \\ &= \|x_n - q\|. \end{aligned} \quad (3.8)$$

This implies that $\{\|x_n - q\|\}$ is decreasing and bounded below, thus $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in \Theta$.

Step 2. Show that $\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0$.

From Lemma 2.1(3), (3.5), (3.7), and $Sq = \{q\}$, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\beta_n w_n + (1 - \beta_n)z_n - q\|^2 \\ &\leq \beta_n \|w_n - q\|^2 + (1 - \beta_n)\|z_n - q\|^2 - \beta_n(1 - \beta_n)\|w_n - z_n\|^2 \\ &\leq \|x_n - q\|^2 - \beta_n(1 - \beta_n)\|w_n - z_n\|^2. \end{aligned} \quad (3.9)$$

This implies that

$$\beta_n(1 - \beta_n)\|w_n - z_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

From Condition (C1) and $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists, we have

$$\lim_{n \rightarrow \infty} \|w_n - z_n\| = 0. \quad (3.10)$$

Step 3. Show that $\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|w_n - u_n\| = 0$.

For $q \in \Theta$, we see that

$$\begin{aligned} \|u_n - q\|^2 &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}q\|^2 \\ &\leq \|(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - q\|^2 \\ &\leq \|x_n - q\|^2 + \gamma^2 \|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 + 2\gamma \langle q - x_n, A^*(I - T_{r_n}^{F_2})Ax_n \rangle \end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - q\|^2 + \gamma^2 \langle Ax_n - T_{r_n}^{F_2} Ax_n, AA^*(I - T_{r_n}^{F_2})Ax_n \rangle \\
&\quad + 2\gamma \langle A(q - x_n), Ax_n - T_{r_n}^{F_2} Ax_n \rangle \\
&\leq \|x_n - q\|^2 + L\gamma^2 \langle Ax_n - T_{r_n}^{F_2} Ax_n, Ax_n - T_{r_n}^{F_2} Ax_n \rangle \\
&\quad + 2\gamma \langle A(q - x_n) + (Ax_n - T_{r_n}^{F_2} Ax_n) \\
&\quad - (Ax_n - T_{r_n}^{F_2} Ax_n), Ax_n - T_{r_n}^{F_2} Ax_n \rangle \\
&\leq \|x_n - q\|^2 + L\gamma^2 \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \\
&\quad + 2\gamma (\langle Ap - T_{r_n}^{F_2} Ax_n, Ax_n - T_{r_n}^{F_2} Ax_n \rangle - \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2) \\
&\leq \|x_n - q\|^2 + L\gamma^2 \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \\
&\quad + 2\gamma \left(\frac{1}{2} \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 - \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \right) \\
&= \|x_n - q\|^2 + \gamma(L\gamma - 1) \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2.
\end{aligned}$$

Thus, by (3.5) and (3.7), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \beta_n \|w_n - q\|^2 + (1 - \beta_n) \|z_n - q\|^2 \\
&\leq \beta_n \|u_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 \\
&\leq \beta_n (\|x_n - q\|^2 + \gamma(L\gamma - 1) \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2) + (1 - \beta_n) \|x_n - q\|^2 \\
&\leq \|x_n - q\|^2 + \gamma(L\gamma - 1) \beta_n \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2.
\end{aligned} \tag{3.11}$$

Therefore, we have

$$-\gamma(L\gamma - 1) \beta_n \|Ax_n - T_{r_n}^{F_2} Ax_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

Since $\gamma(L\gamma - 1) < 0$, it follows by Condition (C1) and the existence of $\lim_{n \rightarrow \infty} \|x_n - q\|$ that

$$\lim_{n \rightarrow \infty} \|Ax_n - T_{r_n}^{F_2} Ax_n\| = 0. \tag{3.12}$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive and $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is nonexpansive, we have

$$\begin{aligned}
\|u_n - q\|^2 &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}q\|^2 \\
&\leq \langle T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}q, (I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - q \rangle \\
&= \langle u_n - q, (I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - q \rangle \\
&= \frac{1}{2} (\|u_n - q\|^2 + \|(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - q\|^2 \\
&\quad - \|u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\|^2) \\
&\leq \frac{1}{2} (\|u_n - q\|^2 + \|x_n - q\|^2 - (\|u_n - x_n\|^2 + \gamma^2 \|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \\
&\quad - 2\gamma \langle u_n - x_n, A^*(I - T_{r_n}^{F_2})Ax_n \rangle)),
\end{aligned}$$

which implies that

$$\begin{aligned}\|u_n - q\|^2 &\leq \|x_n - q\|^2 - \|u_n - x_n\|^2 + 2\gamma \langle u_n - x_n, A^*(I - T_{r_n}^{F_2})Ax_n \rangle \\ &\leq \|x_n - q\|^2 - \|u_n - x_n\|^2 + 2\gamma \|u_n - x_n\| \|A^*(I - T_{r_n}^{F_2})Ax_n\|.\end{aligned}\quad (3.13)$$

This implies by (3.5) and (3.7) that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \beta_n \|w_n - q\|^2 + (1 - \beta_n) \|z_n - q\|^2 \\ &\leq \beta_n \|u_n - q\|^2 + (1 - \beta_n) \|x_n - q\|^2 \\ &\leq \beta_n (\|x_n - q\|^2 - \|u_n - x_n\|^2 + 2\gamma \|u_n - x_n\| \|A^*(I - T_{r_n}^{F_2})Ax_n\|) \\ &\quad + (1 - \beta_n) \|x_n - q\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\beta_n \|u_n - x_n\|^2 &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2\gamma \beta_n \|u_n - x_n\| \|A^*(I - T_{r_n}^{F_2})Ax_n\| \\ &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2\gamma \beta_n M \|A^*(I - T_{r_n}^{F_2})Ax_n\|,\end{aligned}$$

where $M = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. This implies by Condition (C1), (3.12), and the existence of $\lim_{n \rightarrow \infty} \|x_n - q\|$ that

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (3.14)$$

From (3.5), (3.7), and the definition of $\{y_n\}$, we obtain

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \beta_n \|w_n - q\|^2 + (1 - \beta_n) \|z_n - q\|^2 \\ &\leq \beta_n \|x_n - q\|^2 + (1 - \beta_n) \|y_n - q\|^2 \\ &= \beta_n \|x_n - q\|^2 + (1 - \beta_n) (\alpha_n \|x_n - q\|^2 + (1 - \alpha_n) \|w_n - q\|^2 \\ &\quad - \alpha_n (1 - \alpha_n) \|x_n - w_n\|^2) \\ &\leq \beta_n \|x_n - q\|^2 + (1 - \beta_n) (\|x_n - q\|^2 - \alpha_n (1 - \alpha_n) \|x_n - w_n\|^2) \\ &= \|x_n - q\|^2 - \alpha_n (1 - \alpha_n) (1 - \beta_n) \|x_n - w_n\|^2.\end{aligned}$$

This implies that

$$\alpha_n (1 - \alpha_n) (1 - \beta_n) \|x_n - w_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

From Conditions (C1), (C2), and the existence of $\lim_{n \rightarrow \infty} \|x_n - q\|$, we have

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \quad (3.15)$$

By (3.14) and (3.15), we get

$$\|w_n - u_n\| \leq \|w_n - x_n\| + \|x_n - u_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.16)$$

Step 4. Show that $\omega_w(x_n) \subset \Theta$, where $\omega_w(x_n) = \{x \in H_1 : x_{n_i} \rightharpoonup x, \{x_{n_i}\} \subset \{x_n\}\}$. Since $\{x_n\}$ is bounded and H_1 is reflexive, $\omega_w(x_n)$ is nonempty. Let $p \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ converging weakly to p . From (3.14), it implies that $u_{n_i} \rightharpoonup p$ as $i \rightarrow \infty$. By (3.16) and Lemma 2.3, we have $p \in F(S)$.

Next, we show that $p \in MEP(F_1, \varphi)$. Since $u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n$, we have

$$F_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n \rangle \geq 0, \quad \forall y \in C,$$

which implies that

$$F_1(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle - \frac{1}{r_n} \langle y - u_n, \gamma A^*(I - T_{r_n}^{F_2})Ax_n \rangle \geq 0, \quad \forall y \in C.$$

From Assumption 2.4(A2), we have

$$\begin{aligned} \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle - \frac{1}{r_n} \langle y - u_n, \gamma A^*(I - T_{r_n}^{F_2})Ax_n \rangle \\ \geq -F_1(u_n, y) \geq F_1(y, u_n), \quad \forall y \in C, \end{aligned}$$

and hence

$$\begin{aligned} \varphi(y) - \varphi(u_{n_i}) + \frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle - \frac{1}{r_{n_i}} \langle y - u_{n_i}, \gamma A^*(I - T_{r_{n_i}}^{F_2})Ax_{n_i} \rangle \geq F_1(y, u_{n_i}), \\ \forall y \in C. \end{aligned}$$

This implies by $u_{n_i} \rightharpoonup p$, Condition (C3), (3.12), (3.14), Assumption 2.4(A2), and the proper lower semicontinuity of φ that

$$F_1(y, p) + \varphi(p) - \varphi(y) \leq 0, \quad \forall y \in C.$$

Put $y_t = ty + (1 - t)p$ for all $t \in (0, 1]$ and $y \in C$. Consequently, we get $y_t \in C$ and hence $F_1(y_t, p) + \varphi(p) - \varphi(y_t) \leq 0$. So, by Assumption 2.4(A1)-(A4), we have

$$\begin{aligned} 0 &= F_1(y_t, y_t) - \varphi(y_t) + \varphi(y_t) \\ &\leq tF_1(y_t, y) + (1 - t)F_1(y_t, p) + t\varphi(y) + (1 - t)\varphi(p) - \varphi(y_t) \\ &\leq t(F_1(y_t, y) + \varphi(y) - \varphi(y_t)). \end{aligned}$$

Hence, we have

$$F_1(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0, \quad \forall y \in C.$$

Letting $t \rightarrow 0$, by Assumption 2.4(A3) and the proper lower semicontinuity of φ , we have

$$F_1(p, y) + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C.$$

This implies that $p \in MEP(F_1, \varphi)$.

Since A is a bounded linear operator, we have $Ax_{n_i} \rightharpoonup Ap$. Then it follows from (3.12) that

$$T_{r_{n_i}}^{F_2} Ax_{n_i} \rightharpoonup Ap \quad \text{as } i \rightarrow \infty. \quad (3.17)$$

By the definition of $T_{r_{n_i}}^{F_2} Ax_{n_i}$, we have

$$F_2(T_{r_{n_i}}^{F_2} Ax_{n_i}, y) + \phi(y) - \phi(T_{r_{n_i}}^{F_2} Ax_{n_i}) + \frac{1}{r_{n_i}} \langle y - T_{r_{n_i}}^{F_2} Ax_{n_i}, T_{r_{n_i}}^{F_2} Ax_{n_i} - Ax_{n_i} \rangle \geq 0, \quad \forall y \in Q.$$

Since F_2 is upper semicontinuous in the first argument, it implies by (3.17) that

$$F_2(Ap, y) + \phi(y) - \phi(Ap) \geq 0, \quad \forall y \in Q.$$

This shows that $Ap \in \text{MEP}(F_2, \phi)$. Therefore, $p \in \text{SMEP}(F_1, \phi, F_2, \phi)$ and hence $p \in \Theta$.

Step 5. Show that $\{x_n\}$ converges weakly to an element of Θ . It is sufficient to show that $\omega_w(x_n)$ is a singleton set. Let $p, q \in \omega_w(x_n)$ and $\{x_{n_k}\}, \{x_{n_m}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ and $x_{n_m} \rightharpoonup q$. From (3.14), we also have $u_{n_k} \rightharpoonup p$ and $u_{n_m} \rightharpoonup q$. By (3.16) and Lemma 2.3, we see that $p, q \in F(S)$. Applying Lemma 2.2, we obtain $p = q$. This completes the proof. $\square$

If $\varphi = \phi = 0$ in (3.1) and (3.2), then the split mixed equilibrium problem reduces to split equilibrium problem. So, the following result can be obtained from Theorem 3.2 immediately.

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2 . Let $A : H_1 \rightarrow H_2$ be a bounded linear operator and $S : C \rightarrow K(C)$ a λ -hybrid multivalued mapping. Let $F_1 : C \times C \rightarrow \mathbb{R}$, $F_2 : Q \times Q \rightarrow \mathbb{R}$ be bifunctions satisfying Assumption 2.4, and F_2 is upper semicontinuous in the first argument. Assume that $\Theta = F(S) \cap \text{SEP}(F_1, F_2) \neq \emptyset$ and $Sp = \{p\}$ for all $p \in F(S)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} u_n = T_{r_n}^{F_1} (I - \gamma A^* (I - T_{r_n}^{F_2}) A) x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n) w_n, \quad w_n \in Su_n, \\ x_{n+1} = \beta_n w_n + (1 - \beta_n) z_n, \quad z_n \in Sy_n, \forall n \in \mathbb{N}, \end{cases} \quad (3.18)$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$, and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of A . Assume that the following conditions hold:

- (C1) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (C2) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (C3) $0 < \liminf_{n \rightarrow \infty} r_n$.

Then the sequence $\{x_n\}$ generated by (3.18) converges weakly to $p \in \Theta$.

Remark 3.4

- (i) Theorems 3.2 and 3.3 extend the corresponding one of Suantai *et al.* [13] and Kazmi and Rizvi [10] to λ -hybrid multivalued mapping and to a split mixed equilibrium

problem. In fact, we present a new iterative algorithm for finding a common element of the set of solutions of split mixed equilibrium problems and the set of fixed points of λ -hybrid multivalued mappings in a real Hilbert space.

- (ii) It is well known that the class of λ -hybrid multivalued mappings contains the classes of nonexpansive multivalued mappings, nonspreading multivalued mappings. Thus, Theorems 3.2 and 3.3 can be applied to these classes of mappings.

We give an example to illustrate Theorem 3.2 as follows.

Example 3.5 Let $H_1 = \mathbb{R}$, $H_2 = \mathbb{R}$, $C = [-3, 0]$, and $Q = (-\infty, 0]$. Let $A : H_1 \rightarrow H_2$ defined by $Ax = \frac{x}{2}$ for each $x \in H_1$. Then $A^*y = \frac{y}{2}$ for each $y \in H_2$. So, $L = \frac{1}{2}$ is the spectral radius of A^*A . Define a multivalued mapping $S : C \rightarrow K(C)$ by

$$Sx = \begin{cases} [-\frac{|x|}{|x|+1}, 0], & x \in [-3, -2); \\ \{0\}, & x \in [-2, 0]. \end{cases}$$

It easy to see that S is 1-hybrid multivalued mapping with $F(S) = \{0\}$ and $S(0) = \{0\}$. For each $x, y \in C$, define the bifunction $F_1 : C \times C \rightarrow \mathbb{R}$ by $F_1(x, y) = xy + y - x - x^2$ and define $\varphi(x) = 0$ for each $x \in C$. For each $u, v \in Q$, define the bifunction $F_2 : Q \times Q \rightarrow \mathbb{R}$ by $F_2(u, v) = uv + 10v - 10u - u^2$ and define $\phi(u) = 0$ for each $u \in Q$.

Choose $\alpha_n = \frac{n}{5n+1}$, $\beta_n = \frac{n}{9n+1}$, $r_n = \frac{n}{n+1}$, and $\gamma = \frac{1}{15}$. It is easy to check that $F_1, F_2, \{\alpha_n\}, \{\beta_n\}, \{r_n\}$ satisfy all conditions in Theorem 3.2.

For each $x \in C$, we compute $T_r^{F_2}Ax$. Find z such that

$$\begin{aligned} 0 &\leq F_2(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, z - Ax \rangle \\ &= zy + 10y - 10z - z^2 + \frac{1}{r} \left\langle y - z, z - \frac{x}{2} \right\rangle \\ &= (z + 10)(y - z) + \frac{1}{r} (y - z) \left(z - \frac{x}{2} \right) \\ &= (y - z) \left((z + 10) + \frac{1}{r} \left(z - \frac{x}{2} \right) \right) \end{aligned}$$

for all $y \in Q$. Thus, by Lemma 2.5(2), it follows that $z = \frac{x-20r}{2(1+r)}$. That is, $T_r^{F_2}Ax = \frac{x-20r}{2(1+r)}$ for each $x \in C$. Furthermore, we get

$$\begin{aligned} (I - \gamma A^*(I - T_r^{F_2})A)x &= x - \frac{1}{15} A^*(Ax - T_r^{F_2}Ax) \\ &= x - \frac{1}{15} A^* \left(\frac{x}{2} - \frac{x-20r}{2(1+r)} \right) \\ &= x - \frac{1}{15} \left(\frac{x}{4} - \frac{x-20r}{4(1+r)} \right) \\ &= x \left(1 - \frac{\gamma}{60} \right) - \frac{\gamma(x-20r)}{60(1+r)}. \end{aligned}$$

Table 1 Numerical results of Example 3.5 for the algorithm (3.19)

n	x_n	$\ x_n - x_{n-1}\ $
1	-3.0000000e+00	-
2	-6.8786127e-02	2.9312139e+00
3	0.0000000e+00	6.8786127e-02
4	0.0000000e+00	0.0000000e+00

Next, we find $u \in C$ such that $F_1(u, v) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle v - u, u - s \rangle \geq 0$ for all $v \in C$, where $s = (I - \gamma A^*(I - T_r^{F_2})A)x$. Note that

$$\begin{aligned}
 0 \leq F_1(u, v) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle v - u, u - s \rangle &= uv + v - u - u^2 + \frac{1}{r}\langle v - u, u - s \rangle \\
 &= (u + 1)(v - u) + \frac{1}{r}(v - u)(u - s) \\
 &= (v - u)\left((u + 1) + \frac{1}{r}(u - s)\right).
 \end{aligned}$$

Thus, by Lemma 2.5(2), it follows that

$$u = \frac{s - r}{1 + r} = \frac{59x - 60r}{60(1 + r)} - \frac{x - 20r}{60(1 + r)^2}.$$

Then the algorithm (3.3) becomes

$$\begin{cases} u_n = \frac{59x_n - 60r_n}{60(1 + r_n)} - \frac{x_n - 20r_n}{60(1 + r_n)^2}, & r_n = \frac{n}{n+1}, \\ y_n = \frac{n}{5n+1}x_n + \left(1 - \frac{n}{5n+1}\right)w_n, \\ x_{n+1} = \frac{n}{9n+1}w_n + \left(1 - \frac{n}{9n+1}\right)z_n, & \forall n \in \mathbb{N}, \end{cases} \quad (3.19)$$

where

$$w_n = \begin{cases} [-\frac{|u_n|}{|u_n|+1}, 0], & u_n \in [-3, -2); \\ \{0\}, & u_n \in [-2, 0], \end{cases} \quad z_n = \begin{cases} [-\frac{|y_n|}{|y_n|+1}, 0], & y_n \in [-3, -2); \\ \{0\}, & y_n \in [-2, 0]. \end{cases}$$

We choose $w_n = -\frac{|u_n|}{|u_n|+1}$ if $u_n \in [-3, -2)$ and $z_n = -\frac{|y_n|}{|y_n|+1}$ if $y_n \in [-3, -2)$. By using SciLab, we compute the iterates of (3.19) for the initial point $x_1 = -3$. The numerical experiment's results of our iteration for approximating the point 0 are given in Table 1.

References

1. Ceng, LC, Yao, JC: A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. *J. Comput. Appl. Math.* **214**, 186-201 (2008)
2. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **63**, 123-145 (1994)
3. Combettes, Pl, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. *J. Nonlinear Convex Anal.* **6**, 117-136 (2005)
4. Flam, SD, Antipin, AS: Equilibrium programming using proximal-like algorithm. *Math. Program.* **78**, 29-41 (1997)
5. Censor, Y, Elfving, T: A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms* **8**, 221-239 (1994)
6. Censor, Y, Bortfeld, T, Martin, B, Trofimov, A: A unified approach for inversion problems in intensity-modulated radiation therapy. *Phys. Med. Biol.* **51**, 2353-2365 (2006)
7. Censor, Y, Elfving, T, Kopf, N, Bortfeld, T: The multiple-sets split feasibility problem and its applications for inverse problems. *Inverse Probl.* **21**, 2071-2084 (2005)
8. Censor, Y, Motova, A, Segal, A: Perturbed projections and subgradient projections for the multiple-sets split feasibility problem. *J. Math. Anal. Appl.* **327**, 1244-1256 (2007)
9. Chan, T, Shen, J: *Image Processing and Analysis: Variational, PDE, Wavelet, and Stochastic Methods*. SIAM, Philadelphia (2005)
10. Kazmi, KR, Rizvi, SH: Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem. *J. Egypt. Math. Soc.* **21**, 44-51 (2013)
11. Brouhachem, A: Algorithms of common solutions for a variational inequality, a split equilibrium problem and a hierarchical fixed point problem. *Fixed Point Theory Appl.* **2013**, 278 (2013)
12. Brouhachem, A: Strong convergence algorithm for split equilibrium problems and hierarchical fixed point problems. *Sci. World J.* **2014**, 390956 (2014)
13. Suantai, S, Chalamjiak, P, Cho, YJ, Chalamjiak, W: On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces. *Fixed Point Theory Appl.* **2016**, 35 (2016)
14. Deepho, J, Kumam, W, Kumam, P: A new hybrid projection algorithm for solving the split generalized equilibrium problems and the system of variational inequality problems. *J. Math. Model. Algorithms Oper. Res.* **13**(4), 405-423 (2014)
15. Kumam, W, Deepho, J, Kumam, P: Hybrid extragradient method for finding a common solution of the split feasibility and system of equilibrium problems. *Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms* **21**(6), 367-388 (2014)
16. Deepho, J, Martinez-Moreno, J, Kumam, P: A viscosity of Cesaro mean approximation method for split generalized equilibrium, variational inequality and fixed point problems. *J. Nonlinear Sci. Appl.* **9**, 1475-1496 (2016)
17. Opial, Z: Weak convergence of the sequence of successive approximation for nonexpansive mappings. *Bull. Am. Math. Soc.* **73**, 591-597 (1967)
18. Suantai, S: Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings. *J. Math. Anal. Appl.* **311**, 506-517 (2005)
19. Iiduka, H, Takahashi, W: Weak convergence theorem by Cesaro means for nonexpansive mappings and inverse-strongly monotone mappings. *J. Nonlinear Convex Anal.* **7**, 105-113 (2006)
20. Suantai, S, Phuengrattana, W: Existence and convergence theorems for λ -hybrid mappings in Hilbert spaces. *Dyn. Contin. Discrete Impuls. Syst. Ser. A, Math. Anal.* **22**, 177-188 (2015)
21. Ma, Z, Wang, L, Chang, SS, Duan, W: Convergence theorems for split equality mixed equilibrium problems with applications. *Fixed Point Theory Appl.* **2015**, 31 (2015)

4.4 Hybrid method for the equilibrium problem and a family of generalized nonexpansive mappings in Banach spaces

In this section, we introduce a hybrid method for finding a common element of the set of solutions of an equilibrium problem defined on the dual space of a Banach space and the set of common fixed points of a family of generalized nonexpansive mappings and prove strong convergence theorems by using the new hybrid method. Using our main results, we obtain some new strong convergence theorems for finding a solution of an equilibrium problem and a fixed point of a family of generalized nonexpansive mappings in a Banach space.

1. Introduction

Let E be a real Banach space, E^* the dual space of E and C a closed subset of E such that JC is a closed and convex subset of E^* , where J is the duality mapping on E .

Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find

$$\hat{x} \in C \text{ such that } f(J\hat{x}, Jy) \geq 0, \forall y \in C.$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T , that is, $F(T) = \{x \in C : x = Tx\}$. A mapping T of C into itself is called *quasi-nonexpansive* if $F(T)$ is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. It is easy to see that if T is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive.

Numerous problems in physics, optimization and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space, see for instance, Blum and Oettli [1], and Combettes and Hirstoaga [2]. On the other hand, Ibaraki and Takahashi [3] introduced a new resolvent of a maximal monotone operator in a Banach space and the concept of a generalized nonexpansive mapping in a Banach space. Ibaraki and Takahashi [3], and Kohsaka and Takahashi [5] also studied some properties for generalized nonexpansive retractions in Banach spaces. Recently, Takahashi and Zembayashi [12] considered the following equilibrium problem with a bifunction defined on the dual space of a Banach space. Moreover, they proved a strong convergence theorem for finding a solution of the equilibrium problem which generalized the result of Combettes and Hirstoaga [2].

Construction of fixed point iteration of nonlinear mappings is an important subject in the theory of nonlinear mappings and has been widely studied by many mathematicians. In 1953, Mann [6] introduced an algorithm which is used to approximate a fixed point of a nonlinear mapping $T : C \rightarrow C$. Mann's iterative process is defined as follows: $x_0 \in C$

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. However, Mann's algorithm have only weak convergence. For example, Reich [9] proved that if $T : C \rightarrow C$ is a nonexpansive mapping with a fixed point in a closed and convex subset of a uniformly convex Banach space with a Frech'et differentiable norm and $\{\alpha_n\}$ is chosen such that

$\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the Mann's iteration converges weakly to a fixed point of T . Later, Nakajo

and Takahashi [8] attempted to modify the Mann's iteration in order to guarantee strong convergence by using the hybrid method in mathematical programming, called normal hybrid method. For a nonexpansive mapping T in a Hilbert space, it is as follows:

$$\begin{cases} x_1 = x \in C, C_0 = C, \\ u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|z - u_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \quad (1.1)$$

for all $n \in \mathbb{N}$ where $\alpha_n \in [0, a]$ for $a \in [0, 1)$, then sequence $\{x_n\}$ generated by (1.1), converges strongly to $P_{F(T)}x$ which is the metric projection from C onto $F(T)$. Construction the sets C_n and Q_n is difficult to obtain because it has complicated condition. For this reason, Takahashi et al. [11] introduced another hybrid method and proposed the following modification iteration method different from Nakajo and Takahashi's hybrid method [8]. We call such a method the *shrinking projection method*:

$$\begin{cases} x_1 = x \in C, C_0 = C, \\ u_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_n} x \end{cases} \quad (1.2)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$. They proved strong convergence of the sequence $\{x_n\}$ generated by (1.2) under an appropriate control condition on the sequence $\{\alpha_n\}$.

In this paper, motivated by Takahashi et al. [11], we introduce a new hybrid method by using the shrinking projection method and Takahashi and Zembayashi [12] for finding a common element of the set

of solutions of equilibrium problem and the set of common fixed points of a countable family of generalized nonexpansive mappings in a Banach space and prove strong convergence theorems in a Banach space. Using this results, we obtain some new strong convergence results for finding a solution of an equilibrium problem and a fixed point of a generalized nonexpansive mapping or a family of generalized nonexpansive mappings in a Banach space.

2. Preliminaries

Throughout this paper, we assume that all linear spaces are real. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the dual space of E . For a sequence $\{x_n\}$ of E and a point $x \in E$, the *weak* convergence of $\{x_n\}$ to x and the *strong* convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \rightarrow x$, respectively. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E . Then the space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$. We know the following (see [10]):

- (i) if E is smooth, then J is single-valued;
- (ii) if E is reflexive, then J is onto;
- (iii) if E is strictly convex, then J is one-to-one;
- (iv) if E is strictly convex, then J is strictly monotone;
- (v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E .

Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex subset of E . Throughout this paper, define the function $\phi : E \times E \rightarrow \mathbb{R}$ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad \forall y, x \in E. \quad (2.1)$$

Observe that, in a Hilbert space H , (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, for all $x, y \in H$. It is obvious from the definition of the function ϕ that for all $x, y \in E$,

- (P1) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,
- (P2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,
- (P3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Let C be a closed convex subset of a Banach space E , and let T be a mapping from C into itself. Recall that a self-mapping $T : C \rightarrow C$ is *generalized nonexpansive* if $F(T) \neq \emptyset$ and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be a mapping from E onto C . Then R is said to be a *retraction* if $R^2 = R$. The mapping R from E onto C is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$.

A nonempty closed subset C of a smooth Banach space E is said to be a *sunny generalized nonexpansive retract* of E if there exists a sunny generalized nonexpansive retraction R from E onto C . We know the following lemmas for sunny generalized nonexpansive retractions.

Lemma 2.1 ([3]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E and let R be a retraction from E onto C . Then the following assertions are equivalent:*

- (i) *R is sunny generalized nonexpansive;*
- (ii) *$\langle x - Rx, Jy - JRx \rangle \leq 0, \quad \forall x \in E, y \in C$.*

Lemma 2.2 ([3]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E . Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.3 ([3]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C , let $x \in E$ and $z \in C$. Then the following assertions hold:*

- (i) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;*
- (ii) *$\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.*

Lemma 2.4 ([5]). *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E . Then the following items are equivalent:*

- (i) *C is a sunny generalized nonexpansive retract of E ;*
- (ii) *JC is closed and convex.*

Lemma 2.5 ([5]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E . Let R be the sunny generalized nonexpansive retraction from E onto C , let $x \in E$ and $z \in C$. Then the following assertions are equivalent:*

- (i) *$z = Rx$;*
- (ii) *$\phi(x, z) = \min_{y \in C} \phi(x, y)$.*

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. To solve the equilibrium problem, let us assume that a bifunction $f : JC \times JC \rightarrow \mathbb{R}$ satisfies the following conditions:

- (A1) $f(x^*, x^*) = 0$ for all $x^* \in JC$;
- (A2) f is monotone, i.e., $f(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in JC$;
- (A3) for all $x^*, y^*, z^* \in JC$, $\limsup_{t \downarrow 0} f(tz^* + (1-t)x^*, y^*) \leq f(x^*, y^*)$;
- (A4) for all $x^* \in JC$, $f(x^*, \cdot)$ is convex and lower semicontinuous.

Lemma 2.6 ([1]). *Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, and let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)-(A4). Then, for $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(Jz, Jy) + \frac{1}{r} \langle z - x, Jy - Jz \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7 ([12]). *Let C be a nonempty closed subset of a uniformly smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \rightarrow C$ as follows:*

$$T_r(x) = \{z \in C : f(Jz, Jy) + \frac{1}{r} \langle z - x, Jy - Jz \rangle \geq 0, \quad \forall y \in C\}.$$

Then the following statements hold:

- (i) T_r is single-valued;
- (ii) for all $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle x - y, JT_r x - JT_r y \rangle$;
- (iii) $F(T_r) = EP(f)$;
- (iv) $JEP(f)$ is closed and convex.

Lemma 2.8 ([12]). Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then, for $x \in E$ and $p \in F(T_r)$,

$$\phi(x, T_r x) + \phi(T_r x, p) \leq \phi(x, p).$$

The following lemmas are also needed for the proof of our main results.

Lemma 2.9 ([4]). Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 2.10 ([5]). Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a generalized nonexpansive mapping from C into itself. Then $F(T)$ is closed and $JF(T)$ is closed and convex.

Lemma 2.11 ([5]). Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a generalized nonexpansive mapping from C into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E .

Lemma 2.12 ([4]). Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Lemma 2.13 ([13]). Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r(0)$ and $t \in [0, 1]$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Lemma 2.14 ([4]). Let E be a smooth and strictly convex Banach space, $z \in E$, and $\{t_i\}_{i=1}^m \subset (0, 1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in E such that

$$\phi\left(\sum_{i=1}^m t_i x_i, z\right) = \sum_{i=1}^m t_i \phi(x_i, z),$$

then $x_1 = x_2 = \dots = x_m$.

Next, we recall some lemmas for NST-condition.

Let E be a real Banach space and C be a closed convex subset of E . Motivated by Nakajo et al. [7], we give the following definition: Let $\{T_n\}$ and $\mathcal{T}$ be two families of generalized nonexpansive mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. Then, $\{T_n\}$ is said to satisfy the *NST-condition* with $\mathcal{T}$ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0 \Rightarrow \lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0, \text{ for all } T \in \mathcal{T}.$$

In particular, if $\mathcal{T} = \{T\}$, i.e., $\mathcal{T}$ consists of one mapping T , then $\{T_n\}$ is said to satisfy the NST-condition with T . It is obvious that $\{T_n\}$ with $T_n = T$ for all $n \in \mathbb{N}$ satisfies NST-condition with $\mathcal{T} = \{T\}$.

Lemma 2.15. Let C be a closed subset of a uniformly smooth and uniformly convex Banach space E and let T be a generalized nonexpansive mapping from C into E with $F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0, 1]$ satisfies $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n from C into E by

$$T_n x = \beta_n x + (1 - \beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of generalized nonexpansive mappings satisfying the NST-condition with T .

Lemma 2.16. Let C be a closed subset of a uniformly smooth and uniformly convex Banach space E and let S and T be generalized nonexpansive mappings from C into E with $F(S) \cap F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0, 1]$ satisfies $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n from C into E by

$$T_n x = \beta_n Sx + (1 - \beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of generalized nonexpansive mappings satisfying the NST-condition with $\mathcal{T} = \{S, T\}$.

3. Strong convergence theorems

In this section, we introduce and prove a strong convergence theorem of a new hybrid method for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a family of generalized nonexpansive mappings in a Banach space. Recall that an operator T in a Banach space is called *closed*, if $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$.

Before proving our main result, we give the following lemma for non-self generalized nonexpansive mappings in a Banach space.

Lemma 3.1. Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping from C into E such that $F(T) \neq \emptyset$, then $F(T)$ is closed and $JF(T)$ is closed and convex.

Proof. We first prove that $F(T)$ is closed. Let $\{x_n\} \subset F(T)$ with $x_n \rightarrow x$. Since T is generalized nonexpansive, then we have

$$\phi(Tx, x_n) \leq \phi(x, x_n)$$

for each $n \in \mathbb{N}$. This implies

$$\phi(Tx, x) = \lim_{n \rightarrow \infty} \phi(Tx, x_n) \leq \lim_{n \rightarrow \infty} \phi(x, x_n) = \phi(x, x) = 0.$$

Therefore, we have $\phi(Tx, x) = 0$ and hence $x \in F(T)$.

We next show that $JF(T)$ is closed. Let $\{x_n^*\} \subset JF(T)$ such that $x_n^* \rightarrow x^*$ for some $x^* \in E^*$. Note that since JC is closed and convex, we have $x^* \in JC$. Then, there exist $x \in C$ and $\{x_n\} \subset F(T)$ such that $x^* = Jx$ and $x_n^* = Jx_n$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} \phi(Tx, x_n) &\leq \phi(x, x_n) \\ &= \|x\|^2 - 2\langle x, x_n^* \rangle + \|x_n^*\|^2 \\ &\rightarrow \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 = 0. \end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \phi(Tx, x_n) = 0$. Since

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \phi(Tx, x_n) = \lim_{n \rightarrow \infty} (\|Tx\|^2 - 2\langle Tx, x_n^* \rangle + \|x_n^*\|^2) \\ &= \|Tx\|^2 - 2\langle Tx, x^* \rangle + \|x^*\|^2 = \phi(Tx, x), \end{aligned}$$

we have $\phi(Tx, x) = 0$ and hence $x = Tx$. This implies $x^* = Jx \in JF(T)$.

We finally show that $JF(T)$ is convex. Let $x^*, y^* \in JF(T)$ and let $\alpha \in (0, 1)$ and $\beta = 1 - \alpha$. Then we have $x, y \in F(T)$ such that $x^* = Jx$ and $y^* = Jy$. Thus, we have

$$\begin{aligned} & \phi(TJ^{-1}(\alpha Jx + \beta Jy), TJ^{-1}(\alpha Jx + \beta Jy)) \\ &= \|TJ^{-1}(\alpha Jx + \beta Jy)\|^2 - 2\langle TJ^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy \rangle + \|J^{-1}(\alpha Jx + \beta Jy)\|^2 \\ & \quad + \alpha\|x\|^2 + \beta\|y\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\ &= \alpha\phi(TJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(TJ^{-1}(\alpha Jx + \beta Jy), y) + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2). \end{aligned}$$

Since $x, y \in F(T)$ and T is generalized nonexpansive, we have

$$\begin{aligned} & \alpha\phi(TJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(TJ^{-1}(\alpha Jx + \beta Jy), y) + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\ & \leq \alpha\phi(J^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(J^{-1}(\alpha Jx + \beta Jy), y) + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\ &= \alpha\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jx \rangle + \|x\|^2\} \\ & \quad + \beta\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jy \rangle + \|y\|^2\} \\ & \quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\ &= 2\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy \rangle \\ &= 2\|\alpha Jx + \beta Jy\|^2 - 2\|\alpha Jx + \beta Jy\|^2 = 0. \end{aligned}$$

Then we have $TJ^{-1}(\alpha Jx + \beta Jy) = J^{-1}(\alpha Jx + \beta Jy)$ and hence $J^{-1}(\alpha Jx + \beta Jy) \in JF(T)$. This implies that $\alpha Jx + \beta Jy \in JF(T)$. Therefore, $JF(T)$ is convex and the proof is complete. $\square$

Using Lemmas 2.4 and 3.1, we obtain the following lemma.

Lemma 3.2. *Let E be a smooth, strictly convex and reflexive Banach space and C be a closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping from C into E such that $F(T) \neq \emptyset$, then $F(T)$ is a sunny generalized nonexpansive retract of E .*

Theorem 3.3. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)-(A4) and $\{T_n\}$ be a countable family of generalized nonexpansive mappings from C into E , and let $\mathcal{T}$ be a family of closed generalized nonexpansive mappings from C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and $F(\mathcal{T}) \cap EP(f) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n} \langle u_n - y_n, Jy - Ju_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(\mathcal{T}) \cap EP(f)} x$, where $R_{F(\mathcal{T}) \cap EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $F(\mathcal{T}) \cap EP(f)$.

Proof. Since the proof of Theorem 3.3 is very long, so we divide it into 5 steps.

Step1 : We begin by proving that $\{x_n\}$ is well-defined. Putting $u_n = T_{r_n} y_n$ for all $n \in \mathbb{N} \cup \{0\}$, we have from Lemma 2.8 that T_{r_n} is generalized nonexpansive. We first show that $F(\mathcal{T}) \cap EP(f)$ is a sunny generalized nonexpansive retract of E and JC_n is closed and convex. From Lemmas 2.7 and 3.1, we have $JEP(f)$ and $JF(T)$ are closed and convex, respectively. Since E is uniformly convex, J is injective and hence

$$J(F(T) \cap JEP(f)) = JF(T) \cap JEP(f),$$

which is also closed and convex. Using Lemma 2.4, we have $F(\mathcal{T}) \cap EP(f)$ is a sunny generalized nonexpansive retract of E . It is obvious that JC_0 is closed and convex. Since $\phi(u_n, z) \leq \phi(x_n, z)$ is equivalent to

$$0 \leq \|x_n\|^2 - \|u_n\|^2 - 2\langle x_n - u_n, Jz \rangle,$$

which is affine in Jz , hence JC_n is closed and convex. Next, we show by the induction that $F(\mathcal{T}) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. From $C_0 = C$, we have $F(\mathcal{T}) \cap EP(f) \subset C_0$. Suppose that $F(\mathcal{T}) \cap EP(f) \subset C_k$ for some $k \in \mathbb{N} \cup \{0\}$. Since T_{r_k} and T_n are generalized nonexpansive, we have

$$\begin{aligned} \phi(u_k, u) &= \phi(T_{r_k} y_k, u) \leq \phi(y_k, u) \\ &= \phi(\alpha_k x_k + (1 - \alpha_k) T_k x_k, u) \\ &= \|\alpha_k x_k + (1 - \alpha_k) T_k x_k\|^2 - 2\langle \alpha_k x_k + (1 - \alpha_k) T_k x_k, Ju \rangle + \|u\|^2 \\ &\leq \alpha_k \|x_k\|^2 + (1 - \alpha_k) \|T_k x_k\|^2 - 2\alpha_n \langle x_k, Ju \rangle - 2(1 - \alpha_k) \langle T_k x_k, Ju \rangle + \|u\|^2 \\ &= \alpha_k \phi(x_k, u) + (1 - \alpha_k) \phi(T_k x_k, u) \\ &\leq \alpha_k \phi(x_k, u) + (1 - \alpha_k) \phi(x_k, u) = \phi(x_k, u). \end{aligned} \quad (3.1)$$

Hence, we have $u \in C_{k+1}$. This implies that $F(\mathcal{T}) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. So, $\{x_n\}$ is well-defined.

Step2 : We will show that $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$. It follows from Lemma 2.3 (ii) and $x_n = R_{C_n} x$ that

$$\phi(x, x_n) = \phi(x, R_{C_n} x) \leq \phi(x, u) - \phi(R_{C_n} x, u) \leq \phi(x, u)$$

for all $u \in F(\mathcal{T}) \cap EP(f) \subset C_n$. Then, $\{\phi(x, x_n)\}$ is bounded. Moreover, by definition of ϕ , we have that $\{x_n\}$ is bounded. From $C_{n+1} \subset C_n$ and $x_n = R_{C_n} x$, we have

$$\phi(x, x_n) \leq \phi(x, x_{n+1}), \quad n \geq 0.$$

So, the limit of $\{\phi(x, x_n)\}$ exists. From $x_n = R_{C_n} x$, and for any positive integer k , we have

$$\phi(x_n, x_{n+k}) = \phi(R_{C_n} x, x_{n+k}) \leq \phi(x, x_{n+k}) - \phi(x, R_{C_n} x) = \phi(x, x_{n+k}) - \phi(x, x_n).$$

This implies that $\lim_{n \rightarrow \infty} \phi(x_n, x_{n+k}) = 0$. Using Lemma 2.12, we have that, for $m, n \in \mathbb{N}$ with $m > n$,

$$g(\|x_n - x_m\|) \leq \phi(x_n, x_m) \leq \phi(x, x_m) - \phi(x, x_n),$$

where $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. Then the property of the function g yields that $\{x_n\}$ is a Cauchy sequence in C , so there exists $w \in C$ such that $x_n \rightarrow w$. In view of $x_{n+1} = R_{C_{n+1}} x \in C_{n+1}$ and definition of C_{n+1} , we also have

$$\phi(u_n, x_{n+1}) \leq \phi(x_n, x_{n+1}).$$

It follows that $\lim_{n \rightarrow \infty} \phi(u_n, x_{n+1}) = \lim_{n \rightarrow \infty} \phi(x_n, x_{n+1}) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.9 that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = \lim_{n \rightarrow \infty} \|u_n - x_{n+1}\| = 0.$$

So, we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.2)$$

Step3 : We will prove that $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$. Put $r = \max\{\sup_n \|x_n\|, \sup_n \|T x_n\|\}$. Since E is a uniformly convex Banach space, there exists a continuous, strictly increasing and convex function g with $g(0) = 0$ such that

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|)$$

for all $x, y \in B_r(0)$ and $t \in [0, 1]$. So, we have that for $u \in \Omega$,

$$\begin{aligned}
\phi(u_n, u) &= \phi(T_{r_n}y_n, u) \leq \phi(y_n, u) \\
&= \phi(\alpha_n x_n + (1 - \alpha_n)T_n x_n, u) \\
&= \|\alpha_n x_n + (1 - \alpha_n)T_n x_n\|^2 - 2\langle \alpha_n x_n + (1 - \alpha_n)T_n x_n, Ju \rangle + \|u\|^2 \\
&\leq \alpha_n \|x_n\|^2 + (1 - \alpha_n)\|T_n x_n\|^2 - 2\alpha_n \langle x_n, Ju \rangle - 2(1 - \alpha_n)\langle T_n x_n, Ju \rangle + \|u\|^2 \\
&\quad - \alpha_n(1 - \alpha_n)g(\|x_n - T_n x_n\|) \\
&= \alpha_n \phi(x_n, u) + (1 - \alpha_n)\phi(T_n x_n, u) - \alpha_n(1 - \alpha_n)g(\|x_n - T_n x_n\|) \\
&\leq \alpha_n \phi(x_n, u) + (1 - \alpha_k)\phi(x_n, u) - \alpha_n(1 - \alpha_n)g(\|x_n - T_n x_n\|) \\
&= \phi(x_n, u) - \alpha_n(1 - \alpha_n)g(\|x_n - T_n x_n\|).
\end{aligned}$$

Therefore, we have

$$\alpha_n(1 - \alpha_n)g(\|x_n - T_n x_n\|) \leq \phi(x_n, u) - \phi(u_n, u), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Since

$$\begin{aligned}
\phi(x_n, u) - \phi(u_n, u) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle x_n - u_n, Ju \rangle \\
&\leq |\|x_n\|^2 - \|u_n\|^2| + 2|\langle x_n - u_n, Ju \rangle| \\
&\leq |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|x_n - u_n\|\|Ju\| \\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|x_n - u_n\|\|Ju\|,
\end{aligned}$$

it follows that

$$\lim_{n \rightarrow \infty} (\phi(x_n, u) - \phi(u_n, u)) = 0.$$

From $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$, we have $\lim_{n \rightarrow \infty} g(\|x_n - T_n x_n\|) = 0$. By properties of the function g , we have $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$.

Step4 : We will show that $w \in F(T) \cap EP(f)$. Since $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$, we have that

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0, \quad \text{for all } T \in \mathcal{T}.$$

Since $x_n \rightarrow w$ and T is closed, it follows that w is a fixed point of T , that is, $w \in F(\mathcal{T})$ and by (3.2), we have that $u_n \rightarrow w$. On the other hand, from $u_n = T_{r_n}y_n$, Lemma 2.8 and (3.1) we have that

$$\begin{aligned}
\phi(y_n, u_n) &= \phi(y_n, T_{r_n}y_n) \\
&\leq \phi(y_n, u) - \phi(T_{r_n}y_n, u) \\
&\leq \phi(x_n, u) - \phi(T_{r_n}y_n, u) \\
&= \phi(x_n, u) - \phi(u_n, u).
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} (\phi(x_n, u) - \phi(u_n, u)) = 0$, we have that $\lim_{n \rightarrow \infty} \phi(y_n, u_n) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.9 that $\lim_{n \rightarrow \infty} \|y_n - u_n\| = 0$. From $r_n \geq a$, we have

$$\lim_{n \rightarrow \infty} \frac{\|y_n - u_n\|}{r_n} = 0. \quad (3.3)$$

By $u_n = T_{r_n}y_n$, we have

$$f(Ju_n, Jy) + \frac{1}{r_n} \langle u_n - y_n, Jy - Ju_n \rangle \geq 0, \quad \forall y \in C.$$

By (A2), we have that

$$\frac{1}{r_n} \langle u_n - y_n, Jy - Ju_n \rangle \geq -f(Ju_n, Jy) \geq f(Jy, Ju_n), \quad \forall y \in C. \quad (3.4)$$

Since $f(x, \cdot)$ is convex and lower semicontinuous and $u_n \rightarrow w$, it follows from (3.3) and (3.4) that

$$f(Jy, Jw) \leq 0, \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t^* = tJy + (1-t)Jw$. Since JC is convex, we have $y_t^* \in JC$ and hence $f(y_t^*, Jw) \leq 0$. So, from (A1) we have

$$0 = f(y_t^*, y_t^*) \leq tf(y_t^*, Jy) + (1-t)f(y_t^*, Jw) \leq tf(y_t^*, Jy).$$

Hence

$$f(y_t^*, Jy) \geq 0, \quad \forall y \in C.$$

Letting $t \downarrow 0$, from (A3) we have

$$f(Jw, Jy) \geq 0, \quad \forall y \in C.$$

Therefore, we have $Jw \in JEP(f)$ that is $w \in EP(f)$.

Step5 : We will show that x_n converges strongly to $R_{F(\mathcal{T}) \cap EP(f)}x$ by proving $w = R_{F(\mathcal{T}) \cap EP(f)}x$. From Lemma 2.3 (ii), we have

$$\phi(x, R_{F(\mathcal{T}) \cap EP(f)}x) + \phi(R_{F(\mathcal{T}) \cap EP(f)}x, w) \leq \phi(x, w).$$

Since $x_{n+1} = R_{C_{n+1}}x$ and $w \in F(\mathcal{T}) \cap EP(f) \subset C_n$, we get from Lemma 2.3 (ii) that

$$\phi(x, x_{n+1}) + \phi(x_{n+1}, R_{F(\mathcal{T}) \cap EP(f)}x) \leq \phi(x, R_{F(\mathcal{T}) \cap EP(f)}x). \quad (3.5)$$

Since $x_n \rightarrow w$, it follows by definition of ϕ that $\phi(x, x_{n+1}) \rightarrow \phi(x, w)$. This implies by (3.5) that $\phi(x, w) \leq \phi(x, R_{F(\mathcal{T}) \cap EP(f)}x)$. But since $\phi(x, w) \geq \phi(x, R_{F(\mathcal{T}) \cap EP(f)}x)$, we obtain $\phi(x, w) = \phi(x, R_{F(\mathcal{T}) \cap EP(f)}x)$. Therefore, it follows from the uniqueness of $R_{F(\mathcal{T}) \cap EP(f)}x$ that $w = R_{F(\mathcal{T}) \cap EP(f)}x$. This completes the proof. $\square$

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n} \langle u_n - x_n, Jy - Ju_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{EP(f)}x$, where $R_{EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $EP(f)$.

Proof. Putting $T_n = I$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 3.3, we obtain the desired result. $\square$

Corollary 3.5. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $\{T_n\}$ be a countable family of generalized nonexpansive mappings from C into E and, let $\mathcal{T}$ be a family of closed generalized nonexpansive mappings from C into E such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} u_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})}x$, where $R_{F(\mathcal{T})}$ is the sunny generalized nonexpansive retraction from E onto $F(\mathcal{T})$.

Proof. Putting $f(Jx, Jy) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ in Theorem 3.3, we obtain the desired result. $\square$

4. Deduced results

In this section, using Theorem 3.3, we obtain some new convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of one and two of generalized nonexpansive mappings in a Banach space.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)-(A4) and T be a closed generalized nonexpansive mapping from C into E such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n} \langle u_n - y_n, Jy - Ju_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(T) \cap EP(f)}x$, where $R_{F(T) \cap EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $F(T) \cap EP(f)$.

Proof. Put $T_n = T$ for all $n \in \mathbb{N}$. It is obvious that $\{T_n\}$ satisfies the NST-condition with T , so we obtain the desired result by using Theorem 3.3. $\square$

Theorem 4.2. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)-(A4) and T be a closed generalized nonexpansive mapping from C into E such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n), \\ u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n} \langle u_n - y_n, Jy - Ju_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(T) \cap EP(f)}x$, where $R_{F(T) \cap EP(f)}$ is the sunny generalized nonexpansive retraction from E onto $F(T) \cap EP(f)$.

Proof. Define $T_n x = \beta_n x + (1 - \beta_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 2.15, we know that $\{T_n\}$ satisfies the NST-condition with T , so we obtain the desired result by using Theorem 3.3. $\square$

Theorem 4.3. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying (A1)-(A4) and S, T be closed generalized nonexpansive mappings from C into E such that $\Omega = F(S) \cap F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)(\beta_n Sx_n + (1 - \beta_n)Tx_n), \\ u_n \in C, \text{ such that } f(Ju_n, Jy) + \frac{1}{r_n} \langle u_n - y_n, Jy - Ju_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying

$\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $R_\Omega x$, where R_Ω is the sunny generalized nonexpansive retraction from E onto Ω .

Proof. Define $T_n x = \beta_n Sx + (1 - \beta_n)Tx$ for all $n \in \mathbb{N}$ and $x \in C$. By Lemma 2.15, we know that $\{T_n\}$ satisfies the NST-condition with T . So, we obtain the desired result by using Theorem 3.3. $\square$

5. Applications

In this section, we give a strong convergence theorem for a countable family of nonexpansive mappings in a Hilbert space. In a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$, $J = I$, where I is an identity mapping and every nonexpansive mapping is closed generalized nonexpansive. The following two lemmas are directly obtained by Lemmas 2.15 and 2.16, respectively.

Lemma 5.1 ([11, Lemma 2.1]). *Let C be a closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0, 1]$ satisfies $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n of C into itself by*

$$T_n x = \beta_n x + (1 - \beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of nonexpansive mappings satisfying the NST-condition with T .

Lemma 5.2 ([11, Lemma 2.3]). *Let C be a closed and convex subset of a Hilbert space H and let S and T be nonexpansive mappings from C into itself with $F(S) \cap F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0, 1]$ satisfies $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n of C into itself by*

$$T_n x = \beta_n Sx + (1 - \beta_n)Tx$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of nonexpansive mappings satisfying the NST-condition with $\{S, T\}$.

Theorem 5.3. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H . Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and $F(\mathcal{T}) \cap EP(f) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with $\mathcal{T}$. Let $\{x_n\}$ be the sequence generated by $x_0 = x \in C, C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n, \\ u_n \in C, \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle u_n - y_n, y - u_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{T}) \cap EP(f)} x$, where $P_{F(\mathcal{T}) \cap EP(f)}$ is the metric projection from C onto $F(\mathcal{T}) \cap EP(f)$.

Proof. In a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$, $J = I$, where I is the identity mapping. By using Theorem 3.3, we obtain the desired conclusion. $\square$

References

- [1] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145. 1, 2.6
- [2] P. L. Combettes, S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117–136. 1
- [3] T. Ibaraki, W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory, **149** (2007), 1–14. 1, 2.1, 2.2, 2.3
- [4] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim., **13** (2002), 938–945. 2.9, 2.12, 2.14
- [5] F. Kohsaka, W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, J. Nonlinear Convex Anal., **8** (2007), 197–209. 1, 2.4, 2.5, 2.10, 2.11
- [6] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510. 1
- [7] K. Nakajo, K. Shimoji, W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings in Banach spaces*, J. Nonlinear Convex Anal., **8** (2007), 11–34. 2
- [8] K. Nakajo, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279** (2003), 372–379. 1, 1
- [9] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67** (1979), 247–276. 1
- [10] W. Takahashi, *Nonlinear functional analysis*, Fixed point theory and its applications, Yokohama Publ., Yokohama, (2000). 2
- [11] W. Takahashi, Y. Takeuchi, R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **341** (2008), 276–286. 1, 1, 5.1, 5.2
- [12] W. Takahashi, K. Zembayashi, *A strong convergence theorem for the equilibrium problem with a bifunction defined on the dual space of a Banach space*, Fixed point theory and its applications, Yokohama Publ., Yokohama, (2008), 197–209. 1, 1, 2.7, 2.8
- [13] C. Zălinescu, *On uniformly convex functions*, J. Math. Anal. Appl., **95** (1983), 344–374. 2.13

Chapter V

Convex Minimization Problems and Fixed Point Problems in Euclidean Spaces and Complete CAT(0) spaces

In this chapter, we first discuss some algorithms for finding a common element of the set of common minimizers of a finite family of convex and lower semi-continuous functions and the set of common fixed points of a finite family of nonexpansive mappings in complete CAT(0) spaces, and then we study two-stage stochastic quadratic symmetric programming to handle uncertainty in data defining (Deterministic) symmetric programs in which a quadratic function is minimized over the intersection of an affine set and a symmetric cone with finite event space. For the final section, we describe a method of finding the bound of 2-page fixed linear crossing number of G .

5.1 Iterative approaches to solving convex minimization problems and fixed point problems in complete CAT(0) spaces

In this section, we propose a new modified proximal point algorithm for finding a common element of the set of common minimizers of a finite family of convex and lower semi-continuous functions and the set of common fixed points of a finite family of nonexpansive mappings in complete CAT(0) spaces, and prove some convergence theorems of the proposed algorithm under suitable conditions. A numerical example is presented to illustrate the proposed method and convergence result. Our results improve and extend the corresponding results existing in the literature.

1 Introduction

A metric space (X, d) is said to be a *CAT(0) space* if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane (see more details in [1]). It is well known that any complete, simply

connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Euclidean spaces, Hilbert spaces, the Hilbert ball [2], hyperbolic planes [3], and $\mathbb{R}$ -trees [4].

Let X be a CAT(0) space and $h : X \rightarrow (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization is to find $x \in X$ such that

$$h(x) = \min_{u \in X} h(u).$$

We denote by $\operatorname{argmin}_{u \in X} h(u)$ the set of minimizers of h . The proximal point algorithm is an important tool for solving this problem which was initiated by Martinet [5] in 1970. Recently, many convergence results concerning the proximal point algorithm for solving optimization problems have been extended from the classical linear spaces such as Hilbert spaces and Banach spaces to the setting of manifolds (see [6–10]).

In 2013, Bačák [7] considered the minimization problems in complete CAT(0) spaces to prove the following result.

Theorem 1 *Let X be a complete CAT(0) space, and $h : X \rightarrow (-\infty, \infty]$ be a convex and lower semi-continuous function. Suppose that h has a minimizer. For $x_1 \in X$, let $\{x_n\}$ be a sequence in X defined by*

$$x_{n+1} = \operatorname{argmin}_{u \in X} \left[h(u) + \frac{1}{2\lambda_n} d(u, x_n)^2 \right], \quad \forall n \geq 1,$$

where $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then, the sequence $\{x_n\}$ Δ -converges to a minimizer of h .

Recently, Chalamjiak et al. [11] considered the problem of finding a minimizer of a convex and lower semi-continuous function and common fixed points of two nonexpansive mappings in complete CAT(0) spaces. To be more precise, they obtained the following result.

Theorem 2 *Let X be a complete CAT(0) space, and $h : X \rightarrow (-\infty, \infty]$ be a convex and lower semi-continuous function. Let T_1 and T_2 be nonexpansive mappings on X such that $\mathcal{F} = \operatorname{argmin}_{u \in D} h(u) \cap F(T_1) \cap F(T_2)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \geq 1$ and for some a, b , and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and for some λ . For $x_1 \in X$, let $\{x_n\}$ be a sequence in X defined by*

$$\begin{cases} z_n = \operatorname{argmin}_{u \in X} \left[h(u) + \frac{1}{2\lambda_n} d(u, x_n)^2 \right], \\ y_n = \beta_n x_n \oplus (1 - \beta_n) T_1 z_n, \\ x_{n+1} = \alpha_n T_1 x_n \oplus (1 - \alpha_n) T_2 y_n, \quad \forall n \geq 1. \end{cases}$$

Then, the sequence $\{x_n\}$ Δ -converges to a common element of $\mathcal{F}$.

Motivated by [7] and [11], we present the following question arises:

Question I Can we construct an iterative process for finding common minimizers of a finite family of convex and lower semi-continuous functions and common fixed points of a finite family of nonexpansive mappings in CAT(0) spaces?

The aim of this paper is to propose a new modified proximal point algorithm for finding a common element of the set of common minimizers of a finite family of convex and lower semi-continuous functions and the set of common fixed points of a finite family of nonexpansive mappings in a nonempty closed convex subset of a complete CAT(0) space and prove Δ -convergence and strong convergence theorems of the proposed algorithm under suitable conditions. A numerical example to support our main results is also given. Our results not only give an affirmative answer to the question I but also generalize the corresponding results of Bačák [7], Ariza-Ruiz et al. [6], Choleamjiak et al., and many others.

2 Preliminaries and lemmas

Let D be a nonempty subset of a CAT(0) space X . A subset D of X is said to be *convex* if D includes every geodesic segment joining any two of its points, that is, for any $x, y \in D$, we have $[x, y] \subset D$, where $[x, y] := \{\alpha x \oplus (1 - \alpha)y : 0 \leq \alpha \leq 1\}$ is the unique geodesic joining x and y .

Let $T : D \rightarrow D$ be a mapping. An element $x \in D$ is called a *fixed point* of T if $x = Tx$. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in D : x = Tx\}$.

The notion of the asymptotic center can be introduced in a CAT(0) space X as follows: Let $\{x_n\}$ be a bounded sequence in X . For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from [12] that in a complete CAT(0) space, the asymptotic center $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ and call x the Δ -limit of $\{x_n\}$.

We now collect some basic properties of the Δ -convergence which will be used in the sequel.

Lemma 1 ([13]) *Every bounded sequence in a complete CAT(0) space has a Δ -convergent subsequence.*

Lemma 2 ([14]) *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X . If $\{x_n\}$ is a bounded sequence in D , then the asymptotic center of $\{x_n\}$ is in D .*

Lemma 3 ([15]) *Let $\{x_n\}$ be a sequence in a complete $CAT(0)$ space X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 4 ([15]) *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : D \rightarrow D$ be a nonexpansive mapping, i.e., $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$. Let $\{x_n\}$ be a bounded sequence in D such that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$. Then, x is a fixed point of T .*

In 2009, Kopecká and Reich [16] defined convex combinations of a finite members in the Hilbert ball. It can be extended to a $CAT(0)$ space as follows. Let $x_1, \dots, x_n$ be points in a $CAT(0)$ space X and $\gamma_1, \dots, \gamma_n \in (0, 1)$ with $\sum_{i=1}^n \gamma_i = 1$, we write

$$\begin{aligned} & \gamma_1 x_1 \oplus \gamma_2 x_2 \oplus \dots \oplus \gamma_n x_n \\ & := (1 - \gamma_n) \left(\frac{\gamma_1}{1 - \gamma_n} x_1 \oplus \frac{\gamma_2}{1 - \gamma_n} x_2 \oplus \dots \oplus \frac{\gamma_{n-1}}{1 - \gamma_n} x_{n-1} \right) \oplus \gamma_n x_n. \end{aligned} \quad (1)$$

Using (1), we obtain that

$$d(\gamma_1 x_1 \oplus \gamma_2 x_2 \oplus \dots \oplus \gamma_n x_n, y) \leq \sum_{i=1}^n \gamma_i d(x_i, y) \text{ for each } y \in X.$$

In 2014, Chidume et al. [17] proved the following important lemma.

Lemma 5 *Let X be a $CAT(0)$ space and $z \in X$. Let $x_1, \dots, x_N \in X$ and $\gamma_1, \dots, \gamma_N$ be real numbers in $[0, 1]$ such that $\sum_{i=1}^N \gamma_i = 1$. Then, the following inequality holds the following:*

$$d(z, \gamma_1 x_1 \oplus \gamma_2 x_2 \oplus \dots \oplus \gamma_N x_N)^2 \leq \sum_{i=1}^N \gamma_i d(z, x_i)^2 - \sum_{i,j=1, i \neq j}^N \gamma_i \gamma_j d(x_i, x_j)^2.$$

Recall that a function $h : D \rightarrow (-\infty, \infty]$ is *convex* if, for any geodesic $l : [0, 1] \rightarrow D$, the composite function $h \circ l$ is convex. We say that a function h defined on D is *lower semi-continuous* at a point $x \in D$ if

$$h(x) \leq \liminf_{n \rightarrow \infty} h(x_n)$$

for every sequence $\{x_n\}$ in D such that $\lim_{n \rightarrow \infty} x_n = x$. A function h is said to be *lower semi-continuous* on D if it is lower semi-continuous at any point in D . For any $\lambda > 0$, define the *Moreau-Yosida resolvent* of h as follows:

$$J_\lambda x = \operatorname{argmin}_{u \in D} \left[h(u) + \frac{1}{2\lambda} d(u, x)^2 \right]$$

and put $J_0x = x$, for all $x \in D$. This definition in metric spaces with no linear structure first appeared in [18]. The mapping J_λ is well defined for all $\lambda \geq 0$ (see [18]). If h is a proper convex and lower semi-continuous function, then the set of fixed points of the resolvent associated with h coincides with the set of minimizers of h (see [6]). Also, the resolvent J_λ of h is nonexpansive for all $\lambda > 0$ (see [19]).

The following two lemmas are useful for our main results.

Lemma 6 ([19, 20]) *Let X be a complete CAT(0) space and $h : X \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x \in X$ and $\lambda > \mu > 0$, the following identity holds the following:*

$$J_\lambda x = J_\mu \left(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x \right),$$

where J_λ is a Moreau-Yosida resolvent of h .

Lemma 7 ([21]) *Let X be a complete CAT(0) space and $h : X \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x, y \in X$ and $\lambda > 0$, the following inequality holds the following:*

$$\frac{1}{2\lambda} d(J_\lambda x, y)^2 - \frac{1}{2\lambda} d(x, y)^2 + \frac{1}{2\lambda} d(J_\lambda x, x)^2 \leq h(y) - h(J_\lambda x),$$

where J_λ is a Moreau-Yosida resolvent of h .

3 Main results

In this section, we prove Δ -convergence and strong convergence theorems for finding a common solution of the set of a finite family of convex and lower semi-continuous functions and the set of a finite family of nonexpansive mappings in a nonempty closed convex subset of a complete CAT(0) space. In order to prove our main results, the following two lemmas are needed.

Lemma 8 *Let D be a nonempty closed convex subset of a complete CAT(0) space X . Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \bigcap_{i=1}^N \operatorname{argmin}_{u \in D} h_i(u) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by*

$$\begin{cases} y_n^{(i)} = \operatorname{argmin}_{u \in D} \left[h_i(u) + \frac{1}{2\lambda_n^{(i)}} d(u, x_n)^2 \right], \\ z_n = \beta_n^{(0)} x_n \oplus \beta_n^{(1)} y_n^{(1)} \oplus \beta_n^{(2)} y_n^{(2)} \oplus \cdots \oplus \beta_n^{(N)} y_n^{(N)} \\ w_n = \gamma_n^{(0)} z_n \oplus \gamma_n^{(1)} T_1 z_n \oplus \gamma_n^{(2)} T_2 z_n \oplus \cdots \oplus \gamma_n^{(N)} T_N z_n \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) w_n, \quad \forall n \geq 1, \end{cases} \quad (2)$$

where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \beta_n^{(i)} = 1$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \geq 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} > 0$ for all $n \geq 1, i = 1, 2, \dots, N$. Then, $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for $p \in \mathcal{F}$.

Proof Let $p \in \mathcal{F}$. Then, $p = T_i p$ for all $i = 1, 2, \dots, N$ and $h_i(p) \leq h_i(y)$ for all $y \in D$ and $i = 1, 2, \dots, N$. It follows that

$$h_i(p) + \frac{1}{2\lambda_n^{(i)}} d(p, p)^2 \leq h_i(y) + \frac{1}{2\lambda_n^{(i)}} d(y, p)^2$$

for all $y \in D, i = 1, 2, \dots, N$, and hence, $p = J_{\lambda_n^{(i)}} p$ for all $n \geq 1$. Since $y_n^{(i)} = J_{\lambda_n^{(i)}} x_n$ for all $n \geq 1$ and $i = 1, 2, \dots, N$, it implies that

$$d(y_n^{(i)}, p) = d(J_{\lambda_n^{(i)}} x_n, J_{\lambda_n^{(i)}} p) \leq d(x_n, p). \quad (3)$$

By (2), we have that

$$\begin{aligned} d(x_{n+1}, p) &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(w_n, p) \\ &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) (\gamma_n^{(0)} d(z_n, p) + \gamma_n^{(1)} d(T_1 z_n, p) \\ &\quad + \dots + \gamma_n^{(N)} d(T_N z_n, p)) \\ &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(z_n, p) \\ &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) (\beta_n^{(0)} d(x_n, p) + \beta_n^{(1)} d(y_n^{(1)}, p) \\ &\quad + \dots + \beta_n^{(N)} d(y_n^{(N)}, p)). \end{aligned}$$

This implies by (3) that

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Hence, the sequence $\{d(x_n, p)\}$ is nonincreasing and bounded below. It follows that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for $p \in \mathcal{F}$. $\square$

Lemma 9 Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X . Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \bigcap_{i=1}^N \operatorname{argmin}_{u \in D} h_i(u) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \beta_n^{(i)} = 1$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \geq 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ for all $n \geq 1, i = 1, 2, \dots, N$ and some $\lambda^{(i)}$. Then, we have the following:

- (i) $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \dots, N$;
(ii) $\lim_{n \rightarrow \infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0$ for all $i = 1, 2, \dots, N$.

Proof (i): Let $p \in \mathcal{F}$. By Lemma 8, we have $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. Using (2), we have that

$$\begin{aligned}
d(x_{n+1}, p)^2 &\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n) d(w_n, p)^2 - \alpha_n (1 - \alpha_n) d(x_n, w_n)^2 \\
&\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n) \left(\gamma_n^{(0)} d(z_n, p)^2 + \sum_{i=1}^N \gamma_n^{(i)} d(T_i z_n, p)^2 \right. \\
&\quad \left. - \sum_{i=1}^N \gamma_n^{(0)} \gamma_n^{(i)} d(z_n, T_i z_n)^2 - \sum_{i,j=1, i \neq j}^N \gamma_n^{(i)} \gamma_n^{(j)} d(T_i z_n, T_j z_n)^2 \right) \\
&\quad - \alpha_n (1 - \alpha_n) d(x_n, w_n)^2 \\
&\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n) d(z_n, p)^2 - (1 - \alpha_n) \sum_{i=1}^N \gamma_n^{(0)} \gamma_n^{(i)} d(z_n, T_i z_n)^2 \\
&\quad - (1 - \alpha_n) \sum_{i,j=1, i \neq j}^N \gamma_n^{(i)} \gamma_n^{(j)} d(T_i z_n, T_j z_n)^2 - \alpha_n (1 - \alpha_n) d(x_n, w_n)^2 \\
&\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n) \left(\beta_n^{(0)} d(x_n, p)^2 + \sum_{i=1}^N \beta_n^{(i)} d(y_n^{(i)}, p)^2 \right. \\
&\quad \left. - \sum_{i=1}^N \beta_n^{(0)} \beta_n^{(i)} d(x_n, y_n^{(i)})^2 - \sum_{i,j=1, i \neq j}^N \beta_n^{(i)} \beta_n^{(j)} d(y_n^{(i)}, y_n^{(j)})^2 \right) \\
&\quad - (1 - \alpha_n) \sum_{i=1}^N \gamma_n^{(0)} \gamma_n^{(i)} d(z_n, T_i z_n)^2 \\
&\quad - (1 - \alpha_n) \sum_{i,j=1, i \neq j}^N \gamma_n^{(i)} \gamma_n^{(j)} d(T_i z_n, T_j z_n)^2 - \alpha_n (1 - \alpha_n) d(x_n, w_n)^2 \\
&\leq d(x_n, p)^2 - (1 - \alpha_n) \sum_{i=1}^N \beta_n^{(0)} \beta_n^{(i)} d(x_n, y_n^{(i)})^2 \\
&\quad - (1 - \alpha_n) \sum_{i,j=1, i \neq j}^N \beta_n^{(i)} \beta_n^{(j)} d(y_n^{(i)}, y_n^{(j)})^2 - (1 - \alpha_n) \sum_{i=1}^N \gamma_n^{(0)} \gamma_n^{(i)} d(z_n, T_i z_n)^2 \\
&\quad - (1 - \alpha_n) \sum_{i,j=1, i \neq j}^N \gamma_n^{(i)} \gamma_n^{(j)} d(T_i z_n, T_j z_n)^2 - \alpha_n (1 - \alpha_n) d(x_n, w_n)^2.
\end{aligned}$$

This implies by $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$ that

$$d(x_n, p)^2 - d(x_{n+1}, p)^2 \geq a^2(1-b) \sum_{i=1}^N d(x_n, y_n^{(i)})^2 + a^2(1-b) \sum_{i,j=1, i \neq j}^N d(y_n^{(i)}, y_n^{(j)})^2$$

$$\begin{aligned}
& +a^2(1-b) \sum_{i=1}^N d(z_n, T_i z_n)^2 + a^2(1-b) \sum_{i,j=1, i \neq j}^N d(T_i z_n, T_j z_n)^2 \\
& +a(1-b)d(x_n, w_n)^2.
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} d(x_n, p)$ exists and $a, b \in (0, 1)$, it implies that $\lim_{n \rightarrow \infty} d(x_n, w_n) = 0$ and

$$\lim_{n \rightarrow \infty} d(x_n, y_n^{(i)}) = 0, \quad \lim_{n \rightarrow \infty} d(z_n, T_i z_n) = 0, \quad (4)$$

for all $i = 1, 2, \dots, N$, and

$$\lim_{n \rightarrow \infty} d(y_n^{(i)}, y_n^{(j)}) = 0, \quad \lim_{n \rightarrow \infty} d(T_i z_n, T_j z_n) = 0, \quad (5)$$

for all $i, j = 1, \dots, N$, and $i \neq j$. From nonexpansiveness of T_i , we have, for each $i = 1, 2, \dots, N$,

$$\begin{aligned}
d(x_n, T_i x_n) & \leq d(x_n, z_n) + d(z_n, T_i z_n) + d(T_i z_n, T_i x_n) \\
& \leq 2d(x_n, z_n) + d(z_n, T_i z_n) \\
& \leq 2(\beta_n^{(1)} d(x_n, y_n^{(1)}) + \dots + \beta_n^{(N)} d(x_n, y_n^{(N)})) + d(z_n, T_i z_n).
\end{aligned}$$

This implies by (4) that $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \dots, N$.

(ii): Since $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$, by Lemma 6 and nonexpansiveness of $J_{\lambda^{(i)}}$, we have

$$\begin{aligned}
d(x_n, J_{\lambda^{(i)}} x_n) & \leq d(x_n, y_n^{(i)}) + d(y_n^{(i)}, J_{\lambda^{(i)}} x_n) \\
& = d(x_n, y_n^{(i)}) + d(J_{\lambda_n^{(i)}} x_n, J_{\lambda^{(i)}} x_n) \\
& = d(x_n, y_n^{(i)}) + d\left(J_{\lambda^{(i)}} \left(\frac{\lambda_n^{(i)} - \lambda^{(i)}}{\lambda_n^{(i)}} J_{\lambda_n^{(i)}} x_n \oplus \frac{\lambda^{(i)}}{\lambda_n^{(i)}} x_n\right), J_{\lambda^{(i)}} x_n\right) \\
& \leq d(x_n, y_n^{(i)}) + d\left(\left(\frac{\lambda_n^{(i)} - \lambda^{(i)}}{\lambda_n^{(i)}}\right) J_{\lambda_n^{(i)}} x_n \oplus \frac{\lambda^{(i)}}{\lambda_n^{(i)}} x_n, x_n\right) \\
& = d(x_n, y_n^{(i)}) + \left(1 - \frac{\lambda^{(i)}}{\lambda_n^{(i)}}\right) d(J_{\lambda_n^{(i)}} x_n, x_n) \\
& = d(x_n, y_n^{(i)}) + \left(1 - \frac{\lambda^{(i)}}{\lambda_n^{(i)}}\right) d(y_n^{(i)}, x_n) \\
& = \left(2 - \frac{\lambda^{(i)}}{\lambda_n^{(i)}}\right) d(y_n^{(i)}, x_n).
\end{aligned}$$

Thus, by (4), we obtain that $\lim_{n \rightarrow \infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0$. □

We now get the Δ -convergence theorem in complete CAT(0) spaces.

Theorem 3 *Let D be a nonempty closed convex subset of a complete CAT(0) space X . Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \bigcap_{i=1}^N \operatorname{argmin}_{u \in D} h_i(u) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty.*

For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n$, $\beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \beta_n^{(i)} = 1$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \geq 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ for all $n \geq 1$, $i = 1, 2, \dots, N$ and some $\lambda^{(i)}$. Then, $\{x_n\}$ Δ -converges to an element of $\mathcal{F}$.

Proof Lemma 8 shows that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$ and Lemma 9 also implies that $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0$ for all $i = 1, 2, \dots, N$.

Next, we show that $\omega_\Delta(x_n) \subset \mathcal{F}$. Let $u \in \omega_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. From Lemmas 1 and 4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n \rightarrow \infty} v_n = v$ for some $v \in \mathcal{F}$. So, by Lemma 3, we have $u = v$. This shows that $\omega_\Delta(x_n) \subset \mathcal{F}$. Finally, we show that the sequence $\{x_n\}$ Δ -converges to a point in $\mathcal{F}$. To this end, it suffices to show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_\Delta(x_n) \subset \mathcal{F}$ and $\{d(x_n, u)\}$ converges, by Lemma 3, we have $x = u$. Hence, $\omega_\Delta(x_n) = \{x\}$. This completes the proof. $\square$

Next, we establish strong convergence theorems in complete CAT(0) spaces.

Theorem 4 Let D be a nonempty closed convex subset of a complete CAT(0) space X . Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \bigcap_{i=1}^N \text{argmin}_{u \in D} h_i(u) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n$, $\beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \beta_n^{(i)} = 1$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \geq 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ for all $n \geq 1$, $i = 1, 2, \dots, N$ and some $\lambda^{(i)}$. Then, the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$, where $\text{dist}(x, \mathcal{F}) = \inf\{d(x, z) : z \in \mathcal{F}\}$.

Proof The necessity is obvious and then we prove only the sufficiency. Suppose that $\liminf_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$. Let $\varepsilon > 0$ be arbitrarily chosen. So, there exists a positive integer m_0 such that

$$\text{dist}(x_n, \mathcal{F}) < \frac{\varepsilon}{4}, \quad \forall n \geq m_0.$$

In particular, $\inf\{d(x_{m_0}, p) : p \in \mathcal{F}\} < \frac{\varepsilon}{4}$. Thus, there must exist $q \in \mathcal{F}$ such that $d(x_{m_0}, q) < \frac{\varepsilon}{2}$. Then, for all $m, n \geq m_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, q) + d(q, x_n) \\ &\leq 2d(x_{m_0}, q) \\ &< \varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in C . Since C is a closed subset in a complete $\text{CAT}(0)$ space, it is complete. This implies that $\{x_n\}$ converges to some point p^* in C . Since $\mathcal{F}$ is closed and $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$, we have $p^* \in \mathcal{F}$. This completes the proof. $\square$

Theorem 5 *Let D be a nonempty closed convex subset of a complete $\text{CAT}(0)$ space X . Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \bigcap_{i=1}^N \text{argmin}_{u \in D} h_i(u) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \beta_n^{(i)} = 1$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \geq 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ for all $n \geq 1, i = 1, 2, \dots, N$ and some $\lambda^{(i)}$. If there exists a nondecreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r > 0$ such that*

$$g(\text{dist}(x, \mathcal{F})) \leq d(x, T_i x)$$

or

$$g(\text{dist}(x, \mathcal{F})) \leq d(x, J_{\lambda^{(i)}} x)$$

for some $i = 1, 2, \dots, N$ and for all $x \in D$, then the sequence $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Proof From Lemma 8, we know that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$. This implies that $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F})$ exists. By Lemma 9, we have

$$\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$$

and

$$\lim_{n \rightarrow \infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0.$$

By the hypothesis, we see that

$$\lim_{n \rightarrow \infty} g(\text{dist}(x_n, \mathcal{F})) \leq \lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$$

or

$$\lim_{n \rightarrow \infty} g(\text{dist}(x_n, \mathcal{F})) \leq \lim_{n \rightarrow \infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0.$$

So, we have

$$\lim_{n \rightarrow \infty} g(\text{dist}(x_n, \mathcal{F})) = 0.$$

Using the property of g , it follows that $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}) = 0$. It implies by Theorem 4 that $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. This completes the proof. $\square$

Recall that a mapping $T : D \rightarrow D$ is said to be *semi-compact* if for any sequence $\{x_n\}$ in D such that $\lim_{n \rightarrow \infty} d(x_n, T x_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$

such that $\{x_{n_i}\}$ converges strongly to $p \in D$. Now, we prove a strong convergence theorem in complete CAT(0) spaces.

