รายงานวิจัยฉบับสมบูรณ์

โครงการ : วิธีการประมาณค่าแบบซ้ำสำหรับจุดตรึงร่วมของกลุ่มนับได้ของตัวดำเนินการไม่เชิงเส้น

Iterative approximation methods for common fixed points of countable families of nonlinear operators

โดย ผศ.ดร.ระเบียน วังคีรี และคณะ

15 มีนาคม 2554

สัญญาเลขที่ TRG5280011

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ผู้วิจัย : **ผศ.ดร.ระเบียน วังคีรี**

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มหาวิทยาลัยนเรศวร

นักวิจัยที่ปรึกษา : **ศ.ดร.สมยศ พลับเที่ยง**

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์

มหาวิทยาลัยนเรศวร

สนับสนุนโดยสำนักงานคณะกรรมการการอุดมศึกษา และสำนักงานกองทุนสนับสนุนการวิจัย

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Abstract

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Project Title:	Iterative approximation methods for common fixed points of countable families of nonlinear operators
Investigator:	**Assit.**Dr.Rabian Wangkeeree Department of Mathematics, Faculty of Science, Naresuan University
Mentor:	Prof. Dr. Somyot Plubtieng Department of Mathematics, Faculty of Science, Naresuan University
E-mail Address:	rabianw@nu.ac.th (R. Wangkeeree) somyotp@nu.ac.th (S. Plubtieng)
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The purposes of this research are to introduce several new iterative approximation methods for approximating the common fixed points of the countable families of nonlinear mappings and solving many mathematical problems such as variational inequality problems, equilibrium problems and minimization problems in both Hilbert spaces and Banach spaces

Keywords: Iterative approximation methods, Uniformly convex Banach spaces, Nonexpansive mappings, Variational inequality, Equilibrium Problems, Hilbert space.

บทคัดย่อ

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ชื่อนักวิจัย : ผศ.ดร.ระเบียน วังคีรี
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร

นักวิจัยที่ปรึกษา: ศ. ดร.สมยศ พลับเที่ยง
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร

อีเมล์: rabianw@nu.ac.th (ระเบียน วังคีรี)
somyotp@nu.ac.th (สมยศ พลับเที่ยง)

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ในงานวิจัยนี้ ผู้วิจัยได้นำเสนอวิธีการประมาณค่าแบบซ้ำหลายๆแบบ เพื่อใช้สำหรับการประมาณค่าจุดตรึ่งร่วมของกลุ่มนับได้ของการส่งไม่เชิงเส้นแบบต่าง และเพื่อแก้ไขปัญหาในทางคณิตศาสตร์หลายแบบตัวอย่างเช่น ปัญหาอมสมการการแปรผัน ปัญหาเชิงดุลยภาพ และ ปัญหาค่าเหมาะสมที่สุดในทั้งปริภูมิฮิลเบิร์ต และ ปริภูมิบานาค

คำหลัก : วิธีการประมาณค่าแบบซ้ำ; ปริภูมิบานาคแบบเอกรูป; การส่งแบบไม่ขยาย; อมสมการการแปรผัน; ปัญหาดุลยภาพ; ปริภูมิฮิลเบิร์ต

บทนำ

ทฤษฎีจุดตรึง (fixed point theory) นับเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของการวิเคราะห์เชิงฟังก์ชัน (functional analysis) ในปัจจุบันนักคณิตศาสตร์ได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง ในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการ และ การพัฒนาประเทศ เป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ๆ ที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่างๆนั้นแล้ว บางครั้งยังสามารถนำไปประยุกต์ในสาขาอื่นๆ และเป็นพื้นฐานสำคัญในการพัฒนาทางวิทยาศาสตร์พื้นฐาน (basic science) ซึ่งเป็นการวิจัยพื้นฐาน (basic research) เพื่อสร้างองค์ความรู้ใหม่ อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ทฤษฎีจุดตรึงนับว่าเป็นแขนงหนึ่งที่สามารถประยุกต์ได้อย่างกว้างขวาง โดยเฉพาะอย่างยิ่งต่อการศึกษาเกี่ยวกับ **การมีคำตอบ** (existence of solution) และ **การมีเพียงคำตอบเดียว** ของสมการ (uniqueness of solution) ตลอดจนการคิดค้นหาวิธีในการประมาณหาคำตอบของสมการต่างๆ ดังนั้นการศึกษาทฤษฎีต่างๆ ที่เกี่ยวข้องกับการมีจุดตรึงของการส่งต่างๆ และ การหาระเบียบวิธีต่างๆที่ใช้ในการประมาณค่าคำตอบนั้นจึงเป็นหัวข้อที่มีนักคณิตศาสตร์กลุ่มหนึ่งจำนวนมากให้ความสนใจศึกษา เมื่อศึกษาการมีคำตอบของสมการต่างๆแล้ว ปัญหาที่น่าสนใจต่อไปก็คือ เราจะหาคำตอบของสมการต่างๆนั้นได้อย่างไร คำถามดังกล่าวนี้ก็ทำให้มีนักคณิตศาสตร์จำนวนมากสนใจศึกษา คิดค้นระเบียบวิธีการกระทำซ้ำของจุดตรึง(fixed-point iterations) ต่างๆที่ใช้ในการหาคำตอบ และ ประมาณคำตอบ เพื่อนำไปประยุกต์ใช้เกี่ยวกับการแก้ปัญหาในเรื่องของสมการตัวดำเนินการไม่เชิงเส้น (nonlinear operator equations) ในเรื่องของแก้ปัญหาอสมการการแปรผัน (variational inequality problem (VIP)) และแก้สมการหาคำตอบของปัญหาเชิงดุลยภาพ(equilibrium problems (EP)) ปัญหาที่ดีที่สุด(optimizations problems) ปัญหาน้อยที่สุด (minimizations problems) ทั้งในปริภูมิฮิลเบิร์ตและปริภูมิบานาค ซึ่งปัญหาดังกล่าวเป็นปัญหาที่สำคัญที่มีประโยชน์มากมายในสาขาวิชาต่างๆ เช่นสาขาวิชาฟิสิกส์ คณิตศาสตร์ประยุกต์ วิศวกรรม และสาขาทางเศรษฐศาสตร์

จากความสำคัญข้างต้นเป็นผลให้นักคณิตศาสตร์จึงได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง ซึ่งการวิจัยเกี่ยวกับวิธีการทำซ้ำของจุดตรึง และการประมาณค่าจุดตรึงที่สำคัญนั้นสามารถนำมาเป็นเครื่องมือในการแก้สมการหาคำตอบของปัญหาเชิงดุลยภาพ ดังเช่นใน ปี 1997 Combettes และ Hirstoaga ได้เริ่มต้นศึกษาและใช้วิธีประมาณค่าแบบซ้ำในการค้นหาผลเฉลยของปัญหาเชิงดุลยภาพ และได้พิสูจน์ทฤษฎีบทการลู่เข้าอย่างเข้ม (strong convergence theorems) ในปริภูมิฮิลเบิร์ต นอกจากนั้นแล้วยังมีนักคณิตศาสตร์อีกมากมาย นำวิธีการประมาณค่าแบบซ้ำดังกล่าวมาประยุกต์ใช้ในการแก้ปัญหาอสมการการแปรผัน ปัญหาค่าน้อยสุด และปัญหาอื่นๆ ทางคณิตศาสตร์

ดังนั้นการคิดค้นเพื่อให้เกิดวิธีการประมาณค่าแบบซ้ำของจุดตรึงชนิดใหม่ๆ และทฤษฎีการลู่เข้าสู่จุดตรึงจึงเป็นองค์ความรู้ใหม่ที่คาดว่าจะได้รับ นอกจากนั้นแล้วยังสามารถใช้วิธีการประมาณค่าดังกล่าวเพื่อประยุกต์ใช้หาคำตอบของปัญหาเชิงดุลยภาพ และ ปัญหาอสมการการแปรผัน ปัญหาค่าน้อยสุด และปัญหาอื่นๆ ทางคณิตศาสตร์ ซึ่งองค์ความรู้ใหม่ที่ได้นั้นจะเป็นพื้นฐานที่สำคัญในการพัฒนาสาขาวิชาการวิเคราะห์เชิงฟังก์ชันและสาขาวิชาอื่นๆ ที่เกี่ยวข้อง ดังที่ได้กล่าวมาแล้วข้างต้นอันจะเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ผลการวิจัย

1. R. Wangkeeree, U. Kamraksa, An iterative approximation method for solving a general system of variational inequality problems and Mixed equilibrium problems, *Nonlinear Analysis: Hybrid Systems 3 (2009) 615-630.*

In this paper, we first introduce our iterative scheme. Consequently, we will establish strong convergence theorems for this iterative scheme. To be more specific, let $S_1, S_2, \dots$ be infinite mappings of C into itself and $\{\xi_i\}$ be a nonnegative real sequence with $0 \le \xi_i < 1$, $\forall i \ge 1$. For any $n \in \mathbb{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \xi_n S_n U_{n,n+1} + (1-\xi_n)I, \\ U_{n,n-1} &= \xi_{n-1} S_{n-1} U_{n,n} + (1-\xi_{n-1})I, \\ &\vdots \\ U_{n,k} &= \xi_k S_k U_{n,k+1} + (1-\xi_k)I, \\ U_{n,k-1} &= \xi_{k-1} S_{k-1} U_{n,k} + (1-\xi_{k-1})I, \\ &\vdots \\ U_{n,2} &= \mu_2 S_2 U_{n,3} + (1-\xi_2)I, \\ W_n = U_{n,1} &= \xi_1 S_1 U_{n,2} + (1-\xi_1)I. \end{aligned} \tag{0.1}$$

Nonexpansivity of each S_i ensures the nonexpansivity of W_n. The mapping W_n is is called a W-mapping generated by $S_1, S_2, \dots, S_n$ and $\xi_1, \xi_2, \dots, \xi_n$.

The general system of variational inequality problem is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \ge 0, & \forall x \in C, \\ \langle \mu B y^* + y^* - x^*, x - y^* \rangle \ge 0, & \forall x \in C, \end{cases} \tag{0.2}$$

where $\lambda > 0$ and $\mu > 0$ are two constants.

Throughout this paper, we will assume that $0 < \xi_n \le \xi < 1, \forall n \ge 1$. Concerning W_n defined by (0.5), we have the following lemmas which are important to prove our main result. Now we only need the following similar version in Hilbert spaces.

Algorithm 1 Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction satisfying conditions (H1)- (H3) and let $\{S_n\}$ be an infinite family of nonexpansive mappings of C into itself. Let $r, \gamma > 0$ be two constants. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let T be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} < \frac{\gamma}{\alpha}$. Let $A : C \longrightarrow H$ be a L_A-Lipchitzian and relaxed (c,d)-cocoercive mapping and $B : C \longrightarrow H$ be a L_B-Lipchitzian and relaxed (c', d')-cocoercive mapping. Suppose the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ are generated iteratively by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle K'(u_n) - K'(x_n), \eta(x, u_n) \rangle \ge 0, \quad \forall x \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n T) W_n P_C(y_n - \lambda A y_n), \end{cases} \tag{0.3}$$

where $\{\alpha_n\}$and $\{\beta_n\}$ are two sequences in $(0,1)$ and W_n defined by (0.5).

Now we study the strong convergence of the hybrid iterative method (0.52).

Theorem 1. Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let $\Theta : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction satisfying conditions (H1)- (H3), $A : C \longrightarrow H$ be a L_A-Lipchitzian and relaxed (c, d)-cocoercive mapping and $B : C \longrightarrow H$ be a L_B-Lipchitzian and relaxed (c', d')-cocoercive mapping. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $F := \cap_{n=1}^{\infty} F(S_n) \cap \Omega \cap GVI(C, A, B) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and T be a strongly bounded linear operator on H with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that

(i) $\eta : C \times C \longrightarrow H$ is Lipschitz continuous with constant $\lambda > 0$ such that
   (a) $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$,
   (b) $\eta(\cdot, \cdot)$ is affine in the first variable,
   (c) for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;

(ii) $K : C \longrightarrow \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the weak topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$;

(iii) for each $x \in C$; there exist a bounded subset $D_x \subset C$ and $u_x \in C$ such that for any $y \in C \backslash D_x$,
$$\Theta(y, u_x) + \varphi(u_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(u_x, y)\rangle \quad < \quad 0; \tag{0.4}$$

(iv) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$, $\lambda \leq \frac{2(d-cL_A^2)}{L_A^2}$ and $\mu \leq \frac{2(d'-c'L_B^2)}{L_B^2}$.

Given $x_1 \in C$ arbitrary, then the sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ generated iteratively by (0.52) converge strongly to $\tilde{x} \in F$ where $\tilde{x} = P_F(\gamma f + (I - T))(\tilde{x})$, which is the unique solution of the variational inequality
$$\langle(\gamma f - T)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall x \in F,$$
and $(\tilde{x}, \tilde{y})$ is a solution of general system of variational inequality problem (0.2) such that $\tilde{y} = P_C(\tilde{x} - \mu B\tilde{x})$.

2. R. Wangkeeree, U. Kamraksa, A General Iterative Method for Solving the Variational Inequality Problem and Fixed Point Problem of an Infinite Family of Nonexpansive Mappings in Hilbert Spaces, *Fixed Point Theory and Applications ,Volume 2009, Article ID 369215, 23 Pages doi:10.1155/2009/369215*

In this paper, we study the mapping W_n defined by
$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \mu_n T_n U_{n,n+1} + (1 - \mu_n)I,\\
U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1 - \mu_{n-1})I,\\
&\vdots\\
U_{n,k} &= \mu_k T_k U_{n,k+1} + (1 - \mu_k)I,\\
U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1 - \mu_{k-1})I,\\
&\vdots\\
U_{n,2} &= \mu_2 T_2 U_{n,3} + (1 - \mu_2)I,\\
W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1 - \mu_1)I,
\end{aligned} \tag{0.5}$$
where $\{\mu_i\}$ is a nonnegative real sequence with $0 \leq \mu_i < 1$, $\forall i \geq 1$, $T_1, T_2, \ldots$ form a family of infinitely nonexpansive mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Such a W_n is nonexpansive from C to C and it is called a W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

In this paper, we will introduce a new iterative scheme:

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary,} \\ z_n = \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n = \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A) P_C(y_n - \lambda_n B y_n), \end{cases} \tag{0.6}$$

where W_n is a mapping defined by (0.5), f is a contraction , A is strongly positive linear bounded self-adjoint operator, B is a α-inverse strongly monotone and prove that under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$, the sequences $\{x_n\}$ defined by (0.52) converges strongly to a common element of the set of common fixed points of a family of $\{T_n\}$ and the set of solutions of the variational inequality for an inverse-strongly monotone mapping, which solves some variational inequality and is also the optimality condition for the minimization problem :

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{0.7}$$

where h is a potential function for $\gamma f (i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

Theorem 1. Let C be a closed convex subset of a real Hilbert space H, f be a contraction of C into itself, B be an α-inverse strongly monotone mapping of C into H and $\{T_i : C \longrightarrow C\}$ be a family of infinitely nonexpansive mappings with $F := \cap_{n=1}^{\infty} F(T_i) \cap VI(B, C) \neq \emptyset$. Let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma \leq \frac{\overline{\gamma}}{\alpha}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ satisfying the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n \longrightarrow \infty} \delta_n \leq \limsup_{n \longrightarrow \infty} \delta_n < 1$,

(C3) $(1 + \beta_n)\gamma_n - 2\beta_n > d$ for some $d \in (0, 1)$,

(C4) $\lim_{n \longrightarrow \infty} |\beta_{n+1} - \beta_n| = \lim_{n \longrightarrow \infty} |\gamma_{n+1} - \gamma_n| = 0$ and

(C5) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty \; and \; \{\lambda_n\} \subset [a, b]$ for some a,b $\in (0, 2\alpha)$.

Then the sequence $\{x_n\}$ defined by (0.52) converges strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))q$ which solves the following variational inequality

$$\langle \gamma f(q) - Ap, p - q \rangle \leq 0, \qquad \forall p \in F.$$

In this section, we obtain two results by using a special case of the proposed method.

Theorem 2. Let H be a real Hilbert space. Let B be an $\alpha-$inverse strongly monotone mapping on H, $\{T_i : H \longrightarrow H\}$ be a family of infinitely nonexpansive mappings with $F := \cap_{i=1}^{\infty} F(T_i) \cap B^{-1}(0) \neq \emptyset$. Let $f : H \longrightarrow H$ a contraction with coefficient $\alpha \in (0, 1)$ and let A be a strongly positive bounded linear operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Suppose the sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ be generated by

$$\begin{cases} x_1 = x \in H \text{ chosen arbitrary,} \\ z_n = \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n = \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)(y_n - \lambda_n B y_n), \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(C1) $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\longrightarrow\infty} \delta_n \leq \limsup_{n\longrightarrow\infty} \delta_n < 1$,
(C3) $(1+\beta_n)\gamma_n - 2\beta_n > a$ for some a $\in (0,1)$,
(C4) $\lim_{n\longrightarrow\infty} |\beta_{n+1} - \beta_n| = \lim_{n\longrightarrow\infty} |\gamma_{n+1} - \gamma_n| = 0$,
(C5) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ for some a,b $\in (0,2\alpha)$.

Then $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_F(\gamma f + (I - A))(q)$ which solves the variational inequality:

$$\langle (A - \gamma f)q, q - z\rangle \leq 0, \quad z \in F.$$

Theorem 3. Let C be a closed convex subset of a real Hilbert space H. For any integer $N > 1$, assume that, for each $1 \leq i \leq N, S_i : C \longrightarrow H$ is a k_i-strictly pseudocontractive mapping for some $0 \leq k_i < 1$. Let $\{T_i : C \longrightarrow C\}$ be a family of infinitely nonexpansive mappings with $F := \cap_{i=1}^{\infty} F(T_i) \cap \cap_{i=1}^{N} F(S_i) \neq \emptyset$. Let $f : C \longrightarrow C$ a contraction with coefficient $\alpha \in (0,1)$ and let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let the sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ be generated by

$$\begin{cases} x_1 = x \in H \text{ chosen arbitrary}, \\ z_n = \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n = \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A) P_C((1-\lambda_n) y_n - \lambda_n \sum_{i=1}^{N} \varphi_i S_i y_n), \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are the sequences in $[0,1]$ satisfying the following conditions:

(C1) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(C2) $0 < \liminf_{n\longrightarrow\infty} \delta_n \leq \limsup_{n\longrightarrow\infty} \delta_n < 1$,
(C3) $(1+\beta_n)\gamma_n - 2\beta_n > a$ for some a $\in (0,1)$,
(C4) $\lim_{n\longrightarrow\infty} |\beta_{n+1} - \beta_n| = \lim_{n\longrightarrow\infty} |\gamma_{n+1} - \gamma_n| = 0$,
(C5) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ for some a,b $\in (0,2\alpha)$.

Then $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_F(\gamma f + (I - A))(q)$ which solves the variational inequality:

$$\langle (A - \gamma f)q, q - z\rangle \leq 0, \quad z \in F.$$

3. P. Kumam, N. Petrot, R. Wangkeeree , A hybrid iterative scheme for equilibrium problems and fixed point problems of asymptotically k-strict pseudo-contractions, *Journal of Computational and Applied Mathematics, 233,8(2010), 2013-2026*

Let C be a closed convex subset of a real Hilbert space H. Let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\phi : C \times C \longrightarrow \mathbb{R}$ is to find $x \in C$ such that

$$\phi(x,y) \geq 0, \qquad \forall y \in C. \tag{0.8}$$

The set of solutions of (0.8) is denoted by $\mathrm{EP}(\phi)$. Numerous problems in physics, optimization, and economics are reduced to find a solution of (0.8). Some methods have been proposed to solve the equilibrium problem. In 2005, Combettes and Hirstoaga introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(\phi)$ is nonempty and they also proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that the bifunction ϕ satisfies the following conditions:

(A1) $\phi(x,x) = 0$ for all $x \in C$;

(A2) ϕ is monotone, i.e., $\phi(x,y)+\phi(y,x)\leq 0$ for any $x,y\in C$;
(A3) ϕ is upper-hemicontinuous, i.e., for each $x,y,z\in C$,

$$\text{limsup}_{t\longrightarrow 0^+}\phi(tz+(1-t)x,y)\leq\phi(x,y);$$

(A4) $\phi(x,\cdot)$ is convex and lower semicontinuous for each $x\in C$.

Theorem 1. Let C be nonempty closed convex subset of a real Hilbert space H and $N\geq 1$ be an integer, $\phi: C\times C\longrightarrow R$ be a bifunction satisfying $(A1)-(A4)$. Let, for each $1\leq i\leq N, T_i: C\longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0\leq k_i\leq 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i}-1)<\infty$. Let $k=max\{k_i:1\leq i\leq N\}$ and $k_n=max\{k_{n,i}:1\leq i\leq N\}$. Assume that $F:=\cap_{i=1}^N F(T_i)\cap EP(\emptyset)$ is nonempty. For any $x_0\in C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by arbitrary element $x_0\in C$ and then by

$$\begin{cases}\phi(u_{n-1},y)+\frac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle\geq 0;\forall y\in C,\\ x_n=\alpha_{n-1}u_{n-1}+(1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1};\forall n\geq 1.\end{cases}\tag{0.9}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(C1) $\{\alpha_n\}\subset[\alpha,\beta],\alpha,\beta\in(k,1)$, and
(C2) $\{r_n\}\subset(0,\infty)$ and $\liminf_{n\longrightarrow\infty}r_n>0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of F.

Theorem 2. Let C be nonempty closed convex subset of a real Hilbert space H and $N\geq 1$ be an integer, $\phi: C\times C\longrightarrow R$ be a bifunction satisfying $(A1)-(A4)$. Let, for each $1\leq i\leq N, T_i: C\longrightarrow C$ be an asymptotically k_i strictly pseudo-Contractive mapping for some $0\leq u_i\leq 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i}-1)<\infty$. Let $k=max\{k_i:1\leq i\leq N\}$ and $k_n=max\{k_{n,i}:1\leq i\leq N\}$. Assume that $F:=\cap_{i=1}^N F(T_i)\cap EP(\phi)$ is nonempty. For any $x_0\in C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by arbitrary element $x_0\in C$ and

$$\begin{cases}\phi(u_{n-1},y)+\frac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle\geq 0,\forall y\in C\\ x_n=\alpha_{n-1}u_{n-1}+(1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1},\quad\forall n\geq 1,\end{cases}\tag{0.10}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\}\subset[\alpha,\beta]$ for some $\alpha,\beta\in(k,1)$, and
(2) $\{r_n\}\subset(0,\infty)$ and $\liminf_{n\longrightarrow\infty}r_n>0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of F if and only if $\liminf_{n\longrightarrow\infty}d(x_n,F)=0$, where $d(x_n,F)$ denotes the metric distance from the point x_n to F.

Theorem 3. Let C be nonempty closed convex subset of a real Hilbert space H and $N\geq 1$ be an integer, $\phi: C\times C\longrightarrow R$ be a bifunction satisfying $(A1)-(A4)$. Let, for each $1\leq i\leq N, T_i: C\longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0\leq k_i\leq 1$ and a sequence $\{k_{n,i}\}$. Let $k=max\{k_i:1\leq i\leq N\}$ and $k_n=max\{k_{n,i}:1\leq i\leq N\}$. Assume that $F:=\cap_{i=1}^N F(T_i)\cap EP(\phi)$ is nonempty and let $x\in H$. For $C_0=C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:

$$\begin{cases}x_0=P_{C_0}x,\\ u_{n-1}\in C\text{ such that }\phi(u_{n-1},y)+\frac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle\geq 0;\forall y\in C,\\ y_{n-1}=\alpha_{n-1}u_{n-1}+(1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1};\forall n\geq 1,\\ C_n=\{v\in C_{n-1}:\|y_{n-1}-v\|^2\leq\|x_{n-1}-v\|^2+\theta_{n-1}\},\\ x_n=P_{C_n}x,\forall n\geq 1,\end{cases}\tag{0.11}$$

where $\theta_{n-1} = (k^2_{h(n)} - 1)(1 - \alpha_{n-1})\rho^2_{n-1} \longrightarrow \infty$ as $n \longrightarrow \infty$, where $\rho_{n-1} = \sup\{\|x_{n-1} - v\| : v \in F\} < \infty$, and $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$ and $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n \longrightarrow \infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{\cap_{i=1}^N F(T_i) \cap EP(\phi)} x$.

Theorem 4. Let C be nonempty closed convex subset of a real Hilbert space H and $N \geq 1$ be an integer, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let, for each $1 \leq i \leq N, T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \leq k_i \leq 1$ and a sequence $\{k_{n,i}\}$. Let $k = max\{k_i : 1 \leq i \leq N\}$ and $k_n = max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi)$ is nonempty. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:

$$\begin{cases} x_0 = u \in C \text{ chosen arbitrarily }, \\ u_{n-1} \in C \text{ such that } \phi(u_{n-1}, y) + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0; \forall y \in C, \\ y_{n-1} = \alpha_{n-1} u_{n-1} + (1 - \alpha_{n-1}) T^{h(n)}_{i(n)} u_{n-1}; \forall n \geq 1, \\ C_{n-1} = \{v \in C : \|y_{n-1} - v\|^2 \leq \|x_{n-1} - v\|^2 + \theta_{n-1}\}, \\ Q_{n-1} = \{v \in C : \langle x_0 - x_{n-1}, x_{n-1} - v \rangle \geq 0\}, \\ x_n = P_{C_{n-1} \cap Q_{n-1}} x_0, \forall n \geq 1, \end{cases} \tag{0.12}$$

where $\theta_{n-1} = (k^2_{h(n)} - 1)(1 - \alpha_{n-1})\rho^2_{n-1} \longrightarrow \infty$ as $n \longrightarrow \infty$, where $\rho_{n-1} = \sup\{\|x_{n-1} - v\| : v \in F\} < \infty$, and $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$ and $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n \longrightarrow \infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_F x_0$.

4. R. Wangkeeree and R. Wangkeeree, The Shrinking Projection Method for Solving Variational Inequality Problems and Fixed Point Problems in Banach Spaces, *Abstract and Applied Analysis, Volume 2009, Article ID 624798, 26 pages doi:10.1155/2009/624798*

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $A : C \longrightarrow E^*$ be an operator. The classical variational inequality problem [?] for A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \forall y \in C, \tag{0.13}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. The set of all solutions of (0.17) is denoted by $VI(A, C)$. Such a problem is connected with the convex minimization problem, the complementarity, the problem of finding a point $x^* \in E$ satisfying $0 = Ax^*$, and so on.

Theorem 1. Let E be a 2-uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let A be an operator of C into E^* satisfying (C1) and (C2), and let $\{S_i\}_{i \in I}$ and $\{T_i\}_{i \in I}$ be two families of closed quasi-ϕ-nonexpansive mappings of C into itself with $F := \cap_{i \in I} F(T_i) \cap \cap_{i \in I} F(S_i) \cap VI(A, C)$ is nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary} \\ C_{1,i} = C, C_1 = \cap_{i=1}^{\infty} C_{1,i}, x_1 = \Pi_{C_1}(x_0) \quad \forall i \in I, \\ w_{n,i} = \Pi_C J^{-1}(Jx_n - \lambda_{n,i} A x_n), \\ z_{n,i} = J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i}), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 + (1-\alpha_{n,i}) Jz_{n,i}), \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, y_{n,i}) \leqslant \phi(u, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \cap_{i \in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 0, \end{cases} \tag{0.14}$$

where J is the duality mapping on E, $\{\lambda_{n,i}\}, \{\alpha_{n,i}\}$ and $\{\beta_{n,i}^{(j)}\}$ $(j = 1, 2, 3)$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\longrightarrow\infty} \alpha_{n,i} = 0$ for all $i \in I$;

(ii) For all $i \in I$, $\{\lambda_{n,i}\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E;

(iii) $\beta_{n,i}^{(1)} + \beta_{n,i}^{(2)} + \beta_{n,i}^{(3)} = 1$ for all $i \in I$ and if one of the following conditions is satisfied

(a) $\liminf_{n\longrightarrow\infty} \beta_{n,i}^{(1)} \beta_{n,i}^{(l)} > 0$ for all $l = 2, 3$ and for all $i \in I$ and

(b) $\liminf_{n\longrightarrow\infty} \beta_{n,i}^{(2)} \beta_{n,i}^{(3)} > 0$ and $\lim_{n\longrightarrow\infty} \beta_{n,i}^{(1)} = 0$ for all $i \in I$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

From Theorem 1 we can obtain some new and interesting strong convergence theorems. Now we give some examples as follows:

If $\beta_{n,i}^{(1)} = 0$ for all $n \geq 0$, $T_i = S_i$ for all $i \in I$ and $A = 0$ in Theorem **??**, then we have the following result.

Theorem 2. Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i\in I}$ be a family of closed quasi-ϕ-nonexpansive mappings of C into itself with $F := \cap_{i\in I} F(T_i)$ is nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary} \\ C_{1,i} = C, C_1 = \cap_{i=1}^{\infty} C_{1,i}, x_1 = \Pi_{C_1}(x_0) \quad \forall i \in I, \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 + (1-\alpha_{n,i}) JT_i x_n), \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, y_{n,i}) \leqslant \phi(u, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \cap_{i \in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 0, \end{cases} \tag{0.15}$$

where J is the duality mapping on E, $\{\alpha_{n,i}\}$ is a sequence in $(0, 1)$ such that $\limsup_{n\longrightarrow\infty} \alpha_{n,i} = 0, \forall i \in I$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Now we consider the problem of finding a zero point of an inverse-strongly monotone operator of E into E^*. Assume that A satisfies the conditions:

(C1) A is α-inverse-strongly monotone,

(C2) $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$.

Theorem 3. Let E be a 2-uniformly convex and uniformly smooth Banach space. Let A be an operator of E into E^* satisfying (C1) and (C2), and let $\{S_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be two families of closed quasi-ϕ-nonexpansive mappings of E into itself with $F := \cap_{i\in I}F(T_i)\cap\cap_{i\in I}F(S_i)\cap A^{-1}0$ is nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 \in E \text{ chosen arbitrary} \\ C_{1,i} = E, C_1 = \cap_{i=1}^{\infty} C_{1,i}, x_1 = \Pi_{C_1}(x_0) \quad \forall i \in I, \\ w_{n,i} = J^{-1}(Jx_n - \lambda_{n,i}Ax_n), \\ z_{n,i} = J^{-1}(\beta_{n,i}^{(1)}Jx_n + \beta_{n,i}^{(2)}JT_ix_n + \beta_{n,i}^{(3)}JS_iw_{n,i}), \\ y_{n,i} = J^{-1}(\alpha_{n,i}Jx_0 + (1-\alpha_{n,i})Jz_{n,i}), \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, y_{n,i}) \leqslant \phi(u, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \cap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \ \forall n \geq 0, \end{cases} \tag{0.16}$$

where J is the duality mapping on E, $\{\lambda_{n,i}\}, \{\alpha_{n,i}\}$ and $\{\beta_{n,i}^{(j)}\}$ $(j = 1, 2, 3)$ are sequences in $(0, 1)$ such that

(i) $\lim_{n\longrightarrow\infty} \alpha_{n,i} = 0$ for all $i \in I$;
(ii) For all $i \in I$, $\{\lambda_{n,i}\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E;
(iii) $\beta_{n,i}^{(1)} + \beta_{n,i}^{(2)} + \beta_{n,i}^{(3)} = 1$ for all $i \in I$ and if one of the following conditions is satisfied
  (a) $\liminf_{n\longrightarrow\infty} \beta_{n,i}^{(1)}\beta_{n,i}^{(l)} > 0$ for all $l = 2, 3$ and for all $i \in I$ and
  (b) $\liminf_{n\longrightarrow\infty} \beta_{n,i}^{(2)}\beta_{n,i}^{(3)} > 0$ and $\lim_{n\longrightarrow\infty} \beta_{n,i}^{(1)} = 0$ for all $i \in I$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

5. U. Kamraksa, R. Wangkeeree, A hybrid iterative scheme for variational inequality problems for finite families of relatively weak quasi-nonexpansive mappings, *Journal of Inequalities and Applications, Volume 2010, Article ID 724851, 23 pages*

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $A : C \longrightarrow E^*$ be an operator. The classical variational inequality problem for A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C, \tag{0.17}$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ the generalized duality pairing between E and E^*. The set of all solutions of (0.17) is denoted by $VI(A, C)$. Such a problem is connected with the convex minimization problem, the complementarity, the problem of finding a point $x^* \in E$ satisfying $0 = Ax^*$, and so on. First, we recall that a mapping $A : C \longrightarrow E^*$ is said to be:

(i) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0, \ \forall x, y \in C$.
(ii) *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \forall x \in E.$$

It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E

Theorem 1. Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in F$. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from C into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap VI(A, C)$. Assume that T_i and S_i are uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algolithm:

$$\begin{cases} x_0 = x \in C, \text{ chosen arbitrary,} \\ C_1 = C, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n A x_n), \\ z_n = J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n), \\ y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)[\alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)]\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{0.18}$$

where $T_n = T_{n (\text{mod } N)}, S_n = S_{n (\text{mod } N)}$, and J is the normalized duality mapping on E. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0, 1]$ satisfying the restrictions:

- (C1) $\lim_{n \longrightarrow \infty} \delta_n = 0$;
- (C2) $r_n \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;
- (C3) $\alpha_n + \beta_n + \gamma_n = 1$ and if one of the following conditions is satisfied
  - (a) $\liminf_{n \longrightarrow \infty} \alpha_n \beta_n > 0$ and $\liminf_{n \longrightarrow \infty} \alpha_n \gamma_n > 0$ and
  - (b) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\liminf_{n \longrightarrow \infty} \beta_n \gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from C onto F.

Let X be a nonempty closed convex cone in E, and let A be an operator from X into E^*. We define its *polar* in E^* to be the set

$$X^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0 \text{ for all } x \in X\}.$$

Then an element x in X is called a *solution of the complementarity problem* if

$$Ax \in X^* \text{ and } \langle x, Ax \rangle = 0.$$

The set of all solutions of the complementarity problem is denoted by $CP(A, X)$. Several problem arising in different fields, such as mathematical programming, game theory, mechanics, and geometry, are to find solutions of the complementarity problems.

Theorem 2. Let X be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, let A be an α-inverse-strongly monotone mapping of X into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in X$ and $q \in F$. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from X into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap CP(A, X)$. Assume that T_i and S_i are uniformly continuous for all $i \in \{1, 2, \dots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_0 = x \in X, \text{ chosen arbitrary}, \\ C_1 = X, x_1 = \Pi_{C_1} x_0, \\ w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_{n-1} + \beta_n JT_n x_n + \gamma_n JS_n w_n), \\ y_n = J^{-1}(\delta_n Jx_1 + (1-\delta_n)Jz_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \le \delta_n \phi(u, x_1) + (1-\delta_n)[\alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)]\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases}$$

where $T_n = T_{n (\text{mod } N)}, S_n = S_{n (\text{mod } N)}$, and J is the normalized duality mapping on E. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0, 1]$ satisfying the restrictions:

(C1) $\lim_{n \longrightarrow \infty} \delta_n = 0$;

(C2) $r_n \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ and if one of the following conditions is satisfied

(a) $\liminf_{n \longrightarrow \infty} \alpha_n \beta_n > 0$ and $\liminf_{n \longrightarrow \infty} \alpha_n \gamma_n > 0$ and

(b) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\liminf_{n \longrightarrow \infty} \beta_n \gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from X onto F.

6. U. Kamraksa, R. Wangkeeree, Convergence theorems based on the shrinking projection method for variational inequality and equilibrium problems, *J. Appl. Math. Comput, DOI 10.1007/s12190-010-0427-2*

Theorem 1. Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. Assume that A is a continuous operator of C into E^* satisfying conditions (0.19) and (0.20)

$$\langle Ax, J^*(Jx - \beta Ax) \rangle \ge 0, \quad \text{for all } x \in C, \tag{0.19}$$

and

$$\langle Ax, y \rangle \le 0, \quad \forall x \in C, \ y \in VI(A, C). \tag{0.20}$$

and $S, T : C \longrightarrow C$ are relatively weak nonexpansive mappings with $F := F(S) \cap F(T) \cap VI(A, C) \cap EP(f) \ne \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{cases} x_0 = x \in C \text{chosen arbitrary}, C_0 = C, \\ z_n = \Pi_C(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ y_n = J^*(\delta_n Jx_n + (1-\delta_n)J\Pi_C(Jz_n - \beta Az_n)), \\ u_n \in C \ \ such\ that\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} Jx \quad \forall n \ge 0. \end{cases} \tag{0.21}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0, 1]$ satisfying the restrictions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;

(C2) $0 \le \delta_n < 1, \limsup_{n \longrightarrow \infty} \delta_n < 1$;

(C3) $\{r_n\} \subset [a, \infty)$ for some $a > 0$; and

(C4) $\liminf_{n \longrightarrow \infty} \alpha_n \beta_n > 0, \liminf_{n \longrightarrow \infty} \alpha_n \gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x$.

7. R. Wangkeeree and U. Kamraksa, Strong convergence theorems of viscosity iterative methods for a countable family of strict pseudo-contractions in Banach spaces, *Fixed Point Theory and Applications, Volume 2010, Article ID 579725, 21 pages doi:10.1155/2010/579725*

Let E be a real Banach space, and C a nonempty closed convex subset of E. A mapping $f : C \longrightarrow C$ is called k-contraction, if there exists a constant $0 < k < 1$ such that $\|f(x)-f(y)\| \leq k\|x-y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f : f \text{ is a contraction on } C\}$.

