รายงานวิจัยฉบับสมบูรณ์

การสร้างแบบจำลองของตัวดำเนินการ cubic Dirac ของ Kostant ของ Lie superalgebras และ non-compact Lie algebras

โดย ผศ. ดร. เทพอักษร เพ็งพันธ์

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หัวหน้าโครงการวิจัย ผศ. ดร. เทพอักษร เพ็งพันธ์

มหาวิทยาลัยสงขลานครินทร์

นักวิจัยที่ปรึกษา Prof. Dr. Edouard B. Manoukian

มหาวิทยาลัยเทคโนโลยีสุรนารี

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นักวิจัย : ผศ.ดร.เทพอักษร เพ็งพันธ์ มหาวิทยาลัยสงขลานครินทร์
Email Address : teparksorn.p@psu.ac.th

นักวิจัยที่ปรึกษา: Prof. Dr. Edouard B. Manoukian มหาวิทยาลัยเทคโนโลยีสุรนารี

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บทคัดย่อ

ตัวดำเนินการโคสแตนท์บนการหารของพีชคณิตของลี su(2)/u(1) ได้ถูกขยายไปบนการหารของพีชคณิตแบบซูเปอร์ของลี su(2|1)/u(2) และ su(2|1)/(u(1)×u(1)) และตัวดำเนินการบนการหารของพีชคณิตของลี su(3)/(su(2)×u(1)) ได้ถูกขยายไปบน su(3)/(su(2)×u(1)) และ su(3|1)/u(3) พร้อมทั้งได้ทำการหาแก่นคำตอบของตัวดำเนินการในแต่ละกรณี

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Investigator : Assist. Porf. Dr. Teparksorn Pengpan Prince of Songkla University
Email Address : teparksorn.p@psu.ac.th

Mentor: Prof. Dr. Edouard B. Manoukian Suranaree University of Technology

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ABSTRACT

The Kostant operator over the quotient Lie algebra *su*(2)/*u*(1) is extended to the quotient Lie superalgebras *su*(2|1)/*u*(2) and *su*(2|1)/(*u*(1)×*u*(1)), and the one over the quotient Lie algebra *su*(3)/(*su*(2)×*u*(1)) to *su*(3|1)/*u*(3) and *su*(3|1)/(*su*(2|1)×*u*(1)). The lowest lines of kernel solutions of the Kostant operators are obtained for each case.

Keywords : Kostant operator, Lie superalgebras, kernel solutions

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1.1 $su(2|1)$ Lie superalgebra

- Cartan matrix:

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}. \tag{1.1}$$

- Roots in Dynkin basis (ω', ω_1):

$$\omega' \cdot \omega' = -\frac{1}{2}, \quad \omega_1 \cdot \omega_1 = \frac{1}{2}, \quad \omega_1 \cdot \omega' = 0 \tag{1.2}$$

  - Even roots:

$$\alpha_1^{\pm} = \pm(0, 2) \tag{1.3}$$

  - Odd roots:

$$\beta_1^{\pm} = \pm(1, -1), \quad \beta_2^{\pm} = \pm(1, 1) \tag{1.4}$$

- $su(2|1)$ generators
  - The oscillator realization of the $su(2|1)$ fundamental representation

$$[a_i^-, a_j^+] = \delta_{ij}, \quad \{b^+, b^-\} = 1 \tag{1.5}$$

$$\begin{aligned} F_1^+ &\equiv F_{\beta_1^+} = b^+ a_1^-, \quad F_1^- \equiv F_{\beta_1^-} = b^- a_1^+ \\ F_2^+ &\equiv F_{\beta_2^+} = b^+ a_2^-, \quad F_2^- \equiv F_{\beta_2^-} = b^- a_2^+ \\ T_1^+ &\equiv T_{\alpha_1^+} = a_1^+ a_2^-, \quad T_1^- \equiv T_{\alpha_1^-} = a_1^- a_2^+ \\ T_3 &= a_1^+ a_1^- - a_2^+ a_2^-, \quad Z = a_1^+ a_1^- + a_2^+ a_2^- + 2b^+ b^- \end{aligned} \tag{1.6}$$

  - The differential-form realization of the $su(2|1)$ general representation

$$\begin{aligned} F_1^+ &= \theta\partial_1, \quad F_1^- = z_1\partial_\theta, \quad F_2^+ = \theta\partial_2, \quad F_2^- = z_2\partial_\theta, \\ T_1^+ &= z_1\partial_2, \quad T_1^- = z_2\partial_1, \quad T_3 = z_1\partial_1 - z_2\partial_2, \\ Z &= z_1\partial_1 + z_2\partial_2 + 2\theta\partial_\theta \end{aligned} \tag{1.7}$$

- Commutation relations of the $su(2|1)$ generators

$$[Z, T_3] = 0, \quad [T_1^+, T_1^-] = T_3 \tag{1.8}$$

$$[Z, T_1^\pm] = 0, \quad [T_3, T_1^\pm] = \pm 2T_1^\pm \tag{1.9}$$

$$[Z, F_1^\pm] = \pm F_1^\pm, \quad [T_3, F_1^\pm] = \mp F_1^\pm, \tag{1.10}$$

$$[Z, F_2^\pm] = \pm F_2^\pm, \quad [T_3, F_2^\pm] = \pm F_2^\pm, \tag{1.11}$$

$$[T_1^\pm, F_1^\pm] = \mp F_2^\pm, \quad [T_1^\pm, F_2^\mp] = \pm F_1^\mp, \tag{1.12}$$

$$\begin{aligned}\{F_i^+, F_j^-\} &= \frac{1}{2}(T_3 + Z) \cdot \frac{1}{2}(\mathbb{1} + \sigma_3)_{ij} + \frac{1}{2}(-T_3 + Z) \cdot \frac{1}{2}(\mathbb{1} - \sigma_3)_{ij} \\ &\quad + T_1^-(\sigma^+)_{ij} + T_1^+(\sigma^-)_{ij}.\end{aligned} \tag{1.13}$$

- $su(2|1)$ representations
  The representations of $su(2|1)$ are labeled by a pair of numbers in Dynkin basis (b, a_1) and categorized into three types: atypical-1, atypical-2 and typical [12].

  - The atypical-1 representation: $b = 0$

$$\psi_0 = z_1^{a_1} \quad \psi_2 = \theta_2 z_1^{a_1 - 1}. \tag{1.14}$$

  - The atypical-2 representation: $b = 1 + a_1$

$$\psi_0 = z_1^{a_1}, \quad \psi_1 = \theta_1 z_1^{a_1 + 1}. \tag{1.15}$$

  - The typical representation: $b \neq 0,\ 1 + a_1$

$$\psi_0 = z_1^{a_1}, \quad \psi_1 = \theta_1 z_1^{a_1+1}, \quad \psi_2 = \theta_2 z_1^{a_1-1}, \quad \psi_{12} = \theta_1\theta_2 z_1^{a_1}. \tag{1.16}$$

1.2 $su(3|1)$ Lie superalgebra

- Cartan matrix in Dynkin basis:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{1.17}$$

- Roots in Dynkin basis $(\omega', \omega_1, \omega_2)$:

$$\omega' \cdot \omega' = -\frac{1}{6}, \quad \omega_i \cdot \omega_i = \frac{2}{3}, \quad \omega_i \cdot \omega_j = \frac{1}{3}, \quad \omega_i \cdot \omega' = 0 \tag{1.18}$$

  - Even roots:

$$\alpha_1^\pm = \pm(0, 2, -1), \quad \alpha_2^\pm = \pm(0, -1, 2) \quad \alpha_3^\pm = \pm(0, 1, 1) \tag{1.19}$$

  - Odd roots:

$$\beta_1^\pm = \pm(2, -1, 0), \quad \beta_2^\pm = \pm(2, 1, -1), \quad \beta_3^\pm = \pm(2, 0, 1) \tag{1.20}$$

- $su(3|1)$ generators
  - The oscillator realization of the $su(3|1)$ fundamental representation

$$
\begin{aligned}
T_1^+ &\equiv T_{\alpha_1^+} = a_1^+ a_2^-\,, \qquad T_2^+ \equiv T_{\alpha_2^+} = a_2^+ a_3^-\,, \qquad T_3^+ \equiv T_{\alpha_3^+} = a_1^+ a_3^- \\
T_1^- &\equiv T_{\alpha_1^-} = a_1^- a_2^+\,, \qquad T_2^- \equiv T_{\alpha_2^-} = a_2^- a_3^+\,, \qquad T_3^- \equiv T_{\alpha_3^-} = a_1^- a_3^+ \\
F_1^+ &\equiv F_{\beta_1^+} = b^+ a_1^-\,, \qquad F_2^+ \equiv F_{\beta_2^+} = b^+ a_2^-\,, \qquad F_3^+ \equiv F_{\beta_3^+} = b^+ a_3^- \\
F_1^- &\equiv F_{\beta_1^-} = b^- a_1^+\,, \qquad F_2^- \equiv F_{\beta_2^-} = b^- a_2^+\,, \qquad F_3^- \equiv F_{\beta_3^-} = b^- a_3^+ \\
T_3 &= a_1^+ a_1^- - a_2^+ a_2^-\,, \qquad T_8 = a_2^+ a_2^- - a_3^+ a_3^- \\
Z &= a_1^+ a_1^- + a_2^+ a_2^- + a_3^+ a_3^- + 3b^+ b^- \qquad (1.21)
\end{aligned}
$$

  - The differential-form realization of the $su(3|1)$ general representation

$$
\begin{aligned}
T_1^+ &= z_1\partial_2 - \bar{z}_2\bar{\partial}_1, \quad T_1^- = z_2\partial_1 - \bar{z}_1\bar{\partial}_2, \\
T_2^+ &= z_2\partial_3 - \bar{z}_3\bar{\partial}_2, \quad T_2^- = z_3\partial_2 - \bar{z}_2\bar{\partial}_3, \\
T_3^+ &= z_1\partial_3 - \bar{z}_3\bar{\partial}_1, \quad T_3^- = z_3\partial_1 - \bar{z}_1\bar{\partial}_3, \\
F_1^+ &= \theta\partial_1, \quad F_2^+ = \theta\partial_2, \quad F_3^+ = \theta\partial_3 \\
F_1^- &= z_1\partial_\theta, \quad F_2^- = z_2\partial_\theta, \quad F_3^- = z_3\partial_\theta \\
T_3 &= z_1\partial_1 - \bar{z}_1\bar{\partial}_1 - z_2\partial_2 + \bar{z}_2\bar{\partial}_2 \\
T_8 &= z_2\partial_2 - \bar{z}_2\bar{\partial}_2 - z_3\partial_3 + \bar{z}_3\bar{\partial}_3 \\
Z &= z_1\partial_1 + \bar{z}_1\bar{\partial}_1 + z_2\partial_2 + \bar{z}_2\bar{\partial}_2 + z_3\partial_3 + \bar{z}_3\bar{\partial}_3 + 3\theta\partial_\theta \qquad (1.22)
\end{aligned}
$$

- Commutation relations of the $su(3|1)$ generators

$$[T_1^+, T_1^-] = T_3, \quad [T_2^+, T_2^-] = T_8, \quad [T_3^+, T_3^-] = T_3 + T_8 \qquad (1.23)$$

$$[T_1^\pm, T_2^\pm] = \pm T_3^\pm, \quad [T_1^\pm, T_3^\mp] = \mp T_2^\mp, \quad [T_2^\pm, T_3^\mp] = \pm T_1^\mp \qquad (1.24)$$

$$[T_3, T_8] = 0, \quad [Z, T_3] = 0, \quad [Z, T_8] = 0 \qquad (1.25)$$

$$[T_3, T_1^\pm] = \pm 2T_1^\pm, \; [T_8, T_1^\pm] = \mp T_1^\pm, \quad [Z, T_1^\pm] = 0 \qquad (1.26)$$

$$[T_3, T_2^\pm] = \mp T_2^\pm, \quad [T_8, T_2^\pm] = \pm 2T_2^\pm, \quad [Z, T_2^\pm] = 0 \qquad (1.27)$$

$$[T_3, T_3^\pm] = \pm T_3^\pm, \quad [T_8, T_3^\pm] = \pm T_3^\pm, \quad [Z, T_3^\pm] = 0 \qquad (1.28)$$

$$[T_3, F_1^\pm] = \mp F_1^\pm, \quad [T_8, F_1^\pm] = 0, \quad [Z, F_1^\pm] = \pm 2F_1^\pm \qquad (1.29)$$

$$[T_3, F_2^\pm] = \pm F_2^\pm, \quad [T_8, F_2^\pm] = \mp F_2^\pm, \quad [Z, F_2^\pm] = \pm 2F_2^\pm \qquad (1.30)$$

$$[T_3, F_3^\pm] = 0, \quad [T_8, F_3^\pm] = \pm F_3^\pm, \quad [Z, F_3^\pm] = \pm 2F_3^\pm \qquad (1.31)$$

$$[T_1^\pm, F_1^\pm] = \mp F_2^\pm, \quad [T_1^\pm, F_2^\mp] = \pm F_1^\mp, \qquad (1.32)$$

$$[T_2^\pm, F_2^\pm] = \mp F_3^\pm, \quad [T_2^\pm, F_3^\mp] = \pm F_2^\mp, \qquad (1.33)$$

$$[T_3^\pm, F_1^\pm] = \mp F_3^\pm, \quad [T_3^\pm, F_3^\mp] = \pm F_1^\mp, \qquad (1.34)$$

$$
\begin{aligned}
\{F_i^+, F_j^-\} &= \frac{1}{3}(2T_3 + T_8 + Z)(P_1)_{ij} + \frac{1}{3}(-T_3 + T_8 + Z)(P_2)_{ij} \\
&\quad + \frac{1}{3}(-T_3 - 2T_8 + Z)(P_3)_{ij} + T_1^-(\lambda_1^+)_{ij} + T_1^+(\lambda_1^-)_{ij} \\
&\quad + T_2^-(\lambda_2^+)_{ij} + T_2^+(\lambda_2^-)_{ij} + T_3^-(\lambda_3^+)_{ij} + T_3^+(\lambda_3^-)_{ij} \qquad (1.35)
\end{aligned}
$$

- $su(3|1)$ representations
  The representations of $su(3|1)$ are labeled by a triple of numbers in Dynkin basis (b, a_1, a_2) and categorized into four types: atypical-1, atypical-2, atypical-3 and typical.