Theorem 6 Let D be a nonempty closed convex subset of a complete CAT(0) space X . Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \bigcap_{i=1}^N \operatorname{argmin}_{u \in D} h_i(u) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=1}^N \beta_n^{(i)} = 1$ and $\sum_{i=1}^N \gamma_n^{(i)} = 1$ for all $n \geq 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ for all $n \geq 1, i = 1, 2, \dots, N$ and some $\lambda^{(i)}$. If $J_{\lambda^{(i)}} \circ T_i$ is semi-compact for some $i = 1, 2, \dots, N$, then the sequence $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Proof Without loss of generality, we can assume T_1 is semi-compact. By Lemma 9, we have $\lim_{n \rightarrow \infty} d(x_n, T_1 x_n) = 0$. So, by semi-compactness of T_1 , there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow q \in D$ as $k \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0$ for all $i = 1, 2, \dots, N$, we have $d(q, T_i q) = 0$ and $d(q, J_{\lambda^{(i)}} q) = 0$ for all $i = 1, 2, \dots, N$. This shows that $q \in \mathcal{F}$. In other cases, we can prove the strong convergence of $\{x_n\}$ to an element of $\mathcal{F}$. This completes the proof. $\square$

- Remark 1* (i) Theorems 3 and 6 generalizes the results of Chalamjiak et al. [11] from common fixed points of two nonexpansive mapping to common fixed points of a finite family of nonexpansive mappings, and from a minimize of a convex and lower semi-continuous function to common minimizers of a finite family of convex and lower semi-continuous functions.
- (ii) Theorems 3 and 6 extend the main result in Bačák [7], and the corresponding results in Ariza-Ruiz et al. [6] and Chalamjiak et al. [11]. In fact, we construct a new modified proximal point algorithm for solving the constrained convex minimization problem for a finite family of convex and lower semi-continuous functions as well as the fixed point problem for a finite family of nonexpansive mappings in complete CAT(0) spaces.
- (iii) In Hilbert spaces, if we set $\alpha x \oplus (1 - \alpha)y := \alpha x + (1 - \alpha)y$, then Theorems 3—6 can be applied to these spaces.

Finally, we give the numerical example to support our main theorem in a four-dimensional space of real numbers.

Example 1 Let $X = \mathbb{R}^4$ with the Euclidean norm and $D = \{\mathbf{x} = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}) \in \mathbb{R}^4 : -50 \leq x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)} \leq 50\}$. For each $\mathbf{x} = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}) \in D$, we define mappings T_1 and T_2 on D as follows:

$$T_1 \mathbf{x} = \left(\frac{x^{(1)} + 2}{3}, \frac{x^{(2)} - 2}{3}, \frac{x^{(3)} - 4}{3}, \frac{x^{(4)} + 12}{5} \right)$$

Table 1 Numerical results of Example 1 for the algorithm (6)

Number	$\mathbf{x}_n = (x_n^{(1)}, x_n^{(2)}, x_n^{(3)}, x_n^{(4)})^t$	$\ \mathbf{x}_n - \mathbf{x}_{n-1}\ _2$
1	(-2.5000000, 4.6000000, 3.8000000, -5.9000000)	—
2	(0.1948474, -0.1430572, -1.2589906, -0.9740996)	8.92282e+00
3	(0.6858928, -0.8304900, -1.9042613, 1.3402575)	2.54682e+00
4	(0.8644453, -0.9556049, -1.9876400, 2.3146571)	1.00197e+00
5	(0.9421135, -0.9854949, -1.9984688, 2.7173304)	4.11326e-01
$\vdots$	$\vdots$	$\vdots$
16	(0.9999964, -0.9999993, -2.0000000, 2.9999832)	2.44635e-05
17	(0.9999985, -0.9999997, -2.0000000, 2.9999931)	1.01041e-05
18	(0.9999994, -0.9999999, -2.0000000, 2.9999971)	4.17329e-06

and

$$T_2 \mathbf{x} = \left(\frac{x^{(1)} + 9}{10}, \frac{-x^{(2)} - 5}{4}, \frac{3x^{(3)} - 10}{8}, \frac{5x^{(4)} + 3}{6} \right).$$

For each $\mathbf{x} \in D$, we define $h_1, h_2 : D \rightarrow (-\infty, \infty]$ by

$$h_1(\mathbf{x}) = \frac{1}{2} \|A_1 \mathbf{x} - b_1\|^2, \quad h_2(\mathbf{x}) = \frac{1}{2} \|A_2 \mathbf{x} - b_2\|^2,$$

where

$$A_1 = \begin{pmatrix} 1 & -3 & 6 & -1 \\ 4 & -4 & -3 & -2 \\ 3 & 5 & -4 & 1 \\ -1 & -1 & -5 & 0 \end{pmatrix} \quad \text{and} \quad b_1 = \begin{pmatrix} -11 \\ 8 \\ 9 \\ 10 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & 0 & 2 & 3 \\ 4 & -5 & 1 & 3 \\ -1 & 3 & -4 & -2 \\ 3 & 0 & 1 & -4 \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} 4 \\ 16 \\ -2 \\ -11 \end{pmatrix}.$$

We can check that T_1 and T_2 are nonexpansive and h_1 and h_2 are proper convex and lower semi-continuous. For $i = 1, 2$, we know by [22] that

$$\begin{aligned} J_{1(i)} \mathbf{x} &= \underset{\mathbf{u} \in D}{\operatorname{argmin}} \left[h_i(\mathbf{u}) + \frac{1}{2} \|\mathbf{u} - \mathbf{x}\|^2 \right] \\ &= \operatorname{prox}_{h_i} \mathbf{x} \\ &= (I + A_i^t A_i)^{-1} (\mathbf{x} + A_i^t b_i). \end{aligned}$$

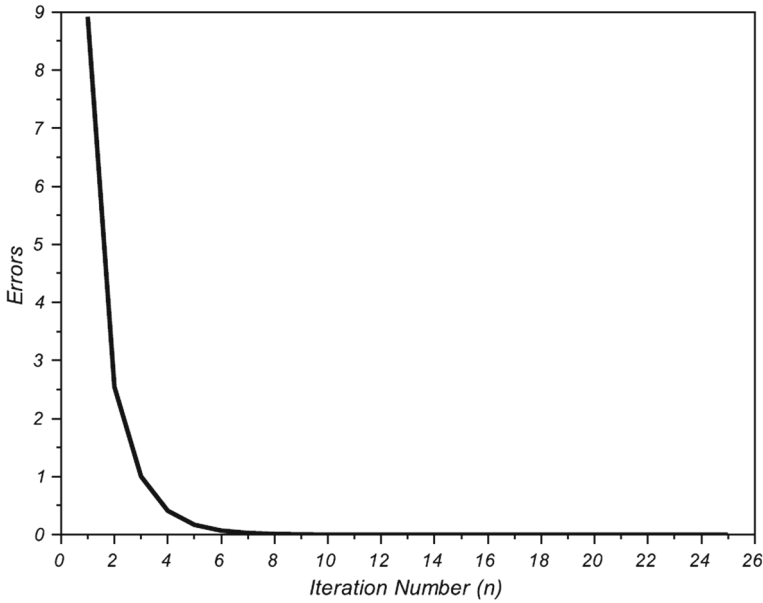


Fig. 1 The error plotting of $\|x_n - x_{n-1}\|_2$ in Table 1

Then, the algorithm (2) becomes:

$$\begin{cases} \mathbf{y}_n^{(1)} = (I + A_1^t A_1)^{-1}(\mathbf{x}_n + A_1^t b_1), \\ \mathbf{y}_n^{(2)} = (I + A_2^t A_2)^{-1}(\mathbf{x}_n + A_2^t b_2), \\ \mathbf{z}_n = \beta_n^{(0)} \mathbf{x}_n + \beta_n^{(1)} \mathbf{y}_n^{(1)} + \beta_n^{(2)} \mathbf{y}_n^{(2)}, \\ \mathbf{w}_n = \gamma_n^{(0)} \mathbf{z}_n + \gamma_n^{(1)} T_1 \mathbf{z}_n + \gamma_n^{(2)} T_2 \mathbf{z}_n, \\ \mathbf{x}_{n+1} = \alpha_n \mathbf{x}_n + (1 - \alpha_n) \mathbf{w}_n, \quad \forall n \geq 1. \end{cases} \quad (6)$$

We choose $\beta_n^{(0)} = \frac{1}{9}$, $\beta_n^{(1)} = \frac{4}{9}$, $\beta_n^{(2)} = \frac{4}{9}$, $\gamma_n^{(0)} = \gamma_n^{(1)} = \gamma_n^{(2)} = \frac{1}{3}$, and $\alpha_n = \frac{25n-1}{625n}$. It can be observed that all the assumptions of Theorems 3 and 6 are satisfied. Using the algorithm (6) with the initial point $\mathbf{x}_1 = (-2.5, 4.6, 3.8, -5.9)^t$, we have numerical results in Table 1 and Fig. 1.

Remark 2 Table 1 and Fig. 1 show that the sequence $\{x_n\}$ converges to a unique point $(1, -1, -2, 3)^t$ which is a common element of the set of common fixed points of T_1 and T_2 and the set of common minimizers of h_1 and h_2 .

Acknowledgements The authors would like to thank the referees for valuable suggestions on the research paper. This paper was supported by the Thailand Research Fund under the project RTA5780007.

References

1. Bridson, M., Haefliger, A.: Metric spaces of non-positive curvature. Springer-Verlag, Berlin, Heidelberg (1999)
2. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)
3. Kohlenbach, U.: Some logical metatheorems with applications in functional analysis. *Trans. Am. Math. Soc.* **357**, 89128 (2015)
4. Tits, J.: A Theorem of Lie-Kolchin for Trees, Contributions to Algebra: a Collection of Papers Dedicated to Ellis Kolchin. Academic Press, New York (1977)
5. Martinet, B.: Regularisation d'inéquations variationnelles par approximations successives. *Rev. Fr. Inform. Rech. Oper.* **4**, 154–158 (1970)
6. Ariza-Ruiz, D., Leustean, L., Lopez, G.: Firmly nonexpansive mappings in classes of geodesic spaces. *Trans. Am. Math. Soc.* **366**, 4299–4322 (2014)
7. Bačák, M.: The proximal point algorithm in metric spaces. *Isr. J. Math.* **194**, 689–701 (2013)
8. Güler, O.: On the convergence of the proximal point algorithm for convex minimization. *SIAM J. Control Optim.* **29**, 403–419 (1991)
9. Marino, G., Xu, H.K.: Convergence of generalized proximal point algorithm. *Commun. Pure. Appl. Anal.* **3**, 791–808 (2004)
10. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.* **14**, 877–898 (1976)
11. Cholamjiak, P., Abdou, A.A., Cho, Y.J.: Proximal point algorithms involving fixed points of nonexpansive mappings in CAT(0) spaces. *Fixed Point Theory Appl.* **2015**, 227 (2015)
12. Dhompongsa, S., Kirk, W.A., Sims, B.: Fixed points of uniformly Lipschitzian mappings. *Nonlinear Anal.* **65**, 762–772 (2006)
13. Kirk, W.A., Panyanak, B.: A concept of convergence in geodesic spaces. *Nonlinear Anal.* **68**, 3689–3696 (2008)
14. Dhompongsa, S., Kirk, W.A., Panyanak, B.: Nonexpansive set-valued mappings in metric and Banach spaces. *J. Nonlinear Convex Anal.* **8**, 35–45 (2007)
15. Dhompongsa, S., Panyanak, B.: On Δ -convergence theorems in CAT(0) spaces. *Comput. Math. Appl.* **56**, 2572–2579 (2008)
16. Kopecká, E., Reich, S.: Asymptotic behavior of resolvents of coaccretive operators in the Hilbert ball. *Nonlinear Anal.* **70**, 3187–3194 (2009)
17. Chidume, C.E., Bello, A.U., Ndambomve, P.: Strong and Δ -convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in CAT(0) spaces, *Abstr. Appl. Anal.* Article ID 805168 (2014)
18. Güler, O.: On the convergence of the proximal point algorithm for convex minimization. *SIAM J. Control Optim.* **40**, 3419, 29 (1991)
19. Jost, J.: Convex functionals and generalized harmonic maps into spaces of nonpositive curvature. *Comment. Math. Helv.* **70**, 659673 (1995)
20. Mayer, U.F.: Gradient flows on nonpositively curved metric spaces and harmonic maps. *Commun. Anal. Geom.* **6**, 199–253 (1998)
21. Ambrosio, L., Gigli, N., Savare, G. *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd edn. Basel, Lectures in Mathematics ETH Zrich. Birkhuser (2008)
22. Combettes, P.L., Pesquet, J.C.: Proximal Splitting Methods in Signal Processing. In: Bauschke, H.H., Burachik, R., Combettes, P.L., Elser, V., Luke, D.R., Wolkowicz, H. (eds.) *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, p. 185212. Springer, New York (2011)

5.2 On solving the minimization problem and the fixed point problem for a finite family of non-expansive mappings in CAT(0) spaces

In this paper, we propose a new modified proximal point algorithm for a finite family of non-expansive mappings in the framework of CAT(0) spaces. We establish Δ -convergence and strong convergence theorems under some mild conditions. Our results extend some known results which appeared in the literature.

1. Introduction

Let (X, d) be a metric space and $f : X \rightarrow (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y).$$

We denote by $\operatorname{argmin}_{y \in X} f(y)$ the set of minimizers of f . The proximal point algorithm is an important tool for solving this problem which was initiated by Martinet [25] in 1970. Recently, many convergence results concerning the proximal point algorithm for solving optimization problems have been extended from the classical linear spaces such as Hilbert spaces and Banach spaces to the setting of manifolds; see [2,3,5,6,8,16,18,24,25,29].

For example, in 2013, Bačák [3] introduced the proximal point algorithm in a CAT(0) space (X, d) as follows: $x_1 \in X$ and

$$x_{n+1} = \operatorname{argmin}_{y \in X} \left(f(y) + \frac{1}{2\lambda_n} d(y, x_n)^2 \right)$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was shown that if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ -converges to its minimizer; see also [4]. Later in 2015, Cholakjiak *et al.* [10] established the strong convergence of the sequence to minimizers of a convex function and to common fixed points of two non-expansive mappings in CAT(0) spaces.

A question arises naturally:

Question A: Can we establish the strong convergence of the sequence to common fixed points of a finite family of non-expansive mappings and to minimizers of a convex function in CAT(0) spaces?

The aim of this paper is to propose a modified proximal point algorithm for a finite family of non-expansive mappings in CAT(0) spaces and prove some convergence theorems of the proposed process under some mild conditions. Our results not only give an affirmative answer to Question A but also generalize the corresponding results of Bačák [3], Cholakjiak *et al.* [10], Dhompsonsa and Panyanak [14], and many others.

2. Preliminaries

A metric space (X, d) is said to be a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane $\mathbb{E}^2$; see more details in [7]. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Euclidean spaces $\mathbb{E}^2$, Hilbert spaces, the Hilbert ball [15], hyperbolic spaces [22], R-trees [30], and many others. The fixed-point theory in CAT(0) spaces was first studied by Kirk [20] in 2003. Since then many authors have published papers on the existence and convergence of fixed points for nonlinear mappings in such spaces (see, e.g. [27,28]).

A subset C of a CAT(0) space X is said to be *convex* if C includes every geodesic segment joining any two of its points, that is, for any $x, y \in C$, we have $[x, y] \subset C$, where $[x, y] := \{\alpha x \oplus (1 - \alpha)y : 0 \leq \alpha \leq 1\}$ is the unique geodesic joining x and y .

Recall that a mapping $T : C \rightarrow C$ is said to be *non-expansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. The fixed-point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

Recall that a function $f : C \rightarrow (-\infty, \infty]$ defined on a convex subset C of a CAT(0) space is *convex* if, for any geodesic $\gamma : [a, b] \rightarrow C$, the function $f \circ \gamma$ is convex, i.e. $f(\alpha x \oplus (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ for all $x, y \in C$. Some important examples can be found in [2]. We say that a function f defined on C is *lower semi-continuous* at a point $x \in C$ if

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$$

for each sequence $\{x_n\}$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. A function f is said to be *lower semi-continuous* on C if it is lower semi-continuous at any point in C . For any $\lambda > 0$, define the *Moreau-Yosida resolvent* of f in CAT(0) spaces as follows:

$$J_\lambda(x) = \operatorname{argmin}_{y \in C} \left(f(y) + \frac{1}{2\lambda} d(y, x)^2 \right)$$

for all $x \in C$. Let $f : C \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. It was shown in [2] that the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $\operatorname{argmin}_{y \in C} f(y)$ of minimizers of f . Also, for any $\lambda > 0$, the resolvent J_λ of f is non-expansive; see [17].

LEMMA 2.1 ([1]) *Let (X, d) be a complete CAT(0) space and $f : X \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}d(J_\lambda x, y)^2 - \frac{1}{2\lambda}d(x, y)^2 + \frac{1}{2\lambda}d(J_\lambda x, x)^2 + f(J_\lambda x) \leq f(y).$$

PROPOSITION 2.2 ([17, 26]) *Let (X, d) be a complete CAT(0) space and $f : X \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then the following identity holds:*

$$J_\lambda x = J_\mu \left(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x \right)$$

for all $x \in X$ and $\lambda > \mu > 0$.

The notion of the asymptotic centre can be introduced in the general setting of a CAT(0) space X as follows: Let $\{x_n\}$ be a bounded sequence in X . For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic centre of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

Another way of stating the asymptotic centre see in [27]. It is known by Dhompongsa et al. [13] that in a complete CAT(0) space, the asymptotic center $A(\{x_n\})$ consists of exactly one point.

We now give the definition and collect some basic properties of the Δ -convergence which will be used in the sequel.

DEFINITION 2.3 ([21]) *A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ -converge to $x \in X$ if x is the unique asymptotic centre of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ and call x the Δ -limit of $\{x_n\}$.*

LEMMA 2.4 ([21]) *Every bounded sequence in a CAT(0) space has a Δ -convergent subsequence.*

LEMMA 2.5 ([12]) *Let C be a nonempty closed convex subset of a CAT(0) space X . If $\{x_n\}$ is a bounded sequence in C , then the asymptotic centre of $\{x_n\}$ is in C .*

LEMMA 2.6 ([14]) *Let $\{x_n\}$ be a sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

LEMMA 2.7 ([14]) *Let C be a closed convex subset of a complete CAT(0) space X and $T : C \rightarrow C$ be a non-expansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$. Then $x = Tx$.*

LEMMA 2.8 ([7]) *Let X be a geodesic metric space. The following are equivalent:*

- (i) X is a CAT(0) space.

(ii) X satisfies the (CN) inequality: If $x, y \in X$ and $x \oplus y/2$ is the midpoint of x and y , then

$$d\left(z, \frac{x \oplus y}{2}\right)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2, \quad \text{for all } z \in X.$$

The following lemma is a generalization of the (CN) inequality which can be found in [14].

LEMMA 2.9 *Let X be a CAT(0) space. Then*

$$d(z, \lambda x \oplus (1 - \lambda)y)^2 \leq \lambda d(z, x)^2 + (1 - \lambda)d(z, y)^2 - \lambda(1 - \lambda)d(x, y)^2,$$

for any $\lambda \in [0, 1]$ and $x, y, z \in X$.

In 2009, Kopecká and Reich [23] defined convex combinations of a finite number of mappings in the Hilbert ball. Later in 2012, Dhompongsa *et al.* [11] introduced the following notation for convex combinations of a finite number of mappings in CAT(0) spaces: Let $x_1, \dots, x_n$ be points in a CAT(0) space X and $\lambda_1, \dots, \lambda_n \in (0, 1)$ with $\sum_{i=1}^n \lambda_i = 1$, we write

$$\bigoplus_{i=1}^n \lambda_i x_i := (1 - \lambda_n) \left(\frac{\lambda_1}{1 - \lambda_n} x_1 \oplus \frac{\lambda_2}{1 - \lambda_n} x_2 \oplus \dots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} x_{n-1} \right) \oplus \lambda_n x_n. \quad (1)$$

The definition of $\bigoplus$ is an ordered one in the sense that it depends on the order of points $x_1, \dots, x_n$. Using (1), we obtain that

$$d\left(\bigoplus_{i=1}^n \lambda_i x_i, y\right) \leq \sum_{i=1}^n \lambda_i d(x_i, y) \quad \text{for each } y \in X.$$

In 2014, Chidume *et al.* [9] proved the following important lemma in CAT(0) spaces.

LEMMA 2.10 *Let (X, d) be a CAT(0) space and $x \in X$. Let $x_0, x_1, \dots, x_N \in X$ and $\gamma_0, \gamma_1, \dots, \gamma_N$ be real numbers in $[0, 1]$ such that $\sum_{i=0}^N \gamma_i = 1$. Then, the following inequality holds:*

$$d\left(x, \bigoplus_{i=0}^N \gamma_i x_i\right)^2 \leq \sum_{i=0}^N \gamma_i d(x, x_i)^2 - \sum_{i,j=0, i \neq j}^N \gamma_i \gamma_j d(x_i, x_j)^2.$$

3. Main results

THEOREM 3.1 *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $f : C \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let $\{T_i\}_{i=1}^N$ be a finite family of non-expansive mappings of C into itself with $\Omega = \bigcap_{i=1}^N F(T_i) \cap \operatorname{argmin}_{y \in C} f(y)$ is nonempty. Assume that $\{\lambda_n\}$ is a sequence such that $\lambda_n > 0$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ be generated in the following manner:*

$$\begin{aligned} y_n &= \operatorname{argmin}_{y \in C} \left[f(y) + \frac{1}{2\lambda_n} d(y, x_n)^2 \right], \\ z_n &= \bigoplus_{i=0}^N \gamma_n^{(i)} T_i y_n, \end{aligned} \quad (2)$$

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n z_n, \quad n \in \mathbb{N},$$

where $T_0 = I$ (identity mapping), $\{\alpha_n\} \subset (0, 1)$, and $\{\gamma_n^{(i)}\} \subset (0, 1)$ for all $i = 0, 1, \dots, N$ with $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \Omega$.

Proof Let $p \in \Omega$. Then $p = T_i p$ for all $i = 1, 2, \dots, N$ and $f(p) \leq f(y)$ for all $y \in C$. It follows that

$$f(p) + \frac{1}{2\lambda_n} d(p, p)^2 \leq f(y) + \frac{1}{2\lambda_n} d(y, p)^2$$

for all $y \in C$ and hence $p = J_{\lambda_n} p$ for all $n \in \mathbb{N}$. Since $y_n = J_{\lambda_n} x_n$ for all $n \in \mathbb{N}$, it implies that

$$d(y_n, p) = d(J_{\lambda_n} x_n, J_{\lambda_n} p) \leq d(x_n, p). \quad (3)$$

By (2), we have, for $p \in \Omega$,

$$\begin{aligned} d(x_{n+1}, p) &\leq (1 - \alpha_n) d(x_n, p) + \alpha_n d(z_n, p) \\ &= (1 - \alpha_n) d(x_n, p) + \alpha_n d\left(\bigoplus_{i=0}^N \gamma_n^{(i)} T_i y_n, p\right) \\ &\leq (1 - \alpha_n) d(x_n, p) + \alpha_n \sum_{i=0}^N \gamma_n^{(i)} d(T_i y_n, p) \\ &\leq (1 - \alpha_n) d(x_n, p) + \alpha_n \sum_{i=0}^N \gamma_n^{(i)} d(y_n, p) \\ &= (1 - \alpha_n) d(x_n, p) + \alpha_n d(y_n, p). \end{aligned} \quad (4)$$

Also, we have, by (3) and (4),

$$d(x_{n+1}, p) \leq d(x_n, p). \quad (5)$$

This shows that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for $p \in \Omega$. ■

THEOREM 3.2 Let C be a nonempty closed convex subset of a complete $CAT(0)$ space (X, d) and $f : C \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let $\{T_i\}_{i=1}^N$ be a finite family of non-expansive mappings of C into itself with $\Omega = \bigcap_{i=1}^N F(T_i) \cap \operatorname{argmin}_{y \in C} f(y)$ is nonempty. Suppose that $\{x_n\}$ is a sequence in C defined by Equation (2) with $\lambda_n > 0$ and $0 < a \leq \alpha_n, \gamma_n^{(i)} \leq b < 1$ for all $i = 0, 1, \dots, N$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \dots, N$.

Proof We first show that $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. By Theorem 3.1, we put

$$\lim_{n \rightarrow \infty} d(x_n, p) = c \quad (6)$$

for some c . From (3), we have

$$\limsup_{n \rightarrow \infty} d(y_n, p) \leq \limsup_{n \rightarrow \infty} d(x_n, p) = c.$$

In fact, from (4), we have

$$d(x_n, p) \leq \frac{1}{\alpha_n} (d(x_n, p) - d(x_{n+1}, p)) + d(y_n, p).$$

Since $d(x_{n+1}, p) \leq d(x_n, p)$ and $\alpha_n \geq a > 0$ for all $n \in \mathbb{N}$, we obtain that

$$d(x_n, p) \leq \frac{1}{a} (d(x_n, p) - d(x_{n+1}, p)) + d(y_n, p).$$

This implies that

$$c = \liminf_{n \rightarrow \infty} d(x_n, p) \leq \liminf_{n \rightarrow \infty} d(y_n, p).$$

Therefore, we have

$$\lim_{n \rightarrow \infty} d(y_n, p) = c. \quad (7)$$

By Lemma 2.1, we obtain that

$$\frac{1}{2\lambda_n} d(y_n, p)^2 - \frac{1}{2\lambda_n} d(x_n, p)^2 + \frac{1}{2\lambda_n} d(x_n, y_n)^2 \leq f(p) - f(y_n).$$

Since $f(p) \leq f(y_n)$ for all $n \in \mathbb{N}$, we have

$$d(x_n, y_n)^2 \leq d(x_n, p)^2 - d(y_n, p)^2.$$

This implies by (6) and (7) that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0. \quad (8)$$

Next, we show that $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \dots, N$.

It follows by Lemma 2.10, (2), and (3) that

$$\begin{aligned} d(x_{n+1}, p)^2 &\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n) d(z_n, p)^2 - \alpha_n (1 - \alpha_n) d(x_n, z_n)^2 \\ &\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n) \left(\sum_{i=0}^N \gamma_n^{(i)} d(T_i y_n, p)^2 - \sum_{i,j=0, i \neq j}^N \gamma_n^{(i)} \gamma_n^{(j)} d(T_i y_n, T_j y_n)^2 \right) \\ &\quad - \alpha_n (1 - \alpha_n) d(x_n, z_n)^2 \\ &\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n) \left(d(y_n, p)^2 - \sum_{i,j=0, i \neq j}^N \gamma_n^{(i)} \gamma_n^{(j)} d(T_i y_n, T_j y_n)^2 \right) \\ &\quad - \alpha_n (1 - \alpha_n) d(x_n, z_n)^2 \\ &\leq d(x_n, p)^2 - (1 - \alpha_n) \sum_{i,j=0, i \neq j}^N \gamma_n^{(i)} \gamma_n^{(j)} d(T_i y_n, T_j y_n)^2 - \alpha_n (1 - \alpha_n) d(x_n, z_n)^2. \end{aligned}$$

This implies that

$$(1 - \alpha_n) \sum_{i,j=0, i \neq j}^N \gamma_n^{(i)} \gamma_n^{(j)} d(T_i y_n, T_j y_n)^2 + \alpha_n (1 - \alpha_n) d(x_n, z_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

By the condition $0 < a \leq \alpha_n, \gamma_n^{(i)} \leq b < 1$, we obtain that

$$(1 - b)a^2 \sum_{i,j=0, i \neq j}^N d(T_i y_n, T_j y_n)^2 + a(1 - b) d(x_n, z_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

Since $\lim_{n \rightarrow \infty} d(x_n, p)$ exists and $a, b \in (0, 1)$, it implies that $\lim_{n \rightarrow \infty} d(x_n, z_n) = 0$ and

$$\sum_{i,j=0, i \neq j}^N \lim_{n \rightarrow \infty} d(T_i y_n, T_j y_n)^2 = 0. \quad (9)$$

Since $T_0 = I$, we rewrite (9) as

$$\sum_{i=0, i \neq j}^N \lim_{n \rightarrow \infty} d(y_n, T_j y_n)^2 + \sum_{i,j=1, i \neq j}^N \lim_{n \rightarrow \infty} d(T_i y_n, T_j y_n)^2 = 0.$$

This implies that

$$\lim_{n \rightarrow \infty} d(y_n, T_j y_n) = 0 \quad \text{for all } j = 1, \dots, N, \quad (10)$$

and

$$\lim_{n \rightarrow \infty} d(T_i y_n, T_j y_n) = 0 \quad \text{for all } i, j = 1, \dots, N \text{ and } i \neq j.$$

From non-expansiveness of T_i , we have, for each $i = 1, 2, \dots, N$,

$$\begin{aligned} d(x_n, T_i x_n) &\leq d(x_n, y_n) + d(y_n, T_i y_n) + d(T_i y_n, T_i x_n) \\ &\leq 2d(x_n, y_n) + d(y_n, T_i y_n). \end{aligned}$$

This implies by (8) and (10) that $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \dots, N$. ■

Next, we prove the Δ -convergence of our iteration.

THEOREM 3.3 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space (X, d) and $f : C \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let $\{T_i\}_{i=1}^N$ be a finite family of non-expansive mappings of C into itself with $\Omega = \bigcap_{i=1}^N F(T_i) \cap \operatorname{argmin}_{y \in C} f(y)$ is nonempty. Suppose that $\{x_n\}$ is a sequence in C defined by (2) with $\lambda_n \geq \lambda > 0$ and $0 < a \leq \alpha_n, \gamma_n^{(i)} \leq b < 1$ for all $i = 0, 1, \dots, N$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ Δ -converges to an element of Ω .*

Proof Since $\lambda_n \geq \lambda > 0$, by Proposition 2.2, we have

$$\begin{aligned} d(J_\lambda x_n, J_{\lambda_n} x_n) &= d\left(J_\lambda x_n, J_\lambda \left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right)\right) \\ &\leq d\left(x_n, \left(\frac{\lambda_n - \lambda}{\lambda_n}\right) J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right) \\ &= \left(1 - \frac{\lambda}{\lambda_n}\right) d(x_n, y_n). \end{aligned}$$

This implies by (8) that

$$\lim_{n \rightarrow \infty} d(J_\lambda x_n, J_{\lambda_n} x_n) = 0. \quad (11)$$

We observe that

$$d(x_n, J_\lambda x_n) \leq d(x_n, y_n) + d(y_n, J_\lambda x_n).$$

It follows by (11) that $\lim_{n \rightarrow \infty} d(x_n, J_\lambda x_n) = 0$. Theorem 3.1 shows that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \Omega$ and Theorem 3.2 also implies that $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \dots, N$.

Next, we show that $\omega_\Delta(x_n) \subset \Omega$. Let $u \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. From Lemmas 2.4 and 2.7, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n \rightarrow \infty} v_n = v$ for some $v \in \Omega$. So, by Lemma 2.6, we have $u = v$. This shows that $\omega_\Delta(x_n) \subset \Omega$. Finally, we show that the sequence $\{x_n\}$ Δ -converges to a point in Ω . To this end, it suffices to show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$

with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_\Delta(x_n) \subset \Omega$ and $\{d(x_n, u)\}$ converges, by Lemma 2.6, we have $x = u$. Hence $\omega_\Delta(x_n) = \{x\}$. This completes the proof. ■

Since every real Hilbert space H is a complete CAT(0) space, the following result can be obtained from Theorem 3.3 immediately.

COROLLARY 3.4 *Let C be a nonempty closed convex subset of a Hilbert space H and $f : C \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let $\{T_i\}_{i=1}^N$ be a finite family of non-expansive mappings of C into itself with $\Omega = \bigcap_{i=1}^N F(T_i) \cap \operatorname{argmin}_{y \in C} f(y)$ is nonempty. Assume that $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ . Let $\{x_n\}$ be generated in the following manner:*

$$\begin{aligned} y_n &= \operatorname{argmin}_{y \in C} \left[f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right], \\ z_n &= \sum_{i=0}^N \gamma_n^{(i)} T_i y_n, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n z_n, \quad n \in \mathbb{N}, \end{aligned}$$

where $T_0 = I$ (identity mapping), and $\{\alpha_n\}, \{\gamma_n^{(i)}\} \subset (0, 1)$ with $0 < a \leq \alpha_n, \gamma_n^{(i)} \leq b < 1$ for all $i = 0, 1, \dots, N$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to an element of Ω .

Remark 3.5

- (i) Theorem 3.3 generalizes the results of Khan and Abbas [19] from one non-expansive mapping to a finite family of non-expansive mappings in CAT(0) spaces.
- (ii) Theorem 3.3 extends the main result in [3], and the corresponding results in [2,10]. In fact, we present a new modified proximal point algorithm for solving the convex minimization problem as well as the fixed-point problem for a finite family of non-expansive mappings in CAT(0) spaces.
- (iii) Corollary 3.4 is an improvement and generalization of the main result in [16,29].

Next, we establish strong convergence theorems of our iteration in CAT(0) spaces.

THEOREM 3.6 *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $f : C \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let $\{T_i\}_{i=1}^N$ be a finite family of non-expansive mappings of C into itself with $\Omega = \bigcap_{i=1}^N F(T_i) \cap \operatorname{argmin}_{y \in C} f(y)$ is nonempty. Suppose that $\{x_n\}$ is a sequence in C defined by (2) with $\lambda_n \geq \lambda > 0$ and $0 < a \leq \alpha_n, \gamma_n^{(i)} \leq b < 1$ for all $i = 0, 1, \dots, N$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \in \mathbb{N}$. If there exists a nondecreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r > 0$ such that $g(\operatorname{dist}(x, \Omega)) \leq d(x, J_\lambda x)$ or $g(\operatorname{dist}(x, \Omega)) \leq d(x, T_i x)$ for some $i = 1, 2, \dots, N$ and for all $x \in C$, then the sequence $\{x_n\}$ converges strongly to an element of Ω , where $\operatorname{dist}(x, \Omega) = \inf\{d(x, z) : z \in \Omega\}$.*

Proof From Theorem 3.1, we know that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \Omega$. This implies that $\lim_{n \rightarrow \infty} \operatorname{dist}(x_n, \Omega)$ exists. By Theorem 3.2, we have $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$. As in the proof of

Theorem 3.3, we obtain $\lim_{n \rightarrow \infty} d(x_n, J_\lambda x_n) = 0$. By the hypothesis, we see that

$$\lim_{n \rightarrow \infty} g(\text{dist}(x_n, \Omega)) \leq \lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$$

or

$$\lim_{n \rightarrow \infty} g(\text{dist}(x_n, \Omega)) \leq \lim_{n \rightarrow \infty} d(x_n, J_\lambda x_n) = 0.$$

So, we have

$$\lim_{n \rightarrow \infty} g(\text{dist}(x_n, \Omega)) = 0.$$

Using the property of g , it follows that $\lim_{n \rightarrow \infty} \text{dist}(x_n, \Omega) = 0$.

Let $\varepsilon > 0$ be arbitrarily chosen. So, there exists a positive integer m_0 such that

$$\text{dist}(x_n, \Omega) < \frac{\varepsilon}{4}, \quad \text{for all } n \geq m_0.$$

In particular, $\inf\{d(x_{m_0}, p) : p \in \Omega\} < \varepsilon/4$. Thus there must exist $q \in \Omega$ such that $d(x_{m_0}, q) < \varepsilon/2$. Then, for all $m, n \geq m_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, q) + d(q, x_n) \\ &\leq 2d(x_{m_0}, q) \\ &< \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C . Since C is a closed subset in a complete CAT(0) space, it is complete. This implies that $\{x_n\}$ converges to some point p^* in C . Since Ω is closed and $\lim_{n \rightarrow \infty} \text{dist}(x_n, \Omega) = 0$, we have $p^* \in \Omega$. This completes the proof. $\blacksquare$

Recall that a mapping $T : C \rightarrow C$ is said to be *semi-compact* if any sequence $\{x_n\}$ in C satisfying $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence.

THEOREM 3.7 *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $f : C \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let $\{T_i\}_{i=1}^N$ be a finite family of non-expansive mappings of C into itself with $\Omega = \bigcap_{i=1}^N F(T_i) \cap \text{argmin}_{y \in C} f(y)$ is nonempty. Suppose that $\{x_n\}$ is a sequence in C defined by (2) with $\lambda_n \geq \lambda > 0$ and $0 < a \leq \alpha_n, \gamma_n^{(i)} \leq b < 1$ for all $i = 0, 1, \dots, N$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \in \mathbb{N}$. If J_λ or T_i is semi-compact for some $i = 1, 2, \dots, N$, then the sequence $\{x_n\}$ converges strongly to an element of Ω .*

Proof Without loss of generality, we can assume T_1 is semi-compact. By Theorem 3.2, we have $\lim_{n \rightarrow \infty} d(x_n, T_1 x_n) = 0$. So, by semi-compactness of T_1 , there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow q \in C$ as $k \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} d(x_n, J_\lambda x_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \dots, N$, we have $d(q, J_\lambda q) = 0$, and $d(q, T_i q) = 0$ for all $i = 1, 2, \dots, N$. This shows that $q \in \Omega$. In other cases, we can prove the strong convergence of $\{x_n\}$ to an element of Ω . This completes the proof. $\blacksquare$

Remark 3.8

- (i) Theorems 3.6 and 3.7 generalize the results of Khan and Abbas [19] from one non-expansive mapping to a finite family of non-expansive mappings in CAT(0) spaces.
- (ii) In Hilbert spaces, if we set $\alpha x \oplus (1 - \alpha)y := \alpha x + (1 - \alpha)y$, then Theorems 3.6 and 3.7 can be applied to these spaces.

References

- [1] L. Ambrosio, N. Gigli, and G. Savare, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2008.
- [2] D. Ariza-Ruiz, L. Leustean, and G. Lopez, *Firmly nonexpansive mappings in classes of geodesic spaces*, Trans. Am. Math. Soc. 366 (2014), pp. 4299–4322.
- [3] M. Bačák, *The proximal point algorithm in metric spaces*, Isr. J. Math. 194 (2013), pp. 689–701.
- [4] M. Bačák and S. Reich, *The asymptotic behavior of a class of nonlinear semigroups in Hadamard spaces*, J. Fixed Point Theory Appl. 16 (2014), pp. 189–202.
- [5] H.H. Bauschke, E. Matoušková, and S. Reich, *Projection and proximal point methods: Convergence results and counterexamples*, Nonlinear Anal. 56 (2004), pp. 715–738.
- [6] O.A. Boikanyo and G. Morosanu, *A proximal point algorithm converging strongly for general errors*, Optim. Lett. 4 (2010), pp. 635–641.
- [7] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, 1999.
- [8] R.E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. 3 (1977), pp. 459–470.
- [9] C.E. Chidume, A.U. Bello, and P. Ndambomve, *Strong and Δ -convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in $CAT(0)$ spaces*, Abstr. Appl. Anal. 2014 (2014), 805168.
- [10] P. Cholamjiak, A.A. Abdou, and Y.J. Cho, *Proximal point algorithms involving fixed points of nonexpansive mappings in $CAT(0)$ spaces*, Fixed Point Theory Appl. 2015 (2015), pp. 1–13.
- [11] S. Dhompongsa, A. Kaewkhao, and B. Panyanak, *On Kirk's strong convergence theorem for multivalued nonexpansive mappings on $CAT(0)$ spaces*, Nonlinear Anal. 75 (2012), pp. 459–468.
- [12] S. Dhompongsa, W.A. Kirk, and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal. 8 (2007), pp. 35–45.
- [13] S. Dhompongsa, W.A. Kirk, and B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal. 65 (2006), pp. 762–772.
- [14] S. Dhompongsa and B. Panyanak, *On Δ -convergence theorems in $CAT(0)$ spaces*, Comput. Math. Appl. 56 (2008), pp. 2572–2579.
- [15] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.
- [16] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. 29 (1991), pp. 403–419.
- [17] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. 70 (1995), pp. 659–673.
- [18] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory 106 (2000), pp. 226–240.
- [19] S.H. Khan and M. Abbas, *Strong and Δ -convergence of some iterative schemes in $CAT(0)$ spaces*, Comput. Math. Appl. 61 (2011), pp. 109–116.
- [20] W.A. Kirk, *Geodesic geometry and fixed point theory*, in: Seminar of Mathematical Analysis, Malaga/Seville, 2002/2003, in: Colecc. Abierta, Vol. 64, Univ. Sevilla Secr. Publ., Seville, 2003, pp. 195–225.
- [21] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. 68 (2008), pp. 3689–3696.
- [22] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Am. Math. Soc. 357 (2015), pp. 89–128.
- [23] E. Kopecká and S. Reich, *Asymptotic behavior of resolvents of coaccretive operators in the Hilbert ball*, Nonlinear Anal. 70 (2009), pp. 3187–3194.
- [24] G. Marino and H.K. Xu, *Convergence of generalized proximal point algorithm*, Commun. Pure Appl. Anal. 3 (2004), pp. 791–808.

- [25] B. Martinet, *Regularisation d'inéquations variationnelles par approximations successives*, Rev. Fr. Inform. Rech. Oper. 4 (1970), pp. 154–158.
- [26] U.F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Commun. Anal. Geom. 6 (1998), pp. 199–253.
- [27] S. Reich and Z. Salinas, *Weak convergence of infinite products of operators in Hadamard spaces*, Rend. Circolo Mat. Palermo 65 (2016), pp. 55–71.
- [28] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. 15 (1990), pp. 537–558.
- [29] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. 14 (1976), pp. 877–898.
- [30] J. Tits, *A Theorem of Lie-Kolchin for Trees*, Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin, Academic Press, New York, 1977.

5.3 An Interior-Point Trust-Region Algorithm for Quadratic Stochastic Symmetric Programming

Stochastic programming is a framework for modeling optimization problems that involve uncertainty. In this section, we study two-stage stochastic quadratic symmetric programming to handle uncertainty in data defining (Deterministic) symmetric programs in which a quadratic function is minimized over the intersection of an affine set and a symmetric cone with finite event space. Two-stage stochastic programs can be modeled as large deterministic programming and we present an interior point trust region algorithm to solve this problem. Numerical results on randomly generated data are available for stochastic symmetric programs. The complexity of our algorithm is proved.

1 Introduction

Deterministic optimization problems are formulated to find optimal solutions in problems with certainty in data. In fact, in some applications we cannot specify the model entirely because it depends on information which is not available at the time of formulation but that will be determined at some point in the future.

Stochastic programming began in the 1950s to handle those problems that involve uncertainty in data. Many real world applications carry an inherent uncertainty within them. In particular, two-stage stochastic programs have been established to formulate many applications to linear, non-linear and integer programming.

Conic programming deals with an important class of tractable convex optimization problems. Symmetric programming is a generalization of linear programming that includes second-order cone programming and semidefinite programming as special cases. All of these problems are treated in a unified theoretical framework for the design of algorithms and the analysis of their complexity. Stochastic symmetric programming may be viewed as an extension of symmetric programming by allowing uncertainty in data.

Interior point methods are considered to be one of the most successful classes of algorithms for solving convex optimization problems. We refer to [1] which developed an interior-point trust-region algorithm for minimizing a quadratic objective function in the intersection of a symmetric cone and an affine subspace by solving sequential trust-region subproblems. Global first-order and second-order convergence results were proved. The techniques and properties in both interior-point algorithms and trust-region methods show the complexity of the algorithm by providing strong theoretical support. Our algorithm is developed as an interior-point trust-region algorithm for minimizing stochastic symmetric programming.

In this paper, we present stochastic symmetric programming with a quadratic function and finite scenarios. And then we will explicitly formulate a problem as a large scale deterministic program and develop an interior-point trust-region algorithm to solve it.

The organization of this paper is as follows. In section 2, we present some concepts and results of symmetric cone and self-concordant barrier function used in the theory of interior-point method. In section 3, we propose a model of two-stage stochastic symmetric programs with linear and quadratic function. We also describe a model in deterministic programming. In section 4, we present an interior point trust-region algorithm for solving stochastic symmetric programming and provide complexity of our algorithm. In section 5, we present computational results on a case study problem. In addition, we compared our results with those of the quadprog function in the MATLAB optimization toolbox in cases of the nonnegative orthant cones. Our algorithm can solve stochastic symmetric programming problems that include variables in second order cone and cones of real symmetric positive semidefinite matrices. The quadprog function is unable to solve problems with these variables. Section 6 has some conclusion and remarks.

2 Symmetric Cone and Self-Concordant Barrier

In this section, we introduce some of the concepts and relevant results that will be useful in section 4. For more detailed exposure to those concepts, see the reference [2] and [3].

Let E be finite-dimensional real Euclidean space with inner product $\langle \cdot, \cdot \rangle$.

Definition 2.1 (Cone). A subset $\mathcal{K}$ of E is said to be a *cone* if for every $x \in \mathcal{K}$ and $\lambda > 0$ imply that $\lambda x \in \mathcal{K}$.

Definition 2.2 (Convex cone). A subset $\mathcal{K}$ of E is a *convex cone* if and only if for every $x, y \in \mathcal{K}$ and $\lambda, \mu > 0$ imply that $\lambda x + \mu y \in \mathcal{K}$.

We assume that $\mathcal{K}$ is a convex cone in a E . The dual of $\mathcal{K}$ is defined as

Definition 2.3 (Dual cone). Let $\mathcal{K}$ be a cone. The set

$$\mathcal{K}^* = \{y \in E \mid \langle x, y \rangle \geq 0, \forall x \in \mathcal{K}\}.$$

is called the *dual cone* of $\mathcal{K}$.

As the name suggests, $\mathcal{K}^*$ is a cone, and is always convex, even when the original cone is not. If cone $\mathcal{K}$ and its dual $\mathcal{K}^*$ coincide, we say that $\mathcal{K}$ is self-dual. In particular, this implies that $\mathcal{K}$ has a nonempty interior and does not contain any straight line (i.e., it is pointed).

We denote by $GL(E)$, the group of general linear transformations on E , and by $Aut(\mathcal{K})$, the automorphism group of convex cone $\mathcal{K}$, that is

$$Aut(\mathcal{K}) = \{g \in GL(E) \mid g(\mathcal{K}) = \mathcal{K}\}.$$

Definition 2.4 (Homogeneity). The convex cone $\mathcal{K}$ is said to be *homogeneous* if for every pair $x, y \in \text{int}\mathcal{K}$, there exists an invertible linear operator g for which $g\mathcal{K} = \mathcal{K}$ and $gx = y$.

Definition 2.5 (Symmetric cone). The convex cone $\mathcal{K}$ is said to be *symmetric* if it is self-dual and homogeneous.

Almost all conic optimization problems in real world applications are associated with symmetric cones such as nonnegative orthant cones, second-order cones and cone of positive semi-definite matrices over the real or complex numbers.

The nonnegative orthant cone

$$R_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, 2, \dots, n\}.$$

The second-order cone (also known as the quadratic, Lorentz, or the ice cream cone) of dimension n

$$\mathcal{Q}^n = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n-1} x_i^2 \leq x_n^2 \text{ and } x_n \geq 0\}.$$

The positive semi-definite cone

$$\mathcal{S}_+^n = \{X \mid X \in \mathbb{R}^{n \times n}, X \text{ is positive semidefinite matrix}\}.$$

A self-concordant barrier function is crucial to an interior-point methods. We introduce some of the fundamentals idea a self-concordant barrier function that will be useful in this paper.

Definition 2.6 (Self-Concordant Function). [3] Let $\mathcal{K}^0 \subset \mathcal{K} \subseteq E$ and $F: \mathcal{K}^0 \rightarrow \mathbb{R}$ be a C^3 smooth convex function such that F is called *self-concordant function* if

$$|F'''(x)[h, h, h]| \leq 2(F''(x)(h, h))^{3/2} \quad (2.1)$$

for all $x \in \mathcal{K}^0$ and for all $h \in S$.

Definition 2.7 (Self-concordant barrier). [3] A self-concordant function F is a *self-concordant barrier* for a closed convex set $\mathcal{K}$ if

$$\vartheta = \sup_{x \in \mathcal{K}^0} \langle F'(x), F''(x)^{-1} F'(x) \rangle \quad (2.2)$$

the value ϑ is called a *barrier parameter* of F .

In principle, every convex cone admits a self-concordant barrier(see [3], section 4). We consider a self-concordant barrier of symmetric cones as follows

$$\begin{aligned} F(x) &= -\sum_{i=1}^n \ln x_i && \text{with } \vartheta = n, \text{ for cone of nonnegative orthant,} \\ F(x) &= -\ln(x_n^2 - \sum_{i=1}^{n-1} x_i^2) && \text{with } \vartheta = 2, \text{ for second-order cone,} \\ F(x) &= -\ln \det(X) && \text{with } \vartheta = n, \text{ for positive semidefinite cone.} \end{aligned}$$

Lemma 2.1. [4] Let $F(x)$ is a self-concordant barrier for $\mathcal{K}$, then

$$F''(x)^{-1} F'(x) = -x, \quad (2.3)$$

$$\langle -F'(x), (x) \rangle = \vartheta. \quad (2.4)$$

Lemma 2.2. [4] If $\mathcal{K}$ is a symmetric cone and F is a self-concordant barrier for $\mathcal{K}$, then $F''(x)$ is a linear automorphism of $\mathcal{K}$ for each $x \in \mathcal{K}^0$.

The strictly convex assumption of $F(x)$ implies that $F''(x)$ is positive definite for every $x \in \mathcal{K}^0$. This allows us to define a norm as follows

$$\|h\|_x^2 = \langle h, F''(x)h \rangle \quad (2.5)$$

is a norm on E induced by $F''(x)$. Let $B_x(y, r)$ denote the open ball of radius r centered at y , where the radius is measured with respect to $\|\cdot\|_x$.

Lemma 2.3. [4] If $F(x)$ is a self-concordant function for $\mathcal{K}$, then we have $B_x(x, 1) \subseteq \mathcal{K}^0$ for all $x \in \mathcal{K}^0$.

Lemma 2.4. [4] Assume $F(x)$ is a self-concordant function for $\mathcal{K}$, $x \in \mathcal{K}^0$, and $y \in B_x(x, 1)$, then

$$\left| F(y) - F(x) - \langle F'(x), y - x \rangle - \frac{\langle y - x, F''(x)(y - x) \rangle}{2} \right| \leq \frac{\|y - x\|_x^3}{3(1 - \|y - x\|_x)}. \quad (2.6)$$

Lemma 2.5. [4] Assume $F(x)$ is a self-concordant function for $\mathcal{K}$. If $\|n(x)\|_x \leq 1/4$, then $F(x)$ has a minimizer z and

$$\|z - x\|_x \leq \|n(x)\|_x + \frac{3\|n(x)\|_x^2}{(1 - \|n(x)\|_x)^3}, \quad (2.7)$$

where $n(x) = -F''(x)^{-1}F'(x)$ be the Newton step of $F(x)$.

Lemma 2.6. [4] Assume $F(x)$ is a self-concordant barrier with barrier parameter ϑ . If $x, y \in \mathcal{K}^0$, then

$$\langle F'(x), y - x \rangle \leq \vartheta. \quad (2.8)$$

3 The Problem and Its Modeling

In a standard two-stage stochastic programming model, decision variables are divided into two subsets. First-stage variables are groups of variables determined before the realizations of random events are known. Once the uncertain events have unfolded, further design or operational adjustments can be made through values of the second-stage known as recourse variables, which are determined after knowing the realized values of the random events.

A standard formulation of the two-stage stochastic program is

$$\min \quad f_1(x) + E[Q(x, \xi)] \quad (3.1)$$

$$\text{st.} \quad A_0 x = b, \quad (3.2)$$

$$x \in \mathcal{K}_1, \quad (3.3)$$

where x is the first-stage decision variable, $Q(x, \xi)$ is the optimal value of the second-stage problem:

$$\min \quad f_2(y, \xi) \quad (3.4)$$

$$\text{st.} \quad T(\xi)x + W(\xi)y = h(\xi), \quad (3.5)$$

$$y \in \mathcal{K}_2 \quad (3.6)$$

where y is the second-stage variable and the cones $\mathcal{K}_1$ and $\mathcal{K}_2$ are symmetric cones. The matrix A_0 and the vectors b and c are deterministic data, the matrices $W(\xi)$ and $T(\xi)$ and the vectors $h(\xi)$ and $d(\xi)$ are random data whose realizations depend on an underlying outcome ξ with a known probability function p .

3.1 The Stochastic Linear Symmetric Programs (SLSP)

The stochastic linear symmetric programs can be stated as:

$$\min \quad \langle c, x \rangle + E[Q(x, \xi)] \quad (3.7)$$

$$\text{st.} \quad A_0 x = b, \quad (3.8)$$

$$x \in \mathcal{K}_1, \quad (3.9)$$

where x is the first-stage decision variable, $Q(x, \xi)$ is the minimum of the problem

$$\min \quad \langle g(\xi), y \rangle \quad (3.10)$$

$$\text{st.} \quad T(\xi)x + W(\xi)y = h(\xi), \quad (3.11)$$

$$y \in \mathcal{K}_2 \quad (3.12)$$

where y is the second-stage variable and the cones $\mathcal{K}_1$ and $\mathcal{K}_2$ are symmetric cones. The formulation of the above problem assumes that the second-stage data ξ can be modeled as a random vector with a known probability distribution. The random vector ξ has a finite number of possible realizations, called scenarios, say $\xi_1, \dots, \xi_K$ with respective probability masses $p_1, \dots, p_K$. Then the expectation in the first-stage problem's objective function can be written as the summation

$$E[Q(x, \xi)] = \sum_{k=1}^K p_k Q(x, \xi_k). \quad (3.13)$$

A SLSP can be written as a deterministic program and represented as follows:

$$\min \quad q(x, y^{(1)}, y^{(2)}, \dots, y^{(K)}) = \langle c, x \rangle + \sum_{k=1}^K p_k \langle g^{(k)}, y^{(k)} \rangle \quad (3.14)$$

$$\text{st.} \quad A_0 x = b, \quad (3.15)$$

$$T^{(k)}x + W^{(k)}y^{(k)} = h^{(k)}, k = 1, 2, \dots, K, \quad (3.16)$$

$$x \in \mathcal{K}_1, y^{(k)} \in \mathcal{K}_2, k = 1, 2, \dots, K, \quad (3.17)$$

where $x \in \mathbb{R}^{n_1}$ is the first-stage decision variable and $y^{(k)} \in \mathbb{R}^{n_{2k}}$ are the second stage variable. The matrix $A_0 \in \mathbb{R}^{m_1 \times n_1}$, the vectors $b \in \mathbb{R}^{m_1}$ and $c \in \mathbb{R}^{n_1}$ are deterministic data. The matrices $W^{(k)} \in \mathbb{R}^{m_{2k} \times n_{2k}}$, $T^{(k)} \in \mathbb{R}^{m_{2k} \times n_1}$ and vectors $h^{(k)} \in \mathbb{R}^{m_{2k}}$, $g^{(k)} \in \mathbb{R}^{n_{2k}}$ are random data associated with probability for $k = 1, 2, \dots, K$. The cones $\mathcal{K}_1$ and $\mathcal{K}_2$ are symmetric cone in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_{2k}}$ for $k = 1, 2, \dots, K$ (n_1, n_{2k} are positive integers).

3.2 The Stochastic Quadratic Symmetric Programs (SQSP)

The stochastic quadratic symmetric programs can also be cast as SLSP. The matrix $H_0 \in \mathbb{R}^{n_1 \times n_1}$, $H_0 \succ 0$ is deterministic data and the matrices $H_1(\xi)$ is random data whose realizations depend on an underlying outcome ξ with a known probability function p . Let $H_1^{(k)} \in \mathbb{R}^{n_{2k} \times n_{2k}}$, $H_1^{(k)} \succ 0$ be random data associated with probability for $k = 1, 2, \dots, K$. A SQSP can be formulated as

$$\min \frac{1}{2} \langle x, H_0 x \rangle + \langle c, x \rangle + E[Q(x, \xi)] \quad (3.18)$$

$$\text{st.} A_0 x = b, \quad (3.19)$$

$$x \in \mathcal{K}_1, \quad (3.20)$$

where x is the first-stage decision variable, and $Q(x, \xi)$ is the minimum of the problem

$$\min \frac{1}{2} \langle y, H_1(\xi)y \rangle + \langle g(\xi), y \rangle \quad (3.21)$$

$$\text{st. } T(\xi)x + W(\xi)y = h(\xi), \quad (3.22)$$

$$y \in \mathcal{K}_2, \quad (3.23)$$

where y is the second-stage variable and the cones $\mathcal{K}_1$ and $\mathcal{K}_2$ are symmetric cones.

A SQSP can be equivalently written as a deterministic symmetric programs as:

$$\begin{aligned} \min \quad & q(x, y^{(1)}, y^{(2)}, \dots, y^{(K)}) = \frac{1}{2} \langle x, H_0 x \rangle + \langle c, x \rangle \\ & + \sum_{k=1}^K p_k \left(\frac{1}{2} \langle y^{(k)}, H_1^{(k)} y^{(k)} \rangle + \langle g^{(k)}, y^{(k)} \rangle \right) \end{aligned} \quad (3.24)$$

$$\text{st.} \quad A_0 x = b, \quad (3.25)$$

$$T^{(k)} x + W^{(k)} y^{(k)} = h^{(k)}, k = 1, 2, \dots, K, \quad (3.26)$$

$$x \in \mathcal{K}_1, y^{(k)} \in \mathcal{K}_2, k = 1, 2, \dots, K. \quad (3.27)$$

Define

$$\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2^{(1)} \times \mathcal{K}_2^{(2)} \times \dots \times \mathcal{K}_2^{(K)} \text{ is the symmetric cone,} \quad (3.28)$$

$$X = [x, y^{(1)}, y^{(2)}, \dots, y^{(K)}]^T \in \mathbb{R}^{(n_1 + \sum_{k=1}^K n_{2_k})}, x \in \mathcal{K}_1, y^{(k)} \in \mathcal{K}_2, \text{ for } k = 1, 2, \dots, K, \quad (3.29)$$

$$B = [b, \quad h^{(1)}, \quad h^{(2)}, \quad \dots, \quad h^{(K)}]^T \in \mathbb{R}^{(m_1 + \sum_{k=1}^K m_{2_k})}, \quad (3.30)$$

$$C = [c, p_1 g^{(1)}, p_2 g^{(2)}, \dots, p_K g^{(K)}]^T \in \mathbb{R}^{(n_1 + \sum_{k=1}^K n_{2_k})}, \quad (3.31)$$

$$\begin{aligned} A &= \begin{bmatrix} A_0 & 0 & 0 & \dots & 0 \\ T^{(1)} & W^{(1)} & 0 & \dots & 0 \\ T^{(2)} & 0 & W^{(2)} & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ T^{(K)} & 0 & 0 & \dots & W^{(K)} \end{bmatrix} \in \mathbb{R}^{(m_1 + \sum_{k=1}^K m_{2_k}) \times (n_1 + \sum_{k=1}^K n_{2_k})}, \quad (3.32) \\ Q &= \begin{bmatrix} H_0 & 0 & 0 & \dots & 0 \\ 0 & p_1 H_1^{(1)} & 0 & \dots & 0 \\ 0 & 0 & p_2 H_1^{(2)} & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & p_K H_1^{(K)} \end{bmatrix} \in \mathbb{R}^{(n_1 + \sum_{k=1}^K n_{2_k}) \times (n_1 + \sum_{k=1}^K n_{2_k})}, \end{aligned} \quad (3.33)$$

we can rewrite a SQSP as a large deterministic program as follow

$$\min q(X) = \frac{1}{2} \langle X, QX \rangle + \langle C, X \rangle \quad (3.34)$$

$$st. AX = B, \quad (3.35)$$

$$X \in \mathcal{K}. \quad (3.36)$$

4 The Interior-Point Trust-Region Algorithm

In this section, we present our interior-point trust-region algorithm for solving (3.34)-(3.36). First, we define the function as

$$\begin{aligned} f_{\eta_i}(X) &= \eta_i q(X) + F(X) \\ &= \eta_i \left(\frac{1}{2} \langle x, H_0 x \rangle + \langle c, x \rangle + \sum_{k=1}^K p_k \left(\frac{1}{2} \langle y^{(k)}, H_1^{(k)} y^{(k)} \rangle + \langle g^{(k)}, y^{(k)} \rangle \right) \right) \\ &\quad + F_1(x) + \sum_{k=1}^K F_2(y^{(k)}) \end{aligned} \quad (4.1)$$

where $F(X) : \mathcal{K} \rightarrow \mathbb{R}$ is a self-concordant barrier for the symmetric cone $\mathcal{K}$ with a barrier parameter ϑ . A self-concordant barrier define by $F(X) = F_1(x) + \sum_{k=1}^K F_2(y^{(k)})$ where $F_1(x)$ is a self-concordant barrier function for cone $\mathcal{K}_1$ with a barrier parameter ϑ_1 and $F_2(y^{(k)})$ is a self-concordant barrier function for cone $\mathcal{K}_2^{(k)}$ with a barrier parameter $\vartheta_2^{(k)}$ for $k = 1, 2, 3, \dots, K$ and $\vartheta = \vartheta_1 + \sum_{k=1}^K \vartheta_2^{(k)}$.

Since $q(X)$ is quadratic and from definition 2.6 then f_{η_i} is also a self-concordant barrier function. Consider interior-point trust-region algorithm in each inner iteration for a fix η_i we need to find the central path for decreased the value of $f_{\eta_i}(X)$. For outer iterations we need increase the value of η_i to positive infinity, which implies that the central path converges to a solution that is optimal for the original problem.

Let $d = (d_1, d_2^{(1)}, d_2^{(2)}, d_2^{(3)}, \dots, d_2^{(K)}) \in E$ be the trail step of f_{η_i} . Consider function f_{η_i} in each inner iteration

$$\begin{aligned} &f_{\eta_i}(x_{i,j} + d_1, y_{i,j}^{(1)} + d_2^{(1)}, y_{i,j}^{(2)} + d_2^{(2)}, \dots, y_{i,j}^{(K)} + d_2^{(K)}) - f_{\eta_i}(x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)}) \\ &= \left(\eta_i q(x_{i,j} + d_1, y_{i,j}^{(1)} + d_2^{(1)}, y_{i,j}^{(2)} + d_2^{(2)}, \dots, y_{i,j}^{(K)} + d_2^{(K)}) + F_1(x_{i,j} + d_1) \right. \\ &\quad \left. + \sum_{k=1}^K F_2(y_{i,j}^{(k)} + d_2^{(k)}) \right) - \left(\eta_i q(x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)}) + F_1(x_{i,j}) + \sum_{k=1}^K F_2(y_{i,j}^{(k)}) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\eta_i \langle x_{i,j} + d_1, H_0(x_{i,j} + d_1) \rangle + \eta_i \langle c, x_{i,j} + d_1 \rangle + F_1(x_{i,j} + d_1) \\
&+ \sum_{k=1}^K \left(\frac{1}{2}\eta_i p_k \langle y_{i,j}^{(k)} + d_2^{(k)}, H_1^{(k)}(y_{i,j}^{(k)} + d_2^{(k)}) \rangle + \eta_i p_k \langle g^{(k)}, y_{i,j}^{(k)} + d_2^{(k)} \rangle + F_2^{(k)}(y_{i,j}^{(k)} + d_2^{(k)}) \right) \\
&- \left(\frac{1}{2}\eta_i \langle x_{i,j}, H_0 x_{i,j} \rangle + \eta_i \langle c, x_{i,j} \rangle + F_1(x_{i,j}) + \sum_{k=1}^K \left(\frac{1}{2}\eta_i p_k \langle y_{i,j}^{(k)}, H_1^{(k)} y_{i,j}^{(k)} \rangle \right. \right. \\
&\left. \left. + \eta_i p_k \langle g^{(k)}, y_{i,j}^{(k)} \rangle + F_2^{(k)}(y_{i,j}^{(k)}) \right) \right) \\
&= \frac{1}{2} \langle d_1, \eta_i H_0 d_1 \rangle + \langle (\eta_i H_0 x_{i,j} + c), d_1 \rangle + \sum_{k=1}^K \frac{1}{2} \langle d_2^{(k)}, \eta_i p_k H_1^{(k)} d_2^{(k)} \rangle \\
&+ \sum_{k=1}^K \langle \eta_i p_k (H_1^{(k)} y_{i,j}^{(k)} + g^{(k)}), d_2^{(k)} \rangle + F_1(x_{i,j} + d_1) - F_1(x_{i,j}) + \\
&\sum_{k=1}^K (F_2(y_{i,j}^{(k)} + d_2^{(k)}) - F_2(y_{i,j}^{(k)})). \tag{4.2}
\end{aligned}$$

We want to find the bound of $f_{\eta_i}(x_{i,j} + d_1, y_{i,j}^{(1)} + d_2^{(1)}, y_{i,j}^{(2)} + d_2^{(2)}, \dots, y_{i,j}^{(K)} + d_2^{(K)})$ then we consider the self-concordant barrier in above equation. From Lemma 2.3, for any $X_{i,j} \in \mathcal{K}^0$, we have $X_{i,j} + d \in \mathcal{K}^0$. Given $\|F_1''(x_{i,j})^{\frac{1}{2}} d_1\| \leq \alpha_{1,i,j} < 1$, $\|F_2''(y_{i,j}^{(k)})^{\frac{1}{2}} d_2^{(k)}\| \leq \alpha_{2,i,j}^{(k)} < 1$ for $k = 1, 2, \dots, K$ and $\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)} < 1$. Implied that $\|F_1''(x_{i,j})^{\frac{1}{2}} d_1\|^2 + \sum_{k=1}^K \|F_2''(y_{i,j}^{(k)})^{\frac{1}{2}} d_2^{(k)}\|^2 \leq \alpha_{1,i,j}^2 + \sum_{k=1}^K \alpha_{2,i,j}^{(k)2} < 1$.

Let $F''(X) = F_1''(x) + \sum_{k=1}^K F_2''(y^{(k)})$ denote the Hessian of a self-concordant function $F(X)$. Since it is positive definite for every $X \in \mathcal{K}^0$ implies that $F''(X)$ is positive definite. For $h = [h_1, h_2^{(1)}, h_2^{(2)}, \dots, h_2^{(K)}] \in E$. Let further we define a norm on E induced by $F''(X)$ as $\|h\|_X^2 = \langle h, F''(X)h \rangle = \langle h_1, F_1''(x)h_1 \rangle + \sum_{k=1}^K \langle h_2^{(k)}, F_2''(y^{(k)})h_2^{(k)} \rangle$

$$\begin{aligned}
&F_1(x_{i,j} + d_1) - F_1(x_{i,j}) + \sum_{k=1}^K (F_2(y_{i,j}^{(k)} + d_2^{(k)}) - F_2(y_{i,j}^{(k)})) \\
&\leq \langle F_1'(x_{i,j}), d_1 \rangle + \frac{1}{2} \langle d_1, F_1''(x_{i,j})d_1 \rangle + \frac{\|d_1\|_{x_{i,j}}^3}{3(1 - \|d_1\|_{x_{i,j}})} + \sum_{k=1}^K \langle F_2'(y_{i,j}^{(k)}), d_2^{(k)} \rangle \\
&+ \sum_{k=1}^K \frac{1}{2} \langle d_2^{(k)}, F_2''(y_{i,j}^{(k)})d_2^{(k)} \rangle + \sum_{k=1}^K \frac{\|d_2^{(k)}\|_{y_{i,j}^{(k)}}^3}{3(1 - \|d_2^{(k)}\|_{y_{i,j}^{(k)}})}
\end{aligned}$$

$$\begin{aligned}
&\leq \langle F'_1(x_{i,j}), d_1 \rangle + \frac{1}{2} \langle d_1, F''_1(x_{i,j})d_1 \rangle + \sum_{k=1}^K \langle F'_2(y_{i,j}^{(k)}), d_2^{(k)} \rangle \\
&+ \sum_{k=1}^K \frac{1}{2} \langle d_2^{(k)}, F''_2(y_{i,j}^{(k)})d_2^{(k)} \rangle + \frac{1}{3} \left(\frac{\alpha_{1,i,j}^3}{1 - \alpha_{1,i,j}} + \sum_{k=1}^K \frac{\alpha_{2,i,j}^{(k)3}}{1 - \alpha_{2,i,j}^{(k)}} \right) \\
&\leq \langle F'_1(x_{i,j}), d_1 \rangle + \frac{1}{2} \langle d_1, F''_1(x_{i,j})d_1 \rangle + \sum_{k=1}^K \langle F'_2(y_{i,j}^{(k)}), d_2^{(k)} \rangle \\
&+ \sum_{k=1}^K \frac{1}{2} \langle d_2^{(k)}, F''_2(y_{i,j}^{(k)})d_2^{(k)} \rangle + \frac{1}{3} \left(\frac{(\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)})^3}{1 - (\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)})} \right). \tag{4.3}
\end{aligned}$$

The third inequality of (4.3) follows from Lemma 2.4. From (4.3) we can rewrite (4.2) as

$$\begin{aligned}
&f_{\eta_i}(x_{i,j} + d_1, y_{i,j}^{(1)} + d_2^{(1)}, y_{i,j}^{(2)} + d_2^{(2)}, \dots, y_{i,j}^{(K)} + d_2^{(K)}) - f_{\eta_i}(x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)}) \\
&\leq \frac{1}{2} \langle d_1, \eta_i H_0 d_1 \rangle + \langle (\eta_i H_0 x_{i,j} + c), d_1 \rangle + \sum_{k=1}^K \frac{1}{2} \langle d_2^{(k)}, \eta_i p_k H_1^{(k)} d_2^{(k)} \rangle \\
&+ \langle F'_1(x_{i,j}), d_1 \rangle + \frac{1}{2} \langle d_1, F''_1(x_{i,j})d_1 \rangle + \sum_{k=1}^K \langle F'_2(y_{i,j}^{(k)}), d_2^{(k)} \rangle \\
&+ \sum_{k=1}^K \frac{1}{2} \langle d_2^{(k)}, F''_2(y_{i,j}^{(k)})d_2^{(k)} \rangle + \frac{1}{3} \left(\frac{(\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)})^3}{1 - (\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)})} \right) \\
&= \frac{1}{2} \langle d_1, (\eta_i H_0 + F''_1(x_{i,j}))d_1 \rangle + \langle \eta_i (H_0 x_{i,j} + c) + F'_1(x_{i,j}), d_1 \rangle \\
&+ \sum_{k=1}^K \frac{1}{2} \langle d_2^{(k)}, (\eta_i p_k H_1^{(k)} + F''_2(y_{i,j}^{(k)}))d_2^{(k)} \rangle + \sum_{k=1}^K \langle \eta_i p_k (H_1^{(k)} y_{i,j}^{(k)} + g^{(k)}) + F'_2(y_{i,j}^{(k)}), d_2^{(k)} \rangle \\
&+ \frac{1}{3} \left(\frac{(\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)})^3}{1 - (\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)})} \right). \tag{4.4}
\end{aligned}$$

Now the inequality (4.4) gives an upper bound for $f_{\eta_i}(x_{i,j} + d_1, y_{i,j}^{(1)} + d_2^{(1)}, y_{i,j}^{(2)} + d_2^{(2)}, \dots, y_{i,j}^{(K)} + d_2^{(K)})$. We can try to minimize this bound given in the right-hand side of inequality. This lead to the following trust-region subproblem

$$\begin{aligned}
& \min \frac{1}{2} \left\langle d_1, (\eta_i H_0 + F_1''(x_{i,j})) d_1 \right\rangle + \left\langle \eta_i (H_0 x_{i,j} + c) + F_1'(x_{i,j}), d_1 \right\rangle \\
& + \sum_{k=1}^K \frac{1}{2} \left\langle d_2^{(k)}, (\eta_i p_k H_1^{(k)} + F_2''(y_{i,j}^{(k)})) d_2^{(k)} \right\rangle \\
& + \sum_{k=1}^K \left\langle \eta_i p_k (H_1^{(k)} y_{i,j}^{(k)} + g^{(k)}) + F_2'(y_{i,j}^{(k)}), d_2^{(k)} \right\rangle \\
& = m_{i,j}(d_1, d_2^{(1)}, d_2^{(2)}, \dots, d_2^{(K)})
\end{aligned} \tag{4.5}$$

$$\text{st. } \|F_1''(x_{i,j})^{\frac{1}{2}} d_1\|^2 + \sum_{k=1}^K \|F_2''(y_{i,j}^{(k)})^{\frac{1}{2}} d_2^{(k)}\|^2 \leq \alpha_{1,i,j}^2 + \sum_{k=1}^K \alpha_{2,i,j}^{(k)2}. \tag{4.6}$$

Let the transformation

$$d'_1 = F_1''(x_{i,j})^{\frac{1}{2}} d_1, \tag{4.7}$$

$$d_2^{(k)} = F_2''(y_{i,j}^{(k)})^{\frac{1}{2}} d_2^{(k)}, \text{ for } k = 1, 2, \dots, K, \tag{4.8}$$

and define

$$Q_{1,i,j} = \eta_i F_1''(x_{i,j})^{-\frac{1}{2}} H_0 F_1''(x_{i,j})^{-\frac{1}{2}} + I, \tag{4.9}$$

$$C_{1,i,j} = F_1''(x_{i,j})^{-\frac{1}{2}} (\eta_i (H_0 x_{i,j} + c) + F_1'(x_{i,j})), \tag{4.10}$$

$$Q_{2,i,j}^{(k)} = \eta_i p_k F_2''(y_{i,j}^{(k)})^{-\frac{1}{2}} H_1^{(k)} F_2''(y_{i,j}^{(k)})^{-\frac{1}{2}} + I^{(k)}, \text{ for } k = 1, 2, \dots, K, \tag{4.11}$$

$$C_{2,i,j}^{(k)} = F_2''(y_{i,j}^{(k)})^{-\frac{1}{2}} (\eta_i p_k (H_1^{(k)} y_{i,j}^{(k)} + g^{(k)}) + F_2'(y_{i,j}^{(k)})), \text{ for } k = 1, 2, \dots, K. \tag{4.12}$$

We can rewrite equations (4.5) and (4.6) as follow

$$\begin{aligned}
& \min q'_{i,j}(d'_1, d_2^{(1)}, d_2^{(2)}, \dots, d_2^{(K)}) \\
& = \frac{1}{2} \langle d'_1, Q_{1,i,j} d'_1 \rangle + \langle C_{1,i,j}, d'_1 \rangle + \sum_{k=1}^K \left(\frac{1}{2} \langle d_2^{(k)}, Q_{2,i,j}^{(k)} d_2^{(k)} \rangle + \langle C_{2,i,j}^{(k)}, d_2^{(k)} \rangle \right)
\end{aligned} \tag{4.13}$$

$$\text{st. } \|d'_1\|^2 + \sum_{k=1}^K \|d_2^{(k)}\|^2 \leq \alpha_{1,i,j}^2 + \sum_{k=1}^K \alpha_{2,i,j}^{(k)2}. \tag{4.14}$$

Once $d'_{i,j}$ is computed then we obtain the trail step

$$d_{1,i,j} = F_1''(x_{i,j})^{-\frac{1}{2}} d'_{1,i,j}, \tag{4.15}$$

$$d_{2i,j}^{(k)} = F_2''(y_{i,j}^{(k)})^{-\frac{1}{2}} d_{2i,j}'^{(k)} \text{ for } k = 1, 2, \dots, K \quad (4.16)$$

and from inequality (4.4) that we have

$$\begin{aligned} & f_{\eta_i}(x_{i,j} + d_{1i,j}^{(1)}, y_{i,j}^{(1)} + d_{2i,j}^{(1)}, \dots, y_{i,j}^{(K)} + d_{2i,j}^{(K)}) - f_{\eta_i}(x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)}) \\ & \leq q_{i,j}'(d_{1i,j}', d_{2i,j}'^{(1)}, \dots, d_{2i,j}'^{(K)}) + \frac{1}{3} \left(\frac{(\alpha_{1i,j} + \sum_{k=1}^K \alpha_{2i,j}^{(k)})^3}{1 - (\alpha_{1i,j} + \sum_{k=1}^K \alpha_{2i,j}^{(k)})} \right). \end{aligned} \quad (4.17)$$

Let $n_{\eta_i}(X_{i,j}) = n_{\eta_i}(x_{i,j}, y_{i,j}^{(1)}, \dots, y_{i,j}^{(K)})$ be the Newton step of $f_{\eta_i}(x, y^{(1)}, \dots, y^{(K)})$ at the point $(x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)})$. We should point out that

$$\begin{aligned} & \|n_{\eta_i}(X_{i,j})\|_{X_{i,j}}^2 \\ & = \left\langle -f_{\eta_i}''(X_{i,j})^{-1} f_{\eta_i}'(X_{i,j}), f_{\eta_i}''(X_{i,j}) \left(-f_{\eta_i}''(X_{i,j})^{-1} f_{\eta_i}'(X_{i,j}) \right) \right\rangle \\ & = \left\langle f_{\eta_i}'(X_{i,j}), f_{\eta_i}''(X_{i,j})^{-1} f_{\eta_i}'(X_{i,j}) \right\rangle \\ & = \left\langle \eta_i(H_0 x_{i,j} + c) + F_1'(x_{i,j}), (\eta_i H_0 + F_1''(x_{i,j}))^{-1} \times (\eta_i(H_0 x_{i,j} + c) + F_1'(x_{i,j})) \right\rangle \\ & + \sum_{k=1}^K \left\langle \eta_i p_k(H_1^{(k)} y_{i,j}^{(k)} + g^{(k)}) + F_2'(y_{i,j}^{(k)}), (\eta_i p_k H_1^{(k)} + F_2''(y_{i,j}^{(k)}))^{-1} \right. \\ & \times \left. (\eta_i p_k(H_1^{(k)} y_{i,j}^{(k)} + g^{(k)}) + F_2'(y_{i,j}^{(k)})) \right\rangle \\ & = \langle C_{1i,j}, Q_{1i,j}^{-1} C_{1i,j} \rangle + \sum_{k=1}^K \langle C_{2i,j}^{(k)}, Q_{2i,j}^{(k)-1} C_{2i,j}^{(k)} \rangle \end{aligned} \quad (4.18)$$

where the last equality follows equalities (4.9) - (4.12).

Now we are ready to present our algorithm. From what we defined in (3.29), (3.30) and (3.32), we define the feasible set by $\mathcal{F} = \{X \in E \mid AX = B, x \in \mathcal{K}_1, y^{(k)} \in \mathcal{K}_2, \text{ for } k = 1, 2, \dots, K\}$

Algorithm : An interior-point trust-region algorithm

Step 0 **Initialization**

set $i = 0, j = 0$, choose starting point $(x_{0,0}, y_{0,0}^{(1)}, y_{0,0}^{(2)}, \dots, y_{0,0}^{(K)}) \in \{\mathcal{F}\}$, an initial trust-region radius $\alpha_{0,0} \in (0, 1)$, and an initial parameter η_0 are given.