A mapping $T : C \longrightarrow C$ is said to be λ-strictly pseudocontractive mapping if there exists a constant $0 \leq \lambda < 1$, such that

$$\|Tx - Ty\|^2 \quad \leq \quad \|x-y\|^2 + \lambda\|(I-T)x - (I-T)y\|^2, \tag{0.22}$$

for all $x, y \in C$. Note that the class of λ-strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mapping T on C such that $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. That is, T is nonexpansive if and only if T is a 0-strict pseudocontraction. A mapping $T : C \longrightarrow C$ is said to be λ-strictly pseudocontractive mapping with respect to p, if, for all $x, y \in C$, there exists a constant $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^p \quad \leq \quad \|x-y\|^p + \lambda\|(I-T)x - (I-T)y\|^p. \tag{0.23}$$

A countable family of mapping $\{T_n : C \longrightarrow C\}_{i=1}^{\infty}$ is called a *family of uniformly λ-strict pseudo-contractions with respect to p*, if there exists a constant $\lambda \in [0, 1)$ such that

$$\|T_n x - T_n y\|^p \leq \|x-y\|^p + \lambda\|(I-T_n)x - (I-T_n)y\|^p, \quad \forall x, y \in C, \quad \forall n \geq 1.$$

We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$.

Let $\{T_n\}$ be a family of mappings from a subset C of a Banach space E into E with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. We say that $\{T_n\}$ satisfies the AKTT-condition if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup_{z \in B} \|T_{n+1}z - T_n z\| < \infty. \tag{0.24}$$

Theorem 1. Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ. Let C be a nonempty closed convex subset of E, $T : C \longrightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Then the net $\{x_t\}$ defined by (0.25)

$$x_t^f = tf(x_t) + (1-t)Tx_t^f. \tag{0.25}$$

converges strongly as $t \longrightarrow 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality :

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T). \tag{0.26}$$

Theorem 2. Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_φ, and C a nonempty closed convex subset of E. Let $\{T_n : C \longrightarrow C\}$ be a family of uniformly λ-strict pseudo-contractions with respect to p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \longrightarrow C$ be a k-contraction with $k \in (0, 1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$;

(ii) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$;

(iii) $0 < \liminf_{n \longrightarrow \infty} \gamma_n \leq \limsup_{n \longrightarrow \infty} \gamma_n < \xi$, where $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ be the sequence generated by the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \geq 1. \end{cases} \tag{0.27}$$

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let T be a mapping of C into itself defined by $Tz = \lim_{n \longrightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality :

$$\langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T). \tag{0.28}$$

Theorem 3. Let E be a real p-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings. Let $\{T_n : C \longrightarrow C\}$ be a family of uniformly λ-strict pseudo-contractions with respect to p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \longrightarrow C$ be a k-contraction with $k \in (0, 1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$;
(ii) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$;
(iii) $0 < \liminf_{n \longrightarrow \infty} \gamma_n \leq \limsup_{n \longrightarrow \infty} \gamma_n < \xi$, where $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ be the sequence generated by the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \geq 1. \end{cases} \tag{0.29}$$

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let T be a mapping of C into itself defined by $Tz = \lim_{n \longrightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point $\tilde{x}$ of $\{T_n\}$.

Theorem 4. Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_φ, and C a nonempty closed convex subset of E. Let Ψ is an m-accretive operator in E such that $\Psi^{-1}0 \neq \emptyset$. Let $f : C \longrightarrow C$ be a k-contraction with $k \in (0, 1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$;
(ii) $\lim_{n \longrightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$;
(iii) $0 < \liminf_{n \longrightarrow \infty} \gamma_n \leq \limsup_{n \longrightarrow \infty} \gamma_n < 1$;
(iv) $\{\lambda_n\}$ is a sequence of $(0, \infty)$ such that $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Let $\{x_n\}$ be the sequence generated by the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n} x_n, \quad n \geq 1. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly $\tilde{x}$ which solves the variational inequality :

$$\langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(J_\lambda).$$

Theorem 5. Let E be a real p-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of E. Let Ψ is an m-accretive operator in E such that $\Psi^{-1}0 \neq \emptyset$. Let $f : C \longrightarrow C$ be a k-contraction with $k \in (0, 1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$;

(ii) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = +\infty$;
(iii) $0 < \liminf_{n\longrightarrow\infty}\gamma_n \leq \limsup_{n\longrightarrow\infty}\gamma_n < 1$;
(iv) $\{\lambda_n\}$ is a sequence of $(0,\infty)$ such that $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$.

Let $\{x_n\}$ be the sequence generated by the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n} x_n, \qquad n \geq 1. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly $\tilde{x}$ in $\Psi^{-1}0$.

8. R. Wangkeeree, N. Petrot, R. Wangkeeree , The general iterative methods for nonexpansive mappings in Banach Spaces, *Journal of Global Optimization, DOI 10.1007/s10898-010-9617-6.*

In Hilbert space H let A be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \text{ for all } x \in H. \tag{0.30}$$

In a Banach space E having a weakly continuous duality mapping J_φ with a gauge function φ, an operator A is said to be *strongly positive* if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x)\rangle \geq \bar{\gamma}\|x\|\varphi(\|x\|) \tag{0.31}$$

and

$$\|\alpha I - \beta A\| = \sup_{\|x\|\leq 1}|\langle(\alpha I - \beta A)x, J_\varphi(x)\rangle|, \quad \alpha \in [0,1], \beta \in [-1,1], \tag{0.32}$$

where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (0.31) reduce to (0.30). The next valuable lemma is proved for applying our main results.

Theorem 1. Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ. Let A be a strong positive linear bounded operator on E with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\bar{\gamma})$.

Theorem 2. Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \longrightarrow E$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Then the net $\{x_t\}$ defined by (0.33)

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t. \tag{0.33}$$

converges strongly as $t \longrightarrow 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality :

$$\langle(A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T). \tag{0.34}$$

Theorem 3. Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\{T_n : E \longrightarrow E\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings with $\cap_{n=1}^{\infty}F(T_n) \neq \emptyset$ and $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following :

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1 - \beta_n)T_n x_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n, \quad n \geq 0 \end{cases} \tag{0.35}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1]$ are real sequences satisfying the following conditions :

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$;
(C2) $\lim_{n\longrightarrow\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$.

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let the mapping $T : E \longrightarrow E$ be defined by (0.36)

$$Tx = \lim_{n\longrightarrow\infty} T_n x, \text{ for all } x \in C. \tag{0.36}$$

and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $\tilde{x}$ that is obtained in Theorem 1.

Theorem 4. Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\{T_n : E \longrightarrow E\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings with $\cap_{n=1}^{\infty}F(T_n) \neq \emptyset$ and $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following :

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1-\beta_n)T_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad n \geq 0 \end{cases} \tag{0.37}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions :

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, and
(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Suppose that $\{T_n\}$ satisfies the PU-condition. Let the mapping $T : E \longrightarrow E$ be defined by (0.36) and suppose that $F(T) = \cap_{n=1}^{\infty}F(T_n)$. Then $\{x_n\}$ converges strongly to $\tilde{x}$ that is obtained in Theorem 1.

9. U. Kamraksa and R. Wangkeeree, Generalized Equilibrium Problems and fixed point problems for nonexpansive semigroups in Hilbert spaces, *Journal of Global Optimization, DOI 10.1007/s 10898-011-9654-9*

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $G : H \times H \longrightarrow \mathbb{R}$ be an equilibrium bifunction, i.e., $G(u,u) = 0$ for each $u \in H$ and $\Psi : H \longrightarrow H$ is a mapping. Then, we consider the following generalized equilibrium problem (for short, GEP):

$$\text{Finding } x^* \in H \text{ such that } G(x^*, y) + \langle \Psi x^*, y - x^* \rangle \geq 0, \forall y \in H. \tag{0.38}$$

The set of solutions for the problem GEP (0.38) is denoted by $GEP(G, \Psi)$.

Special cases.

(1) If $\Psi \equiv 0$, then GEP (0.38) reduces to the following classical equilibrium problem (for short, EP):

$$\text{Finding } x^* \in H \text{ such that } G(x^*, y) \geq 0, \forall y \in H. \tag{0.39}$$

The set of solutions for the problem EP (0.49) is denoted by $EP(G)$.

(2) If $G \equiv 0$, then GEP (0.38) reduces to the following classical variational inequality problem (for short VIP):

$$\text{Finding } x^* \in H \text{ such that } \langle \Psi x^*, y - x^* \rangle \geq 0, \forall y \in H. \tag{0.40}$$

The set of solutions for the problem VIP (0.50) is denoted by $VI(\Psi, H)$.

The problem (0.38) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, the classical equilibrium problems, and others.

Theorem 1. Let $\mathcal{S} = (T(s))_{s\geq 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \longrightarrow H$ be an α-contraction, $A : H \longrightarrow H$ a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$, $G : H \times H \longrightarrow \mathbb{R}$ a mapping satisfying hypotheses (E1)-(E4) and $\Psi : H \longrightarrow H$ an inverse-strongly monotone mappings with coefficients δ such that $F(\mathcal{S}) \cap GEP(G, \Psi) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, 2\delta)$ and $\{s_n\} \subset (0, \infty)$ be the real sequences. Then the following hold.

(i) For any $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, there exists a unique sequence $\{x_n\} \subset H$ such that

$$\begin{cases} G(u_n, y) + \langle \Psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \forall n \geq 1. \end{cases} \quad (0.41)$$

(ii) If $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\liminf_{n\longrightarrow\infty} r_n > 0$, and $\lim_{n\longrightarrow\infty} s_n = +\infty$, then the sequence $\{x_n\}$ defined by (0.41) converges strongly to z, which is a unique solution in $F(\mathcal{S}) \cap GEP(G, \Psi)$ of the variational inequality

$$\langle(\gamma f - A)z, p - z\rangle \leq 0, \qquad \forall p \in F(\mathcal{S}) \cap GEP(G, \Psi). \quad (0.42)$$

Theorem 2. Let $\mathcal{S} = (T(s))_{s\geq 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \longrightarrow H$ be an α-contraction, $A : H \longrightarrow H$ a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma}$ and let γ be a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $G : H \times H \longrightarrow \mathbb{R}$ be a mapping satisfying hypotheses (E1)-(E4) and $\Psi : H \longrightarrow H$ an inverse-strongly monotone mapping with coefficients δ such that $F(\mathcal{S}) \cap GEP(G, \Psi) \neq \emptyset$. Let the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_1 \in H \text{ chosen arbitrary}, \\ G(u_n, y) + \langle \Psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ y_n = \beta_n x_n + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \forall n \geq 1, \end{cases} \quad (0.43)$$

where the real sequences $\{r_n\} \subset (0, 2\delta)$, $\{s_n\} \subset (0, \infty)$ and $\{\alpha_n\}$, $\{\beta_n\}$ in $(0, 1)$ satisfy the following conditions:

(D1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$,
(D2) $\liminf_{n\longrightarrow\infty} r_n > 0$, $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$,
(D3) $\lim_{n\longrightarrow\infty} s_n = +\infty$, $\lim_{n\longrightarrow\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$, and
(D4) $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\longrightarrow\infty} |\beta_n - \beta_{n-1}| = 0$.

Then the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ converge strongly to z which is a unique solution in $F(\mathcal{S}) \cap GEP(G, \Psi)$ of the variational inequality (0.42).

10. R. Wangkeeree, The general hybrid approximation methods for nonexpansive mappings in Banach spaces, *Abstract and Applied Analysis, (Accepted)*

Theorem 1. Let E be a Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0, 1]$ i.e. $T([0, 1]) \subset [0, 1]$. Let $T : E \longrightarrow E$ be a nonexpansive mapping, $f : E \longrightarrow E$ a contraction with coefficient $\alpha \in (0, 1)$. Let A and B be two strongly positive bounded linear operators with coefficients $\bar{\gamma} > 0$ and $\beta > 0$, respectively. Let γ and μ be two constants satisfying the condition (C^*) :

$$(C^*) : 0 < \gamma < \frac{\beta\varphi(1)}{\alpha} \text{ and } \frac{\varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha} < \mu \leq \min\left\{1, \varphi(1)\|B\|^{-1}, \frac{1 + \varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha}\right\}.$$

Then for any $\lambda \in (0, \min\{1, \varphi(1)\|A\|^{-1}\})$, the mapping $S_\lambda : E \longrightarrow E$ defined by

$$S_\lambda(x) = (I - \lambda A)Tx + \lambda[Tx - \mu(BTx - \gamma f(x))], \forall x \in E. \quad (0.44)$$

is a contraction with coefficient $1 - \lambda\tau$, where $\tau := \varphi(1)\bar{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha)$.

Applying the Banach contraction principle to Theorem 1, there exists a unique fixed point x_λ of S_λ in E, that is

$$x_\lambda = (I - \lambda A)Tx_\lambda + \lambda[Tx_\lambda - \mu(BTx_\lambda - \gamma f(x_\lambda))], \text{ for all } \lambda \in (0,1). \tag{0.45}$$

Theorem 2. Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \longrightarrow E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \longrightarrow E$ a contraction with coefficient $\alpha \in (0,1)$, and A, B two strongly positive bounded linear operators with coefficients $\bar{\gamma} > 0$ and $\beta > 0$, respectively. Let γ and μ be two constants satisfying the condition (C^*). Then the net $\{x_\lambda\}$ defined by (0.45) converges strongly as $\lambda \longrightarrow 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality :

$$\langle (A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T). \tag{0.46}$$

Theorem 3. Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \longrightarrow E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \longrightarrow E$ a contraction with coefficient $\alpha \in (0,1)$, and A and B two strongly positive bounded linear operators with coefficients $\bar{\gamma} > 0$ and $\beta > 0$, respectively. Let $x_0 \in E$ be arbitrary and the sequence $\{x_n\}$ be generated by the following iterative scheme :

$$x_{n+1} = (I - \lambda_n A)Tx_n + \lambda_n[Tx_n - \mu(BTx_n - \gamma f(x_n))], \text{ for all } n \geq 0, \tag{0.47}$$

where γ and μ are two constants satisfying the condition (C^*) and $\{\lambda_n\}$ is a real sequence in $(0,1)$ satisfying the following conditions :

(C1) $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$
(C2) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Then the sequence $\{x_n\}$ defined by (0.47) converges strongly to a fixed point $\tilde{x}$ of T that is obtained by Theorem 1.

11. U. Kamraksa and R. Wangkeeree, Iterative Algorithms for solving Mixed Equilibrium Problem and variational inequality problem of a finite family of asymptotically strict pseudo-contractions,*Journal of Computational Analysis and Applications, (Accepted).*

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, C is a nonempty closed convex subset of H. Let $\varphi : C \longrightarrow \mathbb{R}$ be a real-valued function and $F : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction, i.e., $F(u,u) = 0$ for each $u \in C$. The mixed equilibrium problem (for short, MEP) is to find $x^* \in C$ such that

$$MEP : F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \forall y \in C. \tag{0.48}$$

The set of solutions for the problem MEP (0.48) is denoted by $MEP(F,\varphi)$.

Special cases.

(1) If $\varphi \equiv 0$, then MEP (0.48) reduces to the following classical equilibrium problem (for short, EP):

$$\text{Finding } x^* \in C \text{ such that } F(x^*, y) \geq 0, \forall y \in C. \tag{0.49}$$

The set of solutions for the problem EP (0.49) is denoted by $EP(F)$.

(2) If $\varphi \equiv 0$ and $F(x,y) = \langle Ax, y - x\rangle$ for all $x, y \in C$, where A is a mapping from C into H, then MEP (0.48) reduces to the following classical variational inequality problem (for short VIP):

$$\text{Finding } x^* \in C \text{ such that } \langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C. \tag{0.50}$$

The set of solutions for the problem VIP (0.50) is denoted by $VI(C,A)$.

(3) If $F \equiv 0$, then MEP (0.48) becomes the following minimize problem:

$$\text{Finding } x^* \in C \text{ such that } \varphi(y) - \varphi(x^*) \geq 0, \forall y \in C. \tag{0.51}$$

The set of solutions for the problem (0.51) is denoted by $Argmin(\varphi)$.

The problem (0.48) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, the equilibrium problems and others.

Theorem 1. Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let A be a monotone and δ-Lipschitz continuous mapping of C into H. Let, for each $1 \leq i \leq N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strict pseudo-contractive mapping for some $0 \leq k_i < 1$ and a sequence $\{k_{n,i}\}$. Let $k = \max\{k_i : 1 \leq i \leq N\}$ and $k_n = \max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that $\Omega := \cap_{i=1}^N F(T_i) \cap VI(C, A) \cap MEP(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ be sequences generated by

$$\begin{cases} x_0 = x \in H, \text{ chosen arbitrary,} \\ F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C, \\ y_{n-1} = P_C(u_{n-1} - \lambda_{n-1}Au_{n-1}), \\ t_{n-1} = P_C(u_{n-1} - \lambda_{n-1}Ay_{n-1}), \\ z_{n-1} = \alpha_{n-1}t_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)}t_{n-1}, \\ C_{n-1} = \{z \in C : \|z_{n-1} - z\|^2 \leq \|x_{n-1} - z\|^2 + \theta_{n-1}\}, \\ Q_{n-1} = \{z \in H : \langle x_{n-1} - z, x - x_{n-1}\rangle \geq 0\}, \\ x_n = P_{C_{n-1}\cap Q_{n-1}}x, \quad \forall n \geq 1, \end{cases} \tag{0.52}$$

where $\theta_{n-1} = (k_{h(n)}^2 - 1)(1 - \alpha_{n-1})\rho_{n-1}^2 \longrightarrow 0$ as $n \longrightarrow \infty$, where $\rho_{n-1} = \sup\{\|x_{n-1} - z\| : z \in \Omega\} < \infty$. Assume that $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\delta})$, $\{\alpha_n\} \subset [c, d]$ for some c, $d \in (k, 1)$ and $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n\longrightarrow\infty} r_n > 0$. Then $\{x_n\}$,$\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ converge strongly to $w = P_\Omega(x)$.

Theorem 2. Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let A be a monotone and δ-Lipschitz continuous mapping of C into H. Let, for each $1 \leq i \leq N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strict pseudo-contractive mapping for some $0 \leq k_i < 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = \max\{k_i : 1 \leq i \leq N\}$ and $k_n = \max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that $F := \cap_{i=1}^N F(T_i) \cap VI(C, A) \cap MEP(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{t_n\}$, $\{y_n\}$ be sequences generated by

$$\begin{cases} x_0 = x \in H, \text{ chosen arbitrary,} \\ F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C, \\ y_{n-1} = P_C(u_{n-1} - \lambda_{n-1}Au_{n-1}), \\ t_{n-1} = P_C(u_{n-1} - \lambda_{n-1}Ay_{n-1}), \\ x_n = \alpha_{n-1}t_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)}t_{n-1}, \quad \forall n \geq 1, \end{cases} \tag{0.53}$$

where $\theta_{n-1} = (k_{h(n)}^2 - 1)(1 - \alpha_{n-1})\rho_{n-1}^2 \longrightarrow 0$ as $n \longrightarrow \infty$, where $\rho_{n-1} = \sup\{\|x_{n-1} - z\| : z \in F\} < \infty$. Assume that $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some c, $d \in (k, 1)$ and $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n\longrightarrow\infty} r_n > 0$. Then $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ converge weakly to $w \in \Omega$, where $w = \lim_{n\longrightarrow\infty} P_\Omega x_n$.

Out Put จากโครงการวิจัยได้รับทุนจาก สกอ. และ สกว.

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติจำนวน 11 เรื่อง

(1) **<u>R. Wangkeeree</u>**, U. Kamraksa, An iterative approximation method for solving a general system of variational inequality problems and mixed equilibrium problems, (2009), doi:10.1016/j.nahs.2009.05.005
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Corresponding Author : R. Wangkeeree

(2) **<u>R. Wangkeeree</u>**, U. Kamraksa, A General Iterative Method for Solving the Variational Inequality Problem and Fixed Point Problem of an Infinite Family of Nonexpansive Mappings in Hilbert Spaces**",** Volume 2009, Article ID 369215, 23 Pages doi:10.1155/2009/369215
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Corresponding Author : R. Wangkeeree
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(3) P. Kumam, N. Petrot, **<u>R. Wangkeeree</u>**, A hybrid iterative scheme for equilibrium problems and fixed point problems of asymptotically k-strict pseudo-contractions, 233,8(2010), 2013-2026
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วารสาร "Abstract and Applied Analysis" (วารสารระดับนานาชาติ)
Corresponding Author : Rabian Wangkeeree
Impact factor ปี 2009 = 2.221

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Impact factor ปี 2009 = 0.79

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วารสาร "J. Appl. Math. Comput" (วารสารระดับนานาชาติ)
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วารสาร "Fixed Point Theory and Applications" (วารสารระดับนานาชาติ)
Corresponding Author : R. Wangkeeree
Impact factor ปี 2009 = 1.52

(8) **R. Wangkeeree,** N. Petrot, R. Wangkeeree, The general iterative methods for nonexpansive mappings in Banach Spaces, DOI 10.1007/s10898-010-9617-6
วารสาร "Journal of Global Optimization" (วารสารระดับนานาชาติ)
Corresponding Author : R. Wangkeeree
Impact factor ปี 2009 = 1.45

(9) U. Kamraksa and **R. Wangkeeree**, Generalized Equilibrium Problems and fixed point problems for nonexpansive semigroups in Hilbert spaces, DOI 10.1007/s 10898-011-9654-9
วารสาร "Journal of Global Optimization" (วารสารระดับนานาชาติ)
Corresponding Author : R. Wangkeeree

Impact factor ปี 2009 = 1.45

(10) **R. Wangkeeree**, The general hybrid approximation methods for nonexpansive mappings in Banach spaces, (Accepted)
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Corresponding Author : R. Wangkeeree
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(11) U. Kamraksa and **R. Wangkeeree**, Iterative Algorithms for solving Mixed Equilibrium Problem and variational inequality problem of a finite family of asymptotically strict pseudo-contractions, (Accepted)
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2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ในด้านเชิงวิชาการเพื่อให้รู้ถึงวิธีการประมาณค่าคำตอบของปัญหาทางคณิตศาสตร์ในรูปแบบต่างๆ ทั้งในระดับพื้นฐาน และระดับขั้นสูง นอกจากนั้นแล้ว มีการนำไปใช้ประโยชน์ในเชิงสาธารณะ โดยทำให้มีการพัฒนาการเรียนการสอน และ มีเครือข่ายความร่วมมือ สร้างกระแสความสนใจในด้านการพัฒนาวิธีการประมาณค่าประมาณค่า ให้ก้วางขวางมากยิ่งขึ้น

3. การเสนอผลงานในที่ประชุมวิชาการระดับนานาชาติจำนวน 5 ครั้ง

3.1 วันที่ 27 มีนาคม 2552-31 มีนาคม 2552

หัวข้อ: A hybrid iterative scheme for equilibrium problems and fixed point problems of asymptotically k-strictly pseudo-contractions

ชื่อการประชุม: The sixth international conference on Nonlinear Analysis and Convex Analysis (NACA2009), Tokyo Institute of Technology, Tokyo, Japan

ประเทศ Japan

3.2 วันที่ 6 กรกฎาคม 2552-11 กรกฎาคม 2552

หัวข้อ: A general iterative method for variational inequality problems, mixed equilibrium problems and fixed point problems of strictly pseudocontractive mappings in Hilbert spaces

ชื่อการประชุม: Function spaces 2009, Jagiellonian Univeristy, Krakow 6-11 July, 2009., Poland.

ประเทศ Poland

3.3 วันที่ 27 กรกฎาคม 2552-31 กรกฎาคม 2552

หัวข้อ: A general iterative method for variational inequality problems, mixed equilibrium problems and fixed point problems of strictly pseudocontractive mappings in Hilbert spaces

ชื่อการประชุม: The 10th international conference on nonlinear functional analysis and applications, Masan and Chinju, Korea

ประเทศ Korea

3.4 วันที่ 16 กรกฎาคม 2553-19 กรกฎาคม 2553

หัวข้อ: THE SYSTEM OF VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS FOR A COUNTABLE FAMILY OF STRICT PSEUDO-CONTRACTIONS IN BANACH SPACES

ชื่อการประชุม: THE 4th INTERNATIONAL CONFERENCE FIXED POINT THEORY, VARIATIONAL INEQUALITY AND IT'S APPROXIMATION ALGORITHMS, Chaina

ประเทศ Chaina

3.5 วันที่ 9 กันยายน 2553-12 กันยายน 2553

หัวข้อ: A general iterative method for the system of variational Inequalities of a countable family of strict pseudo-contractions

ชื่อการประชุม: The second conference on nonlinear and optimization (NAO-Asian 2010), At Royal Paradise Hotel & Spa, Patong beach, Phuket, THAILAND

ปรพเทศ Thailand

ภาคผนวก 1

An iterative approximation method for solving a general system of variational inequality problems and mixed equilibrium problems

R. Wangkeeree and U. Kamraksa

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An iterative approximation method for solving a general system of variational inequality problems and mixed equilibrium problems ☆

Rabian Wangkeeree *, Uthai Kamraksa

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

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ABSTRACT

In this paper, we introduce an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of a general system of variational inequalities for a cocoercive mapping in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithm converges strongly to a common element of the above three sets. Our results extend and improve the corresponding results of Ceng, Wang, and Yao [L.C. Ceng, C.Y. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Methods Oper. Res. 67 (2008) 375–390], Ceng and Yao [L.C. Ceng, J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, J. Comput. Appl. Math. doi:10.1016/j.cam.2007.02.022], Takahashi and Takahashi [S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515] and many others.

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1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, C is a nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction, i.e., $\Theta(u, u) = 0$ for each $u \in C$. The mixed equilibrium problem (for short, *MEP*) is to find $x^* \in C$ such that

$$MEP : \Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C. \tag{1.1}$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (for short, *EP*), which is to find $x^* \in C$ such that

$$EP : \Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

Denote the set of solutions of *MEP* by Ω. The problem (1.1) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, the equilibrium problems and others; see, e.g., [1–4]. Some methods have been proposed to solve the *MEP* and *EP*, see, e.g., [1,5,6,2,7–9,3,10–18,4,19]. First we recall some relevant important results as follows.

In 1997, Combettes and Hirstoaga [8] introduced an iterative method for finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, Takahashi and Takahashi [13] introduced another iterative scheme for finding a common element of the set of solutions of *EP* and the set of fixed points of a nonexpansive mapping.

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* Corresponding author. Tel.: +66 5963249.
E-mail addresses: rabianw@nu.ac.th (R. Wangkeeree), uthaikam@hotmail.com (U. Kamraksa).

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Using the idea of Takahashi and Takahashi [13], Plubtieng and Punpaeng [10] introduced the general iterative method for finding a common element of the set of solutions of *EP* and the set of fixed points of a nonexpansive mapping which is the optimality condition for the minimization problem in a Hilbert space. Furthermore, Yao, Liou and Yao [18,20] introduced some new iterative schemes for finding a common element of the set of solutions of *EP* and the set of common fixed points of finitely (infinitely) nonexpansive mappings. Very recently, Ceng and Yao [7] considered a new iterative scheme for finding a common element of the set of solutions of *MEP* and the set of common fixed points of finitely many nonexpansive mappings. Their results extend and improve the corresponding results in [8,13,18].

Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

A mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

Denote the set of fixed points of S by $F(S)$.

Recall that

(1) A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

(2) A is called d-strongly monotone, if each $x, y \in C$, we have

$$\langle Ax - Ay, x - y\rangle \ge d\|x - y\|^2,$$

for a constant $d > 0$. This implies that

$$\|Ax - Ay\| \ge d\|x - y\|,$$

that is, A is d-expansive and when $d = 1$, it is expansive.

(3) A is said to be c-cocoercive [21,22], if each $x, y \in C$, we have

$$\langle Ax - Ay, x - y\rangle \ge c\|Ax - Ay\|^2,$$

for a constant $c > 0$. Clearly, every c-cocoercive map A is $\frac{1}{c}$-Lipschitz continuous.

(4) A is said to be $(-c)$-cocoercive, if there exists a constant $c > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge (-c)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

(5) A is said to be relaxed (c, d)-cocoercive, if there exists two constants $c, d > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge (-c)\|Ax - Ay\|^2 + d\|x - y\|^2, \quad \forall x, y \in C.$$

For $c = 0$, A is d-strongly monotone. This class of maps is more general than the class of strongly monotone maps. It is easy to see that we have the following implication: d-strongly monotonicity $\Rightarrow$ relaxed (c, d)-cocoercivity.

(6) An operator T is strongly positive on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Tx, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \ge 0$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [15–17,23,20,24] and the references therein.

For finding an element of $F(S) \cap VI(A, C)$, under the assumption that a set $C \subseteq H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse-strongly monotone, Takahashi and Toyoda [25] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap VI(A, C) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.5) converges weakly to some $z \in F(S) \cap VI(A, C)$. Recently, motivated by the idea of Korpelevich's extragradient method [26], Nadezhkina and Takahashi [27] introduced an iterative scheme for finding an element of $F(S) \cap VI(A, C)$ and the weak convergence theorem is presented. Moreover, Zeng and Yao [28] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$ and obtained the weak convergence theorem for such schemes. Yao, Liou and Yao [20] introduced new iterative scheme for finding an element of $F(S) \cap VI(A, C)$, under some mild conditions, they obtained the strong convergence theorems in a real Hilbert space.

Let $A, B : C \to H$ be two mappings. The general system of variational inequality problem (see [29]) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \ge 0, & \forall x \in C, \\ \langle \mu By^* + y^* - x^*, x - y^*\rangle \ge 0, & \forall x \in C, \end{cases} \tag{1.6}$$

where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.6) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu Ay^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.7}$$

which is defined by Verma [30] (see also Verma [31]), and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (1.7) reduces to the classical variational inequality $VI(A, C)$. In order to find the solutions of the general system of variational inequality problem (1.6), Ceng, Wang, and Yao [29] studied the following approximation method. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping. Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{cases} \tag{1.8}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$. They proved that, under quite mild conditions on the parameters, the iterative sequence defined by the relaxed extragradient method (1.8) converges strongly to a fixed point of S which is a solution of general system of variational inequality (1.6).

On the other hand, Moudafi [32] introduced the viscosity approximation method for nonexpansive mappings (see [33] for further developments in both Hilbert and Banach spaces). Let f be a contraction on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Sx_n + \sigma_n f(x_n), \quad n \geq 0, \tag{1.9}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [32,33] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.9) strongly converges to the unique solution q in C of the variational inequality

$$\langle (I - f)q, p - q \rangle \geq 0, \quad p \in C.$$

Recently, Marino and Xu [34] introduced the following general iterative method:

$$x_{n+1} = (I - \alpha_n T)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0 \tag{1.10}$$

where T is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.10) converges strongly to the unique solution of the variational inequality

$$\langle (T - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C \tag{1.11}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Tx, \quad x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Ceng and Yao [7] introduced a hybrid iterative scheme for finding a common element of the set of solutions of *MEP* and the set of common fixed points of finitely many nonexpansive mappings. Furthermore, they proved that the sequences generated by the hybrid iterative scheme converge strongly to a common element of the set of solutions of *MEP* and the set of common fixed points of finitely many nonexpansive mappings.

In this paper, motivated and inspired by Ceng, Wang, and Yao [29], Moudafi [32], Marino and Xu [34], Ceng and Yao [7], we introduce an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of the general system of variational inequality. Furthermore, we prove that the proposed iterative algorithm converges strongly to a common element of the above three sets. Our results extend and improve the corresponding results of Ceng, Wang, and Yao [29], Ceng and Yao [7], Takahashi and Takahashi [13] and many others.

2. Preliminaries

Let H be a real Hilbert space. It is well known that for any $\lambda \in [0, 1]$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, \quad y - P_C x \rangle \leq 0, \tag{2.2}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.3}$$

for all $x \in H, y \in C$.

Now we collect some useful lemmas for proving the convergence result of this paper.

Lemma 2.1 ([12]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.2 ([33]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 1$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([34]). *Assume that T is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|T\|^{-1}$. Then $\|I - \rho T\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.4 (*Demi-Closedness Principle [9]*). *Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If S has a fixed point, then $I - S$ is demi-closed; that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I-S)x_n\}$ converges strongly to some y (for short, $(I-S)x_n \to y$), it follows that $(I - S)x = y$. Here I is the identity operator of H.*

For solving the mixed equilibrium problem for an equilibrium bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(H1) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0, \forall x, y \in C$;
(H2) for each fixed $y \in C, x \mapsto \Theta(x, y)$ is convex and upper semicontinuous;
(H3) for each $x \in C, y \mapsto \Theta(x, y)$ is convex.

A mapping $\eta : C \times C \to H$ is called Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda\|x - y\|, \quad \forall x, y \in C.$$

A differentiable function $K : C \to \mathbb{R}$ on a convex set C is called:

(K1) η-convex [5] if

$$K(y) - K(x) \geq \langle K'(x), \eta(y, x)\rangle, \quad \forall x, y \in C,$$

where $K'(x)$ is the Fréchet derivative at x;

(K2) η-strongly convex if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x)\rangle \geq \frac{\sigma}{2}\|x - y\|^2, \quad \forall x, y \in C.$$

Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be a real-valued function, and $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction. Let r be a positive number. For a given point $x \in C$, the auxiliary problem for *MEP* consists of finding $y \in C$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \quad \forall z \in C.$$

Let $J_r : C \to C$ be the mapping such that for each $x \in C, J_r(x)$ is the solution set of the auxiliary problem *MEP*, that is

$$J_r(x) = \{y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \forall z \in C\}, \quad \forall x \in C.$$

The following lemmas appear implicitly in [5].

Lemma 2.5 ([5]). *Let C be nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from C to $\mathbb{R}$. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (H1)–(H3). *Assume that*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that;*
  (a) $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in C$,
  (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in C, x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : C \to \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology;*
(iii) *for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0.$$

Then, there exists $y \in C$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.6 ([5]). *Assume that Θ satisfies the same assumptions as Lemma 2.5 for $r > 0$ and $x \in C$, the mapping $J_r : C \to C$ can be defined as follows:*

$$J_r(x) = \left\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \forall z \in C \right\}, \quad \forall x \in C.$$

Then, the following hold:
(i) *J_r is single-valued;*
(ii) (a) *$\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1, u_2)\rangle$, $\forall (x_1, x_2) \in C \times C$, where $u_i = J_r(x_i)$, $i = 1, 2$;*
(b) *J_r is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\sigma > \lambda \upsilon$;*
(iii) *$F(J_r) = \Omega$; and*
(iv) *Ω is closed and convex.*

We also need the following lemma for proving our main results.

Lemma 2.7 ([29]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem (1.6) if and only if x^* is a fixed point the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)], \quad \forall x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$, λ, μ are positive constants and $A, B : C \to H$ are two mappings.

Throughout this paper, the set of fixed points of the mapping G is denoted by $GVI(C, A, B)$.
Now, we prove the following lemma which will be applied in the main theorem.

Lemma 2.8. *Let $G : C \to C$ be defined in Lemma 2.7. If $A : C \to H$ is a L_A-Lipschitzian and relaxed (c, d)-cocoercive mapping and $B : C \to H$ a L_B-Lipschitzian and relaxed (c', d')-cocoercive mapping where $\lambda \leq \frac{2(d - cL_A^2)}{L_A^2}$ and $\mu \leq \frac{2(d' - c'L_B^2)}{L_B^2}$, then G is nonexpansive.*

Proof. For any $x, y \in C$, we have

$$\begin{aligned}
\|G(x) - G(y)\|^2 &= \|P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)] - P_C[P_C(y - \mu By) - \lambda AP_C(y - \mu By)]\|^2 \\
&\leq \|P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx) - [P_C(y - \mu By) - \lambda AP_C(y - \mu By)]\|^2 \\
&= \|[P_C(x - \mu Bx) - P_C(y - \mu By)] - \lambda[AP_C(x - \mu Bx) - AP_C(y - \mu By)]\|^2 \\
&= \|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 - 2\lambda\langle P_C(x - \mu Bx) - P_C(y - \mu By), \\
&\quad AP_C(x - \mu Bx) - AP_C(y - \mu By)\rangle + \lambda^2\|AP_C(x - \mu Bx) - AP_C(y - \mu By)\|^2 \\
&\leq \|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 - 2\lambda\big[-c\|AP_C(x - \mu Bx) - AP_C(y - \mu By)\|^2 \\
&\quad + d\|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2\big] + \lambda^2 L_A^2\|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 \\
&\leq \|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 + 2\lambda c L_A^2\|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 \\
&\quad - 2\lambda d\|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 + \lambda^2 L_A^2\|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 \\
&= (1 + 2\lambda c L_A^2 - 2\lambda d + \lambda^2 L_A^2)\|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 \\
&\leq \|P_C(x - \mu Bx) - P_C(y - \mu By)\|^2 \\
&\leq \|x - \mu Bx - (y - \mu By)\|^2 \\
&\leq \|x - y\|^2 - 2\mu\langle x - y, Bx - By\rangle + \mu^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 - 2\mu[-c'\|Bx - By\|^2 + d'\|x - y\|^2] + \mu^2 L_B^2\|x - y\|^2 \\
&= (1 + 2\mu c' L_B^2 - 2\mu d' + \mu^2 L_B^2)\|x - y\|^2 \\
&\leq \|x - y\|^2.
\end{aligned} \tag{2.4}$$

This show that G is nonexpansive on C. □

3. Main results

In this paper, we first introduce our iterative scheme. Consequently, we will establish strong convergence theorems for this iterative scheme. To be more specific, let $S_1, S_2, \ldots$ be infinite mappings of C into itself and $\{\xi_i\}$ be a nonnegative real sequence with $0 \leq \xi_i < 1, \forall i \geq 1$. For any $n \in \mathbb{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \xi_n S_n U_{n,n+1} + (1-\xi_n)I, \\ U_{n,n-1} &= \xi_{n-1} S_{n-1} U_{n,n} + (1-\xi_{n-1})I, \\ &\vdots \\ U_{n,k} &= \xi_k S_k U_{n,k+1} + (1-\xi_k)I, \\ U_{n,k-1} &= \xi_{k-1} S_{k-1} U_{n,k} + (1-\xi_{k-1})I, \\ &\vdots \\ U_{n,2} &= \mu_2 S_2 U_{n,3} + (1-\xi_2)I, \\ W_n = U_{n,1} &= \xi_1 S_1 U_{n,2} + (1-\xi_1)I. \end{aligned} \tag{3.1}$$

Nonexpansivity of each S_i ensures the nonexpansivity of W_n. The mapping W_n is is called a W-mapping generated by $S_1, S_2, \ldots, S_n$ and $\xi_1, \xi_2, \ldots, \xi_n$.