  – The atypical-1 representation: $b = 0$

$$\begin{aligned} \psi_0 &= z_1^{a_1} \bar{z}_3^{a_2}, \\ \psi_2 &= \theta_2 z_1^{a_1-1} \bar{z}_3^{a_2+1}, \quad \psi_3 = \theta_3 z_1^{a_1} \bar{z}_3^{a_2-1}, \\ \psi_{23} &= \theta_2 \theta_3 z_1^{a_1-1} \bar{z}_3^{a_2}. \end{aligned} \tag{1.36}$$

  – The atypical-2 representation: $b = 1 + a_1$

$$\begin{aligned} \psi_0 &= z_1^{a_1} \bar{z}_3^{a_2}, \\ \psi_1 &= \theta_1 z_1^{a_1+1} \bar{z}_3^{a_2}, \quad \psi_3 = \theta_3 z_1^{a_1} \bar{z}_3^{a_2-1}, \\ \psi_{13} &= \theta_1 \theta_3 z_1^{a_1+1} \bar{z}_3^{a_2-1}. \end{aligned} \tag{1.37}$$

  – The atypical-3 representation: $b = 2 + a_1 + a_2$

$$\begin{aligned} \psi_0 &= z_1^{a_1} \bar{z}_3^{a_2}, \\ \psi_1 &= \theta_1 z_1^{a_1+1} \bar{z}_3^{a_2}, \quad \psi_2 = \theta_2 z_1^{a_1-1} \bar{z}_3^{a_2+1}, \\ \psi_{12} &= \theta_1 \theta_2 z_1^{a_1} \bar{z}_3^{a_2+1}, \end{aligned} \tag{1.38}$$

  – The typical representation: $b \neq 0,\ 1 + a_1,\ 2 + a_1 + a_2$

$$\begin{aligned} \psi_0 &= z_1^{a_1} \bar{z}_3^{a_2}, \\ \psi_1 &= \theta_1 z_1^{a_1+1} \bar{z}_3^{a_2}, \quad \psi_2 = \theta_2 z_1^{a_1-1} \bar{z}_3^{a_2+1}, \quad \psi_3 = \theta_3 z_1^{a_1} \bar{z}_3^{a_2-1}, \\ \psi_{12} &= \theta_1 \theta_2 z_1^{a_1} \bar{z}_3^{a_2+1}, \quad \psi_{13} = \theta_1 \theta_3 z_1^{a_1+1} \bar{z}_3^{a_2-1}, \quad \psi_{23} = \theta_2 \theta_3 z_1^{a_1-1} \bar{z}_3^{a_2}, \\ \psi_{123} &= \theta_1 \theta_2 \theta_3 z_1^{a_1} \bar{z}_3^{a_2}. \end{aligned} \tag{1.39}$$

Chapter 2

Kostant operators for the quotient Lie superalgebras

2.1 Introduction

The boson-fermion equivalence in the supersymmetric theories of elementary particles continues to exercise theoretical physics. One's intuitive unease with this idea has been at least partially overcome by a remarkable success of the associated ideas and formalism in the supergravity theories. This achivement helps sustain interest and spur development in establishing superalgebras (also sometimes called $\mathbb{Z}_2$-graded Lie algebras), the underlying algebras of supersymmetry [1]. Recently, the Lie superalgebras which occur in the Kac's classification scheme [2] have been used in study of odd coset quantum mechanics [3] and of M-theory on the pp wave [4].

The quotient method of Lie algebras has been proved to be the arsenal of mathematical physics in studying supergravity [5, 6] and getting a new kind of algbraic structure for string theory [7].

2.2 Kostant operator over the quotient of Lie algebras

Normally, a Lie algebra g can be decomposed into a direct sum of subalgebra h and the quotient $p \equiv g/h$ whose dimension is equal to dim $g -$ dim h. There exists a special kind of operator on the quotient space, called Kostant operator [8], whose state vector space describing internal degrees of freedom or charges is a tensor product space of the spinor representation of $so(\text{dim } p)$ with an irreducible representation of a Lie algebra [9].

The simplest Kostant's cubic Dirac operator is the one over $su(2)$. By defining $\sigma^{\pm} = (\sigma_1 \pm i\sigma_2)/2$ and the $su(2)$ generators $T_1^{\pm} = T_1 \pm iT_2$ shown in Appendix A as the even part of the $su(2|1)$ generators, one gets the twisted Kostant operator of $su(2)$

$$\not{K}_{su(2)} = \sigma^+ \otimes T_1^- + \sigma^- \otimes T_1^+ + \sigma_3 \otimes T_3 + \frac{1}{2}[\sigma^-, \sigma^+]\sigma_3 \otimes f_{+-3}\mathbb{1}. \tag{2.1}$$

Squaring the $su(2)$ Kostant operator, one yields

$$\not{K}^2_{su(2)} = \mathbb{1} \otimes (T_1^2 + T_2^2 + T_3^2) + \frac{1}{4}\mathbb{1} \otimes \mathbf{1} \tag{2.2}$$

which is positive definite.

The Kostant operator over the quotient $su(2)/u(1)$ is

$$\not{K} = \sigma^+ \otimes T_1^- + \sigma^- \otimes T_1^+. \tag{2.3}$$

The physical state vector space of the Kostant operator is

$$\Psi \equiv \psi^+ \oplus \psi^- = |+\rangle \otimes |j, m_j\rangle \oplus |-\rangle \otimes |j, m_j\rangle \tag{2.4}$$

As explicitly seen, $\not{K}\psi^\pm = c_\pm \psi'^\mp$ look similar to the two spinor-coupled equations of the Dirac equation and $(\not{K})^2\psi^\pm = c_+c_-\psi^\pm$ to the Klein-Gordon equation. It is interesting to see that there exist two kernel solutions, $\psi_{\rm E}^\pm \equiv |\pm\rangle \otimes |j, \pm j\rangle$, such that $\not{K}\psi_{\rm E}^\pm = 0$. One can also define the positive and negative Kostant operators $\not{K}^\pm = \sigma^\mp \otimes T_1^\pm$ which complelely map every state in the quotient vector space $\mathrm{V}(\psi^\pm)/\mathrm{V}(\psi_{\rm E}^\pm)$ to $\mathrm{V}(\psi^\mp)/\mathrm{V}(\psi_{\rm E}^\mp)$. To get a state in terms of the $u(1)$ subalgebra, one needs to act on the state by the $u(1)$ diagonal generator in its Dynkin basis

$$D = \mathbb{1} \otimes T_3 + \frac{1}{2}[\sigma^+, \sigma^-] \otimes f_{+-3}\mathbb{1} \tag{2.5}$$

which yields $D\psi^\pm = (m_j \pm \frac{1}{2})\psi^\pm$ and $D\psi_{\rm E}^\pm = \pm(j + \frac{1}{2})\psi_{\rm E}^\pm$. In terms of a complex variable, an su(2) highest state $|j, j\rangle$ is equivalent to $z_1^{a_1}$, where $a_1 = 2j$ is an $su(2)$ Dynkin label [10]. Alternatively, one obtains $D\psi_{\rm E}^\pm = \pm\frac{1}{2}(a_1 + 1)\psi_{\rm E}^\pm$.

Note that one can also construct a chiral Kostant operator

$$\not{K}_{\rm c} = \sigma^+ \otimes T_1^+ + \sigma^- \otimes T_1^-. \tag{2.6}$$

where its physical vector space is the same as that of the twisted one except that its kernel states are $\psi_{\rm E}^\pm \equiv |\pm\rangle \otimes |j, \mp j\rangle$.

For the quotient $su(3)/(su(2) \times u(1))$, the twisted Kostant operator [6] is

$$\not{K} = \gamma_2^+ \otimes T_2^- + \gamma_2^- \otimes T_2^+ + \gamma_3^+ \otimes T_3^- + \gamma_3^- \otimes T_3^+, \tag{2.7}$$

where $\gamma_2^\pm = \sigma^+ \otimes \sigma^\pm + \sigma^- \otimes \sigma^\pm$, $\gamma_3^\pm = \sigma^+ \otimes \frac{1}{2}(\sigma_3 \pm \mathbb{1}) + \sigma^- \otimes \frac{1}{2}(\sigma_3 \mp \mathbb{1})$ and the $su(3)$ generators $T_{2,3}^\pm$ are defined in Appendix B as the even part of the $su(3|1)$ generators. The physical space of this quotient is the tensor product space of the $so(4)$ positive and negative spinor representations with the $su(3)$ irreducible representation $\mathrm{V}_{(a_1,a_2)}$

$$\Psi = (|++\rangle + |+-\rangle) \otimes \mathrm{V}_{(a_1,a_2)} \oplus (|-+\rangle + |--\rangle) \otimes \mathrm{V}_{(a_1,a_2)}. \tag{2.8}$$

The positive and negative spinor state vector spaces are defined as $\psi^\pm \equiv (\sigma^\mp \otimes \mathbb{1})\Psi$. The $su(2) \times u(1)$ subalgebra consists of two commuting diagonal generators in its Dynkin basis

$$\begin{aligned} D_1 &= \mathbb{1} \otimes T_3 + \frac{1}{2}\left([\gamma_2^+, \gamma_2^-] \otimes f_{+-3}^2\mathbb{1} + [\gamma_3^+, \gamma_3^-] \otimes f_{+-3}^3\mathbb{1}\right) \\ &= \mathbb{1} \otimes T_3 + \left(\frac{1}{2}(\sigma_3 - \mathbb{1}) \otimes \sigma_3\right) \otimes \mathbb{1} \end{aligned} \tag{2.9}$$

$$\begin{aligned} D_2 &= -\mathbb{1} \otimes \frac{1}{6}(T_3 + 2T_8) - \frac{1}{4}\left([\gamma_2^+, \gamma_2^-] \otimes f_{+-8}^2\mathbb{1} + [\gamma_3^+, \gamma_3^-] \otimes f_{+-8}^3\mathbb{1}\right) \\ &= -\mathbb{1} \otimes \frac{1}{6}(T_3 + 2T_8) - \frac{1}{2}\left(\frac{1}{2}(\mathbb{1} + \sigma_3) \otimes \sigma_3\right) \otimes \mathbb{1} \end{aligned} \tag{2.10}$$

which are used to act on a state in the tensor product space of the quotient $su(3)/(su(2)\times u(1))$ to get the state in terms of the diagonal subalgebra. For the trivial one-dimensional representation $\mathrm{V}_{(0,0)}$, one obtains

$$(D_1;D_2)|++>\otimes\mathbf{1} = (0;-\frac{1}{2})|++>\otimes\mathbf{1} \tag{2.11}$$

$$(D_1;D_2)|+->\otimes\mathbf{1} = (0;\frac{1}{2})|+->\otimes\mathbf{1} \tag{2.12}$$

$$(D_1;D_2)|-+>\otimes\mathbf{1} = (-1;0)|-+>\otimes\mathbf{1} \tag{2.13}$$

$$(D_1;D_2)|-->\otimes\mathbf{1} = (1;0)|-->\otimes\mathbf{1} \tag{2.14}$$

Under the $su(2)\times u(1)$ subalgebra, the states $|+\pm>\otimes\mathbf{1}$ are the singlet states $\mathbf{1}_{\pm 1/2}$ with opposite helicity and the states $|-\pm>\otimes\mathbf{1}$ form a doublet $\mathbf{2}_0$ without helicity [11]. All of them are also the kernel solution of the Kostant operator over the quotient $su(3)/(su(2)\times u(1))$ and can be described as the degrees of freedom of the $N=2$ hypermultiplet, when $u(1)\simeq so(2)$ is viewed as the helicity of the four-dimensional Poincaré algebra.

For a general $su(3)$ irreducible representation, the highest state (a_1,a_2) in the vector space $\mathrm{V}_{(a_1,a_2)}$ can be represented in terms of complex variables as $z_1^{a_1}\bar{z}_3^{a_2}$ [6]. The kernel solutions for the positive spinor-state vector space are

$$\psi_E^{++}\equiv|++>\otimes z_1^{a_1}\bar{z}_3^{a_2},\quad \psi_E^{+-}\equiv|+->\otimes z_3^{a_1}\bar{z}_1^{a_2}, \tag{2.15}$$

and for the negative spinor-state vector space are

$$\psi_E^{-+}\equiv|-+>\otimes z_2^{a_1}\bar{z}_1^{a_2},\quad \psi_E^{--}\equiv|-->\otimes z_1^{a_1}\bar{z}_2^{a_2}. \tag{2.16}$$

Note that in the positive spinor space the state ψ_E^{++} is the highest one since $\mathbb{1}\otimes(T_2^+ + [T_1^+,T_2^+])\psi_E^{++}=0$ and the state ψ_E^{+-} is the lowest one since $\mathbb{1}\otimes(T_2^- + [T_1^-,T_2^-])\psi_E^{+-}=0$. While in the negative spinor space the state ψ_E^{-+} is the lowest kernel state of ψ_E^{--} since $\mathbb{1}\otimes(T_2^+ + [T_2^-,T_1^-])\psi_E^{-+}=0$ and $\mathbb{1}\otimes(T_2^- + [T_2^+,T_1^+])\psi_E^{--}=0$.

When acting on the kernel solutions by the commuting pair of subalgebra generator $(D_1;D_2)$, one yields

$$\begin{aligned}
(D_1;D_2)\psi_E^{++} &= \left(a_1;-\frac{1}{6}(a_1+2a_2+3)\right)\psi_E^{++},\\
(D_1;D_2)\psi_E^{+-} &= \left(-a_2;\frac{1}{6}(2a_1+a_2+3)\right)\psi_E^{+-},\\
(D_1;D_2)\psi_E^{-+} &= \left(-(a_1+a_2+1);\frac{1}{6}(a_2-a_1)\right)\psi_E^{-+},\\
(D_1;D_2)\psi_E^{--} &= \left(a_1+a_2+1;\frac{1}{6}(a_2-a_1)\right)\psi_E^{--}
\end{aligned} \tag{2.17}$$

In terms of $su(2)\times u(1)$ highest weights, the eigenvalues of the above equations are the Euler triplets as derived in [11].

2.3 Kostant operator over the quotient Lie superalgebras

Next the Kostant operators for the quotient Lie superalgebras $su(2|1)/u(2)$, $su(2|1)/(su(2)\times u(1))$, $su(3|1)/u(3)$ and $su(3|1)/(su(2|1)\times u(1))$ are constructed and the kernel solutions are obtained in each case, respectively.

2.3.1 Kostant operator over the quotient $su(2|1)/u(2)$

Let $\gamma_1^{\pm} = \sigma^+ \otimes \sigma^{\pm} + \sigma^- \otimes \sigma^{\pm}$, $\gamma_2^{\pm} = \sigma^+ \otimes \frac{1}{2}(\sigma_3 \pm \mathbb{1}) + \sigma^- \otimes \frac{1}{2}(\sigma_3 \mp \mathbb{1})$ be the gamma matrices associated with the odd generators $F_1^{\mp}$ and $F_2^{\mp}$, respectively. The Kostant operator over the quotient $su(2|1)/u(2)$ is

$$\not{K} = \gamma_1^+ F_1^- + \gamma_1^- F_1^+ + \gamma_2^+ F_2^- + \gamma_2^- F_2^+ \tag{2.18}$$

The state vector space of the operator is

$$\Psi = (|++> + |+->) \otimes \mathrm{V}_{(b,a_1)} \oplus (|-+> + |-->) \otimes \mathrm{V}_{(b,a_1)}. \tag{2.19}$$

The $u(2)$ subalgebra consists of two commuting diagonal generators in the $su(2|1)$ orthonormal basis

$$\begin{aligned} D_1 &= \mathbb{1} \otimes T_3 + \frac{1}{4}\left([\gamma_1^+, \gamma_1^-] \otimes g^1_{+-3}\mathbb{1} + [\gamma_2^+, \gamma_2^-] \otimes g^2_{+-3}\mathbb{1}\right) \\ &= \mathbb{1} \otimes T_3 + \frac{1}{2}\left(\frac{1}{2}(-\mathbb{1} + \sigma_3) \otimes \sigma_3\right) \otimes \mathbb{1} \qquad (2.20) \\ D_2 &= \mathbb{1} \otimes Z + \frac{1}{4}\left([\gamma_1^+, \gamma_1^-] \otimes g^1_{+-z}\mathbb{1} + [\gamma_2^+, \gamma_2^-] \otimes g^2_{+-z}\mathbb{1}\right) \\ &= \mathbb{1} \otimes Z + \frac{1}{2}\left(\frac{1}{2}(\mathbb{1} + \sigma_3) \otimes \sigma_3\right) \otimes \mathbb{1} \qquad (2.21) \end{aligned}$$

which are used to act on a state in the tensor product space of the quotient $su(2|1)/u(2)$ to get the state in terms of the diagonal subalgebra.

For the trivial one-dimensional representation $\mathrm{V}_{(0,0)}$ of $su(2|1)$, one obtains the lowest line of the kernel solutions as shown in Table 2.1.