Until convergence repeat.

Step 1 **Test inner iteration termination** compute

$$\|n_{\eta_i}(X_{i,j})\|_{X_{i,j}}^2 \leq \frac{1}{36} \quad (4.19)$$

if $\|n_{\eta_i}(X_{i,j})\|_{X_{i,j}}^2 \leq \frac{1}{36}$ go to step 2, otherwise set
 $(x_{i+1,j}, y_{i+1,j}^{(1)}, y_{i+1,j}^{(2)}, \dots, y_{i+1,j}^{(K)}) = (x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)})$
 and go to step 3.

Step 2 Step calculation

solve (4.13) and (4.14) to obtain $(d'_{1,i,j}, d'_{2,i,j}^{(1)}, d'_{2,i,j}^{(2)}, \dots, d'_{2,i,j}^{(K)})$
 and compute (4.15) and (4.16) to obtain $(d_{1,i,j}^{(1)}, d_{2,i,j}^{(1)}, d_{2,i,j}^{(2)}, \dots, d_{2,i,j}^{(K)})$,
 update $(x_{i,j+1}, y_{i,j+1}^{(1)}, \dots, y_{i,j+1}^{(K)}) = (x_{i,j} + d_{1,i,j}^{(1)}, y_{i,j}^{(1)} + d_{2,i,j}^{(1)}, \dots, y_{i,j}^{(K)} + d_{2,i,j}^{(K)})$
 and go to step 3.

Step 3 Update parameter η

Set $\eta_{i+1} = \theta \eta_i$ for some $\theta > 1$, increase i by 1 and go to step 1.

Lemma 4.1. Any global minimizer $(d'_{1,i,j}, d'_{2,i,j}^{(1)}, d'_{2,i,j}^{(2)}, \dots, d'_{2,i,j}^{(K)})$ of the problems (4.13) and (4.14) satisfies the equation

$$(Q_{1,i,j} + \mu_{i,j}I)d'_{1,i,j} = -C_{1,i,j}, \quad (4.20)$$

$$(Q_{2,i,j}^{(k)} + \mu_{i,j}I^{(k)})d'_{2,i,j}^{(k)} = -C_{2,i,j}^{(k)}, \text{ for } k = 1, 2, \dots, K, \quad (4.21)$$

which $Q_{1,i,j} + \mu_{i,j}I$ and $Q_{2,i,j}^{(k)} + \mu_{i,j}I^{(k)}$ for $k = 1, 2, \dots, K$ are positive semi-definite
 $\mu_{i,j} \geq 0$ and $\mu_{i,j}(\|d'_{1,i,j}\| + \sum_{k=1}^K \|d'_{2,i,j}^{(k)}\| - \alpha_{1,i,j} - \sum_{k=1}^K \alpha_{2,i,j}^{(k)}) = 0$.

This Lemma is well-known in the trust-region literature. For proof see e.g. Section 7.2 of [5].

Theorem 4.2. If we choose $\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)} = \frac{1}{4}$, then we have

$$f_{\eta_i}(x_{i,j+1}, y_{i,j+1}^{(1)}, y_{i,j+1}^{(2)}, \dots, y_{i,j+1}^{(K)}) - f_{\eta_i}(x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)}) < -\frac{1}{145} \quad (4.22)$$

which is independent of i and j .

Proof. If the solution of equation (4.13) and (4.14) line on to boundary of the trust-region, i.e., $\|d'_{1,i,j}\| + \sum_{k=1}^K \|d'_{2,i,j}^{(k)}\| = \alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)}$ we have

$$\begin{aligned} & q'_{i,j}(d'_{1,i,j}, d'_{2,i,j}^{(1)}, d'_{2,i,j}^{(2)}, \dots, d'_{2,i,j}^{(K)}) \\ &= \frac{1}{2} \langle d'_{1,i,j}, Q_{1,i,j} d'_{1,i,j} \rangle + \langle C_{1,i,j}, d'_{1,i,j} \rangle + \sum_{k=1}^K \left(\frac{1}{2} \langle d'_{2,i,j}^{(k)}, Q_{2,i,j}^{(k)} d'_{2,i,j}^{(k)} \rangle + \langle C_{2,i,j}^{(k)}, d'_{2,i,j}^{(k)} \rangle \right), \end{aligned}$$

$$\begin{aligned}
&= \langle d'_{1i,j}, (Q_{1i,j} d'_{1i,j} + C_{1i,j}) \rangle - \frac{1}{2} \langle d'_{1i,j}, Q_{1i,j} d'_{1i,j} \rangle \\
&+ \sum_{k=1}^K \left(\langle d'^{(k)}_{2i,j}, (Q_{2i,j} d'^{(k)}_{2i,j} + C_{2i,j}) \rangle - \frac{1}{2} \langle d'^{(k)}_{2i,j}, Q_{2i,j} d'^{(k)}_{2i,j} \rangle \right) \\
&= -\langle d'_{1i,j}, \mu_{i,j} d'_{1i,j} \rangle - \frac{1}{2} \left\langle d'_{1i,j}, (\eta_i F_1''(x_{i,j})^{-\frac{1}{2}} H_0 F_1''(x_{i,j})^{-\frac{1}{2}} + I) d'_{1i,j} \right\rangle \\
&+ \sum_{k=1}^K \left(\langle -d'^{(k)}_{2i,j}, \mu_{i,j} d'^{(k)}_{2i,j} \rangle - \frac{1}{2} \left\langle d'^{(k)}_{2i,j}, (\eta_i p_k F_2''(y_{i,j})^{-\frac{1}{2}} H_1^{(k)} F_2''(y_{i,j})^{-\frac{1}{2}} + I^{(k)}) d'^{(k)}_{2i,j} \right\rangle \right) \\
&= -\mu_{i,j} (\|d'_{1i,j}\|^2 + \sum_{k=1}^K \|d'^{(k)}_{2i,j}\|^2) - \frac{1}{2} \left\langle d'_{1i,j}, (\eta_i F_1''(x_{i,j})^{-\frac{1}{2}} H_0 F_1''(x_{i,j})^{-\frac{1}{2}}) d'_{1i,j} \right\rangle \\
&- \frac{1}{2} \sum_{k=1}^K \left\langle d'^{(k)}_{2i,j}, (\eta_i p_k F_2''(y_{i,j})^{-\frac{1}{2}} H_1^{(k)} F_2''(y_{i,j})^{-\frac{1}{2}}) d'^{(k)}_{2i,j} \right\rangle - \frac{1}{2} (\|d'_{1i,j}\|^2 + \sum_{k=1}^K \|d'^{(k)}_{2i,j}\|^2) \\
&= -\mu_{i,j} (\alpha_{1i,j}^2 + \sum_{k=1}^K \alpha_{2i,j}^{(k)2}) - \frac{1}{2} \left\langle d'_{1i,j}, (\eta_i F_1''(x_{i,j})^{-\frac{1}{2}} H_0 F_1''(x_{i,j})^{-\frac{1}{2}}) d'_{1i,j} \right\rangle \\
&- \frac{1}{2} \sum_{k=1}^K \left\langle d'^{(k)}_{2i,j}, (\eta_i p_k F_2''(y_{i,j})^{-\frac{1}{2}} H_1^{(k)} F_2''(y_{i,j})^{-\frac{1}{2}}) d'^{(k)}_{2i,j} \right\rangle - \frac{1}{2} (\alpha_{1i,j}^2 + \sum_{k=1}^K \alpha_{2i,j}^{(k)2}) \\
&\leq -\frac{1}{2} (\alpha_{1i,j}^2 + \sum_{k=1}^K \alpha_{2i,j}^{(k)2}) = -\frac{1}{32}. \tag{4.23}
\end{aligned}$$

In the above, the third equality follows from the equalities (4.9), (4.11), (4.20) and (4.21). The inequality follows from the fact that $\eta_i F_1''(x_{i,j})^{-\frac{1}{2}} H_0 F_1''(x_{i,j})^{-\frac{1}{2}}$ and $\eta_i p_k F_2''(y_{i,j})^{-\frac{1}{2}} H_1^{(k)} F_2''(y_{i,j})^{-\frac{1}{2}}$ for $k = 1, 2, \dots, K$ are positive definite or positive semidefinite. Therefore we get the last inequality. From (4.17) and (4.23) we get

$$\begin{aligned}
&f_{\eta_i}(x_{i,j+1}, y_{i,j+1}^{(1)}, y_{i,j+1}^{(2)}, \dots, y_{i,j+1}^{(K)}) - f_{\eta_i}(x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)}) \\
&\leq -\frac{1}{32} + \frac{(1/4)^3}{3(1-1/4)} < -\frac{1}{145}. \tag{4.24}
\end{aligned}$$

If the solution of (4.13) and (4.14) lies in the interior of the trust-region, i.e., $\|d'_{1i,j}\|^2 + \sum_{k=1}^K \|d'^{(k)}_{2i,j}\|^2 < \alpha_{1i,j}^2 + \sum_{k=1}^K \alpha_{2i,j}^{(k)2}$. From Lemma 4.2, we know $\mu_{i,j} = 0$ and consequently $d'_{1i,j} = -Q_{1i,j}^{-1} C_{1i,j}$, $d'^{(k)}_{2i,j} = -Q_{2i,j}^{(k)-1} C_{2i,j}^{(k)}$ for $k = 1, 2, \dots, K$ which gives

$$\begin{aligned}
&q'_{i,j}(d'_{1i,j}, d'^{(1)}_{2i,j}, d'^{(2)}_{2i,j}, \dots, d'^{(K)}_{2i,j}) \\
&= \frac{1}{2} \langle d'_{1i,j}, Q_{1i,j} d'_{1i,j} \rangle + \langle C_{1i,j}, d'_{1i,j} \rangle + \sum_{k=1}^K \left(\frac{1}{2} \langle d'^{(k)}_{2i,j}, Q_{2i,j}^{(k)} d'^{(k)}_{2i,j} \rangle + \langle C_{2i,j}^{(k)}, d'^{(k)}_{2i,j} \rangle \right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \langle C_{1,i,j}, Q_{1,i,j}^{-1} C_{1,i,j} \rangle - \sum_{k=1}^K \frac{1}{2} \langle C_{2,i,j}^{(k)}, Q_{2,i,j}^{(k)-1} C_{2,i,j}^{(k)} \rangle \\
&= -\frac{1}{2} \left(\langle C_{1,i,j}, Q_{1,i,j}^{-1} C_{1,i,j} \rangle + \sum_{k=1}^K \langle C_{2,i,j}^{(k)}, Q_{2,i,j}^{(k)-1} C_{2,i,j}^{(k)} \rangle \right) \\
&= -\frac{1}{2} \|n_{\eta_i}(X_{i,j})\|_{X_{i,j}}^2.
\end{aligned} \tag{4.25}$$

By (4.25) and the mechanism of our algorithm, we know that $\|n_{\eta_i}(X_{i,j})\|_{X_{i,j}}^2 > \frac{1}{36}$ for all i and j we can rewrite (4.17) as

$$\begin{aligned}
&f_{\eta_i}(x_{i,j+1}, y_{i,j+1}^{(1)}, y_{i,j+1}^{(2)}, \dots, y_{i,j+1}^{(K)}) - f_{\eta_i}(x_{i,j}, y_{i,j}^{(1)}, y_{i,j}^{(2)}, \dots, y_{i,j}^{(K)}) \\
&\leq -\frac{1}{2} \|n_{\eta_i}(X_{i,j})\|_{X_{i,j}}^2 + \frac{1}{3} \left(\frac{(\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)})^3}{1 - (\alpha_{1,i,j} + \sum_{k=1}^K \alpha_{2,i,j}^{(k)})} \right) \\
&\leq -\frac{1}{72} + \frac{(1/4)^3}{3(1 - 1/4)} < -\frac{1}{145}.
\end{aligned} \tag{4.26}$$

The proof is completed $\square$

Now, we consider the number of iteration of our algorithm that we can stop the iteration when the reduction of objective function is smaller than some constant.

Lemma 4.3. Let $X^* = \operatorname{argmin}_{X \in \mathcal{K}} q(X)$ and $X^* = [x^*, y^{(1)*}, y^{(2)*}, \dots, y^{(K)*}]^T \in \mathbb{R}^{(n_1 + \sum_{k=1}^K n_{2_k})}$, $x^* \in \mathcal{K}_1, y^{(k)*} \in \mathcal{K}_2$ for $k = 1, 2, \dots, K$. If $\|n_{\eta}(X)\|_X \leq \frac{1}{6}$, then

$$q(X) - q(X^*) \leq \frac{\vartheta + \sqrt{\vartheta}}{\eta}. \tag{4.27}$$

Proof. Let $X(\eta) = \operatorname{argmin}_{X \in \mathcal{K}} f_{\eta}(X)$. From Lemma 2.5

$$\|X - X(\eta)\|_X \leq \frac{1}{6} + \frac{3(\frac{1}{6})^2}{(1 - (\frac{1}{6}))^3} < \frac{1}{3} \tag{4.28}$$

and from Lemma 2.3 we have

$$\|X - X(\eta)\|_{X(\eta)} \leq \frac{\|X - X(\eta)\|_X}{1 - \|X - X(\eta)\|_X} < \frac{1}{2}. \tag{4.29}$$

Consider

$$\begin{aligned}
q(X(\eta)) - q(X^*) &\leq \langle q'(X(\eta)), X(\eta) - X^* \rangle \\
&= \left\langle \frac{-F'(X(\eta))}{\eta}, X(\eta) - X^* \right\rangle \\
&\leq \frac{\vartheta}{\eta}.
\end{aligned} \tag{4.30}$$

The first inequality follows from the convexity of $q(X)$. The equality follows from the fact that $f'(X(\eta)) = 0$ and the last inequality follows from Lemma 2.6. From the inequality (4.28) and (4.29) Then, we have

$$\begin{aligned}
& q(X) - q(X(\eta)) \\
&= \left\langle q'(X(\eta)), X - X(\eta) \right\rangle + \frac{1}{2} \left\langle X - X(\eta), Q(X - X(\eta)) \right\rangle \\
&= \left\langle \frac{-F'(X(\eta))}{\eta}, X - X(\eta) \right\rangle + \frac{1}{2\eta} \left\langle X - X(\eta), \eta Q(X - X(\eta)) \right\rangle \\
&\leq \frac{1}{\eta} \left\langle -F''(X(\eta))^{-\frac{1}{2}} F'(X(\eta)), F''(X(\eta))^{-\frac{1}{2}} (X - X(\eta)) \right\rangle \\
&\quad + \frac{1}{2\eta} \|X - X(\eta)\|_X^2 \\
&\leq \|F''(X(\eta))^{-\frac{1}{2}} F'(X(\eta))\| \|F''(X(\eta))^{-\frac{1}{2}} (X - X(\eta))\| + \frac{1}{18\eta} \\
&= \left\langle F'(X(\eta)), F''(X(\eta))^{-1} F'(X(\eta)) \right\rangle^{\frac{1}{2}} \left\langle (X - X(\eta)), F''(X(\eta))^{-1} (X - X(\eta)) \right\rangle^{\frac{1}{2}} \\
&\quad + \frac{1}{18\eta} \\
&\leq \frac{\sqrt{\vartheta} \|X - X(\eta)\|_{X(\eta)}}{\eta} + \frac{1}{18\eta} \\
&\leq \frac{\sqrt{\vartheta}}{2\eta} + \frac{1}{18\eta} \leq \frac{\sqrt{\vartheta}}{\eta}
\end{aligned} \tag{4.31}$$

where the third last inequality uses the Lemma 2.6 and from the fact that ϑ is always greater than 1. By adding the inequalities (4.30) and (4.31), then we have

$$q(X) - q(X^*) \leq \frac{\vartheta + \sqrt{\vartheta}}{\eta}.$$

The proof is completed $\square$

The above Lemma tells us that to get an ϵ solution. Let $\eta_{i+1} = \theta \eta_i$ for some $\theta > 1$ and, we only need

$$\eta_i = \eta_0 \theta^i \geq \frac{\vartheta + \sqrt{\vartheta}}{\epsilon}, \tag{4.32}$$

provided that the number of outer iterations i satisfies

$$i \geq \frac{\ln(\vartheta + \sqrt{\vartheta}/\epsilon\eta_0)}{\ln\theta}. \tag{4.33}$$

Lemma 4.4. *If $\|n_{\eta_i}(X)\|_X \leq \frac{1}{6}$, then*

$$f_{\eta_{i+1}}(X) - f_{\eta_{i+1}}(X(\eta_{i+1})) \leq \theta(\vartheta + \sqrt{\vartheta}). \tag{4.34}$$

Proof. From the convexity of $f_{\eta_{i+1}}(X)$ and the inequality (4.28), we can show that

$$\begin{aligned}
& f_{\eta_{i+1}}(X) - f_{\eta_{i+1}}(X(\eta_i)) \\
& \leq \langle f'_{\eta_{i+1}}(X), X - X(\eta_i) \rangle \\
& = \langle \eta_{i+1}(QX + C) + F'(X), X - X(\eta_i) \rangle \\
& = \frac{\eta_{i+1}}{\eta_i} \langle \eta_i(QX + C) + F'(X), X - X(\eta_i) \rangle \\
& + \left(\frac{\eta_{i+1}}{\eta_i} - 1 \right) \langle F'(X), X(\eta_i) - X \rangle \\
& = \theta \left\langle f''_{\eta_i}^{-\frac{1}{2}}(X) (\eta_i(QX + C) + F'(X)), f''_{\eta_i}^{\frac{1}{2}}(X) (X - X(\eta_i)) \right\rangle \\
& + (\theta - 1) \left\langle F''(X)^{-\frac{1}{2}} F'(X), F''(X)^{-\frac{1}{2}} (X(\eta_i) - X) \right\rangle \\
& \leq \theta \|f''_{\eta_i}^{-\frac{1}{2}}(X) (\eta_i(QX + C) + F'(X))\| \|f''_{\eta_i}^{\frac{1}{2}}(X) (X - X(\eta_i))\| \\
& + (\theta - 1) \|F''(X)^{-\frac{1}{2}} F'(X)\| \|F''(X)^{\frac{1}{2}} (X(\eta_i) - X)\| \\
& \leq \theta \|n_{\eta_i}(X)\|_X \|X(\eta_i) - X\|_X + (\theta - 1) \sqrt{\vartheta} \|X(\eta_i) - X\|_X \\
& \leq \theta \left(\frac{1}{6} \right) \left(\frac{1}{3} \right) + (\theta - 1) \sqrt{\vartheta} \frac{1}{3} \\
& \leq \theta \sqrt{\vartheta}.
\end{aligned} \tag{4.35}$$

Consider

$$\begin{aligned}
& f_{\eta_{i+1}}(X(\eta_i)) - f_{\eta_{i+1}}(X(\eta_{i+1})) \\
& \leq \langle f'_{\eta_{i+1}}(X(\eta_i)), X(\eta_i) - X(\eta_{i+1}) \rangle \\
& = \langle \eta_{i+1}(QX(\eta_i) + C) + F'(X(\eta_i)), X(\eta_i) - X(\eta_{i+1}) \rangle \\
& = \frac{\eta_{i+1}}{\eta_i} \langle \eta_i(QX(\eta_i) + C) + F'(X(\eta_i)), X(\eta_i) - X(\eta_{i+1}) \rangle \\
& + \left(\frac{\eta_{i+1}}{\eta_i} - 1 \right) \langle F'(X(\eta_i)), X(\eta_{i+1}) - X(\eta_i) \rangle \\
& = \theta \langle f'_{\eta_i}(X(\eta_i)), X(\eta_i) - X(\eta_{i+1}) \rangle \\
& + (\theta - 1) \langle F'(X(\eta_i)), X(\eta_{i+1}) - X(\eta_i) \rangle \\
& = (\theta - 1) \langle F'(X(\eta_i)), X(\eta_{i+1}) - X(\eta_i) \rangle < \theta \vartheta,
\end{aligned} \tag{4.36}$$

where the last equality follows from the fact that $X(\eta_i)$ minimizes $f_{\eta_i}(X)$ (which implies that $f'_{\eta_i}(X) = 0$), the last inequality follows from Lemma 2.6. By adding inequalities (4.35) and (4.36), then we have

$$f_{\eta_{i+1}}(X) - f_{\eta_{i+1}}(X(\eta_{i+1})) \leq \theta(\vartheta + \sqrt{\vartheta}).$$

The proof is completed □

Theorem 4.2 and Lemma 4.4 tell us that the steps in each inner iteration in at most

$$145\theta(\vartheta + \sqrt{\vartheta}) \quad (4.37)$$

steps in each inner iteration.

Theorem 4.5. *If the initial point $X_{0,0}$ satisfies the conditions (4.19), for any $\epsilon > 0$, our algorithm obtains the solution X which satisfies $q(X) - q(X^*) < \epsilon$ in at most*

$$\frac{145\theta(\vartheta - \sqrt{\vartheta}) \ln\left(\frac{\vartheta + \sqrt{\vartheta}}{\epsilon\eta_0}\right)}{\ln\theta} \quad (4.38)$$

steps, here $X^* = \operatorname{argmin}_{X \in \mathcal{K}} q(X)$.

Proof. Inequality (4.33) provides us with the number of outer iterations i and Theorem 4.2 and Lemma 4.4 provides us with the number of steps in each iteration j . The number of iterations is at most the number of outer iterations multiply the number of inner iterations then we have the bound of number of iteration in at most

$$145\theta(\vartheta - \sqrt{\vartheta}) \times \frac{\ln\left(\frac{\vartheta + \sqrt{\vartheta}}{\epsilon\eta_0}\right)}{\ln\theta}$$

steps. □

Consider the case when the initial point $X_{0,0}$ does not satisfy condition (4.19). We can start from the analytic center of the feasible set $\mathcal{K}$, since $\mathcal{K}$ is a bounded convex set. Let $X(\eta_0) = \operatorname{argmin}_{X \in \mathcal{K}} f_{\eta_0}(X)$ and $X^* = \operatorname{argmin}_{X \in \mathcal{K}} q(X)$. Since $X_{0,0} = \operatorname{argmin}_{X \in \mathcal{K}} F(X)$, we have

$$f_{\eta_0}(X_{0,0}) - f_{\eta_0}(X(\eta_0)) \leq \eta_0(q(X_{0,0}) - q(X^*)) \quad (4.39)$$

and we choose $\eta_0 \leq 1/(q(X_{0,0}) - q(X^*))$, from Theorem 4.2 implies that condition (4.19) will be satisfied after at most 145 steps.

5 Implementing the Algorithm

In this section we discuss the performance of our algorithm when the domain of the problem is in nonnegative orthant cones and second-order cones. These cones are all well known examples of symmetric cone. We consider multi dimension real sets. We compare our results with those of the MATLAB optimization toolbox is it called “quadprog function” in cases of the nonnegative orthant cones. Our algorithm can solve stochastic symmetric programming problems that include variables in second- order cone and cones of real symmetric positive semidefinite matrices. The quadprog function is unable to solve problems with these variables.

5.1 Problem Statement

The parameter chosen for deterministic data of SQSP (3.24)-(3.27) are shown below.

$$H_0 = \begin{bmatrix} 64 & 1 & 4 & 9 \\ 1 & 81 & 25 & 16 \\ 4 & 25 & 100 & 9 \\ 9 & 16 & 9 & 144 \end{bmatrix} \quad c = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 5 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 33 \\ 13 \end{bmatrix}$$

The random variables $H_1^{(k)}, W^{(k)}, T^{(k)}, h^{(k)}, d^{(k)}, k = 1, 2, \dots, K$. For simplicity, we consider the simple case $K = 4$ scenarios but the procedure can be easily extended to very large K .

$H_1^{(1)}$	$H_1^{(2)}$	$H_1^{(3)}$
$\begin{bmatrix} 25 & 1 & 4 \\ 1 & 33 & 25 \\ 4 & 25 & 40 \end{bmatrix}$	$\begin{bmatrix} 64 & 1 & 4 & 9 & 2 \\ 1 & 81 & 25 & 16 & 4 \\ 4 & 25 & 100 & 9 & 6 \\ 9 & 16 & 9 & 144 & 8 \\ 2 & 4 & 6 & 8 & 36 \end{bmatrix}$	$\begin{bmatrix} 24 & 1 & 4 & 9 & 2 & 3 \\ 1 & 51 & 25 & 16 & 2 & 4 \\ 4 & 25 & 100 & 9 & 5 & 6 \\ 9 & 16 & 9 & 144 & 3 & 6 \\ 2 & 2 & 5 & 3 & 49 & 1 \\ 3 & 4 & 6 & 6 & 1 & 64 \end{bmatrix}$

$H_1^{(4)}$
$\begin{bmatrix} 20 & 1 & 4 & 9 & 2 & 3 & 4 \\ 1 & 36 & 25 & 16 & 2 & 4 & 5 \\ 4 & 25 & 100 & 9 & 5 & 6 & 6 \\ 9 & 16 & 9 & 144 & 3 & 6 & 7 \\ 2 & 2 & 5 & 3 & 49 & 1 & 8 \\ 3 & 4 & 6 & 6 & 1 & 64 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 40 \end{bmatrix}$

$W^{(1)}$	$W^{(2)}$	$W^{(3)}$	$W^{(4)}$
$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 2 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 3 & 6 & 5 & 4 \\ 2 & 1 & 1 & 1 & 2 & 1 \\ 3 & 4 & 5 & 1 & 2 & 3 \\ 1 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 2 & 1 & 1 \\ 5 & 2 & 1 & 1 & 4 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 2 & 3 & 5 & 8 & 9 \end{bmatrix}$

$h^{(1)}$	$h^{(2)}$	$h^{(3)}$	$h^{(4)}$
$\begin{bmatrix} 33 \\ 42 \end{bmatrix}$	$\begin{bmatrix} 77 \\ 37 \\ 22 \end{bmatrix}$	$\begin{bmatrix} 91 \\ 49 \\ 78 \\ 40 \end{bmatrix}$	$\begin{bmatrix} 91 \\ 65 \\ 60 \\ 140 \\ 163 \end{bmatrix}$

$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$
$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$	$\begin{bmatrix} 3 & 3 & 4 & 5 \\ 2 & 1 & 1 & 2 \\ 3 & 0 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \\ 1 & 1 & 3 & 5 \\ 2 & 2 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 4 & 2 \\ 2 & 4 & 3 & 1 \end{bmatrix}$

$g^{(1)}$	$g^{(2)}$	$g^{(3)}$	$g^{(4)}$
$\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 4 \\ 5 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \\ -5 \\ 1 \\ 1 \end{bmatrix}$

we randomly generate probability function p in 6 cases as shown below.

	p_1	p_2	p_3	p_4
case1	0.1	0.2	0.3	0.4
case2	0.4	0.3	0.2	0.1
case3	0.25	0.25	0.25	0.25
case4	0.5	0.2	0.2	0.1
case5	0.9	0.1	0	0
case6	0.3	0.3	0.3	0.1

5.2 Results of an Interior-Point Trust-Region Algorithm

In order to make a comparison, the problems (3.24)-(3.27) are described in chapter 4. Our algorithm is then implemented under MATLAB environment release 2015a. The results are in table (1) - (2), for case nonnegative orthant cones, we compare our results with those of the quadprog function in MATLAB optimization toolbox and our algorithm as in table in table (1) and (2). In particular, we present our results of stochastic second order cone programming as in Table (3). From the numerical results example, we have the same optimal solutions with those of the quadprog function and our algorithm. The numerical results for stochastic second order cone programming, some case of problem are same optimal solutions and some case are difference because the restrictions of cones.

Table 1: The solution of SQSP when $\mathcal{K}_1$ and $\mathcal{K}_2$ are nonnegative cone by quadprog function for R_+^n .

	Case1	Case2	Case3	Case4	Case5	Case6
x_1	1.0676	0.91286	1.0212	0.87326	0.90544	0.82258
x_2	2.557	2.354	2.4921	2.4486	3.0439	2.1565
x_3	2.1366	2.3107	2.1904	2.2966	2.0392	2.4438
x_4	5.1021	5.1118	5.1058	5.0849	4.9723	5.1332
$y_1^{(1)}$	4.4744	4.4389	4.464	4.4208	4.3938	4.4231
$y_2^{(1)}$	1.1938	0.8827	1.0988	0.86628	1.1694	0.66718
$y_3^{(1)}$	5.6261	5.9737	5.733	5.9665	5.5312	6.2284
$y_1^{(2)}$	1.547	1.5907	1.5593	1.631	1.7314	1.6006
$y_2^{(2)}$	0.77638	0.79472	0.7818	0.80212	0.80853	0.80395
$y_3^{(2)}$	0.79228	0.78553	0.78751	0.88298	1.2586	0.72812
$y_4^{(2)}$	0.97662	0.9977	0.98368	0.9761	0.86986	1.0246
$y_5^{(2)}$	4.5372	4.5741	4.5498	4.527	4.306	4.626
$y_1^{(3)}$	4.51E-01	7.53E-01	5.43E-01	7.96E-01	1.8008	0.94799
$y_2^{(3)}$	3.5633	3.3365	3.4927	3.3736	3.1436	3.1529
$y_3^{(3)}$	0.58643	0.5789	0.58513	0.54237	1.6117	0.59316
$y_4^{(3)}$	2.2359	2.1789	2.2182	2.1884	4.2732	2.1327
$y_5^{(3)}$	2.7175	3.0054	2.8069	2.9676	1.0372	3.2335
$y_6^{(3)}$	5.4149	5.1841	5.3432	5.2154	3.5629	5.0007
$y_1^{(4)}$	1.2744	1.8526	1.4545	1.7549	0.44043	2.3223
$y_2^{(4)}$	1.3673	1.4487	1.387	1.6402	5.7409	1.4042
$y_3^{(4)}$	1.5263	1.5039	1.5222	1.4023	1.3403	1.5427
$y_4^{(4)}$	2.1789	2.472	2.2693	2.4529	0.19677	2.6938
$y_5^{(4)}$	3.3081	2.3796	3.0207	2.4709	2.6141	1.66E+00
$y_6^{(4)}$	4.7635	4.8505	4.7931	4.7462	5.692	4.9694
$y_7^{(4)}$	7.8856	8.1594	7.9684	8.2052	13.714	8.3322
Optimal value	5488.1	4489.4	4991.5	4539.2	4084	4581.5

Table 2: The solution of SQSP when $\mathcal{K}_1$ and $\mathcal{K}_2$ are nonnegative con by an interior-point trust-region algorithm for R_+^n .

	Case1	Case2	Case3	Case4	Case5	Case6
x_1	1.0676	0.91286	1.0212	0.87326	0.90543	0.82258
x_2	2.557	2.354	2.4921	2.4486	3.0439	2.1565
x_3	2.1366	2.3107	2.1904	2.2966	2.0392	2.4438
x_4	5.1021	5.1118	5.1058	5.0849	4.9723	5.1332
$y_1^{(1)}$	4.4744	4.4389	4.464	4.4208	4.3938	4.4231
$y_2^{(1)}$	1.1938	0.88271	1.0988	0.86628	1.1693	0.66719
$y_3^{(1)}$	5.6261	5.9737	5.733	5.9665	5.5312	6.2284
$y_1^{(2)}$	1.547	1.5907	1.5593	1.6309	1.7314	1.6006
$y_2^{(2)}$	0.77639	0.79473	0.78182	0.80213	0.80859	0.80396
$y_3^{(2)}$	0.7923	0.78554	0.78751	0.88298	1.2586	0.72812
$y_4^{(2)}$	0.97665	0.9977	0.98369	0.9761	0.86987	1.0246
$y_5^{(2)}$	4.5372	4.574	4.5498	4.527	4.306	4.626
$y_1^{(3)}$	0.45125	0.7532	0.54272	0.79587	1.7076	0.948
$y_2^{(3)}$	3.5633	3.3365	3.4927	3.3736	3.1237	3.1529
$y_3^{(3)}$	0.58646	0.57891	0.58515	0.54238	1.5184	0.59317
$y_4^{(3)}$	2.2359	2.1789	2.2182	2.1884	4.0644	2.1327
$y_5^{(3)}$	2.7175	3.0054	2.8069	2.9676	1.16	3.2335
$y_6^{(3)}$	5.4149	5.1841	5.3432	5.2154	3.8259	5.0007
$y_1^{(4)}$	1.2744	1.8526	1.4545	1.7549	1.0404	2.3222
$y_2^{(4)}$	1.3674	1.4488	1.387	1.6402	2.4054	1.4042
$y_3^{(4)}$	1.5263	1.5039	1.5222	1.4023	2.1616	1.5427
$y_4^{(4)}$	2.1789	2.472	2.2693	2.4529	1.3539	2.6938
$y_5^{(4)}$	3.3081	2.3796	3.0207	2.4709	3.2819	1.6607
$y_6^{(4)}$	4.7636	4.8505	4.7931	4.7462	3.2029	4.9694
$y_7^{(4)}$	7.8856	8.1594	7.9684	8.2052	9.2244	8.3322
Optimal value	5488.1	4489.4	4991.5	4539.2	4084	4581.5

Table 3: The solution of SQSP when $\mathcal{K}_1$ and $\mathcal{K}_2$ are second-order cone by an interior-point trust-region algorithm for SOCP.

	Case1	Case2	Case3	Case4	Case5	Case6
x_1	1.0658	0.91295	1.0212	0.87326	0.90544	0.87815
x_2	2.5575	2.3541	2.4921	2.4486	3.0439	2.2279
x_3	2.1375	2.3106	2.1904	2.2966	2.0392	2.3819
x_4	5.1017	5.1118	5.1058	5.0849	4.9723	5.13
$y_1^{(1)}$	4.3596	4.4387	4.464	4.4207	4.3938	3.837
$y_2^{(1)}$	1.2485	0.88299	1.0988	0.8663	1.1694	1.0777
$y_3^{(1)}$	5.6473	5.9735	5.733	5.9664	5.5312	6.2044
$y_1^{(2)}$	1.5532	1.5907	1.5593	1.631	1.7314	1.5657
$y_2^{(2)}$	0.77621	0.7947	0.7818	0.80212	0.80856	0.82513
$y_3^{(2)}$	0.79222	0.78552	0.78751	0.88298	1.2586	0.72748
$y_4^{(2)}$	0.97088	0.99765	0.98368	0.97609	0.86985	1.0236
$y_5^{(2)}$	4.5412	4.5741	4.5498	4.527	4.306	4.6022
$y_1^{(3)}$	0.47082	0.75396	0.54272	0.79593	-18.313	0.88651
$y_2^{(3)}$	3.5008	3.3332	3.4926	3.3733	-88.92	3.1116
$y_3^{(3)}$	0.60241	0.57985	0.58517	0.54246	-18.502	0.6428
$y_4^{(3)}$	2.2219	2.1782	2.2181	2.1883	-109.73	2.1538
$y_5^{(3)}$	2.706	3.0045	2.8069	2.9675	40.049	3.0888
$y_6^{(3)}$	5.4647	5.1871	5.3433	5.2157	191.94	5.1488
$y_1^{(4)}$	1.28	1.8524	1.4545	1.7549	4.0762	2.2052
$y_2^{(4)}$	1.3638	1.4483	1.387	1.6402	-0.69272	0.95488
$y_3^{(4)}$	1.5334	1.5043	1.5223	1.4023	10.911	1.6957
$y_4^{(4)}$	2.1791	2.4718	2.2693	2.4529	-1.9777	2.712
$y_5^{(4)}$	3.3016	2.3801	3.0207	2.4709	-0.22866	2.0532
$y_6^{(4)}$	4.7489	4.85	4.7931	4.7462	-7.0953	4.5863
$y_7^{(4)}$	7.8997	8.1596	7.9684	8.2052	19.165	8.4675
Optimal value	5488.2	4489.4	4991.5	4539.2	4084	4585.5

6 Conclusion and Remarks

In this work, we present stochastic symmetric programming with linear function and quadratic function and finite scenarios. We can explicitly formulate the problem as a large scale deterministic programs. The complexity of our algorithm is proved to be as good as the interior-point polynomial algorithm. We have verified our performance of the algorithm on simple case study problems. Numerical results show the effectiveness of our algorithm.

References

- [1] Y. Lu, Y. Yuan, An interior-point trust-region algorithm for general symmetric cone programming, *SIAM Journal on Optimization*, 18 (2007) 65-86.
- [2] J. Faraut, A. Korneyi, *Analysis on Symmetric Cones*, Oxford Univ Press, Oxford, 1994.
- [3] Y.E. Nesterov, A.S. Nemirovski, *Interior-point Polynomial Algorithms in Convex Programming*, SIAM Publications, SIAM, Philadelphia, USA, 1994.
- [4] J. Renegar, *A Mathematical View of Interior-Point Methods in Convex Optimization*, SIAM Publications, SIAM, Philadelphia, USA, 2001.
- [5] A.R. Conn, N.I.M. Gould, Ph.L. Toint, *Trust-region Methods*, SIAM Publications, SIAM, Philadelphia, USA, 2000.

5.4 Bound for the 2-Page Fixed Linear Crossing Number of Hypercube Graph via SDP Relaxation

The crossing number of graph G is the minimum number of edges crossing in any drawing of G in a plane. In this paper we describe a method of finding the bound of 2-page fixed linear crossing number of G . We consider a conflict graph G' of G . Then, instead of minimizing the crossing number of G , we show that it is equivalent to maximize the weight of a cut of G' . We formulate the original problem into the MAXCUT problem. We consider a semidefinite relaxation of the MAXCUT problem. An example of a case where G is hypercube is explicitly shown to obtain an upper bound. The numerical results confirm the effectiveness of the approximation.

1. Introduction

Let G be a simple connected graph with a vertex-set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and an edge-set $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$. The crossing number of graph G , denoted $cr(G)$, is the minimum number of pairwise intersections of edge crossing on the plane drawing of graph G . Clearly, $cr(G) = 0$ if and only if G is planar. It is known that the exact crossing numbers of any graphs are very difficult to compute. In 1973, Erdős and Guy [1] wrote, “Almost all questions that one can ask about crossing numbers remain unsolved.” In fact, Garey and Johnson [2] prove that computing the crossing number is NP-complete.

A 2-page drawing of G is a representation of G on the plane such that its vertices are placed on a straight horizontal line L according to fixed vertex ordering and its edges are drawn as a semicircle above or below L but never cross L .

The n -cube or n -dimensional hypercube Q_n is recursively defined in terms of the Cartesian products. The one-dimension cube Q_1 is simply K_2 where K_2 is a complete graph with 2 vertices. For $n \geq 2$, Q_n is defined recursively as $Q_{n-1} \times K_2$. The order of Q_n is $|V(Q_n)| = 2^n$ and its size is $|E(Q_n)| = n2^{n-1}$. Since Q_n is planar for $n = 1, 2, 3$, so $cr(Q_n) = 0$ for each such n . Eggleton and Guy [3] showed that $cr(Q_4) = 8$ but $cr(Q_n)$ is unknown for $n \geq 5$.

It was declared by Eggleton and Guy [3] that the crossing numbers of the hypercube Q_n (non-2-page) for $n \geq 3$ was

$$cr(Q_n) \leq \frac{5}{32}4^n - \left\lfloor \frac{n^2 + 1}{2} \right\rfloor 2^{n-2}. \quad (1)$$

Then, in 1973, Erdős and Guy [1] conjectured equality in (1). In 1993, a lower bound of $cr(Q_n)$ was proved by Sýkora and Vrt'o [4]:

$$cr(Q_n) \geq \frac{1}{20}4^n + O(n^2 2^n). \quad (2)$$

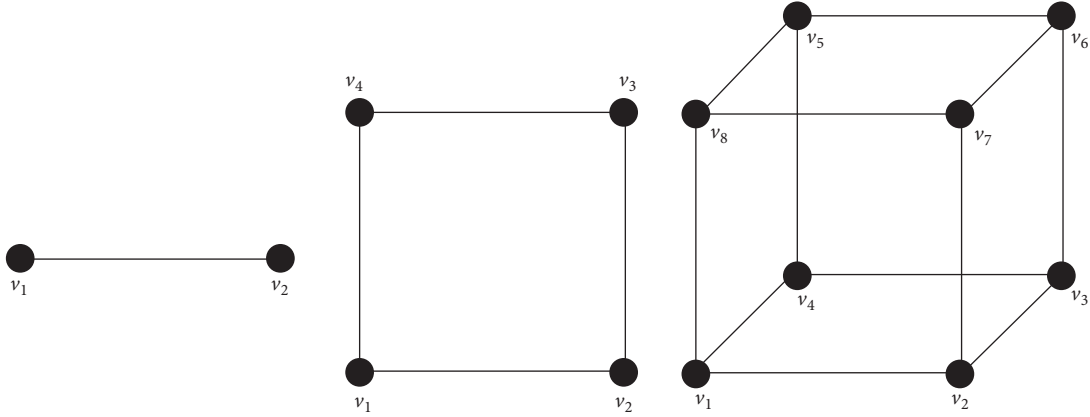
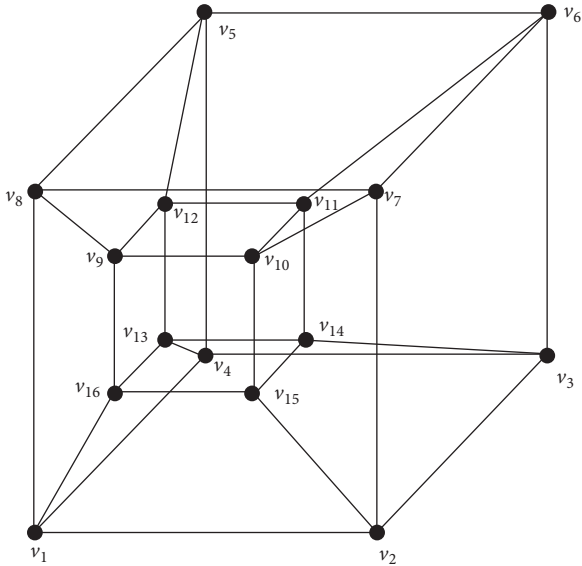
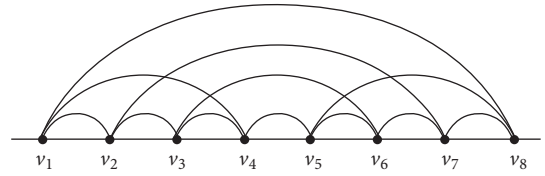
In 2008, Faria et al. [5] constructed a new drawing of Q_n in the plane which led to the conjectured number of crossings

$$\frac{5}{32}4^n - \left\lfloor \frac{n^2 + 1}{2} \right\rfloor 2^{n-2}. \quad (3)$$

To the best of our knowledge, the fixed linear crossing number for Q_n has not been established. In this paper, we discuss a method to obtain an approximation for fixed linear crossing number for hypercube graph.

2. 2-Page Drawings of Hypercube Graph Q_n

Throughout this paper, we consider the ordering of hypercube graph Q_n . Since Q_n is defined recursively as $Q_{n-1} \times K_2$, for

FIGURE 1: n -Cube graphs with fixed vertex ordering for $n = 1, 2, 3$.FIGURE 2: n -Cube graph with fixed vertex ordering for $n = 4$.FIGURE 3: The 2-page drawing of Q_3 with fixed vertex ordering.

other, no two edges cross more than once, and no three edges cross in a point.

For a given 2-page drawing of Q_n with the fixed vertex ordering, a pair of edges $e_{ij} = (v_i, v_j)$ and $e_{lk} = (v_l, v_k)$ are potential crossing if e_{ij} and e_{lk} cross each other when routed on the same side of L . Clearly, e_{ij} and e_{lk} are potential crossing if and only if $v_i < v_l < v_j < v_k$ or $v_l < v_i < v_k < v_j$.

Next we give the definition of conflict graph G' of graph G .

Definition 1. Given a graph G . We define an associated conflict graph $G' = (V', E')$ of a graph $G = (V, E)$. There is corresponding one-to-one and onto mapping between the set of $V'(G')$ and $E(G)$. Two vertices of G' are adjacent if any two edges in G are potential crossing.

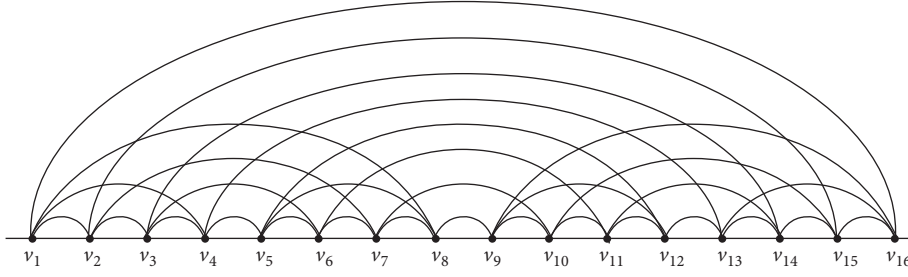
For example, according to the given fixed vertex ordering of Q_3 (see Figure 3), Q'_3 is a graph of $2n^{n-1}$ nodes, $V'(Q'_3) = \{v'_{12}, v'_{23}, v'_{34}, v'_{45}, v'_{56}, v'_{67}, v'_{78}, v'_{14}, v'_{36}, v'_{58}, v'_{27}, v'_{18}\}$. v'_{24} and v'_{35} are adjacent in Q'_3 because e_{24} and e_{35} are potential crossing in a 2-page drawing of Q_3 . A fixed vertex ordering of Q_4 and its potential crossing can be seen in Figure 4.

In this paper, we are interested only in fixed linear embeddings of Q_n . There is a crossing between e_{ij} and e_{lk} if and only if e_{ij} and e_{lk} are potential crossing and embedded on the same side of L . We can see that the number of edge crossings depends on the order of vertices and on the sides to which the edges are assigned.

The 2-page linear crossing number of Q_n , denoted by $\nu_2(Q_n)$, is the minimum number of pairwise intersections of edges crossings determined by a 2-page drawing of Q_n . The 2-page fixed linear crossing number of Q_n is the minimum

$n = 2, \dots$, where Q_1 is a simple graph with 2 vertices together with a single edge incident to both vertices, Q_n has 2 copies of Q_{n-1} with edges connecting between them. Given a fixed ordering on Q_{n-1} , the vertices of the first Q_{n-1} are labeled $1, 2, 3, \dots, 2^{n-1}$ and the vertices of the second Q_{n-1} are labeled $2^{n-1}+1, 2^{n-1}+2, \dots, 2^{n-1}+2^{n-1} = 2^n$. The two vertices between the first Q_{n-1} and the second Q_{n-1} are adjacent if and only if the sum of the labeled is $2^n + 1$. Figures 1 and 2 present the ordering of Q_1, Q_2, Q_3 and Q_4 which we consider throughout this paper. Notice that our method is independent on vertex ordering; therefore, for a fixed n , we can apply the method $(2^n)!$ times so as to obtain the 2-page linear crossing number.

The 2-page drawing of Q_n can be represented by drawing the vertices of Q_n on a straight horizontal line L according a fixed vertex ordering. Each edge fully contained one of the two half-planes (pages) as a semicircle and never cross L . Notice that no edge crosses itself, no adjacent edges cross each

FIGURE 4: The 2-page drawing of Q_4 with fixed vertex ordering.

number of pairwise intersections of edges crossings determined by a 2-page drawing of Q_n with fixed vertex ordering of Q_n . It is known that $\nu_2(Q_n) = 0$ for $n = 1, 2, 3$, $\nu_2(Q_n) > 0$ for $n \geq 4$.

3. Reduction to MAXCUT Problem

In this section, we show that the problem can be reduced to the maximum cut problem. Next, we reduce the fixed linear crossing number problem to the maximum cut problem (MAXCUT). The MAXCUT problem is as follows.

Maximum Cut Problem (MAXCUT). Given an undirected graph $G' = (V', E')$ the edge e_{ij} of the graph is associated with nonnegative weights a_{ij} . The problem is to find a cut of the largest possible weight, that is, to partition the set of V' into disjoint sets V_1 and V_2 such that the total weight of all edges linking V_1 and V_2 (i.e., with one incident node in V_1 and the other one in V_2) is as large as possible.

In the MAXCUT problem, we may assume that the weights $a_{ij} = a_{ji} \geq 0$ are defined for every pair i, j of indices: it suffices to set $a_{ij} = 0$ for pairs i, j of nonadjacent nodes. For the unweighted graph, we assume that $a_{ij} = 1$ for $i, j = 1, 2, \dots, n$.

Let G be a graph with a fixed vertex permutation. Given a vertex partition (V_1, V_2) of its conflict graph G' , the associated cut embedding is the fixed linear embedding of G where edges corresponding to V_1 and V_2 are embedded to the half spaces above and below the vertex line, respectively.

Lemma 2 (see [6]).

$$\nu_2(G) = |E'| - \text{MC}(G'), \quad (4)$$

where $|E'|$ is a number of potential crossing of 2-page drawing of G , which is the number of edges of G' . $\text{MC}(G')$ is the size of the maxcut of G' .

Proof. Given a 2-page (circle) drawing of G , define $W \subset V_n$ as the chords that are drawn inside the circle. The edges of E' with precisely one endpoint in W now correspond to edges of G that do not cross in the drawing. $\square$

Theorem 3 (see [7]). Consider a partition (V_1, V_2) of V' . Then the corresponding cut embedding is a fixed linear embedding of G with a minimum number of crossings if and only if (V_1, V_2) is a maximum cut in G' .

Proof. Let F' be the set of edges in G' with one endpoint in V_1 and one endpoint in V_2 , that is, the cut given by (V_1, V_2) . By definition of G' , we know that every crossing in the cut embedding associated with (V_1, V_2) corresponds to an edge in G' such that either both its endpoint belong to V_1 or both belong to V_2 , that is, to an edge in $E' \setminus F'$. Thus, the number of crossings is $|E'| - |F'|$. As $|E'|$ is constant for a fixed vertex permutation, the result follows. $\square$

Theorem 3 reduces the fixed linear crossing number problem to the maximum cut problem (MAXCUT). In the next section, we describe the relaxation of the MAXCUT problem which leads to semidefinite programming.

3.1. Formulating MAXCUT by Semidefinite Relaxation. In this section, we show that 2-page crossing number of hypercube graph problem can be obtained by computing a semidefinite relaxation of MAXCUT.

First of all, we introduce the adjacency matrix of G denoted $\text{Adj}(G)$ as we know it is an $n \times n$ matrix with the property

$$\text{Adj}(G) \equiv [a_{ij}],$$

$$a_{ij} \in \{0, 1\}, \quad a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

From $\text{Adj}(G)$ we construct the conflict graph of G denoted G' . Finally, we perform MAXCUT on graph G' . We use semidefinite relaxation to approximate the optimal value solution to the MAXCUT problem. Obviously the approximation is larger than the actual MAXCUT optimal value. The feasibility of the relaxation set is strictly larger than the original ones.

According to [2], the MAXCUT problem can be formulated as follows:

$$\begin{aligned} \max \quad & \frac{1}{4} \sum_{i,j=1}^n a_{ij} (1 - x_i x_j) \\ \text{s.t.} \quad & x_i^2 = 1 \\ & i = 1, \dots, n. \end{aligned} \quad (6)$$

FIGURE 5: The adjacency matrix of size $2^6 \times 2^6$ of Q_6 .

We call the optimal value of (6) as “OPT.” Then, the relaxation of (6) can be rewritten as

$$\begin{aligned}
 \max \quad & \frac{1}{4} \sum_{i,j=1}^n a_{ij} (1 - X_{ij}) \\
 \text{s.t.} \quad & X = [X_{ij}]_{i,j=1}^n = X^T \geq 0, \\
 & X_{ii} = 1, \quad i = 1, \dots, n,
 \end{aligned} \tag{7}$$

where $A = [a_{ij}]$ is an adjacency matrix of G' and $X = [X_{ij}]$ is a feasible solution to the semidefinite relaxation. The problem (7) is equivalent to

$$\begin{aligned}
 \min \quad & \frac{1}{4} \text{tr}(AX) \\
 \text{s.t.} \quad & X = [X_{ij}]_{i,j=1}^n = X^T \geq 0, \\
 & X_{ii} = 1, \quad i = 1, \dots, n,
 \end{aligned} \tag{8}$$

where A is a given adjacency matrix of G' and $X = [X_{ij}]$ is a feasible solution to the semidefinite relaxation. We call the optimal value of (8) as “SDP.”

As we have seen from the relation (4), we let $|E'_n|$ be the number of potential crossing of 2-page drawing of Q_n with our fixed vertex ordering (i.e., $|E'_n|$ is the number of edges of Q'_n).

We can determine $|E'_n|$ by considering the upper half of the main diagonal of the adjacency matrix of Q_n .

Definition 4. Let $AG = [a_{ij}]$ be the $n \times n$ adjacency matrix of G . The element a_{ij} where $i + j = n + 1$ is called minor diagonal of adjacency matrix of G and the element a_{ij} where $i + j = n + 1, i < j$ is called semiminor diagonal of adjacency matrix of G , denoted by $\text{smd}(AG)$.

For simplicity, the size of $\text{smd}(AG)$ is a number of elements in $\text{smd}(AG)$. Let AQ_n be the adjacency matrix of graph Q_n . Therefore AQ_n is $2^n \times 2^n$ symmetric matrix. It is clear that the size of $\text{smd}(AQ_n)$ is 2^{n-1} .

Let AG_1 and AG_2 be adjacency matrices of graphs G_1 and G_2 , respectively; we say that the number of potential crossing between AG_1 and AG_2 , denoted by $\text{PC}[AG_1, AG_2]$, is simply the number of potential crossing between 2-page drawing of graph G_1 and G_2 . The adjacency matrix of size $2^6 \times 2^6$ of Q_6 with respect to our ordering is presented in Figure 5.

Lemma 5. For any integer $n \geq 5$,

$$\begin{aligned} \text{PC} [\text{smd}(AQ_{n-1}), \text{smd}(AQ_n)] \\ = \text{PC} [\text{smd}(AQ_{n-1}), \text{smd}(AQ_{n+i})], \end{aligned} \quad (9)$$

where $i = 1, 2, \dots, n$.

Lemma 6. For every adjacency matrix of Q_n , AQ_n , where $n \geq 4$, there exists adjacency matrix of Q_3 , AQ_3 , which is a submatrix embedding in AQ_n . The number of submatrix AQ_3 embedding in AQ_n is 2^{n-3} .

Lemma 7. For any integer $n \geq 5$,

$$\text{PC} [AQ_3, \text{smd}(AQ_n)] = \text{PC} [AQ_3, \text{smd}(AQ_{n+i})], \quad (10)$$

where $i = 1, 2, \dots, n$.

Lemma 8. For any integer $n \geq 5$,

$$\begin{aligned} \text{PC} [AQ_{n-1}, \text{smd}(AQ_n)] \\ = \text{PC} [AQ_{n-1}, \text{smd}(AQ_{n+i})], \end{aligned} \quad (11)$$

where $i = 1, 2, \dots, n$.

Lemma 9. For $n \geq 4$, the number of block $\text{smd}(AQ_i)$ embedding in AQ_n equals 2^{n-i} , $4 \leq i \leq n$.

Lemmas 5–9 follow directly from the definitions.

Lemma 10. For any integer $n \geq 4$,

$$\text{PC} [\text{smd}(AQ_n), \text{smd}(AQ_{n+1})] = \left(\frac{2^{n-1}}{2} \right) (2^n - 2). \quad (12)$$

Proof. The number of potential crossing between $\text{smd}(AQ_n)$ and $\text{smd}(AQ_{n+1})$ is a result of the number of potential crossing between all of the element $a_{1,2^n}, a_{2,2^n-1}, \dots, a_{2^{n-1}, 2^{n-1}+1}$ in AQ_n and $\text{smd}(AQ_{n+1})$. That is,

$$\begin{aligned} \text{PC} [\text{smd}(AQ_n), \text{smd}(AQ_{n+1})] \\ = (2^n - 2) + (2^n - 4) + \dots + 2 + 0 \\ = \left(\frac{2^{n-1}}{2} \right) (2^n - 2). \end{aligned} \quad (13)$$

□

Theorem 11. For any integer $n \geq 5$,

$$\begin{aligned} |E'_n| &= 2 |E'_{n-1}| + 2^{n+1} + 2 \sum_{k=5}^n (2^{n-k}) \left(\frac{2^k}{4} - 1 \right) \left(\frac{2^k}{4} \right), \\ |E'_4| &= 40, \end{aligned} \quad (14)$$

where $|E'_n|$ is the number of potential crossing of 2-page drawing of Q_n with our fixed vertex ordering.

Proof. We prove this lemma by considering the number of potential crossing of 2-page drawing of Q_n with our fixed vertex ordering. Since Q_n has 2 copies of Q_{n-1} with some edges connecting between them, the number of potential crossing of AQ_n is a result of twice of the number of potential crossing within AQ_{n-1} together with $\text{PC}[AQ_3, \text{smd}(AQ_n)]$ and $\text{PC}[\text{smd}(AQ_i), \text{smd}(AQ_n)]$, $i = 4, 5, \dots, n-1$.

Hence, it is enough to show that the number of potential crossing between 2-page drawing of Q_{n-1} and Q_{n-1} is equal to $\gamma(n)$, where

$$\gamma(n) = 2^{n+1} + 2 \sum_{k=5}^n (2^{n-k}) \left(\frac{2^k}{4} - 1 \right) \left(\frac{2^k}{4} \right). \quad (15)$$

Note $\gamma(n)$ is the number of potential crossing between all of submatrices AQ_3 and $\text{smd}(AQ_n)$ and also between $\text{smd}(AQ_i)$ and $\text{smd}(AQ_n)$, $i = 4, 5, \dots, n-1$. By Lemmas 6 and 9,

$$\begin{aligned} \gamma(n) &= 2^{n-3} \cdot \text{PC} [AQ_3, \text{smd}(AQ_n)] + 2^{n-4} \\ &\quad \cdot \text{PC} [\text{smd}(AQ_4), \text{smd}(AQ_n)] + 2^{n-5} \\ &\quad \cdot \text{PC} [\text{smd}(AQ_5), \text{smd}(AQ_n)] + \dots \\ &\quad + 2^{n-(n-2)} \cdot \text{PC} [\text{smd}(AQ_{n-2}), \text{smd}(AQ_n)] \\ &\quad + 2^{n-(n-1)} \cdot \text{PC} [\text{smd}(AQ_{n-1}), \text{smd}(AQ_n)]. \end{aligned} \quad (16)$$

We precede by mathematical induction on n . For $n = 5$, it can be easily seen that $\gamma(5) = 176$ by counting. Assuming (15) holds true, now we consider $\gamma(n+1)$ as a number of potential crossing between all of the submatrices AQ_3 and $\text{smd}(AQ_{n+1})$ and also between $\text{smd}(AQ_i)$ and $\text{smd}(AQ_{n+1})$, $i = 4, 5, \dots, n$. By the Lemmas 5, 6, 7, 9, 8, and 10,

$$\begin{aligned} \gamma(n+1) &= 2^{(n+1)-3} \cdot \text{PC} [AQ_3, \text{smd}(AQ_{n+1})] \\ &\quad + 2^{(n+1)-4} \cdot \text{PC} [\text{smd}(AQ_4), \text{smd}(AQ_{n+1})] \\ &\quad + 2^{(n+1)-5} \cdot \text{PC} [\text{smd}(AQ_5), \text{smd}(AQ_{n+1})] + \dots \\ &\quad + 2^{(n+1)-(n-2)} \cdot \text{PC} [\text{smd}(AQ_{n-2}), \text{smd}(AQ_{n+1})] \\ &\quad + 2^{(n+1)-(n-1)} \cdot \text{PC} [\text{smd}(AQ_{n-1}), \text{smd}(AQ_{n+1})] \\ &\quad + 2^{(n+1)-n} \cdot \text{PC} [\text{smd}(AQ_n), \text{smd}(AQ_{n+1})] \\ &= 2 \left(2^{n-3} \cdot \text{PC} [AQ_3, \text{smd}(AQ_n)] + 2^{n-4} \right. \\ &\quad \cdot \text{PC} [\text{smd}(AQ_4), \text{smd}(AQ_n)] + 2^{n-5} \\ &\quad \cdot \text{PC} [\text{smd}(AQ_5), \text{smd}(AQ_n)] + \dots + 2^{n-(n-2)} \\ &\quad \cdot \text{PC} [\text{smd}(AQ_{n-2}), \text{smd}(AQ_n)] + 2^{n-(n-1)} \\ &\quad \cdot \text{PC} [\text{smd}(AQ_{n-1}), \text{smd}(AQ_n)] \left. \right) + 2^{(n+1)-n} \\ &\quad \cdot \text{PC} [\text{smd}(AQ_n), \text{smd}(AQ_{n+1})] = 2\gamma(n) \end{aligned}$$

$$\begin{aligned}
& + 2^{(n+1)-n} \cdot \text{PC} [\text{smd}(AQ_n), \text{smd}(AQ_{n+1})] \\
& = 2\gamma(n) + 2^{(n+1)-n} \cdot \left(\frac{2^{n-1}}{2}\right) (2^n - 2) = 2\gamma(n) \\
& + (2) (2^{n-1}) (2^{n-1} - 1) = 2\gamma(n) + 2 \left(\frac{2^{n+1}}{4} - 1\right) \\
& \cdot \left(\frac{2^{n+1}}{4}\right) = 2 \left[2^{n+1} \right. \\
& + 2 \sum_{k=5}^n (2^{n-k}) \left(\frac{2^k}{4} - 1\right) \left(\frac{2^k}{4}\right) \left. \right] + 2 \left(\frac{2^{n+1}}{4} - 1\right) \\
& \cdot \left(\frac{2^{n+1}}{4}\right) = 2 \cdot 2^{n+1} + 2 \\
& \cdot \sum_{k=5}^n (2^{n+1-k}) \left(\frac{2^k}{4} - 1\right) \left(\frac{2^k}{4}\right) + 2 \left(\frac{2^{n+1}}{4} - 1\right) \\
& \cdot \left(\frac{2^{n+1}}{4}\right) = 2^{(n+1)+1} \\
& + 2 \left[\sum_{k=5}^n (2^{n+1-k}) \left(\frac{2^k}{4} - 1\right) \left(\frac{2^k}{4}\right) \right. \\
& + \left. \left(2^{(n+1)-(n+1)}\right) \left(\frac{2^{n+1}}{4} - 1\right) \left(\frac{2^{n+1}}{4}\right) \right] = 2^{(n+1)+1} \\
& + 2 \left[\sum_{k=5}^{n+1} (2^{(n+1)-k}) \left(\frac{2^k}{4} - 1\right) \left(\frac{2^k}{4}\right) \right].
\end{aligned} \tag{17}$$

□

The next theorem shows how effective the relaxation is.

Theorem 12 (see [8]). *Let OPT be the optimal value of the MAXCUT problem and SDP be the optimal value of the semidefinite relaxation. Then*

$$1 \geq \frac{OPT}{SDP} \geq 0.87856 \dots \tag{18}$$

Theorem 12 guarantees that the optimal value of the MAXCUT is close to the optimal value of the semidefinite relaxation. From (4), we have

$$\text{AP}(\nu_2(Q_n)) = |E'_n| - \text{AP}(\text{MC}(Q'_n)), \tag{19}$$

where $|E'_n|$ is a number of potential crossing of 2-page drawing of Q_n . $\text{AP}(\nu_2(Q_n))$ is an approximation of 2-page fixed linear crossing number of Q_n and $\text{AP}(\text{MC}(Q'_n))$ is an approximation of $\text{MC}(Q'_n)$.

Corollary 13. *Let $\text{AP}(\nu_2(Q_n))$ be an approximation of $\nu_2(Q_n)$. Then we have*

$$\text{AP}(\nu_2(Q_n)) \leq \nu_2(Q_n) \leq k(n) \text{AP}(\nu_2(Q_n)), \tag{20}$$

where $k(n)$ is a computable quantity depending on n .

TABLE 1: The numerical results of the approximation of 2-page fixed linear crossing number of Q_n , $\text{AP}(\nu_2(Q_n))$ for $n = 4, 5, 6$.

n	$\text{AP}(\text{MC}(Q'_n))$	$ E'_n $	$\text{AP}(\nu_2(Q_n))$
4	35	40	5
5	207	256	48
6	1034	1344	310

TABLE 2: The computable quantity $k(n)$, the bound of $\nu_2(Q_n)$, and the upper bound of $\text{cr}(Q_n)$ for $n = 4, 5, 6$.

n	$k(n)$	$\text{AP}(\nu_2(Q_n))$	$k(n)\text{AP}(\nu_2(Q_n))$	Upper bound of $\text{cr}(Q_n)$
4	1.91	5	9.55	8
5	1.55	48	74.36	56
6	1.43	310	444.42	352

Proof. From (4), (18), and (19), we have

$$\begin{aligned}
\frac{\nu_2(Q_n)}{\text{AP}(\nu_2(Q_n))} &= \frac{|E'_n| - \text{MC}(Q'_n)}{|E'_n| - \text{AP}(\text{MC}(Q'_n))} \\
&\leq \frac{|E'_n| - 0.87856 \text{AP}(\text{MC}(Q'_n))}{|E'_n| - \text{AP}(\text{MC}(Q'_n))}.
\end{aligned} \tag{21}$$

Let $k(n) = (|E'_n| - 0.87856 \text{AP}(\text{MC}(Q'_n))) / (|E'_n| - \text{AP}(\text{MC}(Q'_n)))$ be the computable quantity depending on n . Then,

$$\frac{\nu_2(Q_n)}{\text{AP}(\nu_2(Q_n))} \leq k(n) \tag{22}$$

$$\nu_2(Q_n) \leq k(n) \cdot \text{AP}(\nu_2(Q_n)).$$

□

Corollary 13 shows that the upper bound of $\nu_2(Q_n)$ is $k(n) \cdot \text{AP}(\nu_2(Q_n))$, where $k(n)$ is the computable quantity depending on n .

3.2. Experimental Results. In this section, we consider the hypercube graph Q_n for $n = 4, 5, 6$. Then, we give some examples for approximating the problems of the semidefinite relaxation in the form (8). We approximate this problem via MATLAB program together with an optimization toolbox called “SeDuMi.” The SeDuMi is a package for solving optimization problems with linear, quadratic, and semidefinite constraints.

In Table 1, the second column shows numerical results for the approximation of the MAXCUT on the associated conflict graph Q'_n by using the semidefinite relaxation. It is well known that this problem can be solved in a polynomial time. The third column displays the numbers of potential crossing of 2-page drawing of Q_n referring to our fixed vertex ordering that we evaluate from (14). Notice that this potential crossing of 2-page drawing of Q_n is the exact value. From (19), we calculate the approximation of 2-page fixed linear crossing number of Q_n for $n = 4, 5, 6$. The results are shown in the last column.

In Table 2, we present the lower bound of $\nu_2(Q_n)$, $\text{AP}(\nu_2(Q_n))$ and the upper bound of $\nu_2(Q_n)$, $k(n)\text{AP}(\nu_2(Q_n))$.

The second column shows the values of $k(n)$ for $n = 4, 5, 6$. We see that as n get larger the values of $k(n)$ tend to decline continuously. It is interesting to study the behavior of $k(n)$ as $n \rightarrow \infty$. It does not surprise to see that our approximation is strictly larger than the upper bound of $\text{cr}(Q_n)$ (14) since the latter one does not have a restriction that all vertices must be placed on a line. However, it is surprising to see that these numbers are not so different from each other.

4. Concluding Remarks

In this paper, given graph G , we show how the associating conflict graph G' is constructed. We recharacterize the problem of finding the crossing number of graph G to the MAXCUT problem of G' . We approximate the MAXCUT problem by the semidefinite relaxation which can be solved easily by a standard optimization package; in this case, we use SeDuMi 1.02. The numerical results show reasonable outcome. Clearly, another relaxation method can be explored. Moreover, it would be quite interesting to see the behavior of $k(n)$ as n get larger. One can further study how to estimate $k(n)$ for a larger n .

References

- [1] P. Erdős and R. K. Guy, "Crossing number problems," *The American Mathematical Monthly*, vol. 80, pp. 52–58, 1973.
- [2] M. R. Garey and D. S. Johnson, "Crossing number is NP-complete," *Society for Industrial and Applied Mathematics. Journal on Algebraic and Discrete Methods*, vol. 4, no. 3, pp. 312–316, 1983.
- [3] R. B. Eggleton and R. P. Guy, "The crossing number of the n -cube," *Notices of the American Mathematical Society*, vol. 17, p. 757, 1970.
- [4] O. Sýkora and I. Vrt'o, "On crossing numbers of hypercubes and cube connected cycles," *BIT Numerical Mathematics*, vol. 33, no. 2, pp. 232–237, 1993.
- [5] L. Faria, C. M. H. de Figueiredo, O. Sýkora, and I. Vrt'o, "An improved upper bound on the crossing number of the hypercube," *Journal of Graph Theory*, vol. 59, no. 2, pp. 145–161, 2008.
- [6] E. de Klerk and D. V. Pasechnik, "Improved lower bounds for the 2-page crossing numbers of $K_{m,n}$ and K_n via semidefinite programming," *SIAM Journal on Optimization*, vol. 22, no. 2, pp. 581–595, 2012.
- [7] C. Buchheim and L. Zheng, "Fixed Linear Crossing Minimization by Reduction to The Maximum Cut Problem," in *Computing and combinatorics*, vol. 4112 of *Lecture Notes in Computer Science*, pp. 507–516, Springer, Berlin, Germany, 2006.
- [8] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, MOS-SIAM Series on Optimization, Society for Industrial and Applied Mathematics (SIAM), Pennsylvania, Pa, USA,

Chapter VI

Some Fractional Differential Equations and Integro-difference Equations

6.1 Boundary value problems for impulsive multi-order Hadamard fractional differential equations

In this **section**, we study the existence and uniqueness of solutions for impulsive multi-orders Caputo-Hadamard fractional differential equations equipped with boundary and integral conditions. The Banach, Schaefer, and Rothe fixed point theorems and degree theory are used to establish our main results. Examples illustrating the main results are presented.

1 Introduction

During the last years, fractional calculus has gained considerable importance due to the applications in almost all applied sciences. It was pointed out that fractional derivatives and integrals are more convenient for describing real materials, and some physical problems were treated by using derivatives of non-integer orders. For details, and some recent results on the subject we refer to [1–9] and the references cited therein.

It has been noticed that most of the work on the topic is based on Riemann-Liouville and Caputo type fractional differential equations. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [10], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains a logarithmic function of arbitrary exponent. Details and properties of the Hadamard fractional derivative and integral can be found in [1, 11–15]. However, this calculus with Hadamard derivatives is still studied less than that of Riemann-Liouville.

On the other hand, integer order impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and the social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance [16–20].

Recently in [21], Wang *et al.* studied existence and uniqueness results for the following impulsive multipoint fractional integral boundary value problem involving multi-order

fractional derivatives and a deviating argument:

$$\begin{cases} {}^c D_{t_k^+}^{\alpha_k} u(t) = f(t, u(t), u(\theta(t))), & 1 < \alpha_k \leq 2, \\ \Delta u(t_k) = I_k(u(t_k)), & \Delta u'(t_k) = I_k^*(u(t_k)), & k = 1, 2, \dots, p, \\ u(0) = \sum_{k=0}^p \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} u(\eta_k), & u'(0) = 0, & t_k < \eta_k < t_{k+1}, \end{cases} \quad (1.1)$$

where ${}^c D_{t_k^+}^{\alpha_k}$ is the *Caputo fractional derivative* of order α_k , $\mathcal{J}_{t_k^+}^{\beta_k}$ is Riemann-Liouville fractional integral of order $\beta_k > 0$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\theta \in C(J, J)$, $J = [0, T]$ ($T > 0$), $0 = t_0 < t_1 < \dots < t_k < \dots < t_p < t_{p+1} = T$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, and $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$ where $u(t_k^+)$, $u'(t_k^+)$ and $u(t_k^-)$, $u'(t_k^-)$ denote the right and left hand limits of $u(t)$ and $u'(t)$ at $t = t_k$ ($k = 1, 2, \dots, p$).

In 2015, Wang *et al.* [22] established the existence of solutions for a class of nonlinear impulsive Hadamard fractional differential equations with initial condition of the form

$$\begin{cases} {}_H D_{1^+}^{\alpha} u(t) = f(t, u(t)), & \alpha \in (0, 1), t \in (1, e] \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta u(t_i) = {}_H J_{1^+}^{1-\alpha} u(t_i^+) - {}_H J_{1^+}^{1-\alpha} u(t_i^-) = p_i, & p_i \in \mathbb{R}, i = 1, 2, \dots, m, \\ {}_H J_{1^+}^{1-\alpha} u(1^+) = u_0, & u_0 \in \mathbb{R}, \end{cases} \quad (1.2)$$

where ${}_H D_{1^+}^{\alpha}$ is the left-side *Hadamard fractional derivative* of order α with the lower limit 1 and ${}_H J_{1^+}^{1-\alpha}$ denotes left-side Hadamard fractional integral of order $1 - \alpha$. The existence results were obtained by using the Banach contraction principle and Schauder's fixed point theorem on the weight spaces of piecewise continuous functions.

The Hadamard and Riemann-Liouville fractional derivatives have one similar property, which is the fact that the derivative of a constant is not equal to zero. It is caused by the definitions of them containing the usual derivative outside the integrals. In 2012, Jarad *et al.* [23] presented the modifications of the Hadamard fractional derivative into a more suitable one having physically interpretable initial conditions similar to the Caputo sense. In 2014, Gambo *et al.* [24] proved the fundamental theorem of fractional calculus, some interesting results and also semigroup properties of Caputo-Hadamard operators.

In this paper we are concerned with the existence of solutions for boundary value problems of impulsive Hadamard fractional differential equations of the form

$$\begin{cases} {}^C \mathcal{D}_{t_k}^{p_k} x(t) = f(t, x(t)), & t \in J_k \subset [t_0, T], t \neq t_k, \\ \Delta x(t_k) = \varphi_k(x(t_k)), & k = 1, 2, \dots, m, \\ \alpha x(t_0) + \beta x(T) = \sum_{i=0}^m \gamma_i \mathcal{J}_{t_i}^{q_i} x(t_{i+1}), \end{cases} \quad (1.3)$$

where ${}^C \mathcal{D}_{t_k}^{p_k}$ is the Hadamard fractional derivative of Caputo type of order $0 < p_k \leq 1$ on intervals $J_k := (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, with $J_0 = [t_0, t_1]$, $0 < t_0 < t_1 < t_2 < \dots < t_k < \dots < t_m < t_{m+1} = T$ are the impulse points, $J := [t_0, T]$, $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $\varphi_k \in C(\mathbb{R}, \mathbb{R})$, $\mathcal{J}_{t_i}^{q_i}$ is the Hadamard fractional integral of order $q_i > 0$, $i = 0, 1, \dots, m$. The jump conditions are defined by $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $x(t_k^+) = \lim_{\varepsilon \rightarrow 0^+} x(t_k + \varepsilon)$, $k = 1, 2, 3, \dots, m$.

The paper is organized as follows: Section 2 contains some preliminary notations, definitions and lemmas that we need in the sequel. In Section 3 we present the main results for the problem (1.3), where existence and uniqueness results are proved by using Banach and Rothe fixed point theorems, Leray-Schauder alternative and degree theory. Examples illustrating the obtained results are also presented.

2 Preliminaries

In this section, we introduce some notations and definitions of Hadamard fractional calculus (see [1]) and present preliminary results needed in our proofs later.

Definition 2.1 For an at least n -times differentiable function $g : [a, b] \rightarrow \mathbb{R}$, $a, b > 0$, the Caputo type Hadamard derivative of fractional order α is defined as

$${}^C\mathcal{D}_a^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n g(s) \frac{ds}{s}, \quad n-1 < \alpha < n, n = [\alpha] + 1,$$

where $\delta = t \frac{d}{dt}$, $t \in [a, b]$, and $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 The Hadamard fractional integral of order α is defined as

$$\mathcal{J}_a^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}, \quad \alpha > 0,$$

provided the integral exists on $[a, b]$.

Lemma 2.1 [23] Let $x \in AC_\delta^n[a, b]$ or $C_\delta^n[a, b]$ and $\alpha \in \mathbb{C}$, where $X_\delta^n[a, b] = \{g : [a, b] \rightarrow \mathbb{C} : \delta^{n-1}g(t) \in X[a, b]\}$. Then we have

$$\mathcal{J}_a^\alpha ({}^C\mathcal{D}_a^\alpha)x(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \log\left(\frac{t}{a}\right)^k.$$

The key tools for proving of our results are based on the following fixed point theorems.

Theorem 2.1 [25] Suppose that $A : \bar{\Omega} \rightarrow E$ is a completely continuous operator. If one of the following conditions is satisfied:

- (i) (Altman) $\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2$, for all $x \in \partial\Omega$,
- (ii) (Rothe) $\|Ax\| \leq \|x\|$, for all $x \in \partial\Omega$,
- (iii) (Petryshyn) $\|Ax\| \leq \|Ax - x\|$, for all $x \in \partial\Omega$,

then $\deg(I - A, \Omega, \theta) = 1$, and hence A has at least one fixed point in Ω .

Theorem 2.2 [25] Suppose that $A : \bar{\Omega} \rightarrow E$ is completely continuous operator. If

$$Ax \neq \lambda x, \quad \forall x \in \partial\Omega, \lambda \geq 1,$$

then $\deg(I - A, \Omega, \theta) = 1$ and A has at least one fixed point in $\bar{\Omega}$.

Theorem 2.3 [26] *Let E be a Banach space. Assume that $T : E \rightarrow E$ is a completely continuous operator and the set*

$$V = \{u \in E : u = \lambda Tu, 0 < \lambda < 1\}$$

is bounded. Then T has a fixed point in E .