Throughout this paper, we will assume that $0 < \xi_n \leq \xi < 1, \forall n \geq 1$. Concerning W_n defined by (3.1), we have the following lemmas which are important to prove our main result. Now we only need the following similar version in Hilbert spaces.

Lemma 3.1 ([35]). *Let C be a nonempty closed convex subset of a Hilbert space H, $S_i : C \to C$ be a family of infinitely nonexpansive mapping with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, $\{\xi_i\}$ be a real sequence such that $0 < \xi_i \leq \xi < 1, \forall i \geq 1$. Then:*

(1) *W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{n} F(S_i)$ for each $n \geq 1$;*
(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*
(3) *the mapping $W : C \to C$ define by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C, \tag{3.2}$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\xi_1, \xi_2, \ldots$.

Lemma 3.2 ([36]). *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, $\{\xi_i\}$ be a real sequence such that $0 < \xi_i \leq \xi < 1, \forall i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

Now we introduce the following iteration algorithm.

Algorithm 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3) and let $\{S_n\}$ be an infinite family of nonexpansive mappings of C into itself. Let $r, \gamma > 0$ be two constants. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and let T be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} < \frac{\gamma}{\alpha}$. Let $A : C \to H$ be a L_A-Lipschitzian and relaxed (c, d)-cocoercive mapping and $B : C \to H$ be a L_B-Lipschitzian and relaxed (c', d')-cocoercive mapping. Suppose the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are generated iteratively by

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r}\langle K'(u_n) - K'(x_n), \eta(x, u_n)\rangle \geq 0, \quad \forall x \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n T) W_n P_C(y_n - \lambda A y_n), \end{cases} \tag{3.3}$$

where $\{\alpha_n\}$and $\{\beta_n\}$ are two sequences in $(0, 1)$ and W_n is defined by (3.1).

Now we study the strong convergence of the hybrid iterative method (3.3).

Theorem 3.3. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions* (H1)–(H3), *$A : C \to H$ be a L_A-Lipschitzian and relaxed (c, d)-cocoercive mapping and $B : C \to H$ be a L_B-Lipschitzian and relaxed (c', d')-cocoercive mapping. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $F := \cap_{n=1}^{\infty} F(S_n) \cap \Omega \cap GVI(C, A, B) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and T be a strongly bounded linear operator on H with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*
  (a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$,*

(b) $\eta(\cdot,\cdot)$ *is affine in the first variable,*
(c) *for each fixed* $y \in C$, $x \mapsto \eta(y,x)$ *is sequentially continuous from the weak topology to the weak topology;*
(ii) $K : C \to \mathbb{R}$ *is* η*-strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is not only sequentially continuous from the weak topology to the weak topology but also Lipschitz continuous with constant* $\nu > 0$ *such that* $\sigma \geq \lambda\nu$;
(iii) *for each* $x \in C$; *there exist a bounded subset* $D_x \subset C$ *and* $u_x \in C$ *such that for any* $y \in C \setminus D_x$,

$$\Theta(y,u_x) + \varphi(u_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(u_x,y)\rangle < 0; \tag{3.4}$$

(iv) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\lambda \leq \frac{2(d-cL_A^2)}{L_A^2}$ *and* $\mu \leq \frac{2(d'-c'L_B^2)}{L_B^2}$.

Given $x_1 \in C$ *arbitrarily, then the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *generated iteratively by* (3.3) *converge strongly to* $\tilde{x} \in F$ *where* $\tilde{x} = P_F(\gamma f + (I - T))(\tilde{x})$, *which is the unique solution of the variational inequality*

$$\langle(\gamma f - T)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall x \in F,$$

and $(\tilde{x},\tilde{y})$ *is a solution of the general system of variational inequality problem* (1.6) *such that* $\tilde{y} = P_C(\tilde{x} - \mu B\tilde{x})$.

Proof. Note that from control condition (iv), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|T\|^{-1}$ for all $n \in \mathbb{N}$. Since T is a linear bounded self-adjoint operator on H, then

$$\|T\| = \sup\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I - \alpha_n T)x, x\rangle &= 1 - \beta_n - \alpha_n\langle Tx, x\rangle \\ &\geq 1 - \beta_n - \alpha_n\|T\| \\ &\geq 0,\end{aligned}$$

this shows that $(1-\beta_n)I - \alpha_n T$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n T\| &= \sup\{\langle((1-\beta_n)I - \alpha_n T)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Tx, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma}.\end{aligned}$$

Next we divide the following proofs into several steps.
Step 1. We claim that $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are all bounded.
Let $x^* \in F(W) \cap \Omega \cap GVI(C, A, B)$. Then

$$x^* = P_C[P_C(x^* - \mu Bx^*) - \lambda AP_C(x^* - \mu Bx^*)].$$

Putting $y^* = P_C(x^* - \mu Bx^*)$ and $t_n = P_C(y_n - \lambda Ay_n)$, we have $x^* = P_C(y^* - \mu Ay^*)$ and

$$x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n T)W_n t_n.$$

From the definition of J_r, we know that $u_n = J_r x_n$. It follows that

$$\|u_n - x^*\| = \|J_r x_n - J_r x^*\| \leq \|x_n - x^*\|. \tag{3.5}$$

Since $A : C \to H$ is a L_A-Lipschitzian and relaxed (c,d)-cocoercive mapping and $\lambda \leq \frac{2(d-cL_A^2)}{L_A^2}$, we have

$$\begin{aligned}\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - 2\lambda[-c\|Ax-Ay\|^2 + d\|x-y\|^2] + \lambda^2L_A^2\|x-y\|^2 \\ &\leq \|x-y\|^2 + 2\lambda cL_A^2\|x-y\|^2 - 2\lambda d\|x-y\|^2 + \lambda^2L_A^2\|x-y\|^2 \\ &\leq (1 + 2\lambda cL_A^2 - 2\lambda d + \lambda^2L_A^2)\|x-y\|^2 \\ &\leq \|x-y\|^2, \quad \forall x,y \in C.\end{aligned} \tag{3.6}$$

This show that $I - \lambda A$ is nonexpansive. Similarly, since $B : C \to H$ is a L_B-Lipschitzian and relaxed (c',d')-cocoercive mapping, and $\mu \leq \frac{2(d'-c'L_B^2)}{L_B^2}$, we obtain

$$\begin{aligned}\|(I-\mu B)x - (I-\mu B)y\|^2 &= \|x-y\|^2 - 2\mu\langle x-y, Bx-By\rangle + \mu^2\|Bx-By\|^2 \\ &\leq \|x-y\|^2 - 2\mu[-c'\|Bx-By\|^2 + d'\|x-y\|^2] + \mu^2L_B^2\|x-y\|^2 \\ &\leq \|x-y\|^2 + 2\mu c'L_B^2\|x-y\|^2 - 2\mu d'\|x-y\|^2 + \mu^2L_B^2\|x-y\|^2 \\ &\leq (1 + 2\mu c'L_B^2 - 2\mu d' + \mu^2L_B^2)\|x-y\|^2 \\ &\leq \|x-y\|^2, \quad \forall x,y \in C.\end{aligned} \tag{3.7}$$

Hence $I-\mu B$ is nonexpansive. From (3.5) and nonexpansivities of $I-\lambda A$ and $I-\mu B$, we have

$$\begin{aligned}\|t_n-x^*\| &= \|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\|\\ &\le \|(I-\lambda A)y_n-(I-\lambda A)y^*\|\\ &\le \|y_n-y^*\|\\ &= \|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\|\\ &\le \|(I-\mu B)u_n-(I-\mu B)x^*\|\\ &\le \|u_n-x^*\|\\ &\le \|x_n-x^*\|. \end{aligned} \tag{3.8}$$

Then, we have also

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n(\gamma f(x_n)-Tp)+\beta_n(x_n-p)+((1-\beta_nI-\alpha_nT)(W_nt_n-x^*)\|\\ &\le (1-\beta_n-\alpha_n\overline{\gamma})\|t_n-x^*\|+\beta_n\|x_n-x^*\|+\alpha_n\|\gamma f(x_n)-Tp\|\\ &\le (1-\beta_n-\alpha_n\overline{\gamma})\|x_n-x^*\|+\beta_n\|x_n-x^*\|+\alpha_n\|\gamma f(x_n)-Tp\|\\ &\le (1-\alpha_n\overline{\gamma})\|x_n-x^*\|+\alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Tp\|\\ &\le (1-\alpha_n\overline{\gamma})\|x_n-x^*\|+\alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Tp\|\\ &= (1-(\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n-x^*\|+(\overline{\gamma}-\gamma\alpha)\alpha_n\frac{\|f(p)-Tp\|}{\overline{\gamma}-\gamma\alpha}.\end{aligned}$$

It follows from induction that

$$\|x_n-x^*\|\le\max\left\{\|x_1-x^*\|,\frac{\|f(p)-Tp\|}{\overline{\gamma}-\gamma\alpha}\right\},\quad n\ge 1. \tag{3.9}$$

Thus the sequence $\{x_n\}$ is bounded. Consequently, the sets $\{u_n\}$, $\{t_n\}$, $\{Ay_n\}$, $\{Bx_n\}$ and $\{Wt_n\}$ are bounded.

Step 2. We claim that $\|x_{n+1}-x_n\|\to 0$.

We observe that

$$\begin{aligned}\|t_{n+1}-t_n\| &= \|P_C(y_{n+1}-\lambda Ay_{n+1})-P_C(y_n-\lambda Ay_n)\|\\ &\le \|(y_{n+1}-\lambda Ay_{n+1})-(y_n-\lambda Ay_n)\|\\ &\le \|y_{n+1}-y_n\|\\ &= \|P_C(u_{n+1}-\mu Bu_{n+1})-P_C(u_n-\mu Bu_n)\|\\ &\le \|(u_{n+1}-\mu Bu_{n+1})-(u_n-\mu Bu_n)\|\\ &\le \|u_{n+1}-u_n\|.\end{aligned} \tag{3.10}$$

Since $u_{n+1}=J_rx_{n+1}$ and $u_n=J_rx_n$, from the nonexpansivity of J_r, we get

$$\|u_{n+1}-u_n\|\le\|x_{n+1}-x_n\|. \tag{3.11}$$

Let $x_{n+1}=(1-\beta_n)z_n+\beta_nx_n$. Thus, we get $z_n=\frac{\alpha_nf(x_n)+((1-\beta_nI-\alpha_nT)W_nt_n}{1-\beta_n}$

$$\begin{aligned} z_{n+1}-z_n &= \frac{\alpha_{n+1}f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}T)W_{n+1}t_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nf(x_n)+((1-\beta_n)I-\alpha_nT)W_nt_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)+W_{n+1}t_{n+1}-W_nt_n+\frac{\alpha_n}{1-\beta_n}TW_nt_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}TW_{n+1}t_{n+1}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-TW_{n+1}t_{n+1})+\frac{\alpha_n}{1-\beta_n}(TW_nt_n-\gamma f(x_n))+W_{n+1}t_{n+1}-W_{n+1}t_n+W_{n+1}t_n-W_nt_n.\end{aligned} \tag{3.12}$$

Since S_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}\|W_{n+1}t_n-W_nt_n\| &= \|\xi_1S_1U_{n+1,2}t_n-\xi_1S_1U_{n,2}t_n\|\\ &\le \xi_1\|U_{n+1,2}t_n-U_{n,2}t_n\|\\ &= \xi_1\|\xi_2S_2U_{n+1,3}t_n-\xi_2S_2U_{n,3}t_n\|\\ &\le \xi_1\xi_2\|U_{n+1,3}t_n-U_{n,3}t_n\|\\ &\ \ \vdots\\ &\le \xi_1\xi_2\cdots\xi_n\|U_{n+1,n+1}t_n-U_{n,n+1}t_n\|\\ &\le M_1\prod_{i=1}^n\xi_i,\end{aligned} \tag{3.13}$$

where $M_1 \geq 0$ is a constant such that $\|U_{n+1,n+1}t_n - U_{n,n+1}t_n\| \leq M_1$ for all $n \geq 1$. It follows from (3.10) and (3.12) and (3.13) that

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|TW_{n+1}t_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|TW_nt_n\| + \|\gamma f(x_n)\|) + \|W_{n+1}t_{n+1} - W_{n+1}t_n\| + \|W_{n+1}t_n - W_nt_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|TW_{n+1}t_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|TW_nt_n\| + \|\gamma f(x_n)\|) + \|t_{n+1} - t_n\| \\
&\quad + \|W_{n+1}t_n - W_nt_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|TW_{n+1}t_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|TW_nt_n\| + \|\gamma f(x_n)\|) + M_1\prod_{i=1}^{n}\xi_i,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.1, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.14}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.15}$$

From (iv), (3.10) and (3.11), we also have $\|t_{n+1} - t_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Step 3. We claim that $\|Wt_n - t_n\| \to 0$. To do this, we observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n(\gamma f(x_n) - Tx_n) + ((1-\beta_n)I - \alpha_nT)(W_nt_n - x_n)\| \\
&\leq \alpha_n\|\gamma f(x_n) - Tx_n\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|W_nt_n - x_n\|.
\end{aligned}$$

It follows from $\lim_{n\to\infty}\alpha_n = 0$ and (3.15) that

$$\lim_{n\to\infty}\|W_nt_n - x_n\| = 0.$$

Applying Lemma 3.2 and the last equation, we obtain

$$\|Wt_n - x_n\| \leq \|Wt_n - W_nt_n\| + \|W_nt_n - x_n\| \leq \sup_{t\in\{t_n\}}\|Wt - W_nt\| + \|W_nt_n - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

From (3.3), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|((1-\beta_n)I - \alpha_nT)(W_nt_n - x^*) + \beta_n(x_n - x^*) + \alpha_n(\gamma f(x_n) - Tx^*)\|^2 \\
&= \|((1-\beta_n)I - \alpha_nT)(W_nt_n - x^*) + \beta_n(x_n - x^*)\|^2 + \alpha_n^2\|\gamma f(x_n) - Tx^*\|^2 \\
&\quad + 2\beta_n\alpha_n\langle x_n - x^*, \gamma f(x_n) - Tx^*\rangle + 2\alpha_n\langle((1-\beta_n)I - \alpha_nT)(W_nt_n - x^*), \gamma f(x_n) - Tx^*\rangle \\
&\leq ((1-\beta_n-\alpha_n\bar{\gamma})\|W_nt_n - x^*\| + \beta_n\|x_n - x^*\|)^2 + \alpha_n^2\|\gamma f(x_n) - Tx^*\|^2 \\
&\quad + 2\beta_n\alpha_n\langle x_n - x^*, \gamma f(x_n) - Tx^*\rangle + 2\alpha_n\langle((1-\beta_n)I - \alpha_nT)(W_nt_n - x^*), \gamma f(x_n) - Tx^*\rangle \\
&= (1-\beta_n-\alpha_n\bar{\gamma})^2\|t_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + 2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n\|t_n - x^*\|\|x_n - x^*\| + \alpha_nM_2 \\
&\leq (1-\beta_n-\alpha_n\bar{\gamma})^2\|t_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\beta_n(\|t_n - x^*\|^2 + \|x_n - x^*\|^2) + \alpha_nM_2 \\
&= [(1-\alpha_n\bar{\gamma})^2 - 2(1-\alpha_n\bar{\gamma})\beta_n + \beta_n^2]\|t_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + ((1-\alpha_n\bar{\gamma})\beta_n - \beta_n^2)(\|t_n - x^*\|^2 + \|x_n - x^*\|^2) + \alpha_nM_2 \\
&= (1-\alpha_n\bar{\gamma})^2\|t_n - x^*\|^2 - (1-\alpha_n\bar{\gamma})\beta_n\|t_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + \alpha_nM_2 \\
&= (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|t_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + \alpha_nM_2,
\end{aligned} \tag{3.17}$$

where M_2 is a constant such that

$$\begin{aligned}
M_2 \geq \sup_{n\geq1}\{&\alpha_n\|\gamma f(x_n) - Tx^*\|^2, 2\beta_n\|x_n - x^*\|\|\gamma f(x_n) - Tx^*\|, \\
&2\|((1-\beta_n)I - \alpha_nT)(W_nt_n - x^*)\|\|\gamma f(x_n) - Tx^*\|\}.
\end{aligned}$$

Since J_r is firmly nonexpansive, we have

$$\|u_n - x^*\|^2 = \|J_r x_n - J_r x^*\|^2 \le \langle J_r x_n - J_r x^*, x_n - x^*\rangle$$
$$= \langle u_n - x^*, x_n - x^*\rangle = \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2)$$

and hence

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2.$$

This together with (3.17) implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|u_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + \alpha_n M_2\\ &\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(\|x_n - x^*\|^2 - \|x_n - u_n\|^2) + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + \alpha_n M_2\\ &= (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 - (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|x_n - u_n\|^2 + \alpha_n M_2,\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|x_n - u_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_2\\ &\le \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + \alpha_n M_2.\end{aligned}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and (3.15), we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.18}$$

Moreover, we have

$$\|W_n t_n - u_n\| \le \|W_n t_n - x_n\| + \|x_n - u_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.19}$$

Again from the L_A-Lipschitzian and relaxed (c, d)-cocoercive mapping on A and $\lambda \le \frac{2(d-cL_A^2)}{L_A^2}$, we have

$$\begin{aligned}\|t_n - x^*\|^2 &= \|P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)\|^2\\ &\le \|(y_n - y^*) - \lambda(Ay_n - Ay^*)\|^2\\ &= \|y_n - y^*\|^2 - 2\lambda\langle y_n - y^*, Ay_n - Ay^*\rangle + \lambda^2\|Ay_n - Ay^*\|^2\\ &\le \|y_n - y^*\|^2 - 2\lambda[-c\|Ay_n - Ay^*\|^2 + d\|y_n - y^*\|^2] + \lambda^2\|Ay_n - Ay^*\|^2\\ &= \|x_n - x^*\|^2 + 2\lambda c\|Ay_n - Ay^*\|^2 - 2\lambda d\|y_n - y^*\|^2 + \lambda^2\|Ay_n - Ay^*\|^2\\ &= \|x_n - x^*\|^2 + \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2}\right)\|Ay_n - Ay^*\|^2.\end{aligned} \tag{3.20}$$

Similarly, from the L_B-Lipschitzian and relaxed (c', d')-cocoercive mapping on B and $\mu \le \frac{2(d'-c'L_B^2)}{L_B^2}$, we have

$$\begin{aligned}\|y_n - y^*\|^2 &= \|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\|^2\\ &\le \|(u_n - x^*) - \mu(Bu_n - Bx^*)\|^2\\ &= \|u_n - x^*\|^2 - 2\mu\langle u_n - x^*, Bu_n - Bx^*\rangle + \mu^2\|Bu_n - Bx^*\|^2\\ &\le \|u_n - x^*\|^2 - 2\mu[-c'\|Bu_n - Bx^*\|^2 + d'\|u_n - y^*\|^2] + \mu^2\|Bu_n - Bx^*\|^2\\ &= \|x_n - x^*\|^2 + 2\mu c'\|Bu_n - Bx^*\|^2 - 2\mu d'\|u_n - y^*\|^2 + \mu^2\|Bu_n - Bx^*\|^2\\ &= \|x_n - x^*\|^2 + \left(2\mu c' + \mu^2 - \frac{2\mu d'}{L_B^2}\right)\|Bu_n - By^*\|^2.\end{aligned} \tag{3.21}$$

Substituting (3.20) into (3.17), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\left[\|x_n - x^*\|^2 + \left(2\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2}\right)\|Ay_n - Ay^*\|^2\right]\\ &\quad + (1-\alpha_n\overline{\gamma})\beta_n\|x_n - x^*\|^2 + \alpha_n M_2\\ &\le (1-\alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\left(\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2}\right)\|Ay_n - Ay^*\|^2 + \alpha_n M_2\\ &\le \|x_n - x^*\|^2 + 2\left(\lambda c + \lambda^2 - \frac{2\lambda d}{L_A^2}\right)\|Ay_n - Ay^*\|^2 + \alpha_n M_2.\end{aligned} \tag{3.22}$$

Again from (3.17), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|t_n-x^*\|^2+(1-\alpha_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\alpha_nM_2\\ &\le (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-y^*\|^2+(1-\alpha_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\alpha_nM_2\\ &\le (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\left[\|x_n-x^*\|^2+\left(2\mu c'+\mu^2-\frac{2\mu d'}{L_B^2}\right)\|Bu_n-By^*\|^2\right]\\ &\quad +(1-\alpha_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\alpha_nM_2\\ &\le (1-\alpha_n\overline{\gamma})^2\|x_n-x^*\|^2+2\left(\mu c'+\mu^2-\frac{2\mu d'}{L_B^2}\right)\|Bu_n-By^*\|^2+\alpha_nM_2\\ &\le \|x_n-x^*\|^2+2\left(\mu c'+\mu^2-\frac{2\mu d'}{L_B^2}\right)\|Bu_n-By^*\|^2+\alpha_nM_2.\end{aligned} \tag{3.23}$$

Therefore, by (3.22) and (3.23), we have

$$\begin{aligned}0\le -2\left(\lambda c+\lambda^2-\frac{2\lambda d}{L_A^2}\right)\|Ay_n-Ay^*\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_2\\ &\le \|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+\alpha_nM_2\end{aligned}$$

and

$$\begin{aligned}0\le -2\left(\mu c'+\mu^2-\frac{2\mu d'}{L_B^2}\right)\|Bu_n-By^*\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_2\\ &\le \|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+\alpha_nM_2.\end{aligned}$$

It follows from $\alpha_nM_2\to 0$ as $n\to\infty$ and (3.15) that

$$\lim_{n\to\infty}\|Ay_n-Ay^*\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|Bu_n-Bx^*\|=0. \tag{3.24}$$

From (2.1), we have

$$\begin{aligned}\|y_n-y^*\|^2 &= \|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\|^2\\ &\le \langle (u_n-\mu Bu_n)-(x^*-\mu Bx^*),y_n-y^*\rangle\\ &= \frac{1}{2}\{\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2+\|y_n-y^*\|^2-\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)-(y_n-y^*)\|^2\}\\ &\le \frac{1}{2}\{\|u_n-x^*\|^2+\|y_n-y^*\|^2-\|(u_n-y_n)-\mu(Bu_n-Bx^*)-(x^*-y^*)\|^2\}\\ &= \frac{1}{2}\{\|u_n-x^*\|^2+\|y_n-y^*\|^2-\|(u_n-y_n)-(x^*-y^*)\|^2\\ &\quad +2\mu\langle (u_n-y_n)-(x^*-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\}.\end{aligned}$$

So, we obtain

$$\|y_n-y^*\|^2\le \|u_n-x^*\|^2-\|(u_n-y_n)-(x^*-y^*)\|^2+2\mu\langle (u_n-y_n)-(x^*-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2.$$

Hence

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|t_n-x^*\|^2+(1-\alpha_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\alpha_nM_2\\ &\le (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-y^*\|^2+(1-\alpha_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\alpha_nM_2\\ &\le (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\big[\|u_n-x^*\|^2-\|(u_n-y_n)-(x^*-y^*)\|^2\\ &\quad +2\mu\langle (u_n-y_n)-(x^*-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\big]\\ &\quad +(1-\alpha_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\alpha_nM_2\\ &\le (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})[\|x_n-x^*\|^2-\|(u_n-y_n)-(x^*-y^*)\|^2\\ &\quad +2\mu\langle (u_n-y_n)-(x^*-y^*),Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2]\\ &\quad +(1-\alpha_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\alpha_nM_2\\ &\le (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-x^*\|^2-(1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\ \|\ \|(u_n-y_n)-(x^*-y^*)\|^2\\ &\quad +(1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})2\mu\|(u_n-y_n)-(x^*-y^*)\|\|Bu_n-Bx^*\|\\ &\quad -(1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\mu^2\|Bu_n-Bx^*\|^2+(1-\alpha_n\overline{\gamma})\beta_n\|x_n-x^*\|^2+\alpha_nM_2\end{aligned}$$

$$\begin{aligned}&= (1-\alpha_n\overline{\gamma})\|x_n - x^*\|^2 - (1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|(u_n-y_n)-(x^*-y^*)\|^2\\ &\quad + \mu(1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|Bu_n - Bx^*\|^2[2\|(u_n-y_n)-(x^*-y^*)\| - \mu\|Bu_n-Bx^*\|^2]\\ &\quad + \alpha_n M_2\end{aligned}$$

which implies that

$$\begin{aligned}(1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|(u_n-y_n)-(x^*-y^*)\|^2 &\le (1-\alpha_n\overline{\gamma})\|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \alpha_n M_2\\ &\quad + \mu(1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|Bu_n-Bx^*\|^2\big[2\|(u_n-y_n)-(x^*-y^*)\| - \mu\|Bu_n-Bx^*\|^2\big]\\ &\le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + \alpha_n M_2\\ &\quad + \mu(1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|Bu_n-Bx^*\|^2\big[2\|(u_n-y_n)-(x^*-y^*)\| - \mu\|Bu_n-Bx^*\|^2\big]\\ &\le \|x_n-x_{n+1}\|(\|x_n-x^*\| + \|x_{n+1}-x^*\|) + \alpha_n M_2\\ &\quad + \mu(1-\alpha_n\overline{\gamma})(1-\beta_n-\alpha_n\overline{\gamma})\|Bu_n-Bx^*\|^2\big[2\|(u_n-y_n)-(x^*-y^*)\| - \mu\|Bu_n-Bx^*\|^2\big].\end{aligned}$$

From (3.24) and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we have

$$\|(u_n-y_n)-(x^*-y^*)\| \to 0 \quad \text{as } n \to \infty. \tag{3.25}$$

Observe that, we compute

$$\begin{aligned}\|(y_n-t_n)+(x^*-y^*)\|^2 &= \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-[P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)]+\lambda(Ay_n-Ay^*)\|^2\\ &\le \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-[P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)]\|^2\\ &\quad + 2\lambda\langle Ay_n-Ay^*, (y_n-t_n)+(x^*-y^*)\rangle\\ &\le \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2 - \|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\|^2\\ &\quad + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\ &\le \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2 - \|W_nP_C(y_n-\lambda Ay_n)-W_nP_C(y^*-\lambda Ay^*)\|^2\\ &\quad + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\ &= \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|^2 - \|W_nt_n-W_nx^*\|^2 + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\ &\le \|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)-(W_nt_n-x^*)\|(\|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|+\|W_nt_n-x^*\|)\\ &\quad + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|\\ &= \|u_n-W_nt_n+x^*-y^*-(u_n-y_n)-\lambda(Ay_n-Ay^*)\|(\|y_n-\lambda Ay_n-(y^*-\lambda Ay^*)\|\\ &\quad + \|W_nt_n-x^*\|) + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|.\end{aligned} \tag{3.26}$$

Since $\|W_nt_n - u_n\| \to \infty$, $\|(u_n-y_n)-(x^*-y^*)\| \to 0$ and $\|Ay_n - Ay^*\| \to 0$ as $n \to \infty$, we have

$$\|(y_n-t_n)+(x^*-y^*)\| \to 0 \quad \text{as } n \to \infty.$$

Since

$$\|Wt_n-t_n\| \le \|Wt_n-x_n\| + \|x_n-u_n\| + \|(u_n-y_n)-(x^*-y^*)\| + \|(y_n-t_n)-(x^*-y^*)\|$$

and hence

$$\lim_{n\to\infty}\|Wt_n-t_n\| = 0. \tag{3.27}$$

It is clear that $P_F(\gamma f + (I-T))$ is contractive, then $P_F(\gamma f + (I-T))$ has a unique fixed point, say $\tilde{x} \in H$. That is $\tilde{x} = P_F(\gamma f + (I-T))\tilde{x}$.

Step 4. We claim that

$$\limsup_{n\to\infty}\langle(\gamma f - T)\tilde{x}, x_n - \tilde{x}\rangle \le 0. \tag{3.28}$$

To show this inequality, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - T)\tilde{x}, Wt_n - \tilde{x}\rangle = \lim_{i\to\infty}\langle(\gamma f - T)\tilde{x}, Wt_{n_i} - \tilde{x}\rangle.$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $t_{n_i} \rightharpoonup z$. From $\|Wt_n - t_n\| \to 0$, we obtain $Wt_{n_i} \rightharpoonup z$. By Lemma 2.4, we obtain $z \in F(W) = \cap_{i=1}^{\infty} F(S_i)$.

Next, we show that $z \in GVI(C, A, B)$. Since $\|Wt_n - t_n\| \to 0$, $\|Wt_n - x_n\| \to 0$ as $n \to \infty$ and

$$\|t_n - x_n\| \le \|Wt_n - t_n\| + \|Wt_n - x_n\|,$$

we conclude that $\|t_n - x_n\| \to 0$ as $n \to \infty$. Furthermore, by the nonexpansivity of G in Lemma 2.8, we have

$$\begin{aligned}\|t_n - G(t_n)\| &= \|P_C[P_C(x_n - \mu Bx_n) - \lambda AP_C(x_n - \mu Bx_n)]G(t_n)\|\\ &= \|G(x_n) - G(t_n)\|\\ &\leq \|x_n - t_n\|.\end{aligned} \tag{3.29}$$

Thus $\lim_{n\to\infty} \|t_n - G(t_n)\| = 0$. According to Lemma 2.4 we obtain that $z \in GVI(C, A, B)$. Next we show that $z \in \Omega$. Since $u_n = J_r x_n$, we derive

$$\Theta(u_n, x) + \varphi(x) - \varphi(u_n) + \frac{1}{r}\langle K'(u_n) - K'(x_n), \eta(x, u_n)\rangle \geq 0, \quad \forall x \in C.$$

From the monotonicity of Θ, we have

$$\frac{1}{r}\langle K'(u_n) - K'(x_n), \eta(x, u_n)\rangle + \varphi(x) - \varphi(u_n) \geq -\Theta(u_n, x) \geq \Theta(x, u_n),$$

and hence

$$\left\langle \frac{K'(u_{n_j}) - K'(x_{n_j})}{r}, \eta(x, u_{n_j})\right\rangle + \varphi(x) - \varphi(u_{n_j}) \geq \Theta(x, u_{n_j}).$$

Since $(K'(u_{n_j}) - K'(x_{n_j}))/r \to 0$, and $u_{n_j} \to z$ weakly, from the weak lower semicontinuity of φ and $\Theta(x, y)$ in the second variable y, we have

$$\Theta(x, z) + \varphi(z) - \varphi(x) \leq 0$$

for all $x \in C$. For $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)z$. Since $x \in C$ and $z \in C$, we have $x_t \in C$ and hence $\Theta(x_t, z) + \varphi(z) - \varphi(x_t) \leq 0$. From the convexity of equilibrium bifunction $\Theta(x, y)$ in the second variable y, we have

$$\begin{aligned}0 &= \Theta(x_t, x_t) + \varphi(x_t) - \varphi(x_t)\\ &\leq t\Theta(x_t, x) + (1-t)\Theta(x_t, z) + t\varphi(x) + (1-t)\varphi(z) - \varphi(x_t)\\ &\leq t[\Theta(x_t, x) + \varphi(x) - \varphi(x_t)],\end{aligned}$$

and hence $\Theta(x_t, x) + \varphi(x) - \varphi(x_t) \geq 0$. Then, we have

$$\Theta(z, x) + \varphi(x) - \varphi(z) \geq 0$$

for all $x \in C$ and hence $z \in \Omega$. Hence $z \in F$. Now from (2.2), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f - T)\tilde{x}, x_n - \tilde{x}\rangle &= \limsup_{n\to\infty}\langle(\gamma f - T)\tilde{x}, Wt_n - \tilde{x}\rangle\\ &= \lim_{i\to\infty}\langle(\gamma f - T)\tilde{x}, Wt_{n_i} - \tilde{x}\rangle\\ &= \langle(\gamma f - T)\tilde{x}, z - \tilde{x}\rangle \leq 0.\end{aligned} \tag{3.30}$$

Step 5. We claim that $x_n \to \tilde{x}$. Indeed, we observe from (3.3) that

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n T)W_n t_n - \tilde{x}\|^2\\ &= \|((1-\beta_n)I - \alpha_n T)(W_n t_n - \tilde{x}) + \beta_n(x_n - \tilde{x}) + \alpha_n(\gamma f(x_n) - T\tilde{x})\|^2\\ &= \|((1-\beta_n)I - \alpha_n T)(W_n t_n - \tilde{x}) + \beta_n(x_n - \tilde{x})\|^2 + \alpha_n^2\|\gamma f(x_n) - T\tilde{x}\|^2\\ &\quad + 2\beta_n\alpha_n\langle x_n - \tilde{x}, \gamma f(x_n) - T\tilde{x}\rangle + 2\alpha_n\langle((1-\beta_n)I - \alpha_n T)(W_n t_n - \tilde{x}), \gamma f(x_n) - T\tilde{x}\rangle\\ &\leq ((1-\beta_n-\alpha_n\overline{\gamma})\|W_n t_n - \tilde{x}\| + \beta_n\|x_n - \tilde{x}\|)^2 + \alpha_n^2\|\gamma f(x_n) - T\tilde{x}\|^2\\ &\quad + 2\beta_n\alpha_n\gamma\langle x_n - \tilde{x}, f(x_n) - f(\tilde{x})\rangle + 2\beta_n\alpha_n\langle x_n - \tilde{x}, \gamma f(\tilde{x}) - T(\tilde{x})\rangle\\ &\quad + 2(1-\beta_n)\gamma\alpha_n\langle W_n t_n - \tilde{x}, f(x_n) - f(\tilde{x})\rangle + 2(1-\beta_n)\alpha_n\langle W_n t_n - \tilde{x}, \gamma f(\tilde{x}) - T\tilde{x}\rangle\\ &\quad - 2\alpha_n^2\langle T(W_n t_n - \tilde{x}), \gamma f(\tilde{x}) - T\tilde{x}\rangle\\ &\leq ((1-\beta_n-\alpha_n\overline{\gamma})\|x_n - \tilde{x}\| + \beta_n\|x_n - \tilde{x}\|)^2 + \alpha_n^2\|\gamma f(x_n) - T\tilde{x})\|^2\\ &\quad + 2\beta_n\alpha_n\gamma\alpha\|x_n - \tilde{x}\|^2 + 2\beta_n\alpha_n\langle x_n - \tilde{x}, \gamma f(\tilde{x}) - T\tilde{x}\rangle\\ &\quad + 2(1-\beta_n)\gamma\alpha_n\alpha\|x_n - \tilde{x}\|^2 + 2(1-\beta_n)\alpha_n\langle W_n t_n - \tilde{x}, \gamma f(\tilde{x}) - T\tilde{x}\rangle\\ &\quad - 2\alpha_n^2\langle T(W_n t_n - \tilde{x}), \gamma f(\tilde{x}) - T\tilde{x}\rangle\\ &= [(1-\alpha_n\overline{\gamma})^2 + 2\beta_n\alpha_n\gamma\alpha + 2(1-\beta_n)\gamma\alpha_n\alpha]\|x_n - \tilde{x}\|^2 + \alpha_n^2\|\gamma f(x_n) - T\tilde{x}\|^2\\ &\quad + 2\beta_n\alpha_n\langle x_n - \tilde{x}, \gamma f(\tilde{x}) - T\tilde{x}\rangle + 2(1-\beta_n)\alpha_n\langle W_n t_n - \tilde{x}, \gamma f(\tilde{x}) - T\tilde{x}\rangle\end{aligned}$$

$$
\begin{aligned}
&-2\alpha_n^2\langle T(W_nt_n-\tilde{x}),\gamma f(\tilde{x})-T\tilde{x}\rangle\\
&\le [1-2(\overline{\gamma}-\alpha\gamma)\alpha_n]\|x_n-\tilde{x}\|^2+\overline{\gamma}^2\alpha_n^2\|x_n-\tilde{x}\|^2+\alpha_n^2\|\gamma f(x_n)-T\tilde{x}\|^2\\
&\quad+2\beta_n\alpha_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-T\tilde{x}\rangle+2(1-\beta_n)\alpha_n\langle W_nt_n-\tilde{x},\gamma f(\tilde{x})-T\tilde{x}\rangle\\
&\quad+2\alpha_n^2\|T(W_nt_n-\tilde{x})\|\|\gamma f(\tilde{x})-T\tilde{x}\|\\
&=[1-2(\overline{\gamma}-\alpha\gamma)\alpha_n]\|x_n-\tilde{x}\|^2+\alpha_n\{\alpha_n(\overline{\gamma}^2\|x_n-\tilde{x}\|^2+\|\gamma f(x_n)-T\tilde{x}\|^2\\
&\quad+2\|T(W_nt_n-\tilde{x})\|\|\gamma f(\tilde{x})-T\tilde{x}\|)+2\beta_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-T\tilde{x}\rangle\\
&\quad+2(1-\beta_n)\langle W_nt_n-\tilde{x},\gamma f(\tilde{x})-T\tilde{x}\rangle\}. \qquad (3.31)
\end{aligned}
$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{W_nt_n\}$ are bounded, we can take a constant $M_3>0$ such that

$$\overline{\gamma}^2\|x_n-\tilde{x}\|^2+\|\gamma f(x_n)-T\tilde{x}\|^2+2\|T(W_nt_n-\tilde{x})\|\|\gamma f(\tilde{x})-T\tilde{x}\|\le M_3$$

for all $n\ge 0$. It then follows that

$$\|x_{n+1}-\tilde{x}\|^2\le[1-2(\overline{\gamma}-\alpha\gamma)\alpha_n]\|x_n-\tilde{x}\|^2+\alpha_n\sigma_n, \qquad (3.32)$$

where

$$\sigma_n=2\beta_n\langle x_n-\tilde{x},\gamma f(\tilde{x})-T\tilde{x}\rangle+2(1-\beta_n)\langle W_nt_n-\tilde{x},\quad \gamma f(\tilde{x})-T\tilde{x}\rangle+\alpha_nM_3.$$

Using (iv) and (3.30), we get $\limsup_{n\to\infty}\sigma_n\le 0$. Now applying Lemma 2.2 to (3.32), we conclude that $x_n\to\tilde{x}$. This completes the proof. □

Corollary 3.4. *Let C be a closed convex subset of a real Hilbert space H and let $\varphi: C\to\mathbb{R}$ be a lower semicontinuous and convex function. Let $\Theta: C\times C\to\mathbb{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3) and let $A: C\to H$ be a L_A-Lipschitzian and relaxed (c,d)-cocoercive mapping and $B: C\to H$ be a L_B-Lipschitzian and relaxed (c',d')-cocoercive mapping such that $\Omega\cap GVI(C,A,B)\neq\emptyset$. Let f be a contraction of C into itself with coefficient $\alpha\in(0,1)$ and T be a strongly bounded linear operator on H with coefficient $\overline{\gamma}$ and $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Assume that*

(i) *$\eta: C\times C\to H$ is Lipschitz continuous with constant $\lambda>0$ such that*
 (a) *$\eta(x,y)+\eta(y,x)=0$ for all $x,y\in C$,*
 (b) *$\eta(\cdot,\cdot)$ is affine in the first variable,*
 (c) *for each fixed $y\in C$, $x\mapsto\eta(y,x)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K: C\to\mathbb{R}$ is η-strongly convex with constant $\sigma>0$ and its derivative K' is not only sequentially continuous from the weak topology to the weak topology but also Lipschitz continuous with constant $\nu>0$ such that $\sigma\ge\lambda\nu$;*

(iii) *for each $x\in C$; there exist a bounded subset $D_x\subset C$ and $u_x\in C$ such that for any $y\in C\setminus D_x$,*

$$\Theta(y,u_x)+\varphi(u_x)-\varphi(y)+\frac{1}{r}\langle K'(y)-K'(x),\eta(u_x,y)\rangle<0; \qquad (3.33)$$

(iv) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$, $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, $\lambda\le\frac{2(d-cL_A^2)}{L_A^2}$ *and* $\mu\le\frac{2(d'-c'L_B^2)}{L_B^2}$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ generated iteratively by

$$
\begin{cases}
x_1\in C \text{ chosen arbitrarily},\\
\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r}\langle K'(u_n)-K'(x_n),\eta(x,u_n)\rangle\ge 0,\quad\forall x\in C,\\
y_n=P_C(u_n-\mu Bu_n),\\
x_{n+1}=\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nT)P_C(y_n-\lambda Ay_n),
\end{cases} \qquad (3.34)
$$

converge strongly to $\tilde{x}\in\Omega\cap GVI(C,A,B)$ where $\tilde{x}=P_{\Omega\cap GVI(C,A,B)}(\gamma f+(I-T))(\tilde{x})$, which is the unique solution of the variational inequality

$$\langle(\gamma f-T)\tilde{x},x-\tilde{x}\rangle\le 0,\quad\forall x\in\Omega\cap GVI(C,A,B),$$

and $(\tilde{x},\tilde{y})$ is a solution of the general system of variational inequality problem (1.6) such that $\tilde{y}=P_C(\tilde{x}-\mu B\tilde{x})$.