Table 2.1: The lowest line of $su(2|1)/u(2)$ kernel solutions

|spin states>$\otimes \mathrm{V}_{(0,0)}$	$(D_1; D_2)$ $su(2) \times u(1)$ states	Dimensions
$\|++> \otimes \mathbf{1_0}$	$(0; \frac{1}{2})$	$\mathbf{1}_{1/2}$
$\|-+> \otimes \mathbf{1_0}$	$(-\frac{1}{2}; 0)$	
$\|--> \otimes \mathbf{1_0}$	$(\frac{1}{2}; 0)$	$\mathbf{2}_0$
$\|+-> \otimes \mathbf{1_0}$	$(0; -\frac{1}{2})$	$\mathbf{1}_{-1/2}$

2.3.2 Kostant operator over the quotient $su(2|1)/(u(1) \times u(1))$

Let $\gamma_1^{\pm} = \sigma_1 \otimes \sigma_1 \otimes \sigma^{\pm}$, $\gamma_2^{\pm} = \sigma_1 \otimes (\sigma^+ \otimes \frac{1}{2}(\sigma_3 \pm \mathbb{1}) + \sigma^- \otimes \frac{1}{2}(\sigma_3 \mp \mathbb{1}))$ be the gamma matrices associated with the odd generators $F_1^{\mp}$ and $F_2^{\mp}$ and $\gamma_3^{\pm} = (\sigma^+ \otimes \frac{1}{2}(\sigma_3 \pm 1) + \sigma^- \otimes \frac{1}{2}(\sigma_3 \mp 1)) \otimes \mathbb{1}$ associated with the even generators $T_1^{\mp}$. The Kostant operator over the quotient $su(2|1)/u(2)$ is

$$\not{K} = \gamma_1^+ F_1^- + \gamma_1^- F_1^+ + \gamma_2^+ F_2^- + \gamma_2^- F_2^+ + \gamma_3^+ T_1^- + \gamma_3^- T_1^+ \tag{2.22}$$

The state vector space of the operator is

$$\begin{aligned}\Psi &= (|+++>+|++->+|+-+>+|+-->) \otimes \mathrm{V}_{(b,a_1)} \\ &\oplus(|-++>+|-+->+|--+>+|--->) \otimes \mathrm{V}_{(b,a_1)}. \end{aligned} \quad (2.23)$$

The $u(1)\times u(1)$ subalgebra consists of two commuting diagonal generators in the $su(2|1)$ orthonormal basis

$$\begin{aligned} D_1 &= \mathbb{1}\otimes T_3 + \frac{1}{4}\left([\gamma_1^+,\gamma_1^-]\otimes g^1_{+-3}\mathbb{1} + [\gamma_2^+,\gamma_2^-]\otimes g^2_{+-3}\mathbb{1} + [\gamma_3^+,\gamma_3^-]\otimes f_{+-3}\mathbb{1}\right) \\ &= \mathbb{1}\otimes T_3 + \frac{1}{2}\left(\mathbb{1}\otimes\frac{1}{2}(\mathbb{1}-\sigma_3)\otimes\sigma_3 + \sigma_3\otimes\sigma_3\otimes\mathbb{1}\right)\otimes\mathbb{1} \end{aligned} \quad (2.24)$$

$$\begin{aligned} D_2 &= \mathbb{1}\otimes Z + \frac{1}{4}\left([\gamma_1^+,\gamma_1^-]\otimes g^1_{+-z}\mathbb{1} + [\gamma_2^+,\gamma_2^-]\otimes g^2_{+-z}\mathbb{1}\right) \\ &= \mathbb{1}\otimes Z + \frac{1}{2}\left(\mathbb{1}\otimes\frac{1}{2}(\mathbb{1}+\sigma_3)\otimes\sigma_3\right)\otimes\mathbb{1} \end{aligned} \quad (2.25)$$

which are used to act on a state in the tensor product space of the quotient $su(2|1)/u(1)\times u(1)$ to get the state in terms of the diagonal subalgebra.

For the trivial one-dimensional representation $\mathrm{V}_{(0,0)}$ of $su(2|1)$, one obtains the lowest line of $su(2|1)/(u(1)\times u(1))$ kernel solutions and it is interesting to see that the $u(1)\times u(1)$ lowest line states can be combined to form the $su(2)\times u(1)$ states as shown in Table 2.2.

Table 2.2: The lowest line of $su(2|1)/(u(1)\times u(1))$ kernel solutions

$\vert$spin states$>\otimes \mathrm{V}_{(0,0)}$	$(D_1;D_2)$ $u(1)\times u(1)$ states	$su(2)\times u(1)$ dimensions
$\vert+++>\otimes\mathbf{1_0}$	$(\frac{1}{2};\frac{1}{2})$	$\mathbf{2}_{1/2}$
$\vert++->\otimes\mathbf{1_0}$	$(-\frac{1}{2};\frac{1}{2})$	
$\vert--+>\otimes\mathbf{1_0}$	$(1;0)$	
$\frac{1}{\sqrt{2}}(\vert+-+>+\vert--->)\otimes\mathbf{1_0}$	$\frac{1}{\sqrt{2}}((0;0)_a+(0;0)_b)$	$\mathbf{3}_0$
$\vert+-->\otimes\mathbf{1_0}$	$(-1;0)$	
$\frac{1}{\sqrt{2}}(\vert+-+>-\vert--->)\otimes\mathbf{1_0}$	$\frac{1}{\sqrt{2}}((0;0)_a-(0;0)_b)$	$\mathbf{1}_0$
$\vert-++>\otimes\mathbf{1_0}$	$(\frac{1}{2};-\frac{1}{2})$	
$\vert-+->\otimes\mathbf{1_0}$	$(-\frac{1}{2};-\frac{1}{2})$	$\mathbf{2}_{-1/2}$

2.3.3 Kostant operator over the quotient $su(3|1)/u(3)$

Let $\gamma_1^\pm = \sigma_1\otimes\sigma_1\otimes\sigma^\pm$, $\gamma_2^\pm = \sigma_1\otimes(\sigma^+\otimes\frac{1}{2}(\sigma_3\pm\mathbb{1})+\sigma^-\otimes\frac{1}{2}(\sigma_3\mp\mathbb{1}))$ and $\gamma_3^\pm = (\sigma^+\otimes\frac{1}{2}(\sigma_3\pm 1)+\sigma^-\otimes\frac{1}{2}(\sigma_3\mp 1))\otimes\mathbb{1}$ be the gamma matrices associated with the odd generators $F_1^\mp$, $F_2^\mp$, and $F_3^\mp$, respectively. The Kostant operator over the quotient $su(3|1)/u(3)$ is

$$\not{K} = \gamma_1^+F_1^- + \gamma_1^-F_1^+ + \gamma_2^+F_2^- + \gamma_2^-F_2^+ + \gamma_3^+F_3^- + \gamma_3^-F_3^+ \quad (2.26)$$

The state vector space of the operator is

$$\Psi = (|+++>+|++->+|+-+>+|+-->)\otimes V_{(b,a_1,a_2)} \oplus(|-++>+|-+->+|--+>+|--->)\otimes V_{(b,a_1,a_2)}. \quad (2.27)$$

The $u(3)$ subalgebra consists of three commuting diagonal generators in the $su(3|1)$ orthonormal basis

$$\begin{aligned} D_1 &= \mathbb{1}\otimes T_3 + \frac{1}{4}\left([\gamma_1^+,\gamma_1^-]\otimes g^1_{+-3}\mathbb{1} + [\gamma_2^+,\gamma_2^-]\otimes g^2_{+-3}\mathbb{1} + [\gamma_3^+,\gamma_3^-]\otimes g^3_{+-3}\mathbb{1}\right) \\ &= \mathbb{1}\otimes T_3 + \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3 + \mathbb{1}\otimes(\mathbb{1}-\sigma_3)\otimes\sigma_3 - \sigma_3\otimes\sigma_3\otimes\mathbb{1}\right)\otimes\mathbb{1} \quad (2.28) \\ D_2 &= \mathbb{1}\otimes(T_3+2T_8) + \frac{1}{4}\left([\gamma_1^+,\gamma_1^-]\otimes g^1_{+-8}\mathbb{1} + [\gamma_2^+,\gamma_2^-]\otimes g^2_{+-8}\mathbb{1} + [\gamma_3^+,\gamma_3^-]\otimes g^3_{+-8}\mathbb{1}\right) \\ &= \mathbb{1}\otimes(T_3+2T_8) + \frac{1}{2}\left(\mathbb{1}\otimes\frac{1}{2}(\mathbb{1}+\sigma_3)\otimes\sigma_3 - \sigma_3\otimes\sigma_3\otimes\mathbb{1}\right)\otimes\mathbb{1} \quad (2.29) \\ D_3 &= \mathbb{1}\otimes Z + \frac{1}{4}\left([\gamma_1^+,\gamma_1^-]\otimes g^1_{+-z}\mathbb{1} + [\gamma_2^+,\gamma_2^-]\otimes g^2_{+-z}\mathbb{1} + [\gamma_3^+,\gamma_3^-]\otimes g^3_{+-z}\mathbb{1}\right) \\ &= \mathbb{1}\otimes Z + \frac{1}{2}\left(\mathbb{1}\otimes\frac{1}{2}(\mathbb{1}+\sigma_3)\otimes\sigma_3 + 2\sigma_3\otimes\sigma_3\otimes\mathbb{1}\right)\otimes\mathbb{1} \quad (2.30) \end{aligned}$$

which are used to act on a state in the tensor product space of the quotient $su(3|1)/u(3)$ to get the state in terms of the diagonal subalgebra.

For the trivial one-dimensional representation $V_{(0,0,0)}$ of $su(3|1)$, one obtains the lowest line of $su(3|1)/u(3)$ kernel solutions as shown in Table 2.3.

Table 2.3: The lowest line of $su(3|1)/u(3)$ kernel solutions

$\vert$spin states$>\otimes V_{(0,0,0)}$	$(D_1, D_2; D_3)$ $su(3)\times u(1)$ states	Dimensions
$\vert+++>\otimes\mathbf{1_0}$	$(0,0;\frac{3}{4})$	$\mathbf{1}_{3/4}$
$\vert-++>\otimes\mathbf{1_0}$	$(\frac{1}{2},1;\frac{1}{4})$	$\mathbf{3}_{1/4}$
$\vert--+>\otimes\mathbf{1_0}$	$(\frac{1}{2},-\frac{1}{2};\frac{1}{4})$	
$\vert--->\otimes\mathbf{1_0}$	$(-1,-\frac{1}{2};\frac{1}{4})$	
$\vert++->\otimes\mathbf{1_0}$	$(-\frac{1}{2},-1;-\frac{1}{4})$	$\mathbf{3}_{-1/4}$
$\vert+-+>\otimes\mathbf{1_0}$	$(1,\frac{1}{2};-\frac{1}{4})$	
$\vert+-->\otimes\mathbf{1_0}$	$(-\frac{1}{2},\frac{1}{2};-\frac{1}{4})$	
$\vert-+->\otimes\mathbf{1_0}$	$(0,0;-\frac{3}{4})$	$\mathbf{1}_{-3/4}$

2.3.4 Kostant operator over the quotient $su(3|1)/(su(2|1)\times u(1))$

Let $\gamma_2^\pm = \sigma_1\otimes\sigma_1\otimes\sigma^\pm$ and $\gamma_3^\pm = \sigma_1\otimes(\sigma^+\otimes\frac{1}{2}(\sigma_3\pm\mathbb{1}) + \sigma^-\otimes\frac{1}{2}(\sigma_3\mp\mathbb{1}))$ be the gamma matrices associated with the even generators $T_2^\mp$ and $T_3^\mp$, respectively, and $\gamma_1^\pm = (\sigma^+\otimes\frac{1}{2}(\sigma_3\pm 1) + \sigma^-\otimes\frac{1}{2}(\sigma_3\mp 1))\otimes\mathbb{1}$ associated with the odd generators $F_3^\mp$. The Kostant operator over the quotient $su(3|1)/(su(2|1)\times u(1))$ is

$$\not{K} = \gamma_2^+T_2^- + \gamma_2^-T_2^+ + \gamma_3^+T_3^- + \gamma_3^-T_3^+ + \gamma_1^+F_3^- + \gamma_1^-F_3^+ \quad (2.31)$$

The state vector space of the operator is

$$\Psi = (|+++>+|++->+|+-+>+|+-->)\otimes \mathrm{V}_{(b,a_1,a_2)}$$
$$\oplus(|-++>+|-+->+|--+>+|--->)\otimes \mathrm{V}_{(b,a_1,a_2)}. \quad (2.32)$$

The $su(2|1)\times u(1)$ subalgebra consists of three commuting diagonal generators in the $su(3|1)$ orthonormal basis

$$D_1 = \mathbb{1}\otimes T_3 + \frac{1}{4}\left([\gamma_2^+,\gamma_2^-]\otimes f^2_{+-3}\mathbb{1} + [\gamma_3^+,\gamma_3^-]\otimes f^3_{+-3}\mathbb{1} + [\gamma_1^+,\gamma_1^-]\otimes g^3_{+-3}\mathbb{1}\right)$$
$$= \mathbb{1}\otimes T_3 + \frac{1}{2}\left(\mathbb{1}\otimes\frac{1}{2}(\sigma_3-\mathbb{1})\otimes\sigma_3 - \frac{1}{2}\sigma_3\otimes\sigma_3\otimes\mathbb{1}\right)\otimes\mathbb{1} \quad (2.33)$$
$$D_2 = \mathbb{1}\otimes(T_3+2T_8) + \frac{1}{4}\left([\gamma_2^+,\gamma_2^-]\otimes f^2_{+-8}\mathbb{1} + [\gamma_3^+,\gamma_3^-]\otimes f^3_{+-8}\mathbb{1} + [\gamma_1^+,\gamma_1^-]\otimes g^3_{+-8}\mathbb{1}\right)$$
$$= \mathbb{1}\otimes(T_3+2T_8) + \frac{1}{2}\left(\mathbb{1}\otimes\frac{1}{2}(\mathbb{1}+\sigma_3)\otimes\sigma_3 - \sigma_3\otimes\sigma_3\otimes\mathbb{1}\right)\otimes\mathbb{1} \quad (2.34)$$
$$D_3 = \mathbb{1}\otimes Z + \frac{1}{4}\left([\gamma_1^+,\gamma_1^-]\otimes g^3_{+-z}\mathbb{1}\right)$$
$$= \mathbb{1}\otimes Z + \frac{1}{4}(\sigma_3\otimes\sigma_3\otimes\mathbb{1})\otimes\mathbb{1} \quad (2.35)$$

which are used to act on a state in the tensor product space of the quotient $su(3|1)/(su(2|)\times u(1))$ to get the state in terms of the diagonal subalgebra.

For the trivial one-dimensional representation $\mathrm{V}_{(0,0,0)}$ of $su(3|1)$, one obtains the lowest line of $su(3|1)/(su(2|1)\times u(1))$ kernel solutions as shown in Table 2.4.

Table 2.4: The lowest line of $su(3|1)/(su(2|1)\times u(1))$ kernel solutions

$\vert$spin states$>\otimes\mathrm{V}_{(0,0,0)}$	$(D_1, D_2; D_3)$ $su(2)\times u(1)\times u(1)$ states
$\vert+++>\otimes\mathbf{1_0}$	$(-\frac{1}{4},\frac{1}{4},0)$
$\vert++->\otimes\mathbf{1_0}$	$(-\frac{1}{4},\frac{1}{4},-1)$
$\vert+-+>\otimes\mathbf{1_0}$	$(-\frac{1}{4},-\frac{1}{4},1/2)$
$\vert+-->\otimes\mathbf{1_0}$	$(\frac{3}{4},-\frac{1}{4},1/2)$
$\vert-++>\otimes\mathbf{1_0}$	$(\frac{1}{4},-\frac{1}{4},1)$
$\vert-+->\otimes\mathbf{1_0}$	$(\frac{1}{4},-\frac{1}{4},0)$
$\vert--+>\otimes\mathbf{1_0}$	$(-\frac{3}{4},\frac{1}{4},-1/2)$
$\vert--->\otimes\mathbf{1_0}$	$(\frac{1}{4},\frac{1}{4},-1/2)$

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2.4 Output

International publication

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Appendix A

REPRINT

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Kernel solutions of the Kostant operator on eight-dimensional quotient spaces

Phongpichit Channuie,* Teparksorn Pengpan† and Witsanu Puttawong

Department of Physics, Faculty of Science, Prince of Songkla University
Hatyai 90112, Thailand
E-mail: s4722035@psu.ac.th, teparksorn.p@psu.ac.th, s4622137@psu.ac.th

Abstract: After introducing the generators and irreducible representations of the su(5) and so(6) Lie algebras in terms of the Schwinger's oscillators, the general kernel solutions of the Kostant operators on eight-dimensional quotient spaces su(5)/su(4) × u(1) and so(6)/so(4) × so(2) are derived in terms of the diagonal subalgebras su(4) × u(1) and so(4) × so(2), respectively.