Lemma 2.2 *Assume that $\Phi = \alpha + \beta - \sum_{i=1}^m \frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \neq 0$. Then the solution of the problem (1.3) is equivalent to the following integral equation:*

$$\begin{aligned} x(t) = & \mathcal{J}_{t_k}^{p_k} f(t, x(t)) + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1}))) \\ & + \frac{1}{\Phi} \left[\sum_{i=0}^m \gamma_i \mathcal{J}_{t_i}^{q_i+p_i} f(t_{i+1}, x(t_{i+1})) - \beta \mathcal{J}_{t_m}^{p_m} f(T, x(T)) \right. \\ & - \beta \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1}))) \\ & \left. + \sum_{i=1}^m \left(\frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} f(t_{j+1}, x(t_{j+1})) + \varphi_{j+1}(x(t_{j+1}))) \right) \right]. \end{aligned} \quad (2.1)$$

Proof By Lemma 2.1, the solution of (1.3) on interval J_0 can be written as

$$x(t) = \mathcal{J}_{t_0}^{p_0} f(t, x(t)) + x_0,$$

where $x_0 \in \mathbb{R}$. For $t \in J_1$, by using Lemma 2.1 and the impulse condition $\Delta x(t_1) = \varphi_1(x(t_1))$, we obtain

$$\begin{aligned} x(t) &= \mathcal{J}_{t_1}^{p_1} f(t, x(t)) + x(t_1^+) \\ &= \mathcal{J}_{t_1}^{p_1} f(t, x(t)) + \mathcal{J}_{t_0}^{p_0} f(t_1, x(t_1)) + \varphi_1(x(t_1)) + x_0. \end{aligned}$$

Again, for $t \in J_2$, we have

$$\begin{aligned} x(t) &= \mathcal{J}_{t_2}^{p_2} f(t, x(t)) + x(t_2^+) \\ &= \mathcal{J}_{t_2}^{p_2} f(t, x(t)) + \mathcal{J}_{t_1}^{p_1} f(t_2, x(t_2)) + \varphi_2(x(t_2)) + \mathcal{J}_{t_0}^{p_0} f(t_1, x(t_1)) + \varphi_1(x(t_1)) + x_0. \end{aligned}$$

Repeating the above process, for $t \in J$, we obtain

$$x(t) = \mathcal{J}_{t_k}^{p_k} f(t, x(t)) + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1}))) + x_0. \quad (2.2)$$

Applying the boundary condition of (1.3), it follows that

$$\begin{aligned} \alpha x(t_0) + \beta x(T) &= (\alpha + \beta)x_0 + \beta \mathcal{J}_{t_m}^{p_m} f(T, x(T)) \\ &\quad + \beta \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1}))) \end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=0}^m \gamma_i \mathcal{J}_{t_i}^{q_i} x(t_{i+1}) \\
&= \sum_{i=0}^m \gamma_i \mathcal{J}_{t_i}^{p_i+q_i} f(t_{i+1}, x(t_{i+1})) + x_0 \sum_{i=0}^m \frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \\
&\quad + \sum_{i=1}^m \left(\frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{j-1} f(t_{j+1}, x(t_{j+1})) + \varphi_{j+1}(x(t_{j+1}))) \right),
\end{aligned}$$

which leads to

$$\begin{aligned}
x_0 = \frac{1}{\Phi} & \left[\sum_{i=0}^m \gamma_i \mathcal{J}_{t_i}^{q_i+p_i} f(t_{i+1}, x(t_{i+1})) - \beta \mathcal{J}_{t_m}^{p_m} f(T, x(T)) \right. \\
& - \beta \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1}))) \\
& \left. + \sum_{i=1}^m \left(\frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{j-1} f(t_{j+1}, x(t_{j+1})) + \varphi_{j+1}(x(t_{j+1}))) \right) \right].
\end{aligned}$$

Replacing the constant x_0 into (2.2), we obtain (2.1), as desired. $\square$

3 Main results

Let $PC(J, \mathbb{R}) = \{x : J \rightarrow \mathbb{R}; x(t) \text{ is continuous everywhere except for some } t_k \text{ at which } x(t_k^+) \text{ and } x(t_k^-) \text{ exist and } x(t_k^-) = x(t_k), k = 1, 2, \dots, m\}$. Obviously, $PC(J, \mathbb{R})$ is a Banach space with the norm $\|x\| = \sup\{|x(t)|; t \in J\}$. A function $x \in PC$ is called a solution of the problem (1.3) if it satisfies (1.3).

In this section, we investigate the existence and uniqueness of solutions for the problem (1.3) via a variety of fixed point theorems by defining an operator $\mathcal{K} : PC \rightarrow PC$ as

$$\begin{aligned}
\mathcal{K}x(t) &= \mathcal{J}_{t_k}^{p_k} f(t, x(t)) + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1}))) \\
&+ \frac{1}{\Phi} \left[\sum_{i=0}^m \gamma_i \mathcal{J}_{t_i}^{q_i+p_i} f(t_{i+1}, x(t_{i+1})) - \beta \mathcal{J}_{t_m}^{p_m} f(T, x(T)) \right. \\
&- \beta \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1}))) \\
&\left. + \sum_{i=1}^m \left(\frac{\gamma_i (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} f(t_{j+1}, x(t_{j+1})) + \varphi_{j+1}(x(t_{j+1}))) \right) \right].
\end{aligned}$$

Clearly, the boundary value problem (1.3) becomes a fixed point problem $x = \mathcal{K}x$.

For convenience, we set the notations of constants, thus

$$\begin{aligned}\Lambda_1 &= \frac{(\log(t/t_k))^{p_k}}{\Gamma(p_k+1)} + \sum_{i=0}^{m-1} \frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)} \\ &\quad + \frac{1}{|\Phi|} \left\{ \sum_{i=0}^m \frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i+p_i}}{\Gamma(q_i+p_i+1)} + |\beta| \sum_{i=0}^m \frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)} \right. \\ &\quad \left. + |\beta| \frac{(\log(T/t_m))^{p_m}}{\Gamma(p_m+1)} + \sum_{i=1}^m \sum_{j=0}^{i-1} \left(\frac{(\log(t_{j+1}/t_j))^{p_j}}{\Gamma(p_j+1)} \right) \left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \right) \right\}, \\ \Lambda_2 &= k + \frac{(\log(t/t_k))^{p_k}}{\Gamma(p_k+1)} + \frac{1}{\Phi} \left[|\beta|m + \sum_{i=0}^m \frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} + |\beta| \frac{(\log(T/t_m))^{p_m}}{\Gamma(p_m+1)} \right].\end{aligned}$$

Theorem 3.1 Assume that $f: J \times \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi_k: \mathbb{R} \rightarrow \mathbb{R}, k = 1, 2, \dots, m$, are continuous functions which satisfy the following conditions:

- (H₁) $|f(t, x) - f(t, y)| \leq L_1|x - y|, \forall t \in J, L_1 > 0, x, y \in \mathbb{R};$
(H₂) $|\varphi_k(u) - \varphi_k(v)| \leq L_2|u - v|, L_2 > 0, \text{ for all } u, v \in \mathbb{R}, \forall k = 1, 2, \dots, m.$

If $L_1\Lambda_1 + L_2\Lambda_2 < 1$ then the problem (1.3) has a unique solution on J .

Proof We define a closed ball B_r by $B_r = \{x \in PC; \|x\| \leq r\}$ where $r \geq (M_1\Lambda_1 + M_2\Lambda_2)(1 - L_1\Lambda_1 - L_2\Lambda_2)^{-1}$, where $M_1 = \sup_{t \in J} |f(t, 0)|$ and $M_2 = \max\{|\varphi_i(0)|, i = 1, 2, \dots, m\}$.

We will show that $\mathcal{K}: B_r \rightarrow B_r$. For any $x \in B_r$, we have

$$\begin{aligned}|\mathcal{K}x(t)| &\leq \mathcal{J}_{t_k}^{p_k} |f(t, x(t))| + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|) \\ &\quad + \frac{1}{|\Phi|} \left[\sum_{i=0}^m |\gamma_i| \mathcal{J}_{t_i}^{q_i+p_i} |f(t_{i+1}, x(t_{i+1}))| + |\beta| \mathcal{J}_{t_m}^{p_m} |f(T, x(T))| \right. \\ &\quad \left. + |\beta| \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|) \right. \\ &\quad \left. + \sum_{i=1}^m \left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} |f(t_{j+1}, x(t_{j+1}))| + |\varphi_{j+1}(x(t_{j+1}))|) \right) \right] \\ &\leq \mathcal{J}_{t_k}^{p_k} (|f(t, x(t)) - f(t, 0)| + |f(t, 0)|) \\ &\quad + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} (|f(t_{i+1}, x(t_{i+1})) - f(t_{i+1}, 0)| + |f(t_{i+1}, 0)|) \\ &\quad + |\varphi_{i+1}(x(t_{i+1})) - \varphi_{i+1}(0)| + |\varphi_{i+1}(0)|) \\ &\quad + \frac{1}{|\Phi|} \left[\sum_{i=0}^m |\gamma_i| \mathcal{J}_{t_i}^{q_i+p_i} (|f(t_{i+1}, x(t_{i+1})) - f(t_{i+1}, 0)| + |f(t_{i+1}, 0)|) \right. \\ &\quad \left. + |\beta| \mathcal{J}_{t_m}^{p_m} (|f(T, x(T)) - f(T, 0)| + |f(T, 0)|) \right. \\ &\quad \left. + |\beta| \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} (|f(t_{i+1}, x(t_{i+1})) - f(t_{i+1}, 0)| + |f(t_{i+1}, 0)|) \right.\end{aligned}$$

$$\begin{aligned}
& + |\varphi_{i+1}(x(t_{i+1})) - \varphi_{i+1}(0)| + |\varphi_{i+1}(0)| \\
& + \sum_{i=1}^m \left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} (|f(t_{j+1}, x(t_{j+1})) - f(t_{j+1}, 0)| \right. \\
& \left. + |f(t_{j+1}, 0)|) + |\varphi_{j+1}(x(t_{j+1})) - \varphi_{j+1}(0)| + |\varphi_{j+1}(0)|) \right) \Bigg] \\
& \leq (L_1 r + M) \frac{(\log(t/t_k))^{p_k}}{\Gamma(p_k + 1)} + \sum_{i=0}^{k-1} \left\{ (L_1 r + M_1) \frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i + 1)} + (L_2 r + M_2) \right\} \\
& + \frac{1}{|\Phi|} \left[\sum_{i=0}^m |\gamma_i| (L_1 r + M_1) \frac{(\log(t_{i+1}/t_i))^{q_i + p_i}}{\Gamma(q_i + p_i + 1)} + |\beta| (L_1 r + M_1) \frac{(\log(T/t_m))^{p_m}}{\Gamma(p_m + 1)} \right. \\
& + |\beta| \sum_{i=0}^{m-1} \left\{ (L_1 r + M_1) \frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i + 1)} + (L_2 r + M_2) \right\} \\
& \left. + \sum_{i=1}^m \left\{ \left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (L_1 r + M_1) \frac{(\log(t_{j+1}/t_j))^{p_j}}{\Gamma(p_j + 1)} + (L_2 r + M_2) \right) \right\} \right] \\
& \leq (L_1 \Lambda_1 + L_2 \Lambda_2) r + (M_1 \Lambda_1 + M_2 \Lambda_2) \leq r.
\end{aligned}$$

Then $\mathcal{K}B_r \subseteq B_r$. Next we will show that $\mathcal{K}$ is a contraction mapping. For $x, y \in B_r$, we get

$$\begin{aligned}
& |\mathcal{K}x - \mathcal{K}y| \\
& \leq \mathcal{J}_{t_k}^{p_k} |f(t, x(t)) - f(t, y(t))| + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1})) - f(t_{i+1}, y(t_{i+1}))| \\
& + |\varphi_{i+1}(x(t_{i+1})) - \varphi_{i+1}(y(t_{i+1}))|) \\
& + \frac{1}{|\Phi|} \left[\sum_{i=0}^m |\gamma_i| \mathcal{J}_{t_i}^{q_i + p_i} |f(t_{i+1}, x(t_{i+1})) - f(t_{i+1}, y(t_{i+1}))| \right. \\
& + |\beta| \mathcal{J}_{t_m}^{p_m} |f(T, x(T)) - f(T, y(T))| \\
& + |\beta| \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1})) - f(t_{i+1}, y(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1})) - \varphi_{i+1}(y(t_{i+1}))|) \\
& + \sum_{i=1}^m \left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} |f(t_{j+1}, x(t_{j+1})) - f(t_{j+1}, y(t_{j+1}))| \right. \\
& \left. + |\varphi_{j+1}(x(t_{j+1})) - \varphi_{j+1}(y(t_{j+1}))|) \right) \Bigg] \\
& \leq (L_1 \Lambda_1 + L_2 \Lambda_2) \|x - y\|.
\end{aligned}$$

Since $(L_1 \Lambda_1 + L_2 \Lambda_2) < 1$, the operator $\mathcal{K}$ is contractive. Hence $\mathcal{K}$ has a unique fixed point on B_r . Therefore the problem (1.3) has a unique solution on J . $\square$

Theorem 3.2 *Let f and φ_k , $k = 1, 2, \dots, m$, be continuous functions. Assume that there are two positive real numbers N_1 and N_2 such that:*

(H₃) $|f(t, x)| \leq N_1$ and $|\varphi_k(x)| \leq N_2$, for $t \in J$, $x \in \mathbb{R}$ and $k = 1, 2, \dots, m$.

Then the problem (1.3) has at least one solution on J .

Proof Define a ball $B_\omega = \{x \in PC; \|x\| < \omega\}$. The proof is divided into 3 steps.

Step 1. We will show that $\mathcal{K}$ is continuous. To prove this, we let $\{x_n\}$ be a sequence in PC such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Then we have

$$\begin{aligned}
& |\mathcal{K}x_n(t) - \mathcal{K}x(t)| \\
& \leq \mathcal{J}_{t_k}^{p_k} |f(t, x_n(t)) - f(t, x(t))| \\
& \quad + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x_n(t_{i+1})) - f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x_n(t_{i+1})) - \varphi_{i+1}(x(t_{i+1}))|) \\
& \quad + \frac{1}{|\Phi|} \left[\sum_{i=0}^m |\gamma_i| \mathcal{J}_{t_i}^{q_i + p_i} |f(t_{i+1}, x_n(t_{i+1})) - f(t_{i+1}, x(t_{i+1}))| \right. \\
& \quad + |\beta| \mathcal{J}_{t_m}^{p_m} |f(T, x_n(T)) - f(T, x(T))| \\
& \quad + |\beta| \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x_n(t_{i+1})) - f(t_{i+1}, x(t_{i+1}))| \\
& \quad + |\varphi_{i+1}(x_n(t_{i+1})) - \varphi_{i+1}(x(t_{i+1}))|) \\
& \quad + \sum_{i=1}^m \left(\frac{|\gamma_i| (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} |f(t_{j+1}, x_n(t_{j+1})) - f(t_{j+1}, x(t_{j+1}))| \right. \\
& \quad \left. \left. + |\varphi_{j+1}(x_n(t_{j+1})) - \varphi_{j+1}(x(t_{j+1}))|) \right) \right] \Bigg].
\end{aligned}$$

Using the continuity of f and φ_k for $k = 1, 2, \dots, m$, we have $|f(t, x_n) - f(t, x)|$ and $|\varphi_k(x_n) - \varphi_k(x)|$ vanish as $n \rightarrow \infty$. Therefore $\|\mathcal{K}x_n - \mathcal{K}x\| \rightarrow 0$, which yields the continuity of the operator $\mathcal{K}$.

Step 2. $\mathcal{K}$ maps a bounded set into a bounded set. For each $x \in \bar{B}_\omega$, we have

$$\begin{aligned}
|\mathcal{K}x| & \leq \mathcal{J}_{t_k}^{p_k} |f(t, x(t))| + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|) \\
& \quad + \frac{1}{|\Phi|} \left[\sum_{i=0}^m |\gamma_i| \mathcal{J}_{t_i}^{q_i + p_i} |f(t_{i+1}, x(t_{i+1}))| + |\beta| \mathcal{J}_{t_m}^{p_m} |f(T, x(T))| \right. \\
& \quad + |\beta| \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|) \\
& \quad + \sum_{i=1}^m \left(\frac{|\gamma_i| (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} |f(t_{j+1}, x(t_{j+1}))| + |\varphi_{j+1}(x(t_{j+1}))|) \right) \Bigg] \\
& \leq \Lambda_1 N_1 + \Lambda_2 N_2,
\end{aligned}$$

which yields the boundedness of $\mathcal{K}\bar{B}_\omega$.

Step 3. $\mathcal{K}$ maps a bounded set into an equicontinuous set. Let $\tau_1, \tau_2 \in (t_k, t_{k+1})$, for each $k = 0, 1, 2, \dots, m$, we have

$$|\mathcal{K}x(\tau_1) - \mathcal{K}x(\tau_2)| \leq \mathcal{J}_{t_k}^{p_k} |f(\tau_1, x(\tau_1)) - f(\tau_2, x(\tau_2))|.$$

The continuity of x and f implies that $\mathcal{K}x(\tau_1) \rightarrow \mathcal{K}x(\tau_2)$ as $\tau_1 \rightarrow \tau_2$. Consequently $\mathcal{K}$ is completely continuous by applying the Azelá-Ascoli theorem.

Let $V = \{x \in B_\omega; \mu \mathcal{K}x = x \text{ for } \mu \in (0, 1)\}$. For all $x \in V$, $x = \mu \mathcal{K}x$, we have

$$|x| \leq \mu |\mathcal{K}x| \leq \Lambda_1 N_1 + \Lambda_2 N_2.$$

Hence V is bounded. By Theorem 2.3, the problem (1.3) has at least one solution on J . $\square$

Theorem 3.3 *Assume that*

$$(H_4) \quad \lim_{x \rightarrow 0} \frac{f(t, x)}{x} = 0 \text{ and } \lim_{x \rightarrow 0} \frac{\varphi_k(x)}{x} = 0 \text{ for } k = 1, 2, \dots, m.$$

Then the problem (1.3) has at least one solution on J .

Proof From (H_4) , choosing $\epsilon = 1/(\Lambda_1 + \Lambda_2)$, there exist constants $\delta_1, \delta_2 \in \mathbb{R}^+$ such that

$$|f(t, x)| < \epsilon |x| \quad \text{where } |x| < \delta_1 \quad \text{and} \quad |\varphi(x)| < \epsilon |x| \quad \text{where } |x| < \delta_2.$$

Now, we define an open ball $\Omega = \{u \in PC; \|u\| < \min\{\delta_1, \delta_2\}\}$. By Theorem 3.2, the operator $\mathcal{K} : \bar{\Omega} \rightarrow PC$ is completely continuous. For any $x \in \partial\Omega$, we have

$$\begin{aligned} |\mathcal{K}x| &\leq \mathcal{J}_{t_k}^{p_k} |f(t, x(t))| + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|) \\ &\quad + \frac{1}{|\Phi|} \left[\sum_{i=0}^m |\gamma_i| \mathcal{J}_{t_i}^{q_i + p_i} |f(t_{i+1}, x(t_{i+1}))| + |\beta| \mathcal{J}_{t_m}^{p_m} |f(T, x(T))| \right. \\ &\quad + |\beta| \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|) \\ &\quad \left. + \sum_{i=1}^m \left(\frac{|\gamma_i| (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} |f(t_{j+1}, x(t_{j+1}))| + |\varphi_{j+1}(x(t_{j+1}))|) \right) \right] \\ &\leq (\Lambda_1 \epsilon + \Lambda_2 \epsilon) \|x\| = \|x\|. \end{aligned}$$

It follows from Theorem 2.1, case (ii), that the problem (1.3) has at least one solution on J . $\square$

Theorem 3.4 *Let f and φ_k for $k = 1, 2, \dots, m$, be continuous functions and satisfy the following inequalities:*

$$(H_5) \quad |f(t, x)| \leq a|x| + b, \quad \forall (t, x) \in J \times \mathbb{R} \text{ and } |\varphi_k(x)| \leq c|x| + d, \quad \forall x \in \mathbb{R}, \quad k = 1, \dots, m, \text{ where constants } a, c > 0 \text{ and } b, d \geq 0.$$

Then the problem (1.3) has at least one solution on J .

Proof Define a unit ball as $\mathcal{O} = \{x \in PC; \|x\| < 1\}$. It is straightforward to show that the operator $\mathcal{K} : \bar{\mathcal{O}} \rightarrow PC$ is completely continuous. Suppose that there is $x^* \in \partial\mathcal{O}$. Then we choose $\lambda = (a+c)\Lambda_1 + (b+d)\Lambda_2 + 1$ such that $\mathcal{K}x^* = \lambda x^*$. By taking the norm in both sides of $\|\mathcal{K}x^*\| = \|\lambda x^*\|$, we obtain $\|\mathcal{K}\| \|x^*\| \geq \lambda \|x^*\|$. Then we have

$$\begin{aligned} \|\mathcal{K}\| &= \sup_{\|x\|=1} |\mathcal{K}x| \\ &= \sup_{\|x\|=1} \left\{ \mathcal{J}_{t_k}^{p_k} |f(t, x(t))| + \sum_{i=0}^{k-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|) \right. \\ &\quad + \frac{1}{|\Phi|} \left[\sum_{i=0}^m |\gamma_i| \mathcal{J}_{t_i}^{q_i+p_i} |f(t_{i+1}, x(t_{i+1}))| + |\beta| \mathcal{J}_{t_m}^{p_m} |f(T, x(T))| \right. \\ &\quad + |\beta| \sum_{i=0}^{m-1} (\mathcal{J}_{t_i}^{p_i} |f(t_{i+1}, x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|) \\ &\quad \left. \left. + \sum_{i=1}^m \left(\frac{|\gamma_i| (\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)} \right) \left(\sum_{j=0}^{i-1} (\mathcal{J}_{t_j}^{p_j} |f(t_{j+1}, x(t_{j+1}))| + |\varphi_{j+1}(x(t_{j+1}))|) \right) \right] \right\} \\ &\leq (a+c)\Lambda_1 + (b+d)\Lambda_2 = \lambda - 1, \end{aligned}$$

which contradicts $\|\mathcal{K}\| \geq \lambda$. Hence the assumptions of Theorem 2.2 hold. Therefore the problem (1.3) has at least one solution on J . $\square$

4 Examples

In this section, we present four examples to illustrate our results.

Example 4.1 Consider the boundary value problem for an impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases} {}^C\mathcal{D}_{t_k}^{(\frac{k+1}{k+2})} x(t) = \frac{10-t^2}{8(t^2+24)} \left(\frac{(|x(t)|+2)^2}{|x(t)|+3} \right), & t \in [1, \frac{8e+1}{9}] \setminus \{t_k\}, \\ \Delta x(t_k) = \frac{\sin|x(t_k)|}{5(11-k)}, & t_k = \frac{ke+1}{k+1}, k=1, 2, \dots, 7, \\ \frac{3}{2}x(1) + \frac{4}{5}x(\frac{8e+1}{9}) = \sum_{i=0}^7 (1-e^{-i}) \mathcal{J}_{t_i}^{(\frac{i^2+5i+2}{i^2+4i+3})} x(t_{i+1}). \end{cases} \quad (4.1)$$

Here $\alpha = 3/2$, $\beta = 4/5$, $m = 7$, $p_k = (k+1)/(k+2)$, $\gamma_k = 1 - e^{-k}$, $q_k = (k^2 + 5k + 2)/(k^2 + 4k + 3)$ for $k = 0, 1, \dots, 7$. From the information, we find that $\Phi \approx 2.0961081$, $\Lambda_1 \approx 3.280445$, and $\Lambda_2 \approx 13.552466$. The functions f and φ_k are given by

$$f(t, x) = \frac{10-t^2}{8(t^2+24)} \left(\frac{(|x|+2)^2}{|x|+3} \right), \quad \varphi_k(x) = \frac{\sin|x|}{5(11-k)},$$

which satisfy

$$|f(t, x) - f(t, y)| \leq \frac{2}{25} |x - y| \quad \text{and} \quad |\varphi_k(x) - \varphi_k(y)| \leq \frac{1}{20} |x - y|, \quad \forall k = 1, 2, \dots, 7.$$

Then we get $L_1 = 2/25$ and $L_2 = 1/20$, which implies $L_1\Lambda_1 + L_2\Lambda_2 \approx 0.940059 < 1$. Therefore the problem (4.1) has a unique solution on $[1, (8e+1)/9]$ due to Theorem 3.1.

Example 4.2 Consider the boundary value problem for an impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases} {}^C\mathcal{D}_{t_k}^{\log(\sum_{i=0}^{k+1}(1/(i+1)!))} x(t) = \frac{(2-e^{-t})\log(|x(t)|+1)}{|x(t)|+2} - 2, & t \in [\frac{\pi}{2}, 2\pi] \setminus \{t_k\}, \\ \Delta x(t_k) = e^{-\frac{k}{4}} \cos(kx(t_k)) + e^{\frac{k}{4}} \sin(kx(t_k)), & t_k = 2^{\frac{2k-9}{9}}\pi, k = 1, 2, \dots, 8, \\ -e^{-\frac{\pi}{2}}x(\frac{\pi}{2}) + e^{-2\pi}x(2\pi) = \sum_{i=0}^8 (-2)^i(i^2+1)\mathcal{J}_{t_i}^{(\frac{5i-4}{i+1})} x(t_{i+1}). \end{cases} \quad (4.2)$$

Here $\alpha = -e^{-\pi/2}$, $\beta = e^{-2\pi}$, $m = 8$, $p_k = \log(\sum_{i=0}^{k+1}(1/(i+1)!))$, $\gamma_k = (-2)^k(k^2+1)$, $q_k = |5k-4|/(k+1)$ for $k = 0, 1, \dots, 8$. We find that $\Phi \approx -1.422922 \neq 0$. The functions $f(t, x) = (2 - e^{-t})\log(|x|+1)/(|x|+2) - 2$ and $\varphi_k(x) = e^{-\frac{k}{4}}\cos(kx) + e^{\frac{k}{4}}\sin(kx)$ are bounded as

$$|f(t, x)| \leq 4 \quad \text{and} \quad |\varphi_k(x)| \leq \sqrt{e^{-4} + e^4}.$$

Hence the assumption (H₃) of Theorem 3.2 holds. Therefore the problem (4.2) has at least one solution on $[\pi/2, 2\pi]$.

Example 4.3 Consider the boundary value problem for an impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases} {}^C\mathcal{D}_{t_k}^{(\frac{2k+2}{k^2+2k+2})} x(t) = \frac{e^{tx(t)}(\sin x(t)-x(t))}{2t+1}, & t \in [\frac{4}{3}, 3] \setminus \{t_k\}, \\ \Delta x(t_k) = \frac{kx^3(t_k)}{\log(|x(t_k)|+2)}, & t_k = \frac{k+8}{6}, k = 1, 2, \dots, 9, \\ \sqrt{3}x(\frac{4}{3}) + \frac{3}{5}x(3) = \sum_{i=0}^9 (\frac{i^2+1}{i^2+2})\mathcal{J}_{t_i}^{\arctan i} x(t_{i+1}). \end{cases} \quad (4.3)$$

Here $\alpha = \sqrt{3}$, $\beta = 3/5$, $m = 9$, $p_k = 2(k+1)/(k^2+1)$, $\gamma_k = (k^2+1)/(k^2+2)$, $q_k = \arctan(k)$ for $k = 0, 1, \dots, 9$. We find that $\Phi \approx 2.003684 \neq 0$. The functions $f(t, x) = e^{tx}(\sin x - x)/(2t+1)$ and $\varphi_k(x) = kx^3/\log(|x|+2)$ satisfy

$$\lim_{x \rightarrow 0} \frac{f(t, x)}{x} = \lim_{x \rightarrow 0} \frac{e^{xt}}{2t+1} \left(\frac{\sin x}{x} - 1 \right) = 0$$

and

$$\lim_{x \rightarrow 0} \frac{\varphi_k(x)}{x} = \lim_{x \rightarrow 0} \frac{kx^2}{\log(|x|+2)} = 0, \quad \forall k = 1, 2, \dots, 9.$$

Thus the condition (H₄) of Theorem 3.3 holds. Therefore, we conclude that the problem (4.3) has at least one solution on $[4/3, 3]$.

Example 4.4 Consider the boundary value problem for an impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases} {}^C\mathcal{D}_{t_k}^{\sqrt{1-\sin^2(k+1)}} x(t) = e^{\frac{2t}{3}} \sin x(t) + tx(t) \cos x(t) + 2, & t \in [\frac{3}{2}, 3] \setminus \{t_k\}, \\ \Delta x(t_k) = kx(t_k) - \log(|x(t_k)| + \frac{3}{5}), & t_k = 3 \cdot 2^{\frac{k-11}{11}}, k = 1, 2, \dots, 10, \\ \frac{4}{3}x(\frac{3}{2}) - \frac{3}{4}x(3) = \sum_{i=0}^{10} \frac{(-1)^i}{i+1} \mathcal{J}_{t_i}^{(\frac{3k+2}{2k+3})} x(t_{i+1}). \end{cases} \quad (4.4)$$

Here $\alpha = 4/3$, $\beta = -3/4$, $m = 10$, $p_k = \sqrt{1 - \sin^2(k+1)}$, $\gamma_k = (-1)^k/(k+1)$, $q_k = (3k+2)/(2k+3)$, for $k = 0, 1, \dots, 10$. We find that $\Phi \approx 0.605503 \neq 0$. The functions $f(t, x) = e^{\frac{2t}{3}} \sin x + tx \cos x + 2$ and $\varphi(x) = kx - \log(|x| + (3/5))$ satisfy the inequalities

$$|f(t, x)| \leq t|x| + (2 + e^{\frac{2t}{3}}) \leq 3|x| + (2 + e^2)$$

and

$$|\varphi_k(x)| \leq |x|(k+1) + \frac{3}{5} \leq 11|x| + \frac{3}{5}.$$

Therefore (H_5) holds. According to Theorem 3.4, the problem (4.4) has at least one solution on $[3/2, 3]$.

References

1. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier Science B.V., Amsterdam (2006)
2. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
3. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
4. Ahmad, B, Ntouyas, SK, Alsaedi, A: An existence result for fractional differential inclusions with nonlinear integral boundary conditions. *J. Inequal. Appl.* **2013**, 296 (2013)
5. Nyamoradi, N, Baleanu, D, Agarwal, RP: On a multipoint boundary value problem for a fractional order differential inclusion on an infinite interval. *Adv. Math. Phys.* **2013**, Article ID 823961 (2013)
6. Ntouyas, SK, Tariboon, J, Sudsutad, W: Boundary value problems for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral conditions. *Mediterr. J. Math.* (2015). doi:10.1007/s00009-015-0543-1
7. Alsaedi, A, Ntouyas, SK, Agarwal, RP, Ahmad, B: On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Adv. Differ. Equ.* **2015**, 33 (2015)
8. Yukunthorn, W, Ntouyas, SK, Tariboon, J: Nonlinear fractional Caputo-Langevin equation with nonlocal Riemann-Liouville fractional integral conditions. *Adv. Differ. Equ.* **2014**, 315 (2014)
9. Agarwal, J, Ntouyas, SK, Sudsutad, W: Nonlocal Hadamard fractional integral conditions for nonlinear Riemann-Liouville fractional differential equations. *Bound. Value Probl.* **2014**, 253 (2014)
10. Hadamard, J: Essai sur l'étude des fonctions données par leur développement de Taylor. *J. Mat. Pure Appl. Ser.* **8**, 101-186 (1892)
11. Butzer, PL, Kilbas, AA, Trujillo, JJ: Compositions of Hadamard-type fractional integration operators and the semigroup property. *J. Math. Anal. Appl.* **269**, 387-400 (2002)
12. Butzer, PL, Kilbas, AA, Trujillo, JJ: Fractional calculus in the Mellin setting and Hadamard-type fractional integrals. *J. Math. Anal. Appl.* **269**, 1-27 (2002)
13. Butzer, PL, Kilbas, AA, Trujillo, JJ: Mellin transform analysis and integration by parts for Hadamard-type fractional integrals. *J. Math. Anal. Appl.* **270**, 1-15 (2002)
14. Kilbas, AA: Hadamard-type fractional calculus. *J. Korean Math. Soc.* **38**, 1191-1204 (2001)
15. Kilbas, AA, Trujillo, JJ: Hadamard-type integrals as G-transforms. *Integral Transforms Spec. Funct.* **14**, 413-427 (2003)
16. Agarwal, RP, Ahmad, B: Existence of solutions for impulsive anti-periodic boundary value problems of fractional semilinear evolution equations. *Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal.* **18**, 535-544 (2011)
17. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
18. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions. Hindawi Publishing, New York (2006)
19. Ahmad, B, Wang, G: A study of an impulsive four-point nonlocal boundary value problem of nonlinear fractional differential equations. *Comput. Math. Appl.* **62**, 1341-1349 (2011)
20. Zhang, X, Huang, X, Liu, Z: The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay. *Nonlinear Anal. Hybrid Syst.* **4**, 775-781 (2010)
21. Wang, G, Liu, S, Baleanu, D, Zhang, L: A new impulsive multi-orders fractional differential equation involving multipoint fractional integral boundary conditions. *Abstr. Appl. Anal.* **2014**, Article ID 932747 (2014)
22. Wang, J, Zhang, Y: On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives. *Appl. Math. Lett.* **39**, 85-90 (2015)
23. Jarad, F, Abdeljawad, T, Baleanu, D: Caputo-type modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2012**, 142 (2012)
24. Gambo, YY, Jarad, F, Baleanu, D, Abdeljawad, T: On Caputo modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2014**, 10 (2014)
25. Sun, L, Sun, J, Wang, G: Generalizations of fixed-point theorems of Altman and Rothe types. *Abstr. Appl. Anal.* **2013**, Article ID 639030 (2013)
26. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)

6.2 Existence results for a coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals

In this section, we introduce a new class of coupled systems of boundary value problems for fractional differential equations which contains multiple orders of fractional derivatives and integrals, and discuss the existence and uniqueness of solutions. We apply Leray-Schauder's alternative and Banach's contraction mapping principle to obtain the desired results. Illustrative examples is also included.

1 Introduction

Differential equations of fractional order have played a significant role in engineering, science, and pure and applied mathematics in recent years. Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc. [1]-[4]. Fractional-order boundary value problems involving a variety of classical, nonlocal and integral boundary conditions have been addressed by many authors, for instance, see [5]-[13] and the references cited therein.

Coupled systems of fractional-order differential equations also constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [14], disease models [15]-[18], ecological models [19], synchronization of chaotic systems [20]-[22], etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers [23]-[28].

Recently in [29] a new class of fractional boundary valued problems was introduced, which contains four orders of Riemann-Liouville fractional derivatives, two in fractional differential equation and two

in boundary conditions of the form

$$\begin{cases} (\lambda D^\alpha + (1 - \lambda)D^\beta) x(t) = f(t, x(t)), & t \in (0, T), \\ x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1 - \mu)D^{\gamma_2} x(T) = \gamma_3, \end{cases} \quad (1)$$

where D^ϕ is the Riemann-Liouville fractional derivative of order $\phi \in \{\alpha, \beta, \gamma_1, \gamma_2\}$ such that $1 < \alpha, \beta \leq 2$ and $0 < \gamma_1, \gamma_2 < \alpha - \beta$, $\gamma_3 \in \mathbb{R}$, the given constants $0 < \lambda \leq 1$, $0 \leq \mu \leq 1$ and $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$ is a continuous function. Existence and uniqueness results were obtained by means of Banach's contraction mapping principle, Krasnoselskii's fixed point theorem and Leray-Schauder's nonlinear alternative.

In this paper, we study a coupled system of fractional differential equations

$$\begin{cases} (\lambda D^\alpha + (1 - \lambda)D^\beta) x(t) = f(t, x(t), y(t)), & t \in (0, T), \quad 1 < \alpha, \beta \leq 2 \\ (\lambda_1 D^{\alpha_1} + (1 - \lambda_1)D^{\beta_1}) y(t) = g(t, x(t), y(t)), & t \in (0, T), \quad 1 < \alpha_1, \beta_1 \leq 2, \end{cases} \quad (2)$$

subject to the following type of boundary conditions

$$\begin{cases} x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1 - \mu)D^{\gamma_2} x(T) = \gamma_3, \\ y(0) = 0, \quad \mu_1 I^{\delta_1} y(T) + (1 - \mu_1)I^{\delta_2} y(T) = \delta_3, \end{cases} \quad (3)$$

where D^ϕ denotes the Caputo fractional derivatives of order $\phi \in \{\alpha, \beta, \alpha_1, \beta_1, \gamma_1, \gamma_2\}$, I^χ denotes the Riemann-Liouville fractional integral of order $\chi \in \{\delta_1, \delta_2\}$, $\gamma_3, \delta_3 \in \mathbb{R}$, $0 < \lambda, \lambda_1 \leq 1$, $0 \leq \mu, \mu_1 \leq 1$ and $f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are appropriately chosen functions.

The paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present two auxiliary lemmas. The main results are presented in Section 3. We give two results: the first one derives the existence of solutions via Leray-Schauder's alternative, whereas the second one concerning existence and uniqueness of solutions is established by Banach's contraction principle. We also discuss two examples for illustration of the existence-uniqueness results.

2 Preliminaries

Before presenting two auxiliary lemmas, we recall some basic definitions of fractional calculus [1, 2].

Definition 2.1 For $(n-1)$ -times absolutely continuous function $y : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^c D^q y(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} y^{(n)}(s) ds, \quad n-1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q .

Definition 2.2 The Riemann-Liouville fractional integral of order q is defined as

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{y(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.3 The boundary value problem

$$\begin{cases} (\lambda D^\alpha + (1 - \lambda)D^\beta) x(t) = \omega(t), & t \in (0, T), \\ x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1 - \mu)D^{\gamma_2} x(T) = \gamma_3, \end{cases} \quad (4)$$

is equivalent to the following integral equation

$$x(t) = \frac{\lambda - 1}{\lambda \Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\lambda \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds$$

$$\begin{aligned}
& + \frac{t}{\Lambda_1} \left(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} x(s) ds \right. \\
& - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} \omega(s) ds \\
& - \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} x(s) ds \\
& \left. - \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} \omega(s) ds \right), \quad t \in J := [0, T],
\end{aligned} \tag{5}$$

where the non zero constant Λ_1 is defined by

$$\Lambda_1 = \frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{(1-\mu)T^{1-\gamma_2}}{\Gamma(2-\gamma_2)}. \tag{6}$$

Proof. The first equation of (4) can be rewritten as

$$D^\alpha x(t) = \frac{\lambda-1}{\lambda} D^\beta x(t) + \frac{1}{\lambda} \omega(t), \quad t \in J. \tag{7}$$

Applying the Riemann-Liouville fractional integral of order α to both sides of (7), we obtain

$$x(t) = \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds + C_1 + C_2 t,$$

where constants $C_1, C_2 \in \mathbb{R}$. The first boundary condition of (4) implies that $C_1 = 0$. Hence

$$x(t) = \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s) ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega(s) ds + C_2 t. \tag{8}$$

Taking the Caputo fractional derivative of order $\psi \in \{\gamma_1, \gamma_2\}$ such that $0 < \psi < \alpha - \beta$ to (8), we deduce that

$$\begin{aligned}
D^\psi x(t) &= \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta-\psi)} \int_0^t (t-s)^{\alpha-\beta-\psi-1} x(s) ds \\
&+ \frac{1}{\lambda\Gamma(\alpha-\psi)} \int_0^t (t-s)^{\alpha-\psi-1} \omega(s) ds + C_2 \frac{1}{\Gamma(2-\psi)} t^{1-\psi}.
\end{aligned}$$

Substituting the values $\psi = \gamma_1$ and $\psi = \gamma_2$ to the above relation and using the second condition of (4), we obtain a constant γ_3 as

$$\begin{aligned}
\gamma_3 &= \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} x(s) ds \\
&+ \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} \omega(s) ds + \frac{\mu T^{1-\gamma_1}}{\Gamma(2-\gamma_1)} C_2 \\
&+ \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} x(s) ds \\
&+ \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} \omega(s) ds + \frac{(1-\mu)T^{1-\gamma_2}}{\Gamma(2-\gamma_2)} C_2,
\end{aligned}$$

which yields

$$C_2 = \frac{1}{\Lambda_1} \left[\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} x(s) ds - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} \omega(s) ds \right]$$

$$\left. - \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} x(s) ds - \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} \omega(s) ds \right].$$

Substituting the value of the constant C_2 into (8), we deduce the integral equation (5). The converse follows by direct computation. This completes the proof. $\square$

In the same way, we obtain the following result with the Riemann-Liouville fractional integral boundary conditions.

Lemma 2.4 *The boundary value problem*

$$\begin{cases} (\lambda_1 D^{\alpha_1} + (1-\lambda_1) D^{\beta_1}) y(t) = \omega_1(t), & t \in (0, T), \\ y(0) = 0, \quad \mu_1 I^{\delta_1} y(T) + (1-\mu_1) I^{\delta_2} y(T) = \delta_3, \end{cases} \quad (9)$$

is equivalent to the following integral equation

$$\begin{aligned} y(t) = & \frac{\lambda_1 - 1}{\lambda_1 \Gamma(\alpha_1 - \beta_1)} \int_0^t (t-s)^{\alpha_1 - \beta_1 - 1} y(s) ds + \frac{1}{\lambda_1 \Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} \omega_1(s) ds \\ & + \frac{t}{\Lambda_2} \left(\delta_3 - \frac{\mu_1(\lambda_1 - 1)}{\lambda_1 \Gamma(\delta_1 + \alpha_1 - \beta_1)} \int_0^T (T-s)^{\delta_1 + \alpha_1 - \beta_1 - 1} y(s) ds \right. \\ & - \frac{\mu_1}{\lambda_1 \Gamma(\delta_1 + \alpha_1)} \int_0^T (T-s)^{\delta_1 + \alpha_1 - 1} \omega_1(s) ds \\ & - \frac{(1-\mu_1)(\lambda_1 - 1)}{\lambda_1 \Gamma(\delta_2 + \alpha_1 - \beta_1)} \int_0^T (T-s)^{\delta_2 + \alpha_1 - \beta_1 - 1} y(s) ds \\ & \left. - \frac{1-\mu_1}{\lambda_1 \Gamma(\delta_2 + \alpha_1)} \int_0^T (T-s)^{\delta_2 + \alpha_1 - 1} \omega_1(s) ds \right), \quad t \in J, \end{aligned} \quad (10)$$

where the non zero constant Λ_2 is defined by

$$\Lambda_2 = \frac{\mu_1 T^{1+\delta_1}}{\Gamma(2+\delta_1)} + \frac{(1-\mu_1) T^{1+\delta_2}}{\Gamma(2+\delta_2)}. \quad (11)$$

3 Main Results

Let us introduce the space $X = \{u(t) \mid u(t) \in C(J, \mathbb{R})\}$ endowed with the norm $\|u\| = \sup\{|u(t)|, t \in J\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also $Y = \{v(t) \mid v(t) \in C(J, \mathbb{R})\}$ endowed with the norm $\|v\| = \sup\{|v(t)|, t \in J\}$ is a Banach space. Then the product space $(X \times Y, \|(u, v)\|)$ is also a Banach space equipped with norm $\|(u, v)\| = \|u\| + \|v\|$.

In view of Lemmas 2.3 and 2.4, we define the operator $\mathcal{T} : X \times Y \rightarrow X \times Y$ by

$$\mathcal{T}(u, v)(t) = \begin{pmatrix} \mathcal{T}_1(u, v)(t) \\ \mathcal{T}_2(u, v)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{T}_1(u, v)(t) = & \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} u(s) ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), v(s)) ds \\ & + \frac{t}{\Lambda_1} \left(\gamma_3 - \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} u(s) ds \right. \\ & \left. - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} f(s, u(s), v(s)) ds \right) \end{aligned}$$

$$\begin{aligned} & - \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} u(s) ds \\ & - \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} f(s, u(s), v(s)) ds \Bigg), \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}_2(u, v)(t) &= \frac{\lambda_1 - 1}{\lambda_1 \Gamma(\alpha_1 - \beta_1)} \int_0^t (t-s)^{\alpha_1 - \beta_1 - 1} v(s) ds + \frac{1}{\lambda_1 \Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} g(s, u(s), v(s)) ds \\ &+ \frac{t}{\Lambda_2} \left(\delta_3 - \frac{\mu_1(\lambda_1 - 1)}{\lambda_1 \Gamma(\delta_1 + \alpha_1 - \beta_1)} \int_0^T (T-s)^{\delta_1 + \alpha_1 - \beta_1 - 1} v(s) ds \right. \\ &- \frac{\mu_1}{\lambda_1 \Gamma(\delta_1 + \alpha_1)} \int_0^T (T-s)^{\delta_1 + \alpha_1 - 1} g(s, u(s), v(s)) ds \\ &- \frac{(1-\mu_1)(\lambda_1 - 1)}{\lambda_1 \Gamma(\delta_2 + \alpha_1 - \beta_1)} \int_0^T (T-s)^{\delta_2 + \alpha_1 - \beta_1 - 1} v(s) ds \\ &\left. - \frac{1-\mu_1}{\lambda_1 \Gamma(\delta_2 + \alpha_1)} \int_0^T (T-s)^{\delta_2 + \alpha_1 - 1} g(s, u(s), v(s)) ds \right). \end{aligned}$$

Let us introduce the following hypotheses which are used hereafter.

(H₁) Assume that there exist real constants k_i , $\nu_i \geq 0$ ($i = 1, 2$) and $k_0 > 0, \nu_0 > 0$ such that $\forall x_i \in \mathbb{R}$, ($i = 1, 2$) we have

$$\begin{aligned} |f(t, x_1, x_2)| &\leq k_0 + k_1|x_1| + k_2|x_2|, \\ |g(t, x_1, x_2)| &\leq \nu_0 + \nu_1|x_1| + \nu_2|x_2|. \end{aligned}$$

(H₂) Assume that $f, h : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions and there exist constants m_i, n_i , $i = 1, 2$ such that for all $t \in J$ and $u_i, v_i \in \mathbb{R}$, $i = 1, 2$,

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq m_1|u_1 - v_1| + m_2|u_2 - v_2|$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq n_1|u_1 - v_1| + n_2|u_2 - v_2|.$$

For the sake of convenience, we set constants

$$M_1 = \frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)} + \frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)}, \quad (12)$$

$$N_1 = \frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)} + \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)}, \quad (13)$$

$$M_2 = \frac{T^{\alpha_1}}{\lambda_1\Gamma(\alpha_1+1)} + \frac{T^{\delta_1+\alpha_1+1}\mu_1}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1+1)} + \frac{T^{\delta_2+\alpha_1+1}(1-\mu_1)}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1+1)}, \quad (14)$$

$$N_2 = \frac{T^{\alpha_1-\beta_1}|\lambda_1-1|}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} + \frac{T^{\delta_1+\alpha_1-\beta_1+1}\mu_1|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1-\beta_1+1)} + \frac{T^{\delta_2+\alpha_1-\beta_1+1}(1-\mu_1)|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1-\beta_1+1)} \quad (15)$$

and

$$M_0 = \min\{1 - (M_1k_1 + N_1 + M_2\nu_1), 1 - (M_1k_2 + M_2\nu_2 + N_2)\}, \quad k_i, \nu_i \geq 0 \quad (i = 1, 2). \quad (16)$$

The first result is based on Leray-Schauder alternative.

Lemma 3.1 (Leray-Schauder alternative) ([30] p. 4.) Let $F : E \rightarrow E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2 Assume that (H_1) holds. In addition, it is assumed that

$$M_1 k_1 + N_1 + M_2 \nu_1 < 1 \quad \text{and} \quad M_1 k_2 + M_2 \nu_2 + N_2 < 1,$$

where M_1 and M_2 are given by (13) and (15) respectively. Then the system (2)-(3) has at least one solution.

Proof. First we show that the operator $\mathcal{T} : X \times Y \rightarrow X \times Y$ is completely continuous. By continuity of functions f and g , the operator $\mathcal{T}$ is continuous.

Let $\Omega = \{(u, v) \in X \times Y : \|(u, v)\| \leq r\} \subset X \times Y$ be a bounded set. Then there exist positive constants L_1 and L_2 such that

$$|f(t, u(t), v(t))| \leq L_1, \quad |g(t, u(t), v(t))| \leq L_2, \quad \forall (u, v) \in \Omega.$$

Then for any $(u, v) \in \Omega$, we have

$$\begin{aligned} |\mathcal{T}_1(u, v)(t)| &\leq \frac{|\lambda - 1|}{\lambda \Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} |u(s)| ds + \frac{1}{\lambda \Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} |f(s, u(s), v(s))| ds \\ &\quad + \frac{t}{\Lambda_1} \left(|\gamma_3| + \frac{\mu(\lambda - 1)}{\lambda \Gamma(\alpha - \beta - \gamma_1)} \int_0^T (T - s)^{\alpha - \beta - \gamma_1 - 1} |u(s)| ds \right. \\ &\quad + \frac{\mu}{\lambda \Gamma(\alpha - \gamma_1)} \int_0^T (T - s)^{\alpha - \gamma_1 - 1} |f(s, u(s), v(s))| ds \\ &\quad + \frac{(1 - \mu)|\lambda - 1|}{\lambda \Gamma(\alpha - \beta - \gamma_2)} \int_0^T (T - s)^{\alpha - \beta - \gamma_2 - 1} |u(s)| ds \\ &\quad \left. + \frac{1 - \mu}{\lambda \Gamma(\alpha - \gamma_2)} \int_0^T (T - s)^{\alpha - \gamma_2 - 1} |f(s, u(s), v(s))| ds \right) \\ &\leq L_1 \left[\frac{T^\alpha}{\lambda \Gamma(\alpha + 1)} + \frac{T^{\alpha - \gamma_1 + 1} \mu}{\lambda \Lambda_1 \Gamma(\alpha - \gamma_1 + 1)} + \frac{T^{\alpha - \gamma_2 + 1} (1 - \mu)}{\lambda \Lambda_1 \Gamma(\alpha - \gamma_2 + 1)} \right] \\ &\quad + \|u\| \left[\frac{T^{\alpha - \beta} |\lambda - 1|}{\lambda \Gamma(\alpha - \beta + 1)} + \frac{T^{\alpha - \beta - \gamma_1 + 1} \mu |\lambda - 1|}{\lambda \Lambda_1 \Gamma(\alpha - \beta - \gamma_1 + 1)} \right. \\ &\quad \left. + \frac{T^{\alpha - \beta - \gamma_2 + 1} (1 - \mu) |\lambda - 1|}{\lambda \Lambda_1 \Gamma(\alpha - \beta - \gamma_2 + 1)} \right] + \frac{|\gamma_3| T}{\Lambda_1} \\ &= L_1 M_1 + N_1 r + |\gamma_3| T / \Lambda_1 \end{aligned}$$

and consequently,

$$\|\mathcal{T}_1(u, v)\| \leq L_1 M_1 + N_1 r + |\gamma_3| T / \Lambda_1.$$

Similarly, we get

$$\|\mathcal{T}_2(u, v)\| \leq L_2 M_2 + N_2 r + |\delta_3| T / \Lambda_2.$$

Thus, it follows from the above inequalities that the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that $\mathcal{T}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$|\mathcal{T}_1(u(t_2), v(t_2)) - \mathcal{T}_1(u(t_1), v(t_1))|$$

$$\begin{aligned}
&\leq \frac{|\lambda - 1|}{\lambda\Gamma(\alpha - \beta)} \left[\int_0^{t_2} (t_2 - s)^{\alpha - \beta - 1} u(s) ds - \int_0^{t_1} (t_1 - s)^{\alpha - \beta - 1} u(s) ds \right] \\
&\quad + \frac{1}{\lambda\Gamma(\alpha)} \left[\int_0^{t_2} (t_2 - s)^{\alpha - 1} f(s, u(s), v(s)) ds - \int_0^{t_1} (t_1 - s)^{\alpha - 1} f(s, u(s), v(s)) ds \right] \\
&\quad + \frac{|t_2 - t_1|}{\Lambda_1} \left(|\gamma_3| + \frac{\mu|\lambda - 1|}{\lambda\Gamma(\alpha - \beta - \gamma_1)} \int_0^T (T - s)^{\alpha - \beta - \gamma_1 - 1} u(s) ds \right. \\
&\quad \left. + \frac{\mu}{\lambda\Gamma(\alpha - \gamma_1)} \int_0^T (T - s)^{\alpha - \gamma_1 - 1} f(s, u(s), v(s)) ds \right. \\
&\quad \left. + \frac{(1 - \mu)|\lambda - 1|}{\lambda\Gamma(\alpha - \beta - \gamma_2)} \int_0^T (T - s)^{\alpha - \beta - \gamma_2 - 1} u(s) ds \right. \\
&\quad \left. + \frac{1 - \mu}{\lambda\Gamma(\alpha - \gamma_2)} \int_0^T (T - s)^{\alpha - \gamma_2 - 1} f(s, u(s), v(s)) ds \right) \\
&\leq r \left[\frac{[2(t_2 - t_1)^{\alpha - \beta} + |t_2^{\alpha - \beta} - t_1^{\alpha - \beta}|]|\lambda - 1|}{\lambda\Gamma(\alpha - \beta + 1)} + \frac{|t_2 - t_1|}{\Lambda_1} \left(\frac{|\lambda - 1|\mu T^{\alpha - \beta - \gamma_1}}{\lambda\Gamma(\alpha - \beta - \gamma_1 + 1)} \right. \right. \\
&\quad \left. \left. + \frac{|\lambda - 1|(1 - \mu)T^{\alpha - \beta - \gamma_2}}{\lambda\Gamma(\alpha - \beta - \gamma_2 + 1)} \right) \right] + \frac{|t_2 - t_1||\gamma_3|}{\Lambda_1} \\
&\quad + L_1 \left[\frac{2(t_2 - t_1)^{\alpha - 1} + |t_2^{\alpha - 1} - t_1^{\alpha - 1}|}{\lambda\Gamma(\alpha + 1)} + \frac{|t_2 - t_1|}{\Lambda_1} \left(\frac{\mu T^{\alpha - \gamma_1}}{\lambda\Gamma(\alpha - \gamma_1 + 1)} + \frac{(1 - \mu)T^{\alpha - \gamma_2}}{\lambda\Gamma(\alpha - \gamma_2 + 1)} \right) \right].
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}
&|\mathcal{T}_2(u(t_2), v(t_2)) - \mathcal{T}_2(u(t_1), v(t_1))| \\
&\leq r \left[\frac{[2(t_2 - t_1)^{\alpha_1 - \beta_1} + |t_2^{\alpha_1 - \beta_1} - t_1^{\alpha_1 - \beta_1}|]|\lambda_1 - 1|}{\lambda_1\Gamma(\alpha_1 - \beta_1 + 1)} + \frac{|t_2 - t_1|}{\Lambda_2} \left(\frac{|\lambda_1 - 1|\mu_1 T^{\delta_1 + \alpha_1 - \beta_1}}{\lambda_1\Gamma(\delta_1 + \alpha_1 - \beta_1 + 1)} \right. \right. \\
&\quad \left. \left. + \frac{|\lambda_1 - 1|(1 - \mu)T^{\delta_2 + \alpha_1 - \beta_1}}{\lambda_1\Gamma(\delta_2 + \alpha_1 - \beta_1 + 1)} \right) \right] + \frac{|t_2 - t_1||\delta_3|}{\Lambda_2} \\
&\quad + L_2 \left[\frac{2(t_2 - t_1)^{\alpha_1 - 1} + |t_2^{\alpha_1 - 1} - t_1^{\alpha_1 - 1}|}{\lambda_1\Gamma(\alpha_1 + 1)} + \frac{|t_2 - t_1|}{\Lambda_2} \left(\frac{\mu_1 T^{\delta_1 + \alpha_1}}{\lambda_1\Gamma(\delta_1 + \alpha_1 + 1)} + \frac{(1 - \mu_1)T^{\delta_2 + \alpha_1}}{\lambda_1\Gamma(\delta_2 + \alpha_1 + 1)} \right) \right].
\end{aligned}$$

As $t_2 - t_1 \rightarrow 0$, the right-hand sides of the above inequalities tends to zero independently of $(u, v) \in \Omega$. Therefore, by the Arzelà-Ascoli theorem, the operator $\mathcal{T}(u, v)$ is equicontinuous, and thus the operator $\mathcal{T}(u, v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(u, v) \in X \times Y | (u, v) = \theta\mathcal{T}(u, v), 0 \leq \theta \leq 1\}$ is bounded. Let $(u, v) \in \mathcal{E}$, with $(u, v) = \theta\mathcal{T}(u, v)$. For any $t \in [0, T]$, we have

$$u(t) = \theta\mathcal{T}_1(u, v)(t), \quad v(t) = \theta\mathcal{T}_2(u, v)(t).$$

Then

$$\begin{aligned}
|u(t)| &\leq (k_0 + k_1\|u\| + k_2\|v\|) \left[\frac{T^\alpha}{\lambda\Gamma(\alpha + 1)} + \frac{T^{\alpha - \gamma_1 + 1}\mu}{\lambda\Lambda_1\Gamma(\alpha - \gamma_1 + 1)} + \frac{T^{\alpha - \gamma_2 + 1}(1 - \mu)}{\lambda\Lambda_1\Gamma(\alpha - \gamma_2 + 1)} \right] \\
&\quad + \|u\| \left[\frac{T^{\alpha - \beta}|\lambda - 1|}{\lambda\Gamma(\alpha - \beta + 1)} + \frac{T^{\alpha - \beta - \gamma_1 + 1}\mu|\lambda - 1|}{\lambda\Lambda_1\Gamma(\alpha - \beta - \gamma_1 + 1)} \right. \\
&\quad \left. + \frac{T^{\alpha - \beta - \gamma_2 + 1}(1 - \mu)|\lambda - 1|}{\lambda\Lambda_1\Gamma(\alpha - \beta - \gamma_2 + 1)} \right] + \frac{|\gamma_3|T}{\Lambda_1}
\end{aligned}$$

and

$$\begin{aligned}
|v(t)| \leq & (\nu_0 + \nu_1\|u\| + \nu_2\|v\|) \left[\frac{T^{\alpha_1}}{\lambda_1\Gamma(\alpha_1+1)} + \frac{T^{\delta_1+\alpha_1+1}\mu_1}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1+1)} \right. \\
& \left. + \frac{T^{\delta_2+\alpha_1+1}(1-\mu_1)}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1+1)} \right] + \|v\| \left[\frac{T^{\alpha_1-\beta_1}|\lambda_1-1|}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} \right. \\
& \left. + \frac{T^{\delta_1+\alpha_1-\beta_1+1}\mu_1|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_1+\alpha_1-\beta_1+1)} + \frac{T^{\delta_2+\alpha_1-\beta_1+1}(1-\mu_1)|\lambda_1-1|}{\lambda_1\Lambda_2\Gamma(\delta_2+\alpha_1-\beta_1+1)} \right] + \frac{|\delta_3|T}{\Lambda_2}.
\end{aligned}$$

Hence we have

$$\|u\| \leq M_1(k_0 + k_1\|u\| + k_2\|v\|) + N_1\|u\| + |\gamma_3|T/\Lambda_1$$

and

$$\|v\| \leq M_2(\nu_0 + \nu_1\|u\| + \nu_2\|v\|) + N_2\|v\| + |\delta_3|T/\Lambda_2,$$

which imply that

$$\begin{aligned}
\|u\| + \|v\| = & \left(M_1k_0 + M_2\nu_0 + |\gamma_3|T/\Lambda_1 + |\delta_3|T/\Lambda_2 \right) \\
& + (M_1k_1 + N_1 + M_2\nu_1)\|u\| + (M_1k_2 + M_2\nu_2 + N_2)\|v\|.
\end{aligned}$$

Consequently,

$$\|(u, v)\| \leq \frac{M_1k_0 + M_2\nu_0 + |\gamma_3|T/\Lambda_1 + |\delta_3|T/\Lambda_2}{M_0},$$

for any $t \in [0, T]$, where M_0 is defined by (16), which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Hence the boundary value problem (2)-(3) has at least one solution. The proof is complete. $\square$

In the second result, we prove existence and uniqueness of solutions of the boundary value problem (2)-(3) via Banach's contraction principle.

Theorem 3.3 Assume that (H_2) holds. In addition, assume that

$$M_1(m_1 + m_2) + N_1 + M_2(n_1 + n_2) + N_2 < 1,$$

where M_1 , N_1 , M_2 and N_2 are given by (12) and (15), respectively. Then the system (2)-(3) has a unique solution on J .

Proof. Define $\sup_{t \in J} f(t, 0, 0) = F_0 < \infty$ and $\sup_{t \in J} g(t, 0, 0) = G_0 < \infty$ such that

$$r \geq \frac{N_1M_1 + N_2M_2 + |\gamma_3|T/\Lambda_1 + |\delta_3|T/\Lambda_2}{1 - M_1(m_1 + m_2) - M_2(n_1 + n_2) - (N_1 + N_2)}.$$

We show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{(u, v) \in X \times Y : \|(u, v)\| \leq r\}$.

For $(u, v) \in B_r$, we have

$$\begin{aligned}
|\mathcal{T}_1(u, v)(t)| \leq & \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} |u(s)| ds \\
& + \frac{1}{\lambda\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [|f(s, u(s), v(s)) - f(s, 0, 0)| + |f(s, 0, 0)|] ds \\
& + \frac{T}{\Lambda_1} \left(|\gamma_3| + \frac{\mu|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} |u(s)| ds \right. \\
& \left. + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} [|f(s, u(s), v(s)) - f(s, 0, 0)| + |f(s, 0, 0)|] ds \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} |u(s)| ds \\
& + \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} [|f(s, u(s), v(s)) - f(s, 0)| + |f(s, 0, 0)|] ds \Big) \\
\leq & (m_1\|u\| + m_2\|v\| + F_0) \left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)} + \frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)} \right] \\
& + \|x\| \left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)} \right. \\
& \left. + \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)} \right] + \frac{|\gamma_3|T^{\alpha-1}}{\Lambda_1} \\
\leq & M_1[(m_1+m_2)r + F_0] + N_1r + |\gamma_3|T/\Lambda_1.
\end{aligned}$$

Hence

$$\|\mathcal{T}_1(u, v)\| \leq M_1[(m_1+m_2)r + F_0] + N_1r + |\gamma_3|T/\Lambda_1.$$

In the same way, we can obtain that

$$\|\mathcal{T}_2(u, v)\| \leq M_2[(n_1+n_2)r + G_0] + N_2r + |\delta_3|T/\Lambda_2.$$

Consequently, $\|T(u, v)\| \leq r$.

Now for $(u_2, v_2), (u_1, v_1) \in X \times Y$, and for any $t \in [0, e]$, we get

$$\begin{aligned}
& |\mathcal{T}_1(u_2, v_2)(t) - \mathcal{T}_1(u_1, v_1)(t)| \\
\leq & \frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} |u_2(s) - u_1(s)| ds \\
& + \frac{1}{\lambda\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s, u_2(s), v_2(s)) - f(s, u_1(s), v_1(s))| ds \\
& + \frac{T^{\alpha-1}}{\Lambda_1} \left(\frac{\mu|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta-\gamma_1-1} |u_2(s) - u_1(s)| ds \right. \\
& + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} |f(s, u_2(s), v_2(s)) - f(s, u_1(s), v_1(s))| ds \\
& + \frac{(1-\mu)|\lambda-1|}{\lambda\Gamma(\alpha-\beta-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta-\gamma_2-1} |u_2(s) - u_1(s)| ds \\
& \left. + \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} |f(s, u_2(s), v_2(s)) - f(s, u_1(s), v_1(s))| ds \right) \\
\leq & (m_1\|u_2 - u_1\| + m_2\|v_2 - v_1\|) \left[\frac{T^\alpha}{\lambda\Gamma(\alpha+1)} + \frac{T^{\alpha-\gamma_1+1}\mu}{\lambda\Lambda_1\Gamma(\alpha-\gamma_1+1)} \right. \\
& + \left. \frac{T^{\alpha-\gamma_2+1}(1-\mu)}{\lambda\Lambda_1\Gamma(\alpha-\gamma_2+1)} \right] + \|u_2 - u_1\| \left[\frac{T^{\alpha-\beta}|\lambda-1|}{\lambda\Gamma(\alpha-\beta+1)} + \frac{T^{\alpha-\beta-\gamma_1+1}\mu|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_1+1)} \right. \\
& + \left. \frac{T^{\alpha-\beta-\gamma_2+1}(1-\mu)|\lambda-1|}{\lambda\Lambda_1\Gamma(\alpha-\beta-\gamma_2+1)} \right] \\
\leq & M_1[(m_1\|u_2 - u_1\| + m_2\|v_2 - v_1\|) + N_1\|u_2 - u_1\|] \\
\leq & M_1(m_1+m_2)[\|u_2 - u_1\| + \|v_2 - v_1\|] + N_1[\|u_2 - u_1\| + \|v_2 - v_1\|] \\
\leq & [M_1(m_1+m_2) + N_1][\|u_2 - u_1\| + \|v_2 - v_1\|],
\end{aligned}$$

and consequently we obtain

$$\|\mathcal{T}_1(u_2, v_2) - \mathcal{T}_1(u_1, v_1)\| \leq [M_1(m_1+m_2) + N_1][\|u_2 - u_1\| + \|v_2 - v_1\|]. \quad (17)$$

Similarly,

$$\|\mathcal{T}_2(u_2, v_2) - \mathcal{T}_2(u_1, v_1)\| \leq [M_2(n_1 + n_2) + N_2](\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (18)$$

It follows from (17) and (18) that

$$\|\mathcal{T}(u_2, v_2) - \mathcal{T}(u_1, v_1)\| \leq [M_1(m_1 + m_2) + N_1 + M_2(n_1 + n_2) + N_2](\|u_2 - u_1\| + \|v_2 - v_1\|).$$

Since $M_1(m_1 + m_2) + N_1 + M_2(n_1 + n_2) + N_2 < 1$, therefore, $\mathcal{T}$ is a contraction operator. So, By Banach's fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of problem (2)-(3). This completes the proof. $\square$

Example 3.4 Consider the following coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals boundary conditions of the form

$$\left\{ \begin{array}{l} \left(\frac{33}{38} D^{29/15} + \frac{5}{38} D^{16/15} \right) x(t) = \frac{t}{t+1} + \frac{1}{3} \sin \left(\frac{|x(t)|}{4} \right) + \frac{y^2(t)}{10(1+|y(t)|)}, \quad t \in [0, 3/2], \\ \left(\frac{24}{27} D^{19/13} + \frac{3}{27} D^{15/13} \right) y(t) = \sqrt{t+3} + \frac{x(t)}{10} e^{-|x(t)|} + \frac{t}{3} \tan^{-1} \left(\frac{|y(t)|}{6} \right), \quad t \in [0, 3/2], \\ x(0) = 0, \quad \frac{9}{16} D^{8/15} x \left(\frac{3}{2} \right) + \frac{7}{16} D^{11/15} x \left(\frac{3}{2} \right) = \frac{1}{3}, \\ y(0) = 0, \quad \frac{2}{5} I^{1/2} y \left(\frac{3}{2} \right) + \frac{3}{5} I^{3/2} y \left(\frac{3}{2} \right) = \frac{2}{7}. \end{array} \right. \quad (19)$$

Here $\lambda = 33/38$, $\alpha = 29/15$, $\beta = 16/15$, $T = 3/2$, $\lambda_1 = 24/27$, $\alpha_1 = 19/13$, $\beta_1 = 15/13$, $\mu = 9/16$, $\gamma_1 = 8/15$, $\gamma_2 = 11/15$, $\gamma_3 = 1/3$, $\mu_1 = 2/5$, $\delta_1 = 1/2$, $\delta_2 = 3/2$, $\delta_3 = 2/7$. From all constants, we can compute that $\Lambda_1 = 1.307202573$, $\Lambda_2 = 1.050302214$, $M_1 = 3.248792650$, $N_1 = 0.4373542422$, $M_2 = 2.869543745$ and $N_2 = 0.3962406719$. Clearly,

$$\begin{aligned} |f(t, x, y)| &= \left| \frac{t}{t+1} + \frac{1}{3} \sin \left(\frac{|x|}{4} \right) + \frac{y^2}{10(1+|y|)} \right| \\ &\leq \frac{3}{5} + \frac{1}{12}|x| + \frac{1}{10}|y|, \end{aligned}$$

and

$$\begin{aligned} |g(t, x, y)| &= \left| \sqrt{t+3} + \frac{x}{10} e^{-|x|} + \frac{t}{3} \tan^{-1} \left(\frac{|y|}{6} \right) \right| \\ &\leq \frac{3}{\sqrt{2}} + \frac{1}{10}|x| + \frac{1}{12}|y|. \end{aligned}$$

Setting $k_0 = 3/5$, $k_1 = 1/12$, $k_2 = 1/10$, $\nu_0 = 3/\sqrt{2}$, $\nu_1 = 1/10$ and $\nu_2 = 1/12$, we have

$$M_1 k_1 + M_2 \nu_1 + N_1 = 0.9950413375 < 1 \quad \text{and} \quad M_1 k_2 + M_2 \nu_2 + N_2 = 0.9602485823 < 1.$$

Therefore, by applying Theorem 3.2, the boundary value problem (19) has at least one solution on $[0, 3/2]$.

Example 3.5 Consider the following coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals boundary conditions of the form

$$\left\{ \begin{array}{l} \left(\frac{49}{53} D^{17/9} + \frac{4}{53} D^{10/9} \right) x(t) = \frac{t+1}{2} + \frac{|x(t)|e^{-t^2}}{2(1+|x(t)|)} + \frac{1}{3} \sin |y(t)| \cos 2\pi t, \quad t \in [0, 1/2], \\ \left(\frac{41}{46} D^{13/7} + \frac{5}{46} D^{8/7} \right) y(t) = \frac{t}{4} + \tan^{-1} \left(\frac{|x(t)|}{3} \right) + \frac{1}{8} \left(\frac{y^2(t) + 2|y(t)|}{1+|y(t)|} \right), \quad t \in [0, 1/2], \\ x(0) = 0, \quad \frac{13}{31} D^{5/9} x \left(\frac{1}{2} \right) + \frac{18}{31} D^{4/9} x \left(\frac{1}{2} \right) = \frac{3}{4}, \\ y(0) = 0, \quad \frac{6}{11} I^{5/2} y \left(\frac{1}{2} \right) + \frac{5}{11} I^{7/2} y \left(\frac{1}{2} \right) = \frac{2}{3}. \end{array} \right. \quad (20)$$

Here $\lambda = 49/53$, $\alpha = 17/9$, $\beta = 10/9$, $T = 1/2$, $\lambda_1 = 41/46$, $\alpha_1 = 13/7$, $\beta_1 = 8/7$, $\mu = 13/31$, $\gamma_1 = 5/9$, $\gamma_2 = 4/9$, $\gamma_3 = 3/4$, $\mu_1 = 6/11$, $\delta_1 = 5/2$, $\delta_2 = 7/2$, $\delta_3 = 2/3$, $f(t, x, y) = ((t+1)/2) + ((|x|e^{-t^2})/(2(1+|x|))) + ((\sin |y| \cos 2\pi t)/(3))$ and $g(t, x, y) = (t/4) + \tan^{-1}(|x|/3) + ((y^2 + 2|y|)/(8(1+|y|)))$. From above information, we can calculate that $\Lambda_1 = 0.7921804090$, $\Lambda_2 = 0.004528637717$, $M_1 = 0.3706636539$, $N_1 = 0.09832444532$, $M_2 = 0.4209829845$ and $N_2 = 0.1927580748$. It is easy to see that

$$|f(t, x, y) - f(t, u, v)| \leq \frac{1}{2}|x - u| + \frac{1}{3}|y - v|,$$

and

$$|g(t, x, y) - g(t, u, v)| \leq \frac{1}{3}|x - u| + \frac{1}{4}|y - v|.$$

Putting $m_1 = 1/2$, $m_2 = 1/3$, $n_1 = 1/3$ and $n_2 = 1/4$, we deduce that

$$M_1(m_1 + m_2) + N_1 + M_2(n_1 + n_2) + N_2 = 0.8455423059 < 1.$$

Hence, by using Theorem 3.3, the boundary value problem (20) has a unique solution on $[0, 1/2]$.

References

- [1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.
- [3] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley and Sons, New York, 1993.
- [4] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.
- [5] S. Liang, J. Zhang, Existence of multiple positive solutions for m -point fractional boundary value problems on an infinite interval, *Math. Comput. Modelling* **54** (2011) 1334-1346.
- [6] C. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, *Comput. Math. Appl.* **61** (2011), 191-202.
- [7] J.R. Wang, Y. Zhou, M. Feckan, On recent developments in the theory of boundary value problems for impulsive fractional differential equations, *Comput. Math. Appl.* **64** (2012), 3008-3020.
- [8] K.Q. Lan, W. Lin, Positive solutions of systems of Caputo fractional differential equations, *Commun. Appl. Anal.* **17** (2013), 61-85.
- [9] C. Zhai, L. Xu, Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter, *Commun. Nonlinear Sci. Numer. Simul.* **19** (2014), 2820-2827.
- [10] J. Henderson, N. Kosmatov, Eigenvalue comparison for fractional boundary value problems with the Caputo derivative, *Fract. Calc. Appl. Anal.* **17** (2014), 872-880.
- [11] Y. Ding, Z. Wei, J. Xu, D. O'Regan, Extremal solutions for nonlinear fractional boundary value problems with p -Laplacian, *J. Comput. Appl. Math.* **288** (2015), 151-158.
- [12] L. Peng, Y. Zhou, Bifurcation from interval and positive solutions of the three-point boundary value problem for fractional differential equations, *Appl. Math. Comput.* **257** (2015), 458-466.

- [13] B. Ahmad, S.K. Ntouyas, Some fractional-order one-dimensional semi-linear problems under non-local integral boundary conditions, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM* **110** (2016), 159-172.
- [14] I. M. Sokolov, J. Klafter, A. Blumen, Fractional kinetics, *Phys. Today* **55** (2002), 48-54.
- [15] I. Petras, R.L. Magin, Simulation of drug uptake in a two compartmental fractional model for a biological system, *Commun Nonlinear Sci Numer Simul.* **16** (2011), 4588-4595.
- [16] Y. Ding, Z. Wang, H. Ye, Optimal control of a fractional-order HIV-immune system with memory, *IEEE Trans. Contr. Sys. Techn.* **20** (2012), 763-769.
- [17] A.A.M. Arafa, S.Z. Rida, M. Khalil, Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection, *Nonlinear Biomed Phys.* 2012, 6 (2012).
- [18] A. Carvalho, C.M.A. Pinto, A delay fractional order model for the co-infection of malaria and HIV/AIDS, *Int. J. Dynam. Control* (2016) DOI 10.1007/s40435-016-0224-3.
- [19] M. Javidi, B. Ahmad, Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system, *Ecological Modelling* **318** (2015), 8-18.
- [20] Z.M. Ge, C.Y. Ou, Chaos synchronization of fractional order modified Duffing systems with parameters excited by a chaotic signal, *Chaos Solitons Fractals* **35** (2008), 705-717.
- [21] M. Faieghi, S. Kuntanapreeda, H. Delavari, D. Baleanu, LMI-based stabilization of a class of fractional-order chaotic systems, *Nonlinear Dynam.* **72** (2013), 301-309.
- [22] F. Zhang, G. Chen C. Li, J. Kurths, Chaos synchronization in fractional differential systems, *Phil Trans R Soc A* **371** (2013), 20120155.
- [23] B. Senol, C. Yeroglu, Frequency boundary of fractional order systems with nonlinear uncertainties, *J. Franklin Inst.* **350** (2013), 1908-1925.
- [24] J. Henderson, R. Luca, Nonexistence of positive solutions for a system of coupled fractional boundary value problems, *Bound. Value Probl.* (2015), **2015:138**.
- [25] B. Ahmad, S.K. Ntouyas, Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, *Appl. Math. Comput.* **266** (2015), 615-622.
- [26] J.R. Wang, Y. Zhang, Analysis of fractional order differential coupled systems, *Math. Methods Appl. Sci.* **38** (2015), 3322-3338.
- [27] J. Tariboon, S.K. Ntouyas, W. Sudsutad, Coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions, *J. Nonlinear Sci. Appl.* **9** (2016), 295-308.
- [28] B. Ahmad, S.K. Ntouyas, A. Alsaedi, On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions, *Chaos Solitons Fractals* **83** (2016), 234-241.
- [29] S. Niyom, S.K. Ntouyas, S. Laoprasittichok, J. Tariboon, Boundary value problems with four orders of Riemann-Liouville fractional derivatives, *Adv. Difference Equ.* (2016) **2016:165**.
- [30] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2005.

6.3 Anti-periodic boundary value problems for generalized Sturm-Liouville and Langevin fractional differential inclusions via Hadamard derivative

In this section, we study a new class of generalized Sturm-Liouville and Langevin fractional differential inclusions via Hadamard fractional derivatives with anti-periodic boundary conditions. Some new existence results for convex as well as non-convex multivalued maps are obtained by using standard fixed point theorems. Enlighten examples illustrating the obtained results are also presented.

1 Introduction

Fractional calculus provides an excellent tool for describing the hereditary property of various material and processes. In addition, fractional differential equations have been of great interest recently. It owes to the intensive development of the theory of fractional calculus as well as its applications in various scientific fields such as physics, biomathematics, blood flow phenomena, ecology, environmental issues, viscoelasticity, aerodynamics, electro-dynamics of complex medium, electrical circuits, electron-analytical chemistry, control theory, etc. A comprehensive study of fractional calculus and its applications is introduced in several books (see (Podlubny, I. , 1999), (Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J., 2006), (Diethelm, K., 2010)). For some recent results on fractional differential equations we refer in a series of papers ((Agarwal, R. P., Zhou, Y., and He, Y., 2010), (Ahmad, B., Ntouyas, S. K., and Tariboon, J. , 2015), (Ahmad, B., and Nieto, J. J., 2011), (Ahmad, B., Ntouyas, S. K., and Alsaedi, A., 2011), (Tariboon, J., Ntouyas, S. K., and Thiramanus, P. , 2016), (Ahmad, B., Ntouyas, and S. K., Alsaedi, A. , 2013), (Ahmad, B., and Nieto, J. J. , 2013), (Zhang, L., Ahmad, B., Wang, G.,

and Agarwal, R. P. , 2013), (Liu, X., Jia, M., and Ge, W., 2013)).

Recently, many researchers have focused their attention in fractional Sturm-Liouville problem, see (Al-Mdallal, Q. M., 2009), (Klimek, M., and Agarwal, O. P., 2013), (Rivero, M., Trujillo, J. J., and Velasco, M. P. , 2013) and references therein.

The subject of anti-periodic boundary value problems, has been studied extensively by many authors, for instance, (Ahn, C., and Rim, C., 1999), (Agarwal, R. P., and Ahmad, B., 2011), (Ahmad, B., and Nieto, J. J. , 2010), (Liu, B., 2009), (Wang, G., Ahmad, B., and Zhang, L., 2011), (Chen, T., and Liu, W. , 2012). This kind of problems widely occur in various fields,

such as thermodynamics, wave propagation and others.

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments (Coffey, W. T., Kalmykov, Y. .P., and Waldron, J. T. , 2004). For some new developments on the fractional Langevin equation, see, for example, (Ahmad, B., and Elloe, P. W. , 2010), (Ahmad, B., Nieto, J. J., Alsaedi, A., and El-Shahed, M. , 2012), (Alsaedi, A. Ntouyas, S. K., and Ahmad, B. , 2013), (Tariboon,J., and Ntouyas, S. K., 2014), (Tariboon, J., Ntouyas, S. K., and Thaiprayoon, C., 2014), (Yukunthorn, W., Ntouyas, S. K., and Tariboon, J., 2014).