Proof. $S_nx=x$ for all $n=1,2,3,\ldots$, and for all $x\in C$ in (3.1). Then $W_nx=x$ for all $x\in C$. The conclusion follows immediately from Theorem 3.3. This completes the proof. □

Corollary 3.5. *Let C be a closed convex subset of a real Hilbert space H. Let $A: C\to H$ be a L_A-Lipschitzian and relaxed (c,d)-cocoercive mapping and $B: C\to H$ be a L_B-Lipschitzian and relaxed (c',d')-cocoercive mapping. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $F:=\cap_{n=1}^{\infty}F(S_n)\cap GVI(C,A,B)\neq\emptyset$. Let f be a contraction of C into itself with coefficient $\alpha\in(0,1)$ and T be a strongly bounded linear operator on H with coefficient $\overline{\gamma}$ and $0<\gamma<\frac{\overline{\gamma}}{\alpha}$. Assume that*

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$
(iii) $\lambda \le \frac{2(d-cL_A^2)}{L_A^2}$ and $\mu \le \frac{2(d'-c'L_B^2)}{L_B^2}$.

Given $x_1 \in C$ arbitrarily, then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n T)W_n P_C(y_n - \lambda A y_n), \end{cases} \tag{3.35}$$

converge strongly to $\tilde{x} \in F$ where $\tilde{x} = P_F(\gamma f + (I-T))(\tilde{x})$, which is the unique solution of the variational inequality

$$\langle (\gamma f - T)\tilde{x}, x - \tilde{x}\rangle \le 0, \quad \forall x \in F,$$

and $(\tilde{x}, \tilde{y})$ is a solution of the general system of variational inequality problem (1.6) such that $\tilde{y} = P_C(\tilde{x} - \mu B\tilde{x})$.

Proof. Set $\varphi(x) = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$ and put $r = 1$. Take $K(x) = \frac{\|x\|^2}{2}$ and $\eta(y, x) = y - x$ for all $x, y \in C$. Then we have $u_n = P_C x_n = x_n$. Hence the conclusion follows immediately from Theorem 3.3. This completes the proof. □

Theorem 3.6. *Let H be a real Hilbert space. Let $\varphi : H \to \mathbb{R}$ be a lower semicontinuous and convex function. Let $\Theta : H \times H \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3), $A : C \to H$ be a L_A-Lipschitzian and relaxed (c, d)-cocoercive mapping. Let $\{S_n\}$ be a sequence of nonexpansive mappings on H such that $F := \cap_{n=1}^{\infty} F(S_n) \cap \Omega \cap A^{-1}0 \ne \emptyset$. Let f be a contraction on H with coefficient $\alpha \in (0, 1)$ and T be a strongly bounded linear operator on H with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that*

(i) *$\eta : H \times H \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*
 (a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in H$,*
 (b) *$\eta(\cdot, \cdot)$ is affine in the first variable,*
 (c) *for each fixed $y \in H$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology;*
(ii) *$K : H \to \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the weak topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \ge \lambda\nu$;*
(iii) *for each $x \in H$; there exist a bounded subset $D_x \subset H$ and $u_x \in H$ such that for any $y \in H \setminus D_x$,*

$$\Theta(y, u_x) + \varphi(u_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(u_x, y)\rangle < 0; \tag{3.36}$$

(iv) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, $\lambda \le \frac{2(d-cL_A^2)}{L_A^2}$.

Given $x_1 \in H$ arbitrarily, then the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ generated iteratively by

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle K'(u_n) - K'(x_n), \eta(x, u_n)\rangle \ge 0, \quad \forall x \in H, \\ y_n = u_n - \lambda A u_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n T)W_n(y_n - \lambda A y_n), \end{cases} \tag{3.37}$$

converge strongly to $\tilde{x} \in F$ where $\tilde{x} = P_F(\gamma f + (I-T))(\tilde{x})$, which is the unique solution of the variational inequality

$$\langle (\gamma f - T)\tilde{x}, x - \tilde{x}\rangle \le 0, \quad \forall x \in F.$$

Proof. Setting $\lambda = \mu$, $C = H$, $A = B$, we have $P_H = I$. It follows from the proof of Theorem 4.1 in [29] that $A^{-1}0 = VI(H, A)$. Hence the conclusion follows immediately from Theorem 3.3. This completes the proof. □

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ภาคผนวก 2

A General Iterative Method for Solving the Variational Inequality Problem and Fixed Point Problem of an Infinite Family of Nonexpansive Mappings in Hilbert Spaces

R. Wangkeeree and U. Kamraksa

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Research Article

A General Iterative Method for Solving the Variational Inequality Problem and Fixed Point Problem of an Infinite Family of Nonexpansive Mappings in Hilbert Spaces

Rabian Wangkeeree and Uthai Kamraksa

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Rabian Wangkeeree, rabianw@nu.ac.th

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We introduce an iterative scheme for finding a common element of the set of common fixed points of a family of infinitely nonexpansive mappings, and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space. Under suitable conditions, some strong convergence theorems for approximating a common element of the above two sets are obtained. As applications, at the end of the paper we utilize our results to study the problem of finding a common element of the set of fixed points of a family of infinitely nonexpansive mappings and the set of fixed points of a finite family of k-strictly pseudocontractive mappings. The results presented in the paper improve some recent results of Qin and Cho (2008).

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1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, C is a nonempty closed convex subset of H, and P_C is the metric projection of H onto C. In the following, we denote by $\to$ strong convergence and by $\rightharpoonup$ weak convergence. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tu - Tv\| \le \|u - v\|, \quad \forall u, v \in C. \tag{1.1}$$

We denote by $F(T)$ the set of fixed points of T. Recall that a mapping $B : C \to H$ is said to be

(i) monotone if $\langle Bu - Bv, u - v\rangle \ge 0$, for all $u, v \in C$;

(ii) L-Lipschitz if there exists a constant $L > 0$ such that $\|Bu - Bv\| \le L\|u - v\|$, for all $u, v \in C$;

(iii) α-inverse-strongly monotone [1, 2] if there exists a positive real number α such that

$$\langle Bu - Bv, u - v\rangle \geq \alpha\|Bu - Bv\|^2, \quad \forall u, v \in C. \tag{1.2}$$

Remark 1.1. It is obvious that any α-inverse-strongly monotone mapping B is monotone and $(1/\alpha)$-Lipschitz continuous.

Let $B : C \to H$ be a mapping. The classical variational inequality problem is to find a $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

The set of solutions of variational inequality (1.3) is denoted by $\mathrm{VI}(B, C)$. The variational inequality has been extensively studied in the literature; see, for example, [3, 4] and the references therein.

A self-mapping $f : C \to C$ is a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(u) - f(v)\| \leq \alpha\|u - v\|, \quad \forall u, v \in C. \tag{1.4}$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [5–8] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space:

$$\theta(x) = \min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping T, and b is a given point in H. Let H be a real Hilbert space. Recall that a linear bounded operator B is strongly positive if there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.6}$$

Recently, Marino and Xu [9] introduced the following general iterative scheme based on the viscosity approximation method introduced by Moudafi [10]:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.7}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C, \tag{1.8}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle - h(x), \tag{1.9}$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

On the other hand, two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [11] and is defined as follows:

$$\begin{gathered}x_1 = x\in C \text{ chosen arbitrary,}\\ x_{n+1}=(1-\alpha_n)x_n+\alpha_n Tx_n,\quad n\geq 1,\end{gathered} \tag{1.10}$$

where the sequence $\{\alpha_n\}$ is in the interval $(0,1)$.

The second iteration process is referred to as Ishikawa's iteration process [12] which is defined recursively by

$$\begin{gathered}x_1 = x\in C \text{ chosen arbitrary,}\\ y_n=\beta_n x_n+(1-\beta_n)Tx_n,\\ x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ty_n,\quad n\geq 1,\end{gathered} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $(0,1)$. However, both (1.16) and (1.11) have only weak convergence in general (see [13], e.g.). Very recently, Qin and Cho [14] introduced a composite iterative algorithm $\{x_n\}$ defined as follows:

$$\begin{gathered}x_1 = x\in C \text{ chosen arbitrary,}\\ z_n=\gamma_n x_n+(1-\gamma_n)Tx_n,\\ y_n=\beta_n x_n+(1-\beta_n)Tz_n,\\ x_{n+1}=\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n A)y_n,\quad n\geq 1,\end{gathered} \tag{1.12}$$

where f is a contraction, T is a nonexpansive mapping, and A is a strongly positive linear bounded self-adjoint operator, proved that, under certain appropriate assumptions on the parameters, $\{x_n\}$ defined by (1.12) converges strongly to a fixed point of T, which solves some variational inequality and is also the optimality condition for the minimization problem (1.9).

On the other hand, for finding an element of $F(T)\cap \mathrm{VI}(B,C)$, under the assumption that a set $C\subseteq H$ is nonempty, closed, and convex, a mapping $T:C\to C$ is nonexpansive and a mapping $B:C\to H$ is α-inverse-strongly monotone, Takahashi and Toyoda [15] introduced the following iterative scheme:

$$\begin{gathered}x_1 = x\in C \text{ chosen arbitrary,}\\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)TP_C(x_n-\eta_n Bx_n),\quad n\geq 1,\end{gathered} \tag{1.13}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\eta_n\}$ is a sequence in $(0,2\alpha)$. They proved that if $F(T)\cap \mathrm{VI}(B,C)\neq\emptyset$, then the sequence $\{x_n\}$ generated by (1.13) converges weakly to some $z\in F(T)\cap \mathrm{VI}(B,C)$. Recently, Iiduka and Takahashi [16] proposed another iterative scheme as follows

$$\begin{gathered} x_1 = x\in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) T P_C (x_n - \eta_n B x_n), \quad n\geq 1, \end{gathered} \tag{1.14}$$

where B is an α-inverse strongly monotone mapping, $\{\alpha_n\}\subseteq(0,1)$ and $\{\lambda_n\}\subseteq(0,2\alpha)$ satisfy some parameters controlling conditions. They showed that if $F(T)\cap\mathrm{VI}(B,C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.14) converges strongly to some $z\in F(T)\cap\mathrm{VI}(B,C)$.

The existence of common fixed points for a finite family of nonexpansive mappings has been considered by many authors (see [17–20] and the references therein). The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [21, 22]). The problem of finding an optimal point that minimizes a given cost function over the common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance (see [18, 23]). A simple algorithmic solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation (see [23, 24]).

In this paper, we study the mapping W_n defined by

$$\begin{gathered} U_{n,n+1} = I, \\ U_{n,n} = \mu_n T_n U_{n,n+1} + (1-\mu_n) I, \\ U_{n,n-1} = \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1}) I, \\ \vdots \\ U_{n,k} = \mu_k T_k U_{n,k+1} + (1-\mu_k) I, \\ U_{n,k-1} = \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1}) I, \\ \vdots \\ U_{n,2} = \mu_2 T_2 U_{n,3} + (1-\mu_2) I, \\ W_n = U_{n,1} = \mu_1 T_1 U_{n,2} + (1-\mu_1) I, \end{gathered} \tag{1.15}$$

where $\{\mu_i\}$ is a nonnegative real sequence with $0\leq\mu_i<1$, for all $i\geq 1$, $T_1, T_2, \ldots$, form a family of infinitely nonexpansive mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Such a W_n is nonexpansive from C to C and it is called a W-mapping generated by $T_1, T_2, \ldots, T_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

In this paper, motivated and inspired by Su et al. [25], Marino and Xu [9], Takahashi and Toyoda [15], and Iiduka and Takahashi [16], we will introduce a new iterative scheme:

$$\begin{gathered} x_1 = x\in C \text{ chosen arbitrary}, \\ z_n = \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n = \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n) I - \alpha_n A) P_C (y_n - \lambda_n B y_n), \end{gathered} \tag{1.16}$$

where W_n is a mapping defined by (1.15), f is a contraction, A is strongly positive linear bounded self-adjoint operator, B is a α-inverse strongly monotone, and we prove that under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$, the sequences $\{x_n\}$ defined by (1.16) converge strongly to a common element of the set of common fixed points of a family of $\{T_n\}$ and the set of solutions of the variational inequality for an inverse-strongly monotone mapping, which solves some variational inequality and is also the optimality condition for the minimization problem (1.9).

2. Preliminaries

Let H be a real Hilbert space. It is well known that for any $\lambda \in [0,1]$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.2}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{gathered} \langle x - P_C x, y - P_C x\rangle \le 0, \\ \|x-y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \end{gathered} \tag{2.4}$$

for all $x \in H$, $y \in C$. It is easy to see that the following is true:

$$u \in \mathrm{VI}(B, C) \Longleftrightarrow u = P_C(u - \lambda B u), \quad \lambda > 0. \tag{2.5}$$

A Banach space X is said to satisfy the Opial's condition if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad y \ne x. \tag{2.6}$$

It is well known that each Hilbert space satisfies the Opial's condition.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle u - v, w\rangle \geq 0$, for all $u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{2.7}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(B, C)$; see [26].

Now we collect some useful lemmas for proving the convergence result of this paper.

Lemma 2.1. *In a Hilbert space H. Then the following inequality holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y)\rangle, \quad \forall x, y \in H. \tag{2.8}$$

Lemma 2.2 (see [27]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.3 (see [28]). *Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \geq 1, \tag{2.9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$

(2) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4 (see [9]). *Assume that A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Throughout this paper, we will assume that $0 < \mu_n \leq \mu < 1$, for all $n \geq 1$. Concerning W_n defined by (1.15), we have the following lemmas which are important to prove our main result.

Lemma 2.5 (see [29]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $T_i : C \to C$ be a family of infinitely nonexpansive mapping with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ for each $n \geq 1$;*

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*

(3) *the mapping $W : C \to C$ define by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C, \tag{2.10}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by $T_1, T_2, \ldots,$ and $\mu_1, \mu_2, \ldots.$

Lemma 2.6 (see [30]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0. \tag{2.11}$$

3. Main Results

Now we are in a position to state and prove the main result in this paper.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H, let f be a contraction of C into itself, let B be an α-inverse strongly monotone mapping of C into H, and let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $F := \bigcap_{n=1}^{\infty} F(T_i) \cap \mathrm{VI}(B,C) \neq \emptyset$. Let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma \leq \overline{\gamma}/\alpha$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $[0,1]$ satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$,

(C3) $(1+\beta_n)\gamma_n - 2\beta_n > d$ *for some* $d \in (0,1)$,

(C4) $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = \lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$,

(C5) $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$.

Then the sequence $\{x_n\}$ defined by (1.16) *converges strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))q$ which solves the following variational inequality:*

$$\langle \gamma f(q) - Ap, p - q\rangle \leq 0, \quad \forall p \in F. \tag{3.1}$$

Proof. Since $\alpha_n \to 0$ as $n \to \infty$ by the condition (C1), we may assume, without loss of generality that $\alpha_n < (1-\delta_n)\|A\|^{-1}$ for all $n \geq 0$. First, we will show that $I - \lambda_n B$ is nonexpansive. Indeed, for all $x, y \in C$ and $\lambda_n \in [0, 2\alpha]$,

$$\begin{aligned} \left\|(I-\lambda_n B)x - (I-\lambda_n B)y\right\|^2 &= \left\|(x-y) - \lambda_n(Ax - Ay)\right\|^2 \\ &= \|x-y\|^2 - 2\lambda_n\langle x-y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2, \end{aligned} \tag{3.2}$$

which implies that $I - \lambda_n B$ is nonexpansive. Noticing that A is a linear bounded self-adjoint operator, one has

$$\|A\| = \sup\left\{\left|\langle Ax, x\rangle\right| : x \in H,\ \|x\| = 1\right\}. \tag{3.3}$$

Observing that

$$\begin{aligned}\langle((1-\delta_n)I-\alpha_n A)x,x\rangle &= 1-\delta_n-\alpha_n\langle Ax,x\rangle\\ &\le 1-\delta_n-\alpha_n\|A\|\\ &\le 0,\end{aligned}$$

we obtain $(1-\delta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\delta_n)I-\alpha_n A\| &= \sup\{\langle((1-\delta_n)I-\alpha_n A)x,x\rangle : x\in H,\ \|x\|=1\}\\ &= \sup\{1-\delta_n-\alpha_n\langle Ax,x\rangle : x\in H,\ \|x\|=1\}\\ &\le 1-\delta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

Next, we observe that $\{x_n\}$ is bounded. Indeed, pick $p\in\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{VI}(B,C)$ and notice that

$$\begin{aligned}\|z_n-p\| &\le \gamma_n\|x_n-p\|+(1-\gamma_n)\|W_nx_n-p\|\le\|x_n-p\|,\\ \|y_n-p\| &\le \beta_n\|x_n-p\|+(1-\beta_n)\|W_nz_n-p\|\\ &\le \beta_n\|x_n-p\|+(1-\beta_n)\|z_n-p\|\\ &\le \|x_n-p\|.\end{aligned}\tag{3.4}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\delta_nx_n+((1-\delta_n)I-\alpha_nA)P_C(y_n-\lambda By_n)-p\|\\ &= \|\alpha_n(\gamma f(x_n)-Ap)+\delta_n(x_n-p)+((1-\delta_n)I-\alpha_nA)P_C(y_n-\lambda By_n)-p\|\\ &\le \alpha_n\|\gamma f(x_n)-Ap\|+\delta_n\|x_n-p\|+(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-p\|\\ &\le \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Ap\|+(1-\alpha_n\overline{\gamma})\|x_n-p\|\\ &= [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|\\ &= [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n(\overline{\gamma}-\gamma\alpha)\frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\gamma\alpha}.\end{aligned}\tag{3.5}$$

By simple induction, we have

$$\|x_n-p\|\le\max\left\{\|x_0-p\|,\frac{\|Ap-\gamma f(p)\|}{\overline{\gamma}-\gamma\alpha}\right\},\tag{3.6}$$

which gives that the sequence $\{x_n\}$ is bounded, and so are $\{y_n\}$ and $\{z_n\}$.

Next, we claim that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.7}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_nx_n - W_{n-1}x_n\| &= \|U_{n,1}x_n - U_{n-1,1}x_n\| \\
&= \|\mu_1T_1U_{n,2}x_n - (1-\mu_1)x_n - \mu_1T_1U_{n-1,2}x_n - (1-\mu_1)x_n\| \\
&\le \mu_1\|U_{n,2}x_n - U_{n-1,2}x_n\| \\
&= \mu_1\|\mu_2T_2U_{n,3}x_n - (1-\mu_2)x_n - \mu_2T_2U_{n-1,3}x_n - (1-\mu_2)x_n\| \\
&\le \mu_1\mu_2\|U_{n,3}x_n - U_{n-1,3}x_n\| \\
&\ \ \vdots \\
&\le \left(\prod_{i=1}^{n}\mu_i\right)\|U_{n,n}x_n - U_{n-1,n}x_n\| \\
&\le M_1\left(\prod_{i=1}^{n}\mu_i\right),
\end{aligned} \tag{3.8}$$

where $M_1 \ge 0$ is a constant such that $\|U_{n,n}x_n - U_{n-1,n}x_n\| \le M_1$. Similarly, there exists $M_2 \ge 0$ such that $\|U_{n,n}y_n - U_{n-1,n}y_n\| \le M_2$.

Observing that

$$\begin{aligned}
z_n &= \gamma_nx_n + (1-\gamma_n)W_nx_n, \\
z_{n-1} &= \gamma_{n-1}x_{n-1} + (1-\gamma_{n-1})W_{n-1}x_{n-1},
\end{aligned} \tag{3.9}$$

we obtain that

$$z_n - z_{n-1} = (1-\gamma_n)(W_nx_n - W_{n-1}x_{n-1}) + \gamma_n(x_n - x_{n-1}) + (\gamma_{n-1} - \gamma_n)(W_{n-1}x_{n-1} - x_{n-1}). \tag{3.10}$$

It follows that

$$\begin{aligned}
\|z_n - z_{n-1}\| &\le (1-\gamma_n)\|W_nx_n - W_{n-1}x_{n-1}\| + \gamma_n\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\| \\
&\le (1-\gamma_n)\|W_nx_n - W_{n-1}x_n\| + (1-\gamma_n)\|W_{n-1}x_n - W_{n-1}x_{n-1}\| \\
&\quad + \gamma_n\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\| \\
&\le (1-\gamma_n)\|W_nx_n - W_{n-1}x_n\| + (1-\gamma_n)\|x_n - x_{n-1}\| \\
&\quad + \gamma_n\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\| \\
&= (1-\gamma_n)\|W_nx_n - W_{n-1}x_n\| + \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\| \\
&\le (1-\gamma_n)M_1\prod_{i=1}^{n}\mu_i + \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|W_nx_{n-1} - x_{n-1}\|.
\end{aligned} \tag{3.11}$$

Noticing that

$$y_n = \beta_n x_n + (1-\beta_n) W_n z_n, \\ y_{n-1} = \beta_{n-1} x_{n-1} + (1-\beta_{n-1}) W_{n-1} z_{n-1}, \tag{3.12}$$

we obtain

$$y_n - y_{n-1} = (1-\beta_n)(W_n z_n - W_{n-1} z_{n-1}) + \beta_n (x_n - x_{n-1}) + (W_{n-1} z_{n-1} - x_{n-1})(\beta_{n-1} - \beta_n). \tag{3.13}$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| &\le (1-\beta_n)\|W_n z_n - W_n z_{n-1}\| + \beta_n \|x_n - x_{n-1}\| + \|W_n z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\ &\le (1-\beta_n)\|W_n z_n - W_{n-1} z_n\| + (1-\beta_n)\|W_{n-1} z_n - W_{n-1} z_{n-1}\| \\ &\quad + \beta_n \|x_n - x_{n-1}\| + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\ &\le (1-\beta_n)\|W_n z_n - W_{n-1} z_n\| + (1-\beta_n)\|z_n - z_{n-1}\| \\ &\quad + \beta_n \|x_n - x_{n-1}\| + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\ &\le (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)\|z_n - z_{n-1}\| + \beta_n \|x_n - x_{n-1}\| \\ &\quad + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n|. \end{aligned} \tag{3.14}$$

Substituting (3.11) into (3.14), we get

$$\begin{aligned} \|y_n - y_{n-1}\| &\le (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)(1-\gamma_n) M_1 \prod_{i=1}^{n} \mu_i \\ &\quad + (1-\beta_n)\|x_n - x_{n-1}\| + (1-\beta_n)|\gamma_{n-1} - \gamma_n| \|W_{n-1} x_{n-1} - x_{n-1}\| \\ &\quad + \beta_n \|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| \|W_{n-1} z_{n-1} - x_{n-1}\| \\ &= (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)(1-\gamma_n) M_1 \prod_{i=1}^{n} \mu_i + \|x_n - x_{n-1}\| \\ &\quad + M_3\big((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|\big), \end{aligned} \tag{3.15}$$

where M_3 is an appropriate constant such that

$$M_3 \ge \max\left\{ \sup_{n\ge1} \|W_{n-1} x_{n-1} - x_{n-1}\|, \sup_{n\ge1} \|W_{n-1} z_{n-1} - x_{n-1}\| \right\}. \tag{3.16}$$

Putting $l_n = (x_{n+1} - \delta_n x_n)/(1-\delta_n)$, we get, $x_{n+1} = (1-\delta_n) l_n + \delta_n x_n$.

Now, we compute $l_{n+1} - l_n$. Observing that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\delta_{n+1})I - \alpha_{n+1}A)P_C(y_{n+1} - \lambda_{n+1}By_{n+1})}{1-\delta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(x_n) + ((1-\delta_n)I - \alpha_n A)P_C(y_n - \lambda_n By_n)}{1-\delta_n} \\ &= \frac{\alpha_{n+1}}{1-\delta_{n+1}}(\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})) \\ &\quad + \frac{\alpha_n}{1-\delta_n}(AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)) \\ &\quad + P_C(y_{n+1} - \lambda_{n+1}By_{n+1}) - P_C(y_n - \lambda_n By_n). \end{aligned} \tag{3.17}$$

It follows from (3.15) that

$$\begin{aligned} \|l_{n+1} - l_n\| &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| + \|y_{n+1} - y_n\| \\ &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| \\ &\quad + (1-\beta_n)M_2\prod_{i=1}^{n}\mu_i + (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n}\mu_i \\ &\quad + \|x_n - x_{n-1}\| + M_3((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|). \end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned} \|l_{n+1} - l_n\| - \|x_n - x_{n-1}\| &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| \\ &\quad + (1-\beta_n)M_2\prod_{i=1}^{n}\mu_i + (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n}\mu_i \\ &\quad + M_3((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|). \end{aligned} \tag{3.19}$$

Observing the conditions (C1) and (C4) and taking the superior limit as $n \to \infty$, we get

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_n - x_{n-1}\|) \le 0. \tag{3.20}$$

We can obtain $\lim_{n\to\infty}\|l_n - x_n\| = 0$ easily by Lemma 2.2 since

$$x_{n+1} - x_n = (1-\delta_n)(l_n - x_n), \tag{3.21}$$

one obtains that (3.7) holds. Setting $t_n = P_C(y_n - \lambda_n y_n)$, we have

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)t_n. \tag{3.22}$$

Observing that

$$\begin{aligned} x_n - t_n &= x_n - x_{n+1} + x_{n+1} - t_n \\ &= x_n - x_{n+1} + \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)t_n - t_n \\ &= x_n - x_{n+1} + \alpha_n(\gamma f(x_n) - At_n) + \delta_n(x_n - t_n), \end{aligned} \tag{3.23}$$

we arrive at

$$(1-\delta_n)(x_n - t_n) = x_n - x_{n+1} + \alpha_n(\gamma f(x_n) - At_n). \tag{3.24}$$

This implies

$$(1-\delta_n)\|x_n - t_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - At_n\|. \tag{3.25}$$

From (3.7) and (C1) we obtain that

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.26}$$

Next, we will show that $\|By_n - Bp\| \to 0$ as $n \to \infty$ for any $p \in F$. Observe that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|((1-\delta_n)I - \alpha_n A)(t_n - p) + \delta_n(x_n - p) + \alpha_n(\gamma f(x_n) - Ap)\|^2 \\ &= \|((1-\delta_n)I - \alpha_n A)(t_n - p) + \delta_n(x_n - p)\|^2 + \alpha_n^2\|\gamma f(x_n) - Ap\|^2 \\ &\quad + 2\delta_n\alpha_n\langle x_n - p, \gamma f(x_n) - Ap\rangle + 2\alpha_n\langle((1-\delta_n)I - \alpha_n A)(t_n - p), \gamma f(x_n) - Ap\rangle \\ &\le ((1-\delta_n - \alpha_n\overline{\gamma})\|t_n - p\| + \delta_n\|x_n - p\|)^2 + \alpha_n^2\|\gamma f(x_n) - Ap\|^2 \\ &\quad + 2\delta_n\alpha_n\langle x_n - p, \gamma f(x_n) - Ap\rangle + 2\alpha_n\langle((1-\delta_n)I - \alpha_n A)(t_n - p), \gamma f(x_n) - Ap\rangle \\ &= (1-\delta_n - \alpha_n\overline{\gamma})^2\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + 2(1-\delta_n - \alpha_n\overline{\gamma})\delta_n\|t_n - p\|\|x_n - p\| + c_n \\ &\le (1-\delta_n - \alpha_n\overline{\gamma})^2\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + (1-\delta_n - \alpha_n\overline{\gamma})\delta_n\left(\|t_n - p\|^2 + \|x_n - p\|^2\right) + c_n \\ &= \left[(1-\alpha_n\overline{\gamma})^2 - 2(1-\alpha_n\overline{\gamma})\delta_n + \delta_n^2\right]\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + ((1-\alpha_n\overline{\gamma})\delta_n - \delta_n^2)\left(\|t_n - p\|^2 + \|x_n - p\|^2\right) + c_n \\ &= (1-\alpha_n\overline{\gamma})^2\|t_n - p\|^2 - (1-\alpha_n\overline{\gamma})\delta_n\|t_n - p\|^2 + (1-\alpha_n\overline{\gamma})\delta_n\|x_n - p\|^2 + c_n \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|t_n-p\|^2+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\
&\le (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\left[\|(y_n-\lambda_nBy_n)-(p-\lambda_nBp)\|^2\right]\\
&\quad+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\
&\le (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\left[\|y_n-p\|^2+\lambda_n(\lambda_n-2\alpha)\|By_n-Bp\|^2\right]\\
&\quad+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\
&\le \|x_n-p\|^2+b(b-2\alpha)\|By_n-Bp\|^2+c_n,
\end{aligned}
\tag{3.27}
$$

where

$$
\begin{aligned}
c_n = \alpha_n^2\|\gamma f(x_n)-Ap\|^2+2\delta_n\alpha_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
+2\alpha_n\langle((1-\delta_n)I-\alpha_nA)(t_n-p),\gamma f(x_n)-Ap\rangle.
\end{aligned}
\tag{3.28}
$$

This impies that

$$
\begin{aligned}
-b(b-2\alpha)\|By_n-Bp\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.
\end{aligned}
\tag{3.29}
$$

Since $\lim_{n\to\infty}c_n=0$ and from (3.7), we obtain

$$
\lim_{n\to\infty}\|By_n-Bp\|=0. \tag{3.30}
$$

From (2.3), we have

$$
\begin{aligned}
\|t_n-p\|^2 &= \|P_C(y_n-\lambda_nBy_n)-P_C(p-\lambda_nBp)\|^2\\
&\le \langle (y_n-\lambda_nBy_n)-(p-\lambda_nBp),t_n-p\rangle\\
&= \frac{1}{2}\Big\{\|(y_n-\lambda_nBy_n)-(p-\lambda_nBp)\|^2+\|t_n-p\|^2\\
&\quad-\|(y_n-\lambda_nBy_n)-(p-\lambda_nBp)-(t_n-p)\|^2\Big\}\\
&\le \frac{1}{2}\Big\{\|y_n-p\|^2+\|t_n-p\|^2-\|(y_n-t_n)-\lambda_n(By_n-Bp)\|^2\Big\}\\
&= \frac{1}{2}\Big\{\|y_n-p\|^2+\|t_n-p\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,By_n-Bp\rangle-\lambda_n^2\|By_n-Bp\|^2\Big\},
\end{aligned}
\tag{3.31}
$$

so, we obtain

$$
\|t_n-p\|^2\le\|y_n-p\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,By_n-Bp\rangle-\lambda_n^2\|By_n-Bp\|^2. \tag{3.32}
$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|t_n-p\|^2+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\ &\le (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\\ &\quad\times\left[\|y_n-p\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,By_n-Bp\rangle-\lambda_n^2\|By_n-Bp\|^2\right]\\ &\quad+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\ &\le (1-\alpha_n\overline{\gamma})\|x_n-p\|^2-(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|^2\\ &\quad+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\ &\quad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n,\end{aligned}\tag{3.33}$$

which implies that

$$\begin{aligned}&(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|^2\\ &\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\ &\qquad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n\\ &\quad\le\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\ &\qquad+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\ &\qquad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n.\end{aligned}\tag{3.34}$$

Applying (3.7), (3.30), and $\lim_{n\to\infty}c_n=0$ to the last inequality, we obtain that

$$\lim_{n\to\infty}\|y_n-t_n\|=0.\tag{3.35}$$

It follows from (3.26) and (3.35) that

$$\|x_n-y_n\|\le\|x_n-t_n\|+\|t_n-y_n\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\tag{3.36}$$

On the other hand, one has

$$\begin{aligned}\|W_nx_n-x_n\| &\le\|x_n-y_n\|+\|y_n-W_nx_n\|\\ &\le\|x_n-y_n\|+\|y_n-W_nz_n\|+\|W_nz_n-W_nx_n\|\\ &\le\|x_n-y_n\|+\beta_n\|x_n-W_nz_n\|+\|z_n-x_n\|\\ &\le\|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+\beta_n\|W_nx_n-W_nz_n\|+\|z_n-x_n\|\\ &\le\|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+(1+\beta_n)\|z_n-x_n\|\\ &\le\|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+(1+\beta_n)(1-\gamma_n)\|W_nx_n-x_n\|\\ &=\|x_n-y_n\|-\left[(1+\beta_n)\gamma_n-2\beta_n-1\right]\|W_nx_n-x_n\|,\end{aligned}\tag{3.37}$$

which implies

$$[(1+\beta_n)\gamma_n - 2\beta_n]\,\|W_n x_n - x_n\| \le \|x_n - y_n\|. \tag{3.38}$$

From the conditions (C3), it follows that

$$\lim_{n\to\infty}\|W_n x_n - x_n\| = 0. \tag{3.39}$$

Applying Lemma 2.6 and (3.39), we obtain that

$$\begin{aligned}\|Wx_n - x_n\| &\le \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\| \\ &\le \sup_{x\in\{x_n\}}\|Wx - W_n x\| + \|W_n x_n - x_n\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{3.40}$$

It follows from (3.26) and (3.40) that

$$\begin{aligned}\|Wt_n - t_n\| &\le \|Wt_n - Wx_n\| + \|Wx_n - x_n\| + \|x_n - t_n\| \\ &\le 2\|t_n - x_n\| + \|Wx_n - x_n\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{3.41}$$

We observe that $P_F(\gamma f + (I-A))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}&\|P_F(\gamma f + (I-A))(x) - P_F(\gamma f + (I-A))(y)\| \\ &\quad\le \|(\gamma f + (I-A))(x) - (\gamma f + (I-A))(y)\| \\ &\quad\le \gamma\|f(x) - f(y)\| + \|I-A\|\|x-y\| \\ &\quad\le \gamma\alpha\|x-y\| + (1-\overline{\gamma})\|x-y\| \\ &\quad< \gamma\|x-y\|.\end{aligned} \tag{3.42}$$

Banach's Contraction Mapping Principle guarantees that $P_F(\gamma f + (I-A))$ has a unique fixed point, say $q \in H$. That is, $q = P_F(\gamma f + (I-A))(q)$.