Keywords: Field Theories in Higher Dimensions, Supersymmetric Standard Model, Differential and Algebraic Geometry.

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Contents

1. Introduction

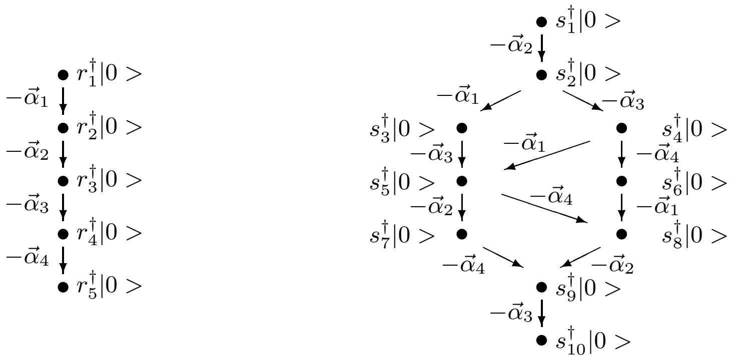
The Dirac operator plays a significant role in quantum field theories. Its natural generalization with a cubic term arose from Kazama and Suzuki's attempt to create a realistic string model [1]. Their cubic Dirac operator appeared in the string model as a supercurrent of a superconformal algebra. Ten years later, this kind of operator was discovered again by Kostant [2]. He understood already that an Euler number multiplet from an equal rank embedding of reductive Lie algebras [3] is nothing more than kernel solutions of the cubic Dirac operator. It is also an accident that the lowest lines of the Euler number multiplets for the 4-, 8-, and 16-dimensional coset spaces match with the known supersymmetric multiplets [4].

Although, the Euler number multiplets are easily derived by the GKRS index formula [3],

$$S^+ \otimes V_\Lambda - S^- \otimes V_\Lambda = \sum_{c\in C} \mathrm{sgn}(c) U_{c\bullet\Lambda},$$

they are not helpful for the formulation of any physical theory. In [5], Brink, Ramond and Xiong used an algebraic method to determine the general kernel solutions or the Euler number multiplets of the Kostant operators on the cosets $\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1)$ and $F_4/\mathrm{SO}(9)$. By realizing the gamma matrices as dynamical variables satisfying Grassmann algebras, the Euler number triplets for $\mathrm{SU}(3)/\mathrm{SU}(2)\times\mathrm{U}(1)$ and $F_4/\mathrm{SO}(9)$ were then written as chiral superfields. A free action in the light-cone frame for both cosets was also formulated.

The intention of this paper is to determine the general kernel solutions of the Kostant operators on the 8-dimensional quotients $\mathrm{su}(5)/\mathrm{su}(4) \times \mathrm{u}(1)$ and $\mathrm{so}(6)/\mathrm{so}(4) \times \mathrm{so}(2)$ by a quantum mechanical method. We will briefly present how to construct the generators of su(5) and so(6) Lie algebras and their irreducible representations (irreps). Only parts that



(a) (b)

Figure 1: The su(5) weight diagrams (a) of a 5-dimensional and (b) of a 10-dimensional irreps.

are used in constructing the Kostant operators will be mentioned. Then, the general kernel solutions will be determined. Their extension to the case of a non-compact Lie algebra was originated in 1999 by Ramond from his curiosity to know the Euler number multiplets. Some comments about them are made in the last section.

2. Kostant operator of the quotient su(5)/su(4) × u(1) and its kernel solutions

2.1 The Schwinger's oscillator realization of the su(5) Lie algebra

To construct the su(5) generators that satisfy Chevalley-Serre relations [6], we introduce four types of Schwinger's oscillators r_i, $\bar{r}_i$, s_j, $\bar{s}_j$, where $i = 1$ to 5 and $j = 1$ to 10, including their adjoints [7]. Action of the raising oscillators $r_i^\dagger$ and $s_j^\dagger$ on the vacuum state in correspondence to the su(5) irreps **5** and **10** is shown in figure 1a and 1b, respectively. By reversing all arrows in figure 1a and 1b, and replacing $r_i^\dagger$ and $s_j^\dagger$ with $\bar{r}_i^\dagger$ and $\bar{s}_j^\dagger$, they become the weight diagrams of the $\overline{\mathbf{5}}$ and $\overline{\mathbf{10}}$ irreps. Although, the **10** and $\overline{\mathbf{10}}$ irreps are not fundamental and can be obtained from anti-symmetrization of **5** and $\overline{\mathbf{5}}$ irreps, respectively, it will be seen later that introducing the oscillators s_j, $\bar{s}_j$ and their adjoints is a convenient way in determining the general kernel solutions.

From the **5**, $\overline{\mathbf{5}}$, **10** and $\overline{\mathbf{10}}$ weight diagrams, all positive root generators are

$$\begin{aligned}
T_1^+ &= r_1^\dagger r_2 + s_2^\dagger s_3 + s_4^\dagger s_5 + s_6^\dagger s_8 + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_2^+ &= r_2^\dagger r_3 + s_1^\dagger s_2 + s_5^\dagger s_7 + s_8^\dagger s_9 + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_3^+ &= r_1^\dagger r_3 - s_1^\dagger s_3 + s_4^\dagger s_7 + s_6^\dagger s_9 + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_4^+ &= r_3^\dagger r_4 + s_2^\dagger s_4 + s_3^\dagger s_5 + s_9^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_5^+ &= r_2^\dagger r_4 + s_1^\dagger s_4 - s_3^\dagger s_7 + s_8^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_6^+ &= r_1^\dagger r_4 - s_1^\dagger s_5 - s_2^\dagger s_7 + s_6^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_7^+ &= r_4^\dagger r_5 + s_4^\dagger s_6 + s_5^\dagger s_8 + s_7^\dagger s_9 + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_8^+ &= r_3^\dagger r_5 + s_2^\dagger s_6 + s_3^\dagger s_8 - s_7^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger,
\end{aligned}$$

$$T_9^+ = r_2^\dagger r_5 + s_1^\dagger s_6 - s_3^\dagger s_9 - s_5^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger,$$
$$T_{10}^+ = r_1^\dagger r_5 - s_1^\dagger s_8 - s_2^\dagger s_9 - s_4^\dagger s_{10} + (r, s \to \bar{r}, \bar{s})^\dagger, \tag{2.1}$$

and all negative ones are

$$T_A^- = \left(T_A^+\right)^\dagger, \qquad A = 1,\ 2,\ 3,\ \ldots,\ 10. \tag{2.2}$$

The Cartan subalgebra generators in the Dynkin basis,

$$\vec{H} \equiv H_1\hat{\omega}_1 + H_2\hat{\omega}_2 + H_3\hat{\omega}_3 + H_4\hat{\omega}_4 = (H_1, H_2, H_3, H_4), \tag{2.3}$$

are obtained from the following commutators:

$$\begin{aligned}
H_1 &\equiv [T_1^+, T_1^-] \\
&= N_1^{(r)} + N_2^{(s)} + N_4^{(s)} + N_6^{(s)} - N_2^{(r)} - N_3^{(s)} - N_5^{(s)} - N_8^{(s)} - (r, s \to \bar{r}, \bar{s}), \\
H_2 &\equiv [T_2^+, T_2^-] \\
&= N_2^{(r)} + N_1^{(s)} + N_5^{(s)} + N_8^{(s)} - N_3^{(r)} - N_2^{(s)} - N_7^{(s)} - N_9^{(s)} - (r, s \to \bar{r}, \bar{s}), \\
H_3 &\equiv [T_4^+, T_4^-] \\
&= N_3^{(r)} + N_2^{(s)} + N_3^{(s)} + N_9^{(s)} - N_4^{(r)} - N_4^{(s)} - N_5^{(s)} - N_{10}^{(s)} - (r, s \to \bar{r}, \bar{s}), \\
H_4 &\equiv [T_7^+, T_7^-] \\
&= N_4^{(r)} + N_4^{(s)} + N_5^{(s)} + N_7^{(s)} - N_5^{(r)} - N_6^{(s)} - N_8^{(s)} - N_9^{(s)} - (r, s \to \bar{r}, \bar{s}),
\end{aligned} \tag{2.4}$$

where $N_i^{r,s,\bar{r},\bar{s}}$ are the number operators. They are related to the Cartan generators in the orthonormal basis,

$$\vec{h} \equiv h_1\hat{e}_1 + h_2\hat{e}_2 + h_3\hat{e}_3 + h_4\hat{e}_4 + h_5\hat{e}_5 = [h_1, h_2, h_3, h_4, h_5], \tag{2.5}$$

as follows:

$$H_1 = h_1 - h_2,\ H_2 = h_2 - h_3,\ H_3 = h_3 - h_4,\ H_4 = h_4 - h_5. \tag{2.6}$$

An su(5) irrep represented by its highest weight Λ in its vector space V_Λ can be generated by action of the raising oscillators, $r_1^\dagger,\ \bar{r}_5^\dagger,\ s_1^\dagger,\ \bar{s}_{10}^\dagger$, on the vacuum state

$$\Lambda = (r_1^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_{10}^\dagger)^{a_3}(\bar{r}_5^\dagger)^{a_4}|0>, \tag{2.7}$$

where $a_{1,2,3,4}$ are non-negative integers, called the Dynkin labels. The action of the Cartan generators in the Dynkin basis on the highest weight gives their eigenvalues as follow:

$$\vec{H}\Lambda = (a_1, a_2, a_3, a_4)\Lambda, \tag{2.8}$$

and in the orthonormal basis as follow:

$$\vec{h}\Lambda = [b_1, b_2, b_3, b_4, b_5]\Lambda, \tag{2.9}$$

where

$$b_1 = \frac{1}{5}(4a_1 + 3a_2 + 2a_3 + a_4),$$

$$\begin{aligned}
b_2 &= \frac{1}{5}(-a_1 + 3a_2 + 2a_3 + a_4),\\
b_3 &= \frac{1}{5}(-a_1 - 2a_2 + 2a_3 + a_4),\\
b_4 &= \frac{1}{5}(-a_1 - 2a_2 - 3a_3 + a_4),\\
b_5 &= \frac{1}{5}(-a_1 - 2a_2 - 3a_3 - 4a_4).
\end{aligned} \tag{2.10}$$

Note that $b_1 + b_2 + b_3 + b_4 + b_5 = 0$ is due to the basis constraint.

Inside the su(5) generators, the generators $T^{\pm}_{1,2,\ldots,6}$ and $H_{1,2,3}$ form the su(4) Lie subalgebra and the generator h_5 is the generator of u(1) subalgebra. The other generators $T^{\pm}_{7,8,9,10}$ lie outside the subalgebra su(4) $\times$ u(1) and they are used to construct the Kostant operator of the quotient su(5)/su(4) $\times$ u(1).

2.2 Kernel solutions of the Kostant operator

To construct the Kostant operator on the 8-dimensional quotient space, the following 16×16 gamma matrices are needed:

$$\begin{aligned}
\Gamma_1 &= \sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma_1, & \Gamma_5 &= \sigma_1\otimes\sigma_1\otimes\sigma_3\otimes\mathbb{1},\\
\Gamma_2 &= \sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma_2, & \Gamma_6 &= \sigma_1\otimes\sigma_2\otimes\mathbb{1}\otimes\mathbb{1},\\
\Gamma_3 &= \sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma_3, & \Gamma_7 &= \sigma_1\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1},\\
\Gamma_4 &= \sigma_1\otimes\sigma_1\otimes\sigma_2\otimes\mathbb{1}, & \Gamma_8 &= \sigma_2\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1},
\end{aligned}$$

where $\sigma_{1,2,3}$ are the Pauli matrices and $\mathbb{1}$ is a 2×2 identity matrix. These gamma matrices satisfy Clifford algebra

$$\{\Gamma_a,\Gamma_b\} = 2\delta_{a,b}\,(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1})\,. \tag{2.11}$$

To associate with the generators of the quotient su(5)/su(4) $\times$ u(1), the gamma matrices are complexified as follows:

$$\begin{aligned}
\gamma_7^{\pm} &= \frac{1}{2}(\Gamma_1 \pm i\Gamma_2) = \sigma_1\otimes\sigma_1\otimes\sigma_1\otimes\sigma^{\pm},\\
\gamma_8^{\pm} &= \frac{1}{2}(\Gamma_3 \pm i\Gamma_4) = \sigma_1\otimes\sigma_1\otimes\left[\sigma^+\otimes\frac{1}{2}(\sigma_3\pm\mathbb{1}) + \sigma^-\otimes\frac{1}{2}(\sigma_3\mp\mathbb{1})\right],\\
\gamma_9^{\pm} &= \frac{1}{2}(\Gamma_5 \pm i\Gamma_6) = \sigma_1\otimes\left[\sigma^+\otimes\frac{1}{2}(\sigma_3\pm\mathbb{1}) + \sigma^-\otimes\frac{1}{2}(\sigma_3\mp\mathbb{1})\right]\otimes\mathbb{1},\\
\gamma_{10}^{\pm} &= \frac{1}{2}(\Gamma_7 \pm i\Gamma_8) = \left[\sigma^+\otimes\frac{1}{2}(\sigma_3\pm\mathbb{1}) + \sigma^-\otimes\frac{1}{2}(\sigma_3\mp\mathbb{1})\right]\otimes\mathbb{1}\otimes\mathbb{1}.
\end{aligned} \tag{2.12}$$

Under these complexification, the positive spinor states of so(8) are $|+\pm\pm\pm>$ and the negative ones $|-\pm\pm\pm>$.