In this paper we study the existence of solutions for the generalized Sturm-Liouville and Langevin fractional differential inclusions of Hadamard type, with anti-periodic boundary conditions of the form

$$\begin{cases} D^\beta([p(t)D^\alpha + r(t)]x(t)) \in F(t, x(t)), & 1 < t < T, \\ x(1) = -x(T), \quad D^\alpha x(1) = -D^\alpha x(T), \end{cases} \quad (1.1)$$

where D^ρ denotes the Caputo-type Hadamard fractional derivative of order ρ , $\rho \in \{\alpha, \beta\}$ with $0 < \alpha, \beta < 1$, $p \in C([1, T], \mathbb{R})$ with $|p(t)| \geq K > 0$, $r \in C([1, T], \mathbb{R})$ and $F : [1, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Observe that:

- If $r(t) \equiv 0$ for all $t \in [1, T]$, then problem (1.1) is reduced to the Sturm-Liouville fractional

boundary value problem of Hadamard type of the form

$$\begin{cases} D^\beta(p(t)D^\alpha x(t)) \in F(t, x(t)), & 1 < t < T, \\ x(1) = -x(T), \quad D^\alpha x(1) = -D^\alpha x(T). \end{cases} \quad (1.2)$$

- If $p(t) \equiv 1$ and $r(t) \equiv \lambda$, $\lambda \in \mathbb{R}$, for $t \in [1, T]$, then problem (1.1) is reduced to

$$\begin{cases} D^\beta(D^\alpha + \lambda)x(t) \in F(t, x(t)), & 1 < t < T, \\ x(1) = -x(T), \quad D^\alpha x(1) = -D^\alpha x(T), \end{cases} \quad (1.3)$$

which is the Langevin fractional boundary value problem.

The present paper is motivated by a recent paper (Kiataramkul, C., Ntouyas, S. K., Tariboon, J., and Kijjathanakorn, A., 2016), where it is considered problem (1.1) with F single valued. Existence and uniqueness results were proved in (Kiataramkul, C., Ntouyas, S. K., Tariboon, J., Kijjathanakorn, A., 2016) by using a variety of fixed point theorems, such as Banach contraction principle, Leray-Schauder nonlinear alternative and Krasnoselskii fixed point theorem. Here we cover the multi-valued case.

We establish some existence results for the problem (1.1), when the right hand side is convex as well as non-convex valued. In the first result we use the nonlinear alternative of Leray-Schauder type while in the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. The third result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. Examples illustrating the obtained results are also presented. The methods used are well known, however their exposition in the framework of problem (1.1) is new.

2 Preliminaries

2.1 Basic material of fractional calculus

In this section, we introduce some notations and definitions of fractional calculus (Podlubny, I., 1999), (Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., 2006) and present preliminary results needed in our proofs later.

Definition 2.1. For an at least n -times differentiable function $f : [1, \infty) \rightarrow \mathbb{R}$, the Caputo-type Hadamard derivative of fractional order α is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{n - \alpha - 1} \delta^n f(s) \frac{ds}{s}, \quad n - 1 < \alpha < n, \quad n = [\alpha] + 1,$$

where $\delta = t \frac{d}{dt}$, $\log(\cdot) = \log_e(\cdot)$, $[\alpha]$ denotes the integer part of the real number α .

Definition 2.2. The Hadamard fractional integral of order α is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha - 1} f(s) \frac{ds}{s}, \quad \alpha > 0,$$

provided the integral exists.

Lemma 2.1. (Jarad, F., Abdeljawad, T., and Baleanu, D., 2012) Let $u \in AC_\delta^n[a, b]$ or $C_\delta^n[a, b]$ and $\alpha \in \mathbb{C}$, where $X_\delta^n[a, b] = \{f : [a, b] \rightarrow \mathbb{C} : \delta^{n-1}f(t) \in X[a, b]\}$. Then, we have

$$I^\alpha(D^\alpha)u(t) = u(t) - \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a} \right)^k,$$

where $c_k \in \mathbb{R}$, $k = 0, 1, 2, \dots, n-1$, ($n = [\alpha] + 1$).

For the sake of convenience, we set constants

$$\mu = \frac{p(1)}{p(T)}, \quad \xi = \mu r(T) - r(1) \quad \text{and} \quad \eta = I^\alpha \left(\frac{1}{p} \right) (T). \quad (2.1)$$

Observe that $\mu > 0$ and $\eta \neq 0$.

Lemma 2.2. (Kiataramkul, C., Ntouyas, S. K., Tariboon, J., Kijjathanakorn, A., 2016) The problem

$$\begin{cases} D^\beta([p(t)D^\alpha + r(t)]x(t)) = g(t), & 1 < t < T, \\ x(1) = -x(T), \quad D^\alpha x(1) = -D^\alpha x(T), \end{cases} \quad (2.2)$$

is equivalent to the following fractional integral equation

$$\begin{aligned} x(t) = & I^\alpha \left(\frac{1}{p} I^\beta(g) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(g)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ & - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(g) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(g)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right], \end{aligned} \quad (2.3)$$

where μ , ξ and η are defined by (2.1).

2.2 Basic material for multivalued maps

Here we outline some basic concepts of multivalued analysis (Deimling, K., 1992), (Hu, Sh., and Papageorgiou, N., 1997).

Let $\mathcal{C} := C([1, T], \mathbb{R})$ denote the Banach space of all continuous functions from $[1, T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|, t \in [1, T]\}$. Also by $L^1([1, T], \mathbb{R})$ we denote the space of functions $x : [1, T] \rightarrow \mathbb{R}$ such that $\|x\|_{L^1} = \int_1^T |x(t)| dt$.

For a normed space $(X, \|\cdot\|)$, let

$$\begin{aligned} \mathcal{P}_{cl}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}, \\ \mathcal{P}_b(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}, \\ \mathcal{P}_{cl,b}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}, \\ \mathcal{P}_{cp}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}, \text{ and} \\ \mathcal{P}_{cp,c}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}. \end{aligned}$$

A multi-valued map $G : X \rightarrow \mathcal{P}(X)$:

- (i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$.
- (ii) is *bounded on bounded sets* if $G(Y) = \cup_{x \in Y} G(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$ (i.e. $\sup_{x \in Y} \{\sup\{|y| : y \in G(x)\}\} < \infty$).

- (iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X , and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$.
- (iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap Y \neq \emptyset\}$ is open for any open set Y in X .
- (v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$; If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.
- (vi) is said to be *measurable* if for every $y \in X$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

- (vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

3 Existence results

To simplify the notations, we use in the following $p_* = \inf_{t \in [1, T]} |p(t)|$, $r^* = \sup_{t \in [1, T]} |r(t)|$ and set

$$\Omega_1 = \frac{3(\log T)^{\alpha+\beta}}{2p_*\Gamma(\alpha+\beta+1)} + \frac{3\mu|\eta|(\log T)^\beta}{2(\mu+1)\Gamma(\beta+1)}, \quad (3.1)$$

$$\Omega_2 = \frac{3r^*(\log T)^\alpha}{2p_*\Gamma(\alpha+1)} + \frac{3|\xi||\eta|}{2(\mu+1)}. \quad (3.2)$$

Since $|p(t)| \geq K > 0$, then we have $p_* > 0$ which implies that Ω_1 and Ω_2 are well defined.

Definition 3.1. A function $x \in C^2([1, T], \mathbb{R})$ is a solution of the problem (1.1) if $x(1) = -x(T)$, $D^\alpha x(1) = -D^\alpha x(T)$, and there exists a function $v \in L^1([1, T], \mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e. on $[1, T]$ and

$$\begin{aligned} x(t) = & I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ & - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right], \end{aligned}$$

where μ , ξ and η are defined by (2.1).

3.1 The Carathéodory case

In this subsection we consider the case when F has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

Definition 3.2. A multivalued map $F : [1, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:

- (i) $t \mapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$;
- (ii) $x \mapsto F(t, x)$ is upper semicontinuous for almost all $t \in [1, T]$.

Further a Carathéodory function F is called L^1 –Carathéodory if

- (iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([1, T], \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [1, T]$.

For each $y \in \mathcal{C}$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([1, T], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ on } [1, T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 3.1. ((Deimling, K., 1992, Proposition 1.2)) *If $G : X \rightarrow \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ and $\{y_n\}_{n \in \mathbb{N}} \subset Y$, if when $n \rightarrow \infty$, $x_n \rightarrow x_*$, $y_n \rightarrow y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 3.2. ((Lasota, A., and Opial, Z., 1965)) *Let X be a Banach space. Let $F : J \times \mathbb{R} \rightarrow \mathcal{P}_{cp,c}(X)$ be an L^1 –Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Theta \circ S_F : C(J, X) \rightarrow \mathcal{P}_{cp,c}(C(J, X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 3.3. (Nonlinear alternative for Kakutani maps)(Granas, A., and Dugundji, J., 2005). *Let E be a Banach space, C a closed convex subset of E , U an open subset of C and $0 \in U$. Suppose that $F : \bar{U} \rightarrow \mathcal{P}_{cp,c}(C)$ is an upper semicontinuous compact map. Then either*

- (i) F has a fixed point in $\bar{U}$, or
- (ii) there is a $u \in \partial U$ and $\nu \in (0, 1)$ with $u \in \nu F(u)$.

Theorem 3.4. *Assume that:*

- (H₁) $F : [1, T] \times \mathbb{R} \rightarrow \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1 –Carathéodory;
- (H₂) there exist a continuous nondecreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ and a function $\phi \in C([1, T], \mathbb{R}^+)$ such that

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \leq \phi(t)\psi(\|x\|) \quad \text{for each } (t, x) \in [1, T] \times \mathbb{R};$$

(H₃) there exists a constant $M > 0$ such that

$$\frac{(1 - \Omega_2)M}{\|\phi\|\psi(M)\Omega_1} > 1, \quad \Omega_2 < 1,$$

where Ω_1 and Ω_2 are defined by (3.1) and (3.2) respectively.

Then the boundary value problem (1.1) has at least one solution on $[1, T]$.

Proof. Define the operator $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ by

$$\mathcal{F}(x) = \left\{ \begin{array}{l} h \in \mathcal{C} : \\ h(t) = \left\{ \begin{array}{l} I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) \\ + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) \right. \\ \left. + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right], \end{array} \right. \end{array} \right. \quad (3.3)$$

for $v \in S_{F,x}$. It is obvious that the fixed points of $\mathcal{F}$ are solutions of the boundary value problem (1.1).

We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder Nonlinear alternative (Lemma 3.3). The proof consists of several steps.

Step 1. $\mathcal{F}(x)$ is convex for each $x \in \mathcal{C}$.

This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Step 2. $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}$.

For $\rho > 0$, we define a bounded ball $B_\rho = \{x \in C([1, T], \mathbb{R}) : \|x\| \leq \rho\}$. Then, for each $h \in \mathcal{F}(x)$, $x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right]. \end{aligned}$$

Then we have

$$\begin{aligned} |h(t)| &\leq I^\alpha \left(\frac{1}{|p|} I^\beta(|v|) \right) (t) + I^\alpha \left(\frac{|r|}{|p|} |x| \right) (t) + \left(\frac{\mu}{(\mu+1)} I^\beta(|v|)(T) \right. \\ &\quad \left. + \frac{|\xi|}{(\mu+1)} |x(T)| \right) I^\alpha \left(\frac{1}{|p|} \right) (t) + \frac{1}{2} \left[I^\alpha \left(\frac{1}{|p|} I^\beta |v| \right) (T) \right. \\ &\quad \left. + I^\alpha \left(\frac{|r|}{|p|} |x| \right) (T) + \left(\frac{\mu}{(\mu+1)} I^\beta(|v|)(T) + \frac{|\xi|}{(\mu+1)} |x(T)| \right) |\eta| \right] \\ &\leq \frac{\|\varphi\|\psi(\|x\|)}{p_*} I^{\alpha+\beta}(1)(T) + \frac{\rho r^*}{p_*} I^\alpha(1)(T) + \frac{\mu|\eta|\|\varphi\|\psi(\|x\|)}{(\mu+1)} I^\beta(1)(T) \\ &\quad + \frac{|\eta||\xi|\rho}{(\mu+1)} + \frac{\|\varphi\|\psi(\|x\|)}{2p_*} I^{\alpha+\beta}(1)(T) + \frac{\rho r^*}{2p_*} I^\alpha(1)(T) \\ &\quad + \frac{\mu|\eta|\|\varphi\|\psi(\|x\|)}{2(\mu+1)} I^\beta(1)(T) + \frac{|\eta||\xi|\rho}{2(\mu+1)} \\ &\leq \|\varphi\|\psi(\rho)\Omega_1 + \rho\Omega_2, \end{aligned}$$

which leads to

$$\|h\| \leq \|\varphi\|\psi(\rho)\Omega_1 + \rho\Omega_2.$$

Step 3. $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.

Let $\tau_1, \tau_2 \in [1, T]$ such that $\tau_1 < \tau_2$ and $x \in B_\rho$. For each $h \in \mathcal{F}(x)$, we obtain

$$\begin{aligned} & |h(\tau_2) - h(\tau_1)| \\ & \leq \frac{\|\varphi\|\psi(\rho)}{p_*\Gamma(\alpha + \beta)} \left[\int_1^{\tau_2} \left(\log \frac{\tau_2}{s} \right)^{\alpha + \beta - 1} \frac{ds}{s} - \int_1^{\tau_1} \left(\log \frac{\tau_1}{s} \right)^{\alpha + \beta - 1} \frac{ds}{s} \right] \\ & \quad + \frac{\rho r^*}{p_*\Gamma(\alpha)} \left[\int_1^{\tau_2} \left(\log \frac{\tau_2}{s} \right)^{\alpha - 1} \frac{ds}{s} - \int_1^{\tau_1} \left(\log \frac{\tau_1}{s} \right)^{\alpha - 1} \frac{ds}{s} \right] \\ & \quad + \left(\frac{\mu\|\varphi\|\psi(\rho)(\log T)^\beta}{|\mu + 1|\Gamma(\beta + 1)} + \frac{\rho|\xi|}{|\mu + 1|} \right) \cdot \frac{1}{p_*\Gamma(\alpha)} \\ & \quad \times \left[\int_1^{\tau_2} \left(\log \frac{\tau_2}{s} \right)^{\alpha - 1} \frac{ds}{s} - \int_1^{\tau_1} \left(\log \frac{\tau_1}{s} \right)^{\alpha - 1} \frac{ds}{s} \right] \\ & = \frac{\|\varphi\|\psi(\rho)}{p_*\Gamma(\alpha + \beta + 1)} \left| (\log \tau_2)^{\alpha + \beta} - (\log \tau_1)^{\alpha + \beta} \right| + \frac{\rho r^*}{p_*\Gamma(\alpha + 1)} \left| (\log \tau_2)^\alpha - (\log \tau_1)^\alpha \right| \\ & \quad + \left(\frac{\mu\|\varphi\|\psi(\rho)(\log T)^\beta}{|\mu + 1|\Gamma(\beta + 1)} + \frac{\rho|\xi|}{|\mu + 1|} \right) \cdot \frac{1}{p_*\Gamma(\alpha + 1)} \cdot \left| (\log \tau_2)^\alpha - (\log \tau_1)^\alpha \right|. \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $\tau_2 - \tau_1 \rightarrow 0$. As $\mathcal{F}$ satisfies the above three assumptions, therefore it follows by the Ascoli-Arzelà theorem that $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ is completely continuous.

Since $\mathcal{F}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph (Lemma 3.1). Thus, in our next step, we show that

Step 4. $\mathcal{F}$ has a closed graph.

Let $x_n \rightarrow x_*$, $h_n \in \mathcal{F}(x_n)$ and $h_n \rightarrow h_*$. Then we need to show that $h_* \in \mathcal{F}(x_*)$. Associated with $h_n \in \mathcal{F}(x_n)$, there exists $v_n \in S_{F, x_n}$ such that for each $t \in [1, T]$,

$$\begin{aligned} h_n(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v_n) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu + 1} I^\beta(v_n)(T) + \frac{\xi}{\mu + 1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v_n) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu + 1} I^\beta(v_n)(T) + \frac{\xi}{\mu + 1} x(T) \right) \eta \right]. \end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F, x_*}$ such that for each $t \in [1, T]$,

$$\begin{aligned} h_*(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v_*) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu + 1} I^\beta(v_*)(T) + \frac{\xi}{\mu + 1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v_*) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu + 1} I^\beta(v_*)(T) + \frac{\xi}{\mu + 1} x(T) \right) \eta \right]. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1([1, T], \mathbb{R}) \rightarrow \mathcal{C}$ given by

$$\begin{aligned} v \mapsto \Theta(v)(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu + 1} I^\beta(v)(T) + \frac{\xi}{\mu + 1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu + 1} I^\beta(v)(T) + \frac{\xi}{\mu + 1} x(T) \right) \eta \right]. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| &= \left\| I^\alpha \left(\frac{1}{p} I^\beta(v_n(s) - v_*(s)) \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v_n(s) - v_*(s)(T)) \right) I^\alpha \left(\frac{1}{p} \right) (t) \right. \\ &\quad \left. - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v_n(s) - v_*(s)) \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v_n(s) - v_*(s)(T)) \right) \eta \right] \right\| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$.

Thus, it follows by Lemma 3.2 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \rightarrow x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v_*) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v_*)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v_*) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v_*)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right], \end{aligned}$$

for some $v_* \in S_{F,x_*}$.

Step 5. We show there exists an open set $U \subseteq \mathcal{C}$ with $x \notin \theta \mathcal{F}(x)$ for any $\theta \in (0, 1)$ and all $x \in \partial U$.

Let $\theta \in (0, 1)$ and $x \in \theta \mathcal{F}(x)$. Then there exists $v \in L^1([1, T], \mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [1, T]$, we have

$$\begin{aligned} x(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right]. \end{aligned}$$

Using the computations of the second step above we have

$$|x(t)| \leq \|\varphi\| \psi(\|x\|) \Omega_1 + \|x\| \Omega_2,$$

which yields

$$\frac{(1 - \Omega_2)\|x\|}{\|\varphi\| \psi(\|x\|) \Omega_1} \leq 1,$$

where constants Ω_1 and Ω_2 are defined by (3.1) and (3.2), respectively.

In view of (H_3) , there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in \mathcal{C} : \|x\| < M\}.$$

Note that the operator $\mathcal{F} : \overline{U} \rightarrow \mathcal{P}(\mathcal{C})$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U , there is no $x \in \partial U$ such that $x \in \theta \mathcal{F}(x)$ for some $\theta \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.3), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. $\square$

If $r(t) \equiv 0$ for $t \in [1, T]$, then $\xi = 0$, $r^* = 0$ which imply $\Omega_2 = 0$. If $p(t) \equiv 1$ and $r(t) \equiv \lambda$ for $t \in [1, T]$, then $p_* = 1$, $r^* = |\lambda|$, $\mu = 1$, $\xi = 0$ and $\eta = (\log T)^\alpha / \Gamma(\alpha + 1)$. By setting constants

$$\omega_1 = \frac{3(\log T)^{\alpha+\beta}}{2\Gamma(\alpha + \beta + 1)} + \frac{3(\log T)^{\alpha+\beta}}{4\Gamma(\alpha + 1)\Gamma(\beta + 1)}, \quad \omega_2 = \frac{3|\lambda|(\log T)^\alpha}{2\Gamma(\alpha + 1)}, \quad (3.4)$$

we obtain the following corollary.

Corollary 3.5. *Assume that the conditions (H_1) and (H_2) of Theorem 3.4 are satisfied.*

(i) *If there exists a constant M such that*

$$\frac{M}{\|\varphi\|\psi(M)\Omega_1} > 1,$$

where Ω_1 is defined in (3.1), then the Sturm-Liouville boundary value problem (1.2) has at least one solution on $[1, T]$.

(ii) *If there exists a constant N such that*

$$\frac{(1 - \omega_2)N}{\|\varphi\|\psi(N)\omega_1} > 1,$$

where ω_1 and ω_2 are defined in (3.4), then the Langevin boundary value problem (1.3) has at least one solution on $[1, T]$.

3.2 The lower semicontinuous case

In the next result, F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo (Bressan, A., and Colombo, G., 1988) for lower semi-continuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and $G : X \rightarrow \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E . Let A be a subset of $[1, T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ -algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[1, T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([1, T], \mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [1, T] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.3. Let Y be a separable metric space and let $N : Y \rightarrow \mathcal{P}(L^1([1, T], \mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [1, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([1, T] \times \mathbb{R}) \rightarrow \mathcal{P}(L^1([1, T], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([1, T], \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [1, T]\},$$

which is called the Nemytskii operator associated with F .

Definition 3.4. Let $F : [1, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 3.6. ((Frigon, M., 1995)) Let Y be a separable metric space and let $N : Y \rightarrow \mathcal{P}(L^1([1, T], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \rightarrow L^1([1, T], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.

Theorem 3.7. Assume that (H_2) , (H_3) and the following condition holds:

(H_4) $F : [1, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that

- (a) $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,
- (b) $x \mapsto F(t, x)$ is lower semicontinuous for each $t \in [1, T]$;

Then the boundary value problem (1.1) has at least one solution on $[1, T]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then from Lemma 3.6, there exists a continuous function $f : C^2([1, T], \mathbb{R}) \rightarrow L^1([1, T], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([1, T], \mathbb{R})$.

Consider the problem

$$\begin{cases} D^\beta([p(t)D^\alpha + r(t)]x(t)) = f(x(t)), & 1 < t < T, \\ x(1) = -x(T), \quad D^\alpha x(1) = -D^\alpha x(T). \end{cases} \quad (3.5)$$

Observe that if $x \in C^2([1, T], \mathbb{R})$ is a solution of (3.5), then x is a solution to the problem (1.1). In order to transform the problem (3.5) into a fixed point problem, we define the operator $\overline{\mathcal{F}}$ as

$$\begin{aligned} \overline{\mathcal{F}}x(t) &= I^\alpha \left(\frac{1}{p} I^\beta(f(x(t))) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) \\ &+ \left(\frac{-\mu}{\mu+1} I^\beta(f(x(t)))(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &- \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(f(x(t))) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) \right. \\ &\left. + \left(\frac{-\mu}{\mu+1} I^\beta(f(x(t)))(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right]. \end{aligned}$$

It can easily be shown that $\overline{\mathcal{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.4. So we omit it. This completes the proof. $\square$

3.3 The Lipschitz case

In this subsection we prove the existence of solutions for the problem (1.1) with a not necessary nonconvex valued right hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler (Covitz, H., and Nadler S. B., Jr., 1970).

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. Then $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space (see (Kisielewicz, M., 1991)).

Definition 3.5. A multivalued operator $N : X \rightarrow \mathcal{P}_{cl}(X)$ is called

(a) γ -Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X;$$

(b) a contraction if and only if it is γ -Lipschitz with $\gamma < 1$.

Lemma 3.8. ((Covitz, H., Nadler S. B., Jr., 1970)) Let (X, d) be a complete metric space. If $N : X \rightarrow \mathcal{P}_{cl}(X)$ is a contraction, then $Fix N \neq \emptyset$.

Theorem 3.9. Assume that:

(A₁) $F : [1, T] \times \mathbb{R} \rightarrow \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [1, T] \rightarrow \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;

(A₂) $H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x - \bar{x}|$ for almost all $t \in [1, T]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([1, T], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq m(t)$ for almost all $t \in [1, T]$.

Then the boundary value problem (1.1) has at least one solution on $[1, T]$ if

$$\frac{3}{2} \left\{ \frac{1}{p_*} I^{\alpha+\beta}(m(s))(T) + \frac{\mu|\eta|}{(\mu+1)} I^\beta(m(s))(T) \right\} < 1.$$

Proof. Consider the operator $\mathcal{F}$ defined by (3.3). Observe that the set $S_{F,x}$ is nonempty for each $x \in \mathcal{C}$ by the assumption (A₁), so F has a measurable selection (see Theorem III.6 (Castaing, C., and Valadier, M., 1977)). Now we show that the operator $\mathcal{F}$ satisfies the assumptions of Lemma 3.8. We show that $\mathcal{F}(x) \in \mathcal{P}_{cl}(\mathcal{C})$ for each $x \in \mathcal{C}$. Let $\{u_n\}_{n \geq 0} \in \mathcal{F}(x)$ be such that $u_n \rightarrow u$ ($n \rightarrow \infty$) in $\mathcal{C}$. Then $u \in \mathcal{C}$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [1, T]$,

$$\begin{aligned} u_n(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v_n) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v_n)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v_n) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v_n)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right]. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([1, T], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [1, T]$, we have

$$\begin{aligned} u_n(t) \rightarrow v(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right]. \end{aligned}$$

Hence, $u \in \mathcal{F}(x)$.

Next we show that there exists $\delta < 1$ such that

$$H_d(\mathcal{F}(x), \mathcal{F}(\bar{x})) \leq \delta \|x - \bar{x}\| \text{ for each } x, \bar{x} \in C^2([1, T], \mathbb{R}).$$

Let $x, \bar{x} \in C^2([1, T], \mathbb{R})$ and $h_1 \in \mathcal{F}(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [1, T]$,

$$\begin{aligned} h_1(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v_1) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v_1)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v_1) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v_1)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right]. \end{aligned}$$

By (A_2) , we have

$$H_d(F(t, x), F(t, \bar{x})) \leq m(t) |x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t) |x(t) - \bar{x}(t)|, \quad t \in [1, T].$$

Define $U : [1, T] \rightarrow \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t) |x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 (Castaing, C., Valadier, M., 1977)), there exists a function $v_2(t)$ which is a measurable selection for U . So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [1, T]$, we have $|v_1(t) - v_2(t)| \leq m(t) |x(t) - \bar{x}(t)|$.

For each $t \in [1, T]$, let us define

$$\begin{aligned} h_2(t) &= I^\alpha \left(\frac{1}{p} I^\beta(v_2) \right) (t) - I^\alpha \left(\frac{r}{p} x \right) (t) + \left(\frac{-\mu}{\mu+1} I^\beta(v_2)(T) + \frac{\xi}{\mu+1} x(T) \right) I^\alpha \left(\frac{1}{p} \right) (t) \\ &\quad - \frac{1}{2} \left[I^\alpha \left(\frac{1}{p} I^\beta(v_2) \right) (T) - I^\alpha \left(\frac{r}{p} x \right) (T) + \left(\frac{-\mu}{\mu+1} I^\beta(v_2)(T) + \frac{\xi}{\mu+1} x(T) \right) \eta \right]. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq I^\alpha \left(\frac{1}{|p|} I^\beta |v_1(s) - v_2(s)| \right) (t) \\ &\quad + \left(\frac{\mu}{(\mu+1)} I^\beta |v_1(s) - v_2(s)|(T) \right) I^\alpha \left(\frac{1}{|p|} \right) (t) \\ &\quad + \frac{1}{2} \left[I^\alpha \left(\frac{1}{|p|} I^\beta |v_1(s) - v_2(s)| \right) (T) + \left(\frac{\mu}{(\mu+1)} I^\beta |v_1(s) - v_2(s)|(T) \right) \eta \right] \\ &\leq \frac{1}{p_*} I^{\alpha+\beta}(m(s) \|x - \bar{x}\|)(T) + \frac{\mu|\eta|}{\mu+1} I^\beta(m(s) \|x - \bar{x}\|)(T) \\ &\quad + \frac{1}{2p_*} I^{\alpha+\beta}(m(s) \|x - \bar{x}\|)(T) + \frac{\mu|\eta|}{2(\mu+1)} I^\beta(m(s) \|x - \bar{x}\|)(T) \\ &\leq \frac{3}{2} \left\{ \frac{1}{p_*} I^{\alpha+\beta}(m(s))(T) + \frac{\mu|\eta|}{\mu+1} I^\beta(m(s))(T) \right\} \|x - \bar{x}\|. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \frac{3}{2} \left\{ \frac{1}{p_*} I^{\alpha+\beta}(m(s))(T) + \frac{\mu|\eta|}{(\mu+1)} I^\beta(m(s))(T) \right\} \|x - \bar{x}\|.$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$H_d(\mathcal{F}(x), \mathcal{F}(\bar{x})) \leq \frac{3}{2} \left\{ \frac{1}{p_*} I^{\alpha+\beta}(m(s))(T) + \frac{\mu|\eta|}{(\mu+1)} I^\beta(m(s))(T) \right\} \|x - \bar{x}\|.$$

So $\mathcal{F}$ is a contraction. Therefore, it follows by Lemma 3.8 that $\mathcal{F}$ has a fixed point x which is a solution of (1.1). This completes the proof. $\square$

Corollary 3.10. Assume that the conditions (A_1) and (A_2) of Theorem 3.9 hold. If

$$\frac{3}{2} \left\{ I^{\alpha+\beta}(m(s))(T) + \frac{(\log T)^\alpha}{2\Gamma(\alpha+1)} I^\beta(m(s))(T) \right\} < 1,$$

Then the boundary value problem (1.3) has at least one solution on $[1, T]$.

3.4 Examples

In this section, we will illustrate our main results with the help of some examples.

Example 3.1. Consider the following anti-periodic boundary value problems for generalized Sturm-Liouville and Langevin fractional differential inclusions via Hadamard derivative of the form

$$\begin{cases} D^{4/5} \left(\left[\left(\sqrt{t} + \frac{1}{2} \right) D^{1/3} + \left(\frac{1}{3} \sqrt{t} - \frac{1}{5} \right) \right] x(t) \right) \in F(t, x(t)), & t \in [1, \pi], \\ x(1) = -x(\pi), \quad D^{1/3}x(1) = -D^{1/3}x(\pi), \end{cases} \quad (3.6)$$

where

$$F(t, x) = \left[\frac{1}{t+2} \left(\frac{|x|e^{-x^2}}{1+|x|} + \frac{1}{5} \right), \frac{1}{t+1} \left(\frac{x^2}{3(1+|x|)} + \frac{1}{4} \right) \right].$$

Here $\alpha = 1/3$, $\beta = 4/5$, $p(t) = \sqrt{t} + (1/2)$, $r(t) = (1/3)\sqrt{t} - (1/5)$ and $T = \pi$. From given information, we find that $\mu = 0.6600794112$, $|\xi| = 0.1246375493$, $\eta = 0.5781789415$, $p_* = 3/2$, $r^* = 0.3908179503$, $\Omega_1 = 1.508061720$ and $\Omega_2 = 0.5229401204$. Observe that $\Omega_2 < 1$. It is easy to see that

$$\|F(t, x)\|_{\mathcal{P}} = \sup\{|y| : y \in F(t, x)\} \leq \left(\frac{1}{t+1} \right) \left(\frac{1}{3}x + \frac{1}{4} \right).$$

Choosing $\phi(t) = 1/(t+1)$ and $\psi(x) = (1/3)x + (1/4)$, we deduce that the conditions (H_1) – (H_2) are satisfied. By direct computation, we have that there exists a constant $M > 0.8351534592$ satisfying condition (H_3) of Theorem 3.4. Hence, by Theorem 3.4, the problem (3.6) has at least one solution on $[1, \pi]$.

Example 3.2. Consider the following anti-periodic boundary value problems for generalized Sturm-Liouville and Langevin fractional differential inclusions via Hadamard derivative of the form

$$\begin{cases} D^{3/4} \left(\left[(2t^{3/4} + 3)D^{2/3} + (2t^2 - 1) \right] x(t) \right) \in F(t, x(t)), & t \in [1, 5/2], \\ x(1) = -x(5/2), \quad D^{2/3}x(1) = -D^{2/3}x(5/2), \end{cases} \quad (3.7)$$

where

$$F(t, x) = \left[\left(\frac{\sqrt{t}}{t^2 + 4} + \frac{1}{3} \right) \left(\frac{x^2 + 2|x|}{5(1 + |x|)} \right), \left(\frac{t}{t + 1} + \frac{1}{2} \right) \left(\frac{x^2 + 2|x|}{(1 + |x|)} \right) \right].$$

Here $\alpha = 2/3$, $\beta = 3/4$, $p(t) = 2t^{3/4} + 3$, $r(t) = 2t^2 - 1$ and $T = 5/2$. From the above data, we can find that $\mu = 0.7167067865$, $\eta = 0.1735552755$, $p_* = 5$. Observe that

$$H_d(F(t, x), F(t, \bar{x})) \leq \left(\frac{t}{t + 1} + \frac{1}{2} \right) |x - \bar{x}|.$$

Setting $m(t) = (t/(t + 1)) + (1/2)$, we have $d(0, F(t, 0)) \leq m(t)$. By direct computation, we get $I^{3/4}(m(s))(5/2) = 0.5736418395$ and $I^{17/12}(m(s))(5/2) = 0.3842174774$. Hence

$$\frac{3}{2} \left\{ \frac{1}{p_*} I^{\alpha+\beta}(m(s))(T) + \frac{\mu|\eta|}{(\mu + 1)} I^{\beta}(m(s))(T) \right\} = 0.1776122044 < 1.$$

By Theorem 3.9, we have that problem (3.7) has at least one solution on $[1, 5/2]$.

4 Conclusions

In this paper we studied the existence of solutions for the generalized Sturm-Liouville and Langevin fractional differential inclusions of Hadamard type, with anti-periodic boundary conditions of the form (1.1). We established some existence results for the problem (1.1), when the right hand side is convex as well as non-convex valued. In the case of convex values we used the nonlinear alternative of Leray-Schauder type while in the case when the multi-function is not necessarily convex valued, we proved two results. In the first we combined the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. In the second result we used the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. In all cases we constructed illustrative examples. We emphasize that our results are new and contribute significantly to the existing literature on the topic.

References

Podlubny, I. 1999. *Fractional Differential Equations*, Academic Press, San Diego.

- Kilbas, A. A., Srivastava, H. M., Trujillo, J. J. 2006. *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam.
- Diethelm, K. 2010. *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg.
- Agarwal, R. P., Zhou, Y., He, Y. 2010. Existence of fractional neutral functional differential equations, *Comput. Math. Appl.* **59**: 1095–1100.
- Ahmad, B., Ntouyas, S. K., Tariboon, J. 2015. Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential equations, *Adv. Difference Equ.* **2015**:293.
- Ahmad, B., Nieto, J. J. 2011. Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Bound. Value Probl.* **2011**:36.
- Ahmad, B., Ntouyas, S. K., Alsaedi, A. 2011. New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Difference Equ.* Art. ID 107384, 11pp.
- Tariboon, J., Ntouyas, S. K., Thiramanus, P. 2016. Riemann-Liouville fractional differential equations with Hadamard fractional integral conditions, *Inter. J. Appl. Math. Stat.* **54**: 119–134.
- Ahmad, B., Ntouyas, S. K., Alsaedi, A. 2013. A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions, *Math. Probl. Eng.*, Art. ID 320415, 9 pp.
- Ahmad, B., Nieto, J. J. 2013. Boundary value problems for a class of sequential integrodifferential equations of fractional order, *J. Funct. Spaces Appl.* Art. ID 149659, 8 pp.
- Zhang, L., Ahmad, B., Wang, G., Agarwal, R. P. 2013. Nonlinear fractional integro-differential equations on unbounded domains in a Banach space, *J. Comput. Appl. Math.* **249**: 51–56.
- Liu, X., Jia, M., Ge, W. 2013. Multiple solutions of a p-Laplacian model involving a fractional derivative, *Adv. Difference Equ.*, **2013**:126.
- Al-Mdallal, Q. M. 2009. An efficient method for solving fractional Sturm-Liouville problems, *Chaos, Solitons & Fractals* **40**: 183–189.
- Klimek, M., Agarwal, O. P. 2013. Fractional Sturm-Liouville problem, *Comput. Math. Appl.* **66**: 795–812.
- Rivero, M., Trujillo, J. J., Velasco, M. P. 2013. A fractional approach to the Sturm-Liouville problem, *Cent. Eur. J. Phys.* **11**: 1246–1254.
- Ahn, C., Rim, C. 1999. Boundary flows in general coset theories, *J. Phys.* **A 32**: 2509–2525.
- Agarwal, R. P., Ahmad, B. 2011. Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.* **62**: 1200–1214.

- Ahmad, B., Nieto, J. J. 2010. Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory, *Topol. Methods Nonlinear Anal.* **35**: 295–304.
- Liu, B. 2009. Anti-periodic solutions for forced Rayleigh-type equations, *Nonlinear Anal. RWA* **10**: 2850–2856.
- Wang, G., Ahmad, B., Zhang, L. 2011. Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order, *Nonlinear Anal. TMA* **74**: 792–804.
- Chen, T., Liu, W. 2012. An anti-periodic boundary value problem for the fractional differential equation with a p -Laplacian operator, *Appl. Math. Lett.* **11**: 1671–1675
- Coffey, W. T., Kalmykov, Y. .P., Waldron, J. T. 2004. *The Langevin Equation*, second ed., World Scientific, Singapore.
- Ahmad, B., Eloe, P. W. 2010. A nonlocal boundary value problem for a nonlinear fractional differential equation with two indices, *Commun. Appl. Nonlinear Anal.* **17**: 69–80.
- Ahmad, B., Nieto, J. J., Alsaedi, A., El-Shahed, M. 2012. A study of nonlinear Langevin equation involving two fractional orders in different intervals, *Nonlinear Anal. Real World Appl.* **13**: 599–606.
- Alsaedi, A., Ntouyas, S. K., Ahmad, B. 2013. Existence results for Langevin fractional differential inclusions involving two fractional orders with four-point multi-term fractional integral boundary conditions, *Abstr. Appl. Anal.*, Article ID 869837, 17 pages.
- Tariboon, J., Ntouyas, S. K. 2014. Nonlinear second-order impulsive q -difference Langevin equation with boundary conditions, *Bound. Value Probl.*, **2014**: 85.
- Tariboon, J., Ntouyas, S. K., Thaiprayoon, C. 2014. Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions, *Adv. Math. Phys.*, Article ID 372749, 15 pages.
- Yukunthorn, W., Ntouyas, S. K., Tariboon, J. 2014. Nonlinear fractional Caputo-Langevin equation with nonlocal Riemann-Liouville fractional integral conditions, *Adv. Difference Equ.*, **2014**:315.
- Kiataramkul, C., Ntouyas, S. K., Tariboon, J., Kijjathanakorn, A. 2016. Generalized Sturm-Liouville and Langevin equations via Hadamard fractional derivatives with anti-periodic boundary conditions, *Bound. Value Probl.*, **2016**:217.
- Jarad, F., Abdeljawad, T., Baleanu, D. 2016. Caputo-type modification of the Hadamard fractional derivatives, *Adv. Difference Equ.*, **2012**:142.
- Deimling, K. 1992. *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York.
- Hu, Sh., Papageorgiou, N. 1997. *Handbook of Multivalued Analysis, Theory I*, Kluwer, Dordrecht.

- Lasota, A., Opial, Z. 1965. An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **13**: 781–786.
- Granas, A., Dugundji, J. 2005. *Fixed Point Theory*, Springer-Verlag, New York.
- Bressan, A., Colombo, G. 1988. Extensions and selections of maps with decomposable values, *Studia Math.* **90**: 69–86.
- Frigon, M. 1995. Théorèmes d'existence de solutions d'inclusions différentielles, *Topological Methods in Differential Equations and Inclusions* (edited by A. Granas and M. Frigon), NATO ASI Series C, Vol. 472, Kluwer Acad. Publ., Dordrecht, 51–87.
- Covitz, H., Nadler S. B., Jr. 1995. Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* **8**: 5–11.
- Kisielewicz, M. 1991. *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands.
- Castaing, C., Valadier, M. 1977. *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin-Heidelberg-New York.

6.4 A coupled system of fractional q -integro-difference equations with nonlocal fractional q -integral boundary conditions

In this **section**, we investigate the existence and the uniqueness of solutions for coupled and uncoupled systems of fractional q -integro-difference equations with nonlocal fractional q -integral boundary conditions. The existence and the uniqueness of the solutions are established by using the Banach contraction principle, while the existence of solutions is derived by applying Leray-Schauder's alternative. Examples illustrating our results are also presented.

1 Introduction

In this paper, we investigate a coupled system of fractional q -integro-difference equations with nonlocal fractional q -integral boundary conditions given by

$$\begin{cases} D_q^\alpha x(t) = f(t, x(t), I_r^\delta y(t)), & t \in [0, T], 1 < \alpha \leq 2, \\ D_p^\beta y(t) = g(t, y(t), I_z^\varepsilon x(t)), & t \in [0, T], 1 < \beta \leq 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa y(\xi), \\ y(0) = 0, & \lambda_2 I_h^\mu y(\theta) = I_k^\nu x(\tau), \end{cases} \quad (1.1)$$

where $0 < p, q, r, z, m, n, h, k < 1$ are quantum numbers, $\eta, \xi, \theta, \tau \in (0, T)$ are fixed points, $\delta, \varepsilon, \gamma, \kappa, \mu, \nu > 0$, and $\lambda_1, \lambda_2 \in \mathbb{R}$ are given constants, D_ω^ρ is the fractional ω -derivative of Riemann-Liouville type of order ρ , when $\rho \in \{\alpha, \beta\}$ and $\omega \in \{p, q\}$, I_ϕ^ψ is the fractional ϕ -integral of order ψ with $\phi \in \{r, z, m, n, h, k\}$ and $\psi \in \{\delta, \varepsilon, \gamma, \kappa, \mu, \nu\}$ and $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

The early work on q -difference calculus or *quantum calculus* dates back to Jackson's paper [1]. Basic definitions and properties of quantum calculus can be found in the book [2]. The fractional q -difference calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. Motivated by recent interest in the study of fractional-order differential equations, the topic of q -fractional equations has attracted the attention of many researchers. The details of some recent development of the subject can be found in [5–18], and the references cited therein, whereas the background material on q -fractional calculus can be found in a recent book [19].

Recently in [20], we have studied the existence and the uniqueness of solutions of a class of boundary value problems for fractional q -integro-difference equations with nonlocal fractional q -integral conditions which have different quantum numbers. Here we extend the results of [20] to a coupled system of fractional q -integro-difference equations with nonlocal fractional q -integral boundary conditions.

The paper is organized as follows: In Section 2 we will present some useful preliminaries and lemmas. Some auxiliary lemmas are presented in Section 3. In Section 4, we establish an existence and a uniqueness result via the Banach contraction principle, and an existence result by applying Leray-Schauder's alternative. Results on the uncoupled integral boundary conditions case are contained in Section 5. Examples illustrating our results are also presented.

2 Preliminaries

To make this paper self-contained, below we recall some well-known facts on fractional q -calculus. The presentation here can be found in, for example, [6, 19].

For $q \in (0, 1)$, define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \quad (2.1)$$

The q -analog of the power function $(a - b)^k$ with $k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ is

$$(a - b)^{(0)} = 1, \quad (a - b)^{(k)} = \prod_{i=0}^{k-1} (a - bq^i), \quad k \in \mathbb{N}, a, b \in \mathbb{R}. \quad (2.2)$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(a - b)^{(\gamma)} = a^\gamma \prod_{i=0}^{\infty} \frac{1 - (b/a)q^i}{1 - (b/a)q^{\gamma+i}}, \quad a \neq 0. \quad (2.3)$$

Note if $b = 0$, then $a^{(\gamma)} = a^\gamma$. We also use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q -gamma function is defined by

$$\Gamma_q(t) = \frac{(1 - q)^{(t-1)}}{(1 - q)^{t-1}}, \quad t \in \mathbb{R} \setminus \{0, -1, -2, \dots\}. \quad (2.4)$$

Obviously, $\Gamma_q(t + 1) = [t]_q \Gamma_q(t)$.

The q -derivative of a function h is defined by

$$(D_q h)(t) = \frac{h(t) - h(qt)}{(1 - q)t} \quad \text{for } t \neq 0 \quad \text{and} \quad (D_q h)(0) = \lim_{t \rightarrow 0} (D_q h)(t), \quad (2.5)$$

and q -derivatives of higher order are given by

$$(D_q^0 h)(t) = h(t) \quad \text{and} \quad (D_q^k h)(t) = D_q(D_q^{k-1} h)(t), \quad k \in \mathbb{N}. \quad (2.6)$$

The q -integral of a function h defined on the interval $[0, b]$ is given by

$$(I_q h)(t) = \int_0^t h(s) d_q s = t(1 - q) \sum_{i=0}^{\infty} h(tq^i) q^i, \quad t \in [0, b]. \quad (2.7)$$

If $a \in [0, b]$ and h is defined in the interval $[0, b]$, then its integral from a to b is defined by

$$\int_a^b h(s) d_qs = \int_0^b h(s) d_qs - \int_0^a h(s) d_qs. \quad (2.8)$$

Similar to derivatives, an operator I_q^k is given by

$$(I_q^0 h)(t) = h(t) \quad \text{and} \quad (I_q^k h)(t) = I_q(I_q^{k-1} h)(t), \quad k \in \mathbb{N}. \quad (2.9)$$

The fundamental theorem of calculus applies to these operators D_q and I_q , i.e.,

$$(D_q I_q h)(t) = h(t), \quad (2.10)$$

and if h is continuous at $t = 0$, then

$$(I_q D_q h)(t) = h(t) - h(0). \quad (2.11)$$

Definition 2.1 Let $\nu \geq 0$ and h be a function defined on $[0, T]$. The fractional q -integral of Riemann-Liouville type is given by $(I_q^\nu h)(t) = h(t)$ and

$$(I_q^\nu h)(t) = \frac{1}{\Gamma_q(\nu)} \int_0^t (t - qs)^{(\nu-1)} h(s) d_qs, \quad \nu > 0, t \in [0, T]. \quad (2.12)$$

Definition 2.2 The fractional q -derivative of Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^\nu h)(t) = h(t)$ and

$$(D_q^\nu h)(t) = (D_q^l I_q^{l-\nu} h)(t), \quad \nu > 0, \quad (2.13)$$

where l is the smallest integer greater than or equal to ν .

Definition 2.3 For any $t, s > 0$,

$$B_q(t, s) = \int_0^1 u^{(t-1)} (1 - qu)^{(s-1)} d_qu \quad (2.14)$$

is called the q -beta function.

The expression of q -beta function in terms of the q -gamma function can be written as

$$B_q(t, s) = \frac{\Gamma_q(t) \Gamma_q(s)}{\Gamma_q(t+s)}.$$

Lemma 2.4 [4] Let $\alpha, \beta \geq 0$ and f be a function defined in $[0, T]$. Then the following formulas hold:

- (1) $(I_q^\beta I_q^\alpha f)(t) = (I_q^{\alpha+\beta} f)(t),$
- (2) $(D_q^\alpha I_q^\alpha f)(t) = f(t).$

Lemma 2.5 [6] *Let $\alpha > 0$ and n be a positive integer. Then the following equality holds:*

$$(I_q^\alpha D_q^n f)(t) = (D_q^n I_q^\alpha f)(t) - \sum_{i=0}^{n-1} \frac{t^{\alpha-n+i}}{\Gamma_q(\alpha+i-n+1)} (D_q^i f)(0). \quad (2.15)$$

3 Some auxiliary lemmas

The following formulas have been modified from Lemmas 3.2 and 7 in [21] and [20], respectively.

Lemma 3.1 *Let $x, y, z > 0$ and $0 < u, v, w < 1$. Then, for $\phi \in \mathbb{R}_+$, we have*

$$\begin{aligned} \text{(i)} \quad I_u^x I_v^y (1)(\phi) &= \frac{\Gamma_u(y+1)}{\Gamma_u(x+y+1)\Gamma_v(y+1)} \phi^{x+y}; \\ \text{(ii)} \quad I_u^x I_v^y I_w^z (1)(\phi) &= \frac{\Gamma_u(y+z+1)\Gamma_v(z+1)}{\Gamma_u(x+y+z+1)\Gamma_v(y+z+1)\Gamma_w(z+1)} \phi^{x+y+z}. \end{aligned}$$

Lemma 3.2 *Given $u, v \in C([0, T], \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D_q^\alpha x(t) = u(t), & t \in [0, T], 1 < \alpha \leq 2, \\ D_p^\beta y(t) = v(t), & t \in [0, T], 1 < \beta \leq 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa y(\xi), \\ y(0) = 0, & \lambda_2 I_h^\mu y(\theta) = I_k^\nu x(\tau), \end{cases} \quad (3.1)$$

is

$$\begin{aligned} x(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s) d_qs + \frac{\lambda_2 \Omega_1}{\Omega} t^{\alpha-1} I_h^\mu I_p^\beta v(\theta) \\ &\quad - \frac{\Omega_1}{\Omega} t^{\alpha-1} I_k^\nu I_q^\alpha u(\tau) + \frac{\lambda_1 \lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_m^\gamma I_q^\alpha u(\eta) \\ &\quad - \frac{\lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_n^\kappa I_p^\beta v(\xi) \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} y(t) &= \frac{1}{\Gamma_p(\beta)} \int_0^t (t-ps)^{(\beta-1)} v(s) d_ps + \frac{\lambda_1 \Omega_2}{\Omega} t^{\beta-1} I_m^\gamma I_q^\alpha u(\eta) \\ &\quad - \frac{\Omega_2}{\Omega} t^{\beta-1} I_n^\kappa I_p^\beta v(\xi) + \frac{\lambda_1 \lambda_2 \Omega_3}{\Omega} t^{\beta-1} I_h^\mu I_p^\beta v(\theta) \\ &\quad - \frac{\lambda_1 \Omega_3}{\Omega} t^{\beta-1} I_k^\nu I_q^\alpha u(\tau), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} \Omega_1 &= \frac{\Gamma_n(\beta)}{\Gamma_n(\beta+\kappa)} \xi^{\beta+\kappa-1}, \\ \Omega_2 &= \frac{\Gamma_k(\alpha)}{\Gamma_k(\alpha+\nu)} \tau^{\alpha+\nu-1}, \\ \Omega_3 &= \frac{\Gamma_m(\alpha)}{\Gamma_m(\alpha+\gamma)} \eta^{\alpha+\gamma-1}, \\ \Omega_4 &= \frac{\Gamma_h(\beta)}{\Gamma_h(\beta+\mu)} \theta^{\beta+\mu-1}, \\ \Omega &= \Omega_1 \Omega_2 - \lambda_1 \lambda_2 \Omega_3 \Omega_4 \neq 0. \end{aligned}$$

Proof From $1 < \alpha \leq 2$, we let $n = 2$. Applying Lemma 2.5, the equations in (3.1) can be expressed as equivalent integral equations

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} u(s) d_qs, \quad (3.4)$$

$$y(t) = d_1 t^{\beta-1} + d_2 t^{\beta-2} + \frac{1}{\Gamma_p(\beta)} \int_0^t (t - ps)^{(\beta-1)} v(s) d_ps \quad (3.5)$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(0) = 0$ and $y(0) = 0$ imply that $c_2 = 0$ and $d_2 = 0$, respectively. Taking the Riemann-Liouville fractional ϕ -integral of order $\psi > 0$ for (3.4) and (3.5), we have the system

$$\begin{aligned} I_\phi^\psi x(t) &= c_1 \frac{\Gamma_\phi(\alpha)}{\Gamma_\phi(\alpha + \psi)} t^{\alpha+\psi-1} \\ &+ \frac{1}{\Gamma_\phi(\psi)\Gamma_q(\alpha)} \int_0^t \int_0^s (t - \phi s)^{(\psi-1)} (s - qw)^{(\alpha-1)} u(w) d_q w d_\phi s, \end{aligned} \quad (3.6)$$

$$\begin{aligned} I_\phi^\psi y(t) &= d_1 \frac{\Gamma_\phi(\beta)}{\Gamma_\phi(\beta + \psi)} t^{\beta+\psi-1} \\ &+ \frac{1}{\Gamma_\phi(\psi)\Gamma_p(\beta)} \int_0^t \int_0^s (t - \phi s)^{(\psi-1)} (s - pw)^{(\beta-1)} v(w) d_p w d_\phi s. \end{aligned} \quad (3.7)$$

Substituting (ψ, ϕ, t) by (γ, m, η) , (ν, k, τ) in (3.6), and (κ, n, ξ) , (μ, h, θ) in (3.7) and using Lemma 2.4 with nonlocal conditions in (3.1), we have

$$\begin{aligned} c_1 &= \frac{\lambda_2 \Omega_1}{\Omega} I_h^\mu I_p^\beta v(\theta) - \frac{\Omega_1}{\Omega} I_k^\nu I_q^\alpha u(\tau) \\ &+ \frac{\lambda_1 \lambda_2 \Omega_4}{\Omega} I_m^\gamma I_q^\alpha u(\eta) - \frac{\lambda_2 \Omega_4}{\Omega} I_n^\kappa I_p^\beta v(\xi) \end{aligned}$$

and

$$\begin{aligned} d_1 &= \frac{\lambda_1 \Omega_2}{\Omega} I_m^\gamma I_q^\alpha u(\eta) - \frac{\Omega_2}{\Omega} I_n^\kappa I_p^\beta v(\xi) \\ &+ \frac{\lambda_1 \lambda_2 \Omega_3}{\Omega} I_h^\mu I_p^\beta v(\theta) - \frac{\lambda_1 \Omega_3}{\Omega} I_k^\nu I_q^\alpha u(\tau). \end{aligned}$$

Substituting the values of c_1 , c_2 , d_1 , and d_2 in (3.4) and (3.5), we obtain the solutions (3.2) and (3.3) as required. $\square$

4 Main results

Let $\mathcal{C} = C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$. Let us introduce the space $X = \{x(t) | x(t) \in \mathcal{C}([0, T], \mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0, T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t) | y(t) \in \mathcal{C}([0, T], \mathbb{R})\}$ be endowed with the norm $\|y\| = \sup\{|y(t)|, t \in [0, T]\}$. Obviously the product space $(X \times Y, \|(x, y)\|)$ is a Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$.

In view of Lemma 3.2, we define an operator $\mathcal{K} : X \times Y \rightarrow X \times Y$ by

$$\mathcal{K}(x, y)(t) = \begin{pmatrix} \mathcal{K}_1(x, y)(t) \\ \mathcal{K}_2(x, y)(t) \end{pmatrix},$$

where

$$\begin{aligned}
\mathcal{K}_1(x, y)(t) &= I_q^\alpha f(s, x(s), I_r^\delta y(s))(t) + \frac{\lambda_2 \Omega_1}{\Omega} t^{\alpha-1} I_h^\mu I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(\theta) \\
&\quad - \frac{\Omega_1}{\Omega} t^{\alpha-1} I_k^\nu I_q^\alpha f(s, x(s), I_r^\delta y(s))(\tau) \\
&\quad + \frac{\lambda_1 \lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_m^\gamma I_q^\alpha f(s, x(s), I_r^\delta y(s))(\eta) \\
&\quad - \frac{\lambda_2 \Omega_4}{\Omega} t^{\alpha-1} I_n^\kappa I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(\xi)
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\mathcal{K}_2(x, y)(t) &= I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(t) + \frac{\lambda_1 \Omega_2}{\Omega} t^{\beta-1} I_m^\gamma I_q^\alpha f(s, x(s), I_r^\delta y(s))(\eta) \\
&\quad - \frac{\Omega_2}{\Omega} t^{\beta-1} I_n^\kappa I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(\xi) \\
&\quad + \frac{\lambda_1 \lambda_2 \Omega_3}{\Omega} t^{\beta-1} I_h^\mu I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(\theta) \\
&\quad - \frac{\lambda_1 \Omega_3}{\Omega} t^{\beta-1} I_k^\nu I_q^\alpha f(s, x(s), I_r^\delta y(s))(\tau).
\end{aligned} \tag{4.2}$$

For the sake of convenience, we set

$$\begin{aligned}
A_1 &= \frac{T^\alpha}{\Gamma_q(\alpha+1)}, & A_2 &= \frac{T^\beta}{\Gamma_p(\beta+1)}, \\
A_3 &= \frac{\Gamma_q(\delta+1)T^{\alpha+\delta}}{\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)}, & A_4 &= \frac{\Gamma_p(\varepsilon+1)T^{\beta+\varepsilon}}{\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)}, \\
A_5 &= \frac{\Gamma_m(\alpha+1)\eta^{\gamma+\alpha}}{\Gamma_m(\gamma+\alpha+1)\Gamma_q(\alpha+1)}, & A_6 &= \frac{\Gamma_n(\beta+1)\xi^{\kappa+\beta}}{\Gamma_n(\kappa+\beta+1)\Gamma_p(\beta+1)}, \\
A_7 &= \frac{\Gamma_h(\beta+1)\theta^{\mu+\beta}}{\Gamma_h(\mu+\beta+1)\Gamma_p(\beta+1)}, & A_8 &= \frac{\Gamma_k(\alpha+1)\tau^{\nu+\alpha}}{\Gamma_k(\nu+\alpha+1)\Gamma_q(\alpha+1)}, \\
A_9 &= \frac{\Gamma_m(\alpha+\delta+1)\Gamma_q(\delta+1)\eta^{\gamma+\alpha+\delta}}{\Gamma_m(\gamma+\alpha+\delta+1)\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)}, \\
A_{10} &= \frac{\Gamma_n(\beta+\varepsilon+1)\Gamma_p(\varepsilon+1)\xi^{\kappa+\beta+\varepsilon}}{\Gamma_n(\kappa+\beta+\varepsilon+1)\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)}, \\
A_{11} &= \frac{\Gamma_h(\beta+\varepsilon+1)\Gamma_p(\varepsilon+1)\theta^{\mu+\beta+\varepsilon}}{\Gamma_h(\mu+\beta+\varepsilon+1)\Gamma_p(\beta+\varepsilon+1)\Gamma_z(\varepsilon+1)}, \\
A_{12} &= \frac{\Gamma_k(\alpha+\delta+1)\Gamma_q(\delta+1)\tau^{\nu+\alpha+\delta}}{\Gamma_k(\nu+\alpha+\delta+1)\Gamma_q(\alpha+\delta+1)\Gamma_r(\delta+1)}.
\end{aligned}$$

Theorem 4.1 Assume that $f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions and there exist positive constants $M_i, N_i, i = 1, 2$, such that for all $t \in [0, T]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$,

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq M_1|u_1 - v_1| + M_2|u_2 - v_2|$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq N_1|u_1 - v_1| + N_2|u_2 - v_2|.$$

In addition, we suppose that

$$B_1 + B_2 + C_1 + C_2 < 1,$$

where

$$\begin{aligned} B_1 &= M_1 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} (\Omega_1 N_2 A_{11} + |\lambda_1| \Omega_4 M_1 A_5 + \Omega_4 N_2 A_{10}) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} M_1 A_8, \\ B_2 &= M_2 A_3 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} (\Omega_1 N_1 A_7 + |\lambda_1| \Omega_4 M_2 A_9 + \Omega_4 N_1 A_6) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} M_2 A_{12}, \\ C_1 &= N_2 A_4 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 M_1 A_5 + |\lambda_2| \Omega_3 N_2 A_{11} + \Omega_3 M_1 A_8) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} N_2 A_{10}, \\ C_2 &= N_1 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 M_2 A_9 + |\lambda_2| \Omega_3 N_1 A_7 + \Omega_3 M_2 A_{12}) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} N_1 A_6. \end{aligned}$$

Then the system (1.1) has a unique solution on $[0, T]$.

Proof Firstly, we define $\sup_{t \in [0, T]} |f(t, 0, 0)| = G_1 < \infty$ and $\sup_{t \in [0, T]} |g(t, 0, 0)| = G_2 < \infty$ such that

$$r \geq \max \left\{ \frac{B_3}{1 - (B_1 + B_2)}, \frac{C_3}{1 - (C_1 + C_2)} \right\},$$

where

$$\begin{aligned} B_3 &= G_1 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} (\Omega_1 G_2 A_7 + |\lambda_1| \Omega_4 G_1 A_5 + \Omega_4 G_2 A_6) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} G_1 A_8, \\ C_3 &= G_2 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 G_1 A_5 + |\lambda_2| \Omega_3 G_2 A_7 + \Omega_3 G_1 A_8) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} G_2 A_6. \end{aligned}$$

We will show that $\mathcal{K}B_r \subset B_r$, where $B_r = \{(x, y) \in X \times Y : \|(x, y)\| \leq r\}$.

For $(x, y) \in B_r$, taking into account Lemma 3.1, we have

$$\begin{aligned} & |\mathcal{K}_1(x, y)(t)| \\ & \leq \sup_{t \in T} \left\{ I_q^\alpha |f(s, x(s), I_r^\delta y(s))|(t) + \frac{|\lambda_2| \Omega_1}{|\Omega|} t^{\alpha-1} I_h^\mu I_p^\beta |g(s, y(s), I_z^\epsilon x(s))|(\theta) \right. \\ & \quad + \frac{\Omega_1}{|\Omega|} t^{\alpha-1} I_k^\nu I_q^\alpha |f(s, x(s), I_r^\delta y(s))|(\tau) + \frac{|\lambda_1| |\lambda_2| \Omega_4}{|\Omega|} t^{\alpha-1} I_m^\gamma I_q^\alpha |f(s, x(s), I_r^\delta y(s))|(\eta) \\ & \quad \left. + \frac{|\lambda_2| \Omega_4}{|\Omega|} t^{\alpha-1} I_n^\kappa I_p^\beta |g(s, y(s), I_z^\epsilon x(s))|(\xi) \right\} \\ & \leq I_q^\alpha (|f(s, x(s), I_r^\delta y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(t) \\ & \quad + \frac{|\lambda_2| \Omega_1}{|\Omega|} T^{\alpha-1} I_h^\mu I_p^\beta (|g(s, y(s), I_z^\epsilon x(s)) - g(s, 0, 0)| + |g(s, 0, 0)|)(\theta) \\ & \quad + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} I_k^\nu I_q^\alpha (|f(s, x(s), I_r^\delta y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\tau) \\ & \quad + \frac{|\lambda_1| |\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} I_m^\gamma I_q^\alpha (|f(s, x(s), I_r^\delta y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\eta) \end{aligned}$$

$$\begin{aligned}
& + \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_p^\kappa I_p^\beta \left(|g(s, y(s), I_z^\varepsilon x(s)) - g(t, 0, 0)| + |g(t, 0, 0)| \right) (\xi) \\
& \leq M_1 \|x\| A_1 + M_2 \|y\| A_3 + G_1 A_1 \\
& + \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1} (N_1 \|y\| A_7 + N_2 \|x\| A_{11} + G_2 A_7) \\
& + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} (M_1 \|x\| A_8 + M_2 \|y\| A_{12} + G_1 A_8) \\
& + \frac{|\lambda_1|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} (M_1 \|x\| A_5 + M_2 \|y\| A_9 + G_1 A_5) \\
& + \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} (N_1 \|y\| A_6 + N_2 \|x\| A_{10} + G_2 A_6) \\
& = B_1 \|x\| + B_2 \|y\| + B_3 \\
& \leq (B_1 + B_2)r + B_3 \leq r.
\end{aligned}$$

In a similar way, we get

$$\begin{aligned}
& |\mathcal{K}_2(x, y)(t)| \\
& \leq I_p^\beta \left(|g(s, y(s), I_z^\varepsilon x(s)) - g(s, 0, 0)| + |g(s, 0, 0)| \right) (t) \\
& + \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} I_m^\gamma I_q^\alpha \left(|f(s, x(s), I_r^\delta y(s)) - f(s, 0, 0)| + |f(s, 0, 0)| \right) (\eta) \\
& + \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^\kappa I_p^\beta \left(|g(s, y(s), I_z^\varepsilon x(s)) - g(s, 0, 0)| + |g(s, 0, 0)| \right) (\xi) \\
& + \frac{|\lambda_1|\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} I_h^\mu I_p^\beta \left(|g(s, y(s), I_z^\varepsilon x(s)) - g(s, 0, 0)| + |g(s, 0, 0)| \right) (\theta) \\
& + \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} I_k^\nu I_q^\alpha \left(|f(s, x(s), I_r^\delta y(s)) - f(s, 0, 0)| + |f(s, 0, 0)| \right) (\tau) \\
& \leq N_1 \|y\| A_2 + N_2 \|x\| A_4 + G_2 A_2 \\
& + \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} (M_1 \|x\| A_5 + M_2 \|y\| A_9 + G_1 A_5) \\
& + \frac{\Omega_2}{|\Omega|} T^{\beta-1} (N_1 \|y\| A_6 + N_2 \|x\| A_{10} + G_2 A_6) \\
& + \frac{|\lambda_1|\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} (N_1 \|y\| A_7 + N_2 \|x\| A_{11} + G_2 A_7) \\
& + \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} (M_1 \|x\| A_8 + M_2 \|y\| A_{12} + G_1 A_8) \\
& = C_1 \|x\| + C_2 \|y\| + C_3 \\
& \leq (C_1 + C_2)r + B_3 \leq r.
\end{aligned}$$

Consequently, $\|\mathcal{K}(x, y)(t)\| \leq r$.

Next, for $(x_2, y_2), (x_1, y_1) \in X \times Y$, and for any $t \in [0, T]$, we have

$$\begin{aligned}
& |\mathcal{K}_1(x_2, y_2)(t) - \mathcal{K}_1(x_1, y_1)(t)| \\
& \leq I_q^\alpha \left(|f(s, x_2(s), I_r^\delta y_2(s)) - f(s, x_1(s), I_r^\delta y_1(s))| \right) (t)
\end{aligned}$$

$$\begin{aligned}
& + \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1} I_h^\mu I_p^\beta (|g(s, y_2(s), I_z^\varepsilon x_2(s)) - g(s, y_1(s), I_z^\varepsilon x_1(s))|)(\theta) \\
& + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} I_k^\nu I_q^\alpha (|f(s, x_2(s), I_r^\delta y_2(s)) - f(s, x_1(s), I_r^\delta y_1(s))|)(\tau) \\
& + \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_m^\gamma I_q^\alpha (|f(s, x_2(s), I_r^\delta y_2(s)) - f(s, x_1(s), I_r^\delta y_1(s))|)(\eta) \\
& + \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} I_n^\kappa I_p^\beta (|g(s, y_2(s), I_z^\varepsilon x_2(s)) - g(s, y_1(s), I_z^\varepsilon x_1(s))|)(\xi) \\
& \leq M_1 \|x_2 - x_1\| I_q^\alpha(1)(T) + M_2 \|y_2 - y_1\| I_r^\alpha I_r^\delta(1)(T) \\
& + \frac{|\lambda_2|\Omega_1}{|\Omega|} T^{\alpha-1} (N_1 \|y_2 - y_1\| I_h^\mu I_p^\beta(1)(\theta) + N_2 \|x_2 - x_1\| I_h^\mu I_p^\beta I_z^\varepsilon(1)(\theta)) \\
& + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} (M_1 \|x_2 - x_1\| I_k^\nu I_q^\alpha(1)(\tau) + M_2 \|y_2 - y_1\| I_k^\nu I_q^\alpha I_r^\delta(1)(\tau)) \\
& + \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} (M_1 \|x_2 - x_1\| I_m^\gamma I_q^\alpha(1)(\eta) + M_2 \|y_2 - y_1\| I_m^\gamma I_q^\alpha I_r^\delta(1)(\eta)) \\
& + \frac{|\lambda_2|\Omega_4}{|\Omega|} T^{\alpha-1} (N_1 \|y_2 - y_1\| I_n^\kappa I_p^\beta(1)(\xi) + N_2 \|x_2 - x_1\| I_n^\kappa I_p^\beta I_z^\varepsilon(1)(\xi)) \\
& = B_1 \|x_2 - x_1\| + B_2 \|y_2 - y_1\|.
\end{aligned}$$

Therefore, we have

$$\|\mathcal{K}_1(x_2, y_2)(t) - \mathcal{K}_1(x_1, y_1)(t)\| \leq (B_1 + B_2)(\|x_2 - x_1\| + \|y_2 - y_1\|). \quad (4.3)$$

In the same way, we have

$$\begin{aligned}
& |\mathcal{K}_2(x_2, y_2)(t) - \mathcal{K}_2(x_1, y_1)(t)| \\
& \leq I_p^\beta (|g(s, y_2(s), I_z^\varepsilon x_2(s)) - g(s, y_1(s), I_z^\varepsilon x_1(s))|)(t) \\
& + \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} I_m^\gamma I_q^\alpha (|f(s, x_2(s), I_r^\delta y_2(s)) - f(s, x_1(s), I_r^\delta y_1(s))|)(\eta) \\
& + \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^\kappa I_p^\beta (|g(s, y_2(s), I_z^\varepsilon x_2(s)) - g(s, y_1(s), I_z^\varepsilon x_1(s))|)(\xi) \\
& + \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} I_h^\mu I_p^\beta (|g(s, y_2(s), I_z^\varepsilon x_2(s)) - g(s, y_1(s), I_z^\varepsilon x_1(s))|)(\theta) \\
& + \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} I_k^\nu I_q^\alpha (|f(s, x_2(s), I_r^\delta y_2(s)) - f(s, x_1(s), I_r^\delta y_1(s))|)(\tau) \\
& \leq N_1 \|y_2 - y_1\| A_2 + N_2 \|x_2 - x_1\| A_4 \\
& + \frac{|\lambda_1|\Omega_2}{|\Omega|} T^{\beta-1} (M_1 \|x_2 - x_1\| A_5 + M_2 \|y_2 - y_1\| A_9) \\
& + \frac{\Omega_2}{|\Omega|} T^{\beta-1} (N_1 \|y_2 - y_1\| A_6 + N_2 \|x_2 - x_1\| A_{10}) \\
& + \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} T^{\beta-1} (N_1 \|y_2 - y_1\| A_7 + N_2 \|x_2 - x_1\| A_{11}) \\
& + \frac{|\lambda_1|\Omega_3}{|\Omega|} T^{\beta-1} (M_1 \|x_2 - x_1\| A_8 + M_2 \|y_2 - y_1\| A_{12}) \\
& = C_1 \|x_2 - x_1\| + C_2 \|y_2 - y_1\|,
\end{aligned}$$

which implies

$$\|\mathcal{K}_2(x_2, y_2)(t) - \mathcal{K}_2(x_1, y_1)(t)\| \leq (C_1 + C_2)(\|x_2 - x_1\| + \|y_2 - y_1\|). \quad (4.4)$$

It follows from (4.3) and (4.4) that

$$\|\mathcal{K}(x_2, y_2)(t) - \mathcal{K}(x_1, y_1)(t)\| \leq (B_1 + B_2 + C_1 + C_2)(\|x_2 - x_1\| + \|y_2 - y_1\|).$$

Since $B_1 + B_2 + C_1 + C_2 < 1$, therefore, $\mathcal{K}$ is a contraction operator. So, by Banach's fixed point theorem, the operator $\mathcal{K}$ has a unique fixed point, which is the unique solution of problem (1.1). The proof is completed. $\square$

In the next result, we prove the existence of solutions for the problem (1.1) by applying the Leray-Schauder alternative.

Lemma 4.2 (Leray-Schauder alternative, see [22], p.4) *Let $F : E \rightarrow E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

For convenience, we set constants

$$\begin{aligned} E_0 &= P_0 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} (\Omega_1 Q_0 A_7 + |\lambda_1| \Omega_4 P_0 A_5 + \Omega_4 Q_0 A_6) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} P_0 A_8, \\ E_1 &= P_1 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} (\Omega_1 Q_2 A_{11} + |\lambda_1| \Omega_4 P_1 A_5 + \Omega_4 Q_2 A_{10}) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} P_1 A_8, \\ E_3 &= P_2 A_3 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} (\Omega_1 Q_1 A_7 + |\lambda_1| \Omega_4 P_2 A_9 + \Omega_4 Q_1 A_6) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} P_2 A_{12}, \\ F_0 &= Q_0 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 P_0 A_5 + |\lambda_2| \Omega_3 Q_0 A_7 + \Omega_3 P_0 A_8) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} Q_0 A_6, \\ F_1 &= Q_2 A_4 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 P_1 A_5 + |\lambda_2| \Omega_3 Q_2 A_{11} + \Omega_3 P_1 A_8) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} Q_2 A_{10}, \\ F_2 &= Q_1 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 P_2 A_9 + |\lambda_2| \Omega_3 Q_1 A_7 + \Omega_3 P_2 A_{12}) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} Q_1 A_6 \end{aligned}$$

and

$$G^* = \max\{1 - (E_1 + F_1), 1 - (E_2 + F_2)\}.$$

Theorem 4.3 *Assume that there exist real constants $P_i, Q_i \geq 0$ ($i = 1, 2$), and $P_0 > 0, Q_0 > 0$ such that for all $u_i, v_i \in \mathbb{R}$ ($i = 1, 2$) we have*

$$|f(t, u_1, u_2)| \leq P_0 + P_1 |u_1| + P_2 |u_2|,$$

$$|g(t, v_1, v_2)| \leq Q_0 + Q_1 |v_1| + Q_2 |v_2|.$$

In addition it is assumed that

$$E_1 + F_1 < 1 \quad \text{and} \quad E_2 + F_2 < 1.$$

Then there exists at least one solution for the system (1.1).

Proof We first prove that the operator $\mathcal{K} : X \times Y \rightarrow X \times Y$ is completely continuous. The continuity of functions f and g imply that the operator $\mathcal{K}$ is continuous. Let $\Phi \subset X \times Y$ be a bounded set. Then there exist positive constants D_1 and D_2 such that

$$|f(t, u_1(t), u_2(t))| \leq D_1, \quad |g(t, v_1(t), v_2(t))| \leq D_2, \quad \forall (u_1, u_2), (v_1, v_2) \in \Phi.$$

Then for any $(u_1, u_2), (v_1, v_2) \in \Phi$, and using Lemma 3.1, we have

$$\begin{aligned} \|\mathcal{K}_1(x, y)\| &\leq I_q^\alpha |f(s, x(s), I_r^\delta y(s))|(t) + \frac{|\lambda_2| \Omega_1}{|\Omega|} T^{\alpha-1} I_h^\mu I_p^\beta |g(s, y(s), I_z^\varepsilon x(s))|(\theta) \\ &\quad + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} I_k^\nu I_q^\alpha |f(s, x(s), I_r^\delta y(s))|(\tau) \\ &\quad + \frac{|\lambda_1| |\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} I_m^\gamma I_q^\alpha |f(s, x(s), I_r^\delta y(s))|(\eta) \\ &\quad + \frac{|\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} I_n^\kappa I_p^\beta |g(s, y(s), I_z^\varepsilon x(s))|(\xi) \\ &\leq D_1 A_1 + \frac{D_2 |\lambda_2| \Omega_1}{|\Omega|} T^{\alpha-1} A_7 + \frac{D_1 \Omega_1}{|\Omega|} T^{\alpha-1} A_8 \\ &\quad + \frac{D_1 |\lambda_1| |\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} A_5 + \frac{D_2 |\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} A_6. \end{aligned}$$

In the same way, we deduce that

$$\begin{aligned} \|\mathcal{K}_2(x, y)\| &\leq I_p^\beta (|g(s, y(s), I_z^\varepsilon x(s))|)(t) + \frac{|\lambda_1| \Omega_2}{|\Omega|} T^{\beta-1} I_m^\gamma I_q^\alpha (|f(s, x(s), I_r^\delta y(s))|)(\eta) \\ &\quad + \frac{\Omega_2}{|\Omega|} T^{\beta-1} I_n^\kappa I_p^\beta (|g(s, y(s), I_z^\varepsilon x(s))|)(\xi) \\ &\quad + \frac{|\lambda_1| |\lambda_2| \Omega_3}{|\Omega|} T^{\beta-1} I_h^\mu I_p^\beta (|g(s, y(s), I_z^\varepsilon x(s))|)(\theta) \\ &\quad + \frac{|\lambda_1| \Omega_3}{|\Omega|} T^{\beta-1} I_k^\nu I_q^\alpha (|f(s, x(s), I_r^\delta y(s))|)(\tau) \\ &\leq D_2 A_2 + \frac{D_1 |\lambda_1| \Omega_2}{|\Omega|} T^{\beta-1} A_5 + \frac{D_2 \Omega_2}{|\Omega|} T^{\beta-1} A_6 \\ &\quad + \frac{D_2 |\lambda_1| |\lambda_2| \Omega_3}{|\Omega|} T^{\beta-1} A_7 + \frac{D_1 |\lambda_1| \Omega_3}{|\Omega|} T^{\beta-1} A_8. \end{aligned}$$

Thus, it follows from the above inequalities that the operator $\mathcal{K}$ is uniformly bounded.