Next, we claim that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, t_n - q\rangle \le 0. \tag{3.43}$$

Indeed, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, Wt_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - Aq, Wt_{n_i} - q\rangle. \tag{3.44}$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $t_{n_i} \rightharpoonup z$. From $\|Wt_{n_i} - t_{n_i}\| \to 0$, we obtain $Wt_{n_i} \rightharpoonup z$. Therefore, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle \gamma f(q) - Aq, Wt_n - q\rangle &= \lim_{i\to\infty}\langle \gamma f(q) - Aq, Wt_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Aq, z - q\rangle. \end{aligned} \tag{3.45}$$

Next we prove that $z \in F := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(B,C)$.

First, we prove that $z \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Suppose the contrary, $z \notin F(W)$, that is, $Wz \neq z$. Since $t_{n_i} \rightharpoonup z$, by the Opial's condition and (3.41), we have

$$\begin{aligned} \liminf_{i\to\infty}\|t_{n_i} - z\| &< \liminf_{i\to\infty}\|t_{n_i} - Wz\| \\ &\leq \liminf_{i\to\infty}\{\|t_{n_i} - Wt_{n_i}\| + \|Wt_{n_i} - Wz\|\} \\ &\leq \liminf_{i\to\infty}\{\|t_{n_i} - Wt_{n_i}\| + \|t_{n_i} - z\|\} \\ &= \liminf_{i\to\infty}\|t_{n_i} - z\|. \end{aligned} \tag{3.46}$$

This is a contradiction, which shows that $z \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Next, we prove $z \in \mathrm{VI}(B,C)$. For this purpose, let T be the maximal monotone mapping defined by (2.7):

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{3.47}$$

For any given $(v,w) \in G(T)$, hence $w - Bv \in N_C(v)$. Since $t_n \in C$, we have

$$\langle v - t_n, w - Bv\rangle \geq 0. \tag{3.48}$$

On the other hand, from $t_n = P_C(y_n - \lambda_n By_n)$, we have

$$\langle v - t_n, t_n - (y_n - \lambda_n By_n)\rangle \geq 0, \tag{3.49}$$

that is,

$$\left\langle v - t_n, \frac{t_n - y_n}{\lambda_n} + By_n\right\rangle \geq 0. \tag{3.50}$$

Therefore, we obtian

$$\begin{aligned}
\langle v-t_{n_i},w\rangle &\geq \langle v-t_{n_i},Bv\rangle \\
&\geq \langle v-t_{n_i},Bv\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle \\
&=\left\langle v-t_{n_i},Bv-By_{n_i}-\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle \\
&=\langle v-t_{n_i},Bv-Bt_{n_i}\rangle+\langle v-t_{n_i},Bt_{n_i}-By_{n_i}\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\geq \langle v-t_{n_i},Bt_{n_i}\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle \\
&=\langle v-t_{n_i},Bt_{n_i}-By_{n_i}\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle.
\end{aligned} \tag{3.51}$$

Noting that $\|t_{n_i}-y_{n_i}\| \to 0$ as $n \to \infty$ and B is Lipschitz continuous, hence from (3.18), we obtain

$$\langle v-z,w\rangle \geq 0. \tag{3.52}$$

Since T is maximal monotone, we have $z \in T^{-1}0$, and hence $z \in \mathrm{VI}(B,C)$.

The conclusion $z \in \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(B,C)$ is proved.

Hence by (3.45), we obtain

$$\limsup_{n\to\infty}\langle \gamma f(q)-Aq,Wt_n-q\rangle=\langle \gamma f(q)-Aq,z-q\rangle \leq 0. \tag{3.53}$$

Since $q=P_F f(q)$, it follows from (3.39), (3.41), and (3.53) that

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(q)-Aq,t_n-q\rangle &=\limsup_{n\to\infty}\langle \gamma f(q)-Aq,(t_n-Wt_n)+(Wt_n-q)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(q)-Aq,Wt_n-q\rangle \\
&\leq 0.
\end{aligned} \tag{3.54}$$

Hence (3.43) holds. Using (3.26) and (3.54), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(q)-Aq,x_n-q\rangle &=\limsup_{n\to\infty}\langle \gamma f(q)-Aq,(x_n-t_n)+(t_n-q)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(q)-Aq,t_n-q\rangle \\
&\leq 0.
\end{aligned} \tag{3.55}$$

Now, from Lemma 2.1, it follows that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n A)t_n-q\|^2\\
&= \|((1-\delta_n)I-\alpha_n A)(t_n-q)+\delta_n(x_n-q)+\alpha_n(\gamma f(x_n)-Aq)\|^2\\
&= \|((1-\delta_n)I-\alpha_n A)(t_n-q)+\delta_n(x_n-q)\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(x_n)-Aq\rangle\\
&\quad +2\alpha_n\langle((1-\delta_n)I-\alpha_n A)(t_n-q),\gamma f(x_n)-Aq\rangle\\
&\le ((1-\delta_n-\alpha_n\overline{\gamma})\|t_n-q\|+\delta_n\|x_n-q\|)^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\gamma\langle x_n-q,f(x_n)-f(q)\rangle+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-A(q)\rangle\\
&\quad +2(1-\delta_n)\gamma\alpha_n\langle t_n-q,f(x_n)-f(q)\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle\\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle\\
&\le ((1-\delta_n-\alpha_n\overline{\gamma})\|x_n-q\|+\delta_n\|x_n-q\|)^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\gamma\alpha\|x_n-q\|^2+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle\\
&\quad +2(1-\delta_n)\gamma\alpha_n\alpha\|x_n-q\|^2+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle\\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle\\
&= \left[(1-\alpha_n\overline{\gamma})^2+2\delta_n\alpha_n\gamma\alpha+2(1-\delta_n)\gamma\alpha_n\alpha\right]\|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle\\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle\\
&\le \left[1-2(\overline{\gamma}-\alpha_n\gamma)\alpha_n\right]\|x_n-q\|^2+\overline{\gamma}^2\alpha_n^2\|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle\\
&\quad +2\alpha_n^2\|A(t_n-q)\|\|\gamma f(q)-Aq\|\\
&= \left[1-2(\overline{\gamma}-\alpha_n\gamma)\alpha_n\right]\|x_n-q\|^2\\
&\quad +\alpha_n\Big\{\alpha_n(\overline{\gamma}^2\|x_n-q\|^2+\|\gamma f(x_n)-Aq\|^2\\
&\qquad +2\|A(t_n-q)\|\|\gamma f(q)-Aq\|)+2\delta_n\langle x_n-q,\gamma f(q)-Aq\rangle\\
&\qquad +2(1-\delta_n)\langle t_n-q,\gamma f(q)-Aq\rangle\Big\}.
\end{aligned} \tag{3.56}$$

Since $\{x_n\}$, $\{f(x_n)\}$, and t_n are bounded, we can take a constant $M_5>0$ such that

$$\overline{\gamma}^2\|x_n-q\|^2+\|\gamma f(x_n)-Aq\|^2+2\|A(t_n-q)\|\|\gamma f(q)-Aq\|\le M_5, \tag{3.57}$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq [1 - 2(\overline{\gamma} - \alpha_n \gamma)\alpha_n] \|x_n - q\|^2 + \alpha_n \sigma_n, \tag{3.58}$$

where

$$\sigma_n = 2\delta_n \langle x_n - q, \gamma f(q) - Aq \rangle + 2(1 - \delta_n)\langle t_n - q, \gamma f(q) - Aq \rangle + \alpha_n M_4. \tag{3.59}$$

Using (C1), (3.54), and (3.55), we get $\limsup_{n\to\infty} \delta_n \leq 0$. Now applying Lemma 2.3 to (3.58), we conclude that $x_n \to q$. This completes the proof. □

Remark 3.2. Theorem 3.1 mainly improve the results of Qin and Cho [14] from a single nonexpansive mapping to an infinite family of nonexpansive mappings.

4. Applications

In this section, we obtain two results by using a special case of the proposed method.

Theorem 4.1. *Let H be a real Hilbert space, let B be an α-inverse strongly monotone mapping on H, let $\{T_i : H \to H\}$ be a family of infinitely nonexpansive mappings with $F := \bigcap_{i=1}^{\infty} F(T_i) \cap B^{-1}(0) \neq \emptyset$. Let $f : H \to H$ a contraction with coefficient $\alpha \in (0,1)$, and let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Suppose the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by*

$$\begin{gathered} x_1 = x \in H \text{ chosen arbitrary}, \\ z_n = \gamma_n x_n + (1 - \gamma_n) W_n x_n, \\ y_n = \beta_n x_n + (1 - \beta_n) W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n) I - \alpha_n A)(y_n - \lambda_n B y_n), \end{gathered} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$,

(C3) $(1 + \beta_n)\gamma_n - 2\beta_n > a$ *for some* $a \in (0,1)$,

(C4) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = \lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$,

(C5) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$.

Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $q = P_F(\gamma f + (I - A))(q)$ which solves the variational inequality:

$$\langle (A - \gamma f)q, q - z \rangle \leq 0, \quad z \in F. \tag{4.2}$$

Proof. We have $B^{-1}(0) = \mathrm{VI}(B,H)$ and $P_H = I$. Applying Theorem 3.1, we obtain the desired result. □

Next, we will apply the main results to the problem for finding a common element of the set of fixed points of a family of infinitely nonexpansive mappings and the set of fixed points of a finite family of k-strictly pseudocontractive mappings.

Definition 4.2. A mappings $S : C \to H$ is said to be a k-strictly pseudocontractive mapping if there exists $k \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{4.3}$$

The following lemmas can be obtained from [31, Proposition 2.6] by Acedo and Xu, easily.

Lemma 4.3. *Let H be a Hilbert space, let C be a closed convex subset of H. For any integer $N \geq 1$, assume that, for each $1 \leq i \leq N$, $S_i : C \to H$ is a k_i-strictly pseudocontractive mapping for some $0 \leq k_i < 1$. Assume that $\{\varphi_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \varphi_i = 1$. Then $\sum_{i=1}^N \varphi_i S_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$.*

Lemma 4.4. *Let $\{S_i\}_{i=1}^N$ and $\{\varphi_i\}_{i=1}^N$ be as in Lemma 4.3. Suppose that $\{S_i\}_{i=1}^N$ has a common fixed point in C. Then $F(\sum_{i=1}^N \varphi_i S_i) = \bigcap_{i=1}^N F(S_i)$.*

Let $S_i : C \to H$ be a k_i-strictly pseudocontractive mapping for some $0 \leq k_i < 1$. We define a mapping $A = I - \sum_{i=1}^N \varphi_i S_i : C \to H$, where $\{\varphi_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \varphi_i = 1$, then A is a $((1-k)/2)$-inverse-strongly monotone mapping with $k = \max\{k_i : 1 \leq i \leq N\}$. In fact, from Lemma 4.3, we have

$$\left\|\sum_{i=1}^N \varphi_i S_i x - \sum_{i=1}^N \varphi_i S_i y\right\|^2 \leq \|x - y\|^2 + k\left\|\left(I - \sum_{i=1}^N \varphi_i S_i\right)x - \left(I - \sum_{i=1}^N \varphi_i S_i\right)y\right\|^2, \quad \forall x, y \in C. \tag{4.4}$$

That is,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2. \tag{4.5}$$

On the other hand

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 - 2\langle x - y, Ax - Ay\rangle + \|Ax - Ay\|^2. \tag{4.6}$$

Hence, we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1-k}{2}\|Ax - Ay\|^2. \tag{4.7}$$

This shows that A is $((1-k)/2)$-inverse-strongly monotone.

Theorem 4.5. *Let C be a closed convex subset of a real Hilbert space H. For any integer $N > 1$, assume that, for each $1 \leq i \leq N$, $S_i : C \to H$ is a k_i-strictly pseudocontractive mapping for* *some* $0 \leq k_i < 1$. *Let* $\{T_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with* $F := \bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. *Let* $f : C \to C$ *a contraction with coefficient* $\alpha \in (0,1)$ *and let* A *be a strongly positive bounded linear operator with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$. *Let the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be generated by*

$$\begin{aligned} x_1 &= x \in H \text{ chosen arbitrary}, \\ z_n &= \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n &= \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n) I - \alpha_n A) P_C \left((1-\lambda_n) y_n - \lambda_n \sum_{i=1}^{N} \varphi_i S_i y_n \right), \end{aligned} \tag{4.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *are the sequences in* $[0,1]$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$,

(C3) $(1+\beta_n)\gamma_n - 2\beta_n > a$ *for some* $a \in (0,1)$,

(C4) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = \lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$,

(C5) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$.

Then $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to* $q = P_F(\gamma f + (I - A))(q)$ *which solves the variational inequality:*

$$\langle (A - \gamma f) q, q - z \rangle \leq 0, \quad z \in F. \tag{4.9}$$

Proof. Taking $B = I - \sum_{i=1}^{N} \varphi_i S_i : C \to H$, we know that $B : C \to H$ is α-inverse strongly monotone with $\alpha = (1-k)/2$. Hence, B is a monotone L-Lipschitz continuous mapping with $L = 2/(1-k)$. From Lemma 4.4, we know that $\sum_{i=1}^{N} \varphi_i S_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$ and then $F(\sum_{i=1}^{N} \varphi_i S_i) = \mathrm{VI}(B, C)$ by Chang [30, Proposition 1.3.5]. Observe that

$$P_C(y_n - \lambda_n B y_n) = P_C \left((1-\lambda_n) y_n - \lambda_n \sum_{i=1}^{N} \varphi_i S_i y_n \right). \tag{4.10}$$

The conclusion of Theorem 4.5 can be obtained from Theorem 3.1. □

Remark 4.6. Theorem 4.5 is a generalization and improvement of the theorems by Qin and Cho [14], Iiduka and Takahashi [16, Thorem 3.1], and Takahashi and Toyoda [15].

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P. Kumam, N. Petrot, R. Wangkeeree

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A hybrid iterative scheme for equilibrium problems and fixed point problems of asymptotically k-strict pseudo-contractions☆

Poom Kumam [a], Narin Petrot [b], Rabian Wangkeeree [b,*]

[a] Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, KMUTT, Bangmod, Bangkok 10140, Thailand

[b] Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

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ABSTRACT

In this paper, we propose an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of asymptotically k-strict pseudo-contractions in the setting of real Hilbert spaces. By using our proposed scheme, we get a weak convergence theorem for a finite family of asymptotically k-strict pseudo-contractions and then we modify these algorithm to have strong convergence theorem by using the two hybrid methods in the mathematical programming. Our results improve and extend the recent ones announced by Ceng, et al.'s result [L.C. Ceng, Al-Homidan, Q.H. Ansari and J.C. Yao, An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings, J. Comput. Appl. Math. 223 (2009) 967–974] Qin, Cho, Kang, and Shang, [X. Qin, Y. J. Cho, S. M. Kang, and M. Shang, A hybrid iterative scheme for asymptotically k-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 70 (2009) 1902–1911] and other authors.

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1. Introduction

Let C be a closed convex subset of a real Hilbert space H. Let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\phi : C \times C \longrightarrow \mathbb{R}$ is to find $x \in C$ such that

$$\phi(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by EP(ϕ). Numerous problems in physics, optimization, and economics are reduced to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem (see [1–3,24–26]). In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when EP(ϕ) is nonempty and they also proved a strong convergence theorem.

Recall that a mapping $T : C \longrightarrow C$ is said to be asymptotically k-strictly pseudo-contractive (the class of asymptotically k-strictly pseudo-contractive maps was first introduced in Hilbert spaces by Qihou [5]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} k_n = 1$ such that there exists $k \in [0, 1)$ such that

$$\|T^n x - T^n y\|^2 \leq k_n^2 \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{1.2}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Note that the class of asymptotically k-strict pseudo-contractions strictly includes the class of asymptotically nonexpansive mappings [6] which are mappings T on C such that

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|, \quad \forall x, y \in C, \tag{1.3}$$

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* Corresponding author. Tel.: +66 859191680; fax: +66 5261025.
E-mail addresses: poom.kum@kmutt.ac.th (P. Kumam), narinp@nu.ac.th (N. Petrot), rabianw@nu.ac.th (R. Wangkeeree).

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where the sequence $\{k_n\} \subset [1, \infty)$ is such that $\lim_{n\longrightarrow\infty} k_n = 1$. That is, T is asymptotically nonexpansive if and only if T is asymptotically 0-strictly pseudo-contractive.

Recall that a mapping $T : C \longrightarrow C$ is called a k-strict pseudo-contraction mapping if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \le \|x - y\|,$$

for all $x, y \in C$. That is, T is nonexpansive if and only if T is 0-strict pseudo-contractive. Note that the class of strict pseudo-contraction mappings strictly includes the class of nonexpansive mappings. Clearly, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. Construction of fixed points of nonexpansive mappings via Mann's algorithm [7] has extensively been investigated in the literature; See, for example [8,7,9–12] and references therein. If T is a nonexpansive self-mapping of C, then Mann's algorithm generates, initializing with an arbitrary $x_1 \in C$, a sequence according to the recursive manner

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \ge 1, \tag{1.5}$$

where $\{\alpha_n\}$ is a real control sequence in the interval $(0, 1)$.

If $T : C \longrightarrow C$ is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm converges weakly to a fixed point of T. Reich [13] showed that the conclusion also holds good in the setting of uniformly convex Banach spaces with a Fréchet differentiable norm. It is well known that Reich's result is one of the fundamental convergence results. Very recently, Marino and Xu [14] extended Reich's result [13] to strict pseudo-contraction mappings in the setting of Hilbert spaces.

Very recently, Motivated and inspired by the research work of Marino and Xu [14] and Takahashi and Takahashi [15], Ceng, Homidan, Ansari and Yao [16], introduced a new implicit iterative scheme for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a strict pseudo-contraction mapping defined in the setting of real Hilbert spaces. They gave some weak and strong convergence theorems for such iterative scheme. More precisely, they proved the following theorems.

Theorem 1.1 (*Ceng, Homidan, Ansari and Yao [16]*). *Let C be a closed convex subset of a Hilbert space H, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and $T : C \longrightarrow C$ be a k-strict pseudo-contraction mapping for some $0 \le k < 1$ such that $F(T) \cap EP(\phi) \ne \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by arbitrary element $x_0 \in H$ and then by*

$$\begin{cases} \phi(u_{n-1}, y) + \dfrac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, & \forall y \in C \\ x_n = \alpha_{n-1}u_{n-1} + (1 - \alpha_{n-1})Tu_{n-1}; & \forall n \ge 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset [\alpha, \beta]$, *for some* $\alpha, \beta \in (k, 1)$, *and*
(2) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\longrightarrow\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(T) \cap EP(\phi)$.

Theorem 1.2 (*Ceng, Homidan, Ansari and Yao [16]*). *Let $C, H, T, \phi, \{x_n\}, \{u_n\}$ and $\{\alpha_n\}, \{r_n\}$ be as in Theorem 1.1. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of F if and only if $\liminf_{n\longrightarrow\infty} d(x_n, F) = 0$, where $d(x_n, F)$ denotes the metric distance from the point x_n to F.*

On the other hand, motivate and inspired in [17,14], very recently, Qin, Cho, Kang and Shang [18] introduced the following algorithm for a finite family of asymptotically k-strict pseudo-contractions. Let $x_0 \in C$ and $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(0, 1)$. The sequence $\{x_n\}$ generated by the following way:

$$\begin{aligned} &x_1 = \alpha_0 x_0 + (1 - \alpha_0)T_1 x_0, \\ &x_2 = \alpha_1 x_1 + (1 - \alpha_1)T_2 x_1 \\ &\dots \\ &x_N = \alpha_{N-1}x_{N-1} + (1 - \alpha_{N-1})T_N x_{N-1} \\ &x_{N+1} = \alpha_N x_N + (1 - \alpha_N)T_1^2 x_N \\ &\dots \\ &x_{2N} = \alpha_{2N-1}x_{2N-1} + (1 - \alpha_{2N-1})T_N^2 x_{2N-1} \\ &x_{2N+1} = \alpha_{2N}x_{2N} + (1 - \alpha_{2N})T_1^3 x_{2N} \\ &\dots \end{aligned}$$

is called the explicit iterative sequence of a finite family of asymptotically k-strict pseudo-contractions $\{T_1, T_2, \ldots, T_N\}$. Since, for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. Hence the above table can be written in the following form:

$$x_n = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \quad \forall n \geq 1. \tag{1.7}$$

Then they proved some weak convergence theorems for a finite family of asymptotically k-strict pseudo-contractions by algorithm (1.7). More precisely, they proved the following theorem.

Theorem 1.3. *Let C be nonempty closed convex subset of a real Hilbert space H and $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \leq k_i \leq 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = max\{k_i : 1 \leq i \leq N\}$ and $k_n = max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that $\cap_{i=1}^{N} F(T_i)$ is nonempty. For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by (1.7). Assume that the control sequence $\{\alpha_n\}$ is chosen such that $k + \varepsilon \leq \alpha_n \leq 1 - \varepsilon$ for all $n \geq 0$ and some $\varepsilon \in (0, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$.*

On the other hand, recently Takahashi, Takeuchi and Kubota [19] introduced the new iterative methods for approximating the common fixed point of a family of nonexpansive mappings $\{T_n : C \longrightarrow C\}$ by using the hybrid method as follows: Let $x \in H$, for $C_0 = C$ and $x_0 = P_{C_0}x$, they define a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_n x_{n-1}, \\ C_n = \{z \in C_{n-1} : \|y_{n-1} - z\| \leq \|x_{n-1} - z\|\}, \\ x_n = P_{C_n}x, \quad \forall n \geq 1 \end{cases} \tag{1.8}$$

where $0 \leq \alpha_n < \alpha < 1$ for all $n \geq 0$. Then, under appropriate conditions on $\{T_n\}$, they obtained a strong convergence theorem for the iterative scheme (1.8) in the setting of a real Hilbert space.

In this paper, inspired and motivated by the above researches, we suggest and analyze an iterative scheme for finding a common element of the set of common fixed points of a finite family of asymptotically k-strict pseudo-contraction and the set of solutions of an equilibrium problem in the framework of Hilbert spaces. Then we modify our iterative scheme to get strong convergence theorems by the hybrid algorithms. Our results extend and improve the corresponding recent results of Ceng, Homidan, Ansari and Yao [16] and Qin, Cho, Kang and Shang [18] and some others.

2. Preliminaries

Throughout the paper, we write $x_n \longrightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x, and $\omega_w(x_n) = \{x : x_{n_j} \rightharpoonup x$ for some subsequence $\{x_{n_j}\}$ of $\{x_n\}\}$ denotes the weak ω-limit set of $\{x_n\}$.

Lemma 2.1 ([14, Lemma 1.1]). *Let H be a real Hilbert space. There hold the following identities*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \ \forall x, y \in H,$
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \forall t \in [0, 1], \forall x, y \in H,$
(iii) *If $\{x_n\}$ is a sequence in H weakly converging to z, then*

$$\limsup_{n\longrightarrow\infty} \|x_n - y\|^2 = \limsup_{n\longrightarrow\infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.$$

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping.

It is also known that H satisfies Opial's condition [20], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| \leq \liminf_{n\longrightarrow\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.2 ([21]). *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and point $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$ the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\},$$

is convex and closed.

Lemma 2.3 ([14, Lemma 1.3]). *Let C be a closed convex subset of H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4 ([21, Lemma 1.5]). *Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition $\|x_n - u\| \leq \|u - q\|$ for all n. Then $x_n \longrightarrow q$.*

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Lemma 2.5 (*Kim and Xu [22]*). *Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H and $T : C \longrightarrow C$ be an asymptotically k-strictly pseudo-contractive mapping for some $0 \le k < 1$ with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and the fixed point set of T is nonempty. Then $(I - T)$ is demiclosed at zero.*

Lemma 2.6 (*[23, Lemma 2]*). *Let the sequences of numbers $\{a_n\}$ and $\{b_n\}$ be satisfy that*

$$a_{n+1} \le (1 + b_n)a_n, \quad a_n \ge 0, b_n \ge 0, \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty, \quad \forall n \ge 1.$$

If $\liminf_{n\longrightarrow\infty} a_n = 0$, then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.7 (*[9]*). *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be the three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \le (1 + s_n)r_n + t_n, \quad \forall n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\longrightarrow\infty} r_n$ exists.

Lemma 2.8 (*Kim and Xu [22]*). *Let H be a real Hilbert space, C a nonempty subset of H and $T : C \longrightarrow C$ be an asymptotically k-strictly pseudo-contractive mapping. Then T is uniformly L-Lipschitzian.*

Lemma 2.9 (*Qin, Cho, Kang, and Shang [18]*). *Let H be a real Hilbert space, C a nonempty subset of H and $T : C \longrightarrow C$ be a k-strictly asymptotically pseudo-contractive mapping. Then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

For solving the equilibrium problem, let us assume that the bifunction ϕ satisfies the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;
(A2) ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \le 0$ for any $x, y \in C$;
(A3) ϕ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\longrightarrow 0^+} \phi(tz + (1 - t)x, y) \le \phi(x, y);$$

(A4) $\phi(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma appears implicitly in [1].

Lemma 2.10 (*[1]*). *Let C be a nonempty closed convex subset of H and let ϕ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \quad \text{for all } y \in C.$$

The following lemma was also given in [4].

Lemma 2.11 (*[4]*). *Assume that $\phi : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $S_r : H \longrightarrow C$ as follows:*

$$S_r(x) = \left\{ z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C \right\},$$

for all $z \in H$. Then, the following hold:

(i) *S_r is single-valued;*
(ii) *S_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|S_r x - S_r y\|^2 \le \langle S_r x - S_r y, x - y\rangle$;*
(iii) *$F(S_r) = EP(\phi)$;*
(iv) *$EP(\phi)$ is closed and convex.*

3. Weak convergence theorems

We are now in a position to prove some weak convergence theorems.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $N \ge 1$ be an integer, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let, for each $1 \le i \le N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive*

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mapping for some $0 \le k_i \le 1$ *and a sequence* $\{k_{n,i}\}$ *such that* $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. *Let* $k = max\{k_i : 1 \le i \le N\}$ *and* $k_n = max\{k_{n,i} : 1 \le i \le N\}$. *Assume that* $F := \cap_{i=1}^{N} F(T_i) \cap EP(\phi)$ *is nonempty. Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated initially by arbitrary element* $x_0 \in C$ *and then by*

$$\begin{cases} \phi(u_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0; & \forall y \in C, \\ x_n = \alpha_{n-1}u_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}; & \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(C1) $\{\alpha_n\} \subset [\alpha, \beta]$, *for some* $\alpha, \beta \in (k, 1)$, *and*
(C2) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\longrightarrow\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to an element of* F.

Proof. We divide the proof into five steps.

Step 1. We claim that $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists, $\forall q \in F$.
Indeed, Let $q \in F := \cap_{i=1}^{N} F(T_i) \cap EP(\phi)$. Thus from the definition of S_r in Lemma 2.11, we have $u_{n-1} = S_{r_{n-1}}x_{n-1}$ and therefore

$$\begin{aligned} \|u_{n-1} - q\| &= \|S_{r_{n-1}}x_{n-1} - S_{r_{n-1}}q\| \\ &\le \|x_{n-1} - q\|, \quad \text{for all } n \ge 1. \end{aligned}$$

Since each $i \in \{1, 2, \dots, N\}$, $T_i : C \longrightarrow C$ is an asymptotically k_i-strictly pseudo-contractive mapping, we have

$$\begin{aligned} \|x_n - q\|^2 &= \|\alpha_{n-1}(u_{n-1} - q) + (1-\alpha_{n-1})(T_{i(n)}^{h(n)}u_{n-1} - q)\|^2 \\ &= \alpha_{n-1}\|u_{n-1} - q\|^2 + (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n+1} - q\|^2 - \alpha_{n-1}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\le \alpha_{n-1}\|u_{n-1} - q\|^2 - \alpha_{n-1}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\quad + (1-\alpha_{n-1})[k_{h(n)}^2\|u_{n-1} - q\|^2 + k\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2] \\ &\le k_{h(n)}^2\|u_{n-1} - q\|^2 - (1-\alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\le k_{h(n)}^2\|x_{n-1} - q\|^2 - (1-\alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\le (1 + (k_{h(n)}^2 - 1))\|x_{n-1} - q\|^2. \end{aligned} \tag{3.2}$$

It follows from Lemma 2.7 that $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists.
Step 2. We claim that $\lim_{n\longrightarrow\infty}\|u_n - u_{n+j}\| = 0; \forall j = 1, 2, \dots, N$. Observing (3.2) again, we have

$$(1-\alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \le k_{h(n)}^2\|x_{n-1} - q\|^2 - \|x_n - q\|^2.$$

It follows from our assumptions that

$$(1-\beta)(\alpha - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \le k_{h(n)}^2\|x_{n-1} - q\|^2 - \|x_n - q\|^2.$$

Taking the limit as $n \longrightarrow \infty$ yields that

$$\lim_{n\longrightarrow\infty}\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\| = 0. \tag{3.3}$$

This implies that

$$\|x_n - u_{n-1}\| = (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.4}$$

Let $q \in F$. Thus as above $u_{n-1} = S_{r_{n-1}}x_{n-1}$ and we have

$$\begin{aligned} \|u_{n-1} - q\|^2 &= \|S_{r_{n-1}}x_{n-1} - S_{r_{n-1}}q\|^2 \\ &\le \langle S_{r_{n-1}}x_n - S_{r_{n-1}}q, x_{n-1} - q\rangle \\ &= \langle u_{n-1} - q, x_{n-1} - q\rangle \\ &= \frac{1}{2}(\|u_{n-1} - q\|^2 + \|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2), \end{aligned}$$

and hence

$$\|u_{n-1} - q\|^2 \le \|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2.$$

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Using (3.2), (C1), and the last inequality, we have

$$\begin{aligned}\|x_n - q\|^2 &= \|\alpha_{n-1}(u_{n-1} - q) + (1-\alpha_{n-1})(T_{i(n)}^{h(n)}u_{n-1} - q)\|^2 \\ &\le k_{h(n)}^2\|u_{n-1} - q\|^2 - (1-\alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}u_{n-1} - u_{n-1}\|^2 \\ &\le k_{h(n)}^2\|u_{n-1} - q\|^2 \\ &= k_{h(n)}^2(\|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2)\end{aligned}$$

and hence

$$k_{h(n)}^2\|x_{n-1} - u_{n-1}\|^2 \le k_{h(n)}^2\|x_{n-1} - q\|^2 - \|x_n - q\|^2.$$

The existence of $\lim_{n\longrightarrow\infty}\|x_n - q\|$ and $\lim_{n\longrightarrow\infty} k_{h(n)} = 1$ imply that

$$\lim_{n\longrightarrow\infty}\|x_{n-1} - u_{n-1}\| = 0. \tag{3.5}$$

Using (3.4) and (3.5), we obtain

$$\|u_n - u_{n-1}\| \le \|u_n - x_n\| + \|x_n - u_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.6}$$

It follows that

$$\lim_{n\longrightarrow\infty}\|u_n - u_{n+j}\| = 0, \quad \forall j = 1, 2, \dots, N. \tag{3.7}$$

Applying (3.5) and (3.6), we obtain that

$$\|x_n - x_{n-1}\| \le \|x_n - u_n\| + \|u_n - u_{n-1}\| + \|u_{n-1} - x_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

This also implies that $\lim_{n\longrightarrow\infty}\|x_n - x_{n+j}\| = 0$; $\forall j = 1, 2, \dots, N$.
Step 3. We claim that $\lim_{n\longrightarrow\infty}\|u_n - T_l u_n\| = 0$, $\lim_{n\longrightarrow\infty}\|x_n - T_l x_n\| = 0$ for all $l = 1, 2, \dots, N$. Since, for any positive integer $n > N$, it can be written as $n = (k(n) - 1)N + i(n)$, where $i(n) \in \{1, 2, \dots, N\}$. Observe that

$$\begin{aligned}\|u_{n-1} - T_n u_{n-1}\| &\le \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + \|T_{i(n)}^{h(n)}u_{n-1} - T_n u_{n-1}\| \\ &= \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + \|T_{i(n)}^{h(n)}u_{n-1} - T_{i(n)}u_{n-1}\| \\ &\le \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + L\|T_{i(n)}^{h(n)-1}u_{n-1} - u_{n-1}\| \\ &\le \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + L[\|T_{i(n)}^{h(n)-1}u_{n-1} - T_{i(n-N)}^{h(n)-1}u_{n-N}\| \\ &\quad + \|T_{i(n-N)}^{h(n)-1}u_{n-N} - u_{(n-N)-1}\| + \|u_{(n-N)-1} - u_{n-1}\|].\end{aligned} \tag{3.8}$$

Since, for each $n > N$, $n = (n - N)(\text{mod}\, N)$ and $n = (k(n) - 1)N + i(n)$, we have

$$n - N = (k(n) - 1)N + i(n) = (k(n - N) - 1)N + i(n - N).$$

That is

$$k(n - N) = k(n) - 1, \qquad i(n - N) = i(n).$$

Observe that

$$\begin{aligned}\|T_{i(n)}^{h(n)-1}u_{n-1} - T_{i(n-N)}^{h(n)-1}u_{n-N}\| &= \|T_{i(n)}^{h(n)-1}u_{n-1} - T_{i(n)}^{h(n)-1}u_{n-N}\| \\ &\le L\|u_{n-1} - u_{n-N}\|,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\|T_{i(n-N)}^{h(n)-1}u_{n-N} - u_{(n-N)-1}\| &= \|T_{i(n-N)}^{h(n-N)}u_{n-N} - u_{(n-N)-1}\| \\ &\le \|T_{i(n-N)}^{h(n-N)}u_{n-N} - T_{i(n-N)}^{h(n)-N}u_{(n-N)-1}\| + \|T_{i(n-N)}^{h(n-N)}u_{(n-N)-1} - u_{n-N-1}\| \\ &\le L\|u_{(n-N)-1} - u_{n-N}\| + \|T_{i(n-N)}^{h(n-N)}u_{(n-N)-1} - u_{n-N-1}\|.\end{aligned} \tag{3.10}$$

It follows from (3.8)–(3.10) that

$$\begin{aligned}\|u_{n-1} - T_n u_{n-1}\| &\le \|u_{n-1} - T_{i(n)}^{h(n)}u_{n-1}\| + L(L\|u_{n-1} - u_{n-N}\| + L\|u_{(n-N)-1} - u_{n-N}\| \\ &\quad + \|T_{i(n-N)}^{h(n-N)}u_{(n-N)-1} - u_{n-N-1}\| + \|u_{(n-N)-1} - u_{n-1}\|).\end{aligned} \tag{3.11}$$

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Applying (3.7) and (3.3) to (3.11), we obtain

$$\lim_{n\longrightarrow\infty} \|u_{n-1} - T_n u_{n-1}\| = 0. \tag{3.12}$$

Notice that

$$\begin{aligned}\|u_n - T_n u_n\| &\le \|u_n - u_{n-1}\| + \|u_{n-1} - T_n u_{n-1}\| + \|T_n u_{n-1} - T_n u_n\|.\\ &\le (1+L)\|u_n - u_{n-1}\| + \|u_{n-1} - T_n u_{n-1}\|.\end{aligned}$$

From (3.6) and (3.12), one can easily see that

$$\lim_{n\longrightarrow\infty} \|u_n - T_n u_n\| = 0.$$

We also have

$$\begin{aligned}\|u_n - T_{n+j} u_n\| &\le \|u_n - u_{n+j}\| + \|u_{n+j} - T_{n+j} u_{n+j}\| + \|T_{n+j} u_{n+j} - T_{n+j} u_n\|\\ &\le (1+L)\|u_n - u_{n+j}\| + \|u_{n+j} - T_{n+j} u_{n+j}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned}$$

for any $j = 1, 2, \ldots, N$, which give that

$$\lim_{n\longrightarrow\infty} \|u_n - T_l u_n\| = 0; \quad \forall l = 1, 2, \ldots, N.$$

Moreover, for each $l \in \{1, 2, \ldots, N\}$, we obtain that

$$\begin{aligned}\|x_n - T_l x_n\| &\le \|x_n - u_n\| + \|u_n - T_l u_n\| + \|T_l u_n - T_l x_n\|\\ &\le (1+L)\|x_n - u_n\| + \|u_n - T_l u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.13}$$

Step 4. We claim that

$$\omega_w(x_n) \subset F := \cap_{i=1}^{N} F(T_i) \cap EP(\phi), \tag{3.14}$$

where $\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}$.
Indeed, since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Let $w \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Applying (3.5), we can obtain that $u_{n_i} \rightharpoonup w$ as $i \longrightarrow \infty$. It follows from $\|u_n - T_l u_n\| \longrightarrow 0$ that

$$T_l u_{n_i} \rightharpoonup w, \quad \text{for all } l = 1, 2, \ldots, N.$$

Let us show $w \in EP(\phi)$. Since $u_n = T_{r_n} u_n$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge \phi(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge \phi(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$ and $u_{n_i} \rightharpoonup w$, from (A4) have

$$0 \ge \phi(y, w), \quad \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $\phi(y_t, w) \le 0$. So, from (A1) and (A4) we have

$$0 = \phi(y_t, y_t) \le t\phi(y_t, y) + (1-t)\phi(y_t, w) \le t\phi(y_t, y)$$

and hence $0 \le \phi(y_t, y)$. From (A3), we have

$$0 \le \phi(w, y), \quad \forall y \in C$$

and hence $w \in EP(\phi)$.

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Next, we prove that $w \in \cap_{i=1}^N F(T_i)$. Assume that $w \notin \cap_{i=1}^N F(T_i)$. Thus there exists $l \in \{1, \ldots, N\}$ such that $w \notin F(T_l)$. From (3.13) and Opial's condition,

$$\begin{aligned}\liminf_{i\longrightarrow\infty} \|x_{n_i} - w\| &< \liminf_{i\longrightarrow\infty} \|x_{n_i} - T_l w\| \\ &\le \liminf_{i\longrightarrow\infty} \{\|x_{n_i} - T_l x_{n_i}\| + \|T_l x_{n_i} - T_l w\|\} \\ &\le \lim_{i\longrightarrow\infty} L\|x_{n_i} - w\|,\end{aligned}$$

which derives a contradiction. This implies that $w \in \cap_{i=1}^N F(T_i)$.