From the commutators of the generators of the quotient,

$$\begin{aligned}
[T_7^+,T_7^-] &= h_4 - h_5, & [T_8^+,T_8^-] &= h_3 - h_5,\\
[T_9^+,T_9^-] &= h_2 - h_5, & [T_{10}^+,T_{10}^-] &= h_1 - h_5,
\end{aligned} \tag{2.13}$$

the generators $T^{\pm}_{a=7,8,9,10}$ are not generated. The structure constants of these transformations are zero. Hence, there are no cubic terms, which are composed of a product of three gamma matrices associated with the structure constants. The Kostant operator of the quotient su(5)/su(4) × u(1) is just

$$\not{K} = \sum_{a=7}^{10} (\gamma_a^+ T_a^- + \gamma_a^- T_a^+). \tag{2.14}$$

This Kostant operator acts on a tensor-product space of the so(8) spinor representations and the su(5) irrep

$$\psi^{\pm}_{\Lambda} \equiv | \pm \pm \pm \pm > V_\Lambda, \tag{2.15}$$

and there exist kernel solutions such that

$$\not{K} \psi^{\pm}_{\lambda_i} = 0, \tag{2.16}$$

where λ_i is a weight in the vector space V_Λ. Equation (2.16) can be decomposed into sixteen, independent equations as follows:

$$\begin{aligned}
(T_7^+ + T_8^+ + T_9^+ + T_{10}^+)\psi^+_{\lambda_1} &= 0,\\
(T_7^- - T_8^- + T_9^+ + T_{10}^+)\psi^+_{\lambda_2} &= 0,\\
(T_7^+ + T_8^- - T_9^- + T_{10}^+)\psi^+_{\lambda_3} &= 0,\\
(T_7^- - T_8^+ - T_9^- + T_{10}^+)\psi^+_{\lambda_4} &= 0,\\
(T_7^+ + T_8^+ + T_9^- - T_{10}^-)\psi^+_{\lambda_5} &= 0,\\
(T_7^- - T_8^- + T_9^- - T_{10}^-)\psi^+_{\lambda_6} &= 0,\\
(T_7^+ + T_8^- - T_9^+ - T_{10}^-)\psi^+_{\lambda_7} &= 0,\\
(T_7^- - T_8^+ - T_9^+ - T_{10}^-)\psi^+_{\lambda_8} &= 0,\\
(T_7^+ + T_8^+ + T_9^+ + T_{10}^-)\psi^-_{\lambda'_1} &= 0,\\
(T_7^- - T_8^- + T_9^+ + T_{10}^-)\psi^-_{\lambda'_2} &= 0,\\
(T_7^+ + T_8^- - T_9^- + T_{10}^-)\psi^-_{\lambda'_3} &= 0,\\
(T_7^- - T_8^+ - T_9^- + T_{10}^-)\psi^-_{\lambda'_4} &= 0,\\
(T_7^+ + T_8^+ + T_9^- - T_{10}^+)\psi^-_{\lambda'_5} &= 0,\\
(T_7^- - T_8^- + T_9^- - T_{10}^+)\psi^-_{\lambda'_6} &= 0,\\
(T_7^+ + T_8^- - T_9^+ - T_{10}^+)\psi^-_{\lambda'_7} &= 0,\\
(T_7^- - T_8^+ - T_9^+ - T_{10}^+)\psi^-_{\lambda'_8} &= 0.
\end{aligned} \tag{2.17}$$

One of the possible kernel solutions in the positive spinor space is as follows:

$$\begin{aligned}
\psi^+_{\lambda_1} &= |++++> (r_1^\dagger)^{a_1} (s_1^\dagger)^{a_2} (\bar{s}_{10}^\dagger)^{a_3} (\bar{r}_5^\dagger)^{a_4} |0>,\\
\psi^+_{\lambda_2} &= |+++-> (r_1^\dagger)^{a_1} (s_1^\dagger)^{a_2} (\bar{s}_7^\dagger)^{a_3} (\bar{r}_4^\dagger)^{a_4} |0>,\\
\psi^+_{\lambda_3} &= |++-+> (r_1^\dagger)^{a_1} (s_4^\dagger)^{a_2} (\bar{s}_3^\dagger)^{a_3} (\bar{r}_3^\dagger)^{a_4} |0>,
\end{aligned}$$

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$$
\begin{aligned}
\psi^+_{\lambda_4} &= |++--- > (r_1^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_5^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^+_{\lambda_5} &= |+-++> (r_4^\dagger)^{a_1}(s_7^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}(\bar{r}_1^\dagger)^{a_4}|0>,\\
\psi^+_{\lambda_6} &= |+-+-> (r_5^\dagger)^{a_1}(s_6^\dagger)^{a_2}(\bar{s}_7^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^+_{\lambda_7} &= |+--+> (r_2^\dagger)^{a_1}(s_5^\dagger)^{a_2}(\bar{s}_2^\dagger)^{a_3}(\bar{r}_1^\dagger)^{a_4}|0>,\\
\psi^+_{\lambda_8} &= |+---> (r_3^\dagger)^{a_1}(s_3^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,
\end{aligned} \tag{2.18}
$$

and in the negative spinor space as follows:

$$
\begin{aligned}
\psi^-_{\lambda'_1} &= |-+++> (r_4^\dagger)^{a_1}(s_7^\dagger)^{a_2}(\bar{s}_6^\dagger)^{a_3}(\bar{r}_1^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_2} &= |-++-> (r_2^\dagger)^{a_1}(s_8^\dagger)^{a_2}(\bar{s}_7^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_3} &= |-+-+> (r_4^\dagger)^{a_1}(s_{10}^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}(\bar{r}_1^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_4} &= |-+--> (r_3^\dagger)^{a_1}(s_9^\dagger)^{a_2}(\bar{s}_5^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_5} &= |--++> (r_4^\dagger)^{a_1}(s_7^\dagger)^{a_2}(\bar{s}_8^\dagger)^{a_3}(\bar{r}_2^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_6} &= |--+-> (r_1^\dagger)^{a_1}(s_6^\dagger)^{a_2}(\bar{s}_7^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_7} &= |---+> (r_4^\dagger)^{a_1}(s_5^\dagger)^{a_2}(\bar{s}_9^\dagger)^{a_3}(\bar{r}_3^\dagger)^{a_4}|0>,\\
\psi^-_{\lambda'_8} &= |----> (r_1^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_{10}^\dagger)^{a_3}(\bar{r}_4^\dagger)^{a_4}|0>.
\end{aligned} \tag{2.19}
$$

To get the kernel solutions in terms of su(4) × u(1), it needs to act on them by the Cartan subalgebra generators, which in the Dynkin basis are

$$
\begin{aligned}
D_1 &= h_1 - h_2 + \frac{1}{2}\left(f^{10}_{+-1}[\gamma^+_{10},\gamma^-_{10}] - f^{9}_{+-2}[\gamma^+_{9},\gamma^-_{9}]\right)\\
&= H_1 + \frac{1}{2}\left(\sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1}\right),\\
D_2 &= h_2 - h_3 + \frac{1}{2}\left(f^{9}_{+-2}[\gamma^+_{9},\gamma^-_{9}] - f^{8}_{+-3}[\gamma^+_{8},\gamma^-_{8}]\right)\\
&= H_2 + \frac{1}{2}\left(\mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1} - \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3\right),\\
D_3 &= h_3 - h_4 + \frac{1}{2}\left(f^{8}_{+-3}[\gamma^+_{8},\gamma^-_{8}] - f^{7}_{+-4}[\gamma^+_{7},\gamma^-_{7}]\right)\\
&= H_3 + \frac{1}{2}\left(\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3 - \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\right),\\
D_4 &= \frac{1}{2}h_5 + \frac{1}{4}\left(f^{7}_{+-5}[\gamma^+_{7},\gamma^-_{7}] + f^{8}_{+-5}[\gamma^+_{8},\gamma^-_{8}] + f^{9}_{+-5}[\gamma^+_{9},\gamma^-_{9}] + f^{10}_{+-5}[\gamma^+_{10},\gamma^-_{10}]\right)\\
&= \frac{1}{2}h_5 - \frac{1}{4}(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3 + \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3 + \mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1}\\
&\qquad + \sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1}).
\end{aligned} \tag{2.20}
$$

The structure constants in (2.20) are read directly from (2.13). The generators D_1, D_2 and D_3 are the Cartan generators of su(4) and the generator D_4 is the Cartan generator of u(1). When the Cartan generators act on the kernel solutions, they give

$$
(D_1, D_2, D_3; D_4)\psi^+_{\lambda_1} = (a_1, a_2, a_3; (b_5-2)/2)\psi^+_{\lambda_1},
$$

$$
\begin{aligned}
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_2} &= (a_1,a_2+a_3+1,a_4;b_3/2)\psi^+_{\lambda_2},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_3} &= (a_1+a_2+a_3+1,a_4,-a_2-a_3-a_4-1;b_3/2)\psi^+_{\lambda_3},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_4} &= (a_1+a_2+a_3+1,-a_2-a_3-1,a_2+a_3+a_4+1;b_3/2)\psi^+_{\lambda_4},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_5} &= (-a_4,-a_2-a_3-1,-a_1;b_3/2)\psi^+_{\lambda_5},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_6} &= (a_2,a_3,a_4;(b_1+2)/2)\psi^+_{\lambda_6},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_7} &= (-a_1-a_2-a_3-a_4-1,a_1+a_2+a_3+1,-a_2-a_3-1;b_3/2)\psi^+_{\lambda_7},\\
(D_1,D_2,D_3;D_4)\psi^+_{\lambda_8} &= (-a_2-a_3-1,-a_1,a_1+a_2+a_3+a_4+1;b_3/2)\psi^+_{\lambda_8},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_1} &= (-a_3-a_4-1,-a_2,-a_1;(b_4-1)/2)\psi^-_{\lambda'_1},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_2} &= (-a_1-a_2-1,a_1+a_2+a_3+1,a_4;(b_2+1)/2)\psi^-_{\lambda'_2},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_3} &= (-a_4,-a_3,-a_1-a_2-1;(b_2+1)/2)\psi^-_{\lambda'_3},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_4} &= (a_3,-a_1-a_2-a_3-1,a_1+a_2+a_3+a_4+1;(b_2+1)/2)\psi^-_{\lambda'_4},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_5} &= (a_3+a_4+1,-a_2-a_3-a_4-1,-a_1;(b_4-1)/2)\psi^-_{\lambda'_5},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_6} &= (a_1+a_2+1,a_3,a_4;(b_2+1)/2)\psi^-_{\lambda'_6},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_7} &= (-a_2,a_2+a_3+a_4+1,-a_1-a_2-a_3-a_4-1;(b_4-1)/2)\psi^-_{\lambda'_7},\\
(D_1,D_2,D_3;D_4)\psi^-_{\lambda'_8} &= (a_1,a_2,a_3+a_4+1;(b_4-1)/2)\psi^-_{\lambda'_8}.
\end{aligned}
\tag{2.21}
$$

In case $a_1=a_2=a_3=a_4=0$, the kernel solutions (2.21) can be grouped in terms of su(4) dimensions as follows:

$$
\mathbf{1}_{-1}\equiv\psi^+_{\lambda_1}\sim(0,0,0)_{-1},\quad \mathbf{6}_0\equiv\begin{cases}\psi^+_{\lambda_2}\sim(\ 0,\ 1,\ 0)_0\\ \psi^+_{\lambda_4}\sim(\ 1,-1,\ 1)_0\\ \psi^+_{\lambda_8}\sim(-1,\ 0,\ 1)_0\\ \psi^+_{\lambda_3}\sim(\ 1,\ 0,-1)_0\\ \psi^+_{\lambda_7}\sim(-1,\ 1,-1)_0\\ \psi^+_{\lambda_5}\sim(\ 0,-1,\ 0)_0\end{cases},\quad \mathbf{1}_1\equiv\psi^+_{\lambda_6}\sim(0,0,0)_1,
$$

$$
\mathbf{4}_{-1/2}\equiv\begin{cases}\psi^-_{\lambda'_1}\sim(-1,\ 0,\ 0)_{-1/2}\\ \psi^-_{\lambda'_5}\sim(\ 1,-1,\ 0)_{-1/2}\\ \psi^-_{\lambda'_7}\sim(\ 0,\ 1,-1)_{-1/2}\\ \psi^-_{\lambda'_8}\sim(\ 0,\ 0,\ 1)_{-1/2}\end{cases},\qquad \mathbf{4}_{1/2}\equiv\begin{cases}\psi^-_{\lambda'_6}\sim(\ 1,\ 0,\ 0)_{1/2}\\ \psi^-_{\lambda'_2}\sim(-1,\ 1,\ 0)_{1/2}\\ \psi^-_{\lambda'_4}\sim(\ 0,-1,\ 1)_{1/2}\\ \psi^-_{\lambda'_3}\sim(\ 0,\ 0,-1)_{1/2}\end{cases}.
$$

Since the Dynkin labels $a_{1,2,3,4}$ are non-negative, the direct sum of the su(4) highest weights

$$
\psi^+_{\lambda_1}\oplus\psi^+_{\lambda_2}\oplus\psi^+_{\lambda_6}\oplus\psi^-_{\lambda'_8}\oplus\psi^-_{\lambda'_6},
\tag{2.22}
$$

or in terms of its Dynkin labels,

$$
\begin{aligned}
&(a_1,a_2,a_3)_{(b_5-2)/2}\oplus(a_1,a_2+a_3+1,a_4)_{b_3/2}\oplus(a_2,a_3,a_4)_{(b_1+2)/2}\\
&\oplus(a_1,a_2,a_3+a_4+1)_{(b_4-1)/2}\oplus(a_1+a_2+1,a_3,a_4)_{(b_2+1)/2},
\end{aligned}
\tag{2.23}
$$

forms the Euler number multiplet.

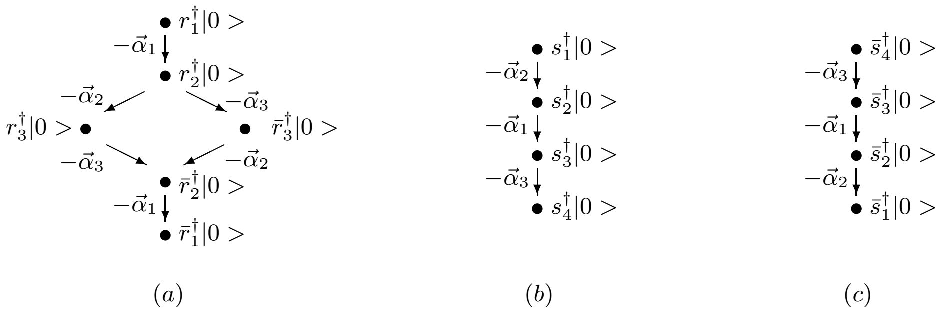

Figure 2: The so(6) weight diagrams (a) of a 6-dimensional vector (b) of a 4-dimensional co-spinor and (c) of a 4-dimensional spinor representations.

3. Kostant operator of the quotient so(6)/so(4) × so(2) and its kernel solutions

3.1 The Schwinger's oscillator realization of the so(6) Lie algebra

To construct the generators for so(6), we introduce four types of Schwinger's oscillators $r_i,\ \bar{r}_i,\ s_j,\ \bar{s}_j$, where $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, including their adjoints. Action of the raising oscillators $r_i^\dagger,\ \bar{r}_i^\dagger,\ s_i^\dagger$ and $\bar{s}_j^\dagger$ on the vacuum state in correspondence to the so(6) irreps **6**, $\mathbf{4}_c$ and $\mathbf{4}_s$ is shown in figure 2a, 2b and 2c, respectively. Although, the **6** irrep is not fundamental and can be obtained from an anti-symmetric product of two copies of either $\mathbf{4}_c$ or $\mathbf{4}_s$ irrep, it will be seen later that introducing the oscillators r_j and $\bar{r}_j$ is an easy way to determine the kernel solutions of the Kostant operator.