Next, we show that $\mathcal{K}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned} &|\mathcal{K}_1(x, y)(t_2) - \mathcal{K}_1(x, y)(t_1)| \\ &\leq |I_q^\alpha f(s, x(s), I_r^\delta y(s))(t_2) - I_q^\alpha f(s, x(s), I_r^\delta y(s))(t_1)| \end{aligned}$$

$$\begin{aligned}
& + \frac{|\lambda_2|\Omega_1}{|\Omega|} |t_2^{\alpha-1} - t_1^{\alpha-1}| I_h^\mu I_p^\beta |g(s, y_2(s), I_z^\varepsilon x_2(s))|(\theta) \\
& + \frac{\Omega_1}{|\Omega|} |t_2^{\alpha-1} - t_1^{\alpha-1}| I_k^\nu I_q^\alpha |f(s, x(s), I_r^\delta y(s))|(\tau) \\
& + \frac{|\lambda_1||\lambda_2|\Omega_4}{|\Omega|} |t_2^{\alpha-1} - t_1^{\alpha-1}| I_m^\nu I_q^\alpha |f(s, x(s), I_r^\delta y(s))|(\eta) \\
& + \frac{|\lambda_2|\Omega_4}{|\Omega|} |t_2^{\alpha-1} - t_1^{\alpha-1}| I_n^\kappa I_p^\beta |g(s, y(s), I_z^\varepsilon x(s))|(\xi) \\
& \leq \frac{D_1}{\Gamma_q(\alpha)} \int_0^{t_1} [(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}] d_qs \\
& + \frac{D_1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)} d_qs \\
& + \frac{|\lambda_2|\Omega_1 D_2}{|\Omega|} |t_2^{\alpha-1} - t_1^{\alpha-1}| A_7 + \frac{\Omega_1 D_1}{|\Omega|} |t_2^{\alpha-1} - t_1^{\alpha-1}| A_8 \\
& + \frac{|\lambda_1||\lambda_2|\Omega_4 D_1}{|\Omega|} |t_2^{\alpha-1} - t_1^{\alpha-1}| A_5 + \frac{|\lambda_2|\Omega_4 D_2}{|\Omega|} |t_2^{\alpha-1} - t_1^{\alpha-1}| A_6.
\end{aligned}$$

Analogously, we can get

$$\begin{aligned}
& |\mathcal{K}_2(x, y)(t_2) - \mathcal{K}_2(x, y)(t_1)| \\
& \leq |I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(t_2) - I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(t_1)| \\
& + \frac{|\lambda_1|\Omega_2}{|\Omega|} |t_2^{\beta-1} - t_1^{\beta-1}| I_m^\nu I_q^\alpha |f(s, x_2(s), I_r^\delta y_2(s))|(\eta) \\
& + \frac{\Omega_2}{|\Omega|} |t_2^{\beta-1} - t_1^{\beta-1}| I_n^\kappa I_p^\beta |g(s, y_2(s), I_z^\varepsilon x_2(s))|(\xi) \\
& + \frac{|\lambda_1||\lambda_2|\Omega_3}{|\Omega|} |t_2^{\beta-1} - t_1^{\beta-1}| I_h^\mu I_p^\beta |g(s, y_2(s), I_z^\varepsilon x_2(s))|(\theta) \\
& + \frac{|\lambda_1|\Omega_3}{|\Omega|} |t_2^{\beta-1} - t_1^{\beta-1}| I_k^\nu I_q^\alpha |f(s, x_2(s), I_r^\delta y_2(s))|(\tau) \\
& \leq \frac{D_2}{\Gamma_p(\beta)} \int_0^{t_1} [(t_2 - ps)^{(\beta-1)} - (t_1 - ps)^{(\beta-1)}] d_ps \\
& + \frac{D_2}{\Gamma_p(\beta)} \int_{t_1}^{t_2} (t_2 - ps)^{(\beta-1)} d_ps \\
& + \frac{|\lambda_1|\Omega_2 D_1}{|\Omega|} |t_2^{\beta-1} - t_1^{\beta-1}| A_5 + \frac{\Omega_2 D_2}{|\Omega|} |t_2^{\beta-1} - t_1^{\beta-1}| A_6 \\
& + \frac{|\lambda_1||\lambda_2|\Omega_3 D_2}{|\Omega|} |t_2^{\beta-1} - t_1^{\beta-1}| A_7 + \frac{|\lambda_1|\Omega_3 D_1}{|\Omega|} |t_2^{\beta-1} - t_1^{\beta-1}| A_8.
\end{aligned}$$

Therefore, the operator $\mathcal{K}(x, y)$ is equicontinuous, and thus the operator $\mathcal{K}(x, y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(x, y) \in X \times Y : (x, y) = \lambda \mathcal{K}(x, y), 0 \leq \lambda \leq 1\}$ is bounded. Let $(x, y) \in \mathcal{E}$, then $(x, y) = \lambda \mathcal{K}(x, y)$. For any $t \in [0, T]$, we have

$$x(t) = \lambda \mathcal{K}_1(x, y)(t), \quad y(t) = \lambda \mathcal{K}_2(x, y)(t).$$

Then we have

$$\begin{aligned}
|x(t)| &\leq P_0 I_q^\alpha(1)(t) + P_1 \|x\| I_q^\alpha(1)(t) + P_2 \|y\| I_q^\alpha I_r^\delta(1)(t) \\
&\quad + \frac{|\lambda_2| \Omega_1}{|\Omega|} T^{\alpha-1} (Q_0 I_h^\mu I_p^\beta(1)(\theta) + Q_1 \|y\| I_h^\mu I_p^\beta(1)(\theta) + Q_2 \|x\| I_h^\mu I_p^\beta I_z^\varepsilon(1)(\theta)) \\
&\quad + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} (P_0 I_k^\nu I_q^\alpha(1)(\tau) + P_1 \|x\| I_k^\nu I_q^\alpha(1)(\tau) + P_2 \|y\| I_k^\nu I_q^\alpha I_r^\delta(1)(\tau)) \\
&\quad + \frac{|\lambda_1| |\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} (P_0 I_m^\gamma I_q^\alpha(1)(\eta) + P_1 \|x\| I_m^\gamma I_q^\alpha(1)(\eta) + P_2 \|y\| I_m^\gamma I_q^\alpha I_r^\delta(1)(\eta)) \\
&\quad + \frac{|\lambda_2| \Omega_4}{|\Omega|} T^{\alpha-1} (Q_0 I_n^\kappa I_p^\beta(1)(\xi) + Q_1 \|y\| I_n^\kappa I_p^\beta(1)(\xi) + Q_2 \|x\| I_n^\kappa I_p^\beta I_z^\varepsilon(1)(\xi)) \\
&\leq E_0 + E_1 \|x\| + E_2 \|y\|
\end{aligned}$$

and

$$\begin{aligned}
|y(t)| &\leq Q_0 I_p^\beta(1)(t) + Q_1 \|y\| I_p^\beta(1)(t) + Q_1 \|x\| I_p^\beta I_z^\varepsilon(1)(t) \\
&\quad + \frac{|\lambda_1| \Omega_2}{|\Omega|} T^{\beta-1} (P_0 I_m^\gamma I_q^\alpha(1)(\eta) + P_1 \|x\| I_m^\gamma I_q^\alpha(1)(\eta) + P_2 \|y\| I_m^\gamma I_q^\alpha I_r^\delta(1)(\eta)) \\
&\quad + \frac{\Omega_2}{|\Omega|} T^{\beta-1} (Q_0 I_n^\kappa I_p^\beta(1)(\xi) + Q_1 \|y\| I_n^\kappa I_p^\beta(1)(\xi) + Q_2 \|x\| I_n^\kappa I_p^\beta I_z^\varepsilon(1)(\xi)) \\
&\quad + \frac{|\lambda_1| |\lambda_2| \Omega_3}{|\Omega|} T^{\beta-1} (Q_0 I_h^\mu I_p^\beta(1)(\theta) + Q_1 \|y\| I_h^\mu I_p^\beta(1)(\theta) + Q_2 \|x\| I_h^\mu I_p^\beta I_z^\varepsilon(1)(\theta)) \\
&\quad + \frac{|\lambda_1| \Omega_3}{|\Omega|} T^{\beta-1} (P_0 I_k^\nu I_q^\alpha(1)(\tau) + P_1 \|x\| I_k^\nu I_q^\alpha(1)(\tau) + P_2 \|y\| I_k^\nu I_q^\alpha I_r^\delta(1)(\tau)) \\
&\leq F_0 + F_1 \|x\| + F_2 \|y\|,
\end{aligned}$$

which yields

$$\|x\| \leq E_0 + E_1 \|x\| + E_2 \|y\|$$

and

$$\|y\| \leq F_0 + F_1 \|x\| + F_2 \|y\|.$$

Therefore, we have

$$\|x\| + \|y\| \leq (E_0 + F_0) + (E_1 + F_1) \|x\| + (E_2 + F_2) \|y\|,$$

and, consequently,

$$\|(x, y)\| \leq \frac{E_0 + F_0}{G^*}$$

for any $t \in [0, T]$, which proves that $\mathcal{E}$ is bounded. Thus, by Lemma 4.2, the operator $\mathcal{K}$ has at least one fixed point. Hence the system (1.1) has at least one solution. The proof is complete. $\square$

4.1 Examples

In this subsection, we present some examples to illustrate our results.

Example 4.4 Consider the following coupled system of fractional q -integro-difference equations:

$$\begin{cases} D_{1/2}^{3/2}x(t) = \frac{\cos^2 \pi t}{(e^t+4)^2} \cdot \frac{|x(t)|}{4+|x(t)|} + \frac{e^{-t^2}}{(t+8)^2} \cdot I_{1/4}^{\pi}y(t) + \frac{\sqrt{2}}{2}, & 0 < t < 2, \\ D_{1/3}^{4/3}y(t) = \frac{\sin^2 \pi t}{(11+t)^2} \cdot \frac{|y(t)|}{1+|y(t)|} + \frac{1}{(e^t+8)^2} \cdot I_{1/5}^{\pi/2}x(t) + \sqrt{3}, \\ x(0) = 0, & \sqrt{2}I_{1/8}^{7/6}x(\frac{3}{2}) = I_{1/6}^{\sqrt{2}}y(\frac{1}{2}), \\ y(0) = 0, & \frac{\sqrt{3}}{2}I_{1/9}^ey(\frac{1}{3}) = I_{1/7}^{\sqrt{3}}x(\frac{5}{3}). \end{cases} \quad (4.5)$$

Here $\alpha = 3/2$, $\delta = \pi$, $\beta = 4/3$, $\varepsilon = \pi/2$, $\gamma = 7/6$, $\kappa = \sqrt{2}$, $\mu = e$, $\nu = \sqrt{3}$, $q = 1/2$, $r = 1/4$, $p = 1/3$, $z = 1/5$, $m = 1/8$, $n = 1/6$, $h = 1/9$, $k = 1/7$, $\eta = 3/2$, $\xi = 1/2$, $\theta = 1/3$, $\tau = 5/3$, $\lambda_1 = \sqrt{2}$, $\lambda_2 = \sqrt{3}/2$, $T = 2$, $f(t, x, I_r^\delta y) = (|x| \cos^2 \pi t) / ((e^t + 4)^2(4 + |x|)) + (e^{-t^2} / ((t + 8)^2)) I_{1/4}^\pi y + \sqrt{2}/2$, and $g(t, y, I_r^\delta x) = (|y| \sin^2 \pi t) / ((11 + t)^2(1 + |y|)) + (1 / (e^t + 8)^2) I_{1/5}^{\pi/2} x + \sqrt{3}$. Since

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq \frac{1}{100}|u_1 - v_1| + \frac{1}{64}|u_2 - v_2|$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq \frac{1}{121}|u_1 - v_1| + \frac{1}{81}|u_2 - v_2|,$$

then the assumptions of Theorem 4.1 are satisfied with $M_1 = 1/100$, $M_2 = 1/64$, $N_1 = 1/121$, and $N_2 = 1/81$. By using the Maple program, we can find that

$$\Omega = \Omega_1 \Omega_2 - \lambda_1 \lambda_2 \Omega_3 \Omega_4 \approx 0.61154471 \neq 0$$

and

$$\begin{aligned} B_1 &= M_1 A_1 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} (\Omega_1 N_2 A_{11} + |\lambda_1| \Omega_4 M_1 A_5 + \Omega_4 N_2 A_{10}) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} M_1 A_8 \\ &\approx 0.05777709, \\ B_2 &= M_2 A_3 + \frac{|\lambda_2| T^{\alpha-1}}{|\Omega|} (\Omega_1 N_1 A_7 + |\lambda_1| \Omega_4 M_2 A_9 + \Omega_4 N_1 A_6) + \frac{\Omega_1}{|\Omega|} T^{\alpha-1} M_2 A_{12} \\ &\approx 0.1489994, \\ C_1 &= N_2 A_4 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 M_1 A_5 + |\lambda_2| \Omega_3 N_2 A_{11} + \Omega_3 M_1 A_8) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} N_2 A_{10} \\ &\approx 0.22629179, \\ C_2 &= N_1 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 M_2 A_9 + |\lambda_2| \Omega_3 N_1 A_7 + \Omega_3 M_2 A_{12}) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} N_1 A_6 \\ &\approx 0.28656994. \end{aligned}$$

Therefore, we get

$$B_1 + B_2 + C_1 + C_2 \approx 0.71963204 < 1.$$

Hence, by Theorem 4.1, the problem (4.5) has a unique solution on $[0, 2]$.

Example 4.5 Consider the following coupled system of fractional q -integro-difference equations with fractional q -integral conditions:

$$\begin{cases} D_{1/3}^{4/3}x(t) = \frac{4e^{-t}}{(t+11)^2} \cdot \frac{|x(t)|}{2+|x(t)|} + \frac{1}{(e^{-t^2}+8)^2} \cdot I_{1/5}^{\sqrt{3}}y(t) + \sqrt{2}, & 0 < t < \pi, \\ D_{1/4}^{7/5}y(t) = \frac{\cos^2 2\pi t}{(10+t)^2} \cdot \frac{|y(t)|}{1+|y(t)|} + \frac{1}{(e^t+7)^2} \cdot I_{1/6}^{\sqrt{2}}x(t) + \frac{1}{2}, \\ x(0) = 0, & I_{1/3}^{1/3}y(\frac{2\pi}{5}) + \sqrt{5}I_{1/2}^{\sqrt{5}}x(\frac{\pi}{5}) = 0, \\ y(0) = 0, & I_{1/8}^{\sqrt{2}/2}x(\frac{4\pi}{5}) + \sqrt{2}I_{1/7}^{\sqrt{\pi}}y(\frac{3\pi}{5}) = 0. \end{cases} \quad (4.6)$$

Here $\alpha = 4/3$, $\delta = \sqrt{3}$, $\beta = 7/5$, $\varepsilon = \sqrt{2}$, $\gamma = \sqrt{5}$, $\kappa = 1/3$, $\mu = \sqrt{\pi}$, $\nu = \sqrt{2}/2$, $q = 1/3$, $r = 1/5$, $p = 1/4$, $z = 1/6$, $m = 1/2$, $n = 1/3$, $h = 1/7$, $k = 1/8$, $\eta = \pi/5$, $\xi = 2\pi/5$, $\theta = 3\pi/5$, $\tau = 4\pi/5$, $\lambda_1 = -\sqrt{5}$, $\lambda_2 = -\sqrt{2}$, $T = \pi$, $f(t, x, I_r^\delta y) = (4e^{-t}/(t+11)^2)(|x|/(2+|x|)) + (1/(e^{-t^2}+8)^2)I_{1/5}^{\sqrt{3}}y + \sqrt{2}$, and $g(t, y, I_r^\delta x) = (\cos^2 2\pi t/(10+t)^2)(|y|/(1+|y|)) + (1/(e^t+7)^2)I_{1/6}^{\sqrt{2}}x + (1/2)$. Since

$$|f(t, u_1, u_2)| \leq \sqrt{2} + \frac{1}{72}|u_1| + \frac{1}{81}|u_2|$$

and

$$|g(t, v_1, v_2)| \leq \frac{1}{2} + \frac{1}{100}|v_1| + \frac{1}{81}|v_2|,$$

then the assumptions of Theorem 4.3 are satisfied with $P_0 = \sqrt{2}$, $P_1 = 1/72$, $P_2 = 1/81$, $Q_0 = 1/2$, $Q_1 = 1/100$, and $Q_2 = 1/81$. By using the Maple program, we can find that

$$\Omega = \Omega_1\Omega_2 - \lambda_1\lambda_2\Omega_3\Omega_4 \approx -4.18385985 \neq 0$$

and

$$\begin{aligned} E_0 &= P_0A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}(\Omega_1Q_0A_7 + |\lambda_1|\Omega_4P_0A_5 + \Omega_4Q_0A_6) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_0A_8 \\ &\approx 11.08984581, \\ E_1 &= P_1A_1 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}(\Omega_1Q_2A_{11} + |\lambda_1|\Omega_4P_1A_5 + \Omega_4Q_2A_{10}) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_1A_8 \\ &\approx 0.1629615871, \\ E_2 &= P_2A_3 + \frac{|\lambda_2|T^{\alpha-1}}{|\Omega|}(\Omega_1Q_1A_7 + |\lambda_1|\Omega_4P_2A_9 + \Omega_4Q_1A_6) + \frac{\Omega_1}{|\Omega|}T^{\alpha-1}P_2A_{12} \\ &\approx 0.34091157, \\ F_0 &= Q_0A_2 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}(\Omega_2P_0A_5 + |\lambda_2|\Omega_3Q_0A_7 + \Omega_3P_0A_8) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_0A_6 \\ &\approx 25.68580671, \\ F_1 &= Q_2A_4 + \frac{|\lambda_1|T^{\beta-1}}{|\Omega|}(\Omega_2P_1A_5 + |\lambda_2|\Omega_3Q_2A_{11} + \Omega_3P_1A_8) + \frac{\Omega_2}{|\Omega|}T^{\beta-1}Q_2A_{10} \\ &\approx 0.68153261, \end{aligned}$$

$$F_2 = Q_1 A_2 + \frac{|\lambda_1| T^{\beta-1}}{|\Omega|} (\Omega_2 P_2 A_9 + |\lambda_2| \Omega_3 Q_1 A_7 + \Omega_3 P_2 A_{12}) + \frac{\Omega_2}{|\Omega|} T^{\beta-1} Q_1 A_6$$

$$\approx 0.5902944$$

and

$$G^* = \max\{1 - (E_1 + F_1), 1 - (E_2 + F_2)\} = \max\{0.1555058, 0.06879403\} = 0.1555058.$$

Therefore, we get

$$E_1 + F_1 \approx 0.84444942 < 1 \quad \text{and} \quad E_2 + F_2 \approx 0.93120597 < 1.$$

Hence, by Theorem 4.3, the problem (4.6) has at least one solution on $[0, \pi]$.

5 Uncoupled integral boundary conditions case

In this section we consider the following system:

$$\begin{cases} D_q^\alpha x(t) = f(t, x(t), I_r^\delta y(t)), & t \in [0, T], 1 < \alpha \leq 2, \\ D_p^\beta y(t) = g(t, y(t), I_z^\varepsilon x(t)), & t \in [0, T], 1 < \beta \leq 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa x(\xi), \\ y(0) = 0, & \lambda_2 I_h^\mu y(\theta) = I_k^\nu y(\tau). \end{cases} \quad (5.1)$$

Lemma 5.1 (Auxiliary lemma, see [20]) *For $h \in C([0, T], \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} D_q^\alpha x(t) = h(t), & t \in [0, T], 1 < \alpha \leq 2, \\ x(0) = 0, & \lambda_1 I_m^\gamma x(\eta) = I_n^\kappa x(\xi), \end{cases} \quad (5.2)$$

is given by

$$x(t) = I_q^\alpha h(t) + \frac{\lambda_1 t^{\alpha-1}}{\Lambda} I_m^\gamma I_q^\alpha h(\eta) - \frac{t^{\alpha-1}}{\Lambda} I_n^\kappa I_q^\alpha h(\xi), \quad (5.3)$$

where

$$\Lambda = \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa + \alpha)} \xi^{\kappa + \alpha - 1} - \lambda_1 \frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma + \alpha)} \eta^{\gamma + \alpha - 1} \neq 0.$$

5.1 Existence results for uncoupled case

In view of Lemma 5.1, we define an operator $\mathcal{T} : X \times Y \rightarrow X \times Y$ by

$$\mathcal{T}(x, y)(t) = \begin{pmatrix} \mathcal{T}_1(x, y)(t) \\ \mathcal{T}_2(x, y)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{T}_1(x, y)(t) &= I_q^\alpha f(s, x(s), I_r^\delta y(s))(t) + \frac{\lambda_1 t^{\alpha-1}}{\Lambda} I_m^\gamma I_q^\alpha f(s, x(s), I_r^\delta y(s))(\eta) \\ &\quad - \frac{t^{\alpha-1}}{\Lambda} I_n^\kappa I_q^\alpha f(s, x(s), I_r^\delta y(s))(\xi) \end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_2(x, y)(t) &= I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(t) + \frac{\lambda_2 t^{\beta-1}}{\Psi} I_h^\mu I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(\theta) \\ &\quad - \frac{t^{\beta-1}}{\Psi} I_k^\nu I_p^\beta g(s, y(s), I_z^\varepsilon x(s))(\tau),\end{aligned}$$

where

$$\Psi = \frac{\Gamma_k(\beta)}{\Gamma_k(\nu + \beta)} \tau^{\nu+\beta-1} - \lambda_2 \frac{\Gamma_h(\beta)}{\Gamma_h(\mu + \beta)} \theta^{\mu+\beta-1} \neq 0.$$

In the sequel, we set constants

$$\begin{aligned}A_{13} &= \frac{\Gamma_n(\alpha + 1) \xi^{\kappa+\alpha}}{\Gamma_n(\kappa + \alpha + 1) \Gamma_q(\alpha + 1)}, \\ A_{14} &= \frac{\Gamma_n(\alpha + \delta + 1) \Gamma_q(\delta + 1) \xi^{\kappa+\alpha+\delta}}{\Gamma_n(\kappa + \alpha + \delta + 1) \Gamma_q(\alpha + \delta + 1) \Gamma_r(\delta + 1)}, \\ A_{15} &= \frac{\Gamma_k(\beta + 1) \tau^{\nu+\beta}}{\Gamma_k(\nu + \beta + 1) \Gamma_p(\beta + 1)}, \\ A_{16} &= \frac{\Gamma_k(\beta + \varepsilon + 1) \Gamma_p(\varepsilon + 1) \tau^{\nu+\beta+\varepsilon}}{\Gamma_k(\nu + \beta + \varepsilon + 1) \Gamma_p(\beta + \varepsilon + 1) \Gamma_z(\varepsilon + 1)}, \\ H_1 &= \overline{M}_1 A_1 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \overline{M}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|} \overline{M}_1 A_{13}, \\ H_2 &= \overline{M}_2 A_3 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \overline{M}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|} \overline{M}_2 A_{14}, \\ L_1 &= \overline{N}_2 A_4 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \overline{N}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|} \overline{N}_2 A_{16}, \\ L_2 &= \overline{N}_1 A_2 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \overline{N}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|} \overline{N}_1 A_{15}.\end{aligned}$$

Now we present the existence and the uniqueness result for the problem (5.1). We do not provide the proof of this result as it is similar to the one for Theorem 4.1.

Theorem 5.2 Assume that $f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions and there exist constants $\overline{K}_i, \overline{L}_i, i = 1, 2$ such that for all $t \in [0, T]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$,

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq \overline{M}_1 |u_1 - v_1| + \overline{M}_2 |u_2 - v_2|$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq \overline{N}_1 |u_1 - v_1| + \overline{N}_2 |u_2 - v_2|.$$

In addition, assume that

$$H_1 + H_2 + L_1 + L_2 < 1.$$

Then the boundary value problem (5.1) has a unique solution on $[0, T]$.

The second result deals with the existence of solutions for the problem (5.1), is analogous to Theorem 4.3 and is given below.

Theorem 5.3 Assume that there exist real constants $\bar{m}_i, \bar{n}_i \geq 0$ ($i = 1, 2$), and $\bar{m}_0 > 0, \bar{n}_0 > 0$ such that $\forall x_i \in \mathbb{R}$ ($i = 1, 2$) we have

$$\begin{aligned} |f(t, x_1, x_2)| &\leq \bar{m}_0 + \bar{m}_1|x_1| + \bar{m}_2|x_2|, \\ |g(t, x_1, x_2)| &\leq \bar{n}_0 + \bar{n}_1|x_1| + \bar{n}_2|x_2|. \end{aligned}$$

In addition it is assumed that

$$U_1 + V_1 < 1 \quad \text{and} \quad U_2 + V_2 < 1,$$

where $U_i, V_i, i = 1, 2$, are given by

$$\begin{aligned} U_1 &= \bar{m}_1 A_1 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \bar{m}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|} \bar{m}_1 A_{13}, \\ U_2 &= \bar{m}_2 A_3 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \bar{m}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|} \bar{m}_2 A_{14}, \\ V_1 &= \bar{n}_2 A_4 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \bar{n}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|} \bar{n}_2 A_{16}, \\ V_2 &= \bar{n}_1 A_2 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \bar{n}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|} \bar{n}_1 A_{15}. \end{aligned}$$

Then the boundary value problem (5.1) has at least one solution on $[0, T]$.

Proof By setting

$$G^* = \min\{1 - (U_1 + V_1), 1 - (U_2 + V_2)\},$$

the proof is similar to that of Theorem 4.3. So we omit it. $\square$

5.2 Examples

In this subsection, we present two examples of uncoupled case of nonlocal conditions.

Example 5.4 Consider the following system of fractional q -integro-difference equations with q -integral conditions:

$$\begin{cases} D_{1/9}^{3/2} x(t) = \frac{e^{-t} \sin \pi t}{(t+3)^2} \cdot \frac{|x(t)|}{9+|x(t)|} + \frac{\cos^2 \pi t}{\pi(t+7)^2} \cdot I_{1/8}^{\sqrt{2}} y(t) - \frac{1}{3}, & 0 < t < 3, \\ D_{1/7}^{5/4} y(t) = \frac{2\pi e^{-2t}}{(7\pi+t)^2} \cdot \frac{|y(t)|}{2+|y(t)|} + \frac{\sin 2\pi t}{(3e^t+5)^2} \cdot I_{1/6}^{\sqrt{3}} x(t) + \frac{1}{3}, \\ x(0) = 0, & \frac{1}{2} I_{1/3}^{\sqrt{2}} x(\frac{3}{4}) = I_{1/4}^{\sqrt{3}} x(\frac{9}{4}), \\ y(0) = 0, & I_{1/6}^{\sqrt{\pi}} y(3) + \frac{1}{3} I_{1/5}^{\pi} y(\frac{3}{2}) = 0. \end{cases} \quad (5.4)$$

Here $\alpha = 3/2, \delta = \sqrt{2}, \beta = 5/4, \varepsilon = \sqrt{3}, \gamma = \sqrt{2}/2, \kappa = \sqrt{3}/2, \mu = \pi, v = \sqrt{\pi}, q = 1/9, r = 1/8, p = 1/7, z = 1/6, m = 1/3, n = 1/4, h = 1/5, k = 1/6, \eta = 3/4, \xi = 9/4, \theta = 3/2, \tau = 3, \lambda_1 =$

$1/2$, $\lambda_2 = -1/3$, $T = 3$, $f(t, x, I_r^\delta y) = (e^{-t} \sin \pi t / (t+3)^2)(|x|/(9+|x|)) + (\cos^2 \pi t / \pi(t+7)^2) I_{1/8}^{\sqrt{2}} y - (1/3)$, and $g(t, y, I_r^\delta x) = (2\pi e^{-2t} / (7\pi + t^2))(|y|/(2+|y|)) + (\sin 2\pi t / (3e^t + 5)^2) I_{1/6}^{\sqrt{3}} x + (1/3)$. Since

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq \frac{1}{81}|u_1 - v_1| + \frac{1}{49\pi}|u_2 - v_2|$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq \frac{1}{36\pi}|u_1 - v_1| + \frac{1}{64}|u_2 - v_2|,$$

then the assumptions of Theorem 5.2 are satisfied with $\overline{M}_1 = 1/81$, $\overline{M}_2 = 1/49\pi$, $\overline{N}_1 = 1/36\pi$, and $\overline{N}_2 = 1/64$. By using the Maple program, we can find that

$$\begin{aligned}\Lambda &= \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa + \alpha)} \xi^{\kappa + \alpha - 1} - \lambda_1 \frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma + \alpha)} \eta^{\gamma + \alpha - 1} \approx 1.9245172 \neq 0, \\ \Psi &= \frac{\Gamma_k(\beta)}{\Gamma_k(\nu + \beta)} \tau^{\nu + \beta - 1} - \lambda_2 \frac{\Gamma_h(\beta)}{\Gamma_h(\mu + \beta)} \theta^{\mu + \beta - 1} \approx 8.37494759 \neq 0\end{aligned}$$

and

$$\begin{aligned}H_1 &= \overline{M}_1 A_1 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \overline{M}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|} \overline{M}_1 A_{13} \approx 0.11268247, \\ H_2 &= \overline{M}_2 A_3 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \overline{M}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|} \overline{M}_2 A_{14} \approx 0.19713212, \\ L_1 &= \overline{N}_2 A_4 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \overline{N}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|} \overline{N}_2 A_{16} \approx 0.58490031, \\ L_2 &= \overline{N}_1 A_2 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \overline{N}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|} \overline{N}_1 A_{15} \approx 0.06286768.\end{aligned}$$

Therefore, we get

$$H_1 + H_2 + L_1 + L_2 \approx 0.95758259 < 1.$$

Hence, by Theorem 5.2, the problem (5.4) has a unique solution on $[0, 3]$.

Example 5.5 Consider the following system of fractional q -integro-difference equations:

$$\begin{cases} D_{\sqrt{2}/2}^{\sqrt{\pi}} x(t) = \frac{25e^{-t}}{(e^{-t}+4)^2} \cdot \frac{|x(t)|}{5+|x(t)|} + \frac{3\pi^2}{(t+3\pi)^2} \cdot I_{\sqrt{3}/2}^{1/2} y(t) + \frac{1}{\sqrt{5}}, & 0 < t < 1, \\ D_{\pi/4}^{\pi/2} y(t) = \frac{9e^{-t} \cos^2 \pi t}{(t+6)^2} \cdot \frac{|y(t)|}{1+|y(t)|} + \frac{6}{(t+6)^2} \cdot I_{\pi/5}^{3/2} x(t) + \frac{\sqrt{2}}{3}, \\ x(0) = 0, & \frac{\sqrt{3}}{2} I_{\pi/6}^{4/5} x(1) + I_{\pi/7}^{2/3} x(\frac{3}{4}) = 0, \\ y(0) = 0, & 5 I_{\pi/8}^{\sqrt{3}} y(\frac{1}{2}) = I_{\pi/9}^{1/3} y(\frac{1}{4}). \end{cases} \quad (5.5)$$

Here $\alpha = \sqrt{\pi}$, $\delta = 1/2$, $\beta = \pi/2$, $\varepsilon = 3/2$, $\gamma = 4/5$, $\kappa = 2/3$, $\mu = \sqrt{3}$, $\nu = 1/3$, $q = \sqrt{2}/2$, $r = \sqrt{3}/2$, $p = \pi/4$, $z = \pi/5$, $m = \pi/6$, $n = \pi/7$, $h = \pi/8$, $k = \pi/9$, $\eta = 1$, $\xi = 3/4$, $\theta = 1/2$, $\tau = 1/4$, $\lambda_1 = -\sqrt{3}/2$, $\lambda_2 = 5$, $T = 1$, $f(t, x, I_r^\delta y) = (25e^{-t} / (e^{-t} + 4)^2)(|x|/(5+|x|)) + (3\pi^2 / (t+3\pi)^2) I_{\sqrt{3}/2}^{1/2} y + (1/\sqrt{5})$, and $g(t, y, I_r^\delta x) = (9e^{-t} \cos^2 \pi t / (t+6)^2)(|y|/(1+|y|)) + (6/(t+6)^2) I_{\pi/5}^{3/2} x +$

$(\sqrt{2}/3)$. Since

$$|f(t, x_1, x_2)| \leq \frac{1}{\sqrt{5}} + \frac{1}{5}|x_1| + \frac{1}{3}|x_2|$$

and

$$|g(t, x_1, x_2)| \leq \frac{\sqrt{2}}{3} + \frac{1}{4}|x_1| + \frac{1}{6}|x_2|,$$

then the assumptions of Theorem 5.3 are satisfied with $\bar{m}_0 = 1/\sqrt{5}$, $\bar{m}_1 = 1/5$, $\bar{m}_2 = 1/3$, $\bar{n}_0 = \sqrt{2}/3$, $\bar{n}_1 = 1/4$, and $\bar{n}_2 = 1/6$. By using the Maple program, we can find that

$$\begin{aligned}\Lambda &= \frac{\Gamma_n(\alpha)}{\Gamma_n(\kappa + \alpha)} \xi^{\kappa + \alpha - 1} - \lambda_1 \frac{\Gamma_m(\alpha)}{\Gamma_m(\gamma + \alpha)} \eta^{\gamma + \alpha - 1} \approx 1.21235918 \neq 0, \\ \Psi &= \frac{\Gamma_k(\beta)}{\Gamma_k(\nu + \beta)} \tau^{\nu + \beta - 1} - \lambda_2 \frac{\Gamma_h(\beta)}{\Gamma_h(\mu + \beta)} \theta^{\mu + \beta - 1} \approx -0.32647283 \neq 0\end{aligned}$$

and

$$\begin{aligned}U_1 &= \bar{m}_1 A_1 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \bar{m}_1 A_5 + \frac{T^{\alpha-1}}{|\Lambda|} \bar{m}_1 A_{13} \approx 0.23965603, \\ U_2 &= \bar{m}_2 A_3 + \frac{|\lambda_1| T^{\alpha-1}}{|\Lambda|} \bar{m}_2 A_9 + \frac{T^{\alpha-1}}{|\Lambda|} \bar{m}_2 A_{14} \approx 0.27471434, \\ V_1 &= \bar{n}_2 A_4 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \bar{n}_2 A_{11} + \frac{T^{\beta-1}}{|\Psi|} \bar{n}_2 A_{16} \approx 0.04758258, \\ V_2 &= \bar{n}_1 A_2 + \frac{|\lambda_2| T^{\beta-1}}{|\Psi|} \bar{n}_1 A_7 + \frac{T^{\beta-1}}{|\Psi|} \bar{n}_1 A_{15} \approx 0.36424461.\end{aligned}$$

Therefore, we get

$$U_1 + V_1 \approx 0.28723861 < 1 \quad \text{and} \quad U_2 + V_2 \approx 0.63895895 < 1.$$

Hence, by Theorem 5.3, the problem (5.5) has at least one solution on $[0, 1]$.

References

1. Jackson, FH: q -Difference equations. Am. J. Math. **32**, 305-314 (1910)
2. Kac, V, Cheung, P: Quantum Calculus. Springer, New York (2002)
3. Al-Salam, WA: Some fractional q -integrals and q -derivatives. Proc. Edinb. Math. Soc. **15**(2), 135-140 (1966/1967)
4. Agarwal, RP: Certain fractional q -integrals and q -derivatives. Proc. Camb. Philos. Soc. **66**, 365-370 (1969)
5. Ernst, T: The history of q -calculus and a new method. UUDM Report 2000:16, Department of Mathematics, Uppsala University (2000)
6. Ferreira, RAC: Nontrivial solutions for fractional q -difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2010**, 70 (2010)
7. Goodrich, CS: Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. Comput. Math. Appl. **61**, 191-202 (2011)
8. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
9. Srivastava, HM: Some generalizations and basic (or q -) extensions of the Bernoulli, Euler and Genocchi polynomials. Appl. Math. Inf. Sci. **5**, 390-444 (2011)
10. Ma, J, Yang, J: Existence of solutions for multi-point boundary value problem of fractional q -difference equation. Electron. J. Qual. Theory Differ. Equ. **2011**, 92 (2011)
11. Graef, JR, Kong, L: Positive solutions for a class of higher order boundary value problems with fractional q -derivatives. Appl. Math. Comput. **218**, 9682-9689 (2012)
12. Ahmad, B, Ntouyas, SK, Purnaras, IK: Existence results for nonlocal boundary value problems of nonlinear fractional q -difference equations. Adv. Differ. Equ. **2012**, 140 (2012)
13. Ahmad, B, Ntouyas, SK: Existence of solutions for nonlinear fractional q -difference inclusions with nonlocal Robin (separated) conditions. Mediterr. J. Math. **10**, 1333-1351 (2013)
14. Ahmad, B, Ntouyas, SK, Alsaedi, A, Al-Hutami, H: Nonlinear q -fractional differential equations with nonlocal and sub-strip type boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2014**, 26 (2014)
15. Ahmad, B, Nieto, JJ, Alsaedi, A, Al-Hutami, H: Existence of solutions for nonlinear fractional q -difference integral equations with two fractional orders and nonlocal four-point boundary conditions. J. Franklin Inst. **351**, 2890-2909 (2014)
16. Alsaedi, A, Ahmad, B, Al-Hutami, H: A study of nonlinear fractional q -difference equations with nonlocal integral boundary conditions. Abstr. Appl. Anal. **2013**, Article ID 410505 (2013)
17. Li, X, Han, Z, Sun, S: Existence of positive solutions of nonlinear fractional q -difference equation with parameter. Adv. Differ. Equ. **2013**, 260 (2013)
18. Ahmad, B, Nieto, JJ, Alsaedi, A, Al-Hutami, H: Boundary value problems of nonlinear fractional q -difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions. Filomat **28**(8), 1719-1736 (2014)
19. Annaby, MH, Mansour, ZS: q -Fractional Calculus and Equations. Lecture Notes in Mathematics, vol. 2056. Springer, Berlin (2012)
20. Asawasamrit, S, Tariboon, J, Ntouyas, SK: Existence of solutions for fractional q -integrodifference equations with nonlocal fractional q -integral conditions. Abstr. Appl. Anal. **2014**, Article ID 474138 (2014)
21. Pongarm, N, Asawasamrit, S, Tariboon, J, Ntouyas, SK: Multi-strip fractional q -integral boundary value problems for nonlinear fractional q -difference equations. Adv. Differ. Equ. **2014**, 193 (2014)
22. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)

Chapter VII

Fixed Point Theorems and Null Point Theorems in b-Metric Spaces and CAT(0) spaces

7.1 Dislocated Quasi-b-Metric Spaces and Point Theorems for Cyclic Weakly Contractions

In this section, we introduce the notions of type q-cyclic-weak Banach contraction, q-cyclic- ϕ -contraction and derive the existence of fixed point theorems on dislocated quasi-b-metric spaces. Our main theorem extends and unifies existing results in the recent literature.

1. Introduction and Preliminaries

Banach contraction principle was introduced in 1922 by Banach [3]. In 2001, Rhoades [7] introduced weakly contractive as follows:

(i) A mapping $T: X \rightarrow X$ is said to be a *weakly contractive* if for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

where $\phi: [0, \infty) \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

If one takes $\phi(t) = (1 - k)t$, where $0 < k < 1$, a weak contraction reduces to a Banach contraction.

Now, we recall the definition of cyclic map. Let A and B be nonempty subsets of a metric space (X, d) and $T: A \cup B \rightarrow A \cup B$, then T is called a *cyclic map* iff $T(A) \subseteq B$ and $T(B) \subseteq A$. In 2003, Kirk et al. [4]

introduced cyclic contraction as follows:

(ii) A cyclic map $T: A \cup B \rightarrow A \cup B$ is said to be a *cyclic contraction* if there exists $a \in [0, 1)$ such that

$$d(Tx, Ty) \leq ad(x, y),$$

for all $x \in A$ and $y \in B$.

In 2013, K. Zoto [9] introduced d-cyclic- ϕ -contraction follows:

(iii) A cyclic map $T: A \cup B \rightarrow A \cup B$ is said to be a *d-cyclic- ϕ -contraction* if $\phi \in \Phi$ such that

$$d(Tx, Ty) \leq \phi(d(x, y)),$$

for all $x \in A, y \in B$, where Φ the family of non-decreasing functions: $\phi: [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$, where n is the n -th iterate of ϕ .

Lemma 1.1. *Suppose that the function $\phi: [0, \infty) \rightarrow [0, \infty)$ is non-decreasing, then for each $t > 0$, $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ implies $\phi(t) < t$.*

If (X, d) is complete metric spaces, at least one of (i), (ii) and (iii) holds, then T has a unique fixed point (see[7]-[9]). Recently, Klin-eam and Suanoom [5] introduced dislocated quasi b-metric spaces, which is a new generalization of quasi b-metric space (see[8]), b-metric-like space (see[1]), b-metric space (see[2]), metric space, etc. as follows:

Definition 1.2. [5] Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

- (d1) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;
- (d2) $d(x, y) \leq s[d(x, z) + d(z, y)]$, for all $x, y, z \in X$.

The pair (X, d) is then called a *dislocated quasi b-metric space (or simply dqb-metric)*. The number s is called to be the coefficient of (X, d) .

Remark 1.3. When, in addition, the condition of $d(x, y) = d(y, x)$ and $d(x, x) = 0$ are true, then d is a b-metric.

Definition 1.4. Let $\{x_n\}$ be a sequence in a dqb-metric space (X, d) .

- (1) A sequence $\{x_n\}$ *dislocated quasi-b-converges (for short, dqb-converges)* to $x \in X$ if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0 = \lim_{n \rightarrow \infty} d(x, x_n).$$

In this case x is called a dqb-limit of $\{x_n\}$ and we write $(x_n \rightarrow x)$.

- (2) A sequence $\{x_n\}$ is called *dislocated quasi-b-Cauchy (for short, dqb-Cauchy)*, if

$$\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0 = \lim_{n, m \rightarrow \infty} d(x_m, x_n).$$

- (3) A dqb-metric space (X, d) is complete if every dqb-Cauchy sequence is dqb-convergent in X . Moreover, they introduced the notion of dqb-cyclic-Banach, dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such space.

In this paper, we study the properties of dislocated quasi-b-metric spaces and introduce dqb-cyclic-weak Banach contraction, dqb-cyclic- ϕ -contraction and derive the existence of fixed point theorems in dislocated quasi-b-metric spaces. Our main theorem extends and unifies existing results in the recent literature.

2. Main results

Every dislocated quasi-b-metric space (X, d) can be considered as a topological space on which the topology is introduced by taking, for any $x \in X$, the collection $\{B_r(x) | r > 0\}$ as a base of the neighbourhood filter of the point x . Here the ball $B_r(x)$ is defined by the equality $B_r(x) = \{y \in X | \max\{d(x, y), d(y, x)\} < r\}$.

Definition 2.1. [6] Let X be topological space. Then X is said to be *Hausdorff topological space* if for any distinct points $x, y \in X$, there exists two open sets G and H such that $x \in G$, and $y \in H$ and $G \cap H = \emptyset$.

Proposition 2.2. *Every dqb-metric space is Hausdorff topological space.*

Proof. Let x and y be two distinct points in X . Then $d(x, y) > 0$ and $d(y, x) > 0$. Choose $\delta = \frac{d(x, y)}{2s}$. Then, there exists

$$B_\delta(x) = \{z \in X | \max\{d(x, z), d(z, x)\} < \delta\}$$

and

$$B_\delta(y) = \{z \in X | \max\{d(y, z), d(z, y)\} < \delta\}$$

such that $x \in B_\delta(x)$ and $y \in B_\delta(y)$.

To show that $B_\delta(x) \cap B_\delta(y) = \emptyset$, suppose that $B_\delta(x) \cap B_\delta(y) \neq \emptyset$. Then, there exists $z \in B_\delta(x) \cap B_\delta(y)$. We have

$$\begin{aligned} d(x, y) &\leq sd(x, z) + sd(z, y) \\ &\leq s \max\{d(x, z), d(z, x)\} + s \max\{d(y, z), d(z, y)\} \\ &< s\delta + s\delta \\ &= d(x, y) \end{aligned}$$

So, $d(x, y) < d(x, y)$ a contradiction. Therefore $B_\delta(x) \cap B_\delta(y) = \emptyset$. □

Proposition 2.3. *Every dqb-convergent sequence in a dqb-metric space (X, d) is dqb-Cauchy sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-convergent. Then there exists $x \in X$ such that $x_n \rightarrow x$, that is

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0 = \lim_{n \rightarrow \infty} d(x, x_n).$$

Consider,

$$d(x_n, x_m) \leq sd(x_n, x) + sd(x, x_m).$$

Taking limit as $n, m \rightarrow \infty$ we obtain

$$\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0.$$

Similarly,

$$\lim_{n, m \rightarrow \infty} d(x_m, x_n) = 0.$$

Therefore $\{x_n\}$ is dqb-Cauchy. □

Definition 2.4. A subset S of a dqb-metric space (X, d) is bounded if there exists $\bar{x}$, $M \in (0, \infty)$ such that $d(x, \bar{x}) \leq M$ for all $x \in S$.

Proposition 2.5. *Every dqb-convergent sequence in a dqb-metric space (X, d) is bounded sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-convergent. Then there exists $x \in X$ such that $x_n \rightarrow x$, that is

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0 = \lim_{n \rightarrow \infty} d(x, x_n).$$

Let $\epsilon = 1$. Then there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ and $d(x, x_n) < \epsilon$, for all $n \geq n_0$. Choose

$$K = \max\{d(x_1, x), d(x_2, x), \dots, d(x_{n_0-1}, x), 1\}.$$

Thus, $d(x_n, x) \leq K$, for all $n \in \mathbb{N}$, and so $\{x_n\}$ is bounded sequence. $\square$

Proposition 2.6. *Every dqb-Cauchy sequence in a dqb-metric space (X, d) is bounded sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-Cauchy. Then

$$\lim_{n \rightarrow \infty} d(x_n, x_m) = 0 = \lim_{n \rightarrow \infty} d(x_m, x_n).$$

Let $\epsilon = 1$. Then there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ and $d(x_m, x_n) < 1$ for all $n, m \geq n_0$. Let p be any point in the space, and let

$$k = \max_{i \leq m} d(x_i, p),$$

The maximum exists, since $\{x_i : i \leq m\}$ is a finite set. If $n \leq m$, then $d(x_n, p) \leq k$. If $n > m$, then $d(x_n, p) \leq d(x_n, x_m) + d(x_m, p) \leq 1 + k$ for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded sequence. $\square$

The next two propositions for subsequence follow immediately from definitions of dqb-convergent sequence and dqb-Cauchy sequence respectively.

Proposition 2.7. *Every subsequence of dqb-convergent sequence in a dqb-metric space (X, d) is dqb-convergent sequence.*

Proposition 2.8. *Every subsequence of dqb-Cauchy sequence in a dqb-metric space (X, d) is dqb-Cauchy sequence.*

Proposition 2.9. *Let $\{x_n\}$ be sequence in a dqb-metric space (X, d) . Then $x_n \rightarrow x$ if and only if $d(x_n, x) \rightarrow 0$ and $d(x, x_n) \rightarrow 0$.*

Proof. Suppose that $x_n \rightarrow x$. Then

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0$$

Thus $d(x_n, x) \rightarrow 0$ and $d(x, x_n) \rightarrow 0$.

Conversely, Suppose that $d(x_n, x) \rightarrow 0$ and $d(x, x_n) \rightarrow 0$. Then

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0$$

By definition of dqb-convergent sequence, we get $x_n \rightarrow x$. $\square$

Proposition 2.10. *Let $\{x_n\}$ be sequence in a dqb-metric space (X, d) . If $x_n \rightarrow x$ and $x_n \rightarrow y$, then $x = y$.*

Proof. Suppose that $x_n \rightarrow x$ and $x_n \rightarrow y$. Then

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = \lim_{n \rightarrow \infty} d(x_n, y) = \lim_{n \rightarrow \infty} d(y, x_n) = 0$$

Consider,

$$0 \leq d(x, y) \leq sd(x, x_n) + sd(x_n, y)$$

and

$$0 \leq d(y, x) \leq sd(y, x_n) + sd(x_n, x).$$

Taking limit as $n, m \rightarrow \infty$, we obtain

$$d(x, y) = d(y, x) = 0.$$

Therefore $x = y$. $\square$

Now, we begin with introducing the property of a continuous functions.

Definition 2.11. Suppose that (X, d_X) and (Y, d_Y) are dislocated quasi-b-metric spaces, $E \subset X$, $f : E \rightarrow Y$ and $p \in E$. Then f is continuous at p iff for all $\epsilon > 0$ there exists $\delta > 0$ such that $\max\{d_Y(fx, fp), d_Y(fp, fx)\} < \epsilon$ for all $x \in E$, when $\max\{d_X(x, p), d_X(p, x)\} < \delta$.

Theorem 2.12. Let (X, d_X) and (Y, d_Y) be dislocated quasi-b-metric spaces, $E \subset X$, $f : E \rightarrow Y$ and $p \in E$. Then f is continuous at p if and only if for every dislocated quasi-b-converges sequence $\{x_n\}$ in X , $\lim_{n \rightarrow \infty} fx_n = fp$.

Proof. Suppose that f is continuous at p and $\{x_n\}$ converges to p .

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $\max\{d_Y(fx, fp), d_Y(fp, fx)\} < \epsilon$, when $\max\{d_X(x, p), d_X(p, x)\} < \delta$, for all $x \in E$.

Since $\{x_n\}$ converges to p , there exists $N \in \mathbb{N}$ such that $\max\{d_X(x_n, p), d_X(p, x_n)\} < \delta$, for all $n \geq N$.

Since f is continuous at p , we have $\max\{d_Y(fx_n, fp), d_Y(fp, fx_n)\} < \epsilon$, for all $n \geq N$.

Hence $\lim_n fx_n = fp$.

Conversely, let $x \in X$ and assume in the contrary that

$$\exists \epsilon > 0 \forall \delta > 0 : \max\{d_X(x, p), d_X(p, x)\} < \delta, \max\{d_Y(fx, fp), d_Y(fp, fx)\} \geq \epsilon.$$

Applying these successively for all $\delta = \frac{1}{k}$, we find a sequence $\{x_k\}$ such that $\max\{d_X(x_k, p), d_X(p, x_k)\} < \frac{1}{k}$ and $\max\{d_Y(fx_k, fp), d_Y(fp, fx_k)\} \geq \epsilon$. Thus

$$\lim_{k \rightarrow \infty} x_k = p.$$

By assumption, we have

$$\lim_{k \rightarrow \infty} fx_k = fp.$$

Hence, there exists a k_0 such that for all $k > k_0$

$$\max\{d_Y(fx_k, fp), d_Y(fp, fx_k)\} < \epsilon$$

a contradiction. □

Definition 2.13. Suppose that (X, d_X) and (Y, d_Y) are dislocated quasi-b-metric spaces, $E \subset X$, $f : E \rightarrow Y$ and $p \in E$. Then f is continuous on E iff f is continuous at p for all $p \in E$.

Next, we begin with prove fixed point theorems.

Definition 2.14. Let A and B be nonempty closed subsets of a dislocated quasi-b-metric spaces (X, d) . A cyclic map $T : A \cup B \rightarrow A \cup B$ is said to be a *dqb-cyclic-weak contraction* or *dqb-cyclic-weakly contraction* and if, for all $x \in A, y \in B$ such that

$$sd(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \quad (2.1)$$

where $\psi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

Lemma 2.15. Let (X, d_X) and (Y, d_Y) be dislocated quasi-b-metric spaces, A and B be nonempty closed subsets of a dislocated quasi-b-metric spaces (X, d) . A cyclic map $T : A \cup B \rightarrow A \cup B$. If T is *dqb-cyclic-weak contraction*, then T is continuous.

Proof. Let $\epsilon > 0$, all $x \in A \cup B$ and fixed $p \in A \cup B$. Suppose that $\max\{d_X(x, p), d_C(p, x)\} < \delta$. Choose $\epsilon = \frac{\delta}{s}$. Since T is dqb-cyclic-weak contraction, we have

$$\begin{aligned} sd(Tx, Tp) &\leq d(x, p) - \psi(d(x, p)) \\ &\leq d(x, p) \\ &< \delta \end{aligned}$$

and

$$\begin{aligned} sd(Tp, Tx) &\leq d(p, x) - \psi(d(p, x)) \\ &\leq d(p, x) \\ &< \delta. \end{aligned}$$

So, $d(Tx, Tp) < \epsilon$ and $d(Tp, Tx) < \epsilon$. Thus T is continuous at p , and hence T is continuous on $A \cup B$. $\square$

Now, we present a fixed point theorem related to dqb-cyclic-weak contraction.

Theorem 2.16. *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d) . Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-weak contraction. Then, T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$ be fixed. Using contractive condition in assumptions, we have

$$\begin{aligned} d(T^2x, Tx) &\leq sd(T^2x, Tx) \\ &= sd(T(Tx), Tx) \\ &\leq d(Tx, x) - \psi(d(Tx, x)), \\ &\leq d(Tx, x), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(Tx, T^2x) &\leq sd(Tx, T^2x) \\ &= sd(Tx, T(Tx)) \\ &\leq d(x, Tx) - \psi(d(x, Tx)), \\ &\leq d(x, Tx). \end{aligned} \tag{2.3}$$

So,

$$d(T^3x, T^2x) \leq d(T^2x, Tx) - \psi(d(T^2x, Tx)), \tag{2.4}$$

and

$$d(T^2x, T^3x) \leq d(Tx, T^2x) - \psi(d(Tx, T^2x)). \tag{2.5}$$

For all $n \in \mathbb{N}$, we get

$$d(T^{n+2}x, T^{n+1}x) \leq d(T^{n+1}x, T^n x) - \psi(d(T^{n+1}x, T^n x)) \tag{2.6}$$

and

$$d(T^{n+1}x, T^{n+2}x) \leq d(T^n x, T^{n+1}x) - \psi(d(T^n x, T^{n+1}x)). \tag{2.7}$$

Set $\varsigma_n = d(T^{n+1}x, T^n x)$ and $\tau_n = d(T^n x, T^{n+1}x)$.

By inequalities (2.6) and (2.7), we get

$$\varsigma_{n+1} \leq \varsigma_n - \psi(\varsigma_n) \leq \varsigma_n \quad (2.8)$$

and

$$\tau_{n+1} \leq \tau_n - \psi(\tau_n) \leq \tau_n. \quad (2.9)$$

Thus $\{\varsigma_n\}$ and $\{\tau_n\}$ are decreasing sequences of non-negative real numbers, and hence possess a $\lim_{n \rightarrow \infty} \varsigma_n = \varsigma \geq 0$ and $\lim_{n \rightarrow \infty} \tau_n = \tau \geq 0$. Suppose that $\varsigma > 0$. Since ψ is nondecreasing, $\psi(\varsigma_n) \geq \psi(\varsigma) > 0$. By inequality (2.8), we have $\varsigma_{n+1} \leq \varsigma_n - \psi(\varsigma)$. Thus $\varsigma_{N+m} \leq \varsigma_N - N\psi(\varsigma)$, a contradiction for N large enough. Therefore $\varsigma = 0$.

Similarly, $\tau = 0$. Next, we prove that the sequence is a $\{T^n x\}$ Cauchy. Suppose that $\{T^n x\}$ is not Cauchy, then there exist $\epsilon > 0$ and subsequence $\{T^{m_k} x\}$ and $\{T^{n_k} x\}$ with $m_k > n_k \geq n$ such that $d(T^{m_k} x, T^{n_k} x) \geq \epsilon$ and $d(T^{m_k-1} x, T^{n_k-1} x) < \epsilon$. Now, we consider

$$sd(T^{m_k} x, T^{n_k} x) \leq d(T^{m_k-1} x, T^{n_k-1} x) - \psi(d(T^{m_k-1} x, T^{n_k-1} x)) \quad (2.10)$$

$$\leq d(T^{m_k-1} x, T^{n_k-1} x) \quad (2.11)$$

which implies that

$$s\epsilon \leq d(T^{m_k-1} x, T^{n_k-1} x). \quad (2.12)$$

Take limit inferior in (2.12) as $k \rightarrow \infty$, we get

$$\epsilon s \leq \liminf d(T^{m_k-1} x, T^{n_k-1} x). \quad (2.13)$$

We have

$$\begin{aligned} d(T^{m_k-1} x, T^{n_k-1} x) &\leq sd(T^{m_k-1} x, T^{n_k} x) + sd(T^{n_k} x, T^{n_k-1} x) \\ &< s\epsilon + sd(T^{n_k} x, T^{n_k-1} x). \end{aligned} \quad (2.14)$$

Take limit superior in (2.14) as $k \rightarrow \infty$, we get

$$\limsup d(T^{m_k-1} x, T^{n_k-1} x) \leq s\epsilon. \quad (2.15)$$

By (2.13) and (2.15), we get

$$\lim d(T^{m_k-1} x, T^{n_k-1} x) = s\epsilon. \quad (2.16)$$

Letting $k \rightarrow \infty$ in (2.10), by property of ψ and (2.16), we get

$$s\epsilon \leq s\epsilon - \psi(s\epsilon) < s\epsilon \quad (2.17)$$

which is a contradiction. Hence $\{T^n x\}$ is a dqb-Cauchy sequence. Since (X, d) is complete, we have $\{T^n x\}$ converges to some $z \in X$. We note, that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to same limit z . Since A and B are closed, we have $z \in A \cap B$, and hence $A \cap B \neq \emptyset$. The continuity of T implies that the limit is a fixed point. Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$.

Then, we have

$$d(z, z^*) = d(Tz, Tz^*) \leq sd(Tz, Tz^*) \leq d(z, z^*) - \psi(d(z, z^*)) \leq d(z, z^*). \quad (2.18)$$

On the other hand,

$$d(z^*, z) = d(Tz^*, Tz) \leq sd(Tz^*, Tz) \leq d(z^*, z) - \psi(d(z^*, z)) \leq d(z^*, z). \quad (2.19)$$

By forms (2.22) and (2.19), we obtain that $d(z, z^*) = d(z^*, z) = 0$, this implies that $z^* = z$. Therefore z is a unique fixed point of T . This completes the proof. $\square$

Example 2.17. Let $X = [-1, 1]$ and $T : A \cup B \rightarrow A \cup B$ be defined by $Tx = \frac{-x}{3}$, and $\psi(t) = \frac{t}{50}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Defined the function $d : X^2 \rightarrow [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X (see[??]).

Let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{3} \leq \frac{1}{3}$. Thus, $Tx \in B$. On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{3} \leq \frac{-x}{3} \leq 0$. Thus, $Tx \in A$.

Hence the map T is cyclic on X , because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$\begin{aligned} 2d(Tx, Ty) &= 2(|Tx - Ty|^2 + \frac{1}{10}|Tx| + \frac{1}{11}|Ty|) \\ &= 2(|\frac{-x}{3} - \frac{-y}{3}|^2 + \frac{1}{10}|\frac{-x}{3}| + \frac{1}{11}|\frac{-y}{3}|) \\ &= \frac{49}{50}(\frac{100}{441}|x - y|^2 + \frac{50}{1470}|x| + \frac{100}{539}|y|) \\ &\leq \frac{49}{50}(|x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|) \\ &= |x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y| - \psi(|x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|) \\ &= d(x, y) - \psi(d(x, y)). \end{aligned}$$

Thus T satisfies dqb-cyclic-weak contraction of Theorem 2.16 and 0 is the unique fixed point of T .

Definition 2.18. Let A and B be nonempty subsets of a dislocated quasi-b-metric spaces. (X, d) . A cyclic map $T : A \cup B \rightarrow A \cup B$ is said to be a *dqb-cyclic- ϕ -contraction* and if there exists $k \in [0, 1)$ and $s \geq 1$ such that

$$sd(Tx, Ty) \leq \phi(d(x, y)), \quad (2.20)$$

for all $x \in A, y \in B$, where Φ the family of non-decreasing functions: $\phi : [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$, where n is the n -th iterate of ϕ .

Theorem 2.19. Let A and B be nonempty closed subsets of a complete dislocated quasi-b-metric space (X, d) . Let T be a cyclic mapping that satisfies the condition a dqb-cyclic- ϕ -contraction. Then, T has a unique fixed point in $A \cap B$.

Proof. Let $x \in A$ be fixed, then using contractive condition of theorem, we have

$$\begin{aligned} sd(T^2x, Tx) &= sd(T(Tx), Tx) \\ &\leq \phi(d(Tx, x)), \end{aligned}$$

and

$$\begin{aligned} sd(Tx, T^2x) &= sd(Tx, T(Tx)) \\ &\leq \phi(d(x, Tx)). \end{aligned}$$

Inductively, we have for all $n \in \mathbb{N}$, we get

$$s^n d(T^{n+1}x, T^n x) \leq \phi^n(d(Tx, x)),$$

and

$$s^n d(T^n x, T^{n+1}x) \leq \phi^n(d(x, Tx)).$$

Let $\epsilon > 0$ be fixed and $n(\epsilon) \in \mathbb{N}$, such that

$$\Sigma_{n \geq n(\epsilon)} \phi^n(d(Tx, x)) < \epsilon,$$

and

$$\Sigma_{n \geq n(\epsilon)} \phi^n(d(x, Tx)) < \epsilon.$$

Let $n, m \in \mathbb{N}$ with $m > n > n(\epsilon)$, using the triangular inequality, we have:

$$\begin{aligned} d(T^m x, T^n x) &\leq s^{m-n} d(T^m x, T^{m-1} x) + s^{m-n-1} d(T^{m-1} x, T^{m-2} x) + \dots + s d(T^{n+1} x, T^n x) \\ &\leq s^{m-1} d(T^m x, T^{m-1} x) + s^{m-2} d(T^{m-1} x, T^{m-2} x) + \dots + s^n d(T^{n+1} x, T^n x) \\ &\leq \phi^{m-1}(d(Tx, x)) + \phi^{m-2}(d(Tx, x)) + \phi^{m-3}(d(Tx, x)) + \dots + \phi^n(d(Tx, x)) \\ &= \Sigma_{k=n}^{m-1} \phi^k(d(x, Tx)) \\ &\leq \Sigma_{n \geq n(\epsilon)} \phi^n(d(x, Tx)) < \epsilon. \end{aligned}$$

Similarly,

$$d(T^n x, T^m x) < \epsilon.$$

Thus $\{T^n x\}$ is a Cauchy sequence. Since (X, d) is complete, we have $\{T^n x\}$ converges to some $z \in X$. We note that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to same limit z . Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$. Now, we will show that $Tz = z$. By using (2.20), consider

$$\begin{aligned} d(z, Tz) &\leq sd(z, T^{2n} x) + sd(T^{2n} x, Tz) \\ &\leq sd(z, T^{2n} x) + d(T^{2n-1} x, z) \end{aligned}$$

Taking limit as $n \rightarrow \infty$ in above inequality, we have

$$d(z, Tz) = 0.$$

Similarly considering form (2.20), we get

$$\begin{aligned} d(Tz, z) &\leq sd(Tz, T^{2n} x) + sd(T^{2n} x, z) \\ &\leq d(z, T^{2n-1} x) + sd(T^{2n} x, z) \end{aligned}$$

Taking limit as $n \rightarrow \infty$ in above inequality, we have

$$d(Tz, z) = 0.$$

Hence $d(z, Tz) = d(Tz, z) = 0$, this implies that $Tz = z$ that is z is a fixed point of T .

Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$. Then, we have

$$d(z^*, z) \leq sd(Tz^*, T^n x) + sd(T^n x, Tz) \leq \phi(d(Tz^*, T^n x)) + \phi(d(T^n x, Tz)), \quad (2.21)$$

and on the other hand,

$$d(z, z^*) \leq sd(Tz, T^n x) + sd(T^n x, Tz^*) \leq \phi(d(Tz, T^n x)) + \phi(d(T^n x, Tz^*)). \quad (2.22)$$

Letting $n \rightarrow \infty$ we obtain that $d(z, z^*) = d(z^*, z) = 0$, this implies that $z^* = z$.

Therefore z is a unique fixed point of T . This completes the proof. $\square$

Example 2.20. Let $X = [-1, 1]$ and $T : A \cup B \rightarrow A \cup B$ be defined by $Tx = \frac{-x}{5}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Defined the function $d : X^2 \rightarrow [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X , where $s = 2$. Let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{5} \leq \frac{1}{5}$. Thus, $Tx \in B$. On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{5} \leq \frac{-x}{5} \leq 0$. Thus, $Tx \in A$. Hence the map T is cyclic on X , because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$\begin{aligned} sd(Tx, Ty) &= 2d(Tx, Ty) \\ &= 2(|Tx - Ty|^2 + \frac{1}{10}|Tx| + \frac{1}{11}|Ty|) \\ &= 2(|\frac{-x}{5} - \frac{-y}{5}|^2 + \frac{1}{10}|\frac{-x}{5}| + \frac{1}{11}|\frac{-y}{5}|) \\ &= \frac{2}{3}(\frac{3}{25}|x - y|^2 + \frac{3}{50}|x| + \frac{3}{55}|y|) \\ &\leq \frac{2}{3}(|x - y|^2 + \frac{5}{50}|x| + \frac{5}{55}|y|) \\ &= \frac{2}{3}(|x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|) \\ &= \phi(d(x, y)), \end{aligned}$$

where the function $\phi \in \Phi$ is $\phi(t) = \frac{2t}{3}$. Clearly, 0 is the unique fixed point of T .

The following corollary can be taken as a particular case of Theorem 2.19 if we take $\phi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$. That is the dqb-cyclic-Banach contraction, in the setting of dislocated quasi-b-metric spaces.

Corollary 2.21. Let A and B be nonempty closed subsets of a complete dislocated quasi-b-metric space (X, d) . Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-Banach contraction; that is, if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y). \quad (2.23)$$

for all $x \in A, y \in B$ and $s \geq 1$ and $sk \leq 1$. Then, T has a unique fixed point in $A \cap B$.

References

- [1] MA. Alghamdi, N. Hussain and P. Salimi, *Fixed point and coupled fixed point theorems on b-metric-like spaces*, J. Inequal. Appl., **402** (2013). 1
- [2] IA. Bakhtin, *The contraction principle in quasimetric spaces*, Functional Analysis., **30** (1989), 26–37. 1
- [3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1
- [4] W.A. Kirk, P.S. Srinivasan and P. Veeramani, *Fixed Points for mapping satisfying Cyclic contractive conditions*, Fixed Point Theory., **4** (2003), 79–89. 1
- [5] C. Klin-eam and C. Suanoom, *Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions*, Fixed point theory and applications., **74** (2015). 1, 1.2
- [6] E. Kreyszig, *Introductory functional analysis with applications with applications*, Singapore: John Wiley and Sons., (1978). 2.1
- [7] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., TMA **47**(4) (2001), 2683–2693. 1, 1
- [8] MH. Shah and N. Hussain *Nonlinear contractions in partially ordered quasi b-metric spaces*, Commun. Korean Math. Soc., **27**(1) (2012), 117–128. 1
- [9] K. Zoto, *Fixed Point Theorems for Cyclic Contractions in Dislocated Metric Spaces*, Advanced Research in

7.2 EXISTENCE THEOREMS OF A NEW SET-VALUED MT-CONTRACTION IN b -METRIC SPACES ENDOWED WITH GRAPHS AND APPLICATIONS

In this section, a new concept of set-valued Mizoguchi-Takahashi G -contractions is introduced and some fixed point theorems for such mappings in b -metric spaces endowed with directed graphs are established under some sufficient conditions. Our results improve and extend those of [20] and [24]. We also give some examples supporting our main results. As an applications, we prove the existence of fixed points for multivalued mappings satisfying generalized MT-contractive condition in ϵ -chainable b -metric spaces and the existence of a solution for some integral equations.

1. INTRODUCTION

In 1969, Nadler [21] extended the Banach contraction principle to a multi-valued contraction mapping and proved the following theorem.

Theorem 1.1 ([21]). *Let (X, d) be a complete metric space and T be a mapping X into $CB(X)$. Assume that there exists $k \in [0, 1)$ such that*

$$H(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

In 1988, Mizoguchi and Takahashi [20] extended the Nadler's theorem by using some auxiliary functions. They introduced the following contractive condition,

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y)$$

for all $x, y \in X$, where $\varphi : [0, \infty) \rightarrow [0, 1)$ is a function satisfying

$$\lim_{r \rightarrow t^+} \sup \varphi(r) < 1, \text{ for each } t \in [0, \infty) \quad (1)$$

and proved fixed point theorem for this type of mappings. We call a function satisfying (1) a MT-function.

After that, many authors extended and generalized Nadler's fixed point theorem for set-valued mappings in several directions (see [4], [5], [6], [13], [16], [23]).

In 2012, Du [14] gave characterizations of MT-functions and proved the followings fixed point theorem for some contractive set-valued mappings in a complete metric space.

Theorem 1.2 ([14]). *Let (X, d) be a complete metric space, $T : X \rightarrow CB(X)$ be a set-valued mapping, $g : X \rightarrow X$ be a continuous self-map and $\varphi : [0, \infty) \rightarrow [0, 1)$ be an MT-function. Assume that*

- (a) Tx is g -invariant (i.e. $g(Tx) \subseteq Tx$) for each $x \in X$;
- (b) there exists a function $h : X \rightarrow [0, \infty)$ such that

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + h(gy)d(gy, Tx) \quad \text{for all } x, y \in X.$$

Then there exists $v \in X$ such that $gv \in Tv$ and $v \in Tv$.

Afterward, Pathak, Agarwal and Cho [22] introduced a concept of P -functions and gave characterizations of the mappings in this class. By using this concept, they proved several fixed point and coincidence point theorems for a set-valued mapping satisfying some contractive conditions.

Recently, in 2014, Javahernia et al. [19] introduced the notion of a generalized Mizoguchi-Takahashi function as follows.

Definition 1.3 ([19]). A function $\beta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is called a *generalized Mizoguchi-Takahashi function* (or *generalized MT-function*) if the following conditions hold:

- (1) $0 < \beta(u, v) < 1$ for all $u, v > 0$;
- (2) $\limsup_{n \rightarrow \infty} \beta(u_n, v_n) < 1$, for any bounded sequence $\{u_n\} \subset (0, \infty)$ and any non-increasing sequence $\{v_n\} \subset (0, \infty)$.

We denote by Λ the set of all generalized MT-functions. By using the concept of generalized MT-function, they also proved some fixed point theorems for some contractive set-valued mappings.

On the other hand, the concept of a metric space has been generalized in many ways. In 1989, Bakhtin [2] introduced the concept of b -metric spaces which is a metric space satisfying a relaxed form of the triangle inequality and proved Banach's contraction principle in this space. Since Bakhtin's results, many authors have followed this concept and proved fixed point results for several types of single-valued and set-valued mappings in b -metric spaces (see [2],[7]–[12]).

On the other hand, in 2008, Jachymski [17] combined two concepts in fixed point theory and graph theory to study fixed point theorems for G -contraction mappings in a metric space endowed a directed graph. These results have been generalized by some authors in several ways (see [4], [13], [16], [24], [22], [23])

Let $G = (V(G), E(G))$ be a directed graph, where $V(G)$ is a set of vertices of graph and $E(G)$ is a set of its edges. Assume that G has no parallel edges. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges, that is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

Let x and y be two vertices in G , a *path* in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^n$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for each $i = 1, 2, \dots, n$. A graph G is said to be *connected* if there exists a (directed) path between any two vertices of G . We denote

$$[x]_G^N = \{y \in X : \text{there is a path in } G \text{ of length } N \text{ from } x \text{ to } y\}.$$

In 2014, Sultana and Vetrivel [24] introduced the notion of a Mizoguchi-Takahashi G -contraction as follows:

Definition 1.4 ([24]). Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. A mapping $T : X \rightarrow CB(X)$ is called a *Mizoguchi-Takahashi G -contraction* if, for any $x, y \in X$ with $x \neq y$ and $(x, y) \in E(G)$,

$$(1) \quad H(Tx, Ty) \leq \alpha(d(x, y))d(x, y), \text{ where } \alpha : [0, \infty) \rightarrow [0, 1) \text{ with}$$

$$\limsup_{s \rightarrow t^+} \alpha(s) < 1$$

for all $t \in [0, \infty)$;

$$(2) \quad \text{if } u \in Tx \text{ and } v \in Ty \text{ are such that } d(u, v) \leq d(x, y), \text{ then } (u, v) \in E(G).$$

They obtained some fixed point theorems for a Mizoguchi-Takahashi G -contraction in a metric space as follows.

Theorem 1.5 ([24]). *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. Let $T : X \rightarrow CB(X)$ be a Mizoguchi-Takahashi G -contraction. Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that:*

- (1) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;
- (2) *for any sequence $\{x_n\} \subset X$, if $x_n \rightarrow x$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{k_n}\}_{k \in \mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$ for all $k \in \mathbb{N}$.*

Then there exists a sequence $\{x_n\}$ in X with $x_n \in [x_0]_G^N \cap T(x_{n-1})$ for all $n \in \mathbb{N}$ converging to a fixed point of T .

Motivated by the result of Jachymski, Javahernia et al. and Sultana et al., we introduced the concept of a new generalized Mizoguchi-Takahashi G -contraction in b -metric spaces and establish some fixed point theorems for this contraction in b -metric spaces endowed with a directed graph. Also, we give some examples to illustrate our main results and apply our main result to obtain some fixed point theorems for some contractions in ϵ -chainable metric spaces.

2. PRELIMINARIES

Now, we give some basic definitions, lemmas and notations concerning b -metric spaces.

Definition 2.1. Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is said to be a b -metric if, for all $x, y, z \in X$, the following conditions are satisfied:

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$;
- (3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Then (X, d) is called a b -metric space with coefficient s .

It is noted that every metric space is a b -metric space with $s = 1$, but the converse is not generally true.

Example 2.2. Let $X = [0, 1]$ and a mapping $d : X \times X \rightarrow \mathbb{R}^+$ be defined by

$$d(x, y) = |x - y|^2$$

for all $x, y \in X$. Clearly, (X, d) is a b -metric space with coefficient $s = 2$.

Example 2.3. Let $0 < p < 1$ and $L_p[0, 1]$ be the set of all real functions x on $[0, 1]$ such that $\int_0^1 |x(t)|^p dt < 1$. Define a mapping $d : L_p[0, 1] \times L_p[0, 1] \rightarrow \mathbb{R}^+$ by

$$d(x, y) = \int_0^1 |x(t) - y(t)|^p dt$$

for all $x, y \in L_p[0, 1]$. Then (X, d) is a b -metric space with coefficient $s = 2^p$.

Definition 2.4. Let (X, d) be a b -metric space.

- (1) A sequence $\{x_n\}$ is said to be *convergent* to a point $x \in X$ if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0;$$

- (2) A sequence $\{x_n\}$ is called a *Cauchy sequence* if, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon$$

for all $m, n > N$;

- (3) A b -metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point $x \in X$.

Next, we give some notions and lemmas concerning set-valued mappings on b -metric spaces. We denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X and $CL(X)$ the class of all nonempty closed subset of X . For any $A, B \in CB(X)$, define the function $H : CB(X) \times CB(X) \rightarrow \mathbb{R}^+$ by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where

$$\delta(A, B) = \sup\{d(a, B) : a \in A\},$$

$$\delta(B, A) = \sup\{d(b, A) : b \in B\},$$

$$d(a, C) = \inf\{d(a, x) : x \in C\}.$$

Note that H is called the *Hausdorff b -metric* induced by the b -metric d .

Now, we recall the following properties from ([10], [11], [12]).

Lemma 2.5 ([10]). *Let (X, d) be a b -metric space with coefficient $s \geq 1$. For any $A, B, C \in CB(X)$ and $x, y \in X$, one has the following:*

- (1) $d(x, B) \leq d(x, b)$ for any $b \in B$;
- (2) $\delta(A, B) \leq H(A, B)$;

- (3) $d(x, B) \leq H(A, B)$ for any $x \in A$;
- (4) $H(A, A) = 0$;
- (5) $H(A, B) = H(B, A)$;
- (6) $H(A, C) \leq s[H(A, B) + H(B, C)]$;
- (7) $d(x, A) \leq s[d(x, y) + d(y, A)]$.

Remark 2.6 ([10]). The function $H : CL(X) \times CL(X) \rightarrow \mathbb{R}^+$ is a generalized Pompeiu-Hausdorff b -metric, that is, $H(A, B) = +\infty$ if $\max\{\delta(A, B), \delta(B, A)\}$ does not exist.

Lemma 2.7 ([11]). Let (X, d) be a b -metric space with coefficient $s \geq 1$. For any $A \in CL(X)$ and $x \in X$, one has

$$d(x, A) = 0 \iff x \in \bar{A} = A,$$

where $\bar{A}$ denote the closure of the set A .

Lemma 2.8 ([12]). Let (X, d) be a b -metric space with coefficient $s \geq 1$ and $A, B \in CL(X)$ with $H(A, B) > 0$. Then, for each $h > 1$ and $a \in A$, there exists $b \in B$ such that $d(a, b) \leq hH(A, B)$.

3. MAIN RESULTS

We first need the following class of functions for our main results.

Definition 3.1. Let $s \geq 1$. A function $\beta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is called a s -generalized Mizoguchi-Takahashi function (or s -generalized MT-function) if the following conditions hold:

- (1) $0 < \beta(u, v) < 1$ for all $u, v > 0$;
- (2) for any bounded sequence $\{u_n\} \subset (0, \infty)$ and any non-increasing sequence $\{v_n\} \subset (0, \infty)$, we have

$$\limsup_{n \rightarrow \infty} \beta(u_n, v_n) < \frac{1}{s}.$$

We denote by Λ_s the set of all s -generalized MT-functions.

Example 3.2. Let $\beta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a generalized MT-function, then β is 1-generalized MT-function.

Example 3.3. Let $s \geq 1$ and $g(x) = \frac{\ln(x+5)}{s(x+4)}$ for all $x > -4$. Define a function $\beta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\beta(u, v) = \begin{cases} \frac{u}{s(v^2+1)}, & 1 < u < v, \\ g(v), & \text{otherwise.} \end{cases}$$

It is easily to see that β is a s -generalized MT-function.

Definition 3.4. Let (X, d) be a b -metric space with coefficient $s \geq 1$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A mapping $T : X \rightarrow CB(X)$ is called a s -generalized Mizoguchi-Takahashi G -contraction if there exists $\beta \in \Lambda_s$ such that, for any $x, y \in X$ with $x \neq y$ and $(x, y) \in E(G)$,

- (1) $H(Tx, Ty) \leq \beta(H(Tx, Ty), d(x, y)) \cdot d(x, y)$;
- (2) if $u \in Ty$ and $v \in Ty$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$.

Now, we prove our main results.

Theorem 3.5. Let (X, d) be a b -metric space with coefficient $s \geq 1$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that $T : X \rightarrow CB(X)$ is a s -generalized Mizoguchi-Takahashi G -contraction. Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that

- (1) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;
- (2) for any sequence $\{x_n\}$ in X , if $x_n \rightarrow x$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Then there exists $x \in X$ such that $x \in Tx$.

Proof. By (1), there exists $x_1 \in [x_0]_G^N \cap Tx_0$. Then there is a path $(y^i)_{i=0}^N$ in G from x_0 to x_1 , i.e., $y^0 = x_0$, $y^N = x_1$ and $(y^{i-1}, y^i) \in E(G)$ for each $i = 1, 2, \dots, N$. We can choose $k_1^1 > 0$ such that

$$\begin{aligned} & \beta^{k_1^1}(H(Ty^0, Ty^1), d(y^0, y^1)) + \beta(H(Ty^0, Ty^1), d(y^0, y^1)) \cdot d(y^0, y^1) \\ & < d(y^0, y^1). \end{aligned} \quad (3.1)$$

Since $(y^0, y^1) \in E(G)$, we obtain

$$H(Ty^0, Ty^1) \leq \beta(H(Ty^0, Ty^1), d(y^0, y^1)) \cdot d(y^0, y^1). \quad (3.2)$$

Since $x_1 \in Ty^0$, by Lemma 2.8, there exists $y_1^1 \in Ty^1$ such that

$$d(x_1, y_1^1) \leq H(Ty^0, Ty^1) + \beta^{k_1^1}(H(Ty^0, Ty^1), d(y^0, y^1)),$$

which implies, by (3.2), that

$$\begin{aligned} & d(x_1, y_1^1) \\ & \leq \beta(H(Ty^0, Ty^1), d(y^0, y^1)) \cdot d(y^0, y^1) + \beta^{k_1^1}(H(Ty^0, Ty^1), d(y^0, y^1)). \end{aligned} \quad (3.3)$$

From (3.2), it follows that $d(x_1, y_1^1) < d(y^0, y^1)$ and $(x_1, y_1^1) \in E(G)$.

In the same argument, set $y_1^0 = x_1$, for each $i = 1, 2, \dots, N$, we can choose k_1^i with $k_1^{i-1} < k_1^i$ and $y_1^i \in Ty^i$ such that

$$\begin{aligned} & d(y_1^{i-1}, y_1^1) \\ & \leq H(Ty^{i-1}, Ty^i) + \beta^{k_1^i}(H(Ty^{i-1}, Ty^i), d(y^{i-1}, y^i)) \\ & \leq \beta(H(Ty^{i-1}, Ty^i), d(y^{i-1}, y^i)) \cdot d(y^{i-1}, y^i) + \beta^{k_1^i}(H(Ty^{i-1}, Ty^i), d(y^{i-1}, y^i)) \\ & < d(y^{i-1}, y^i) \end{aligned}$$

and so $(y_1^{i-1}, y^i) \in E(G)$ for each $i = 2, 3, \dots, N$. If we denote $x_2 = y_1^N \in Ty^N$, then $(y_1^i)_{i=0}^N$ is a path from x_1 to x_2 . Hence $x_2 \in [x_1]_G^N \cap Tx_1$.