Step 5. We show that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $\cap_{i=1}^N F(T_i) \cap EP(\phi)$.

Indeed, to verify that the assertion is valid, it is sufficient to show that $\omega_w(x_n)$ is a single-point set. We take $w_1, w_2 \in \omega_w(x_n)$ arbitrarily and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \rightharpoonup w_1$ and $x_{m_j} \rightharpoonup w_2$, respectively. Since $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists for each $q \in \cap_{i=1}^N F(T_i) \cap EP(\phi)$ and $w_1, w_2 \in \cap_{i=1}^N F(T_i) \cap EP(\phi)$, by Lemma 2.1(iii), we obtain

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|x_n - w_1\|^2 &= \lim_{j\longrightarrow\infty} \|x_{m_j} - w_1\|^2 \\ &= \lim_{j\longrightarrow\infty} \|x_{m_j} - w_2\|^2 + \|w_2 - w_1\|^2 \\ &= \lim_{j\longrightarrow\infty} \|x_{k_i} - w_2\|^2 + \|w_2 - w_1\|^2 \\ &= \lim_{j\longrightarrow\infty} \|x_{k_i} - w_1\|^2 + 2\|w_2 - w_1\|^2 \\ &= \lim_{n\longrightarrow\infty} \|x_n - w_1\|^2 + 2\|w_2 - w_1\|^2.\end{aligned}$$

Hence $w_1 = w_2$. This shows that $\omega_w(x_n)$ is a single-point set. This completes the proof. □

Remark 3.2. Theorem 3.1 mainly improve Theorem 3.1 of Ceng, Homidan, Ansari and Yao [16], from a k-strictly pseudo-contractive mapping to an finite family of the asymptotically k_i-strictly pseudo-contractive mappings.

A direct consequence of Theorem 3.1, we derive the following theorem of Qin, Cho, Kang and Shang [18].

Theorem 3.3 *([18, Theorem 2.1]). Let C be nonempty closed convex subset of a real Hilbert space H and $N \ge 1$ be an integer. Let, for each $1 \le i \le N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \le k_i \le 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = max\{k_i : 1 \le i \le N\}$ and $k_n = max\{k_{n,i} : 1 \le i \le N\}$. Assume that $\cap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated initially by arbitrary element $x_0 \in C$ and then by*

$$x_n = \alpha_{n-1}x_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)} x_{n-1}; \quad \forall n \ge 1,$$

where $\{\alpha_n\}$ is a sequence in $[\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$. Then, the sequences $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^N$.

Proof. Put $\phi(x, y) = 0$ for all $x, y \in C$, $r_n = 1$ for all $n \ge 0$ in Theorem 3.1. Thus, we have $u_n = x_n$. Then the sequence $\{x_n\}$ generated in Theorem 3.3 converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^N$. □

4. Strong convergence theorems

Theorem 4.1. *Let C be nonempty closed convex subset of a real Hilbert space H and $N \ge 1$ be an integer, $\phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let, for each $1 \le i \le N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0 \le k_i \le 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = max\{k_i : 1 \le i \le N\}$ and $k_n = max\{k_{n,i} : 1 \le i \le N\}$. Assume that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi)$ is nonempty. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by arbitrary element $x_0 \in C$ and*

$$\begin{cases}\phi(u_{n-1}, y) + \dfrac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, & \forall y \in C \\ x_n = \alpha_{n-1}u_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)} u_{n-1}, & \forall n \ge 1,\end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) *$\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$, and*
(2) *$\{r_n\} \subset (0, \infty)$ and $\liminf_{n\longrightarrow\infty} r_n > 0$.*

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of F if and only if $\liminf_{n\longrightarrow\infty} d(x_n, F) = 0$, where $d(x_n, F)$ denotes the metric distance from the point x_n to F.

Proof. From the proof of Theorem 3.1, we know that $\lim_{n\longrightarrow\infty}\|x_n-q\|$ exists for each $q\in F$ and $\lim_{n\longrightarrow\infty}\|x_n-u_n\|=0$. Hence $\{x_n\}$ is bounded. The necessity is apparent. We show the sufficiency. Indeed we suppose that $\liminf_{n\longrightarrow\infty} d(x_n,F)=0$.

Since $x_n=\alpha_{n-1}u_{n-1}+(1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}$, from (3.2), we have

$$\|x_n-q\|^2\le(1+(k_{h(n)}^2-1))\|x_{n-1}-q\|^2.$$

From the fact that $x\ge 0$, $1+x\le e^x$, we can calculate

$$\begin{aligned}\|x_{n+m}-q\|^2 &= (1+(k_{h(n+m)}^2-1))\|x_{(n+m)-1}-q\|^2\\ &\le e^{(k_{h(n+m)}^2-1)}\|x_{(n+m)-1}-q\|^2\\ &\vdots\\ &\le e^{\sum_{j=n}^{n+m}(k_{h(n)}^2-1)}\|x_n-q\|^2\\ &\le e^{\sum_{j=1}^{\infty}(k_{h(j)}^2-1)}\|x_n-q\|^2,\quad \text{for all } n,m\in\mathbb{N}.\end{aligned}\tag{4.2}$$

Let $e^{\sum_{j=1}^{\infty}(k_{h(j)}^2-1)}=M$, for some nonnegative number M. Thus,

$$\|x_{n+m}-q\|^2\le M\|x_n-q\|^2,\quad \text{for all } n,m\in\mathbb{N}.$$

This gives that

$$\|x_{n+m}-q\|\le\sqrt{M}\|x_n-q\|,\quad \text{for all } n,m\in\mathbb{N}.\tag{4.3}$$

From Lemma 2.6, we have

$$\lim_{n\longrightarrow\infty} d(x_n,F)=0.$$

Here after, we will prove that $\{x_n\}$ is a Cauchy sequence. For any $\varepsilon>0$ there exists $N_1\in\mathbb{N}$ such that,

$$d(x_n,F)\le\frac{\varepsilon}{3\sqrt{M}},\quad \forall n\ge N_1.$$

In particular, we obtain that $d(x_{N_1},F)\le\frac{\varepsilon}{3\sqrt{M}}$. This implies that there exists $q_1\in F$ such that

$$\|x_{N_1}-q_1\|=d(x_{N_1},F)\le\frac{\varepsilon}{3\sqrt{M}}.$$

It follows from (4.3) that $n>N_1$,

$$\begin{aligned}\|x_{n+m}-x_n\|^2 &= \|x_{n+m}-q_1\|+\|x_n-q_1\|\\ &\le\sqrt{M}\|x_{N_1}-q_1\|+\sqrt{M}\|x_{N_1}-q_1\|\\ &\le\sqrt{M}\frac{\varepsilon}{2\sqrt{M}}+\sqrt{M}\frac{\varepsilon}{2\sqrt{M}}=\varepsilon.\end{aligned}\tag{4.4}$$

Thus $\{x_n\}$ is a Cauchy sequence. Suppose that $x_n\longrightarrow x^*\in H$. Then

$$d(x^*,F)=\lim_{n\longrightarrow\infty} d(x_n,F)=0.$$

Since for each $i=1,2,\ldots,N$, T_i is an asymptotically k_i-strictly pseudo-contractive mapping, we know form Lemma 2.9 that $\cap_{i=1}^N F(T_i)$ is closed and convex. Note that $EP(\phi)$ is closed and convex according to Lemma 2.11. Thus $F:=\cap_{i=1}^N F(T_i)\cap EP(\phi)$ is closed and convex. Consequently, $x^*\in F$. By using $\|u_n-x_n\|\longrightarrow 0$ as $n\longrightarrow\infty$, we conclude that both $\{x_n\}$ and $\{u_n\}$ converge strongly to an element x^* of F. □

Remark 4.2. Theorem 4.1 mainly improve Theorem 3.2 of Ceng, Homidan, Ansari and Yao [16], from a k-strictly pseudo-contractive mapping to an finite family of the asymptotically k_i-strictly pseudo-contractive mappings.

Theorem 4.3. *Let C be nonempty closed convex subset of a real Hilbert space H and $N\ge 1$ be an integer, $\phi: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let, for each $1\le i\le N$, $T_i: C\longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0\le k_i\le 1$ and a sequence $\{k_{n,i}\}$. Let $k=\max\{k_i: 1\le i\le N\}$ and $k_n=\max\{k_{n,i}: 1\le i\le N\}$. Assume that $F:=\cap_{i=1}^N F(T_i)\cap EP(\phi)$ is nonempty and let $x\in H$. For $C_0=C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

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$$\begin{cases} x_0 = P_{C_0}x, \\ u_{n-1} \in C \text{ such that } \phi(u_{n-1}, y) + \dfrac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C, \\ y_{n-1} = \alpha_{n-1}u_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}, \\ C_n = \{v \in C_{n-1} : \|y_{n-1} - v\|^2 \leq \|x_{n-1} - v\|^2 + \theta_{n-1}\}, \\ x_n = P_{C_n}x, \quad \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\theta_{n-1} = (k_{h(n)}^2 - 1)(1-\alpha_{n-1})\rho_{n-1}^2 \longrightarrow \infty$ *as* $n \longrightarrow \infty$, *where* $\rho_{n-1} = \sup\{\|x_{n-1} - v\| : v \in F\} < \infty$, *and* $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (k, 1)$ *and* $\{r_n\} \subset (0, \infty)$ *such that* $\liminf_{n\longrightarrow\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_F x$.

Proof. We first show by induction that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C_n$ for all $n \geq 0$. It is obvious that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C = C_0$. Suppose that $F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C_{j-1}$ for some $j \in \mathbb{N}$. Hence, for any $q \in F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C_{j-1}$ and by $u_{j-1} = S_{r_{j-1}}x_{j-1}$, we have

$$\begin{aligned} \|u_{j-1} - q\| &= \|S_{r_{j-1}}x_{n-1} - S_{r_{j-1}}q\| \\ &\leq \|x_{j-1} - q\|. \end{aligned}$$

Hence

$$\begin{aligned} \|y_{j-1} - q\|^2 &= \|\alpha_{j-1}(u_{j-1} - q) + (1-\alpha_{j-1})(T_{i(j)}^{h(j)}u_{j-1} - q)\|^2 \\ &= \alpha_{j-1}\|u_{j-1} - q\|^2 + (1-\alpha_{j-1})\|T_{i(j)}^{h(j)}u_{j-1} - q\|^2 \\ &\quad - \alpha_{j-1}(1-\alpha_{j-1})\|T_{i(j)}^{h(j)}u_{j-1} - u_{j-1}\|^2 \\ &\leq \alpha_{j-1}\|u_{j-1} - q\|^2 - \alpha_{j-1}(1-\alpha_{j-1})\|T_{i(j)}^{h(j)}u_{j-1} - u_{j-1}\|^2 \\ &\quad + (1-\alpha_{j-1})[k_{h(j)}^2\|u_{j-1} - q\|^2 + k\|T_{i(j)}^{h(j)}u_{j-1} - u_{j-1}\|^2] \\ &\leq \|x_{j-1} - q\|^2 + \theta_{j-1} - (1-\alpha_{j-1})(\alpha_{j-1} - k)\|T_{i(j)}^{h(j)}u_{j-1} - u_{j-1}\|^2 \\ &\leq \|x_{j-1} - q\|^2 + \theta_{j-1}. \end{aligned} \tag{4.6}$$

Therefore, $q \in C_j$. This implies that

$$F := \cap_{i=1}^N F(T_i) \cap EP(\phi) \subset C = C_n, \quad \text{for all } n \geq 0.$$

Next, we prove that C_n is closed and convex for all $n \geq 0$. It is obvious that $C_0 = C$ is closed and convex. Suppose that C_{k-1} is closed and convex for some $k \in \mathbb{N}$. For $v \in C_{k-1}$, we know that $\|y_{k-1} - v\|^2 \leq \|x_{k-1} - v\|^2 + \theta_{k-1}$ is equivalent to

$$2\langle x_{k-1} - y_{k-1}, v\rangle \leq \|x_{k-1}\|^2 - \|y_{k-1}\|^2 + \theta_{k-1}.$$

So, C_k is closed and convex. Then, for any $n \geq 0$, C_n is closed and convex. This implies that $\{x_n\}$ is well defined. From Lemma 2.10, the sequence $\{u_n\}$ is also well defined. From $x_n = P_{C_n}x$, we have

$$\langle x - x_n, x_n - y\rangle \geq 0,$$

for each $y \in C_n$. Using $F \subset C_n$, we also have

$$\langle x - x_n, x_n - p\rangle \geq 0 \quad \text{for each } p \in F \text{ and } n \in \mathbb{N}.$$

Hence, for $p \in F$, we have

$$\begin{aligned} 0 &\leq \langle x - x_n, x_n - p\rangle \\ &= \langle x - x_n, x_n - x + x - p\rangle \\ &= -\langle x_n - x, x_n - x\rangle + \langle x - x_n, x - p\rangle \\ &= -\|x_n - x\|^2 + \|x - x_n\|\|x - p\|. \end{aligned}$$

This implies that

$$\|x - x_n\| \leq \|x - p\|, \quad \text{for all } p \in F \text{ and } n \in \mathbb{N}. \tag{4.7}$$

Hence $\{x_n\}$ is bounded. It follows that $\{y_n\}$ is also bounded. From $x_{n-1} = P_{C_{n-1}}x$ and $x_n = P_{C_n}x \in C_n \subset C_{n-1}$, we obtain

$$\langle x - x_{n-1}, x_{n-1} - x_n\rangle \geq 0. \tag{4.8}$$

It follow that, for $n \in \mathbb{N}$,

$$0 \leq \langle x - x_{n-1}, x_{n-1} - x_n\rangle$$

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$$
\begin{aligned}
&= \langle x - x_{n-1}, x_{n-1} - x + x - x_n \rangle \\
&= -\|x - x_{n-1}\|^2 + \langle x - x_{n-1}, x - x_n \rangle \\
&\le -\|x - x_{n-1}\|^2 + \|x - x_{n-1}\|\|x - x_n\|,
\end{aligned}
$$

and hence

$$\|x - x_{n-1}\| \le \|x - x_n\|.$$

Hence $\{\|x_n - x\|\}$ is nondecreasing, so $\lim_{n\longrightarrow\infty} \|x_n - x\|$ exists.

Next we can show that $\lim_{n\longrightarrow\infty} \|x_n - x_{n-1}\| = 0$. Indeed, from (4.8) we get

$$
\begin{aligned}
\|x_{n-1} - x_n\|^2 &= \|x_{n-1} - x + x - x_n\|^2 \\
&= \|x_{n-1} - x\|^2 + 2\langle x_{n-1} - x, x - x_n \rangle + \|x - x_n\|^2 \\
&= -\|x_{n-1} - x\|^2 + 2\langle x_{n-1} - x, x - x_{n-1} + x_{n-1} + x_n \rangle + \|x - x_n\|^2 \\
&\le -\|x_{n-1} - x\|^2 + \|x - x_n\|^2.
\end{aligned}
$$

Since $\lim_{n\longrightarrow\infty} \|x - x_n\|$ exists, we have

$$\lim_{n\longrightarrow\infty} \|x_{n-1} - x_n\| = 0. \tag{4.9}$$

On the other hand, $x_n \in C_n$, we have

$$\|y_{n-1} - x_n\|^2 \le \|x_{n-1} - x_n\|^2 + \theta_{n-1}. \tag{4.10}$$

So, we have $\lim_{n\longrightarrow\infty} \|y_{n-1} - x_n\| = 0$. It follows that

$$\|y_{n-1} - x_{n-1}\| \le \|y_{n-1} - x_n\| + \|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.11}$$

Next, we claim that $\lim_{n\longrightarrow\infty} \|x_n - u_n\| = 0$. Indeed, let $q \in F$. Thus as above $u_{n-1} = S_{r_{n-1}} x_{n-1}$ and we have

$$
\begin{aligned}
\|u_{n-1} - q\|^2 &= \|S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} q\|^2 \\
&\le \langle S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} q, x_{n-1} - q \rangle \\
&= \langle u_{n-1} - q, x_{n-1} - q \rangle \\
&= \frac{1}{2}(\|u_{n-1} - q\|^2 + \|\alpha_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2).
\end{aligned}
$$

and hence

$$\|u_{n-1} - q\|^2 \le \|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2.$$

This implies that

$$
\begin{aligned}
\|y_{n-1} - q\|^2 &= \|\alpha_{n-1}(u_{n-1} - q) + (1 - \alpha_{n-1})(T_{i(n)}^{h(n)} u_{n-1} - q)\|^2 \\
&= \alpha_{n-1}\|u_{n-1} - q\|^2 + (1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} u_{n-1} - q\|^2 - \alpha_{n-1}(1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} u_{n-1} - u_{n-1}\|^2 \\
&\le \alpha_{n-1}\|u_{n-1} - q\|^2 - \alpha_{n-1}(1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} u_{n-1} - u_{n-1}\|^2 \\
&\quad + (1 - \alpha_{n-1})[k_{h(n)}^2 \|u_{n-1} - q\|^2 + k\|T_{i(n)}^{h(n)} u_{n-1} - u_{n-1}\|^2] \\
&\le \|u_{n-1} - q\|^2 + \theta_{n-1} - (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)} u_{n-1} - u_{n-1}\|^2 \\
&\le \|u_{n-1} - q\|^2 + \theta_{n-1} \\
&\le \|x_{n-1} - q\|^2 - \|x_{n-1} - u_{n-1}\|^2 + \theta_{n-1}.
\end{aligned} \tag{4.12}
$$

Hence

$$
\begin{aligned}
\|x_{n-1} - u_{n-1}\|^2 &\le \|x_{n-1} - q\|^2 - \|y_{n-1} - q\|^2 + \theta_{n-1}. \\
&\le \|x_{n-1} - y_{n-1}\|\{\|x_{n-1} - q\| - \|y_{n-1} - q\|\} + \theta_{n-1}.
\end{aligned} \tag{4.13}
$$

It follows from (4.11) and the boundedness of the sequences $\{x_n\}$ and $\{y_n\}$ that

$$\lim_{n\longrightarrow\infty} \|x_{n-1} - u_{n-1}\| = 0. \tag{4.14}$$

From (4.9) and (4.14), we have

$$\|u_n - u_{n-1}\| \le \|u_n - x_n\| + \|x_n - x_{n-1}\| + \|x_{n-1} - u_{n-1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

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This implies that

$$\lim_{n\longrightarrow\infty}\|u_n-u_{n+j}\|=0,\quad \text{for all } j\in\{1,2,\ldots,N\}.$$

Form $y_{n-1}=\alpha_{n-1}u_{n-1}+(1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1}$, $\forall n\geq 1$, we have

$$\begin{aligned}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1}-x_{n-1}\| &= \|y_{n-1}-\alpha_{n-1}u_{n-1}-(1-\alpha_{n-1})x_{n-1}\|\\ &\leq \|y_{n-1}-x_{n-1}\|+\alpha_{n-1}\|x_{n-1}-u_{n-1}\|.\end{aligned}$$

Applying (4.11) and (4.14) to the last inequality, we obtain

$$\lim_{n\longrightarrow\infty}\|T_{i(n)}^{h(n)}u_{n-1}-x_{n-1}\|=0. \tag{4.15}$$

From (4.14) and (4.15), we have

$$\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|\leq\|T_{i(n)}^{h(n)}u_{n-1}-x_{n-1}\|+\|x_{n-1}-u_{n-1}\|\longrightarrow 0. \tag{4.16}$$

By using the same method as in the proof of the Theorem 3.1, we easily obtain

$$\lim_{n\longrightarrow\infty}\|T_lu_n-u_n\|=0\quad\text{for all } l\in\{1,2,\ldots,N\} \tag{4.17}$$

and

$$\omega_w(x_n)\subset F:=\cap_{i=1}^N F(T_i)\cap EP(\phi). \tag{4.18}$$

This, together with (4.7) and Lemma 2.4 guarantees the strong convergence of $\{x_n\}$ to $p=P_Fx$. From (4.14), we also have the strong convergence of $\{u_n\}$ to $p=P_Fx$. This completes the proof. □

5. The CQ method for asymptotically *k*-strictly pseudo-contractive mappings

Theorem 5.1. *Let C be nonempty closed convex subset of a real Hilbert space H and $N\geq 1$ be an integer, $\phi: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let, for each $1\leq i\leq N$, $T_i: C\longrightarrow C$ be an asymptotically k_i-strictly pseudo-contractive mapping for some $0\leq k_i\leq 1$ and a sequence $\{k_{n,i}\}$. Let $k=max\{k_i: 1\leq i\leq N\}$ and $k_n=max\{k_{n,i}: 1\leq i\leq N\}$. Assume that $F:=\cap_{i=1}^N F(T_i)\cap EP(\phi)$ is nonempty. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated the following algorithm:*

$$\begin{cases} x_0=u\in C \text{ chosen arbitrarily},\\ u_{n-1}\in C \text{ such that } \phi(u_{n-1},y)+\dfrac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle\geq 0;\quad \forall y\in C,\\ y_{n-1}=\alpha_{n-1}u_{n-1}+(1-\alpha_{n-1})T_{i(n)}^{h(n)}u_{n-1},\\ C_{n-1}=\{v\in C:\|y_{n-1}-v\|^2\leq\|x_{n-1}-v\|^2+\theta_{n-1}\},\\ Q_{n-1}=\{v\in C:\langle x_0-x_{n-1},x_{n-1}-v\rangle\geq 0\},\\ x_n=P_{C_{n-1}\cap Q_{n-1}}x_0,\quad \forall n\geq 1,\end{cases} \tag{5.1}$$

where $\theta_{n-1}=(k_{h(n)}^2-1)(1-\alpha_{n-1})\rho_{n-1}^2\longrightarrow\infty$ as $n\longrightarrow\infty$, where $\rho_{n-1}=\sup\{\|x_{n-1}-v\|: v\in F\}<\infty$, and $\{\alpha_n\}\subset[\alpha,\beta]$ for some $\alpha,\beta\in(k,1)$ and $\{r_n\}\subset(0,\infty)$ such that $\liminf_{n\longrightarrow\infty}r_n>0$. Then $\{x_n\}$ converges strongly to P_Fx_0.

Proof. We show first that the sequence $\{x_n\}$ is well defined. From the definition of C_{n-1} and Q_{n-1}, it is obvious that C_{n-1} is closed and Q_{n-1} is closed and convex for each $n\in\mathbb{N}\cup\{0\}$. We prove that C_{n-1} is convex. For any $v_1,v_2\in C_{n-1}$ and $t\in(0,1)$, put $v=tv_1+(1-t)v_2$. It is sufficient to show that $v\in C_{n-1}$. Since

$$\|y_{n-1}-v\|^2\leq\|x_{n-1}-v\|^2+\theta_{n-1}$$

is equivalent to

$$2\langle x_{n-1}-y_{n-1},v\rangle\leq\|x_{n-1}\|^2-\|y_{n-1}\|^2+\theta_{n-1}.$$

One can easily see that $v\in C_{n-1}$. Therefore we can obtain that C_{n-1} is convex. So, $C_{n-1}\cap Q_{n-1}$ is a closed convex subset of H for any $n\in\mathbb{N}$.

Next, we show that $\cap_{i=1}^N F(T_i)\cap EP(\phi)\subseteq C_{n-1}$. Indeed, let $q\in\cap_{i=1}^N F(S_i)\cap EP(\phi)$ and let $\{S_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.11. Then $q=S_{r_n}q$. From $u_{n-1}=S_{r_{n-1}}x_{n-1}$, we have

$$\begin{aligned}\|u_{n-1}-q\| &= \|S_{r_{n-1}}x_{n-1}-S_{r_{n-1}}q\|\\ &\leq \|x_{n-1}-q\|.\end{aligned} \tag{5.2}$$

By our assumptions, we have

$$\begin{aligned}\|y_{n-1}-q\|^2 &= \|\alpha_{n-1}(u_{n-1}-q)+(1-\alpha_{n-1})(T_{i(n)}^{h(n)}u_{n-1}-q)\|^2\\ &= \alpha_{n-1}\|u_{n-1}-q\|^2+(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n+1}-q\|^2\\ &\quad -\alpha_{n-1}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|^2\\ &\le \alpha_{n-1}\|u_{n-1}-q\|^2-\alpha_{n-1}(1-\alpha_{n-1})\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|^2\\ &\quad +(1-\alpha_{n-1})[k_{h(n)}^2\|u_{n-1}-q\|^2+k\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|^2]\\ &\le \|x_{n-1}-q\|^2+\theta_{n-1}-(1-\alpha_{n-1})(\alpha_{n-1}-k)\|T_{i(n)}^{h(n)}u_{n-1}-u_{n-1}\|^2\\ &\le \|x_{n-1}-q\|^2+\theta_{n-1} \end{aligned} \tag{5.3}$$

Therefore, $q \in C_{n-1}$ for all $n \ge 1$.

Next, we show that

$$\cap_{i=1}^N F(T_i)\cap EP(\phi)\subseteq Q_{n-1}, \quad \forall n\ge 1. \tag{5.4}$$

We prove this by induction. For $n = 1$, we have $\cap_{i=1}^N F(T_i)\cap EP(\phi)\subset C = Q_0$. Assume that $\cap_{i=1}^N F(T_i)\cap EP(\phi)\subset Q_{n-1}$. Since x_n is the projection of x_0 onto $C_{n-1}\cap Q_{n-1}$, by Lemma 2.3, we have

$$\langle x_0-x_n, x_n-v\rangle \ge 0, \quad \forall v\in C_{n-1}\cap Q_{n-1}.$$

In particular, we have

$$\langle x_0-x_n, x_n-q\rangle \ge 0$$

for each $q\in \cap_{i=1}^N F(T_i)\cap EP(\phi)$ and hence $q\in Q_n$. Hence (5.4) holds for all $n\ge 1$. Therefore, we obtain that

$$\cap_{i=1}^N F(T_i)\cap EP(\phi)\subset C_{n-1}\cap Q_{n-1}, \quad \forall n\ge 1.$$

Next, we show that $\lim_{n\longrightarrow\infty}\|x_n-x_{n-1}\|=0$.

Since $\cap_{i=1}^N F(T_i)\cap EP(\phi)$ is a nonempty closed convex subset of H, there exists a unique $z'\in \cap_{i=1}^N F(T_i)\cap EP(\phi)$ such that

$$z' = P_{\cap_{i=1}^N F(T_i)\cap EP(\phi)}x_0.$$

From $x_n = P_{C_{n-1}\cap Q_{n-1}}x_0$, we have

$$\|x_n-x_0\|\le\|z-x_0\| \quad \text{for all } z\in C_{n-1}\cap Q_{n-1} \quad \text{and all } n\in\mathbb{N}.$$

Since $z'\in \cap_{i=1}^N F(T_i)\cap EP(\phi)\subset C_{n-1}\cap Q_{n-1}$ we have

$$\|x_n-x_0\|\le\|z'-x_0\| \ \text{ all } n\in\mathbb{N}. \tag{5.5}$$

Therefore, $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$. From the definition of Q_{n-1}, we have $x_{n-1} = P_{Q_{n-1}}x_0$, which together with the fact that $x_n\in C_{n-1}\cap Q_{n-1}\subset Q_{n-1}$ implies that

$$\|x_0-x_{n-1}\|\le\|x_0-x_n\|. \tag{5.6}$$

This show that the sequence $\{x_n-x_0\}$ is nondecreasing. So, we have $\lim_{n\longrightarrow\infty}\|x_n-x_0\|$ exists. Notice again that $x_{n-1} = P_{Q_{n-1}}x_0$ and $x_n\in Q_{n-1}$, which give that $\langle x_n-x_{n-1}, x_{n-1}-x_0\rangle\ge 0$. Therefore, we have

$$\begin{aligned}\|x_n-x_{n-1}\|^2 &= \|(x_n-x_0)-(x_{n-1}-x_0)\|^2\\ &= \|x_n-x_0\|^2-\|x_{n-1}-x_0\|^2-2\langle x_n-x_{n-1}, x_{n-1}-x_0\rangle\\ &\le \|x_n-x_0\|^2-\|x_{n-1}-x_0\|^2. \end{aligned} \tag{5.7}$$

This together with the existence of $\lim_{n\longrightarrow\infty}\|x_n-x_0\|$ implies that $\lim_{n\longrightarrow\infty}\|x_n-x_{n-1}\|=0$. Since $x_n\in C_{n-1}$, we have

$$\|y_{n-1}-x_n\|^2\le\|x_{n-1}-x_n\|^2+\theta_{n-1}. \tag{5.8}$$

So, we have $\lim_{n\longrightarrow\infty}\|y_{n-1}-x_n\|=0$. It follows that

$$\|y_{n-1}-x_{n-1}\|\le\|y_{n-1}-x_n\|+\|x_n-x_{n-1}\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{5.9}$$

Similar to the proof of Theorem 4.1, we have

$$\lim_{n\longrightarrow\infty}\|x_{n-1}-u_{n-1}\|=0 \tag{5.10}$$

and $\omega_w(x_n)\subset F$. This, together with (5.5) and Lemma 2.4 guarantees the strong convergence of $\{x_n\}$ to $P = P_F x$. From (5.10), we also have the strong convergence of $\{u_n\}$ to $p = P_F x$. This completes the proof. □

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ภาคผนวก 4

The Shrinking Projection Method for Solving Variational Inequality Problems and Fixed Point Problems in Banach Spaces

R. Wangkeeree, R. Wangkeeree

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Research Article

The Shrinking Projection Method for Solving Variational Inequality Problems and Fixed Point Problems in Banach Spaces

Rabian Wangkeeree and Rattanaporn Wangkeeree

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Rabian Wangkeeree, rabianw@nu.ac.th

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We consider a hybrid projection algorithm based on the shrinking projection method for two families of quasi-ϕ-nonexpansive mappings. We establish strong convergence theorems for approximating the common element of the set of the common fixed points of such two families and the set of solutions of the variational inequality for an inverse-strongly monotone operator in the framework of Banach spaces. As applications, at the end of the paper we first apply our results to consider the problem of finding a zero point of an inverse-strongly monotone operator and we finally utilize our results to study the problem of finding a solution of the complementarity problem. Our results improve and extend the corresponding results announced by recent results.

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1. Introduction

Let E be a Banach space and let C be a nonempty, closed, and convex subset of E. Let $A : C \to E^*$ be an operator. The classical variational inequality problem [1] for A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. The set of all solutions of (1.1) is denoted by $VI(A, C)$. Such a problem is connected with the convex minimization problem, the complementarity, the problem of finding a point $x^* \in E$ satisfying $0 = Ax^*$, and so on. First, we recall that a mapping $A : C \to E^*$ is said to be

(i) *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$, for all $x, y \in C$,

(ii) *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.2}$$

In this paper, we assume that the operator A satisfies the following conditions:

(C1) A is α-inverse-strongly monotone,

(C2) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$.

Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \quad \forall x \in E. \tag{1.3}$$

It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Some properties of the duality mapping are given in [2–4].

Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

If C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is a nonexpansive mapping. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [5] recently introduced a generalized projection operator C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \tag{1.5}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Observe that, in a Hilbert space H, (1.5) reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y, x). \tag{1.6}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, e.g., [2, 5–7]). In Hilbert spaces, $\Pi_C = P_C$, where P_C is the metric projection. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$,

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$ for all $x, y, z \in E$,

(3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$,

(4) if E is a reflexive, strictly convex, and smooth Banach space, then for all $x, y \in E$,

$$\phi(x, y) = 0 \quad \text{iff } x = y. \tag{1.7}$$

For more details see [2, 3]. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed point of T. A point p in C is said to be an *asymptotic fixed point* of T [8] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [9–11] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings which was studied in [9–11] is of special interest in the convergence analysis of feasibility, optimization, and equilibrium methods for solving the problems of image processing, rational resource allocation, and optimal control. The most typical examples in this regard are the Bregman projections and the Yosida type operators which are the cornerstones of the common fixed point and optimization algorithms discussed in [12] (see also the references therein).

The mapping T is said to be *ϕ-nonexpansive* if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$. T is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 1.1. The class of quasi-ϕ-nonexpansive is more general than the class of relatively nonexpansive mappings [9, 10, 13–15] which requires the strong restriction $\widehat{F}(T) = F(T)$.

Next, we give some examples which are closed quasi-ϕ-nonexpansive [16].

Example 1.2. (1) Let E be a uniformly smooth and strictly convex Banach space and let A be a maximal monotone mapping from E to E such that its zero set $A^{-1}0$ is nonempty. The resolvent $J_r = (J + rA)^{-1}J$ is a closed quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

(2) Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed and quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Iiduka and Takahashi [17] introduced the following algorithm for finding a solution of the variational inequality for an operator A that satisfies conditions (C1)-(C2) in a 2 uniformly convex and uniformly smooth Banach space E. For an initial point $x_0 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n x_n), \quad \forall n \geq 0. \tag{1.8}$$

where J is the duality mapping on E, and Π_C is the generalized projection of E onto C. Assume that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$ where $1/c$ is the 2 uniformly convexity constant of E. They proved that if J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to some element z in $VI(A, C)$ where $z = \lim_{n\to\infty}\Pi_{VI(A,C)}(x_n)$.

The problem of finding a common element of the set of the variational inequalities for monotone mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors; see, for instance, [18–20] and the references cited therein.

On the other hand, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping (see [21]). More precisely, let $t \in (0,1)$ and define a contraction $G_t : C \to C$ by $G_t x = tx_0 + (1-t)Tx$ for all $x \in C$, where $x_0 \in C$ is a fixed point in C. Applying Banach's Contraction Principle, there exists a unique fixed point x_t of G_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$ even if T has a fixed point. However, in the case of T having a fixed point, Browder [21] proved that the net $\{x_t\}$ defined by $x_t = tx_0 + (1-t)Tx_t$ for all $t \in (0,1)$ converges strongly to an element of $F(T)$ which is nearest to x_0 in a real Hilbert space. Motivated by Browder [21], Halpern [22] proposed the following iteration process:

$$x_0 \in C, \qquad x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad n \geqslant 0 \tag{1.9}$$

and proved the following theorem.

Theorem H. *Let C be a bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on C. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}, 0 < \theta < 1$. Define a sequence $\{x_n\}$ by (1.9). Then $\{x_n\}$ converges strongly to the element of $F(T)$ which is the nearest to u.*

Recently, Martinez-Yanes and Xu [23] have adapted Nakajo and Takahashi's [24] idea to modify the process (1.9) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{gathered} x_0 = x \in C \text{ chosen arbitrary}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \left\{ v \in C : \left\| y_n - v \right\|^2 \leqslant \|x_n - v\|^2 + \alpha_n \left(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle \right) \right\}, \\ Q_n = \{ v \in C : \langle x_n - v, x_0 - x_n \rangle \geqslant 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{gathered} \tag{1.10}$$

where P_C denotes the metric projection of H onto a closed convex subset C of H. They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$, then the sequence $\{x_n\}$ generated by (1.10) converges strongly to $P_{F(T)}x$.

In [15] (see also [13]), Qin and Su improved the result of Martinez-Yanes and Xu [23] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem.

Theorem QS. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let $T : C \to C$ be a relatively nonexpansive mapping. Assume that*

$\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{aligned}
&x_0 = x \in C \text{ chosen arbitrary},\\
&y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T x_n),\\
&C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, y_n) + (1-\alpha_n)\phi(v, x_n)\},\\
&Q_n = \{v \in C : \langle x_n - v, J x_0 - J x_n\rangle \geqslant 0\},\\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_0,
\end{aligned} \tag{1.11}$$

where J is the single-valued duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.

In [14], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings:

$$\begin{aligned}
&x_0 = x \in C \text{ chosen arbitrary},\\
&z_n = J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n\right),\\
&y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J z_n),\\
&H_n = \left\{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n) + \alpha_n\left(\|x_0\|^2 + 2\langle z, J x_n - J x\rangle\right)\right\},\\
&W_n = \{z \in C : \langle x_n - z, J x - J x_n\rangle \geqslant 0\},\\
&x_{n+1} = P_{H_n \cap W_n} x, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{1.12}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, and $\{\beta_n^{(3)}\}$ are sequences in $[0,1]$ satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and T, S are relatively nonexpansive mappings and J is the single-valued duality mapping on E. They proved, under appropriate conditions on the parameters, that the sequence $\{x_n\}$ generated by (1.12) converges strongly to a common fixed point of T and S.

Very recently, Qin et al. [25] introduced a new hybrid projection algorithm for two families of quasi-ϕ-nonexpansive mappings which are more general than relatively nonexpansive mappings to have strong convergence theorems in the framework of Banach spaces. To be more precise, they proved the following theorem.