From the weight diagrams of so(6), all positive root generators are

$$\begin{aligned}
T_1^+ &= r_1^\dagger r_2 + s_2^\dagger s_3 + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_2^+ &= r_2^\dagger r_3 + s_1^\dagger s_2 + (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_3^+ &= r_2^\dagger \bar{r}_3 + s_3^\dagger s_4 + (r, \bar{r}, s \to \bar{r}, r, \bar{s})^\dagger, \\
T_4^+ &= r_1^\dagger r_3 - s_1^\dagger s_3 - (r, s \to \bar{r}, \bar{s})^\dagger, \\
T_5^+ &= r_1^\dagger \bar{r}_3 + s_2^\dagger s_4 - (r, \bar{r}, s \to \bar{r}, r, \bar{s})^\dagger, \\
T_6^+ &= -r_1^\dagger \bar{r}_2 + s_1^\dagger s_4 + (r, \bar{r}, s \to \bar{r}, r, \bar{s})^\dagger,
\end{aligned} \tag{3.1}$$

and all negative root generators are

$$T_A^- = \left(T_A^+\right)^\dagger, \qquad A = 1,\ 2,\ 3,\ \ldots,\ 6. \tag{3.2}$$

The Cartan subalgebra generators in the Dynkin basis,

$$\vec{H} \equiv H_1 \hat{\omega}_1 + H_2 \hat{\omega}_2 + H_3 \hat{\omega}_3 = (H_1, H_2, H_3), \tag{3.3}$$

are obtained from the following commutators:

$$\begin{aligned}
H_1 &\equiv [T_1^+, T_1^-] = N_1^{(r)} + N_2^{(s)} - N_2^{(r)} - N_3^{(s)} - (r, s \to \bar{r}, \bar{s}), \\
H_2 &\equiv [T_2^+, T_2^-] = N_2^{(r)} + N_1^{(s)} - N_3^{(r)} - N_2^{(s)} - (r, s \to \bar{r}, \bar{s}),
\end{aligned}$$

$$H_3 \equiv [T_3^+, T_3^-] = N_2^{(r)} + N_3^{(s)} - N_3^{(\bar{r})} - N_4^{(s)} - (r, \bar{r}, s \to \bar{r}, r, \bar{s}). \tag{3.4}$$

They are related to the Cartan generators in the orthonormal basis,

$$\vec{h} \equiv h_1 \hat{e}_1 + h_2 \hat{e}_2 + h_3 \hat{e}_3 = [h_1, h_2, h_3], \tag{3.5}$$

as follows:

$$H_1 = h_1 - h_2, \;\; H_2 = h_2 - h_3, \;\; H_3 = h_2 + h_3. \tag{3.6}$$

For an so(6) irrep, its highest weight is

$$\Lambda = (r_1^\dagger)^{a_1} (s_1^\dagger)^{a_2} (\bar{s}_4^\dagger)^{a_3} |0>, \tag{3.7}$$

where $a_{1,2,3}$ are non-negative integers. Action of the so(6) Cartan generators in the Dynkin basis on it yields

$$\vec{H}\Lambda = (a_1, a_2, a_3)\Lambda, \tag{3.8}$$

and in the orthonormal basis

$$\vec{h}\Lambda = [b_1, b_2, b_3]\Lambda, \tag{3.9}$$

where

$$\begin{aligned} b_1 &= \frac{1}{2}(a_3 + a_2) + a_1, \\ b_2 &= \frac{1}{2}(a_3 + a_2), \\ b_3 &= \frac{1}{2}(a_3 - a_2). \end{aligned} \tag{3.10}$$

Inside the so(6) generators, the generators $T_{1,6}^\pm$ and $H_{1,6}$ form the so(4) Lie subalgebra and the generator h_3 is the generator of so(2) subalgebra. The other generators $T_{2,3,4,5}^\pm$ lie outside the subalgebra so(4) $\times$ so(2) and they are used to construct the Kostant operator of the quotient so(6)/so(4) $\times$ so(2).

3.2 Kernel solutions of the Kostant operator

To construct the Kostant operator of the quotient so(6)/so(4) $\times$ so(2), the gamma matrices used here are

$$\begin{aligned} \gamma_2^\pm &= \frac{1}{2}(\Gamma_1 \pm i\Gamma_2), \qquad & \gamma_3^\pm &= \frac{1}{2}(\Gamma_3 \pm i\Gamma_4), \\ \gamma_4^\pm &= \frac{1}{2}(\Gamma_5 \pm i\Gamma_6), \qquad & \gamma_5^\pm &= \frac{1}{2}(\Gamma_7 \pm i\Gamma_8). \end{aligned} \tag{3.11}$$

From the commutator of the generators of the quotient,

$$\begin{aligned} [T_2^+, T_2^-] &= h_2 - h_3, \qquad & [T_3^+, T_3^-] &= h_2 + h_3, \\ [T_4^+, T_4^-] &= h_1 - h_3, \qquad & [T_5^+, T_5^-] &= h_1 + h_3, \end{aligned} \tag{3.12}$$

the generators $T_{a=2,3,4,5}^\pm$ are not generated. The structure constants associated with these transformations are zero. Hence, the Kostant operator is just

$$\not{K} = \sum_{a=2}^{5} (\gamma_a^+ T_a^- + \gamma_a^- T_a^+). \tag{3.13}$$

A vector space of the Kostant operator is $\psi_\Lambda^\pm \equiv |\pm\pm\pm\pm> \otimes V_\Lambda$. Here, V_Λ is the vector space of the so(6) irrep with its highest weight Λ. For the kernel solutions

$$\not{K}\psi^{\pm}_{\lambda_i} = 0, \tag{3.14}$$

where λ_i is a weight in the vector space V_Λ. It is noted that the derivation of the kernel solutions $\psi^+_{\lambda_3}$ and $\psi^+_{\lambda_8}$ in this quotient is not straightforward as the one in su(5)/su(4)×u(1). At first glance, the following two equations,

$$\begin{aligned}(T_2^+ + T_3^- - T_4^- + T_5^+)\psi^+_{\lambda_3} &= 0,\\ (T_2^- - T_3^+ - T_4^+ - T_5^-)\psi^+_{\lambda_8} &= 0,\end{aligned} \tag{3.15}$$

have kernel solutions as follows:

$$\begin{aligned}\psi^+_{\lambda_3} &= |++-+>|0>,\\ \psi^+_{\lambda_8} &= |+--->|0>.\end{aligned} \tag{3.16}$$

These solutions are true only when $a_1 = a_2 = a_3 = 0$. We fix this problem by twisting their spinor states and obtain the general kernel solutions in the positive spinor space as follows:

$$\begin{aligned}
\psi^+_{\lambda_1} &= |++++>(r_1^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>,\\
\psi^+_{\lambda_2} &= |+++->(r_1^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\\
\psi^+_{\lambda_3} &= |+--->(r_2^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>,\\
\psi^+_{\lambda_4} &= |++-->(r_3^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>,\\
\psi^+_{\lambda_5} &= |+-++>(\bar{r}_1^\dagger)^{a_1}(s_3^\dagger)^{a_2}(\bar{s}_2^\dagger)^{a_3}|0>,\\
\psi^+_{\lambda_6} &= |+-+->(\bar{r}_1^\dagger)^{a_1}(s_4^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}|0>,\\
\psi^+_{\lambda_7} &= |+--+>(\bar{r}_3^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\\
\psi^+_{\lambda_8} &= |++-+>(\bar{r}_2^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,
\end{aligned} \tag{3.17}$$

and in the negative spinor space as follows:

$$\begin{aligned}
\psi^-_{\lambda'_1} &= |-+++>(r_2^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_2^\dagger)^{a_3}|0>,\\
\psi^-_{\lambda'_2} &= |-++->(\bar{r}_2^\dagger)^{a_1}(s_4^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\\
\psi^-_{\lambda'_3} &= |-+-+>(\bar{r}_1^\dagger)^{a_1}(s_4^\dagger)^{a_2}(\bar{s}_2^\dagger)^{a_3}|0>,\\
\psi^-_{\lambda'_4} &= |-+-->(\bar{r}_1^\dagger)^{a_1}(s_3^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}|0>,\\
\psi^-_{\lambda'_5} &= |--++>(r_2^\dagger)^{a_1}(s_3^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>,\\
\psi^-_{\lambda'_6} &= |--+->(\bar{r}_2^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_1^\dagger)^{a_3}|0>,\\
\psi^-_{\lambda'_7} &= |---+>(r_1^\dagger)^{a_1}(s_1^\dagger)^{a_2}(\bar{s}_3^\dagger)^{a_3}|0>,\\
\psi^-_{\lambda'_8} &= |---->(r_1^\dagger)^{a_1}(s_2^\dagger)^{a_2}(\bar{s}_4^\dagger)^{a_3}|0>.
\end{aligned} \tag{3.18}$$

To get the kernel solutions in terms of so(4) × so(2), it needs to act on them by the Cartan generators, which in the Dynkin basis are

$$
\begin{aligned}
D_1 &= h_1 - h_2 + \frac{1}{2}\left(f^4_{+-1}[\gamma_4^+,\gamma_4^-] + f^5_{+-1}[\gamma_5^+,\gamma_5^-] - f^2_{+-2}[\gamma_2^+,\gamma_2^-] - f^3_{+-2}[\gamma_3^+,\gamma_3^-]\right)\\
&= H_1 + \frac{1}{2}(\mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1} + \sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1} - \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3 - \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3),\\
D_2 &= h_1 + h_2 + \frac{1}{2}\left(f^4_{+-1}[\gamma_4^+,\gamma_4^-] + f^5_{+-1}[\gamma_5^+,\gamma_5^-] + f^2_{+-2}[\gamma_2^+,\gamma_2^-] + f^3_{+-2}[\gamma_3^+,\gamma_3^-]\right)\\
&= H_1 + H_2 + H_3 + \frac{1}{2}(\mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1} + \sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\\
&\qquad + \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3),\\
D_3 &= \frac{1}{2}h_3 + \frac{1}{4}\left(f^2_{+-3}[\gamma_2^+,\gamma_2^-] + f^3_{+-3}[\gamma_3^+,\gamma_3^-] + f^4_{+-3}[\gamma_4^+,\gamma_4^-] + f^5_{+-3}[\gamma_5^+,\gamma_5^-]\right)\\
&= \frac{1}{2}h_3 - \frac{1}{4}(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma_3 - \mathbb{1}\otimes\mathbb{1}\otimes\sigma_3\otimes\sigma_3 + \mathbb{1}\otimes\sigma_3\otimes\sigma_3\otimes\mathbb{1}\\
&\qquad - \sigma_3\otimes\sigma_3\otimes\mathbb{1}\otimes\mathbb{1}).
\end{aligned}
\tag{3.19}
$$

The structure constants in (3.19) are read directly from (3.12). The generators D_1 and D_2 are the Cartan generators of so(4) and the generator D_3 is the Cartan generator of so(2). When the Cartan generators act on the kernel solutions, they give

$$
\begin{aligned}
(D_1,D_2;D_3)\psi^+_{\lambda_1} &= (a_1, a_1+a_2+a_3+2; b_3/2)\psi^+_{\lambda_1},\\
(D_1,D_2;D_3)\psi^+_{\lambda_2} &= (a_1+a_2+a_3+2, a_1; -b_3/2)\psi^+_{\lambda_2},\\
(D_1,D_2;D_3)\psi^+_{\lambda_3} &= (-a_1, a_1+a_2+a_3; b_3/2)\psi^+_{\lambda_3},\\
(D_1,D_2;D_3)\psi^+_{\lambda_4} &= (a_2, a_3; (b_1+2)/2)\psi^+_{\lambda_4},\\
(D_1,D_2;D_3)\psi^+_{\lambda_5} &= (-a_1-a_2-a_3-2, -a_1; -b_3/2)\psi^+_{\lambda_5},\\
(D_1,D_2;D_3)\psi^+_{\lambda_6} &= (-a_1, -a_1-a_2-a_3-2; b_3/2)\psi^+_{\lambda_6},\\
(D_1,D_2;D_3)\psi^+_{\lambda_7} &= (a_3, a_2; -(b_1+2)/2)\psi^+_{\lambda_7},\\
(D_1,D_2;D_3)\psi^+_{\lambda_8} &= (a_1+a_2+a_3, -a_1; -b_3/2)\psi^+_{\lambda_8},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_1} &= (-a_1-a_3-1, a_1+a_2+1; -(b_2+1)/2)\psi^-_{\lambda'_1},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_2} &= (a_1+a_3+1, -a_1-a_2-1; -(b_2+1)/2)\psi^-_{\lambda'_2},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_3} &= (-a_1-a_3-1, -a_1-a_2-1; -(b_2+1)/2)\psi^-_{\lambda'_3},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_4} &= (-a_1-a_2-1, -a_1-a_3-1; (b_2+1)/2)\psi^-_{\lambda'_4},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_5} &= (-a_1-a_2-1, a_1+a_3+1; (b_2+1)/2)\psi^-_{\lambda'_5},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_6} &= (a_1+a_2+1, -a_1-a_3-1; (b_2+1)/2)\psi^-_{\lambda'_6},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_7} &= (a_1+a_3+1, a_1+a_2+1; -(b_2+1)/2)\psi^-_{\lambda'_7},\\
(D_1,D_2;D_3)\psi^-_{\lambda'_8} &= (a_1+a_2+1, a_1+a_3+1; (b_2+1)/2)\psi^-_{\lambda'_8}.
\end{aligned}
\tag{3.20}
$$

In case $a_1 = a_2 = a_3 = 0$, the kernel solutions (3.20) can be grouped in terms of so(4) dimensions as follows:

$$(\mathbf{1},\mathbf{1})_1 \equiv \psi^+_{\lambda_4} \sim (\ 0,\ 0)_1,$$

$$(\mathbf{1},\mathbf{3})_0 \equiv \begin{cases} \psi^+_{\lambda_1} \sim (\ 0,\ 2)_0 \\ \psi^+_{\lambda_8} \sim (\ 0,\ 0)_0 \\ \psi^+_{\lambda_6} \sim (\ 0,-2)_0 \end{cases} , \qquad (\mathbf{3},\mathbf{1})_0 \equiv \begin{cases} \psi^+_{\lambda_2} \sim (\ 2,\ 0)_0 \\ \psi^+_{\lambda_3} \sim (\ 0,\ 0)_0 \\ \psi^+_{\lambda_5} \sim (-2,\ 0)_0 \end{cases} ,$$

$$(\mathbf{1},\mathbf{1})_{-1} \equiv \psi^+_{\lambda_7} \sim (\ 0,\ 0)_{-1},$$

$$(\mathbf{2},\mathbf{2})_{1/2} \equiv \begin{cases} \psi^-_{\lambda'_8} \sim (\ 1,\ 1)_{1/2} \\ \psi^-_{\lambda'_6} \sim (\ 1,-1)_{1/2} \\ \psi^-_{\lambda'_5} \sim (-1,\ 1)_{1/2} \\ \psi^-_{\lambda'_4} \sim (-1,-1)_{1/2} \end{cases} , \qquad (\mathbf{2},\mathbf{2})_{-1/2} \equiv \begin{cases} \psi^-_{\lambda'_7} \sim (\ 1,\ 1)_{-1/2} \\ \psi^-_{\lambda'_2} \sim (\ 1,-1)_{-1/2} \\ \psi^-_{\lambda'_1} \sim (-1,\ 1)_{-1/2} \\ \psi^-_{\lambda'_3} \sim (-1,-1)_{-1/2} \end{cases} . \qquad (3.21)$$

Since the Dynkin labels $a_{1,2,3}$ are non-negative, the direct sum of the so(4) highest weights,

$$\psi^+_{\lambda_4} \oplus \psi^+_{\lambda_1} \oplus \psi^+_{\lambda_2} \oplus \psi^+_{\lambda_7} \oplus \psi^-_{\lambda'_8} \oplus \psi^-_{\lambda'_7}, \qquad (3.22)$$

or in terms of its Dynkin labels,

$$(a_2, a_3)_{(b_1+2)/2} \oplus (a_1, a_1 + a_2 + a_3 + 2)_{b_3/2} \oplus (a_1 + a_2 + a_3 + 2, a_1)_{-b_3/2} \oplus (a_3, a_2)_{-(b_1+2)/2}$$
$$(a_1 + a_2 + 1, a_1 + a_3 + 1)_{(b_2+1)/2} \oplus (a_1 + a_3 + 1, a_1 + a_2 + 1)_{-(b_2+1)/2}, \qquad (3.23)$$

forms the Euler number multiplet.