Next, we can choose $k_2^1 > k_1^1$ such that

$$\begin{aligned} & \beta^{k_2^1}(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)) + \beta(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)) \cdot d(y_1^0, y_1^1) \\ & < d(y_1^0, y_1^1). \end{aligned} \quad (3.4)$$

Since $(y_1^0, y_1^1) \in E(G)$, we obtain

$$H(Ty_1^0, Ty_1^1) \leq \beta(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)) \cdot d(y_1^0, y_1^1). \quad (3.5)$$

By Lemma 2.8, there exists $y_2^1 \in Ty_1^1$ such that

$$d(x_2, y_2^1) \leq H(Ty_1^0, Ty_1^1) + \beta^{m_1^1}(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)),$$

which implies, by (3.5), that

$$\begin{aligned} & d(x_2, y_2^1) \\ & \leq \beta(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)) \cdot d(y_1^0, y_1^1) + \beta^{k_2^1}(H(Ty_1^0, Ty_1^1), d(y_1^0, y_1^1)). \end{aligned} \quad (3.6)$$

From (3.5), it follows that $d(x_2, y_2^1) < d(y_1^0, y_1^1)$ and $(x_2, y_2^1) \in E(G)$.

In the same argument, set $y_2^0 = x_2$, for each $i = 1, 2, \dots, N$, we can choose $k_2^i > 0$ and $y_2^i \in Ty_1^i$ such that

$$\begin{aligned} & d(y_2^{i-1}, y_2^1) \\ & \leq H(Ty_1^{i-1}, Ty_1^i) + \beta^{k_2^i}(H(Ty_1^{i-1}, Ty_1^i), d(y_1^{i-1}, y_1^i)) \\ & \leq \beta(H(Ty_1^{i-1}, Ty_1^i), d(y_1^{i-1}, y_1^i)) \cdot d(y_1^{i-1}, y_1^i) + \beta^{k_2^i}(H(Ty_1^{i-1}, Ty_1^i), d(y_1^{i-1}, y_1^i)) \\ & < d(y_1^{i-1}, y_1^i). \end{aligned}$$

Then $(y_2^{i-1}, y_2^i) \in E(G)$ for each $i = 2, 3, \dots, N$. If we denote $x_3 = y_2^N \in Ty_1^N = Tx_2$, then $(y_2^i)_{i=0}^N$ is a path from x_2 to x_3 . Hence $x_3 \in [x_2]_G^N \cap Tx_2$.

Continuing this process for each $n \in \mathbb{N}$, we get $x_{n+1} \in [x_n]_G^N \cap Tx_n$ by producing a path $(y_n^i)_{i=0}^N$ from x_n to x_{n+1} , i.e., $y_n^0 = x_n$, $y_n^N = x_{n+1}$ and $y_n^i \in Ty_{n-1}^i$ and

$$\begin{aligned} & d(y_n^{i-1}, y_n^i) \\ & \leq H(Ty_{n-1}^{i-1}, Ty_{n-1}^i) + \beta^{k_n^i}(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i)) \\ & \leq \beta(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i)) + \beta^{k_n^i}(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i)) \\ & < d(y_{n-1}^{i-1}, y_{n-1}^i), \end{aligned}$$

where $k_n^i > k_{n-1}^i$ for each $i = 1, 2, \dots, N$.

For each $i = 1, 2, \dots, N$, we denote $d_0^i = d(y^{i-1}, y^i)$ and $d_n^i = d(y_{n-1}^{i-1}, y_{n-1}^i)$ for each $n \geq 1$. From the above inequality, it follows that, for each $i = 1, 2, \dots, N$, $\{d_n^i\}_{n \in \mathbb{N}}$ is a monotone non-increasing sequence of nonnegative real numbers. Then $d_n^i \rightarrow r^i \geq 0$ as $n \rightarrow \infty$. From (1) of Definition 3.1 and

$$H(Ty_{n-1}^{i-1}, Ty_{n-1}^i) \leq \beta(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i)) \cdot d(y_{n-1}^{i-1}, y_{n-1}^i),$$

it follows that $\{H(Ty_{n-1}^{i-1}, Ty_{n-1}^i)\}_{n \in \mathbb{N}}$ is a bounded sequence. By (2) of Definition 3.1, we obtain

$$\limsup_{n \rightarrow \infty} \beta(H(Ty_{n-1}^{i-1}, Ty_{n-1}^i), d(y_{n-1}^{i-1}, y_{n-1}^i)) < \frac{1}{s}. \quad (3.7)$$

We denote $\pi_n^i = H(Ty_{n-1}^{i-1}, Ty_{n-1}^i)$ for each $i = 1, 2, \dots, N$. Then there exist positive integer n_0^i and $\gamma^i \in [0, 1)$ such that $\beta(\pi_n^i, d_n^i) < \gamma^i$ for all $n \geq n_0^i$, where

$$\limsup_{n \rightarrow \infty} \beta(\pi_n^i, d_n^i) < \gamma^i < \frac{1}{s}.$$

Thus, for each $i = 1, 2, \dots, N$, we have

$$\beta(\pi_n^i, d_n^i) < \gamma < \frac{1}{s}$$

for all $n \geq n_0$, where $\gamma = \max_{1 \leq i \leq N} \gamma^i$ and $n_0 = \max_{1 \leq i \leq N} n_0^i$. For each $i = 1, 2, \dots, N$ and $n \geq n_0 + 1$, we obtain

$$\begin{aligned}
d_n^i &\leq \beta(\pi_{n-1}^i, d_{n-1}^i) \cdot d_{n-1}^i + \beta^{k_n^i}(\pi_{n-1}^i, d_{n-1}^i) \\
&\leq \dots \\
&\leq \prod_{j=0}^{n-1} \beta(\pi_j^i, d_j^i) \cdot d_0^i + \sum_{m=1}^{n-1} \prod_{j=m+1}^n \beta(\pi_{j-1}^i, d_{j-1}^i) \beta^{k_m^i}(\pi_{j-1}^i, d_{j-1}^i) + \beta^{m_n^i}(\pi_{n-1}^i, d_{n-1}^i) \\
&\leq \gamma^{n-n_0} \prod_{j=0}^{n_0} \beta(\pi_j^i, d_j^i) \cdot d_0^i \\
&\quad + \sum_{m=1}^{n-1} \prod_{j=\max\{n_0+1, m+1\}}^n \beta(\pi_{j-1}^i, d_{j-1}^i) \beta^{k_m^i}(\pi_{j-1}^i, d_{j-1}^i) + \gamma^n.
\end{aligned}$$

We denote by B the second term of the last inequality and so

$$\begin{aligned}
B &= \sum_{m=1}^{n-1} \prod_{j=\max\{n_0+1, m+1\}}^n \beta(\pi_{j-1}^i, d_{j-1}^i) \beta^{k_m^i}(\pi_{j-1}^i, d_{j-1}^i) \\
&\leq \gamma^{n-n_0} \sum_{m=1}^{n_0} \beta^{k_m^i}(\pi_{m-1}^i) + \sum_{m=n_0+1}^{n-1} \gamma^{n-m} \beta^{k_m^i}(\pi_{m-1}^i, d_{m-1}^i) \\
&\leq Q_1^i \gamma^n + \sum_{m=n_0+1}^{n-1} \gamma^{n-m+k_m^i} \\
&\leq Q_1^i \gamma^n + \gamma^n \frac{\gamma^{k_{n_0}^i - n_0 - 1}}{1 - \gamma} \leq Q_2^i \gamma^n,
\end{aligned}$$

where Q_1^i and Q_2^i are nonnegative real numbers. Thus we have

$$d_n^i \leq \gamma^{n-n_0} \prod_{j=0}^{n_0} \beta(\pi_j^i, d_0^i) \cdot d_0^i + Q_2^i \gamma^n + \gamma^n \leq Q^i \gamma^n, \quad (3.8)$$

where Q^i is a nonnegative real number. Hence it follows that, for any $n \geq n_0 + 1$,

$$d(x_n, x_{n+1}) = d(y_n^0, y_n^N) \leq \sum_{i=1}^N d_n^i \leq \sum_{i=1}^N Q^i \gamma^n.$$

This implies from $s\gamma < 1$ that, for any $n \geq n_0 + 1$ and $m \in \mathbb{N}$,

$$\begin{aligned}
 d(x_n, x_{n+m}) &\leq s \cdot d(x_n, x_{n+1}) + s^2 \cdot d(x_{n+1}, x_{n+2}) + \cdots \\
 &\quad + s^{n+m-1} \cdot d(x_{n+m-2}, x_{n+m-1}) + s^{n+m} \cdot d(x_{n+m-1}, x_{n+m}) \\
 &\leq \sum_{i=1}^N Q^i \left[s\gamma^n + s^2\gamma^{n+1} + \cdots + s^{n+m}\gamma^{n+m-1} \right] \\
 &= \sum_{i=1}^N Q^i \frac{s\gamma^n (1 - (s\gamma)^m)}{1 - s\gamma} \\
 &\leq \sum_{i=1}^N Q^i \frac{s\gamma^n}{1 - s\gamma}.
 \end{aligned}$$

Since $s\gamma^n \rightarrow 0$ and $n \rightarrow \infty$, $\{x_n\}$ is a b -Cauchy sequence in X . By the b -completeness of X , there exists $x^* \in X$ such that $x_n \rightarrow x^*$. We now denote

$$E = \{n \in \mathbb{N} : x_n = x^*\} \text{ and } \text{card}(E) = \text{the cardinal number of } E.$$

If $\text{card}(E) = \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \subset E$ i.e., $x_{n_k} = x^*$ for all $k \in \mathbb{N}$. Since $\{x_{n_k+1}\}$ is a sequence in Tx^* and $x_{n_k+1} \rightarrow x^*$, it follows that $x^* \in Tx^*$.

If $\text{card}(E) < \infty$, then there exists $l \in \mathbb{N}$ such that $x_n \neq x^*$ for all $n \in \mathbb{N}$ with $n \geq l$, which implies that $H(Tx_n, Tx^*) > 0$ and $d(x_n, x^*) > 0$ for any $n \geq l$. Since $x_n \rightarrow x^*$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$, by (2), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x^*) \in E(G)$ for any $k \in \mathbb{N}$. Hence we have

$$\begin{aligned}
 d(x^*, Tx^*) &\leq s \left[d(x^*, x_{n_k+1}) + d(x_{n_k+1}, Tx^*) \right] \\
 &\leq sd(x^*, x_{n_k+1}) + sH(Tx_{n_k}, Tx^*) \\
 &\leq sd(x^*, x_{n_k+1}) + s\beta(H(Tx_{n_k}, Tx^*), d(x_{n_k}, x^*)) \cdot d(x_{n_k}, x^*).
 \end{aligned}$$

Taking the limit supremum as $k \rightarrow \infty$, we have $d(x^*, Tx^*) = 0$. Since Tx^* is b -closed, it follows that $x^* \in Tx^*$. This complete the proof. $\square$

Since every metric space is b -metric space with $s = 1$, following result is directly obtained by Theorem 3.5.

Corollary 3.6. *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that $T : X \rightarrow CB(X)$ is a generalized Mizoguchi-Takahashi G -contraction. Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that*

$$(1) [x_0]_G^N \cap Tx_0 \neq \emptyset;$$

(2) for any sequence $\{x_n\}$ in X , if $x_n \rightarrow x$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Then there exists $x \in X$ such that $x \in Tx$.

Remark 3.7. If we put $\beta(u, v) = \alpha(v)$ in Corollary 3.6, then we have Theorem 1.5 (Sultana et al. [24]).

Remark 3.8. If we put $\beta(u, v) = \alpha(v)$, $E(G) = X \times X$ and $N = 1$, then we obtain Mizoguchi's Theorem (Mizoguchi et al. [20]) as a corollary.

By putting $\beta(u, v) = \frac{\varphi(v)}{v}$ in Theorem 3.5, we obtain the following.

Corollary 3.9. Let (X, d) be a complete b -metric space with coefficient $s \geq 1$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that $T : X \rightarrow CB(X)$ is a set-valued mapping such that

(1) for all $x, y \in X$, $x \neq y$ and $(x, y) \in E(G)$

$$H(Tx, Ty) \leq \varphi(d(x, y)),$$

where $\varphi : [0, \infty) \rightarrow [0, 1)$ such that $\varphi(v) < v$ and $\limsup_{v \rightarrow u^+} \frac{\varphi(v)}{v} < 1$ for all $v \in [0, \infty)$, and if $u \in Tx$ and $v \in Ty$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$.

Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that

(2) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;

(3) for any sequence $\{x_n\}$ in X , if $x_n \rightarrow x$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Then there exists $x \in X$ such that $x \in Tx$.

Definition 3.10 ([19]). A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is said to be a *weakly lower semi-continuous function* (shortly, a w.l.s.c. function) if, for any bounded sequence $\{u_n\} \subset (0, \infty)$, we have

$$\liminf_{n \rightarrow \infty} \phi(u_n) > 0.$$

We denote by $W_{lsc}(R)$ the collection of all w.l.s.c. function.

Corollary 3.11. Let (X, d) be a complete b -metric space with coefficient $s \geq 1$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that $T : X \rightarrow CB(X)$ is a set-valued mapping such that

- (1) for all $x, y \in X$, $x \neq y$ and $(x, y) \in E(G)$

$$H(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

where $\phi \in W_{lsc}(R)$ and $\phi(0) = 0$, $\phi(v) < sv$ for all $v \in (0, \infty)$;

- (2) if $u \in Tx$ and $v \in Ty$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$.

Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that

- (3) $[x_0]_G^N \cap Tx_0 \neq \emptyset$;

- (4) for any sequence $\{x_n\}$ in X , if $x_n \rightarrow x$ and $x_n \in [x_{n-1}]_G^N \cap Tx_{n-1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Then there exists $x \in X$ such that $x \in Tx$.

Proof. Define $\beta(u, v) = \frac{1}{s} - \frac{\phi(u)}{u}$ for all $u, v > 0$ and let $\{u_n\} \subseteq [0, \infty)$ be a bounded sequence. Then we have $\liminf_{n \rightarrow \infty} \phi(u_n) > 0$ and so $\liminf_{n \rightarrow \infty} \frac{\phi(u_n)}{u_n} > 0$. This implies that

$$\limsup_{n \rightarrow \infty} \left(\frac{1}{s} - \frac{\phi(u_n)}{u_n} \right) = \frac{1}{s} - \liminf_{n \rightarrow \infty} \frac{\phi(u_n)}{u_n} < \frac{1}{s},$$

which means that $\beta \in \Lambda_s$. Also, we obtain

$$H(Tx, Ty) \leq \beta(H(Tx, Ty), d(x, y)) \cdot d(x, y).$$

Therefore, all conditions of Theorem 3.5 are satisfied and so T has a fixed point. This completes the proof. $\square$

By using Theorem 3.5, we get the following results for single-valued mappings.

Corollary 3.12. Let (X, d) be a b -metric space with coefficient $s \geq 1$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Suppose that $f : X \rightarrow X$ satisfies the following condition: there exists $\beta \in \Lambda_s$ such that, for any $x, y \in X$, $x \neq y$ with $(x, y) \in E(G)$,

$$(f(x), f(y)) \in E(G), \quad d(f(x), f(y)) \leq \beta(d(f(x), f(y)), d(x, y))d(x, y).$$

Assume that there exist $N \in \mathbb{N}$ and $x_0 \in X$ such that

- (1) $Tx_0 \in [x_0]_G^N$;

- (2) if $f^n(x_0) \rightarrow x$ and $f^n(x_0) \in [x_{n-1}]_G^N$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{f^{n_k}(x_0)\}$ of $\{f^n(x_0)\}$ such that $(f^{n_k}(x_0), x) \in E(G)$ for all $k \in \mathbb{N}$.

Then $\{f^n(x_0)\}$ converges to a fixed point of f .

Next, we give an example to illustrate Theorem 3.5.

Example 3.13. Let $X = [0, \infty)$ and a b -metric $d : X \times X \rightarrow [0, \infty)$ be defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. It is easy to see that (X, d) be a complete b -metric space with coefficient $s = 2$. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and

$$E(G) = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{n+1}} \right) : n \in \mathbb{N} \right\} \cup \left\{ \left(0, \frac{1}{2^n} \right) : n \in \mathbb{N} \right\} \cup \{(0, 0)\}.$$

Define a mapping $T : X \rightarrow X$ by

$$Tx = \begin{cases} \{0, \frac{x}{2}\} & \text{if } 0 \leq x < 1; \\ \{x, x + 1\} & \text{if } x > 0. \end{cases}$$

Choosing $x_0 = \frac{1}{4}$ and $x_1 = \frac{1}{8} \in Tx_0$, we have $(x_0, x_1) = \left(\frac{1}{4}, \frac{1}{8}\right)$. We choose $\beta(u, v) = \frac{1}{2}$, then $\beta \in \Lambda_s$. To prove that T is a s -generalized MT G -contraction, let $x, y \in X$ be such that $(x, y) \in E(G)$.

If $(x, y) = (0, 0)$, it is obvious that T satisfies (1) and (2) of Definition 3.1.

If $(x, y) = \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right)$, then we obtain

$$H(Tx, Ty) = \frac{\left|\frac{1}{2^n} - \frac{1}{2^{n+1}}\right|^2}{4} \leq \frac{1}{2}d(x, y) = \beta(H(Tx, Ty), d(x, y))d(x, y)$$

and, for any $y \in Tx$ and $v \in Ty$, we have $(u, v) \in E(G)$.

If $(x, y) = \left(\frac{1}{2^n}, 0\right)$, then we obtain

$$H(Tx, Ty) = \frac{\left|\frac{1}{2^n} - 0\right|^2}{4} \leq \frac{1}{2}d(x, y) = \beta(H(Tx, Ty), d(x, y))d(x, y)$$

and, for any $y \in Tx$ and $v \in Ty$, we have $(u, v) \in E(G)$. Hence T is a s -generalized Mizoguchi-Takahashi G -contraction. It is easy to see that the condition (2) of Theorem 3.5 are satisfied with $N = 1$. Therefore, all conditions of Theorem 3.5 are satisfied and we see that 0 is a fixed point of T .

Remark 3.14. It is noted that Nadler's Theorem [21] cannot be applied in Example 3.13 with usual metric space, because for any $k \in [0, 1]$,

$$H(T0, T2) = 3 \not\leq kd(0, 2).$$

4. APPLICATIONS TO ϵ -CHAINABLE b -METRIC SPACES

In 1961, Eldelstein [15] introduced a uniformly locally contractive mapping on an ϵ -chainable metric space and prove some fixed point theorems for this type of mappings.

In this section, we show the existence of fixed points for generalized Mizoguchi-Takahashi contractions on an ϵ -chainable b -metric space. The following definitions are needed.

Definition 4.1. A b -metric space (X, d) with coefficient $s \geq 1$ is said to be ϵ -chainable if, for any $a, b \in X$, there exist $N \in \mathbb{N}$ and a sequence $\{y^i\}_{i=0}^N$ such that $y^0 = a$ and $y^N = b$ and $d(y^{i-1}, y^i) < \epsilon$ for each i with $1 \leq i \leq N$.

Now, we establish the following theorem for set-valued mappings on ϵ -chainable b -metric spaces as an application of Theorem 3.5.

Theorem 4.2. Let (X, d) be a complete ϵ -chainable b -metric space with coefficient $s \geq 1$ and $T : X \rightarrow CB(X)$ be a set-valued mapping such that there exists $\beta \in \Lambda_s$ satisfying

$$H(Tx, Ty) \leq \beta(H(Tx, Ty), d(x, y)) \cdot d(x, y)$$

for all $x, y \in X$ with $x \neq y$ and $d(x, y) < \epsilon$. Then T has a fixed point.

Proof. We define $G = (V(G), E(G))$ by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : d(x, y) < \epsilon\}.$$

Then $E(G) \supseteq \Delta$ and G has no parallel edges. It is easily to see that T is a s -generalized Mizoguchi-Takahasji G -contraction. Let $x_0 \in X$ and $x_1 \in Tx_0$. Since (X, d) is an ϵ -chainable, there exists a path $(y^i)_{i=0}^N$ from x_0 to x_1 , i.e., $y^0 = x_0$, $y^N = x_1$ and $(y^{i-1}, y^i) \in E(G)$ for each $i = 1, 2, \dots, N$. This implies that there exists $N \in \mathbb{N}$ such that $[x_0]_G^N \cap Tx_1 \neq \emptyset$.

Next, let $\{x_n\}$ be a sequence in X such that $x_n \rightarrow z \in X$. Then there exists a positive integer M such that

$$d(x_n, z) < \epsilon$$

for all $n \geq M$. Thus we can obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, z) \in E(G)$ for all $k \in \mathbb{N}$. So all the conditions of Theorem 3.5 are satisfied. Therefore, T has a fixed point. This completes the proof. $\square$

The following result is obtained immediately by Theorem 4.2.

Corollary 4.3. Let (X, d) be a complete ϵ -chainable metric space and $T : X \rightarrow CB(X)$ be a set-valued mapping such that there exists $\beta \in \Lambda$ satisfying

$$H(Tx, Ty) \leq \beta(H(Tx, Ty), d(x, y)) \cdot d(x, y)$$

for all $x, y \in X$ with $x \neq y$ and $d(x, y) < \epsilon$. Then T has a fixed point.

The following results is obtained by setting the function β in Corollary 4.3 to be $\beta(u, v) = \alpha(u)$.

Corollary 4.4 ([24]). *Let (X, d) be a complete ϵ -chainable metric space and $T : X \rightarrow CB(X)$ be a set-valued mapping such that there exists $\alpha : [0, \infty) \rightarrow [0, 1)$ such that $\limsup_{r \rightarrow t^+} \alpha(r) < 1$ for all $t \geq 0$ satisfying*

$$H(Tx, Ty) \leq \alpha(d(x, y)) \cdot d(x, y)$$

for all $x, y \in X$ with $x \neq y$ and $d(x, y) < \epsilon$. Then T has a fixed point.

5. APPLICATIONS TO INTEGRAL EQUATIONS

In this section, we give an application of fixed point method to study the existence of solutions for some integral equations.

Let $X = C([0, I], \mathbb{R})$ be the set of real continuous functions defined on closed interval $[0, I]$ where $I > 0$, and define $d : X \times X \rightarrow [0, \infty)$ by

$$d(x, y) = \|(x - y)^2\|_\infty = \sup_{t \in [0, I]} (x(t) - y(t))^2$$

for all $x, y \in X$. Then (X, d) is a complete b -metric space with coefficient $s = 2$.

Consider the integral equation

$$x(t) = p(t) + \int_0^t S(t, u) f(u, x(u)) du, \quad (9.1)$$

where $f : [0, I] \times \mathbb{R} \rightarrow \mathbb{R}$ and $p : [0, I] \rightarrow \mathbb{R}$ are two continuous functions and $S : [0, I] \times [0, I] \rightarrow [0, \infty)$ is a function such that $S(t, \cdot) \in L^1([0, I])$ for all $t \in [0, I]$.

Let $T : X \rightarrow X$ be the operator defined by

$$T(x)(t) = p(t) + \int_0^t S(t, u) f(u, x(u)) du. \quad (9.2)$$

Now, we prove the existence of a solution of the integral equation (9.1) by using Corollary 3.12.

Theorem 5.1. *Let $X = C([0, I], \mathbb{R})$ and $G = (V(G), E(G))$ be a directed graph with $V(G) = X$. Suppose that the following conditions hold:*

(1) *for all $u \in [0, I]$ and $x, y \in X$ with $(x, y) \in E(G)$, we have*

$$|f(u, x(u)) - f(u, y(u))| \leq \eta(x, y) |x(u) - y(u)| \quad (9.3)$$

where

$$\left\| \int_0^I S(t, u) \eta(x, y) du \right\|_\infty \leq \sqrt{\varphi(\|(x - y)^2\|_\infty)}$$

- and $\varphi : [0, \infty) \rightarrow [0, 1)$ such that $\limsup_{r \rightarrow t^+} \varphi(r) < \frac{1}{s}$ for all $t \in [0, \infty)$;
- (2) for any $x, y \in X$ with $(x, y) \in E(G)$, $(Tx, Ty) \in E(G)$;
- (3) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;
- (4) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.

Then the integral equation (9.1) has a solution in X .

Proof. We see that a fixed point of (9.2) is a solution of (9.1). Let $x, y \in X$ be such that $(x, y) \in E(G)$. From the condition (1), it follows that, for all $t \in [0, I]$,

$$\begin{aligned}
 |T(x)(t) - T(y)(t)|^2 &= \left[\int_0^I S(t, u) [f(u, x(u)) - f(u, y(u))] du \right]^2 \\
 &\leq \left[\int_0^I S(t, u) |f(u, x(u)) - f(u, y(u))| du \right]^2 \\
 &\leq \left[\int_0^I S(t, u) \eta(x, y) \sqrt{|x(u) - y(u)|^2} du \right]^2 \\
 &\leq \left[\int_0^I S(t, u) \eta(x, y) \sqrt{\|(x(u) - y(u))^2\|_\infty} du \right]^2 \\
 &= \|(x(u) - y(u))^2\|_\infty \left[\int_0^I S(t, u) \eta(x, y) du \right]^2.
 \end{aligned}$$

Then we have

$$\left\| (T(x) - T(y))^2 \right\|_\infty \leq \|(x - y)^2\|_\infty \left\| \int_0^I S(t, u) \eta(x, y) du \right\|_\infty^2$$

and so, for all $x, y \in X$ with $(x, y) \in E(G)$, we obtain

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y),$$

where $\varphi \in \Lambda_s$. Therefore, all the conditions of Corollary 3.12 are satisfied and hence the operator T has a fixed point, that is, a solution of the integral equation (9.1) exists in X . This completes the proof. $\square$

REFERENCES

- [1] M. U. Ali, T. Kamran, M. Postolache, Fixed point theorems for multivalued G -contractions in Hausdorff b -gauge spaces. *J. Nonlinear Sci. Appl.* 8 (2015), no. 5, 847–855.
- [2] I. A. Bakhtin, The contraction mapping principle in quasi-metric spaces, *Functional Anal.* 30(1989), 26–37 (Russian).
- [3] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* 3 (1922), 133–181.
- [4] I. Beg, A. R. Butt, The contraction principle for set valued mappings on a metric space with graph, *Comput. Math. Appl.* 60(2010), 1214–1219.
- [5] M. Berinde, V. Berinde, On a general class of multi-valued weakly Picard mappings, *J. Math. Anal. Appl.* 326(2007), 772–782.
- [6] V. Berinde, M. Pacurar, The role of Pompeiu-Hausdorff metric in fixed point theory, *Creat. Math. Inform.* 22, 143–150.
- [7] V. Berinde, Generalized contractions in quasi metrics spaces, *Seminar on Fixed Point Theory* 1993, 3–9.
- [8] M. Boriceanu, Fixed point theory for multivalued generalized contraction on a set with two b -metrics, *Studia Univ. Babes-Bolyai-Series Math.* 54(2009), 314.
- [9] M. Boriceanu, M. Bota, A. Petrusel, Multivalued fractals in b -metric spaces, *Center. Europ. J. Math.* 8 (2010), 367–377.
- [10] S. Czerwick, Nonlinear set-valued contraction mappings in b -metric spaces, *Atti del Semin. Mat. e Fisico dell'Univ. di Modena* 46(1998), 263–276.
- [11] S. Czerwick, K. Dłutek, S. L. Singh, Round-off stability of iteration procedures for set-valued operators in b -metric spaces, *J. Natural Physical Sci.* 11(2007), 87–94.
- [12] S. Czerwick, Contraction mappings in b -metric spaces, *Acta Math. Inform. Univ. Ostraviensis* 1(1993), 5–11.
- [13] T. Dinevari, M. Frigon, Fixed point results for multivalued contractions on a metric space with a graph, *J. Math. Anal. Appl.* 405(2013), 507–517.
- [14] W. S. Du, On coincidence point and fixed point theorems for nonlinear multivalued maps, *Topology and its Applications* 159 (2012) 49–56.
- [15] M. Edelstein, An extension of Banach's contraction principle, *Proc. Amer. Math. Soc.* 12 (1961), 7–10.
- [16] A. Erduran, M. Abbas, Fixed point theory for Suzuki type (θ, L) -weak multivalued operators, *Fixed Point Theory* 16 (2) (2015) 303–312.
- [17] J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.* 136 (2008), 1359–1373.
- [18] C. Klanarong, S. Suantai, Coincidence point theorems for some multi-valued mappings in complete metric spaces endowed with a graph, *Fixed Point Theory Appl.* 2015, 2015:129, 16 pp.
- [19] M. Javahernia, A. Razani, F. Khojasteh, Common fixed point of generalized Mizoguchi-Takahashi's type contractions, *Fixed Point Theory Appl.* 2014, 2014:195.
- [20] N. Mizoguchi, W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, *J. Math. Anal. Appl.* 141(1989), 177–188.
- [21] S. Nadler, Jr., Multi-valued contraction mappings, *Pacific J. Math.* 20(1969), 475–488.

- [22] H. K. Pathak, R. P. Agarwal, Y. J. Cho, Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions. *J. Comput. Appl. Math.* 283 (2015) 201–217.
- [23] S. Reich, Fixed points of contractive functions, *Boll. Un. Mat. Ital.* 5(1972), 26–42.
- [24] A. Sultana, V. Vetrivel, Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applicatiobs, *J. Math. Anal. Appl.* 417 (2014), 336–344.
- [25] J. Tiammee, S. Suantai, Coincidence point theorems for graph-preserving multi-valued mappings, *Fixed Point Theory Appl.* 2014, 2014:70, 11 pp.
- [26] J. Tiammee, S. Suantai, Coincidence point theorems for multi-valued mappings of Reich-type on metric spaces endowed with a graph, *J. Nonlinear Convex Anal.* 16 (2) (2015) 365–373.

7.3 Fuzzy fixed point theorems on the complete fuzzy spaces under supremum metric

In this section, by using the concept of a class of functions, the $\mathcal{R}$ -functions, we provide some fuzzy fixed point theorems on a space of fuzzy sets equipped with the supremum metric. By presenting a technique of constructing a sequence of successive approximations, we obtain some interesting results that improve many existing results. The related cases are also shown and discussed.

1 Introduction

By using a natural generalization of the concept of a set, the fuzzy set, which was introduced initially by Zadeh [1], considering mathematical programming problems which are expressed as optimizing some goal function given certain constraints, this be relaxed by means of a subjective gradation. In 1981, Heilpern [2] used the concept of fuzzy sets and introduced a class of fuzzy mappings, which is a generalization of the set-valued mapping, and proved a fixed point theorem for fuzzy contraction mappings in metric linear spaces. It is worth noting that the result announced by Heilpern [2] is a fuzzy extension of the Banach contraction principle. Subsequently, several other authors have studied the existence of fixed points of fuzzy mappings; for example, Estruch and Vidal [3] proved a fixed point theorem for fuzzy contraction mappings over a complete metric space, which is a generalization of the given Heilpern fixed point theorem, and Sedghi *et al.* [4] gave an extended version of the Estruch and Vidal [3] theorem (for more examples, see [5–14]).

Although many kinds of fixed point theorems for fuzzy contraction mappings in complete metric spaces have been studied extensively in recent years, we have to point out that one has given most attention to the class of fuzzy sets with nonempty compact α -cut sets in the metric space X , but little attention to the class of fuzzy sets with nonempty bounded or closed, or even bounded closed, α -cut sets. However, it is well known that all compact sets are bounded closed sets in a general metric space and the converse is not always true.

In 2008, Qui and Shu [12] established the completeness of $\mathcal{CB}(X)$ with respect to the completeness of the metric space X , where $\mathcal{CB}(X)$ denotes the class of fuzzy sets with

nonempty bounded closed α -cut sets equipped with the generalized Hausdorff metric d_∞ , which takes the supremum on the Hausdorff distances between the corresponding α -cut sets. Also they proved the following common fixed point theorem for a family of fuzzy mappings.

Theorem 1.1 [12] *Let (X, d) be a complete metric space and let $\{F_i\}$ be a sequence of self-mappings of $\mathcal{CB}(X)$. If there exists a constant $q \in (0, 1)$ such that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and j , $i \neq j$,*

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq qM_{(i,j)}(\mu_1, \mu_2),$$

where

$$M_{(i,j)}(\mu_1, \mu_2) = \max \left\{ d_\infty(\mu_1, \mu_2), \rho_\infty(\mu_1, F_i(\mu_1)), \rho_\infty(\mu_2, F_j(\mu_2)), \frac{\rho_\infty(\mu_2, F_i(\mu_1)) + \rho_\infty(\mu_1, F_j(\mu_2))}{2} \right\}.$$

Then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Later, by using the concept of d_∞ metric, Qiu *et al.* [15] proved the following common fixed point theorem, but under the assumption of a compact cut set $\mathcal{C}(X)$ instead of a closed bounded cut set $\mathcal{CB}(X)$.

Theorem 1.2 [15] *Let (X, d) be a compact metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of self-mappings of $\mathcal{C}(X)$. Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing function satisfying the following condition: Φ is continuous from the right and*

$$\sum_{n=1}^{\infty} \Phi^n(t) < \infty, \quad \text{for all } t > 0,$$

where Φ^n denotes the n th iterative function of Φ . Suppose that for arbitrary positive integers i and j , $i \neq j$,

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq \Phi(M_{(i,j)}(\mu_1, \mu_2)),$$

where $M_{(i,j)}(\mu_1, \mu_2)$ is defined as in Theorem 1.1. Then there exists a $\mu_* \in \mathcal{C}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Notice that Theorem 1.2 can be used to apply to a larger class of mappings than that of Theorem 1.1. However, after careful consideration, one may see that Theorem 1.2 is relevant when the considered space is a compact metric space instead of a complete metric space, which has been considered in Theorem 1.1.

Based on the above remarks, here we will present an extension of Theorem 1.1, but in a complete metric space setting. In fact, we will use the concept of a class of functions, so-called $\mathcal{R}$ -functions, to show some fixed point theorems for self-mappings of $\mathcal{CB}(X)$ with the supremum metric for fuzzy sets. Of course, our results improve and extend those results which have been presented in [12] and [15].

2 Preliminaries

In this section, we will provide some important basic concepts and useful results. Let (X, d) be a metric space, and let $CB(X)$ be the set of all nonempty bounded closed subsets of X . Recall that the Hausdorff metric is a function H on $CB(X)$ defined by

$$\begin{aligned} H(A, B) &= \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\} \\ &:= \max \{ \rho(B, A), \rho(A, B) \}, \quad \text{for all } A, B \in CB(X), \end{aligned}$$

where $\rho(A, B) = \sup_{x \in A} d(x, B)$ is the Hausdorff separation of A from B .

2.1 Fuzzy sets and fuzzy mappings

Let $I = [0, 1]$. A fuzzy set μ of a metric space X is defined by its membership function $\mu(x)$, which is a mapping from X into I . We denote by $\mathfrak{F}(X)$ the set of all fuzzy mappings on X , that is, $\mathfrak{F}(X) := \{\mu | \mu : X \rightarrow [0, 1]\}$. For any $\alpha \in (0, 1]$, the α -cut of the fuzzy set μ is defined by

$$[\mu]_\alpha = \{x \in X : \mu(x) \geq \alpha\},$$

where $\alpha \in (0, 1]$, and we separately specify the support $[\mu]_0$ of μ to be the closure of the union of $[\mu]_\alpha$ for $0 < \alpha \leq 1$. We denote by $\mathcal{CB}(X)$ the totality of fuzzy sets $\mu : X \rightarrow I$ for which, for each $\alpha \in I$, the α -cut of μ is a nonempty closed bounded subset of X .

Let $\mu_1, \mu_2 \in \mathfrak{F}(X)$. Then μ_1 is said to be included in μ_2 , denoted by $\mu_1 \subseteq \mu_2$, if and only if $\mu_1(x) \leq \mu_2(x)$ for each $x \in X$. Thus we have $\mu_1 \subseteq \mu_2$ if and only if $[\mu_1]^\alpha \subseteq [\mu_2]^\alpha$ for all $\alpha \in I$. Let X, Y be any underling sets and $\mathfrak{U}, \mathfrak{V}$ are subsets of $\mathfrak{F}(X)$ and $\mathfrak{F}(Y)$, respectively. A mapping $F : \mathfrak{U} \rightarrow \mathfrak{V}$ is said to be a fuzzy mapping, *i.e.*, $F(\mu) \in \mathfrak{V}$ for each $\mu \in \mathfrak{U}$. An element $\mu_* \in \mathfrak{U}$ is said to be a fixed point of a fuzzy self-mapping F on $\mathfrak{U}$ if and only if $\mu_* \subseteq F(\mu_*)$.

The d_∞ -metric (called supremum or generalized Hausdorff metric) is a metric on $\mathcal{CB}(X)$ which is defined as follows:

$$\begin{aligned} d_\infty(\mu_1, \mu_2) &= \sup_{0 \leq \alpha \leq 1} H([\mu_1]^\alpha, [\mu_2]^\alpha) \\ &= \max \{ \rho_\infty(\mu_1, \mu_2), \rho_\infty(\mu_2, \mu_1) \}, \end{aligned} \tag{2.1}$$

where $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and

$$\rho_\infty(\mu_1, \mu_2) = \sup_{0 \leq \alpha \leq 1} \rho([\mu_1]^\alpha, [\mu_2]^\alpha)$$

is the Hausdorff separation of μ_1 from μ_2 . Notice that the supremum in (2.1) may be not attained, and so it cannot be replaced by a maximum. To clarify this, we include the following example, which can be found in [16].

Example 2.1 Let X be a set of real numbers and $\mu, \nu \in \mathfrak{F}(X)$ be fuzzy subsets of X such that the corresponding level sets are

$$[\mu]^\alpha = [\nu]^\alpha = [0, 1] \quad \text{for } 0 \leq \alpha \leq \frac{1}{2},$$

and

$$[\mu]^\alpha = \{0\}, \quad [v]^\alpha = [0, 2(1-\alpha)] \quad \text{for } \frac{1}{2} \leq \alpha \leq 1.$$

It follows that

$$H([\mu]^\alpha, [v]^\alpha) = \begin{cases} 0 & \text{for } 0 \leq \alpha \leq \frac{1}{2}, \\ 2(1-\alpha) & \text{for } \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

Then $d_\infty(\mu, v) = \sup_{0 \leq \alpha \leq 1} H([\mu]^\alpha, [v]^\alpha) = 1$, but this is not attained.

Note that if $\{\mu_n\}$ be a sequence in $\mathcal{CB}(X)$, then it follows from the definition of d_∞ that $\{\mu_n\}$ converges with respect to the d_∞ -metric if and only if $[\mu_n]^\alpha$ converges uniformly in $\alpha \in I$ with respect to the Hausdorff metric. Further, we know that the metric space $(\mathcal{CB}(X), d_\infty)$ is complete provided (X, d) is complete (see [12]). Here, we collect some useful properties of the d_∞ -metric, which will be used in order to obtain our results.

Lemma 2.2 [12] *Let $\mu_1, \mu_2, \mu_3 \in \mathcal{CB}(X)$. The following items are true:*

- (i) $\rho_\infty(\mu_1, \mu_2) = 0$ if and only if $\mu_1 \subseteq \mu_2$,
- (ii) if $\mu_1 \subseteq \mu_2$, then $\rho_\infty(\mu_1, \mu_3) \leq d_\infty(\mu_2, \mu_3)$,
- (iii) $\rho_\infty(\mu_1, \mu_3) \leq d_\infty(\mu_1, \mu_2) + \rho_\infty(\mu_2, \mu_3)$.

Theorem 2.3 [12] *Let (X, d) be a metric space and $\mu_1, \mu_2 \in \mathcal{CB}(X)$. Then for any $\beta > 1$ and any $\mu_3 \in \mathcal{CB}(X)$ satisfying $\mu_3 \subseteq \mu_1$, there exists a $\mu_4 \in \mathcal{CB}(X)$ such that $\mu_4 \subseteq \mu_2$ and $d_\infty(\mu_3, \mu_4) \leq \beta d_\infty(\mu_1, \mu_2)$.*

2.2 $\mathcal{R}$ -Functions

In this subsection, we will recall an important tool related to our considered class of mappings. A function $\varphi : [0, \infty) \rightarrow [0, 1)$ is said to be an $\mathcal{R}$ -function if

$$\limsup_{s \rightarrow t^+} \varphi(s) < 1 \quad \text{for all } t \in [0, \infty).$$

Note that if $\varphi : [0, \infty) \rightarrow [0, 1)$ is a non-decreasing function or a non-increasing function, then φ is an $\mathcal{R}$ -function. This means the set of $\mathcal{R}$ -functions is a rich class. In [17], Du proved some of the following characterizations for the class of $\mathcal{R}$ -functions.

Theorem 2.4 [17] *Let $\varphi : [0, \infty) \rightarrow [0, 1)$ be a function. Then the following statements are equivalent.*

- (a) φ is an $\mathcal{R}$ -function.
- (b) For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1$.

3 Fixed point theorems for fuzzy mappings induced by $\mathcal{R}$ -functions

Now, we are in a position to present our main results.

Theorem 3.1 *Let (X, d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $\mathcal{CB}(X)$. Assume that there exists an $\mathcal{R}$ -function $\varphi : [0, \infty) \rightarrow [0, 1)$ such*

that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and j , $i \neq j$,

$$d_\infty(F_i(\mu_1), F_j(\mu_2)) \leq \varphi(d_\infty(\mu_1, \mu_2))M_{(i,j)}(\mu_1, \mu_2), \quad (3.1)$$

where $M_{(i,j)}(\mu_1, \mu_2)$ is defined as in Theorem 1.1. Then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all positive integers i .

Proof Let us define a function $k : [0, \infty) \rightarrow [0, 1)$ by

$$k(t) = \frac{1 + \varphi(t)}{2}, \quad \text{for all } t \in [0, \infty).$$

Note that we have $0 \leq \varphi(t) < k(t) < 1$ for all $t \in [0, \infty)$.

We will start by picking a fuzzy set $\mu_0 \in \mathcal{CB}(X)$. We subsequently choose $\mu_1 \subseteq F_1(\mu_0)$ and a positive real number ε_0 such that $\varepsilon_0 \in (\frac{1-k(d_\infty(\mu_0, \mu_1))}{2}, 1 - k(d_\infty(\mu_0, \mu_1)))$. Next, by using this ε_0 , we can find a positive real number β_0 such that $\beta_0 \in (1, \frac{1-\varepsilon_0}{k(d_\infty(\mu_0, \mu_1))})$. Now, by Theorem 2.3, there exists $\mu_2 \in \mathcal{CB}(X)$ such that $\mu_2 \subseteq F_2(\mu_1)$ and

$$d_\infty(\mu_1, \mu_2) \leq \beta_0 d_\infty(F_1(\mu_0), F_2(\mu_1)).$$

Next, let us to choose a positive real number ε_1 such that $\varepsilon_1 \in (\frac{1-k(d_\infty(\mu_1, \mu_2))}{2}, 1 - k(d_\infty(\mu_1, \mu_2)))$, and then pick a positive real number β_1 such that $\beta_1 \in (1, \frac{1-\varepsilon_1}{k(d_\infty(\mu_1, \mu_2))})$. Similarly to the above, by Theorem 2.3, we can find $\mu_3 \in \mathcal{CB}(X)$ such that $\mu_3 \subseteq F_3(\mu_2)$ and

$$d_\infty(\mu_2, \mu_3) \leq \beta_1 d_\infty(F_2(\mu_1), F_3(\mu_2)).$$

By continuing this process, we obtain two sequences of positive real numbers $\{\varepsilon_n\}$, $\{\beta_n\}$ and a sequence $\{\mu_n\}$ in $\mathcal{CB}(X)$ such that

$$\begin{cases} \mu_{n+1} \subseteq F_{n+1}(\mu_n), \\ d_\infty(\mu_{n+1}, \mu_{n+2}) \leq \beta_n d_\infty(F_{n+1}(\mu_n), F_{n+2}(\mu_{n+1})), \\ \varepsilon_n \in (\frac{1-k(d_\infty(\mu_n, \mu_{n+1}))}{2}, 1 - k(d_\infty(\mu_n, \mu_{n+1}))), \\ \beta_n \in (1, \frac{1-\varepsilon_n}{k(d_\infty(\mu_n, \mu_{n+1}))}), \end{cases} \quad (3.2)$$

for each $n \in \mathbb{N}$.

In order to complete the proof, we will divide it into three steps.

Step 1. We show that $\sup_{n \in \mathbb{N}} \beta_n k(d_\infty(\mu_n, \mu_{n+1})) < 1$.

Note that, by Lemma 2.2(iii), we have

$$\rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})) \leq d_\infty(\mu_{n-1}, \mu_n) + \rho_\infty(\mu_n, F_n(\mu_{n-1})),$$

for each $n \in \mathbb{N}$. Subsequently, since $\mu_n \subseteq F_n(\mu_{n-1})$, in view of Lemma 2.2(i) we obtain

$$\rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})) \leq d_\infty(\mu_{n-1}, \mu_n),$$

for each $n \in \mathbb{N}$.

Using this fact, we now derive

$$\begin{aligned}
M_{(n+1,n+2)}(\mu_n, \mu_{n+1}) &= \max\{d_\infty(\mu_n, \mu_{n+1}), \rho_\infty(\mu_n, F_{n+1}(\mu_n)), \rho_\infty(\mu_{n+1}, F_{n+2}(\mu_{n+1})), \\
&\quad 1/2[\rho_\infty(\mu_{n+1}, F_{n+1}(\mu_n)) + \rho_\infty(\mu_n, F_{n+2}(\mu_{n+1}))]\} \\
&\leq \max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2}), 1/2[d_\infty(\mu_n, \mu_{n+2}) \\
&\quad + \rho_\infty(\mu_{n+2}, F_{n+2}(\mu_{n+1}))]\} \\
&\leq \max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2}), 1/2[d_\infty(\mu_n, \mu_{n+1}) + d_\infty(\mu_{n+1}, \mu_{n+2})]\} \\
&= \max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2})\}.
\end{aligned}$$

Using this one, in view of the inequalities (3.1) and (3.2), we obtain

$$d_\infty(\mu_{n+1}, \mu_{n+2}) < \beta_n k(d_\infty(\mu_n, \mu_{n+1})) \max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2})\}. \quad (3.3)$$

Then, by inequality (3.3), we must have

$$\max\{d_\infty(\mu_n, \mu_{n+1}), d_\infty(\mu_{n+1}, \mu_{n+2})\} = d_\infty(\mu_n, \mu_{n+1}). \quad (3.4)$$

Indeed, if (3.4) is not true, we would have

$$d_\infty(\mu_{n+1}, \mu_{n+2}) < \beta_n k(d_\infty(\mu_n, \mu_{n+1})) d_\infty(\mu_{n+1}, \mu_{n+2}),$$

by inequality (3.3). This leads to a contradiction, since $0 < \beta_n k(d_\infty(\mu_n, \mu_{n+1})) < 1 - \varepsilon_n < 1$ for all $n \in \mathbb{N}$. This shows that (3.4) holds. Subsequently, since $\beta_n k(d_\infty(\mu_n, \mu_{n+1})) \in (0, 1)$, we have

$$\begin{aligned}
d_\infty(\mu_{n+1}, \mu_{n+2}) &< \beta_n k(d_\infty(\mu_n, \mu_{n+1})) d_\infty(\mu_n, \mu_{n+1}) \\
&< d_\infty(\mu_n, \mu_{n+1}).
\end{aligned} \quad (3.5)$$

This means the sequence $\{d_\infty(\mu_n, \mu_{n+1})\}$ is strictly decreasing in $[0, \infty)$. Thus, by applying Theorem 2.4, we know that

$$0 \leq \sup_{n \in \mathbb{N}} \varphi(d_\infty(\mu_n, \mu_{n+1})) < 1.$$

This implies

$$\sup_{n \in \mathbb{N}} k(d_\infty(\mu_n, \mu_{n+1})) = \frac{1}{2} \left[1 + \sup_{n \in \mathbb{N}} \varphi(d_\infty(\mu_n, \mu_{n+1})) \right] < 1. \quad (3.6)$$

On the other hand, we observe that

$$\begin{aligned}
\beta_n k(d_\infty(\mu_n, \mu_{n+1})) &< 1 - \varepsilon_n \\
&< 1 - \left(\frac{1 - k(d_\infty(\mu_n, \mu_{n+1}))}{2} \right) \\
&= \frac{1 + k(d_\infty(\mu_n, \mu_{n+1}))}{2}.
\end{aligned}$$

Using this together with (3.6), we can conclude that

$$\sup_{n \in \mathbb{N}} \beta_n k(d_\infty(\mu_n, \mu_{n+1})) < 1,$$

as required.

Step 2. We show that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{CB}(X)$.

According to step 1, let us put $c := \sup_{n \in \mathbb{N}} \beta_n k(d_\infty(\mu_n, \mu_{n+1})) \in (0, 1)$. Then, by (3.5), we have

$$\begin{aligned} d_\infty(\mu_{n+1}, \mu_{n+2}) &< \beta_n k(d_\infty(\mu_n, \mu_{n+1})) d_\infty(\mu_n, \mu_{n+1}) \\ &\leq c d_\infty(\mu_n, \mu_{n+1}), \end{aligned}$$

for each $n \in \mathbb{N}$. By using this relation, we deduce that

$$d_\infty(\mu_{n+1}, \mu_{n+2}) < c d_\infty(\mu_n, \mu_{n+1}) < \cdots < c^{n+1} d_\infty(\mu_0, \mu_1),$$

for each $n \in \mathbb{N}$.

So, for arbitrary positive integers m and k , we see that

$$\begin{aligned} d_\infty(\mu_k, \mu_{k+m}) &\leq \sum_{i=k}^{k+m-1} d_\infty(\mu_i, \mu_{i+1}) \\ &< \sum_{i=k}^{k+m-1} c^i d_\infty(\mu_0, \mu_1) \\ &\leq \frac{c^k}{1-c} d_\infty(\mu_0, \mu_1). \end{aligned}$$

Since $c \in (0, 1)$, we can conclude that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{CB}(X)$, as required.

Step 3. We show that there is $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subset F_i(\mu_*)$, for all $i \in \mathbb{N}$.

Since $\{\mu_n\}$ is a Cauchy sequence in a complete metric space $(\mathcal{CB}(X), d_\infty)$, there is $\mu_* \in \mathcal{CB}(X)$ such that $\mu_n \rightarrow \mu_*$ as $n \rightarrow \infty$. We now show that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Let $i \in \mathbb{N}$ be arbitrary. Firstly, by Lemma 2.2(iii) and (iv), let us notice that

$$\begin{aligned} \rho_\infty(\mu_*, F_i(\mu_*)) &\leq d_\infty(\mu_*, \mu_j) + \rho_\infty(\mu_j, F_i(\mu_*)) \\ &\leq d_\infty(\mu_*, \mu_j) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), \end{aligned} \tag{3.7}$$

since $\mu_j \subset F_j(\mu_{j-1})$ for arbitrary natural numbers j such that $i \neq j$.

Subsequently, by using (3.7) and Lemma 2.2(iii), we derive

$$\begin{aligned} d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \\ &\quad \rho_\infty(\mu_*, F_i(\mu_*)), 1/2[\rho_\infty(\mu_*, F_j(\mu_{j-1})) + \rho_\infty(\mu_{j-1}, F_i(\mu_*))]\} \\ &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \\ &\quad d_\infty(\mu_*, \mu_j) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), 1/2[d_\infty(\mu_*, \mu_j) \\ &\quad + \rho_\infty(\mu_j, F_j(\mu_{j-1})) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\} \end{aligned}$$

$$\begin{aligned}
&\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \\
&\quad d_\infty(\mu_j, \mu_*) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), 1/2[d_\infty(\mu_j, \mu_*) \\
&\quad + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\}.
\end{aligned} \tag{3.8}$$

We now consider the following two possible cases:

Case 1. If

$$\begin{aligned}
&d_\infty(\mu_j, \mu_*) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \\
&= \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), d_\infty(\mu_j, \mu_*) \\
&\quad + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\},
\end{aligned}$$

then, by inequality (3.8), we have

$$d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) [d_\infty(\mu_j, \mu_*) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*))].$$

This is equivalent to

$$d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \leq \frac{\varphi(d_\infty(\mu_{j-1}, \mu_*)) d_\infty(\mu_j, \mu_*)}{1 - \varphi(d_\infty(\mu_{j-1}, \mu_*))}.$$

Using, the previous argument together with inequality (3.7), we have

$$\begin{aligned}
\rho_\infty(\mu_*, F_i(\mu_*)) &\leq d_\infty(\mu_*, \mu_j) + \frac{\varphi(d_\infty(\mu_{j-1}, \mu_*)) d_\infty(\mu_j, \mu_*)}{1 - \varphi(d_\infty(\mu_{j-1}, \mu_*))} \\
&= \frac{d_\infty(\mu_*, \mu_j)}{1 - \varphi(d_\infty(\mu_{j-1}, \mu_*))}.
\end{aligned} \tag{3.9}$$

Note that, since $d_\infty(\mu_j, \mu_*) \rightarrow 0$ as $j \rightarrow \infty$, without loss of generality (passing to a subsequence if necessary), we may assume that $\{d_\infty(\mu_j, \mu_*)\}_{j=1}^\infty$ is a nonincreasing sequence. Subsequently, by Theorem 2.4, we have

$$0 \leq \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_j, \mu_*)) < 1.$$

Subsequently, by inequality (3.9), we obtain

$$\begin{aligned}
\rho_\infty(\mu_*, F_i(\mu_*)) &\leq \frac{d_\infty(\mu_j, \mu_*)}{1 - \varphi(d_\infty(\mu_{j-1}, \mu_*))} \\
&\leq \frac{d_\infty(\mu_j, \mu_*)}{1 - \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_{j-1}, \mu_*))}.
\end{aligned} \tag{3.10}$$

Letting $j \rightarrow \infty$ in inequality (3.10), we have

$$\rho_\infty(\mu_*, F_i(\mu_*)) \leq 0.$$

This implies, by Lemma 2.2(i), that $\mu_* \subseteq F_i(\mu_*)$.

Case 2. Assume

$$\begin{aligned}
 & d_\infty(\mu_j, \mu_*) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \\
 & \neq \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), d_\infty(\mu_j, \mu_*) \\
 & \quad + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) \\
 & \quad + \rho_\infty(\mu_*, F_i(\mu_*))]\}.
 \end{aligned}$$

In this situation, by inequality (3.8), we have

$$\begin{aligned}
 & d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \\
 & \leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), \rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})), \\
 & \quad 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\} \\
 & \leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), d_\infty(\mu_{j-1}, \mu_j), 1/2[d_\infty(\mu_j, \mu_*) \\
 & \quad + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\}.
 \end{aligned}$$

So, it follows by inequality (3.7) that

$$\begin{aligned}
 \rho_\infty(\mu_*, F_i(\mu_*)) & \leq d_\infty(\mu_*, \mu_j) + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), d_\infty(\mu_{j-1}, \mu_j), \\
 & \quad 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\} \\
 & \leq d_\infty(\mu_*, \mu_j) + \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_{j-1}, \mu_*)) \max\{d_\infty(\mu_{j-1}, \mu_*), d_\infty(\mu_{j-1}, \mu_j), \\
 & \quad 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\} \\
 & < d_\infty(\mu_*, \mu_j) + \max\{d_\infty(\mu_{j-1}, \mu_*), d_\infty(\mu_{j-1}, \mu_j), \\
 & \quad 1/2[d_\infty(\mu_j, \mu_*) + d_\infty(\mu_{j-1}, \mu_*) + \rho_\infty(\mu_*, F_i(\mu_*))]\}. \tag{3.11}
 \end{aligned}$$

Letting $j \rightarrow \infty$ in inequality (3.11), we obtain

$$\rho_\infty(\mu_*, F_i(\mu_*)) \leq \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2},$$

which implies that $\rho_\infty(\mu_*, F_i(\mu_*)) = 0$. Again, by Lemma 2.2(i), it follows that $\mu_* \subseteq F_i(\mu_*)$. Hence, by Cases 1 and 2, the proof is completed. $\square$

Remark 3.2 Theorem 3.1 recovers Theorem 1.1 as a special case. Meanwhile, it improves Theorem 1.2 since we are considering a larger class of metric space settings.

Using Theorem 3.1, we also obtain the following result.

Corollary 3.3 Let (X, d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $CB(X)$. If there exist nonnegative constants a, b, c , $a + 2b + 2c < 1$, such

that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and j , $i \neq j$,

$$\begin{aligned} d_\infty(F_i(\mu_1), F_j(\mu_2)) &\leq ad_\infty(\mu_1, \mu_2) + b\{\rho_\infty(\mu_1, F_i(\mu_1)) + \rho_\infty(\mu_2, F_j(\mu_2))\} \\ &\quad + c\{\rho_\infty(\mu_1, F_i(\mu_2)) + \rho_\infty(\mu_2, F_j(\mu_1))\}, \end{aligned} \quad (3.12)$$

then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Proof Let us define an $\mathcal{R}$ function $\varphi : [0, \infty) \rightarrow [0, 1)$ by

$$\varphi(t) = a + 2b + 2c \quad \text{for all } t \in [0, \infty).$$

Then one can derive that the relation (3.12) is transformed to (3.1), and so the required result follows immediately from Theorem 3.1. $\square$

Next, motivated by the idea of Berinde [18], we now present another fuzzy fixed point theorem.

Theorem 3.4 *Let (X, d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $\mathcal{CB}(X)$. If there exist an $\mathcal{R}$ -function $\varphi : [0, \infty) \rightarrow [0, 1)$ and $L \geq 0$ such that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and j , $i \neq j$,*

$$\begin{aligned} d_\infty(F_i(\mu_1), F_j(\mu_2)) &\leq \varphi(d_\infty(\mu_1, \mu_2)) \frac{[\rho_\infty(\mu_1, F_i(\mu_1)) + \rho_\infty(\mu_2, F_j(\mu_2))]}{2} \\ &\quad + L\rho_\infty(\mu_2, F_i(\mu_1)), \end{aligned} \quad (3.13)$$

then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Proof Let us construct and consider again a function k and the sequences which we have defined in (3.2). Recall that, by our constructive method, we also know that

$$\rho_\infty(\mu_{n-1}, F_n(\mu_{n-1})) \leq d_\infty(\mu_{n-1}, \mu_n),$$

for each $n \in \mathbb{N}$. Using this together with (3.13), we derive

$$\begin{aligned} d_\infty(\mu_{n+1}, \mu_{n+2}) &\leq \beta_n d_\infty(F_{n+1}(\mu_n), F_{n+2}(\mu_{n+1})) \\ &\leq \beta_n \varphi(d_\infty(\mu_n, \mu_{n+1})) \frac{[\rho_\infty(\mu_n, F_{n+1}(\mu_n)) + \rho_\infty(\mu_{n+1}, F_{n+2}(\mu_{n+1}))]}{2} \\ &\quad + \beta_n L\rho_\infty(\mu_{n+1}, F_{n+1}(\mu_n)) \\ &\leq \beta_n \varphi(d_\infty(\mu_n, \mu_{n+1})) \frac{[d_\infty(\mu_n, \mu_{n+1}) + d_\infty(\mu_{n+1}, \mu_{n+2})]}{2} \\ &< \beta_n k(d_\infty(\mu_n, \mu_{n+1})) \frac{[d_\infty(\mu_n, \mu_{n+1}) + d_\infty(\mu_{n+1}, \mu_{n+2})]}{2}, \end{aligned}$$

for each $n \in \mathbb{N}$. This is equivalent to

$$d_\infty(\mu_{n+1}, \mu_{n+2}) < \frac{\beta_n k(d_\infty(\mu_n, \mu_{n+1}))}{2 - \beta_n k(d_\infty(\mu_n, \mu_{n+1}))} d_\infty(\mu_n, \mu_{n+1}), \quad (3.14)$$

for each $n \in \mathbb{N}$. Observe that, since $\beta_n k(d_\infty(\mu_n, \mu_{n+1})) \in (0, 1)$, we also have

$$\frac{\beta_n k(d_\infty(\mu_n, \mu_{n+1}))}{2 - \beta_n k(d_\infty(\mu_n, \mu_{n+1}))} \in (0, 1),$$

for all $n \in \mathbb{N}$. Thus, by (3.14), we see that $\{d_\infty(\mu_n, \mu_{n+1})\}$ is a strictly decreasing sequence in $[0, \infty)$. Since φ is an $\mathcal{R}$ -function, by applying Theorem 2.4, we have

$$0 \leq \sup_{n \in \mathbb{N}} \varphi(d_\infty(\mu_n, \mu_{n+1})) < 1.$$

Next, as we have done in order to prove Theorem 3.1, we write

$$\lambda := \sup_{n \in \mathbb{N}} \beta_n k(d_\infty(\mu_n, \mu_{n+1})) < 1.$$

Let us take $c := \frac{\lambda}{2-\lambda}$. It follows that $c \in (0, 1)$ and, by (3.14), we also have

$$\begin{aligned} d_\infty(\mu_{n+1}, \mu_{n+2}) &< \frac{\beta_n k(d_\infty(\mu_n, \mu_{n+1}))}{2 - \beta_n k(d_\infty(\mu_n, \mu_{n+1}))} d_\infty(\mu_n, \mu_{n+1}) \\ &\leq c d_\infty(\mu_n, \mu_{n+1}). \end{aligned} \quad (3.15)$$

Now, again by reasoning along the lines of proving Theorem 3.1, we can show that there is a $\mu_* \in \mathcal{CB}(X)$ which is the limit point of the considered sequence $\{\mu_n\}$, under the supremum metric d_∞ , and it satisfies

$$0 \leq \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_j, \mu_*)) < 1. \quad (3.16)$$

Finally, we will show that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be arbitrary. Note again that

$$\begin{aligned} \rho_\infty(\mu_*, F_i(\mu_*)) &\leq d_\infty(\mu_*, \mu_j) + \rho_\infty(\mu_j, F_i(\mu_*)) \\ &\leq d_\infty(\mu_*, \mu_j) + d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)), \end{aligned} \quad (3.17)$$

for each $j \in \mathbb{N}$ such that $j \neq i$. Consider

$$\begin{aligned} &d_\infty(F_j(\mu_{j-1}), F_i(\mu_*)) \\ &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{[\rho_\infty(\mu_{j-1}, F_j(\mu_{j-1})) + \rho_\infty(\mu_*, F_i(\mu_*))]}{2} + L \rho_\infty(\mu_*, F_j(\mu_{j-1})) \\ &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2} \\ &\quad + L(d_\infty(\mu_*, \mu_j) + \rho_\infty(\mu_j, F_j(\mu_{j-1}))) \\ &\leq \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2} \\ &\quad + L d_\infty(\mu_*, \mu_j) + L \rho_\infty(\mu_j, F_j(\mu_{j-1})), \end{aligned}$$

where $j \in \mathbb{N}$ with $j \neq i$. Subsequently, from (3.17) together with (3.16), we obtain

$$\begin{aligned}
\rho_\infty(\mu_*, F_i(\mu_*)) &\leq d_\infty(\mu_*, \mu_j) + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} \\
&\quad + \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2} \\
&\quad + Ld_\infty(\mu_*, \mu_j) + L\rho_\infty(\mu_j, F_j(\mu_{j-1})) \\
&\leq (1+L)d_\infty(\mu_*, \mu_j) + \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_j, \mu_*)) \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} \\
&\quad + \sup_{j \in \mathbb{N}} \varphi(d_\infty(\mu_{j-1}, \mu_*)) \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2} \\
&< (1+L)d_\infty(\mu_*, \mu_j) + \frac{d_\infty(\mu_{j-1}, \mu_j)}{2} + \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2}. \tag{3.18}
\end{aligned}$$

By letting $j \rightarrow \infty$ on the right of inequality (3.18), we have

$$\rho_\infty(\mu_*, F_i(\mu_*)) \leq \frac{\rho_\infty(\mu_*, F_i(\mu_*))}{2},$$

which implies that $\rho_\infty(\mu_*, F_i(\mu_*)) = 0$. Then, by Lemma 2.2(i), we have $\mu_* \subseteq F_i(\mu_*)$. Since $i \in \mathbb{N}$ is arbitrary, we complete the proof. $\square$

The following result can be deduced from Theorem 3.4.

Corollary 3.5 *Let (X, d) be a complete metric space and let $\{F_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $\mathcal{CB}(X)$. If there exist an $\mathcal{R}$ -function $\varphi : [0, \infty) \rightarrow [0, 1)$ and $L \geq 0$ such that for each $\mu_1, \mu_2 \in \mathcal{CB}(X)$, and for arbitrary positive integers i and j , $i \neq j$,*

$$\begin{aligned}
d_\infty(F_i(\mu_1), F_j(\mu_2)) &\leq \varphi(d_\infty(\mu_1, \mu_2)) [\rho_\infty(\mu_1, F_i(\mu_1)) \rho_\infty(\mu_2, F_j(\mu_2))]^{1/2} \\
&\quad + L\rho_\infty(\mu_2, F_i(\mu_1)), \tag{3.19}
\end{aligned}$$

then there exists a $\mu_* \in \mathcal{CB}(X)$ such that $\mu_* \subseteq F_i(\mu_*)$ for all $i \in \mathbb{N}$.

Proof Since for each $a, b \in \mathbb{R}^+$, we have $\sqrt{ab} \leq \frac{a+b}{2}$, the required result follows immediately from our Theorem 3.4. $\square$

Remark 3.6 Corollary 3.5 recovers a result which has presented in [12], when $L = 0$.

4 Conclusions

In this paper we have presented fuzzy fixed point theorems on the space of fuzzy sets under a kind of supremum metric setting. We would like to point out that this kind of space is very general and interesting. Moreover, as one can observe, a fixed point in this situation is, in fact, a fixed (fuzzy) set. This means that our presented results are very general and we recover many existing results on fixed point theory as regards both single-valued and set-valued mappings.

References

1. Zadeh, LA: Fuzzy sets. *Inf. Control* **8**, 338-353 (1965)
2. Heilpern, S: Fuzzy mappings and fixed point theorem. *J. Math. Anal. Appl.* **83**, 566-569 (1981)
3. Estruch, VD, Vidal, A: A note on fixed fuzzy points for fuzzy mappings. *Rend. Ist. Mat. Univ. Trieste* **32**, 39-45 (2001)
4. Sedghi, S, Shobe, N, Altun, I: A fixed fuzzy point for fuzzy mappings in complete metric spaces. *Math. Commun.* **13**, 289-294 (2008)
5. Anastassiou, GA: Multivariate fuzzy perturbed neural network operators approximation. *J. Nonlinear Sci. Appl.* **7**(6), 383-406 (2014)
6. Azama, A, Beg, I: Common fuzzy fixed points for fuzzy mappings. *Fixed Point Theory Appl.* **2013**, 14 (2013)
7. Bose, RK, Sahani, D: Fuzzy mappings and fixed point theorems. *Fuzzy Sets Syst.* **21**, 53-58 (1987)
8. Butnariu, D: Fixed point for fuzzy mapping. *Fuzzy Sets Syst.* **7**, 191-207 (1982)
9. Kaur, S, Kaur, J: Integrability and L^1 -convergence of fuzzy trigonometric series with special fuzzy coefficients. *J. Nonlinear Sci. Appl.* **8**(1), 23-39 (2015)
10. Lee, BS, Cho, SJ: A fixed point theorem for contractive type fuzzy mappings. *Fuzzy Sets Syst.* **61**, 309-312 (1994)
11. Moreno, JM, Roldan, A, Roldan, C, Cho, YJ: Multi-dimensional coincidence point theorems for weakly compatible mappings with the CLRg-property in (fuzzy) metric spaces. *Fixed Point Theory Appl.* **2015**, 53 (2015)
12. Qiu, D, Shu, L: Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings. *Inf. Sci.* **178**, 3595-3604 (2008)
13. Rashwan, RA, Ahmad, MA: Common fixed point theorems for fuzzy mappings. *Arch. Math.* **38**, 219-226 (2002)
14. Turkoglu, D, Rhoades, BE: A fixed fuzzy point for fuzzy mapping in complete metric spaces. *Math. Commun.* **10**, 115-121 (2005)
15. Qiu, D, Shu, L, Guan, J: Common fixed point theorems for fuzzy mappings under Φ -contraction condition. *Chaos Solitons Fractals* **41**, 360-367 (2009)
16. Diamond, P, Kloeden, P: *Metric Spaces of Fuzzy Sets Theory and Applications*. World Scientific, Singapore (1994)
17. Du, W-S: On coincidence point and fixed point theorems for nonlinear multivalued maps. *Topol. Appl.* **159**, 49-56 (2012)
18. Berinde, V: Approximating fixed points of weak contractions using Picard iteration. *Nonlinear Anal. Forum* **9**(1), 43-53 (2004)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)

7.4 A modified regularization method for finding zeros of monotone operators in Hilbert spaces

In this section, we study the regularization method for solving the variational inclusion problem of the sum of two monotone operators in Hilbert spaces. The strong convergence theorem is then established under some relaxed conditions which mainly improves and recovers that of Qin *et al.* (Fixed Point Theory Appl. 2014:75, 2014). We also apply our main result to the convex minimization problem, the fixed point problem and the variational inequality problem. Finally we provide numerical examples for supporting the main result.

1 Introduction

Let C be a nonempty subset of a real Hilbert space H . Define the domain and the range of an operator $B : H \rightarrow 2^H$ by $D(B) = \{x \in H : Bx \neq \emptyset\}$ and $R(B) = \bigcup \{Bx : x \in D(B)\}$, respectively. The inverse of B , denoted by B^{-1} , is defined by $x \in B^{-1}y$ if and only if $y \in Bx$. We study the problem of finding $\hat{x}$ such that

$$0 \in A\hat{x} + B\hat{x},$$

where $A : C \rightarrow H$ is an operator and $B : D(B) \subset H \rightarrow 2^H$ is a set-valued operator. This problem is called the variational inclusion problem. Some typical problems arising in various branches of science, applied sciences, economics, and engineering such as machine learning, image restoration, and signal recovery can be viewed as this form. To be more precise, it includes, as special cases, the variational inequality problem, the split feasibility problem, the linear inverse problem, and the following convex minimization problem:

$$\min_{x \in H} F(x) + G(x),$$

where $F : H \rightarrow \mathbb{R}$ is a smooth convex function, and $G : H \rightarrow \mathbb{R}$ is a non-smooth convex function. That is,

$$F(\hat{x}) + G(\hat{x}) = \min_{x \in H} F(x) + G(x) \quad \Leftrightarrow \quad 0 \in \nabla F(\hat{x}) + \partial G(\hat{x}),$$

where ∇F is the gradient of F and ∂G is the subdifferential of G defined by

$$\partial G(x) = \{z \in H : \langle y - x, z \rangle + G(x) \leq G(y), \forall y \in H\}.$$

For $r > 0$, define the mapping $T_r : C \rightarrow D(B)$ as follows:

$$T_r = (I + rB)^{-1}(I - rA). \quad (1.1)$$

We see that

$$T_r x = x \Leftrightarrow x = (I + rB)^{-1}(x - rAx) \Leftrightarrow x - rAx \in x + rBx \Leftrightarrow 0 \in Ax + Bx,$$

which shows that the fixed points set of T_r coincides with the solutions set of $(A + B)^{-1}(0)$. This suggests the following iteration process: $x_0 \in C$ and

$$x_{n+1} = (I + r_n B)^{-1}(x_n - r_n A x_n) = T_{r_n} x_n, \quad n \geq 0,$$

where $\{r_n\} \subset (0, \infty)$ and $D(B) \subset C$. This method is called a forward-backward splitting algorithm [1, 2]. If $A \equiv 0$, then we obtain the proximal point algorithm [3–6] and if $B \equiv 0$, then we obtain the gradient method [7]. However, it is noted that the sequences generated by these schemes converge weakly in general. In the literature, many methods have been suggested to solve the variational inclusion problem for maximal monotone operators; see, e.g., [8–12].

Very recently, Qin *et al.* [13] proved the following theorem in Hilbert spaces.

Theorem Q *Let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping and let B be a maximal monotone operator on H . Assume that $D(B) \subset C$ and $(A + B)^{-1}(0)$ is nonempty. Let $f : C \rightarrow C$ be a fixed k -contraction and let $J_{r_n} = (I + r_n B)^{-1}$. Let $\{z_n\}$ be a sequence in C in the following process: $z_0 \in C$ and*

$$\begin{aligned} w_n &= \alpha_n f(z_n) + (1 - \alpha_n) z_n, \\ z_{n+1} &= J_{r_n}(w_n - r_n A w_n + e_n), \quad n \geq 0, \end{aligned} \quad (1.2)$$

where $\{\alpha_n\} \subset (0, 1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0, 2\alpha)$. If the control sequences satisfy the following restrictions:

- (a) $\alpha_n \rightarrow 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
- (b) $0 < a \leq r_n \leq b < 2\alpha$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$;
- (c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{z_n\}$ converges strongly to a point $\bar{x} \in (A + B)^{-1}(0)$, where $\bar{x} = P_{(A+B)^{-1}(0)} f(\bar{x})$.

In this paper, motivated by Qin *et al.* [13], we prove that the above theorem still holds even if the additional conditions that $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ are removed. As a direct consequence, we obtain some results concerning the fixed point problem of strict pseudocontractions, the convex minimization problem and the variational inequality problem. We also provide examples as well as numerical results.

2 Preliminaries and lemmas

We now provide some basic concepts, definitions and lemmas which will be used in the sequel.

Let C be a nonempty, closed, and convex subset of a real Hilbert space H with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. For each $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. Then P_C is called the metric projection of H on to C . For $x \in H$, we know that

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \quad (2.1)$$

for all $y \in C$. Recall that the mapping $T : C \rightarrow C$ is said to be

- (i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
- (ii) k -contractive if there exists $0 < k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\|$$

for all $x, y \in C$;

- (iii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$.

- (iv) monotone if $\langle Tx - Ty, x - y \rangle \geq 0$ for all $x, y \in C$;
- (v) α -inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2$$

for all $x, y \in C$. We denote by $F(T)$ the fixed points set of T , that is,

$$F(T) = \{x \in C : x = Tx\}.$$

A set-valued operator B is said to be *monotone* if, for each $x, y \in D(B)$,

$$\langle u - v, x - y \rangle \geq 0, \quad u \in Bx, v \in By.$$

A monotone operator A is said to be *maximal* if $R(I + rB) = H$ for all $r > 0$ (see Minty [14]). For a maximal monotone operator B on H , and $r > 0$, we define the single-valued resolvent $J_r : H \rightarrow D(B)$ by $J_r = (I + rB)^{-1}$. It is well known that J_r is firmly nonexpansive, and $F(J_r) = B^{-1}(0)$.

We now collect some crucial lemmas.

Lemma 2.1 [15] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H . Let the mapping $A : C \rightarrow H$ be α -inverse strongly monotone and $r > 0$ be a constant. Then we have*

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2$$

for all $x, y \in C$. In particular, if $0 < r \leq 2\alpha$, then $I - rA$ is nonexpansive.

Lemma 2.2 [9] *Let $A : C \rightarrow H$ be a mapping and $B : D(B) \subset H \rightarrow 2^H$ a monotone operator. Then $\|x - T_s x\| \leq 2\|x - T_r x\|$ for all $0 < s \leq r$ and $x \in C$.*

Lemma 2.3 [16] *Let C be a nonempty, closed, and convex subset of a Hilbert space H , and $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $x_n \rightarrow x$ and $\|x_n - Tx_n\| \rightarrow 0$, then $x \in F(T)$.*

Lemma 2.4 [17] *Let $\{a_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

- (i) *If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
- (ii) *If $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\limsup_{n \rightarrow \infty} b_n / \delta_n \leq 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.*

We need the following crucial lemma proved by He-Yang [18].

Lemma 2.5 [18] *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n \delta_n, \quad n \geq 0$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$, and $\{\rho_n\}$ are real sequences such that

- (i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
- (ii) $\lim_{n \rightarrow \infty} \rho_n = 0$,
- (iii) $\lim_{k \rightarrow \infty} \eta_{n_k} = 0$ implies $\limsup_{n \rightarrow \infty} \delta_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

3 Main results

In this section, we present the main theorem of this paper.

Theorem 3.1 *Let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping and let B be a maximal monotone operator on H such that $D(B) \subset C$ and $(A + B)^{-1}(0)$ is nonempty. Let $f : C \rightarrow C$ be a k -contraction. Assume that $\{\alpha_n\} \subset (0, 1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0, 2\alpha)$ with the following restrictions:*

- (a) $\alpha_n \rightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (b) $0 < a \leq r_n \leq b < 2\alpha$;
- (c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\|e_n\|/\alpha_n \rightarrow 0$.

Then the sequence $\{z_n\}$ generated by (1.2) converges strongly to a point $\bar{x} \in (A + B)^{-1}(0)$, where $\bar{x} = P_{(A+B)^{-1}(0)}f(\bar{x})$.

Proof Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and

$$y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n,$$

$$x_{n+1} = J_{r_n}(y_n - r_n A y_n), \quad n \geq 0.$$

Firstly, we shall show that $\{x_n\}$ and $\{z_n\}$ are equivalent. Indeed,

$$\begin{aligned}\|y_n - w_n\| &= \|\alpha_n(f(x_n) - f(z_n)) + (1 - \alpha_n)(x_n - z_n)\| \\ &\leq \alpha_n k \|x_n - z_n\| + (1 - \alpha_n) \|x_n - z_n\| \\ &= (1 - \alpha_n(1 - k)) \|x_n - z_n\|.\end{aligned}$$

Using Lemma 2.1, condition (b), and the fact that J_{r_n} is nonexpansive, we obtain

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &= \|J_{r_n}(y_n - r_n A y_n) - J_{r_n}(w_n - r_n A w_n + e_n)\| \\ &\leq \|(y_n - r_n A y_n) - (w_n - r_n A w_n + e_n)\| \\ &\leq \|y_n - w_n\| + \|e_n\| \\ &\leq (1 - \alpha_n(1 - k)) \|x_n - z_n\| + \|e_n\|.\end{aligned}$$

Applying Lemma 2.4(ii) with conditions (a) and (c), we conclude that $\|x_n - z_n\| \rightarrow 0$.

On the other hand, it can be checked that $P_{(A+B)^{-1}(0)}f$ is a contraction. So there exists a unique point $\bar{x} \in C$ such that

$$\bar{x} = P_{(A+B)^{-1}(0)}f(\bar{x}). \quad (3.1)$$

To finish our proof, it suffices to show that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$.