Theorem QCKZ. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{S_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be two families of closed quasi-ϕ-nonexpansive mappings of C into itself with $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$ being nonempty, where*

I is an index set. Let the sequence $\{x_n\}$ *be generated by the following manner:*

$$
\begin{aligned}
x_0 &= x \in C \text{ chosen arbitrary}, \\
z_{n,i} &= J^{-1}\left(\beta_{n,i}^{(1)} J x_n + \beta_{n,i}^{(2)} J T_i x_n + \beta_{n,i}^{(3)} J S_i x_n\right), \\
y_{n,i} &= J^{-1}(\alpha_{n,i} J x_0 + (1-\alpha_{n,i}) J z_{n,i}), \\
C_{n,i} &= \left\{u \in C : \phi(u, y_{n,i}) \leqslant \phi(u, x_n) + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle u, J x_n - J x_n\rangle\right)\right\}, \\
C_n &= \bigcap_{i\in I} C_{n,i}, \\
Q_0 &= C, \\
Q_n &= \{u \in Q_{n-1} : \langle x_n - u, J x_0 - J x_n\rangle \geq 0\}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{1.13}
$$

where J *is the duality mapping on* E, *and* $\{\alpha_{n,i}\}$ *and* $\{\beta_{n,i}^{(i)}\}$ $(i = 1, 2, 3, \ldots)$ *are sequences in* $(0, 1)$ *satisfying*

(i) $\beta_{n,i}^{(1)} + \beta_{n,i}^{(2)} + \beta_{n,i}^{(3)} = 1$ *for all* $i \in I$,

(ii) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for all* $i \in I$,

(iii) $\liminf_{n\to\infty} \beta_{n,i}^{(2)} \beta_{n,i}^{(3)} > 0$ *and* $\lim_{n\to\infty} \beta_{n,i}^{(1)} = 0$ *for all* $i \in I$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

On the other hand, recently, Takahashi et al. [26] introduced the following hybrid method (1.14) which is different from Nakajo and Takahashi's [24] hybrid method. It is called the shrinking projection method. They obtained the following result.

Theorem NT. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* T *be a nonexpansive mapping of* C *into* H *such that* $F(T) \neq \emptyset$ *and let* $x_0 \in H$. *For* $C_1 = C$ *and* $x_1 = P_{C_1} x_0$, *define a sequence* $\{x_n\}$ *of* C *as follows:*

$$
\begin{aligned}
y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\
C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} &= P_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{aligned}
\tag{1.14}
$$

where $0 \leq \alpha_n < a < 1$ *for all* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to* $z_0 = P_{F(T)}(x_0)$.

Motivated and inspired by Iiduka and Takahashi [17], Martinez-Yanes and Xu [23], Matsushita and Takahashi [13], Plubtieng and Ungchittrakool [14], Qin and Su [15], Qin et al. [25], and Takahashi et al. [26], we introduce a new hybrid projection algorithm basing on the shrinking projection method for two families of quasi-ϕ-nonexpansive mappings which are more general than relatively nonexpansive mappings to have strong convergence theorems for approximating the common element of the set of common fixed points of two families of quasi-ϕ-nonexpansive mappings and the set of solutions of the variational inequality for an inverse-strongly monotone operator in the framework of Banach spaces. As applications, the problem of finding a zero point of an inverse-strongly monotone operator and the problem of finding a solution of the complementarity problem are studied. Our results improve and extend the corresponding results announced by recent results.

2. Preliminaries

Let E be a real Banach space with duality mapping J. We denote strong convergence of $\{x_n\}$ to x by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T)$ and range $R(T)$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i, i = 1, 2$. A monotone operator T is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operators.

A Banach space E is said to be strictly convex if $\|(x + y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided that

$$\lim_{t\to 0}\frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subset of E. Some properties of the duality mapping are given in [2, 3, 27–29]. We define the function $\delta : [0, 2] \to [0, 1]$ which is called the modulus of convexity of E as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x, y \in C, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}. \tag{2.2}$$

Then E is said to be 2 uniformly convex if there exists a constant $c > 0$ such that constant $\delta(\varepsilon) > c\varepsilon^2$ for all $\varepsilon \in (0, 2]$. Constant $1/c$ is called the 2 uniformly convexity constant of E. A 2 uniformly convex Banach space is uniformly convex; see [30, 31] for more details. We know the following lemma of 2 uniformly convex Banach spaces.

Lemma 2.1 (see [32, 33]). *Let E be a 2 uniformly convex Banach, then for all x, y from any bounded set of E and $jx \in Jx, jy \in Jy$,*

$$\langle x - y, jx - jy \rangle \geq \frac{c^2}{2}\|x - y\|^2, \tag{2.3}$$

where $1/c$ is the 2 uniformly convexity constant of E.

Now we present some definitions and lemmas which will be applied in the proof of the main result in the next section.

Lemma 2.2 (Kamimura and Takahashi [7]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty}\phi(y_n, z_n) = 0$, then $\lim_{n\to\infty}\|y_n - z_n\| = 0$.*

Lemma 2.3 (Alber [5]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for any $y \in C$.*

Lemma 2.4 (Alber [5]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x) \tag{2.4}$$

for all $y \in C$.

Lemma 2.5 (Qin et al. [25]). *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed quasi-ϕ-nonexpansive mapping of C into itself. Then $F(T)$ is a closed convex subset of C.*

Let E be a reflexive strictly convex, smooth, and uniformly Banach space and the duality mapping from E to E^*. Then J^{-1} is also single valued, one to one, and surjective, and it is the duality mapping from E^* to E. We need the following mapping V which is studied in Alber [5]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x\|^2 \tag{2.5}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma.

Lemma 2.6 (Kamimura and Takahashi [7]). *Let E be a reflexive, strictly convex, and smooth Banach space, and let V be as in* (2.5). *Then*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^*\right\rangle \leq V(x, x^* + y^*) \tag{2.6}$$

for all $x \in E$ and $x^*,\ y^* \in E^*$.

Lemma 2.7 (see [34, Lemma 1.4]). *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|) \tag{2.7}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

An operator A of C into E^* is said to be hemicontinuous if, for all $x, y \in C$, the mapping F of $[0,1)$ into E^* defined by $F(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C\}. \tag{2.8}$$

Lemma 2.8 (see [35]). *Let C be a nonempty closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{2.9}$$

Then T is maximal monotone and $T^{-1}0 = VI(A, C)$.

3. Main Results

In this section, we prove strong convergence theorem which is our main result.

Theorem 3.1. *Let E be a 2 uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let A be an operator of C into E^* satisfying (C1) and (C2), and let $\{S_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be two families of closed quasi-ϕ-nonexpansive mappings of C into itself with $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap VI(A, C)$ being nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered} x_0 \in C \text{ chosen arbitrary}, \\ C_{1,i} = C, \qquad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \qquad x_1 = \Pi_{C_1}(x_0) \quad \forall i \in I, \\ w_{n,i} = \Pi_C J^{-1}(Jx_n - \lambda_{n,i} A x_n), \\ z_{n,i} = J^{-1}\left(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i}\right), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 + (1-\alpha_{n,i}) Jz_{n,i}), \\ C_{n+1,i} = \left\{u \in C_{n,i} : \phi(u, y_{n,i}) \leqslant \phi(u, x_n) + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle\right)\right\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{gathered} \tag{3.1}$$

where J is the duality mapping on E, and $\{\lambda_{n,i}\}$, $\{\alpha_{n,i}\}$, and $\{\beta_{n,i}^{(j)}\}$ $(j = 1, 2, 3)$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for all $i \in I$;*

(ii) *for all $i \in I$, $\{\lambda_{n,i}\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2 uniformly convexity constant of E;*

(iii) $\beta_{n,i}^{(1)} + \beta_{n,i}^{(2)} + \beta_{n,i}^{(3)} = 1$ *for all* $i \in I$ *and if one of the following conditions is satisfied:*

(a) $\liminf_{n\to\infty} \beta_{n,i}^{(1)}\beta_{n,i}^{(l)} > 0$ *for all* $l = 2, 3$ *and for all* $i \in I$,

(b) $\liminf_{n\to\infty} \beta_{n,i}^{(2)}\beta_{n,i}^{(3)} > 0$ *and* $\lim_{n\to\infty} \beta_{n,i}^{(1)} = 0$ *for all* $i \in I$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* Π_F *is the generalized projection from* C *onto* F.

Proof. We divide the proof into six steps.

Step 1. Show that $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well defined.

To this end, we prove first that F is closed and convex. It is obvious that $VI(A, C)$ is a closed convex subset of C. By Lemma 2.5, we know that $\bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$ is closed and convex. Hence $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap VI(A, C)$ is a nonempty, closed, and convex subset of C. Consequently, $\Pi_F x_0$ is well defined.

We next show that C_{n+1} is convex for each $n \geq 0$. From the definition of C_n, it is obvious that C_n is closed for each $n \geq 0$. Notice that

$$C_{n+1,i} = \left\{u \in C_{n,i} : \phi(u, y_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle\right)\right\} \tag{3.2}$$

is equivalent to

$$C'_{n+1,i} = \left\{u \in C_{n,i} : 2\langle u, Jx_n - Jy_{n,i}\rangle - 2\alpha_{n,i}\langle u, Jx_n - Jx_0\rangle \leq \|x_n\|^2 - \|y_{n,i}\|^2 + \alpha_{n,i}\|x_0\|^2\right\}. \tag{3.3}$$

It is easy to see that $C'_{n+1,i}$ is closed and convex for all $n \geq 0$ and $i \in I$. Therefore, $C_{n+1} = \bigcap_{i\in I} C_{n+1,i} = \bigcap_{i\in I} C'_{n+1,i}$ is closed and convex for every $n \geq 0$. This shows that $\Pi_{C_{n+1}} x_0$ is well defined.

Step 2. Show that $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap VI(A, C) \subset C_n$ for all $n \geq 0$.

Put $v_{n,i} = J^{-1}(Jx_n - \lambda_{n,i}Ax_n)$. We have to show that $F \subset C_n$ for all $n \geq 0$. For all $u \in F$, we know from Lemmas 2.4 and 2.6 that

$$\begin{aligned}
\phi(u, w_{n,i}) &= \phi(u, \Pi_C v_{n,i}) \\
&\leq \phi(u, v_{n,i}) \\
&= \phi\left(u, J^{-1}(Jx_n - \lambda_{n,i}Ax_n)\right) \\
&= V(u, Jx_n - \lambda_{n,i}Ax_n) \\
&\leq V(u, (Jx_n - \lambda_{n,i}Ax_n) + \lambda_{n,i}Ax_n) - 2\left\langle J^{-1}(Jx_n - \lambda_{n,i}Ax_n) - u, \lambda_{n,i}Ax_n\right\rangle \\
&= V(u, Jx_n) - 2\lambda_{n,i}\langle v_{n,i} - u, Ax_n\rangle \\
&= \phi(u, x_n) - 2\lambda_{n,i}\langle x_n - u, Ax_n\rangle + 2\langle v_{n,i} - x_n, -\lambda_{n,i}Ax_n\rangle.
\end{aligned} \tag{3.4}$$

Since $u \in VI(A,C)$ and from condition (C1), we have

$$\begin{aligned} -2\lambda_{n,i}\langle x_n - u, Ax_n\rangle &= -2\lambda_{n,i}\langle x_n - u, Ax_n - Au\rangle - 2\lambda_{n,i}\langle x_n - u, Au\rangle \\ &\le -2\alpha\lambda_{n,i}\|Ax_n - Au\|^2. \end{aligned} \tag{3.5}$$

From Lemma 2.1, and condition (C2), we also have

$$\begin{aligned} 2\langle v_{n,i} - x_n, -\lambda_{n,i}Ax_n\rangle &= 2\left\langle J^{-1}(Jx_n - \lambda_{n,i}Ax_n) - J^{-1}(Jx_n), -\lambda_{n,i}Ax_n\right\rangle \\ &\le 2\left\|J^{-1}(Jx_n - \lambda_{n,i}Ax_n) - J^{-1}(Jx_n)\right\|\|\lambda_{n,i}Ax_n\| \\ &\le \frac{4}{c^2}\left\|JJ^{-1}(Jx_n - \lambda_{n,i}Ax_n) - JJ^{-1}(Jx_n)\right\|\|\lambda_{n,i}Ax_n\| \\ &= \frac{4}{c^2}\|(Jx_n - \lambda_{n,i}Ax_n) - (Jx_n)\|\|\lambda_{n,i}Ax_n\| \\ &\le \frac{4}{c^2}\lambda_{n,i}^2\|Ax_n\|^2 \\ &\le \frac{4}{c^2}\lambda_{n,i}^2\|Ax_n - Au\|^2. \end{aligned} \tag{3.6}$$

Subtituting (3.6) and (3.5) into (3.4) and using the assumption (ii), we obtain

$$\begin{aligned} \phi(u, w_{n,i}) &\le \phi(u, x_n) - 2\alpha\lambda_{n,i}\|Ax_n - Au\|^2 + \frac{4}{c^2}\lambda_{n,i}^2\|Ax_n - Au\|^2. \\ &\le \phi(u, x_n) + 2\lambda_{n,i}\left(\frac{2}{c^2}\lambda_{n,i} - \alpha\right)\|Ax_n - Au\|^2 \\ &\le \phi(u, x_n). \end{aligned} \tag{3.7}$$

It follows from the convexity of $\|\cdot\|^2$ and (3.7) that

$$\begin{aligned} \phi(u, z_{n,i}) &= \phi\left(u, J^{-1}\left(\beta_{n,i}^{(1)}Jx_n + \beta_{n,i}^{(2)}JT_ix_n + \beta_{n,i}^{(3)}JS_iw_{n,i}\right)\right) \\ &= \|u\|^2 - 2\left\langle u, \beta_{n,i}^{(1)}Jx_n + \beta_{n,i}^{(2)}JT_ix_n + \beta_{n,i}^{(3)}JS_iw_{n,i}\right\rangle \\ &\quad + \left\|\beta_{n,i}^{(1)}Jx_n + \beta_{n,i}^{(2)}JT_ix_n + \beta_{n,i}^{(3)}JS_iw_{n,i}\right\|^2 \\ &\le \|u\|^2 - 2\beta_{n,i}^{(1)}\langle u, Jx_n\rangle - 2\beta_{n,i}^{(2)}\langle u, JT_ix_n\rangle - 2\beta_{n,i}^{(2)}\langle u, JS_iw_{n,i}\rangle \\ &\quad + \beta_{n,i}^{(1)}\|Jx_n\|^2 + \beta_{n,i}^{(2)}\|JT_ix_n\|^2 + \beta_{n,i}^{(3)}\|JS_iw_{n,i}\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \beta_{n,i}^{(1)}\phi(u, x_n) + \beta_{n,i}^{(2)}\phi(u, T_i x_n) + \beta_{n,i}^{(3)}\phi(u, S_i w_{n,i}) \\
&\le \beta_{n,i}^{(1)}\phi(u, x_n) + \beta_{n,i}^{(2)}\phi(u, x_n) + \beta_{n,i}^{(3)}\phi(u, w_{n,i}) \\
&\le \phi(u, x_n),
\end{aligned} \tag{3.8}
$$

and hence

$$
\begin{aligned}
\phi(u, y_{n,i}) &= \phi\Big(u, J^{-1}(\alpha_{n,i}Jx_0 + (1-\alpha_{n,i})Jz_{n,i})\Big) \\
&= \|u\|^2 - 2\langle u, \alpha_{n,i}Jx_0 + (1-\alpha_{n,i})Jz_{n,i}\rangle + \|\alpha_{n,i}Jx_0 + (1-\alpha_{n,i})Jz_{n,i}\|^2 \\
&\le \|u\|^2 - 2\alpha_{n,i}\langle u, Jx_0\rangle - 2(1-\alpha_{n,i})\langle u, Jz_{n,i}\rangle + \alpha_{n,i}\|x_0\|^2 + (1-\alpha_{n,i})\|z_{n,i}\|^2 \\
&\le \alpha_{n,i}\phi(u, x_0) + (1-\alpha_{n,i})\phi(u, z_{n,i}) \\
&\le \alpha_{n,i}\phi(u, x_0) + (1-\alpha_{n,i})\phi(u, x_n) \\
&= \phi(u, x_n) + \alpha_{n,i}\left[\phi(u, x_0) - \phi(u, x_n)\right] \\
&\le \phi(u, x_n) + \alpha_{n,i}\Big(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle\Big).
\end{aligned} \tag{3.9}
$$

This show that $u \in C_{n+1,i}$ for each $i \in I$. That is, $u \in C_n = \bigcap_{i\in I} C_{n,i}$ for all $n \ge 0$. This show that

$$
F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap VI(A, C) \subset C_n, \quad \forall n \ge 0. \tag{3.10}
$$

Step 3. Show that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

We note that $C_{n+1,i} \subset C_{n,i}$ for all $n \ge 0$ and for all $i \in I$. Hence

$$
C_{n+1} = \bigcap_{i\in I} C_{n+1,i} \subset C_n = \bigcap_{i\in I} C_{n,i}. \tag{3.11}
$$

From $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n} x_0 \in C_n$, we have

$$
\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \quad \forall n \ge 1. \tag{3.12}
$$

This shows that $\{\phi(x_n, x_0)\}$ is nondecreasing. On the other hand, from Lemma 2.4, we have

$$
\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(w, x_0) - \phi(w, x_n) \le \phi(w, x_0) \tag{3.13}
$$

for each $w \in F \subset C_n$. This show that $\{\phi(x_n, x_0)\}$ is bounded. Consequently, $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

Step 4. Show that $\{x_n\}$ is a convergent sequence in C.

Since $x_m = \Pi_{C_m} x_0 \in C_n$ for any $m \geq n$. It follows that

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0)\\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0)\\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.14}$$

Letting $m, n \to \infty$ in (3.14), we have $\phi(x_m, x_n) \to 0$. It follows from Lemma 2.2 that

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0. \tag{3.15}$$

Hence $\{x_n\}$ is a Cauchy sequence in C. By the completeness of E and the closedness of C, we can assume that

$$x_n \longrightarrow p \in C \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

Step 5. We show that $p \in F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap VI(A, C)$.

(I) We first show that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$. Taking $m = n+1$ in (3.14), one arrives that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.17}$$

From Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.18}$$

Noticing that $x_{n+1} = \Pi_{C_{n+1}} x_0$, from the definition of $C_{n,i}$ for every $i \in I$, we obtain

$$\phi\left(x_{n+1}, y_{n,i}\right) \leqslant \phi(x_{n+1}, x_n) + \alpha_{n,i}\Big(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle\Big). \tag{3.19}$$

It follows from (3.17) and $\lim_{n\to\infty} \alpha_{n,i} = 0$ and the fact that $\{Jx_n\}$ is bounded that

$$\lim_{n\to\infty} \phi\left(x_{n+1}, y_{n,i}\right) = 0, \quad \forall i \in I. \tag{3.20}$$

From Lemma 2.2, we obtain

$$\lim_{n\to\infty} \left\|x_{n+1} - y_{n,i}\right\| = 0, \quad \forall i \in I. \tag{3.21}$$

It follows from (3.18) that

$$\lim_{n\to\infty} \left\|x_n - y_{n,i}\right\| = 0, \quad \forall i \in I. \tag{3.22}$$

Since J is uniformly norm-to-norm continuity on bounded sets, for every $i \in I$, one has

$$\lim_{n\to\infty} \|Jx_n - Jy_{n,i}\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0, \quad \forall i \in I. \tag{3.23}$$

For every $i \in I$, we obtain from the properties of ϕ that

$$\begin{aligned}\phi(z_{n,i}, x_n) &= \phi(z_{n,i}, y_{n,i}) + \phi(y_{n,i}, x_n) + 2\langle z_{n,i} - y_{n,i}, Jy_{n,i} - Jx_n\rangle \\ &\le \phi(z_{n,i}, y_{n,i}) + \phi(y_{n,i}, x_n) + 2\|z_{n,i} - y_{n,i}\| \|Jy_{n,i} - Jx_n\|.\end{aligned} \tag{3.24}$$

On the other hand, for all $i \in I$, we have

$$\begin{aligned}\phi(z_{n,i}, y_{n,i}) &= \|z_{n,i}\|^2 - 2\langle z_{n,i}, \alpha_{n,i}Jx_0 + (1-\alpha_{n,i})Jz_{n,i}\rangle + \|\alpha_{n,i}Jx_0 + (1-\alpha_{n,i})Jz_{n,i}\|^2 \\ &\le \|z_{n,i}\|^2 - 2\alpha_{n,i}\langle z_{n,i}, Jx_0\rangle - 2(1-\alpha_{n,i})\langle z_{n,i}, Jz_{n,i}\rangle + \alpha_{n,i}\|x_0\|^2 + (1-\alpha_{n,i})\|z_{n,i}\|^2 \\ &= \alpha_{n,i}\left(\|z_{n,i}\|^2 - 2\langle z_{n,i}, Jx_0\rangle + \|x_0\|^2\right) = \alpha_{n,i}\phi(z_{n,i}, x_0).\end{aligned} \tag{3.25}$$

It follows form (ii) that

$$\lim_{n\to\infty} \phi(z_{n,i}, y_{n,i}) = 0, \quad \forall i \in I. \tag{3.26}$$

Notice that

$$\begin{aligned}\phi(y_{n,i}, x_n) &= \|y_{n,i}\|^2 - 2\langle y_{n,i}, Jx_n\rangle + \|x_n\|^2 \\ &= \|y_{n,i}\|^2 - 2\langle y_{n,i}, Jx_n\rangle + \|x_n\|^2 + \|x_{n+1}\|^2 - \|x_{n+1}\|^2 \\ &\quad - 2\langle x_{n+1}, Jy_{n,i}\rangle + 2\langle x_{n+1}, Jy_{n,i}\rangle \\ &= \phi(x_{n+1}, y_{n,i}) - 2\langle y_{n,i}, Jx_n\rangle + \|x_n\|^2 - \|x_{n+1}\|^2 + 2\langle x_{n+1}, Jy_{n,i}\rangle \\ &= \phi(x_{n+1}, y_{n,i}) + (\|x_n - x_{n+1}\|)(\|x_n\| + \|x_{n+1}\|) \\ &\quad - 2\langle y_{n,i}, Jx_n - Jy_{n,i}\rangle - 2\langle y_{n,i}, Jy_{n,i}\rangle + 2\langle x_{n+1}, Jy_{n,i}\rangle \\ &= \phi(x_{n+1}, y_{n,i}) + (\|x_n - x_{n+1}\|)(\|x_n\| + \|x_{n+1}\|) \\ &\quad + 2\langle y_{n,i}, Jy_{n,i} - Jx_n\rangle + 2\langle x_{n+1} - y_{n,i}, Jy_{n,i}\rangle \\ &\le \phi(x_{n+1}, y_{n,i}) + (\|x_n - x_{n+1}\|)(\|x_n\| + \|x_{n+1}\|) \\ &\quad + 2\|y_{n,i}\| \|Jy_{n,i} - Jx_n\| + 2\|x_{n+1} - y_{n,i}\| \|Jy_{n,i}\|.\end{aligned} \tag{3.27}$$

Applying (3.18), (3.20), (3.21), and (3.23) to the last inequality, we obtain

$$\lim_{n\to\infty} \phi(y_{n,i}, x_n) = 0, \quad \forall i \in I. \tag{3.28}$$

Combining (3.26) with (3.28) in (3.24), we have

$$\lim_{n\to\infty}\phi(z_{n,i},x_n)=0,\quad \forall i\in I. \tag{3.29}$$

From Lemma 2.2, we have

$$\lim_{n\to\infty}\|z_{n,i}-x_n\|=0,\quad \forall i\in I. \tag{3.30}$$

Since J is uniformly norm-to-norm continuity on bounded sets, for every $i\in I$, one has

$$\lim_{n\to\infty}\|Jz_{n,i}-Jx_n\|=0,\quad \forall i\in I. \tag{3.31}$$

Let $r=\sup_{n\geq 1}\{\|x_n\|,\|T_ix_n\|,\|S_ix_n\|\}$ for every $i\in I$. Therefore Lemma 2.7 implies that there exists a continuous strictly increasing convex function $g:[0,\infty)\to[0,\infty)$ satisfying $g(0)=0$ and (2.7).

Case I. Assume that (a) holds. Applying (2.7), we can calculate

$$\begin{aligned}
\phi(u,z_{n,i})&=\phi\Big(u,J^{-1}\Big(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_iw_{n,i}\Big)\Big)\\
&=\|u\|^2-2\Big\langle u,\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_iw_{n,i}\Big\rangle\\
&\quad+\Big\|\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_iw_{n,i}\Big\|^2\\
&\leq\|u\|^2-2\beta_{n,i}^{(1)}\langle u,Jx_n\rangle-2\beta_{n,i}^{(2)}\langle u,JT_ix_n\rangle-2\beta_{n,i}^{(2)}\langle u,JS_iw_{n,i}\rangle\\
&\quad+\beta_{n,i}^{(1)}\|x_n\|^2+\beta_{n,i}^{(2)}\|T_ix_n\|^2+\beta_{n,i}^{(3)}\|S_iw_{n,i}\|^2-\beta_{n,i}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_ix_n\|)\\
&=\beta_{n,i}^{(1)}\phi(u,x_n)+\beta_{n,i}^{(2)}\phi(u,T_ix_n)+\beta_{n,i}^{(3)}\phi(u,S_iw_{n,i})-\beta_{n,i}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_ix_n\|)\\
&\leq\beta_{n,i}^{(1)}\phi(u,x_n)+\beta_{n,i}^{(2)}\phi(u,x_n)+\beta_{n,i}^{(3)}\phi(u,w_{n,i})-\beta_{n,i}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_ix_n\|)\\
&\leq\beta_{n,i}^{(1)}\phi(u,x_n)+\beta_{n,i}^{(2)}\phi(u,x_n)+\beta_{n,i}^{(3)}\phi(u,x_n)-\beta_{n,i}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_ix_n\|)\\
&=\phi(u,x_n)-\beta_{n,i}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_ix_n\|).
\end{aligned}\tag{3.32}$$

This implies that

$$\beta_{n,i}^{(1)}\beta_{n,i}^{(2)}g(\|Jx_n-JT_ix_n\|)\leq\phi(u,x_n)-\phi(u,z_{n,i}),\quad \forall i\in I. \tag{3.33}$$

On the other hand, for every $i\in I$, one has

$$\begin{aligned}
\phi(u,x_n)-\phi(u,z_{n,i})&=\|x_n\|^2-\|z_{n,i}\|^2-2\langle u,Jx_n-Jz_{n,i}\rangle\\
&\leq\|x_n-z_{n,i}\|(\|x_n\|+\|z_{n,i}\|)+2\|u\|\|Jx_n-Jz_{n,i}\|.
\end{aligned}\tag{3.34}$$

It follows from (3.30) and (3.31) that

$$\phi(u, x_n) - \phi(u, z_{n,i}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \ \forall i \in I. \tag{3.35}$$

Applying $\liminf_{n\to\infty} \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} > 0$ and (3.35) in (3.33) we get

$$g(\|Jx_n - JT_i x_n\|) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \ \forall i \in I. \tag{3.36}$$

It follows from the property of g that

$$\|Jx_n - JT_i x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \ \forall i \in I. \tag{3.37}$$

Since J^{-1} is also uniformly norm-to-norm continuity on bounded sets, for every $i \in I$, one has

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \forall i \in I. \tag{3.38}$$

In a similar way, one has

$$\lim_{n\to\infty} \|x_n - S_i w_{n,i}\| = 0, \quad \forall i \in I. \tag{3.39}$$

On the other hand, we observe from (3.7) that

$$\begin{aligned}
\phi(u, z_{n,i}) &= \phi\left(u, J^{-1}\left(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i}\right)\right) \\
&= \|u\|^2 - 2\left\langle u, \beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i} \right\rangle \\
&\quad + \left\| \beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i} \right\|^2 \\
&\leq \|u\|^2 - 2\beta_{n,i}^{(1)} \langle u, Jx_n \rangle - 2\beta_{n,i}^{(2)} \langle u, JT_i x_n \rangle - 2\beta_{n,i}^{(2)} \langle u, JS_i w_{n,i} \rangle \\
&\quad + \beta_{n,i}^{(1)} \|Jx_n\|^2 + \beta_{n,i}^{(2)} \|JT_i x_n\|^2 + \beta_{n,i}^{(3)} \|JS_i w_{n,i}\|^2 - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|Jx_n - JT_i x_n\|) \\
&= \beta_{n,i}^{(1)} \phi(u, x_n) + \beta_{n,i}^{(2)} \phi(u, T_i x_n) + \beta_{n,i}^{(3)} \phi(u, S_i w_{n,i}) - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|Jx_n - JT_i x_n\|) \\
&\leq \beta_{n,i}^{(1)} \phi(u, x_n) + \beta_{n,i}^{(2)} \phi(u, x_n) + \beta_{n,i}^{(3)} \phi(u, w_{n,i}) - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|Jx_n - JT_i x_n\|) \\
&\leq \beta_{n,i}^{(1)} \phi(u, x_n) + \beta_{n,i}^{(2)} \phi(u, x_n) + \beta_{n,i}^{(3)} \left[\phi(u, x_n) + 2\lambda_{n,i} \left(\frac{2}{c^2} \lambda_{n,i} - \alpha \right) \|Ax_n - Au\|^2 \right] \\
&= \phi(u, x_n) + 2\beta_{n,i}^{(3)} \lambda_{n,i} \left(\frac{2}{c^2} \lambda_{n,i} - \alpha \right) \|Ax_n - Au\|^2.
\end{aligned} \tag{3.40}$$

Hence

$$2a\left(\alpha - \frac{2}{c^2}b\right)\|Ax_n - Au\|^2 \leq \phi(u, x_n) - \phi(u, z_{n,i}). \tag{3.41}$$

Using (3.35), we can conclude that

$$\lim_{n\to\infty} \|Ax_n - Au\| = 0, \quad \forall i \in I. \tag{3.42}$$

From (3.6), we can calculate

$$\begin{aligned}
\phi(x_n, w_{n,i}) &= \phi(x_n, \Pi_C v_{n,i}) \\
&\leq \phi(x_n, v_{n,i}) \\
&= \phi\left(x_n, J^{-1}(Jx_n - \lambda_{n,i}Ax_n)\right) \\
&= V(x_n, Jx_n - \lambda_{n,i}Ax_n) \\
&\leq V(x_n, (Jx_n - \lambda_{n,i}Ax_n) + \lambda_{n,i}Ax_n) \\
&\quad - 2\left\langle J^{-1}(Jx_n - \lambda_{n,i}Ax_n) - u, \lambda_{n,i}Ax_n\right\rangle \\
&= V(x_n, Jx_n) + 2\langle v_{n,i} - x_n, -\lambda_{n,i}Ax_n\rangle \\
&= \phi(x_n, x_n) + 2\langle v_{n,i} - x_n, -\lambda_{n,i}Ax_n\rangle \\
&= 2\langle v_{n,i} - x_n, -\lambda_{n,i}Ax_n\rangle \\
&\leq \frac{4}{c^2}\lambda_{n,i}^2\|Ax_n - Au\|.
\end{aligned} \tag{3.43}$$

It follows from (3.42) and the fact that $\{\lambda_{n,i}\}$ is bounded that

$$\lim_{n\to\infty} \phi(x_n, w_{n,i}) = 0, \quad \forall i \in I. \tag{3.44}$$

From Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - w_{n,i}\| = 0, \quad \forall i \in I. \tag{3.45}$$

Hence $w_{n,i} \to p$ as $n \to \infty$ for each $i \in I$. From (3.39) and (3.45), we have

$$\lim_{n\to\infty} \|w_{n,i} - S_i w_{n,i}\| = 0, \quad \forall i \in I. \tag{3.46}$$

The closedness of T_i and S_i implies that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$.

Case II. Assume that (b) holds. We observe that

$$\begin{aligned}
\phi(u, z_{n,i}) &= \phi\left(u, J^{-1}\left(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i}\right)\right) \\
&= \|u\|^2 - 2\langle u, \beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i}\rangle \\
&\quad + \left\|\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i w_{n,i}\right\|^2 \\
&\leq \|u\|^2 - 2\beta_{n,i}^{(1)}\langle u, Jx_n\rangle - 2\beta_{n,i}^{(2)}\langle u, JT_i x_n\rangle - 2\beta_{n,i}^{(2)}\langle u, JS_i w_{n,i}\rangle \\
&\quad + \beta_{n,i}^{(1)}\|Jx_n\|^2 + \beta_{n,i}^{(2)}\|JT_i x_n\|^2 + \beta_{n,i}^{(3)}\|JS_i w_{n,i}\|^2 - \beta_{n,i}^{(2)}\beta_{n,i}^{(3)} g(\|JS_i w_{n,i} - JT_i x_n\|) \\
&= \beta_{n,i}^{(1)}\phi(u, x_n) + \beta_{n,i}^{(2)}\phi(u, T_i x_n) + \beta_{n,i}^{(3)}\phi(u, S_i w_{n,i}) - \beta_{n,i}^{(2)}\beta_{n,i}^{(3)} g(\|JS_i w_{n,i} - JT_i x_n\|) \\
&\leq \beta_{n,i}^{(1)}\phi(u, x_n) + \beta_{n,i}^{(2)}\phi(u, x_n) + \beta_{n,i}^{(3)}\phi(u, w_{n,i}) - \beta_{n,i}^{(2)}\beta_{n,i}^{(3)} g(\|JS_i w_{n,i} - JT_i x_n\|) \\
&\leq \beta_{n,i}^{(1)}\phi(u, x_n) + \beta_{n,i}^{(2)}\phi(u, x_n) + \beta_{n,i}^{(3)}\phi(u, x_n) - \beta_{n,i}^{(2)}\beta_{n,i}^{(3)} g(\|JS_i w_{n,i} - JT_i x_n\|) \\
&= \phi(u, x_n) - \beta_{n,i}^{(2)}\beta_{n,i}^{(3)} g(\|JS_i w_{n,i} - JT_i x_n\|).
\end{aligned} \tag{3.47}$$

This implies that

$$\beta_{n,i}^{(2)}\beta_{n,i}^{(3)} g(\|JS_i w_{n,i} - JT_i x_n\|) \leq \phi(u, x_n) - \phi(u, z_{n,i}), \quad \forall i \in I. \tag{3.48}$$

On the other hand, for every $i \in I$, one has

$$\begin{aligned}
\phi(u, x_n) - \phi(u, z_{n,i}) &= \|x_n\|^2 - \|z_{n,i}\|^2 - 2\langle u, Jx_n - Jz_{n,i}\rangle \\
&\leq \|x_n - z_{n,i}\|(\|x_n\| + \|z_{n,i}\|) + 2\|u\|\|Jx_n - Jz_{n,i}\|.
\end{aligned} \tag{3.49}$$

It follows from (3.30) and (3.31) that

$$\phi(u, x_n) - \phi(u, z_{n,i}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \ \forall i \in I. \tag{3.50}$$

Applying $\liminf_{n\to\infty}\beta_{n,i}^{(2)}\beta_{n,i}^{(3)} > 0$ and (3.50) we get

$$g(\|JS_i w_{n,i} - JT_i x_n\|) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \ \forall i \in I. \tag{3.51}$$

It follows from the property of g that

$$\|JS_i w_{n,i} - JT_i x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \ \forall i \in I. \tag{3.52}$$

Since J^{-1} is also uniformly norm-to-norm continuity on bounded sets, for every $i \in I$, one has

$$\lim_{n\to\infty} \|T_i x_n - S_i w_{n,i}\| = 0, \quad \forall i \in I. \tag{3.53}$$

On the other hand, we can calculate

$$\begin{aligned}
\phi(T_i x_n, z_{n,i}) &= \phi\left(T_i x_n, J^{-1}\left(\beta_{n,i}^{(1)} J x_n + \beta_{n,i}^{(2)} J T_i x_n + \beta_{n,i}^{(3)} J S_i w_{n,i}\right)\right) \\
&= \|T_i x_n\|^2 - 2\left\langle T_i x_n, \beta_{n,i}^{(1)} J x_n + \beta_{n,i}^{(2)} J T_i x_n + \beta_{n,i}^{(3)} J S_i w_{n,i} \right\rangle \\
&\quad + \left\|\beta_{n,i}^{(1)} J x_n + \beta_{n,i}^{(2)} J T_i x_n + \beta_{n,i}^{(3)} J S_i w_{n,i})\right\|^2 \\
&\le \|T_i x_n\|^2 - 2\beta_{n,i}^{(1)} \langle T_i x_n, J x_n\rangle - 2\beta_{n,i}^{(2)} \langle T_i x_n, J T_i x_n\rangle - 2\beta_{n,i}^{(3)} \langle T_i x_n, J S_i w_{n,i}\rangle \\
&\quad + \beta_{n,i}^{(1)} \|x_n\|^2 + \beta_{n,i}^{(2)} \|T_i x_n\|^2 + \beta_{n,i}^{(3)} \|S_i w_{n,i}\|^2 \\
&\le \beta_{n,i}^{(1)} \phi(T_i x_n, x_n) + \beta_{n,i}^{(3)} \phi(T_i x_n, S_i w_{n,i}).
\end{aligned} \tag{3.54}$$

Observe that

$$\begin{aligned}
\phi(T_i x_n, S_i w_{n,i}) &= \|T_i x_n\|^2 - 2\langle T_i x_n, J S_i w_{n,i}\rangle + \|S_i w_{n,i}\|^2 \\
&= \|T_i x_n\|^2 - 2\langle T_i x_n, J T_i x_n\rangle + 2\langle T_i x_n, J T_i x_n - J S_i w_{n,i}\rangle + \|S_i w_{n,i}\|^2 \\
&\le \|S_i w_{n,i}\|^2 - \|T_i x_n\|^2 + 2\|T_i x_n\| \|J T_i x_n - J S_i w_{n,i}\| \\
&\le \|S_i w_{n,i} - T_i x_n\| (\|S_i w_{n,i}\| + \|T_i x_n\|) + 2\|T_i x_n\| \|J T_i x_n - J S_i w_{n,i}\|.
\end{aligned} \tag{3.55}$$

It follows from (3.52) and (3.53) that

$$\lim_{n\to\infty} \phi(T_i x_n, S_i w_{n,i}) = 0, \quad \forall i \in I. \tag{3.56}$$

Applying $\lim_{n\to\infty}\beta_{n,i}^{(1)} = 0$ and (3.56) and the fact that $\{\phi(T_i x_n, x_n)\}$ is bounded to (3.54), we obtain

$$\lim_{n\to\infty} \phi(T_i x_n, z_{n,i}) = 0, \quad \forall i \in I. \tag{3.57}$$

From Lemma 2.2, one obtains

$$\lim_{n\to\infty} \|T_i x_n - z_{n,i}\| = 0, \quad \forall i \in I. \tag{3.58}$$

We observe that

$$\|T_i x_n - x_n\| \le \|T_i x_n - z_{n,i}\| + \|z_{n,i} - x_n\|. \tag{3.59}$$

It follows from (3.30) and (3.58) that

$$\lim_{n\to\infty} \|T_i x_n - x_n\| = 0, \quad \forall i \in I. \tag{3.60}$$

By the same proof as in Case I, we obtain that

$$\lim_{n\to\infty} \|x_n - w_{n,i}\| = 0, \quad \forall i \in I. \tag{3.61}$$

Hence $w_{n,i} \to p$ as $n \to \infty$ for each $i \in I$ and

$$\lim_{n\to\infty} \|Jx_n - Jw_{n,i}\| = 0, \quad \forall i \in I. \tag{3.62}$$

Combining (3.53), (3.60), and (3.61), we also have

$$\lim_{n\to\infty} \|S_i w_{n,i} - w_{n,i}\| = 0, \quad \forall i \in I. \tag{3.63}$$

It follows from the closedness of T_i and S_i that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$.