4. Remarks

Kernel solutions of the Kostant operator of the 8-dimensional quotients can be easily determined by the quantum mechanical method. The Euler number multiplet obtained in terms of the diagonal subalgebra is the direct sum of the highest weights of the kernel solutions, which appear only once. The Euler number multiplets presented in this paper are exactly the same as derived by using the Weyl group elements of su(5) and so(6) that are not in their subalgebras [3]. The lowest line of the Euler number multiplet for the quotient su(5)/su(4) × u(1) is

$$1_1 \oplus 4_{1/2} \oplus 6_0 \oplus 4_{-1/2} \oplus 1_{-1}, \qquad (4.1)$$

and for the quotient so(6)/so(4) × so(2)

$$(1,1)_1 \oplus (2,2)_{1/2} \oplus (3,1)_0 \oplus (1,3)_0 \oplus (2,2)_{-1/2} \oplus (1,1)_{-1}. \qquad (4.2)$$

There are many possible ways to interpret these Euler number multiplets. If so(2), which is locally isomorphic to u(1), is viewed as a light-cone little group of ISO(3, 1), then they correspond to degrees of freedom of $N = 4$ Yang-Mills massless representation in 3+1 space-time. Similarly, if so(6), which is locally isomorphic to su(4), is viewed as a light-cone little group of ISO(7, 1), then they correspond to degrees of freedom of the massless representation in 7+1 space-time. Lastly, if so(6) × so(2), which is locally isomorphic to su(4) × u(1), is viewed as a subgroup of SO(6, 2), the anti-de Sitter group and the conformal group, then they correspond to the massless representations in the 6+1 and 5+1 space-time, respectively.

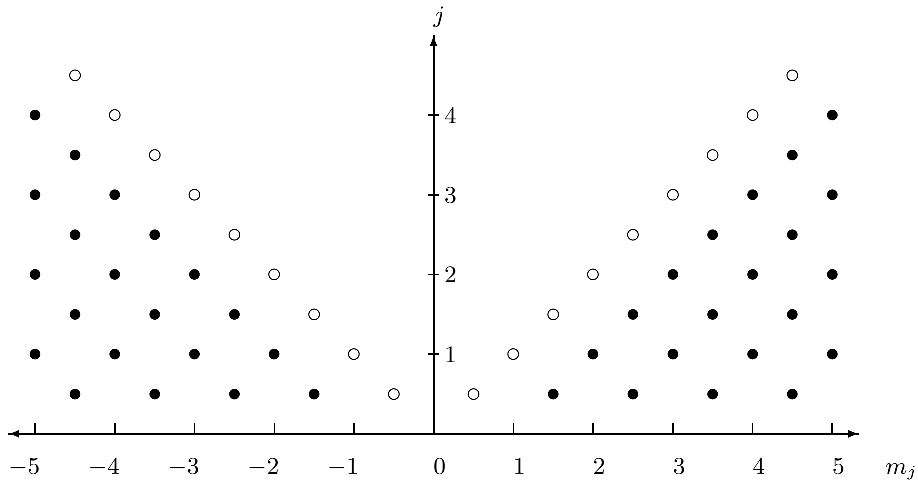

Figure 3: The so(2, 1) weight diagram associated with the discrete representations. Open and solid circles along a horizontal line are the so(2, 1) weights of a j representation. In each horizontal line, only the open circles, the lowest weight in the V_j^+ and the highest weight in the V_j^-, are the non-trivial kernel solutions of the Kostant operator of the quotient so(2, 1)/so(2).

The Kostant operator can be extended from a compact Lie algebra to a non-compact one. Methods to construct the Kostant operator are similar in both the compact and the non-compact Lie algebras. The simplest quotient of the non-compact Lie algebras is so(2, 1)/so(2). For details of the so(2, 1) generators, commutation relations and representations, see [8]. The Kostant operator,

$$\not{K} = \sigma^+ T^- + \sigma^- T^+, \tag{4.3}$$

acts on its vector space $\psi_j^\pm = |\pm> |j, m_j>$, where in each discrete representation j, $|m_j| \geq j$. Its non-trivial kernel solutions, whose corresponding states are shown as open circles in figure 3, are

$$\psi_j^+ = |+> |j, -j>, \qquad \psi_j^- = |-> |j, j> . \tag{4.4}$$

These solutions are similar to the kernel solutions of su(2)/u(1). Another interesting non-compact Lie algebra is so(4, 2), the conformal group in the 3+1 space-time, whose spinors are twistors [9]. For the case so(4, 2)/so(4)×so(2), it is found that its lowest line of the Euler number multiplet for the discrete representation is similar to that of so(6)/so(4) × so(2).

Finally, it is just a hope that the constructions of the Kostant operators and the derivations of their kernel solutions presented here will be useful when someone wants to oxidize a low-dimensional field theory to a higher-dimensional one or to reduce a high-dimensional field theory to a lower-dimensional one [10], or even to connect the Kostant operators to the string theory [11, 12].

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Computer simulation of the electron energy levels in a tetrahedral-shaped quantum dot

Chalongrat Daengngam and Teparksorn Pengpan

Department of Physics, Prince of Songkla University, PO Box 3, Hat Yai 90112, Thailand

E-mail: chalongrat.d@psu.ac.th and teparksorn.p@psu.ac.th

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Abstract

We report on a three-month undergraduate research project to compute energy levels and their corresponding wavefunctions of an electron confined in a tetrahedral-shaped quantum dot heterostructure. A typical example of such a quantum system is an InAs tetrahedral-shaped quantum dot embedded in a cuboid GaAs matrix. For the simulation we used the Schrödinger equation in three-dimensional Cartesian space. After discretizing the Schrödinger equation by using the finite volume method, the resulting large-scale eigenvalue matrix is solved for eigenvalues and eigenvectors.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Semiconductor quantum dots (QD) have recently become greatly interesting because of their unique electronic characteristics and the possibility of fabricating realistic quantum dots in laboratories by applying nanotechnology. Quantum dots have discrete energy levels like those of a single atom. Therefore, novel optoelectronic devices such as the quantum dot laser [1] and the quantum dot infrared photodetector (QDIP) [2] have become feasible.

Several studies of semiconductor quantum dots were focused on both experimental and theoretical aspects. Experiments indicate that quantum dots can have several shapes depending on the fabrication methodology. Self-assembled quantum dots usually have a pyramidal shape [3] or an island-like shape [4]. However, quantum dots grown in a specific recess, for example, in a tetrahedral-shaped confinement, have a shape similar to the recess shape [5]. A typical example is an InAs tetrahedral-shaped quantum dot embedded in a cuboid GaAs matrix. The shape of quantum dots is an important property, since calculations have demonstrated that the energy levels of an electron confined in a quantum dot depend on its internal shape.

In our project, we used the finite volume method to discretize the Schrödinger equation for an electron confined in a one-dimensional (1D) quantum well, a two-dimensional (2D) quantum wire, and in three-dimensional (3D) cubic-shaped and tetrahedral-shaped quantum dots. This method transforms the differential form of the Schrödinger equation into a large-scale matrix eigenvalue problem. The eigenvalues and eigenvectors—the energy levels and wavefunctions of the electron, respectively—are then calculated. We show in detail how to discretize the Schrödinger equation for the electron in a 3D tetrahedral-shaped quantum dot by assuming that the effective mass of the electron is constant. Resulting examples are presented in the form of energy levels and wavefunctions.

2. The discretization of the Schrödinger equation

Consider an electron confined in a 3D semiconductor tetrahedral-shaped quantum dot. In a real situation, the electron's effective mass m^* depends both on its position and its energy, which can be derived from the eight-band **k·p** analysis and the effective mass theory [6]. We simplify this situation by assuming that the electron's effective mass is constant

$$m^*(x, y, z, E) = m_c^*. \tag{1}$$

For convenience, we define the notation

$$\varepsilon \equiv -\frac{\hbar^2}{2m_c^*}, \tag{2}$$

where $\hbar$ is the reduced Plank constant. The Schrödinger equation, to be solved for the relevant energy levels and their corresponding wavefunctions, is

$$\varepsilon\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)+V(x, y, z)\Psi = E\Psi, \tag{3}$$

where E is an energy and Ψ is a wavefunction. The tetrahedral-shaped potential $V(x, y, z)$ is given by

$$V(x, y, z) = \begin{cases} V_{\text{inside}}, & x+y+z \leqslant 2\ell \\ & \text{and } x-y-z \leqslant 0 \\ & \text{and } -x+y-z \leqslant 0 \\ & \text{and} -x-y+z \leqslant 0 \\ V_{\text{outside}}, & \text{otherwise.} \end{cases} \tag{4}$$

In this paper, we also assume that V_{inside} and V_{outside} are constant. As shown in figure 1, the tetrahedral-shaped potential is properly placed in a cuboid matrix having the size $\ell \times \ell \times \ell$. So, we can simplify the computations using a uniform mesh in Cartesian coordinates.

The Schrödinger equation is then calculated by the finite volume method over a control volume rather than at a single node [7]. As shown in figure 2, the control volume surrounded by grid nodes has limits from x to $x+\Delta x$, y to $y+\Delta y$ and z to $z+\Delta z$ and its centre is located at a node (i, j, k). For the first term on the left-hand side of (3), integration over the control volume yields

$$\int_z^{z+\Delta z}\int_y^{y+\Delta y}\int_x^{x+\Delta x} \varepsilon\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \varepsilon\left\{\left(\frac{\partial\Psi}{\partial x}\right)_{x+\Delta x}-\left(\frac{\partial\Psi}{\partial x}\right)_x\right\}\Delta y\Delta z$$
$$+\varepsilon\left\{\left(\frac{\partial\Psi}{\partial y}\right)_{y+\Delta y}-\left(\frac{\partial\Psi}{\partial y}\right)_y\right\}\Delta x\Delta z+\varepsilon\left\{\left(\frac{\partial\Psi}{\partial z}\right)_{z+\Delta z}-\left(\frac{\partial\Psi}{\partial z}\right)_z\right\}\Delta x\Delta y. \tag{5}$$

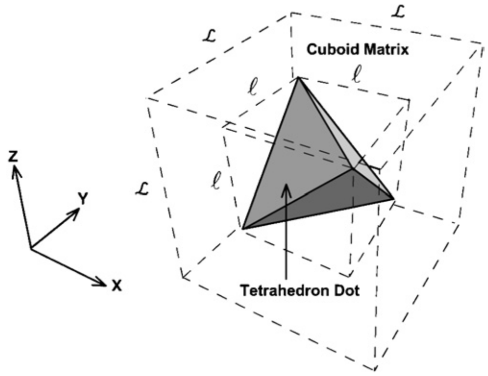

Figure 1. The structure model of the tetrahedral-shaped quantum dot.

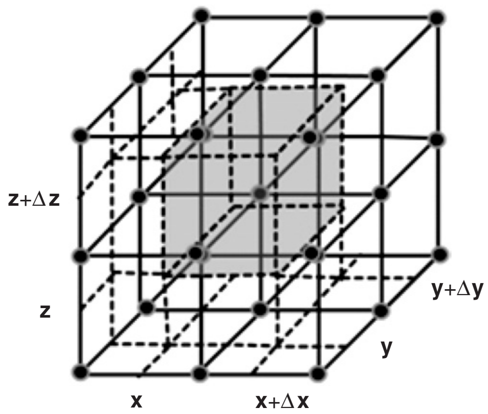

Figure 2. The control volume centred at a node (i, j, k).

For the second term, integration yields

$$\int_{z}^{z+\Delta z}\int_{y}^{y+\Delta y}\int_{x}^{x+\Delta x} V(x, y, z)\Psi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \Psi_{(i,j,k)}\bar{V}_{(i,j,k)}\Delta x\Delta y\Delta z, \qquad (6)$$

where Ψ is supposed to be constant inside the control volume and equal to $\Psi_{(i,j,k)}$, and $\bar{V}_{(i,j,k)}$ denotes a volume average of $V(x, y, z)$. To get $\bar{V}_{(i,j,k)}$, it is necessary to consider five types of nodes located in five different parts of a tetrahedron as shown in figure 3:

- For the first type, the node is inside the tetrahedron. The whole control volume is also inside the tetrahedron. Then

$$\bar{V}_{(i,j,k)} = V_{\text{inside}}. \qquad (7)$$

- For the second one, the node is located on a surface of the tetrahedron. The ratio of the control volume inside to outside is 1:1,

$$\bar{V}_{(i,j,k)} = \tfrac{1}{2}V_{\text{inside}} + \tfrac{1}{2}V_{\text{outside}}. \qquad (8)$$

- For the third one, the node lies on an edge of the tetrahedron. The ratio of the control volume inside to outside is 5:19,

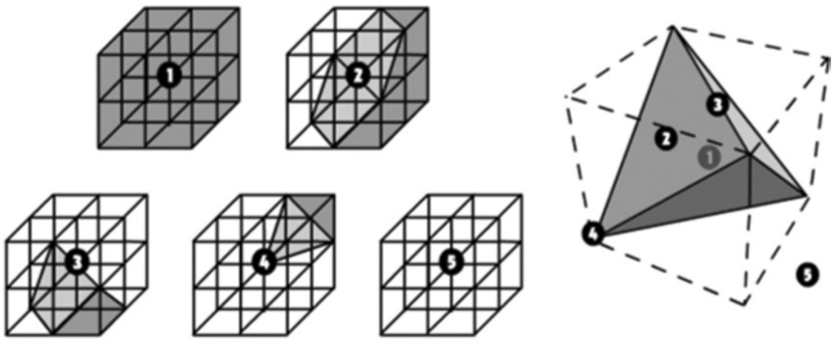

Figure 3. Five types of nodes located in five different parts of a tetrahedron.

$$\bar{V}_{(i,j,k)} = \tfrac{5}{24} V_{\text{inside}} + \tfrac{19}{24} V_{\text{outside}}. \tag{9}$$

- For the fourth one, the node is situated at the corner of the tetrahedron. The ratio of the control volume inside to outside is 1:15,

$$\bar{V}_{(i,j,k)} = \tfrac{1}{16} V_{\text{inside}} + \tfrac{15}{16} V_{\text{outside}}. \tag{10}$$

- For the last one, the node is positioned outside the tetrahedron. The whole control volume is also outside the tetrahedron:

$$\bar{V}_{(i,j,k)} = V_{\text{outside}}. \tag{11}$$

For the right-hand side of equation (3), integration yields

$$\int_{z}^{z+\Delta z}\int_{y}^{y+\Delta y}\int_{x}^{x+\Delta x} E\Psi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = E\Psi_{(i,j,k)}\Delta x\Delta y\Delta z. \tag{12}$$

Up to here, we had

$$\begin{aligned} &\varepsilon\left\{\left(\frac{\partial\Psi}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial\Psi}{\partial x}\right)_{x}\right\}\Delta y\Delta z + \varepsilon\left\{\left(\frac{\partial\Psi}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial\Psi}{\partial y}\right)_{y}\right\}\Delta x\Delta z \\ &\quad + \varepsilon\left\{\left(\frac{\partial\Psi}{\partial z}\right)_{z+\Delta z} - \left(\frac{\partial\Psi}{\partial z}\right)_{z}\right\}\Delta x\Delta y \\ &\quad + \bar{V}_{(i,j,k)}\Psi_{(i,j,k)}\Delta x\Delta y\Delta z = E\Psi_{(i,j,k)}\Delta x\Delta y\Delta z. \end{aligned} \tag{13}$$

The gradients at surfaces of the control volume in equation (13) are approximated at first order by taking the difference of wavefunctions between adjacent control volumes as follows:

$$\left(\frac{\partial\Psi}{\partial x}\right)_{x+\Delta x} \simeq \frac{\Psi_{(i+1,j,k)} - \Psi_{(i,j,k)}}{\Delta x}, \tag{14a}$$

$$\left(\frac{\partial\Psi}{\partial y}\right)_{y+\Delta y} \simeq \frac{\Psi_{(i,j+1,k)} - \Psi_{(i,j,k)}}{\Delta y}, \tag{14b}$$

$$\left(\frac{\partial\Psi}{\partial z}\right)_{z+\Delta z} \simeq \frac{\Psi_{(i,j,k+1)} - \Psi_{(i,j,k)}}{\Delta z}, \tag{14c}$$

$$\left(\frac{\partial\Psi}{\partial x}\right)_{x} \simeq \frac{\Psi_{(i,j,k)} - \Psi_{(i-1,j,k)}}{\Delta x}, \tag{14d}$$

$$\left(\frac{\partial\Psi}{\partial y}\right)_y \simeq \frac{\Psi_{(i,j,k)} - \Psi_{(i,j-1,k)}}{\Delta y}, \tag{14e}$$

$$\left(\frac{\partial\Psi}{\partial z}\right)_z \simeq \frac{\Psi_{(i,j,k)} - \Psi_{(i,j,k-1)}}{\Delta z}. \tag{14f}$$