We next show that $\{x_n\}$ is bounded. Fixing $p \in (A + B)^{-1}(0)$, we obtain

$$\begin{aligned}\|y_n - p\| &= \|\alpha_n(f(x_n) - f(p)) + \alpha_n(f(p) - p) + (1 - \alpha_n)(x_n - p)\| \\ &\leq \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &= (1 - \alpha_n(1 - k)) \|x_n - p\| + \alpha_n \|f(p) - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|J_{r_n}(y_n - r_n A y_n) - J_{r_n}(p - r_n A p)\| \\ &\leq \|y_n - p\| \\ &\leq (1 - \alpha_n(1 - k)) \|x_n - p\| + \alpha_n \|f(p) - p\|.\end{aligned}$$

Hence $\{x_n\}$ is bounded by Lemma 2.4(i). So are $\{f(x_n)\}$ and $\{y_n\}$. Observing

$$\begin{aligned}\|y_n - \bar{x}\|^2 &= \alpha_n \langle f(x_n) - f(\bar{x}), y_n - \bar{x} \rangle + \alpha_n \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle + (1 - \alpha_n) \langle x_n - \bar{x}, y_n - \bar{x} \rangle \\ &\leq \alpha_n k \|x_n - \bar{x}\| \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle + (1 - \alpha_n) \|x_n - \bar{x}\| \|y_n - \bar{x}\| \\ &= (1 - \alpha_n(1 - k)) \|x_n - \bar{x}\| \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle \\ &\leq \frac{1}{2} (1 - \alpha_n(1 - k)) (\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2) + \alpha_n \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle,\end{aligned}$$

we have

$$\|y_n - \bar{x}\|^2 \leq \left(1 - \frac{2\alpha_n(1 - k)}{1 + \alpha_n(1 - k)}\right) \|x_n - \bar{x}\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1 - k)} \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle.$$

So, by Lemma 2.1 and the firm nonexpansiveness of J_{r_n} , we have

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \|J_{r_n}(y_n - r_n A y_n) - J_{r_n}(\bar{x} - r_n A \bar{x})\|^2 \\
&\leq \|(y_n - r_n A y_n) - (\bar{x} - r_n A \bar{x})\|^2 \\
&\quad - \|(I - J_{r_n})(y_n - r_n A y_n) - (I - J_{r_n})(\bar{x} - r_n A \bar{x})\|^2 \\
&\leq \|y_n - \bar{x}\|^2 - r_n(2\alpha - r_n)\|A y_n - A \bar{x}\|^2 \\
&\quad - \|(I - J_{r_n})(y_n - r_n A y_n) - (I - J_{r_n})(\bar{x} - r_n A \bar{x})\|^2 \\
&\leq \left(1 - \frac{2\alpha_n(1-k)}{1 + \alpha_n(1-k)}\right) \|x_n - \bar{x}\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1-k)} \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle \\
&\quad - r_n(2\alpha - r_n)\|A y_n - A \bar{x}\|^2 \\
&\quad - \|(I - J_{r_n})(y_n - r_n A y_n) - (I - J_{r_n})(\bar{x} - r_n A \bar{x})\|^2.
\end{aligned}$$

This implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq \left(1 - \frac{2\alpha_n(1-k)}{1 + \alpha_n(1-k)}\right) \|x_n - \bar{x}\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1-k)} \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle \quad (3.2)$$

and

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\leq \|x_n - \bar{x}\|^2 - r_n(2\alpha - r_n)\|A y_n - A \bar{x}\|^2 \\
&\quad - \|(I - J_{r_n})(y_n - r_n A y_n) - (I - J_{r_n})(\bar{x} - r_n A \bar{x})\|^2 \\
&\quad + \frac{2\alpha_n}{1 + \alpha_n(1-k)} \|f(\bar{x}) - \bar{x}\| \|y_n - \bar{x}\|.
\end{aligned} \quad (3.3)$$

We set, for all $n \geq 1$, $s_n = \|x_n - \bar{x}\|^2$, $\gamma_n = \frac{2\alpha_n(1-k)}{1 + \alpha_n(1-k)}$, $\delta_n = \frac{1}{1-k} \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle$, $\rho_n = \frac{2\alpha_n}{1 + \alpha_n(1-k)} \|f(\bar{x}) - \bar{x}\| \|y_n - \bar{x}\|$, and $\eta_n = r_n(2\alpha - r_n)\|A y_n - A \bar{x}\|^2 + \|(I - J_{r_n})(y_n - r_n A y_n) - (I - J_{r_n})(\bar{x} - r_n A \bar{x})\|^2$. We can check that all sequences satisfies conditions (i) and (ii) in Lemma 2.5. Then (3.2) and (3.3) can be rewritten as the following inequalities:

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n \delta_n, \quad n \geq 0$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \quad n \geq 0.$$

To complete the proof, we verify that the condition (iii) in Lemma 2.5 is satisfied. Let $\{n_k\} \subset \{n\}$ be such that $\eta_{n_k} \rightarrow 0$. Then, by condition (b), we have

$$\lim_{n \rightarrow \infty} \|A y_{n_k} - A \bar{x}\| = 0$$

and

$$\lim_{n \rightarrow \infty} \|(I - J_{r_{n_k}})(y_{n_k} - r_{n_k} A y_{n_k}) - (I - J_{r_{n_k}})(\bar{x} - r_{n_k} A \bar{x})\| = 0.$$

Hence we obtain

$$\lim_{n \rightarrow \infty} \|y_{n_k} - J_{r_{n_k}}(y_{n_k} - r_{n_k}Ay_{n_k})\| = 0.$$

By Lemma 2.2(ii) and (b), we have

$$\|J_a(y_{n_k} - aAy_{n_k}) - y_{n_k}\| \leq 2\|J_{r_{n_k}}(y_{n_k} - r_{n_k}Ay_{n_k}) - y_{n_k}\| \rightarrow 0,$$

where $J_a = (I + aB)^{-1}$. Since $\{y_n\}$ is bounded, by Lemma 2.3, we have $\omega_w(y_{n_k}) \subset (A + B)^{-1}(0)$. Hence

$$\limsup_{k \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, y_{n_k} - \bar{x} \rangle = \langle f(\bar{x}) - \bar{x}, y - \bar{x} \rangle \leq 0,$$

where $y \in \omega_w(y_{n_k})$. It follows that $\limsup_{n \rightarrow \infty} \delta_{n_k} \leq 0$. So, by Lemma 2.5, we conclude that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. We thus complete the proof. $\square$

Remark 3.2 We remove the additionally required conditions: $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$ proposed in the main theorem of Qin *et al.* [13].

4 Applications and numerical examples

In this section, we give some applications of our result to the variational inequality problem, the fixed point problem of strict pseudocontractions and the convex minimization problem.

4.1 Variational inequality problem

Let C be a nonempty subset of a Hilbert space H . The variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \quad (4.1)$$

The solution set of (4.1) is denoted by $\text{VI}(A, C)$. It is well known that $F(P_C(I - rA)) = \text{VI}(A, C)$ for all $r > 0$. Define the indicator function of C , denoted by i_C , as $i_C(x) = 0$ if $x \in C$ and $i_C(x) = \infty$ if $x \notin C$. We see that ∂i_C is maximal monotone. So, for $r > 0$, we can define $J_r = (I + r\partial i_C)^{-1}$. Moreover, $x = J_r y$ if and only if $x = P_C y$. Hence we obtain the following result.

Theorem 4.1 *Let $A : C \rightarrow H$ be an α -inverse strongly monotone mapping such that $\text{VI}(A, C)$ is nonempty. Let $f : C \rightarrow C$ be a k -contraction. Let $\{z_n\}$ be a sequence in C defined by $z_0 \in C$ and*

$$\begin{aligned} w_n &= \alpha_n f(z_n) + (1 - \alpha_n)z_n, \\ z_{n+1} &= P_C(w_n - r_n A w_n + e_n), \quad n \geq 0, \end{aligned} \quad (4.2)$$

where $\{\alpha_n\} \subset (0, 1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0, 2\alpha)$. Assume that

- (a) $\alpha_n \rightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (b) $0 < a \leq r_n \leq b < 2\alpha$;
- (c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\|e_n\|/\alpha_n \rightarrow 0$.

Then $\{z_n\}$ converges strongly to a point $\bar{x} \in \text{VI}(A, C)$, where $\bar{x} = P_{\text{VI}(A, C)} f(\bar{x})$.

4.2 Fixed point problem of strict pseudocontractions

A mapping $T : C \rightarrow C$ is called β -strictly pseudocontractive if there exists $\beta \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$. It is well known that if T is β -strictly pseudocontractive, then $I - T$ is $\frac{1-\beta}{2}$ -inverse strongly monotone. Moreover, by putting $A = I - T$, we have $F(T) = \text{VI}(A, C)$. So we immediately obtain the following result.

Theorem 4.2 *Let $T : C \rightarrow C$ be a β -strict pseudocontraction such that $F(T) \neq \emptyset$ and let $f : C \rightarrow C$ be a k -contraction. Let $\{z_n\}$ be a sequence in C defined by $z_0 \in C$ and*

$$\begin{aligned} w_n &= \alpha_n f(z_n) + (1 - \alpha_n)z_n, \\ z_{n+1} &= P_C((1 - r_n)w_n + r_n T w_n + e_n), \quad n \geq 0, \end{aligned} \tag{4.3}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0, 1 - \beta)$. Assume that

- (a) $\alpha_n \rightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (b) $0 < a \leq r_n \leq b < 1 - \beta$;
- (c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\|e_n\|/\alpha_n \rightarrow 0$.

Then $\{z_n\}$ converges strongly to a point $\bar{x} \in F(T)$, where $\bar{x} = P_{F(T)}f(\bar{x})$.

4.3 Convex minimization problem

We next consider the following convex minimization problem:

$$\min_{x \in H} F(x) + G(x),$$

where $F : H \rightarrow \mathbb{R}$ is a convex and differentiable function and $G : H \rightarrow \mathbb{R}$ is a convex function. It is well known that if ∇F is $(1/L)$ -Lipschitz continuous, then it is L -inverse strongly monotone [19]. Moreover, ∂G is maximal monotone [20]. Putting $A = \nabla F$ and $B = \partial G$, we then obtain the following result.

Theorem 4.3 *Let H be a Hilbert space. Let $F : H \rightarrow \mathbb{R}$ be a convex and differentiable function with $(1/L)$ -Lipschitz continuous gradient ∇F and $G : H \rightarrow \mathbb{R}$ be a convex and lower semi-continuous function such that $\Omega := (\nabla F + \partial G)^{-1}(0) \neq \emptyset$. Let $f : H \rightarrow H$ be a k -contraction. Let $\{z_n\}$ be generated by $z_0 \in H$ and*

$$\begin{aligned} w_n &= \alpha_n f(z_n) + (1 - \alpha_n)z_n, \\ z_{n+1} &= J_{r_n}(w_n - r_n \nabla F(w_n) + e_n), \quad n \geq 0, \end{aligned} \tag{4.4}$$

where $J_{r_n} = (I + r_n \partial G)^{-1}$, $\{\alpha_n\} \subset (0, 1)$, $\{e_n\} \subset H$, and $\{r_n\} \subset (0, 2L)$. Assume that

- (a) $\alpha_n \rightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (b) $0 < a \leq r_n \leq b < 2L$;
- (c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\|e_n\|/\alpha_n \rightarrow 0$.

Then $\{z_n\}$ converges strongly to a minimizer $\bar{x}$ of $F + G$, where $\bar{x} = P_{\Omega}f(\bar{x})$.

Table 1 Numerical results of Example 4.4 for iteration process (4.4)

n	$z_n = (t_n, u_n, v_n)^T$	$F(z_n) + G(z_n)$	$\ z_{n+1} - z_n\ _2$
1	(-0.9999988000, 4.9999960000, 0.9999984000)	21.499964000010	3.6464868709E+00
2	(-0.8749992000, 1.6249970000, -0.3750010000)	-1.601574899991	1.8007236270E+00
3	(-0.9004624463, 0.0000000000, -1.1504634130)	-7.134189931344	4.1056538071E-01
4	(-0.9346061331, 0.0000000000, -1.5596065914)	-7.400888643932	2.1362893808E-01
5	(-0.9593029918, 0.0000000000, -1.7718031709)	-7.473134980367	1.1059145541E-01
6	(-0.9750218023, 0.0000000000, -1.8812718377)	-7.492639856561	5.7254600304E-02
7	(-0.9845953935, 0.0000000000, -1.9377203595)	-7.497941972237	2.9747437021E-02
8	(-0.9903445225, 0.0000000000, -1.9669069578)	-7.499405811158	1.5563253898E-02
9	(-0.9938004751, 0.0000000000, -1.9820816494)	-7.499820249302	8.2314060025E-03
10	(-0.9959001978, 0.0000000000, -1.9900407454)	-7.499942002435	4.4231489769E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	(-0.9999828882, 0.0000000000, -1.9999828718)	-7.499999999707	1.4278683289E-06
51	(-0.9999838977, 0.0000000000, -1.9999838817)	-7.499999999740	1.3175686638E-06
52	(-0.9999848292, 0.0000000000, -1.9999848135)	-7.499999999770	1.2177300519E-06
53	(-0.9999856901, 0.0000000000, -1.9999856747)	-7.499999999795	1.1271827509E-06
54	(-0.9999864870, 0.0000000000, -1.9999864719)	-7.499999999817	1.0449069970E-06
55	(-0.9999872257, 0.0000000000, -1.9999872109)	-7.499999999837	9.7001140362E-07

We next provide the example as well as its numerical results.

Example 4.4 Let $H = \mathbb{R}^3$. Minimize the following ℓ_1 -least square problem:

$$\min_{x \in \mathbb{R}^3} \|x\|_1 + \frac{1}{2} \|x\|_2^2 + (2, 1, 3)x - 5,$$

where $x = (t, u, v)^T$.

Let $F(x) = \frac{1}{2} \|x\|_2^2 + (2, 1, 3)x - 5$ and $G(x) = \|x\|_1$. Then $\nabla F(x) = (t + 2, u + 1, v + 3)^T$. Moreover, ∇F is 1-Lipschitz continuous and hence it is 1-inverse strongly monotone.

From [21] we know that, for $r > 0$,

$$\begin{aligned} & (I + r\partial G)^{-1}(x) \\ &= (\max\{|t| - r, 0\} \operatorname{sign}(t), \max\{|u| - r, 0\} \operatorname{sign}(u), \max\{|v| - r, 0\} \operatorname{sign}(v))^T. \end{aligned}$$

Let $z_n = (t_n, u_n, v_n)^T$. Set $f(x) = \frac{x}{5}$ and choose $\alpha_n = \frac{10^{-6}}{n+1}$, $r_n = 0.5$, and $e_n = \frac{1}{(n+1)^3}(1, 1, 1)^T$. For the initial point $z_0 = (t_0, u_0, v_0)^T = (-3, 10, 4)^T$, computing $\{z_n\}$ by the algorithm (4.4), we obtain numerical results with an error 10^{-7} in Table 1.

From Table 1, we see that $z_\infty = (-1, 0, -2)$ is the minimizer of $F + G$ and its minimum value is -7.5 .

References

1. Lions, PL, Mercier, B: Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.* **16**, 964-979 (1979)
2. Passty, GB: Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. *J. Math. Anal. Appl.* **72**, 383-390 (1979)
3. Brézis, H, Lions, PL: Produits infinis de resolventes. *Isr. J. Math.* **29**, 329-345 (1978)
4. Güler, O: On the convergence of the proximal point algorithm for convex minimization. *SIAM J. Control Optim.* **29**, 403-419 (1991)
5. Martinet, B: Régularisation d'inéquations variationnelles par approximations successives. *Rev. Fr. Inform. Rech. Oper.* **4**, 154-158 (1970)
6. Rockafellar, RT: Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.* **14**, 877-898 (1976)
7. Dunn, JC: Convexity, monotonicity, and gradient processes in Hilbert space. *J. Math. Anal. Appl.* **53**, 145-158 (1976)
8. Combettes, PL: Iterative construction of the resolvent of a sum of maximal monotone operators. *J. Convex Anal.* **16**, 727-748 (2009)
9. López, G, Martín-Márquez, V, Wang, F, Xu, HK: Forward-backward splitting methods for accretive operators in Banach spaces. *Abstr. Appl. Anal.* **2012**, Article ID 109236 (2012)
10. Takahashi, W: Viscosity approximation methods for resolvents of accretive operators in Banach spaces. *J. Fixed Point Theory Appl.* **1**, 135-147 (2007)
11. Wang, F, Cui, H: On the contraction-proximal point algorithms with multi-parameters. *J. Glob. Optim.* **54**, 485-491 (2012)
12. Xu, HK: A regularization method for the proximal point algorithm. *J. Glob. Optim.* **36**, 115-125 (2006)
13. Qin, X, Cho, SY, Wang, L: A regularization method for treating zero points of the sum of two monotone operators. *Fixed Point Theory Appl.* **2014**, 75 (2014)
14. Minty, GJ: On the maximal domain of a monotone function. *Mich. Math. J.* **8**, 135-137 (1961)
15. Nadezhkina, N, Takahashi, W: Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. *J. Optim. Theory Appl.* **128**, 191-201 (2006)
16. Browder, FE: Nonexpansive nonlinear operators in a Banach space. *Proc. Natl. Acad. Sci. USA* **54**, 1041-1044 (1965)
17. Xu, HK: Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66**, 240-256 (2002)
18. He, S, Yang, C: Solving the variational inequality problem defined on intersection of finite level sets. *Abstr. Appl. Anal.* **2013**, Article ID 942315 (2013)
19. Baillon, JB, Haddad, G: Quelques propriétés des opérateurs angle-bornes et cycliquement monotones. *Isr. J. Math.* **26**, 137-150 (1977)
20. Rockafellar, RT: On the maximal monotonicity of subdifferential mappings. *Pac. J. Math.* **33**, 209-216 (1970)
21. Hale, ET, Yin, W, Zhang, Y: A fixed-point continuation method for ℓ_1 -regularized minimization with applications to compressed sensing. *Tech. rep., CAAM TR07-07* (2007)

7.5 Attractive points and convergence theorems for normally generalized hybrid mappings in CAT(0) spaces

In this section, we prove Δ -convergence theorems of Mann type to the set of attractive points of normally generalized hybrid mappings in CAT(0) spaces. Consequently, our main result can be applied to the result of Takahashi, Wong and Yao (Journal of Nonlinear and Convex Analysis 15:1087-1103, 2014, Theorem 5.1).

1 Introduction

In 2011, Takahashi and Takeuchi [1] introduced the notion of attractive points for nonlinear mappings in a Hilbert space: Let H be a Hilbert space and C be a nonempty subset of H . Let T be a mapping from C into H . Let $A(T)$ denote the set of all attractive points of T , i.e.

$$A(T) = \{z \in H : \|z - Ty\| \leq \|z - y\|, \forall y \in C\}.$$

In 2012, Takahashi *et al.* [2] introduced the class of normally generalized hybrid mappings in a Hilbert space which contains the class of generalized hybrid mappings and the class of contractive mappings.

Definition 1.1 A mapping $T : C \rightarrow H$ is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

- (1) $\alpha + \beta + \gamma + \delta \geq 0$;
- (2) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$; and
- (3) $\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0, \forall x, y \in C$.

Such a mapping T can be called an $(\alpha, \beta, \gamma, \delta)$ -normally generalized hybrid mapping.

The authors also proved the weak convergence theorems of Mann type for normally generalized hybrid mappings in Hilbert spaces without convexity assumption on the domain of mappings.

Theorem 1.2 ([2], Theorem 5.1) *Let H be a Hilbert space and C be a convex subset of H . Let $T : C \rightarrow C$ be a normally generalized hybrid mapping with $A(T) \neq \emptyset$. Let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$, $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to v in $A(T)$, where $v = \lim_{n \rightarrow \infty} P_{A(T)}x_n$.*

In 2013, Kakavandi [3] introduced the so-called half-space topology such that convergence in this topology is equivalent to Δ -convergence for any sequence in a complete CAT(0) space. The author's results answered positively open questions in [4]. Moreover, the concepts of (S) property and $(\overline{Q}_4)$ condition are also introduced in a complete CAT(0) space.

In this paper, motivated by [2], we consider the concept of attractive points in CAT(0) spaces and prove Δ -convergence theorems of Mann type for normally generalized hybrid mappings in such spaces satisfying the (S) property and the $(\overline{Q}_4)$ condition.

2 Preliminaries

Let (X, d) be a metric space and C be a nonempty subset of X . A mapping $T : C \rightarrow X$ is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C.$$

In 2008, Kohsaka and Takahashi [5] introduced the class of nonspreading mappings in a Hilbert space H , i.e. a mapping $T : C \rightarrow H$ is called nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In 2010, Takahashi [6] and Kocourek *et al.* [7] introduced the wider class of nonspreading mappings in Hilbert spaces as follows. A mapping $T : C \rightarrow H$ is called hybrid [6] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

A mapping $T : C \rightarrow H$ is called an (α, β) -generalized hybrid [7] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2, \quad \forall x, y \in C.$$

We can see that $(1, 0)$, $(2, 1)$ and $(\frac{3}{2}, \frac{1}{2})$ -generalized hybrid mappings are nonexpansive mappings, nonspreading mappings and hybrid mappings, respectively. Moreover, if $\alpha + \beta = -\gamma - \delta = 1$, then an $(\alpha, \beta, \gamma, \delta)$ -normally generalized hybrid mapping is a generalized hybrid mapping.

In 2003, Takahashi and Toyoda [8] proved the following result.

Lemma 2.1 ([8]) *Let C be a nonempty closed convex subset of a Hilbert space H . Let P be the metric projection from H onto C . Let $\{u_n\}$ be a sequence in H . If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for all $u \in C$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in C$.*

The following result is obtained by Takahashi *et al.* [2].

Lemma 2.2 ([2]) *Let C be a nonempty subset of a real Hilbert space H . Let $T : C \rightarrow C$ be an $(\alpha, \beta, \gamma, \delta)$ -normally generalized hybrid mapping. If $\{x_n\}$ converges weakly to z and $x_n - Tx_n \rightarrow 0$, then $z \in A(T)$.*

It is worth mentioning that the related recent results are obtained in [9, 10] and [11].

We recall the concept of Banach limit which plays a major role in our results. Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of the dual metric space $(l^\infty)^*$ of the space l^∞ . We denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. We sometimes denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. Moreover, a mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$ and we know that there exists a Banach limit on l^∞ (see [12] for more details). If μ is a Banach limit on l^∞ , then for any $(x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

Let (X, d) be a metric space. A geodesic path (or geodesic) joining x to y in X is a map c from a closed interval $[0, l] \subseteq \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. In particular, the mapping c is an isometry and $d(x, y) = l$. The image of c is called a geodesic segment joining x and y when it is unique and denoted by $[x, y]$. We denote the unique point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ and $d(y, z) = (1 - \alpha)d(x, y)$ by $(1 - \alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$.

The metric space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of points (the edges of Δ). A comparison triangle for $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.

A geodesic space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle in $\mathbb{R}^2$. Then the triangle Δ is said to satisfy the CAT(0) inequality if

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}),$$

for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$.

It is well known that any complete simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Hilbert spaces, pre-Hilbert spaces, $\mathbb{R}$ -trees (see [13]), Euclidean buildings (see [14]), the complex Hilbert ball with a hyperbolic metric (see [15]), and many others.

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment y_1, y_2 , then the CAT(0) inequality implies the so-called (CN) inequality of Bruhat and Tits [16], *i.e.*

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

Moreover, a uniquely geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [13] for more details).

The following results are important to our theorems.

Lemma 2.3 ([13]) *Let X be a CAT(0) space, $x_1, x_2, y_1, y_2 \in X$ and $\alpha \in [0, 1]$. Then*

$$d(\alpha x_1 \oplus (1 - \alpha)y_1, \alpha x_2 \oplus (1 - \alpha)y_2) \leq \alpha d(x_1, x_2) + (1 - \alpha)d(y_1, y_2).$$

From the above lemma, it is easy to see that for $x, y, z \in X$ and $\alpha \in [0, 1]$,

$$d(\alpha x \oplus (1 - \alpha)y, z) \leq \alpha d(x, z) + (1 - \alpha)d(y, z).$$

Lemma 2.4 ([17]) *Let X be a CAT(0) space, $x, y, z \in X$ and $\alpha \in [0, 1]$. Then*

$$d^2(\alpha x \oplus (1 - \alpha)y, z) \leq \alpha d^2(x, z) + (1 - \alpha)d^2(y, z) - \alpha(1 - \alpha)d^2(x, y).$$

In 2008, Kirk and Panyanak [4] specialized Lim's concept [18] of Δ -convergence in a general metric spaces to CAT(0) spaces and showed that many results in Banach spaces involving weak convergence have precise analogs in this setting.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d) . For $x \in X$, we set

$$r(x, \{x_n\}) := \limsup_n d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) := \inf_{x \in X} r(x, \{x_n\}).$$

The asymptotic center of $\{x_n\}$ is given by

$$A(\{x_n\}) := \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from [19] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to Δ -converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$. The uniqueness of asymptotic center implies that the CAT(0) space X satisfies Opial's property, i.e. for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ -converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n \rightarrow \infty} d(x_n, x) < \limsup_{n \rightarrow \infty} d(x_n, y).$$

We also know the following results.

Lemma 2.5 ([4]) *Every bounded sequence in a complete CAT(0) space always has a Δ -convergent subsequence.*

Lemma 2.6 ([20]) *If C is a closed convex subset of a complete CAT(0) space (X, d) and $\{x_n\}$ is a bounded sequence in C , then $A(\{x_n\}) \in C$.*

It is well known that a norm linear space satisfies the CAT(0) inequality if and only if it satisfies the parallelogram identity, *i.e.* it is a pre-Hilbert space; hence it is usual to have an inner product-like notion in complete CAT(0) spaces. In 2008, Berg and Nikolaev [21] introduced the concept of quasilinearization along these lines.

Let $\vec{ab}$ denote a pair $(a, b) \in X \times X$, and it is called a vector. Then the quasilinearization mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbb{R}$ is defined by

$$\langle \vec{ab}, \vec{uv} \rangle = \frac{1}{2} (d^2(a, v) + d^2(b, u) - d^2(a, u) - d^2(b, v))$$

for any $a, b, u, v \in X$.

We say that (X, d) satisfies the Cauchy-Schwarz inequality if

$$\langle \vec{ab}, \vec{uv} \rangle \leq d(a, b)d(u, v) \quad \text{for all } a, b, u, v \in X.$$

The following lemma was also proved in [21].

Lemma 2.7 ([21]) *A geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.*

Consider the map $\Theta : \mathbb{R} \times X \times X \rightarrow C(X; \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t \langle \vec{ab}, \vec{ax} \rangle \quad \text{for all } x \in X,$$

where $C(X; \mathbb{R})$ is the space of all continuous real-valued functions on X . Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is the Lipschitz function with the Lipschitz semi-norm $L(\Theta(t, a, b)) = td(a, b)$ ($t \in \mathbb{R}, a, b \in X$), where $L(\varphi) = \sup\{\frac{\varphi(x) - \varphi(y)}{d(x, y)} : x, y \in X, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \rightarrow \mathbb{R}$.

In 2010, Kakavandi and Amini [22] defined a pseudometric D on $\mathbb{R} \times X \times X$ by

$$D((t, a, b), (s, u, v)) = L(\Theta(t, a, b) - \Theta(s, u, v)).$$

Lemma 2.8 ([22]) *Let X be a complete CAT(0) space. Let $(t, a, b), (s, u, v) \in \mathbb{R} \times X \times X$. Then $D((t, a, b), (s, u, v)) = 0$ if and only if $t \langle \vec{ab}, \vec{xy} \rangle = s \langle \vec{uv}, \vec{xy} \rangle$ for all $x, y \in X$.*

Therefore, D defines an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalent class of (t, a, b) is

$$[t\vec{ab}] = \{s\vec{uv} : t \langle \vec{ab}, \vec{xy} \rangle = s \langle \vec{uv}, \vec{xy} \rangle, \forall x, y \in X\}.$$

Let $X^* = \{[t\vec{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$. Then (X^*, D) is a metric space and it is called the dual metric space of (X, d) .

In 2013, Kakavandi [3] introduced the concept of (S) property for a complete CAT(0) space as follows.

Definition 2.9 ([3]) *A complete CAT(0) space (X, d) satisfies the (S) property if for any $(x, y) \in X \times X$ there exists a point $y_x \in X$ such that $[\vec{xy}] = [\vec{y_xx}]$.*

There are many CAT(0) spaces satisfying the (S) property; for example, Hilbert spaces and symmetric Hadamard manifolds including hyperbolic spaces [3]. Moreover, Kakavandi also proved the characterization of Δ -convergence for CAT(0) spaces satisfying the (S) property as follows.

Lemma 2.10 ([3]) *Let (X, d) be a complete CAT(0) space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. If X satisfies the (S) property, then $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ if and only if $\lim_{n \rightarrow \infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0$ for all $y \in X$.*

In 2008, Kirk and Panyanak [4] introduced a geometric condition on CAT(0) spaces called the (Q_4) condition.

(Q_4) A CAT(0) space (X, d) is said to satisfy the (Q_4) condition if $\forall x, y, p, q \in X$

$$d(p, x) < d(x, q) \quad \text{and} \quad d(p, y) < d(y, q) \quad \text{imply} \quad d(p, m) < d(m, q), \quad \forall m \in [x, y].$$

The authors mentioned that this condition holds in many CAT(0) spaces including Hilbert spaces and $\mathbb{R}$ -trees. Furthermore, in 2009, Espínola and Fernández-León [23] proved that any CAT(0) space of constant curvature satisfies the (Q_4) condition but any CAT(0) gluing space containing two spaces of constant but different curvatures fails to satisfy this condition.

In 2013, Kakavandi [3] modified the (Q_4) condition as follows.

$(\overline{Q}_4)$ A CAT(0) space (X, d) is said to satisfy the $(\overline{Q}_4)$ condition if for any $x, y, p, q \in X$,

$$d(p, x) \leq d(x, q) \quad \text{and} \quad d(p, y) \leq d(y, q) \quad \text{imply} \quad d(p, m) \leq d(m, q), \quad \forall m \in [x, y].$$

We can see that Hilbert spaces, $\mathbb{R}$ -trees and every CAT(0) space of constant curvature satisfy the $(\overline{Q}_4)$ condition. Anyway, since $(\overline{Q}_4)$ implies (Q_4) , there are some complete CAT(0) spaces that do not satisfy such a condition.

3 Main results

In this section, we prove Δ -convergence theorems of Mann type [24] to the set of attractive points of normally generalized hybrid mappings in CAT(0) spaces satisfying the (S) property and the $(\overline{Q}_4)$ condition. First of all, we consider the notion of the set of attractive points for any mapping $T : C \rightarrow X$, where X is a metric space and C is a nonempty subset of X defined as

$$A(T) = \{z \in X : d(z, Ty) \leq d(z, y), \forall y \in C\}.$$

Moreover, in metric spaces, a normally generalized hybrid mapping is defined analogously to Definition 1.1 as follows:

A mapping $T : C \rightarrow X$ is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

- (1) $\alpha + \beta + \gamma + \delta \geq 0$;
- (2) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$; and
- (3) $\alpha d(Tx, Ty)^2 + \beta d(x, Ty)^2 + \gamma d(Tx, y)^2 + \delta d(x, y)^2 \leq 0, \forall x, y \in C$.

Before proving our main theorem, we need the following lemmas which we proved in [25], but for the sake of completeness, we give a short proof.

Lemma 3.1 ([25]) *Let (X, d) be a complete CAT(0) space satisfying the $(\overline{Q_4})$ condition. Let C be a nonempty subset of X . Then, for any mapping $T : C \rightarrow X$, $A(T)$ is closed and convex.*

Proof Let $\{x_n\}$ be a sequence in $A(T)$ such that $x_n \rightarrow x \in X$ as $n \rightarrow \infty$.

For any $n \in \mathbb{N}$ and $y \in C$, we have $d(x_n, Ty) \leq d(x_n, y)$. Then

$$\lim_{n \rightarrow \infty} d(x_n, Ty) \leq \lim_{n \rightarrow \infty} d(x_n, y),$$

$$d(x, Ty) \leq d(x, y).$$

This implies that $A(T)$ is a closed subset of X . Let $z_1, z_2 \in A(T)$, then for any $y \in C$, we have $d(z_1, Ty) \leq d(z_1, y)$ and $d(z_2, Ty) \leq d(z_2, y)$. Since X satisfies the $(\overline{Q_4})$ condition, we have $d(m, Ty) \leq d(m, y)$ for all $m \in [z_1, z_2]$. This implies that $A(T)$ is a convex subset of X . Therefore, $A(T)$ is closed and convex. $\square$

Lemma 3.2 ([25]) *Let C be a nonempty subset of a CAT(0) space X . Let $\{x_n\}$ be a bounded sequence in C . Let T be a mapping from C into itself such that $d(x_n, Tx_n) \rightarrow 0$. Then*

- (1) *the sequences $\{d(x_n, y)\}$ and $\{d(Tx_n, y)\}$ are bounded for all $y \in C$;*
- (2) *$\mu_n d(x_n, y) = \mu_n d(Tx_n, y)$ for any Banach limit μ on l^∞ .*

Proof Let $n \in \mathbb{N}$ and $y \in C$. By the triangle inequality, we have

$$d(x_n, y) \leq d(x_n, x_1) + d(x_1, y).$$

Since $\{x_n\}$ is bounded, the sequence $\{d(x_n, y)\}$ is bounded. Since $d(x_n, Tx_n) \rightarrow 0$ and $\{d(x_n, y)\}$ is bounded, the sequence $\{d(Tx_n, y)\}$ is also bounded.

Let $y \in C$. Then

$$d(x_n, y) \leq d(x_n, Tx_n) + d(Tx_n, y) \tag{3.1}$$

and

$$d(Tx_n, y) \leq d(Tx_n, x_n) + d(x_n, y). \tag{3.2}$$

Since $d(x_n, Tx_n) \rightarrow 0$, by applying a Banach limit μ to both sides of (3.1) and (3.2), we can conclude that

$$\mu_n d(x_n, y) = \mu_n d(Tx_n, y). \tag{3.3}$$

$\square$

Next, we prove the analogous result to Lemma 2.2 in complete CAT(0) spaces satisfying the (S) property.

Lemma 3.3 *Let (X, d) be a complete CAT(0) space satisfying the (S) property. Let C be a nonempty subset of X and let $T : C \rightarrow C$ be an $(\alpha, \beta, \gamma, \delta)$ -normally generalized hybrid*

mapping. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \xrightarrow{\Delta} z$ and $d(x_n, Tx_n) \rightarrow 0$. Then $z \in A(T)$.

Proof Since T is an $(\alpha, \beta, \gamma, \delta)$ -normally generalized hybrid mapping,

$$\alpha d^2(Tx, Ty) + \beta d^2(x, Ty) + \gamma d^2(Tx, y) + \delta d^2(x, y) \leq 0, \quad \forall x, y \in C.$$

We have from Lemma 3.2 that $\{d(x_n, y)\}$ and $\{d(Tx_n, y)\}$ are bounded for all $y \in C$.

If $\alpha + \beta > 0$, then

$$\alpha d^2(Tx_n, Ty) + \beta d^2(x_n, Ty) + \gamma d^2(Tx_n, y) + \delta d^2(x_n, y) \leq 0,$$

$$(\alpha + \beta)\mu_n d^2(x_n, Ty) \leq -(\gamma + \delta)\mu_n d^2(x_n, y)$$

for all $y \in C$. Since $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta > 0$,

$$\begin{aligned} \mu_n d^2(x_n, Ty) &\leq \frac{-(\gamma + \delta)}{\alpha + \beta} \mu_n d^2(x_n, y) \\ &\leq \mu_n d^2(x_n, y). \end{aligned}$$

If $\alpha + \gamma > 0$, then

$$\alpha d^2(Ty, Tx_n) + \beta d^2(y, Tx_n) + \gamma d^2(Ty, x_n) + \delta d^2(y, x_n) \leq 0,$$

$$(\alpha + \gamma)\mu_n d^2(Ty, x_n) \leq -(\beta + \delta)\mu_n d^2(y, x_n)$$

for all $y \in C$. It follows from $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma > 0$ that

$$\begin{aligned} \mu_n d^2(Ty, x_n) &\leq \frac{-(\beta + \delta)}{\alpha + \gamma} \mu_n d^2(y, x_n) \\ &\leq \mu_n d^2(y, x_n). \end{aligned}$$

Therefore,

$$\mu_n d^2(x_n, Ty) \leq \mu_n d^2(x_n, y) \tag{3.3}$$

for all $y \in C$. Furthermore, Lemma 2.10 and $x_n \xrightarrow{\Delta} z$ imply that

$$\lim_{n \rightarrow \infty} (d^2(x_n, z) - d^2(x_n, y) + d^2(z, y)) = 0$$

for all $y \in X$. Thus,

$$\mu_n (d^2(x_n, z) - d^2(x_n, y) + d^2(z, y)) = 0 \tag{3.4}$$

for all $y \in X$. From (3.3), we have

$$-\mu_n d^2(x_n, y) \leq -\mu_n d^2(x_n, Ty).$$

By adding $\mu_n(d^2(x_n, z) + d^2(z, y) + d^2(z, Ty))$ to the both sides of the above inequality, we assert that

$$\begin{aligned} & -\mu_n d^2(x_n, y) + \mu_n(d^2(x_n, z) + d^2(z, y) + d^2(z, Ty)) \\ & \leq -\mu_n d^2(x_n, Ty) + \mu_n(d^2(x_n, z) + d^2(z, y) + d^2(z, Ty)), \\ & d^2(z, Ty) + \mu_n(d^2(x_n, z) - d^2(x_n, y) + d^2(z, y)) \\ & \leq d^2(z, y) + \mu_n(d^2(x_n, z) - d^2(x_n, Ty) + d^2(z, Ty)). \end{aligned}$$

From (3.4), we have $d(z, Ty) \leq d(z, y)$ and hence $z \in A(T)$. $\square$

We prove the analogous result to Lemma 2.1 in complete CAT(0) spaces as follows.

Lemma 3.4 *Let (X, d) be a complete CAT(0) space and C be a closed convex subset of X . Let $\{x_n\}$ be a bounded sequence in X . If $d(x_{n+1}, z) \leq d(x_n, z)$ for all $z \in C$, then $P_C x_n \rightarrow z_0 \in C$, where P_C is the metric projection from X onto C .*

Proof Since $d(x_{n+1}, z) \leq d(x_n, z)$ for all $z \in C$,

$$\begin{aligned} d(x_{n+1}, P_C x_{n+1}) & \leq d(x_{n+1}, P_C x_n) \\ & \leq d(x_n, P_C x_n). \end{aligned}$$

Then $\{d(x_n, P_C x_n)\}$ is decreasing and bounded, which implies that $\lim_{n \rightarrow \infty} d(x_n, P_C x_n)$ exists.

In order to show that $\{P_C x_n\}$ is a Cauchy sequence in C , let $m, n \in \mathbb{N}$ be such that $m < n$.

Put $y = \frac{1}{2}P_C x_m \oplus \frac{1}{2}P_C x_n$. By the (CN) inequality, we have

$$d^2(x_n, y) \leq \frac{1}{2}d^2(x_n, P_C x_m) + \frac{1}{2}d^2(x_n, P_C x_n) - \frac{1}{4}d^2(P_C x_m, P_C x_n).$$

Then

$$d^2(P_C x_m, P_C x_n) \leq 2d^2(x_n, P_C x_m) + 2d^2(x_n, P_C x_n) - 4d^2(x_n, y).$$

Since $-d(x_n, y) \leq -d(x_n, P_C x_n)$,

$$\begin{aligned} d^2(P_C x_m, P_C x_n) & \leq 2d^2(x_n, P_C x_m) + 2d^2(x_n, P_C x_n) - 4d^2(x_n, P_C x_n) \\ & = 2d^2(x_n, P_C x_m) - 2d^2(x_n, P_C x_n). \end{aligned}$$

By the assumption, we can conclude that

$$d^2(P_C x_m, P_C x_n) \leq 2d^2(x_m, P_C x_m) - 2d^2(x_n, P_C x_n).$$

Since $\lim_{n \rightarrow \infty} d(x_n, P_C x_n)$ exists, $\{P_C x_n\}$ is a Cauchy sequence.

Therefore, $P_C x_n \rightarrow z_0$ for some $z_0 \in C$. $\square$

In 2012, Dehghan and Rooin [26] proved the following lemma. Now, we give another proof.

Lemma 3.5 ([26]) *Let (X, d) be a complete CAT(0) space and C be a closed convex subset of X . Let $x \in X$ and $y \in C$. Then $y = P_C x$ if and only if $\langle \vec{xy}, \vec{yz} \rangle \geq 0$ for all $z \in C$.*

Proof Let $z \in C$. Suppose that $y = P_C x$. Let $0 < \lambda < 1$ and $m = \lambda z \oplus (1 - \lambda)y$.

By Lemma 2.4, we have

$$d^2(x, m) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(x, y) - \lambda(1 - \lambda)d^2(z, y).$$

Then

$$\frac{d^2(x, m)}{\lambda} - \frac{1 - \lambda}{\lambda} d^2(x, y) + (1 - \lambda)d^2(z, y) \leq d^2(x, z). \quad (3.5)$$

From (3.5) and $2\langle \vec{xy}, \vec{yz} \rangle = d^2(x, z) - d^2(x, y) - d^2(z, y)$, we have

$$\begin{aligned} 2\langle \vec{xy}, \vec{yz} \rangle &\geq \frac{d^2(x, m)}{\lambda} - \frac{1 - \lambda}{\lambda} d^2(x, y) + (1 - \lambda)d^2(z, y) - d^2(x, y) - d^2(z, y) \\ &= \frac{d^2(x, m)}{\lambda} - \frac{d^2(x, y)}{\lambda} - \lambda d^2(z, y). \end{aligned}$$

Since $y = P_C x$, $d(x, y) \leq d(x, m)$. Therefore,

$$\begin{aligned} 2\langle \vec{xy}, \vec{yz} \rangle &\geq \frac{d^2(x, y)}{\lambda} - \frac{d^2(x, y)}{\lambda} - \lambda d^2(z, y) \\ &= -\lambda d^2(z, y). \end{aligned}$$

By letting $\lambda \rightarrow 0$, we assert that $\langle \vec{xy}, \vec{yz} \rangle \geq 0$ for all $z \in C$.

Conversely, suppose that $\langle \vec{xy}, \vec{yz} \rangle \geq 0$ for all $z \in C$.

For any $z \in C$, we have

$$\begin{aligned} 0 &\leq 2\langle \vec{xy}, \vec{yz} \rangle \\ &= d^2(x, z) - d^2(x, y) - d^2(y, z) \\ &\leq d^2(x, z) - d^2(x, y). \end{aligned}$$

Then $d(x, y) \leq d(x, z)$ and hence $y = P_C x$. □

We are now in a position to prove our main theorem.

Theorem 3.6 *Let (X, d) be a complete CAT(0) space satisfying the $(\mathbb{S})$ property and the $(\overline{Q}_4)$ condition. Let C be a convex subset of X and $T : C \rightarrow C$ be a normally generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T x_n, \quad n \in \mathbb{N}.$$

Then $x_n \xrightarrow{\Delta} v \in A(T)$, where $v = \lim_{n \rightarrow \infty} P_{A(T)} x_n$ and $P_{A(T)}$ is the metric projection from X onto $A(T)$.

Proof Claim 1. We show that $\lim_{n \rightarrow \infty} d(u, x_n)$ exists for all $u \in A(T)$.

By Lemma 2.3, we have

$$\begin{aligned} d(u, x_{n+1}) &= d(u, \alpha_n x_n \oplus (1 - \alpha_n)Tx_n) \\ &\leq \alpha_n d(u, x_n) + (1 - \alpha_n)d(u, Tx_n). \end{aligned}$$

Since $u \in A(T)$,

$$\begin{aligned} d(u, x_{n+1}) &\leq \alpha_n d(u, x_n) + (1 - \alpha_n)d(u, x_n) \\ &= d(u, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Then $\{d(u, x_n)\}$ is a decreasing and bounded sequence.

Therefore, $\lim_{n \rightarrow \infty} d(u, x_n)$ exists and $\{x_n\}$ is bounded.

Claim 2. We show that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. Choose $z \in A(T)$.

By Lemma 2.4, we have

$$\begin{aligned} d^2(z, x_{n+1}) &= d^2(z, \alpha_n x_n \oplus (1 - \alpha_n)Tx_n) \\ &\leq \alpha_n d^2(z, x_n) + (1 - \alpha_n)d^2(z, Tx_n) - \alpha_n(1 - \alpha_n)d^2(x_n, Tx_n). \end{aligned}$$

Then

$$\alpha_n(1 - \alpha_n)d^2(x_n, Tx_n) \leq \alpha_n d^2(z, x_n) + (1 - \alpha_n)d^2(z, Tx_n) - d^2(z, x_{n+1}).$$

Since $z \in A(T)$,

$$\begin{aligned} \alpha_n(1 - \alpha_n)d^2(x_n, Tx_n) &\leq \alpha_n d^2(z, x_n) + (1 - \alpha_n)d^2(z, x_n) - d^2(z, x_{n+1}) \\ &= d^2(z, x_n) - d^2(z, x_{n+1}). \end{aligned}$$

By the existence of $\lim_{n \rightarrow \infty} d(z, x_n)$, it follows that

$$\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n)d^2(x_n, Tx_n) \leq \liminf_{n \rightarrow \infty} (d^2(z, x_n) - d^2(z, x_{n+1})) = 0.$$

Since $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$, we have $d(x_n, Tx_n) \rightarrow 0$.

Claim 3. The sequence $\{x_n\}$ Δ -converges to an element in $A(T)$. Indeed, since $\{x_n\}$ is bounded, we can assume that $A(\{x_n\}) = \{v\}$ for some $v \in X$. It is sufficient to show that $A(\{x_{n_k}\}) = \{v\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\{x_{n_{k'}}\}$ be a subsequence of $\{x_{n_k}\}$ with $A(\{x_{n_{k'}}\}) = \{w\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k''}}\}$ of $\{x_{n_{k'}}\}$ such that $x_{n_{k''}} \xrightarrow{\Delta} z$ for some $z \in X$.

By Lemma 3.3 and Claim 2, we have $z \in A(T)$ and hence $\lim_{n \rightarrow \infty} d(z, x_n)$ exists. If $z \neq w$, then it follows from the uniqueness of asymptotic center that

$$\begin{aligned} \lim_{n \rightarrow \infty} d(z, x_n) &= \limsup_{k' \rightarrow \infty} d(z, x_{n_{k'}}) \\ &< \limsup_{k' \rightarrow \infty} d(w, x_{n_{k'}}) \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{k \rightarrow \infty} d(w, x_{n_k}) \\
&< \limsup_{k \rightarrow \infty} d(z, x_{n_k}) \\
&= \lim_{n \rightarrow \infty} d(z, x_n),
\end{aligned}$$

which is a contradiction. Therefore, $w = z \in A(T)$.

Suppose that $v \neq w$. Then

$$\begin{aligned}
\lim_{n \rightarrow \infty} d(w, x_n) &= \limsup_{k \rightarrow \infty} d(w, x_{n_k}) \\
&< \limsup_{k \rightarrow \infty} d(v, x_{n_k}) \\
&\leq \limsup_{n \rightarrow \infty} d(v, x_n) \\
&< \limsup_{n \rightarrow \infty} d(w, x_n) \\
&= \lim_{n \rightarrow \infty} d(w, x_n).
\end{aligned}$$

This leads to a contradiction and hence $v = w \in A(T)$.

Therefore, $x_n \xrightarrow{\Delta} v \in A(T)$.

Claim 4. We show that $v = \lim_{n \rightarrow \infty} P_{A(T)} x_n$.

We can conclude from Claim 1 that $d(x_{n+1}, z) \leq d(x_n, z)$ for all $z \in A(T)$ and $n \in \mathbb{N}$. Furthermore, we obtain from Lemma 3.4 that $P_{A(T)} x_n \rightarrow p$ for some $p \in A(T)$. It is sufficient to show that $\langle \vec{v}p, \vec{p}z \rangle \geq 0$ for all $z \in A(T)$.

First, we show that $\lim_{n \rightarrow \infty} d(x_n, p) = \lim_{n \rightarrow \infty} d(x_n, P_{A(T)} x_n)$.

From Claim 1, we have that $\lim_{n \rightarrow \infty} d(p, x_n)$ exists.

Since

$$\begin{aligned}
d(x_{n+1}, P_{A(T)} x_{n+1}) &\leq d(x_{n+1}, P_{A(T)} x_n) \\
&\leq d(x_n, P_{A(T)} x_n)
\end{aligned}$$

for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} d(x_n, P_{A(T)} x_n)$ exists. By the triangle inequality, we have

$$\begin{aligned}
d(x_n, p) &\leq d(x_n, P_{A(T)} x_n) + d(P_{A(T)} x_n, p) \\
&\leq d(x_n, p) + d(P_{A(T)} x_n, p).
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} d(P_{A(T)} x_n, p) = 0$, we have

$$\lim_{n \rightarrow \infty} d(x_n, p) \leq \lim_{n \rightarrow \infty} d(x_n, P_{A(T)} x_n) \leq \lim_{n \rightarrow \infty} d(x_n, p).$$

This implies that

$$\lim_{n \rightarrow \infty} d(x_n, p) = \lim_{n \rightarrow \infty} d(x_n, P_{A(T)} x_n).$$

Moreover, we can apply Lemma 2.10 to conclude that

$$\lim_{n \rightarrow \infty} (d^2(x_n, v) - d^2(x_n, z) + d^2(v, z)) = 0$$

for all $z \in X$. Consider

$$\begin{aligned}
 0 &= \lim_{n \rightarrow \infty} (d^2(x_n, v) - d^2(x_n, p) + d^2(v, p)), \\
 -d^2(v, p) &= \lim_{n \rightarrow \infty} (d^2(x_n, v) - d^2(x_n, p)), \\
 -2d^2(v, p) &= \lim_{n \rightarrow \infty} (d^2(x_n, v) - d^2(x_n, p) - d^2(v, p)) \\
 &= \lim_{n \rightarrow \infty} (d^2(x_n, v) - d^2(x_n, P_{A(T)}x_n) - d^2(v, P_{A(T)}x_n)) \\
 &= \lim_{n \rightarrow \infty} \overrightarrow{\langle x_n P_{A(T)}x_n, P_{A(T)}x_n v \rangle}.
 \end{aligned}$$

By Lemma 3.5, we have

$$\overrightarrow{\langle x_n P_{A(T)}x_n, P_{A(T)}x_n v \rangle} \geq 0,$$

which implies that $d^2(v, p) \leq 0$ and hence $v = p$.

Therefore, $x_n \xrightarrow{\Delta} v$, where $v = \lim_{n \rightarrow \infty} P_{A(T)}x_n$. □

It is known that a Hilbert space satisfies both the (S) property and the $(\overline{Q_4})$ condition. Furthermore, Δ -convergence and weak convergence are the same in a Hilbert space. Consequently, we can conclude that the result of Takahashi, Wong and Yao [2], Theorem 5.1, is an application of Theorem 3.6.

Moreover, the following example shows that there is a CAT(0) space satisfying both the (S) property and the $(\overline{Q_4})$ condition but it is not a Hilbert space. Thus, our main theorem is a strict generalization of the result of Takahashi, Wong and Yao [2], Theorem 5.1.

Example 3.7 Consider the subset $\mathcal{H}$ of $\mathbb{R}^2$

$$\mathcal{H} = \{(x, y) \in \mathbb{R}^2 | y^2 - x^2 = 1 \text{ and } y > 0\}.$$

In other words, $\mathcal{H}$ is the upper curve of the hyperbolic $\{(x, y) \in \mathbb{R}^2 | y^2 - x^2 = 1\}$. Let d be a metric defined by the function $d : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ that assigns to each pair of vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ the unique nonnegative number $d(u, v) \geq 0$ such that

$$\cosh d(u, v) = u_2 v_2 - u_1 v_1.$$

It is known that, in general, the metric space $(\mathcal{H}, d)$ is a CAT(0) space and also a one-dimensional hyperbolic space viewed from a hyperboloid model (for more details, see [13]). Then $(\mathcal{H}, d)$ satisfies the (S) property. Furthermore, it is easy to see that $(\mathcal{H}, d)$ is an $\mathbb{R}$ -tree. Hence $(\mathcal{H}, d)$ satisfies the $(\overline{Q_4})$ condition.

References

1. Takahashi, W, Takeuchi, Y: Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space. *J. Nonlinear Convex Anal.* **12**, 399-406 (2011)
2. Takahashi, W, Wong, N-C, Yao, J-C: Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces. *J. Nonlinear Convex Anal.* **13**, 745-757 (2012)
3. Kakavandi, BA: Weak topologies in complete CAT(0) metric spaces. *Proc. Am. Math. Soc.* **141**, 1029-1039 (2013)
4. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. *Nonlinear Anal.* **68**, 3689-3696 (2008)
5. Kohsaka, F, Takahashi, W: Fixed point theorem for a class of nonlinear mappings related to maximal monotone operators in Banach spaces. *Arch. Math.* **91**, 166-177 (2008)
6. Takahashi, W: Fixed point theorems for new nonexpansive mappings in a Hilbert space. *J. Nonlinear Convex Anal.* **11**, 79-88 (2010)
7. Kocourek, P, Takahashi, W, Yao, J-C: Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces. *Taiwan. J. Math.* **14**, 2497-2511 (2010)
8. Takahashi, W, Toyoda, M: Weak convergence theorems for nonexpansive mappings and monotone mappings. *J. Optim. Theory Appl.* **118**, 417-428 (2003)
9. Karapinar, E, Vaezpour, SM, Salahifard, H: Demiclosedness principle for total asymptotically nonexpansive mappings in CAT(0) spaces. *J. Appl. Math.* **2013**, Article ID 738150 (2013)
10. Naraghirad, E, Wong, NC, Yao, JC: Approximating fixed points of α -nonexpansive mappings in uniformly convex Banach spaces and CAT(0) spaces. *Fixed Point Theory Appl.* **2013**, 57 (2013)
11. Takahashi, W, Wong, NC, Yao, JC: Attractive point and mean convergence theorems for semigroups of mappings without continuity in Hilbert spaces. *J. Nonlinear Convex Anal.* **15**, 1087-1103 (2014)
12. Takahashi, W: *Nonlinear Functional Analysis. Fixed Point Theory and Its Applications.* Yokohama Publishers, Yokohama (2000)
13. Bridson, M, Haefliger, A: *Metric Spaces of Nonpositive Curvature.* Springer, Berlin (1999)
14. Brown, KS: *Buildings.* Springer, New York (1989)
15. Goebel, K, Reich, S: *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings.* Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 83. Dekker, New York (1983)
16. Bruhat, F, Tits, J: Groupes réductifs sur un corps local: I. Données radicielles valuées. *Publ. Math. Inst. Hautes Études Sci.* **41**, 5-251 (1972)
17. Dhompongsa, S, Panyanak, B: On Δ -convergence theorems in CAT(0) spaces. *Comput. Math. Appl.* **56**, 2572-2579 (2008)
18. Lim, TC: Remark on some fixed point theorems. *Proc. Am. Math. Soc.* **60**, 179-182 (1976)
19. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. *Nonlinear Anal.* **65**, 762-772 (2006)
20. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. *J. Nonlinear Convex Anal.* **8**, 35-45 (2007)
21. Berg, ID, Nikolaev, IG: Quasilinearization and curvature of Alexandrov spaces. *Geom. Dedic.* **133**, 195-218 (2008)
22. Kakavandi, BA, Amini, M: Duality and subdifferential for convex functions on complete CAT(0) metric spaces. *Nonlinear Anal.* **73**, 3450-3455 (2010)
23. Espínola, R, Fernández-León, A: CAT(k)-spaces, weak convergence and fixed points. *J. Math. Anal. Appl.* **353**, 410-427 (2009)
24. Mann, WR: Mean value methods in iteration. *Proc. Am. Math. Soc.* **4**, 506-510 (1953)
25. Kunwai, K, Kaewkhao, A, Inthakon, W: Properties of attractive points in CAT(0) spaces. *Thai J. Math.* **13**, 109-121 (2015)
26. Dehghan, H, Roojin, J: A characterization of metric projection in CAT(0) spaces. In: *International Conference on Functional Equation, Geometric Functions and Applications (ICFGA 2012) 11-12th May 2012*, pp. 41-43. Payame University, Tabriz (2012)

Output ที่ได้จากโครงการ

(ก) ผลงานวิจัยจำนวน 46 เรื่อง จำแนกดังต่อไปนี้

(ก1) ผลงานที่ตีพิมพ์แล้วในวารสารวิชาการระดับนานาชาติ (published papers) จำนวน 36 เรื่อง ดังต่อไปนี้

- (1) Jukrapong Tiammee, Attapol Kaewkhao and Suthep Suantai, On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, Tiammee et al. *Fixed Point Theory and Applications* (2015) 2015:187 DOI 10.1186/s13663-015-0436-9.
- (2) B. Panyanak and S. Suantai, Viscosity Approximation Methods for Multivalued Nonexpansive Mappings in Geodesic Spaces, *Fixed Point Theory and Applications* (2015) 2015 :114 DOI 10.1186/s 13663-015-0356-8
- (3) Attapol Kaewkhao, Bancha Panyanak and Suthep Suantai, Viscosity iteration method in CAT(0) spaces without the nice projection property, *Journal of Inequalities and Applications*, (2015) DOI 10.1186/s13660-015-0801-6
- (4) Weera Yukunthorn, Suthep Suantai, Sotiris K Ntouyas and Jessada Tariboon, Boundary value problems for impulsive multi-order Hadamard fractional differential equations, *Boundary Value Problems* (2015) 2015:148, DOI 10.1186/s13661-015-0414-5.
- (5) Suthep Suantai, Sotiris K Ntouyas, Suphawat Asawasamrit and Jessada Tariboon, A coupled system of fractional q -integro-difference equations with nonlocal fractional q -integral boundary conditions, *Advances in Difference Equations* (2015) 2015:124 DOI 10.1186/s13662-015-0462-2.
- (6) Suthep Suantai and Withun Phuengrattana, A new iterative process for a hybrid pair of generalized asymptotically nonexpansive single-valued and generalized nonexpansive multi-valued mappings in Banach spaces, *Fixed Point Theory and Applications* (2015) 2015:58 DOI 10.1186/s13663-015-0304-7.

- (7) A Kaewkhao, W Inthakon and K Kunwai, Attractive points and convergence theorems for normally generalized hybrid mappings in CAT(0) spaces, *Fixed Point Theory and Applications* (2015) 2015:96, DOI 10.1186/s13663-015-0336-.
- (8) Chalongchai Klanarong and Suthep Suantai, Coincidence point theorems for some multi-valued mappings in complete metric spaces endowed with a graph, *Fixed Point Theory and Applications* (2015) 2015:129, DOI 10.1186/s13663-015-0379-1.
- (9) Suthep Suantai, Phakdi Charoensawan and Tatjana Aleksic Lampert, Common coupled fixed point theorems for $\theta\psi$ -contraction mappings endowed with a directed graph, *Fixed Point Theory and Applications* (2015) 2015:224, DOI 10.1186/s13663-015-0473-4.
- (10) Suthep Suantai and Withun Phuengrattana, Existence and convergence theorems for lambda-hybrid set-valued mappings in Hilbert spaces, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* 22 (2015) 177-188.
- (11) Suthep Suantai, Narin Petrot and Warut Saksirikun, Fuzzy fixed point theorems on the complete fuzzy spaces under supremum metric, *Fixed Point Theory and Applications* (2015) 2015:167, DOI 10.1186/s13663-015-0418-y.
- (12) Chakkrid Klin-eam, Cholatis Suanoom and Suthep Suantai, Generalized multi-valued mappings satisfy some inequalities conditions on metric spaces, *Journal of Inequalities and Applications* (2015) 2015:343, DOI 10.1186/s13660-015-0864-4.
- (13) Prasit Chalamjiak, Watcharaporn Chalamjiak and Suthep Suantai, A modified regularization method for finding zeros of monotone operators in Hilbert spaces, *Journal of Inequalities and Applications* (2015) 2015:220, DOI 10.1186/s13660-015- 0739-8.
- (14) Kanyarat Cheawchan, Suthep Suantai and Atid Kangtunyakarn, A new technique for convergence theorem of fixed point problem of quasi-nonexpansive mapping, *Fixed Point Theory and Applications* (2015) 2015:216, DOI 10.1186/s13663-015-0453-8.

- (15) Suthep Suantai, Prasit Chalamjiak, Yeol Je Cho and Watcharaporn Chalamjiak, On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces, *Fixed Point Theory and Applications* (2016) 2016:35, DOI 10.1186/s13663-016-0509-4.
- (16) Chakkrid Klin-eam, Prondanai Kaskasem, Suthep Suantai, Hybrid method for the equilibrium problem and a family of generalized nonexpansive mappings in Banach spaces, *J. Nonlinear Sci. Appl.* 9 (2016), 4963 - 4975.
- (17) Cholatis Suanoom, Chakkrid Klin-eam, Suthep Suantai, Dislocated Quasi-b-Metric Spaces and Fixed Point Theorems for Cyclic Weakly Contractions, *J. Nonlinear Sci. Appl.* (2016), 152 – 161.
- (18) Phikul Sridarat and Suthep Suantai, Caristi Fixed Point Theorem in Metric Spaces with a Graph and Its Applications, *Journal of Nonlinear Analysis and Convex Analysis*, Volume 17, Number 7, 2016, 1417 – 1428.
- (19) Jukrapong Tiammee, Yeol Je Cho, Suthep Suantai, Fixed point theorems for nonself G-almost contractive mappings in Banach spaces endowed with graphs, *Carpathian Journal of Mathematics*, 32 (2016), No. 3, 375 – 382.
- (20) Pornpimon, Narin Petrot and Suthep Suantai, Fixed point theorems for Prešić almost contraction mappings in orbitally complete metric spaces endowed with directed graphs, *CARPATHIAN JOURNAL OF MATHEMATICS*, 32 (2016), No. 3, 295 – 305.
- (21) Suthep Suantai and Withun Phuengrattana, A hybrid shrinking projection method for common fixed points of a finite family of demicontractive mappings with variational inequality problems, *Banach J. Math. Anal.* 11 (2017), no. 3, 661- 675 .
- (22) Raweerote Suparatulatorn, Watcharaporn Chalamjiak and Suthep Suantai, A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs, *Numer Algor* (2017) DOI 10.1007/s11075-017-0324-y .
- (23) Phannipa Kabcome and Thanasak Mouktonglang, An Interior-Point Trust-Region Algorithm for Quadratic Stochastic Symmetric Programming, *Thai Journal of Mathematics*, Volume 15 (2017) Number 1 : 237–260.

- (24) Raweerote Suparatulatorn, Suthep Suantai, Watcharaporn Chalamjiak, Hybrid methods for a finite family of G -nonexpansive mappings in Hilbert spaces endowed with graphs, *AKCE International Journal of Graphs and Combinatorics* 14 (2017) 101–111.
- (25) Kritsada Lerkchaiyaphum, Withun Phuengrattana, Iterative approaches to solving convex minimization problems and fixed point problems in complete $CAT(0)$ spaces, *Numer Algor*, DOI 10.1007/s11075-017-0337-6.
- (26) Nawitcha Onjai-uea and Withun Phuengrattana, On solving split mixed equilibrium problems and fixed point problems of hybrid-type multivalued mappings in Hilbert spaces, *Journal of Inequalities and Applications* (2017) 2017:137 DOI 10.1186/s13660-017-1416-x.
- (27) Suthep Suantai and Withun Phuengrattana, Proximal point algorithms for a hybrid pair of nonexpansive single-valued and multi-valued mappings in geodesic metric spaces, *Mediterr. J. Math.* (2017) 14:62 DOI 10.1007/s00009-017-0876-z 1660-5446/17/020001-17.
- (28) Pachara Jailoka & Suthep Suantai, Split common fixed point and null point problems for demicontractive operators in Hilbert spaces, *Optimization Methods & Software*, 2017 <https://doi.org/10.1080/10556788.2017.1359265>.
- (29) J. Tiammee, P. Charoensawan and S. Suantai, Fixed point theorems for multivalued nonself G -Almost contractions in Banach spaces endowed with graphs, *Journal of Function Spaces*, Volume 2017, Article ID 7053849, 5 pages <https://doi.org/10.1155/2017/7053849>.
- (30) A. Suebsriwichai and T. Mouktonglang, Bound for the 2-page fixed linear crossing number of hypercube graph via SDP relaxation, *Journal of Applied Mathematics* Volume 2017, Article ID 7640347, 7 pages <https://doi.org/10.1155/2017/7640347>.
- (31) Asawathep Cuntavepanit, Withun Phuengrattana, On solving the minimization problem and the fixed-point problem for a finite family of nonexpansive mappings in $CAT(0)$ spaces, *Optimization Methods & Software*, 2017 <http://dx.doi.org/10.1080/10556788.2017.1312397>.

- (32) Montira Suwannaprapa, Narin Petrot and Suthep Suantai, Weak convergence theorems for split feasibility problems on zeros of the sum of monotone operators and fixed point sets in Hilbert spaces, *Fixed Point Theory and Applications* (2017) 2017:6 DOI 10.1186/s13663-017-0599-7.
- (33) Raweerote Suparatulatorn, Prasit Cholamjiak and Suthep Suantai, On solving the minimization problem and the fixed-point problem for nonexpansive mappings in CAT(0) spaces, *Optimization Methods & Software*, 2017, Vol. 32, No. 1, 182–192, <http://dx.doi.org/10.1080/10556788.2016.1219908>.
- (34) Chalongchai Klanarong and Suthep Suantai, Best Proximity Point Theorems for G-Proximal Generalized Contraction in Complete Metric Spaces endowed with Graphs, *Thai Journal of Mathematics* Volume 15 (2017) Number 1 : 261–276.
- (35) Supaporn Nirunruttanakit and Suthep Suantai, Some Iterative Methods for Coincidence Points of Two Continuous Functions on Closed Interval, *Thai Journal of Mathematics* Volume 15 (2017) Number 2 : 551 - 563.
- (36) Suthep Suantai , S.K. Ntouyas and Jessada Tariboon, Existence results for a coupled system of fractional differential equations with multiple orders of fractional derivatives and integrals, *J. COMPUTATIONAL ANALYSIS AND APPLICATIONS*, VOL. 24, NO.7, 2018.

(ก2) ผลงานที่ได้รับการตอบรับ (accepted) ให้ตีพิมพ์แล้วในวารสารวิชาการระดับนานาชาติ จำนวน 5 เรื่อง ดังต่อไปนี้

- (1) Sotiris K. Ntouyas, Suthep Suantai and Jessada Tariboon, Anti-periodic boundary value problems for generalized Sturm-Liouville and Langevin fractional differential inclusions via Hadamard derivative, (accepted) *International Journal of Applied Mathematics and Statistics*.
- (2) Jukrapong Tiammee , Suthep Suantai and Yeol Je Cho, Existence theorems of a new set-valued MT-contraction in b- metric spaces endowed with graphs and applications, (accepted) *Fixed Point Theory*, (2017) http://www.math.ubbcluj.ro/_nodeacj/sfptcj.html.
- (3) Jukrapong Tiammee, Suthep Suantai, Endpoints of Multi-valued Weakly Contraction in Complete Metric Spaces Endowed with Graphs, (accepted) *Filomat*.

- (4) Suthep Suantai, Bancha Panyanak, and Withun Phuengrattana, A new one-step iterative process for approximating common fixed points of a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces, (accepted) Bulletin of the Iranian Mathematical Society, In press.
- (5) Suthep Suantai, Kittipong Srisap , Natthapong Naprang, Manatsawin Mamat , Vithoon Yundon and Prasit Chalamjiak, Convergence theorems for finding the split common null point in Banach spaces, (accepted) Applied General Topology.

**(ก3) ผลงานที่ส่งไปตีพิมพ์ (submitted) ในวารสารวิชาการระดับนานาชาติ
จำนวน 5 เรื่อง ดังต่อไปนี้**

- (1) Phikul Sridarat¹, Raweerote Suparaturatorn¹, Suthep Suantai¹ and Yeol Je Cho, Convergence Analysis of SP-iteration for G -nonexpansive mappings with directed graphs, (submitted) Bulletin Malaysian Journal of Mathematics.
- (2) Adisak Hanjing and Suthep Suantai, The Split Equality Common Fixed Point Problem for Infinite Families of Multi-valued Demicontractive Mappings, (submitted) Journal of Optimization.
- (3) Pachara Jailoka and Suthep Suantai, Split common null point problems and fixed point problems for demicontractive multivalued mappings, (submitted) Applicable Analysis.
- (4) Pachara Jailoka and Suthep Suantai, The split common fixed point problem for multivalued demicontractive mappings and its Applications, (submitted) Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A. Matemáticas,
- (5) Raweerote Suparatulatorn, Suthep Suantai, Global minimization of common best proximity points for generalized cyclic φ -contractions in metric spaces, (submitted) Journal of Fixed Point Theory and Applications.

ภาคผนวก

(ต่อที่เล่มที่ 2)

(ก) ผลงานวิจัยที่เป็น Research Articles ทั้งหมด 46 เรื่อง

ประกอบด้วย

- (1) ผลงานที่ตีพิมพ์แล้ว จำนวน 36 เรื่อง
- (2) ผลงานที่ได้รับการตอบรับให้ตีพิมพ์ (accepted) จำนวน 5 เรื่อง
- (3) ผลงานที่อยู่ระหว่างการพิจารณา (submitted) จำนวน 5 เรื่อง

(ข) ผลงานที่เป็นการสร้างนักวิจัยรุ่นใหม่ และ รุ่นกลาง จำนวน 32 คน

- (ข1) นักวิจัยระดับบัณฑิตศึกษา จำนวน 22 คน แบ่งเป็น
 - (1) นักศึกษาระดับปริญญาเอก จำนวน 16 คน
 - (2) นักศึกษาระดับปริญญาโท จำนวน 6 คน
- (ข2) นักวิจัยอาจารย์รุ่นใหม่ จำนวน 5 คน
- (ข3) นักวิจัยรุ่นกลาง จำนวน 5 คน

(ค) การสร้างเครือข่ายความร่วมมือกับต่างประเทศ

- (1) ความร่วมมือกับ North University of Baia Mare ประเทศโรมาเนีย
- (2) ความร่วมมือกับ Gyeongsang National University, ประเทศเกาหลีใต้
- (3) ความร่วมมือกับนักวิจัยชาวต่างประเทศ

(ง) การบริการวิชาการระดับนานาชาติ

ทางกลุ่มวิจัยได้จัดทำโครงการ SEAMS School 2017 ภายใต้หัวข้อ “Fixed Point Theory and Optimization” สำหรับให้พื้นฐานและทิศทางการวิจัยแก่อาจารย์และนักศึกษาในภูมิภาคเอเชียตะวันออกเฉียงใต้ ระหว่าง 24 – 31 กรกฎาคม 2560

(ข) ผลงานที่เป็นการสร้างนักวิจัยรุ่นใหม่ และ รุ่นกลาง

(ข1) นักวิจัยระดับบัณฑิตศึกษา จำนวน 22 คน แบ่งเป็น

(1) นักศึกษาระดับปริญญาเอก จำนวน 16 คน

(2) นักศึกษาระดับปริญญาโท จำนวน 6 คน

(ข2) นักวิจัยอาจารย์รุ่นใหม่ จำนวน 5 คน

(ข3) นักวิจัยรุ่นกลาง จำนวน 5 คน

นักวิจัยรุ่นใหม่ ระดับปริญญาเอก 16 คน

ลำดับ	รายนามนักศึกษา	อาจารย์ที่ปรึกษา วิทยานิพนธ์	มหาวิทยาลัย	หมายเหตุ
1	นายจักรพงศ์ เตียมมี	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	สำเร็จการศึกษาแล้ว
2	นาย ฉลองชัย กล้าณรงค์	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	สำเร็จการศึกษาแล้ว
3	นาย คณิติน สมานมิตร	ผศ. ดร.บัญชา ปัญญานาค	มหาวิทยาลัยเชียงใหม่	สำเร็จการศึกษาแล้ว
4	นายชลธิศ เสือนุ่ม	ผศ.ดร.จักรกฤษ กลิ่นเอี่ยม	มหาวิทยาลัยนเรศวร	สำเร็จการศึกษาแล้ว
5	นาง พิกุล ศรีดาร์ตัน	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา
6	นาย พชร ใจโลกา	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา
7	นางสาว มนต์ธีรา สุวรรณ ประภา	รศ.ดร.นรินทร์ เพ็ชรโรจน์	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา
8	นาย อติศักดิ์ หาญจริง	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา
9	นาย ณัฐวุฒิ บุญลือ	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา
10	นาย ปณิธาน สานเมทา	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา
11	นางสาว พรพิมล บริวัล	รศ.ดร.นรินทร์ เพ็ชรโรจน์	มหาวิทยาลัยนเรศวร	กำลังศึกษา
12	นายพรดนัย กาศเกษม	ผศ.ดร.จักรกฤษ กลิ่นเอี่ยม	มหาวิทยาลัยนเรศวร	กำลังศึกษา
13	นายวาธิศ ศรีวิราช	ผศ.ดร.จักรกฤษ กลิ่นเอี่ยม	มหาวิทยาลัยนเรศวร	กำลังศึกษา
14	นาย เจนวิทย์	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา
15	นายพชร ใจโลกา	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา
16	นายรวีโรจน์ ศุภรตุลธร	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา

นักวิจัยรุ่นใหม่ ระดับปริญญาโท 6 คน

ลำดับ	รายนามนักศึกษา	อาจารย์ที่ปรึกษา วิทยานิพนธ์	มหาวิทยาลัย	หมายเหตุ
1	นาย อติศักดิ์ หาญจริง	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	สำเร็จการศึกษาแล้ว
2	นาย ณัฐวุฒิ บุญลือ	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	สำเร็จการศึกษาแล้ว
3	นาย ปณิธาน สานเมทา	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	สำเร็จการศึกษาแล้ว
4	นางสาว ดาวลย์ ชมพูนาม	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	สำเร็จการศึกษาแล้ว
5	นาย เจนวิทย์	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	สำเร็จการศึกษาแล้ว
6	นาย วรยุทธ บุญยะศรี	ศ.ดร.สุเทพ สอนใต้	มหาวิทยาลัยเชียงใหม่	กำลังศึกษา

(ข2) นักวิจัยอาจารย์รุ่นใหม่ จำนวน 5 คน ดังนี้

- (1) ผศ.ดร. จักรกฤษณ์ กลิ่นเอี่ยม มหาวิทยาลัยนเรศวร
ตำแหน่งอาจารย์ตอนเข้าร่วมโครงการ และได้ตำแหน่งผู้ช่วยศาสตราจารย์หลังจากร่วมสร้าง
ผลงานวิจัยในโครงการประมาณ 2 ปี
- (2) ผศ. ดร. อาทิตย์ แข็งธัญการ สถาบันเทคโนโลยีพระจอมเกล้าเจ้าคุณทหารลาดกระบัง
ตำแหน่งอาจารย์ตอนเข้าร่วมโครงการ และได้ตำแหน่งผู้ช่วยศาสตราจารย์หลังจากร่วมสร้าง
ผลงานวิจัยในโครงการประมาณ 2 ปี
- (3) ผศ. ดร. ภัคดี เจริญสุวรรณ มหาวิทยาลัยเชียงใหม่
ตำแหน่งอาจารย์ตอนเข้าร่วมโครงการ และได้ตำแหน่งผู้ช่วยศาสตราจารย์หลังจากร่วมสร้าง
ผลงานวิจัยในโครงการประมาณ 2 ปี
- (4) อ. ดร. วิฑูรย์ พึ่งรัตนา มหาวิทยาลัยราชภัฏนครปฐม
ตำแหน่งอาจารย์ตอนเข้าร่วมโครงการ และได้ตำแหน่งผู้ช่วยศาสตราจารย์หลังจากร่วมสร้าง
ผลงานวิจัยในโครงการประมาณ 2 ปี
- (5) อ. ดร. วัชรภรณ์ ขอลำเจียก มหาวิทยาลัยพะเยา
ตำแหน่งอาจารย์ตอนเข้าร่วมโครงการ และได้เสนอขอตำแหน่งผู้ช่วยศาสตราจารย์แล้ว
อยู่ระหว่างรอผลพิจารณา

(ข3) นักวิจัยรุ่นกลาง จำนวน 5 คน ดังต่อไปนี้

- (1) รศ.ดร. นรินทร์ เพ็ชรโรจน์ มหาวิทยาลัยนเรศวร
ปัจจุบันได้ทุนนักวิจัยรุ่นกลาง สกว. และ กำลังเขียนตำราเพื่อเสนอตำแหน่งศาสตราจารย์
- (2) รศ. ดร. บัญชา ปัญญานาค มหาวิทยาลัยนเรศวร
ตอนเริ่มเข้าโครงการดำรงตำแหน่งผู้ช่วยศาสตราจารย์ และ ได้รับตำแหน่ง
รองศาสตราจารย์ หลังร่วมโครงการ 3 ปี ปัจจุบันได้รับทุนนักวิจัยรุ่นกลางจาก สกว.
- (3) รศ. ดร. เจษฎา อีราบุญ มหาวิทยาลัยเทคโนโลยีพระจอมเกล้าพระนครเหนือ
ตอนเริ่มเข้าโครงการดำรงตำแหน่งผู้ช่วยศาสตราจารย์ และ ได้รับตำแหน่ง
รองศาสตราจารย์ หลังร่วมโครงการ 2 ปี
- (4) ผศ. ดร. อรรถพล แก้วขาว มหาวิทยาลัยเชียงใหม่
สร้างผลงานวิจัยได้ดีมีคุณภาพ และกำลังเขียนตำราเพื่อเสนอตำแหน่งรองศาสตราจารย์
- (5) ผศ. ดร. ธนศักดิ์ หมวกทองกลาง มหาวิทยาลัยเชียงใหม่
ตอนเข้าร่วมโครงการเป็นอาจารย์นักวิจัยทางด้าน Optimization ที่มีคุณภาพ
เพิ่งได้รับตำแหน่งหัวหน้าภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่
เมื่อเดือน เมษายน 2560

Reprints / Manuscripts

ผลงานวิจัยที่เป็น Research Articles ทั้งหมด 46 เรื่อง

ประกอบด้วย

- (1) ผลงานที่ตีพิมพ์แล้ว จำนวน 36 เรื่อง
- (2) ผลงานที่ได้รับการตอบรับให้ตีพิมพ์ (accepted) จำนวน 5 เรื่อง
- (3) ผลงานที่อยู่ระหว่างการพิจารณา (submitted) จำนวน 5 เรื่อง