For a uniform mesh in 3D Cartesian coordinates, we set

$$\Delta x = \Delta y = \Delta z = \Delta h. \tag{15}$$

With this simplification, the Schrödinger equation (13) finally becomes

$$\beta(\Psi_{(i+1,j,k)} + \Psi_{(i-1,j,k)} + \Psi_{(i,j+1,k)} + \Psi_{(i,j-1,k)} + \Psi_{(i,j,k+1)} + \Psi_{(i,j,k-1)}) \\ + \alpha_{(i,j,k)}\Psi_{(i,j,k)} = E\Psi_{(i,j,k)}, \tag{16}$$

where

$$\beta \equiv \frac{\varepsilon}{(\Delta h)^2}, \tag{17}$$

and

$$\alpha_{(i,j,k)} \equiv -\frac{6\varepsilon}{(\Delta h)^2} + \bar{V}_{(i,j,k)}. \tag{18}$$

By using Kronecker's delta-function $\delta_{a,b}$ defined by

$$\delta_{a,b} = \begin{cases} 1, & a = b \\ 0, & a \neq b, \end{cases} \tag{19}$$

equation (16) can be written as

$$[\beta(\delta_{(i,j,k),(i+1,j,k)} + \delta_{(i,j,k),(i-1,j,k)} + \delta_{(i,j,k),(i,j+1,k)} + \delta_{(i,j,k),(i,j-1,k)} + \delta_{(i,j,k),(i,j,k+1)} \\ + \delta_{(i,j,k),(i,j,k-1)}) + \alpha_{(i,j,k)}\delta_{(i,j,k),(i,j,k)}]\Psi_{(i,j,k)} = E\Psi_{(i,j,k)}. \tag{20}$$

In the computational box, the number of nodes on each side is set to be $N+2$, starting from 0 to $N+1$, and the wavefunctions at nodes on surfaces of the box are set to be zero. Hence, there are only N^3 unknown wavefunctions. For convenience in constructing a matrix equation, the triple (i, j, k) is transformed to be a number as follows:

$$(i, j, k) \to N^2(i-1) + N(j-1) + k, i, j, k = 1, 2, 3, \ldots, N. \tag{21}$$

Then, the discretized Schrödinger equation (20) can be written in matrix form

$$[H][\Psi] = E[\Psi], \tag{22}$$

where the column matrix $[\Psi]$ is the matrix of unknown wavefunctions $\Psi_{N^2(i-1)+N(j-1)+k}$ and the square matrix $[H]$ is the sparse or Hamiltonian matrix whose all nonzero entries locate along seven diagonal lines as follows:

$$\begin{aligned}
&H_{N^2(i-1)+N(j-1)+k,N^2(i)+N(j-1)+k} = \beta, && i = 1 \text{ to } N-1, && j,k = 1 \text{ to } N,\\
&H_{N^2(i-1)+N(j-1)+k,N^2(i-2)+N(j-1)+k} = \beta, && i = 2 \text{ to } N, && j,k = 1 \text{ to } N,\\
&H_{N^2(i-1)+N(j-1)+k,N^2(i-1)+N(j)+k} = \beta, && j = 1 \text{ to } N-1, && i,k = 1 \text{ to } N,\\
&H_{N^2(i-1)+N(j-1)+k,N^2(i-1)+N(j-2)+k} = \beta, && j = 2 \text{ to } N, && i,k = 1 \text{ to } N,\\
&H_{N^2(i-1)+N(j-1)+k,N^2(i-1)+N(j-1)+k+1} = \beta, && k = 1 \text{ to } N-1, && i,j = 1 \text{ to } N,\\
&H_{N^2(i-1)+N(j-1)+k,N^2(i-1)+N(j-1)+k-1} = \beta, && k = 2 \text{ to } N, && i,j = 1 \text{ to } N,\\
&H_{N^2(i-1)+N(j-1)+k,N^2(i-1)+N(j-1)+k} = \alpha_{(i,j,k)}, && i,j,k = 1 \text{ to } N.
\end{aligned} \tag{23}$$

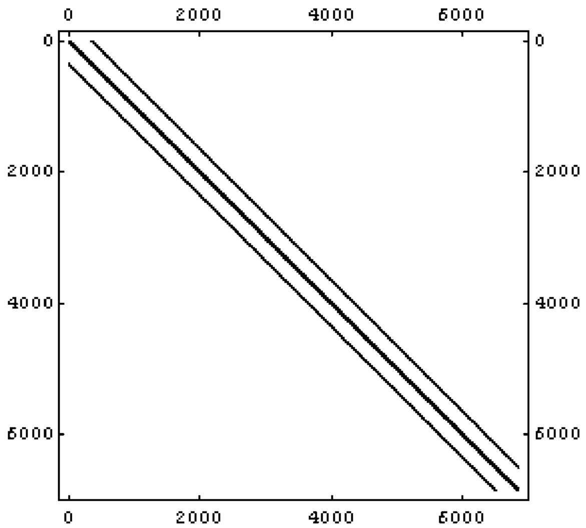

Figure 4. The sparsity pattern of the matrix $[H]$. The blank space represents zero entries and the diagonal lines represent nonzero entries.

The number of eigenvalues of $[H]$ is also equal to N^3, the number of unknown wavefunctions. However, the eigenvalues that can be energy levels of quantum dots are those which have values lower than V_{outside}.

To obtain the energy levels as well as the corresponding wavefunctions of the electron, the eigenvalues and eigenvectors of the matrix $[H]$ are calculated by a specially developed modification of the existing Mathematica standard algorithm. The main algorithm of the simulation consists of the following steps: (i) input all necessary initial data such as sizes of the computational box and the tetrahedron quantum dot, the values of inside and outside potentials, and the number of nodes; (ii) build up the sparse matrix $[H]$; (iii) compute eigenvalues of $[H]$; (iv) select the eigenvalues that are lower than V_{outside}; (v) compute and plot 3D graphics of the corresponding eigenvectors.

3. Numerical results

In this section, we present some of the numerical results. In our simulation, all physical parameters are in the same order as in [8]. The electron effective mass in the quantum dot is presumably equal to 0.028 of the electron rest mass $m_0 = 9.1 \times 10^{-31}$ kg. For the InAs tetrahedral-shaped QD in the GaAs matrix, the confinement potential is 0.77 eV. Therefore, the value of $V_{\text{inside}} = -0.77$ eV and $V_{\text{outside}} = 0$ eV.

We firstly generate a computational box of $20 \times 20 \times 20$ nm^3 with at its centre a tetrahedral-shaped quantum dot, which is properly imbedded in a cuboid matrix of $14 \times 14 \times 14$ nm^3. Due to the PC's memory limit, the number of nodes used on each side of the computational box has been restricted to 21, starting from 0 to 20 (including 20). As the values of wavefunctions at the surfaces of the computational box are set to zero, the number of unknown wavefunctions on each side of the computational box is equal to the number of nodes minus 2, thus 19. Although this seems to be fairly small, the number of unknown wavefunctions is still $19^3 = 6859$. Nevertheless, the computational program generates the sparse matrix $[H]$ of dimension 6859×6859 as shown in figure 4.

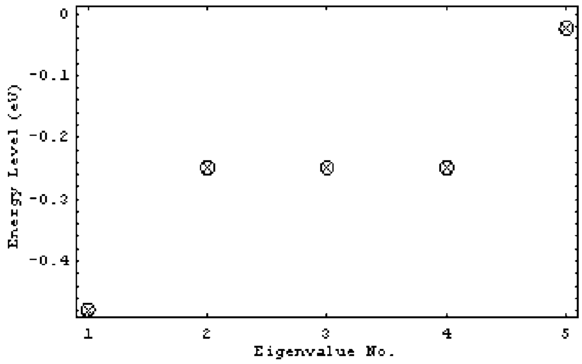

Figure 5. Distribution of the negative eigenvalues of the matrix $[H]$.

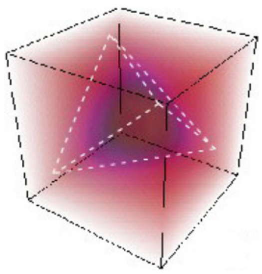

Figure 6. The wavefunction of the ground state.

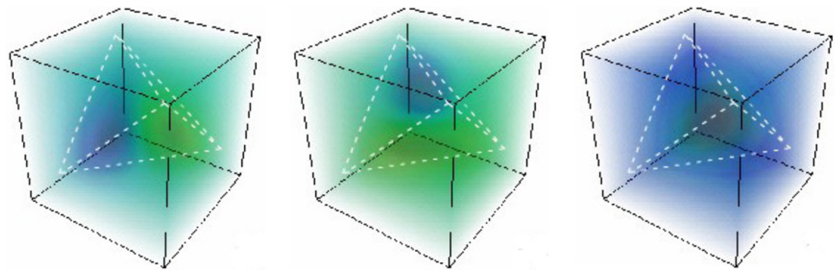

Figure 7. Threefold degenerate wavefunctions of the first-excited state.

In the next step, the eigenvalues of $[H]$ have been computed. With the settings, there exist only five eigenvalues, which are all negative, and their distribution is shown in figure 5. The rest, which is not shown, is positive. The smallest eigenvalue, approximately equal to -0.48 eV, is the energy level of the ground state of the electron confined in the tetrahedral-shaped quantum dot. The next three values, all approximately equal to -0.25 eV, are the energy levels of the first-excited state. Their energy levels are threefold degenerated. All these degeneracies are due to the symmetry of the tetrahedron. The last one, approximately equal to -0.02 eV, is the energy level of the second-excited state.

In the last step, our computational program calculates the corresponding wavefunctions and plots the ground state, first-excited state and second-excited state wavefunctions as shown in figures 6, 7 and 8, respectively. Note that the wavefunctions of the first-excited state look similar to each other except for their alignment. By observing nodal areas that separate different shaded regions, the wavefunctions of the 3D tetrahedral-shaped quantum dot look

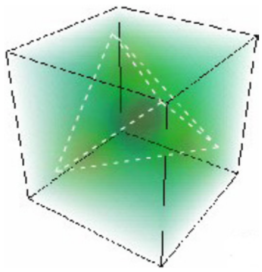

Figure 8. The wavefunction of the second-excited state.

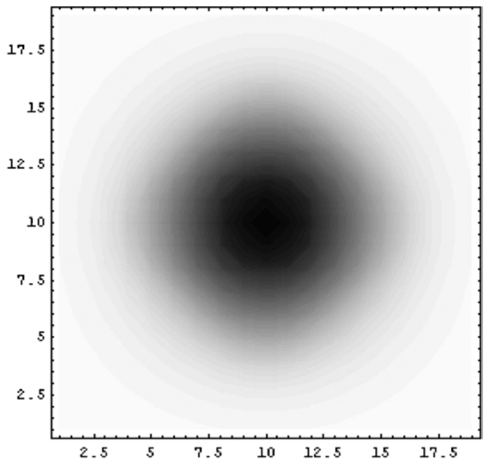

Figure 9. The cross section of the ground state wavefunction at $z = 10$ nm. The tetrahedral-shaped quantum dot is imbedded in the cuboid matrix ranging from 3 to 17 nm.

compatible with those of the 1D quantum well. For the ground state wavefunction, the shade variation from the centre to the surfaces of the tetrahedron corresponds to a decrease of amplitude of the wavefunction. A nodal area of the ground state wavefunction lies near the surfaces of the tetrahedron as shown in figure 9. For the first-excited state wavefunctions, there exists a nodal area that separates two different shaded regions as shown in figure 10. For the second-excited state wavefunction, when looking on each side, there exist two nodal areas between different shaded regions as shown in figure 11.

4. Discussions

It is hard to solve the Schrödinger equation analytically for an electron confined in a tetrahedral-shaped quantum dot, to derive its energy levels and corresponding wavefunctions. The numerical simulation presented here is a simple and an effective tool for deriving them. The finite volume method is used to discretize the Schrödinger equation. This discretization gives a large-scale eigenvalue matrix equation, which is solved by the Mathematica program.

The program has been tested by using realistic physical parameters of the InAs/GaAs quantum dot. The difference between the inside and outside potential of tetrahedron is 0.77 eV. Due to our hardware limits, we have set the number of nodes in each direction of the computational box equal to 21 and the wavefunctions at nodes on the surfaces of the

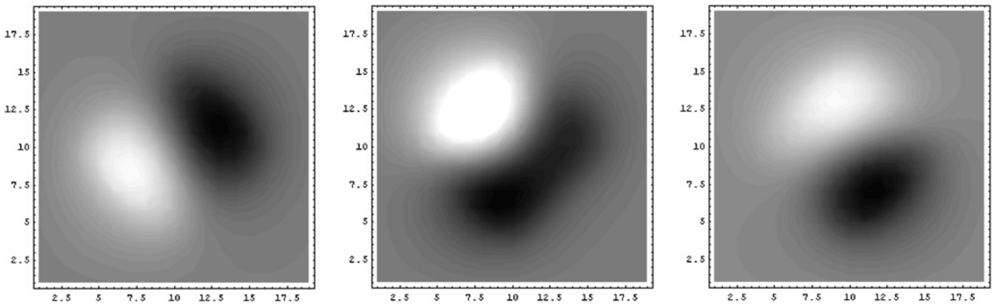

Figure 10. The cross section of the first-excited state wavefunctions at $z = 10$ nm, corresponding to figure 7 in the same order.

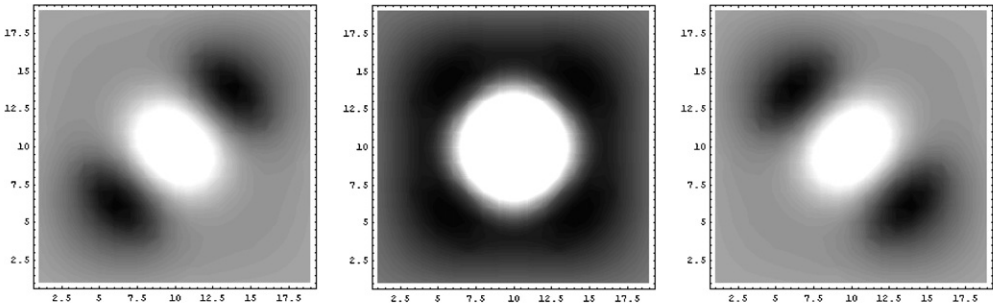

Figure 11. The cross section of the second-excited state wavefunction, from left to right, at $z = 8$, 10, 12 nm.

computational box equal to zero. From these settings, the number of unknown wavefunctions is just $19^3 = 6859$. It takes an amount of time for our program to get the eigenvalues and corresponding wavefunctions of the 6859×6859 matrix $[H]$. The resulting eigenvalues which can be the energy levels of the electron confined in the tetrahedral potential are those which have values less than the outside potential.

The ground state, first-excited state and second-excited state wavefunctions of the 3D tetrahedral-shaped quantum dot are plotted in the 3D graphics. The first-excited state wavefunctions are threefold degenerate due to the symmetry of the tetrahedron. Clearly, looking at the nodal areas between different shaded regions, the wavefunctions of the 3D tetrahedral-shaped quantum dot are compatible with those of the 1D quantum well.

Finally, it is noted that electron energy levels also depend on the size of the quantum dot and the electron effective mass. If the dot size increases, the electron energy level decreases. Conversely, if the dot size decreases, the electron energy level increases. Also, variation of the electron effective mass gives the same result as that of the dot size.

Acknowledgments

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