(II) Now, we show that $p \in VI(A, C)$.
Let $T \subset E \times E^*$ be an operator defined by

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.64}$$

By Lemma 2.8, we have that T is maximal monotone and $T^{-1}0 = VI(A, C)$. Let $(v, w) \in G(T)$. Since $w \in Tv = Av + N_C(v)$, we obtain that $w - Av \in N_C(v)$. From $x_n = \Pi_{C_n} x_0 \subset C_n \subset C$, we have

$$\langle v - x_n, w - Av \rangle \ge 0. \tag{3.65}$$

Since A is α-inverse strongly monotone, we can calculate

$$\begin{aligned} \langle v - x_n, w \rangle &\ge \langle v - x_n, Av \rangle \\ &= \langle v - x_n, Av - Ax_n \rangle + \langle v - x_n, Ax_n \rangle \\ &\ge \langle v - x_n, Ax_n \rangle. \end{aligned} \tag{3.66}$$

From $w_{n,i} = \Pi_C J^{-1}(Jx_n - \lambda_{n,i}Ax_n)$ and by Lemma 2.3, we have

$$\langle v - w_{n,i}, Jw_{n,i} - Jx_n - \lambda_{n,i}Ax_n \rangle \geq 0. \tag{3.67}$$

This implies that

$$\left\langle v - w_{n,i}, \frac{Jx_n - Jw_{n,i}}{\lambda_{n,i}} - Ax_n \right\rangle \leq 0. \tag{3.68}$$

Since A is α-inverse strongly monotone, we have also that A is $1/\alpha$-Lipschitzian. Hence

$$\begin{aligned}
\langle v - x_n, w \rangle &\geq \langle v - x_n, Ax_n \rangle + \left\langle v - w_{n,i}, \frac{Jx_n - Jw_{n,i}}{\lambda_{n,i}} - Ax_n \right\rangle \\
&= \langle v - w_{n,i}, Ax_n \rangle + \langle w_{n,i} - x_n, Ax_n \rangle \\
&\quad - \langle v - w_{n,i}, Ax_n \rangle + \left\langle v - w_{n,i}, \frac{Jx_n - Jw_{n,i}}{\lambda_{n,i}} \right\rangle \\
&= \langle w_{n,i} - x_n, Ax_n \rangle + \left\langle v - w_{n,i}, \frac{Jx_n - Jw_{n,i}}{\lambda_{n,i}} \right\rangle \\
&\geq -\|w_{n,i} - x_n\| \|Ax_n\| - \|v - w_{n,i}\| \left\| \frac{Jx_n - Jw_{n,i}}{a} \right\|
\end{aligned} \tag{3.69}$$

for all $n \geq 0$. By Taking the limit as $n \to \infty$ and by (3.61) and (3.62), we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of T we obtain $p \in T^{-1}0$ and hence $p \in VI(A, C)$. Hence $p \in F$.

Step 6. Finally, we show that $p = \Pi_F x_0$.

From $x_n = \Pi_{C_n} x_0$, we have

$$\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0, \quad \forall z \in C_n. \tag{3.70}$$

Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - u \rangle \geq 0, \quad \forall u \in F. \tag{3.71}$$

By taking limit in (3.71), we obtain that

$$\langle Jx_0 - Jp, p - u \rangle \geq 0, \quad \forall u \in F. \tag{3.72}$$

By Lemma 2.3, we can conclude that $p = \Pi_F x_0$. This completes the proof.

Remark 3.2. Theorem 3.1 improves and extends main results of Iiduka and Takahashi [17], Martinez-Yanes and Xu [23], Matsushita and Takahashi [13], Plubtieng and Ungchittrakool [14], Qin and Su [15], and Qin et al. [25] because it can be applied to solving the problem of finding the common element of the set of common fixed points of two families of quasi-ϕ-nonexpansive mappings and the set of solutions of the variational inequality for an inverse-strongly monotone operator.

4. Applications

From Theorem 3.1 we can obtain some new and interesting strong convergence theorems. Now we give some examples as follows.

If $\beta_{n,i}^{(1)} = 0$ for all $n \geq 0$, $T_i = S_i$ for all $i \in I$ and $A = 0$ in Theorem 3.1, then we have the following result.

Corollary 4.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i\in I}$ be a family of closed quasi-ϕ-nonexpansive mappings of C into itself with $F := \bigcap_{i\in I} F(T_i)$ being nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered}
x_0 \in C \text{ chosen arbitrary}, \\
C_{1,i} = C, \qquad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \qquad x_1 = \Pi_{C_1}(x_0) \quad \forall i \in I, \\
y_{n,i} = J^{-1}(\alpha_{n,i} J x_0 + (1-\alpha_{n,i}) J T_i x_n), \\
C_{n+1,i} = \left\{ u \in C_{n,i} : \phi(u, y_{n,i}) \leqslant \phi(u, x_n) + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0 \rangle \right) \right\}, \\
C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{4.1}$$

where J is the duality mapping on E, and $\{\alpha_{n,i}\}$ is a sequence in $(0,1)$ such that $\limsup_{n\to\infty} \alpha_{n,i} = 0$, for all $i \in I$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Now we consider the problem of finding a zero point of an inverse-strongly monotone operator of E into E^*. Assume that A satisfies the following conditions:

(C1) A is α-inverse-strongly monotone,

(C2) $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$.

Corollary 4.2. *Let E be a 2 uniformly convex and uniformly smooth Banach space. Let A be an operator of E into E^* satisfying (C1) and (C2), and let $\{S_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be two families of closed* *quasi-ϕ-nonexpansive mappings of E into itself with $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap A^{-1}0$ being nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$
\begin{gathered}
x_0 \in E \text{ chosen arbitrary}, \\
C_{1,i} = E, \qquad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \qquad x_1 = \Pi_{C_1}(x_0) \quad \forall i \in I, \\
w_{n,i} = J^{-1}(Jx_n - \lambda_{n,i} A x_n), \\
z_{n,i} = J^{-1}\left(\beta_{n,i}^{(1)} J x_n + \beta_{n,i}^{(2)} J T_i x_n + \beta_{n,i}^{(3)} J S_i w_{n,i}\right), \\
y_{n,i} = J^{-1}(\alpha_{n,i} J x_0 + (1-\alpha_{n,i}) J z_{n,i}), \\
C_{n+1,i} = \left\{u \in C_{n,i} : \phi(u, y_{n,i}) \leqslant \phi(u, x_n) + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle\right)\right\}, \\
C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{gathered}
\tag{4.2}
$$

where J is the duality mapping on E, and $\{\lambda_{n,i}\}$, $\{\alpha_{n,i}\}$, and $\{\beta_{n,i}^{(j)}\}$ $(j = 1, 2, 3)$ are sequences in $(0, 1)$ such that

(i) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for all $i \in I$;*

(ii) *for all $i \in I$, $\{\lambda_{n,i}\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2 uniformly convexity constant of E;*

(iii) $\beta_{n,i}^{(1)} + \beta_{n,i}^{(2)} + \beta_{n,i}^{(3)} = 1$ *for all $i \in I$ and if one of the following conditions is satisfied:*

(a) $\liminf_{n\to\infty} \beta_{n,i}^{(1)} \beta_{n,i}^{(l)} > 0$ *for all $l = 2, 3$ and for all $i \in I$,*

(b) $\liminf_{n\to\infty} \beta_{n,i}^{(2)} \beta_{n,i}^{(3)} > 0$ *and* $\lim_{n\to\infty} \beta_{n,i}^{(1)} = 0$ *for all $i \in I$.*

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Proof. Setting $C = E$ in Theorem 3.1, we get that Π_E is the identity mapping, that is, $\Pi_E x = x$ for all $x \in E$. We also have $VI(A, E) = A^{-1}0$. From Theorem 3.1, we can obtain the desired conclusion easily. □

Let X be a nonempty closed convex cone in E, and let A be an operator from X into E^*. We define its *polar* in E^* to be the set

$$
X^* = \left\{y^* \in E^* : \langle x, y^*\rangle \geq 0 \ \forall x \in X\right\}. \tag{4.3}
$$

Then an element x in X is called a *solution of the complementarity problem* if

$$
Ax \in X^*, \quad \langle x, Ax\rangle = 0. \tag{4.4}
$$

The set of all solutions of the complementarity problem is denoted by $CP(A,X)$. Several problems arising in different fields, such as mathematical programming, game theory, mechanics, and geometry, are to find solutions of the complementarity problems.

Corollary 4.3. *Let E be a 2 uniformly convex and uniformly smooth Banach space, and let X be a nonempty closed convex subset of E. Let A be an operator of X into E^* satisfying (C1) and (C2), and let $\{S_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be two families of closed quasi-ϕ-nonexpansive mappings of X into itself with $F := \bigcap_{i\in I}F(T_i)\cap\bigcap_{i\in I}F(S_i)\cap CP(A,X)$ being nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered}
x_0 \in X \text{ chosen arbitrary}, \\
C_{1,i} = X, \qquad C_1 = \bigcap_{i=1}^{\infty} C_{1,i}, \qquad x_1 = \Pi_{C_1}(x_0) \quad \forall i \in I, \\
w_{n,i} = \Pi_X J^{-1}(Jx_n - \lambda_{n,i}Ax_n), \\
z_{n,i} = J^{-1}\left(\beta_{n,i}^{(1)}Jx_n + \beta_{n,i}^{(2)}JT_ix_n + \beta_{n,i}^{(3)}JS_iw_{n,i}\right), \\
y_{n,i} = J^{-1}(\alpha_{n,i}Jx_0 + (1-\alpha_{n,i})Jz_{n,i}), \\
C_{n+1,i} = \left\{u \in C_{n,i} : \phi(u, y_{n,i}) \leqslant \phi(u, x_n) + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle\right)\right\}, \\
C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\
x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0,
\end{gathered} \tag{4.5}$$

where J is the duality mapping on E, and $\{\lambda_{n,i}\}, \{\alpha_{n,i}\}$, and $\{\beta_{n,i}^{(j)}\}$ $(j = 1,2,3)$ are sequences in $(0,1)$ such that

(i) $\lim_{n\to\infty}\alpha_{n,i} = 0$ *for all $i \in I$;*

(ii) *for all $i \in I$, $\{\lambda_{n,i}\} \subset [a,b]$ for some a,b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2 uniformly convexity constant of E;*

(iii) $\beta_{n,i}^{(1)} + \beta_{n,i}^{(2)} + \beta_{n,i}^{(3)} = 1$ *for all $i \in I$ and if one of the following conditions is satisfied:*

(a) $\liminf_{n\to\infty}\beta_{n,i}^{(1)}\beta_{n,i}^{(l)} > 0$ *for all $l = 2,3$ and for all $i \in I$,*

(b) $\liminf_{n\to\infty}\beta_{n,i}^{(2)}\beta_{n,i}^{(3)} > 0$ *and* $\lim_{n\to\infty}\beta_{n,i}^{(1)} = 0$ *for all $i \in I$.*

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Proof. From [29, Lemma 7.1.1], we have $VI(A,X) = CP(A,X)$. From Theorem 3.1, we can obtain the desired conclusion easily. □

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ภาคผนวก 5

A hybrid iterative scheme for variational inequality problems for finite families of relatively weak quasi-nonexpansive mappings

U. Kamraksa, R. Wangkeeree

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Research Article

A Hybrid Iterative Scheme for Variational Inequality Problems for Finite Families of Relatively Weak Quasi-Nonexpansive Mappings

Uthai Kamraksa and Rabian Wangkeeree

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Rabian Wangkeeree, rabianw@nu.ac.th

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We consider a hybrid projection algorithm basing on the shrinking projection method for two families of relatively weak quasi-nonexpansive mappings. We establish strong convergence theorems for approximating the common fixed point of the set of the common fixed points of such two families and the set of solutions of the variational inequality for an inverse-strongly monotone operator in the framework of Banach spaces. At the end of the paper, we apply our results to consider the problem of finding a solution of the complementarity problem. Our results improve and extend the corresponding results announced by recent results.

1. Introduction

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be an operator. The classical variational inequality problem [1] for A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. The set of all solutions of (1.1) is denoted by $VI(A, C)$. Such a problem is connected with the convex minimization problem, the complementarity, the problem of finding a point $x^* \in E$ satisfying $0 = Ax^*$, and so on. First, we recall that a mapping $A : C \to E^*$ is said to be

(i) *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$, for all $x, y \in C$.

(ii) *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.2}$$

Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \quad \forall x \in E. \tag{1.3}$$

It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Some properties of the duality mapping are given in [2–4].

Recall that a mappings $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

If C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is a nonexpansive mapping. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [5] recently introduced a generalized projection operator C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \tag{1.5}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Observe that, in a Hilbert space H, (1.5) reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y, x). \tag{1.6}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, e.g., [3, 5–7]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$,

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$ for all $x, y, z \in E$,

(3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$,

(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \quad \text{iff } x = y. \tag{1.7}$$

For more detail see [2, 3]. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed point of T. A point p in C is said to be an *asymptotic fixed point* of T [8] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called relatively nonexpansive [7, 9, 10] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings were studied in [7, 9]. A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of S will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called relatively weak nonexpansive if $\widetilde{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. If E is a smooth strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}0 \neq \emptyset$, then it is proved in [11] that $J_r = (J + rA)^{-1}J$, for $r > 0$ is relatively weak nonexpansive. T is called relatively weak quasi-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 1.1. The class of relatively weak quasi-nonexpansive mappings is more general than the class of relatively weak nonexpansive mappings [7, 9, 12–14] which requires the strong restriction $\widehat{F}(T) = F(T)$.

Remark 1.2. If $T : C \to C$ is relatively weak quasi-nonexpansive, then using the definition of ϕ (i.e., the same argument as in the proof of [12, page 260]) one can show that $F(T)$ is closed and convex. It is obvious that relatively nonexpansive mapping is relatively weak nonexpansive mapping. In fact, for any mapping $T : C \to C$ we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is a relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

Iiduka and Takahashi [15] introduced the following algorithm for finding a solution of the variational inequality for an α-inverse-strongly monotone mapping A with $\|Ay\| \le \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$ in a 2-uniformly convex and uniformly smooth Banach space E. For an initial point $x_0 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad \forall n \ge 0. \tag{1.8}$$

where J is the duality mapping on E, and Π_C is the generalized projection of E onto C. Assume that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$ where $1/c$ is the 2-uniformly convexity constant of E. They proved that if J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to some element z in $VI(A, C)$ where $z = \lim_{n\to\infty} \Pi_{VI(A,C)}(x_n)$.

The problem of finding a common element of the set of the variational inequalities for monotone mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors; see, for instance, [16–18] and the references cited therein.

On the other hand, in 2001, Xu and Ori [19] introduced the following implicit iterative process for a finite family of nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$, with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in C$:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1)T_1 x_1,\\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2,\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_1 x_{N+1},\\
&\vdots
\end{aligned} \tag{1.9}$$

which can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad \forall n \geq 1, \tag{1.10}$$

where $T_n = T_{n(\bmod N)}$ (here the mod N function takes values in $\{1, 2, \ldots, N\}$). They obtained the following result in a real Hilbert space.

Theorem XO

Let H be a real Hilbert space, C a nonempty closed convex subset of H, and let $T : C \to C$ be a finite family of nonexpansive self-mappings on C such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (1.10). If $\{\alpha_n\}$ is chosen so that $\alpha_n \to 0$, as $n \to \infty$, then $\{x_n\}$ converges weakly to a common fixed point of the family of $\{T_i\}_{i=1}^{N}$.

On the other hand, Halpern [20] considered the following explicit iteration:

$$x_0 \in C, \qquad x_{n+1} = \alpha_n u + (1-\alpha_n)T x_n, \quad \forall n \geq 0, \tag{1.11}$$

where T is a nonexpansive mapping and $u \in C$ is a fixed point. He proved the strong convergence of $\{x_n\}$ to a fixed point of T provided that $\alpha_n = n^{-\theta}$, where $\theta \in (0,1)$.

Very recently, Qin et al. [21] proposed the following modification of the Halpern iteration for a single relatively quasi-nonexpansive mapping in a real Banach space. More precisely, they proved the following theorem.

Theorem QCKZ. *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed and quasi-ϕ-nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{aligned}
&x_0 = x \in C \quad \text{chosen arbitrary},\\
&C_1 = C, \quad x_1 = \Pi_{C_1} x_0,\\
&y_n = J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J T x_n),\\
&C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \quad n \geq 1,\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \geq 1.
\end{aligned} \tag{1.12}$$

Assume that $\{\alpha_n\}$ satisfies the restriction: $\lim_{n\to\infty} \alpha_n = 0$, *then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.*

Motivated and inspired by the above results, Cai and Hu [22] introduced the hybrid projection algorithm to modify the iterative processes (1.10), (1.11), and (1.12) to have strong convergence for a finite family of relatively weak quasi-nonexpansive mappings in Banach spaces. More precisely, they obtained the following theorem.

Theorem CH

Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E and $\{T_1, T_2, \ldots, T_N\}$ be finite family of closed relatively weak quasi-nonexpansive mappings of C into itself with $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Assume that T_i is uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned}
&x_0 = x \in C \quad \text{chosen arbitrary,} \\
&C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\
&z_n = J^{-1}\left(\beta_n J x_{n-1} + (1-\beta_n) J T_n x_n\right), \quad T_n = T_{n(\operatorname{mod} N)}, \\
&y_n = J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J z_n), \\
&C_{n+1} = \left\{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\left[\beta_n \phi(z, x_{n-1}) + (1-\beta_n)\phi(z, x_n)\right]\right\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \geq 1.
\end{aligned} \tag{1.13}$$

Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $[0, 1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from C onto F.

Motivated and inspired by Iiduka and Takahashi [15], Xu and Ori [19], Qin et al.[21], and Cai and Hu [22], we introduce a new hybrid projection algorithm basing on the shrinking projection method for two finite families of closed relatively weak quasi-nonexpansive mappings to have strong convergence theorems for approximating the common element of the set of common fixed points of two finite families of such mappings and the set of solutions of the variational inequality for an inverse-strongly monotone operator in the framework of Banach spaces. Our results improve and extend the corresponding results announced by recent results.

2. Preliminaries

A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [3, 23–25]. A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [3, 23, 25] for more details.

We define the function $\delta : [0,2] \to [0,1]$ which is called the modulus of convexity of E as following

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in C, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}. \quad (2.2)$$

Then E is said to be 2-uniformly convex if there exists a constant $c > 0$ such that constant $\delta(\varepsilon) > c\varepsilon^2$ for all $\varepsilon \in (0,2]$. Constant $1/c$ is called the 2-uniformly convexity constant of E. A 2-uniformly convex Banach space is uniformly convex, see [26, 27] for more details. We know the following lemma of 2-uniformly convex Banach spaces.

Lemma 2.1 (see [28, 29]). *Let E be a 2-uniformly convex Banach, then for all x, y from any bounded set of E and $jx \in Jx, jy \in Jy$,*

$$\langle x-y, jx-jy\rangle \geq \frac{c^2}{2}\|x-y\|^2 \quad (2.3)$$

where $1/c$ is the 2-uniformly convexity constant of E.

Now we present some definitions and lemmas which will be applied in the proof of the main result in the next section.

Lemma 2.2 (Kamimura and Takahashi [30]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty}\phi(y_n, z_n) = 0$, then $\lim_{n\to\infty}\|y_n - z_n\| = 0$.*

Lemma 2.3 (Alber [5]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for any $y \in C$.*

Lemma 2.4 (Alber [5]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x) \quad (2.4)$$

for all $y \in C$.

Let E be a reflexive strictly convex, smooth and uniformly Banach space and the duality mapping J from E to E^*. Then J^{-1} is also single-valued, one to one, surjective, and it is the duality mapping from E^* to E. We need the following mapping V which studied in Alber [5]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x\|^2 \quad (2.5)$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma.

Lemma 2.5 (Kamimura and Takahashi [30]). *Let E be a reflexive, strictly convex and smooth Banach space, and let V be as in* (2.5). *Then*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \leq V(x, x^* + y^*) \tag{2.6}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 ([31, Lemma 1.4]). *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leqslant \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|), \tag{2.7}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping F of $[0, 1)$ into E^* defined by $F(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C\}. \tag{2.8}$$

Lemma 2.7 (see [32]). *Let C be a nonempty closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{2.9}$$

Then T is maximal monotone and $T^{-1}0 = VI(A, C)$.

3. Main Results

In this section, we prove strong convergence theorem which is our main result.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, let A be an α-inverse-strongly monotone mapping of C into E^* with $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in F$. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from C into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap VI(A, C)$. Assume that T_i and S_i are uniformly continuous for all*

$i \in \{1,2,\ldots,N\}$. Let $\{x_n\}$ be a sequence generated by the following algolithm:

$$\begin{aligned}
&x_0 = x \in C, \quad \text{chosen arbitrary},\\
&C_1 = C, \quad x_1 = \Pi_{C_1} x_0,\\
&w_n = \Pi_C J^{-1}(Jx_n - r_n A x_n),\\
&z_n = J^{-1}\left(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n\right),\\
&y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n),\\
&C_{n+1} = \left\{u \in C_n : \phi(u,y_n) \le \delta_n \phi(u,x_1) + (1-\delta_n)\left[\alpha_n \phi(u,x_{n-1}) + (1-\alpha_n)\phi(u,x_n)\right]\right\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1,
\end{aligned} \tag{3.1}$$

where $T_n = T_{n(\bmod N)}, S_n = S_{n(\bmod N)}$, and J is the normalized duality mapping on E. Assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ are the sequences in $[0,1]$ satisfying the restrictions:

(C1) $\lim_{n\to\infty} \delta_n = 0$;

(C2) $r_n \subset [a,b]$ *for some a, b with $0 < a < b < c^2\alpha/2$, where $1/c$ is the 2-uniformly convexity constant of E;*

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

(a) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ *and*

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from C onto F.

Proof. By the same method as in the proof of Cai and Hu [22], we can show that C_n is closed and convex. Next, we show $F \subset C_n$ for all $n \ge 1$. In fact, $F \subset C_1 = C$ is obvious. Suppose $F \subset C_n$ for some $n \in \mathbb{N}$. Then, for all $q \in F \subset C_n$, we know from Lemma 2.5 that

$$\begin{aligned}
\phi(q,w_n) &= \phi\left(q, \Pi_C J^{-1}(Jx_n - r_n A x_n)\right)\\
&\le \phi\left(q, J^{-1}(Jx_n - r_n A x_n)\right)\\
&= V(q, Jx_n - r_n A x_n)\\
&\le V(q, (Jx_n - r_n A x_n) + r_n A x_n) - 2\left\langle J^{-1}(Jx_n - r_n A x_n) - q, r_n A x_n\right\rangle\\
&= V(q, Jx_n) - 2r_n\left\langle J^{-1}(Jx_n - r_n A x_n) - q, A x_n\right\rangle\\
&= \phi(q,x_n) - 2r_n\langle x_n - q, A x_n\rangle + 2\left\langle J^{-1}(Jx_n - r_n A x_n) - x_n, -r_n A x_n\right\rangle.
\end{aligned} \tag{3.2}$$

Since $q \in VI(A,C)$ and A is α-inverse-strongly monotone, we have

$$-2r_n\langle x_n - q, Ax_n\rangle = -2r_n\langle x_n - q, Ax_n - Aq\rangle - 2r_n\langle x_n - q, Aq\rangle \le -2\alpha r_n \|Ax_n - Aq\|^2. \tag{3.3}$$

Therefore, from Lemma 2.1 and the assumption that $\|Ay\| \leq \|Ay - Aq\|$ for all $y \in C$ and $q \in F$, we obtain that

$$\begin{aligned}
2\left\langle J^{-1}(Jx_n - r_nAx_n) - x_n, -r_nAx_n \right\rangle &= 2\left\langle J^{-1}(Jx_n - r_nAx_n) - J^{-1}(Jx_n), -r_nAx_n \right\rangle \\
&\leq 2\left\| J^{-1}(Jx_n - r_nAx_n) - J^{-1}(Jx_n)\right\| \|r_nAx_n\| \\
&\leq \frac{4}{c^2}\left\| JJ^{-1}(Jx_n - r_nAx_n) - JJ^{-1}(Jx_n)\right\| \|r_nAx_n\| \\
&= \frac{4}{c^2}\|(Jx_n - r_nAx_n) - Jx_n\| \|r_nAx_n\| \\
&= \frac{4}{c^2}r_n^2\|Ax_n\|^2 \leq \frac{4}{c^2}r_n^2\|Ax_n - Aq\|^2.
\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) and using the condition that $r_n < c^2\alpha/2$, we get

$$\phi(q, w_n) \leq \phi(q, x_n) + 2r_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Aq\|^2 \leq \phi(q, x_n). \tag{3.5}$$

Using (3.5) and the convexity of $\|\cdot\|^2$, for each $q \in F \subset C_n$, we obtain

$$\begin{aligned}
\phi(q, z_n) &= \phi\left(q, J^{-1}(\alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n)\right) \\
&= \|q\|^2 - 2\alpha_n\langle q, Jx_{n-1}\rangle - 2\beta_n\langle q, JT_nx_n\rangle - 2\gamma_n\langle q, JS_nw_n\rangle \\
&\quad + \|\alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_n\langle q, Jx_{n-1}\rangle - 2\beta_n\langle q, JT_nx_n\rangle - 2\gamma_n\langle q, JS_nw_n\rangle \\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT_nx_n\|^2 + \gamma_n\|JS_nw_n\|^2 \\
&= \alpha_n\phi(q, x_{n-1}) + \beta_n\phi(q, T_nx_n) + \gamma_n\phi(q, S_nw_n) \\
&\leq \alpha_n\phi(q, x_{n-1}) + \beta_n\phi(q, x_n) + \gamma_n\phi(q, w_n) \\
&\leq \alpha_n\phi(q, x_{n-1}) + \beta_n\phi(q, x_n) + \gamma_n\phi(q, x_n) \\
&= \alpha_n\phi(q, x_{n-1}) + (1 - \alpha_n)\phi(q, x_n).
\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned}
\phi(q, y_n) &= \phi\Big(q, J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n)\Big) \\
&= \|q\|^2 - 2\delta_n \langle q, J x_1\rangle - 2(1-\delta_n)\langle q, J z_n\rangle + \|\delta_n J x_1 + (1-\delta_n) J z_n\|^2 \\
&\le \|q\|^2 - 2\delta_n \langle q, J x_1\rangle - 2(1-\delta_n)\langle q, J z_n\rangle + \delta_n \|x_1\|^2 + (1-\delta_n)\|z_n\|^2 \\
&= \delta_n \phi(q, x_1) + (1-\delta_n)\phi(q, z_n) \\
&\le \delta_n \phi(q, x_1) + (1-\delta_n)\big[\alpha_n \phi(q, x_{n-1}) + (1-\alpha_n)\phi(q, x_n)\big].
\end{aligned} \tag{3.7}$$

So, $q \in C_{n+1}$. Then by induction, $F \subset C_n$ for all $n \ge 1$ and hence the sequence $\{x_n\}$ generated by (3.1) is well defined. Next, we show that $\{x_n\}$ is a convergent sequence in C. From $x_n = \Pi_{C_n} x_1$, we have

$$\langle x_n - u, J x_1 - J x_n\rangle \ge 0, \quad \forall u \in C_n. \tag{3.8}$$

It follows from $F \subset C_n$ for all $n \ge 1$ that

$$\langle x_n - z, J x_1 - J x_n\rangle \ge 0, \quad \forall z \in F. \tag{3.9}$$

From Lemma 2.4, we have

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \le \phi(u, x_1) - \phi(u, x_n) \le \phi(u, x_1), \tag{3.10}$$

for each $u \in F \subset C_n$ and for all $n \ge 1$. Therefore, the sequence $\{\phi(x_n, x_1)\}$ is bounded. Furthermore, since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \le \phi(x_{n+1}, x_1), \quad \forall n \ge 1. \tag{3.11}$$

This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing and hence $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. Similarly, by Lemma 2.4, we have, for any positive integer m, that

$$\begin{aligned}
\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_1) \le \phi(x_{n+m}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\
&= \phi(x_{n+m}, x_1) - \phi(x_n, x_1), \quad \forall n \ge 1.
\end{aligned} \tag{3.12}$$

The existence of $\lim_{n\to\infty} \phi(x_n, x_1)$ implies that $\phi(x_{n+m}, x_n) \to 0$ as $n \to \infty$. From Lemma 2.2, we have

$$\|x_{n+m} - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.13}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now, we will show that $p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \cap VI(A, C)$.

(I) We first show that $p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i)$. Indeed, taking $m = 1$ in (3.12), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.14}$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

This implies that

$$\lim_{n\to\infty} \|x_{n+l} - x_n\| = 0, \quad \forall l \in \{1, 2, \ldots, N\}. \tag{3.16}$$

The property of the function ϕ implies that

$$\lim_{n\to\infty} \phi(x_{n+l}, x_n) = 0, \quad \forall l \in \{1, 2, \ldots, N\}. \tag{3.17}$$

Since $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, y_n) \le \delta_n \phi(x_{n+1}, x_n) + (1 - \delta_n)\left[\alpha_n \phi(x_{n+1}, x_{n-1}) + (1 - \alpha_n)\phi(x_{n+1}, x_n)\right]. \tag{3.18}$$

It follows from the condition (3.14) and (3.17) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.19}$$

From Lemma 2.2, we have

$$\lim_{n\to\infty} \left\|x_{n+1} - y_n\right\| = 0. \tag{3.20}$$

Combining (3.15) and (3.20), we have

$$\left\|x_n - y_n\right\| \le \|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \left\|Jx_n - Jy_n\right\| = 0. \tag{3.22}$$

On the other hand, noticing

$$\left\|Jy_n - Jz_n\right\| = \delta_n \|Jx_1 - Jz_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.23}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.24}$$

Using (3.15), (3.20), and (3.24) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.25}$$

Taking the constant $r = \sup_{n\geq 1}\{\|x_{n+1}\|, \|T_n x_n\|, \|S_n w_n\|\}$, we have, from Lemma 2.6, that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ satisfying the inequality (2.7) and $g(0) = 0$.

Case 1. Assume that (a) holds. Applying (2.7) and (3.5), we can calculate

$$\begin{aligned}
\phi(u, z_n) &= \phi\Big(u, J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n)\Big) \\
&= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&\quad + \left\|\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n\right\|^2 \\
&\leq \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle \\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT_n x_n\|^2 + \gamma_n\|JS_n w_n\|^2 - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, T_n x_n) + \gamma_n\phi(u, S_n w_n) - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, w_n) - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + \beta_n\phi(u, x_n) + \gamma_n\phi(u, x_n) \\
&\quad + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \\
&\leq \alpha_n\phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 \\
&\quad - \alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|).
\end{aligned} \tag{3.26}$$

This implies that

$$\alpha_n\beta_n g(\|Jx_{n-1} - JT_n x_n\|) \leq \alpha_n\left[\phi(u, x_{n-1}) - \phi(u, x_n)\right] + \phi(u, x_n) - \phi(u, z_n). \tag{3.27}$$

We observe that

$$\begin{aligned}
&\alpha_n\left[\phi(u,x_{n-1})-\phi(u,x_n)\right]+\phi(u,x_n)-\phi(u,z_n)\\
&\quad\le \alpha_n\left[\|x_{n-1}\|^2-\|x_n\|^2-2\langle u,Jx_{n-1}-Jx_n\rangle\right]\\
&\qquad+\|x_n\|^2-\|z_n\|^2-2\langle u,Jx_n-Jz_n\rangle\\
&\quad\le \alpha_n\left[\|x_{n-1}-x_n\|(\|x_{n-1}\|+\|x_n\|)+2\|u\|\|Jx_{n-1}-Jx_n\|\right]\\
&\qquad+\|x_n-z_n\|(\|x_n\|+\|z_n\|)+2\|u\|\|Jx_n-Jz_n\|.
\end{aligned} \tag{3.28}$$

It follows from (3.15), (3.22), (3.23) and (3.25) that

$$\lim_{n\to\infty}\alpha_n\left[\phi(u,x_{n-1})-\phi(u,x_n)\right]+\phi(u,x_n)-\phi(u,z_n)=0. \tag{3.29}$$

From $\liminf_{n\to\infty}\alpha_n\beta_n>0$ and (3.27), we get

$$\lim_{n\to\infty}g(\|Jx_{n-1}-JT_nx_n\|)=0. \tag{3.30}$$

By the property of function g, we obtain that

$$\lim_{n\to\infty}\|Jx_{n-1}-JT_nx_n\|=0. \tag{3.31}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|x_{n-1}-T_nx_n\|=\lim_{n\to\infty}\left\|J^{-1}(Jx_{n-1})-J^{-1}(JT_nx_n)\right\|=0. \tag{3.32}$$

From (3.15) and (3.32), we have

$$\lim_{n\to\infty}\|x_n-T_nx_n\|=0. \tag{3.33}$$

Noticing that

$$\|x_n-T_{n+l}x_n\|\le\|x_n-x_{n+l}\|+\|x_{n+l}-T_{n+l}x_{n+l}\|+\|T_{n+l}x_{n+l}-T_{n+l}x_n\|, \tag{3.34}$$

for all $l\in\{1,2,\dots,N\}$. By the uniformly continuity of T_l, (3.16) and (3.33), we obtain

$$\lim_{n\to\infty}\|x_n-T_{n+l}x_n\|=0,\quad \forall l\in\{1,2,\dots,N\}. \tag{3.35}$$

Thus

$$\lim_{n\to\infty}\|x_n-T_lx_n\|=0,\quad \forall l\in\{1,2,\dots,N\}. \tag{3.36}$$

From the closeness of T_i, we get $p = T_i p$. Therefore $p \in \bigcap_{i=1}^N F(T_i)$. In the same manner, we can apply the condition $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ to conclude that

$$\lim_{n\to\infty} \|x_n - S_n w_n\| = 0. \tag{3.37}$$

Again, by (C2) and (3.26), we have

$$\begin{aligned} 2\gamma_n\left(\alpha - \frac{2}{c^2}b\right)\|Ax_n - Au\|^2 &\le \frac{1}{a}\left[\alpha_n\phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n) - \phi(u, z_n)\right] \\ &= \frac{1}{a}\left[\alpha_n\left(\phi(u, x_{n-1}) - \phi(u, x_n)\right) + \phi(u, x_n) - \phi(u, z_n)\right]. \end{aligned} \tag{3.38}$$

It follows from (3.29) and $\liminf_{n\to\infty} \gamma_n \ge \liminf_{n\to\infty} \beta_n\gamma_n > 0$ that

$$\liminf_{n\to\infty} \|Ax_n - Au\| \le 0. \tag{3.39}$$

Since $\liminf_{n\to\infty}\|Ax_n - Au\| \ge 0$, we have

$$\lim_{n\to\infty} \|Ax_n - Au\| = 0. \tag{3.40}$$

From Lemmas 2.4, 2.5, and (3.4), we have

$$\begin{aligned} \phi(x_n, w_n) &= \phi\left(x_n, \Pi_C J^{-1}(Jx_n - r_n Ax_n)\right) \le \phi\left(x_n, J^{-1}(Jx_n - r_n Ax_n)\right) = V(x_n, Jx_n - r_n Ax_n) \\ &\le V(x_n, (Jx_n - r_n Ax_n) + r_n Ax_n) - 2\left\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, r_n Ax_n\right\rangle \\ &= \phi(x_n, x_n) + 2\left\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, -r_n Ax_n\right\rangle \\ &= 2\left\langle J^{-1}(Jx_n - r_n Ax_n) - x_n, -r_n Ax_n\right\rangle \le \frac{4}{c^2}b^2\|Ax_n - Au\|^2. \end{aligned} \tag{3.41}$$

It follows from (3.40) that

$$\lim_{n\to\infty} \phi(x_n, w_n) = 0. \tag{3.42}$$

Lemma 2.2 implies that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.43}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jw_n\| = 0. \tag{3.44}$$

Combining (3.37) and (3.43), we also obtain

$$\lim_{n\to\infty}\|w_n - S_n w_n\| = 0. \tag{3.45}$$

Moreover

$$\|w_n - w_{n+1}\| \le \|w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - w_{n+1}\|. \tag{3.46}$$

By (3.43), (3.15), we have

$$\lim_{n\to\infty}\|w_n - w_{n+1}\| = 0. \tag{3.47}$$

This implies that

$$\lim_{n\to\infty}\|w_n - w_{n+l}\| = 0, \quad \forall l \in \{1,2,\dots,N\}. \tag{3.48}$$

Noticing that

$$\|w_n - S_{n+l}w_n\| \le \|w_n - w_{n+l}\| + \|w_{n+l} - S_{n+l}w_{n+l}\| + \|S_{n+l}w_{n+l} - S_{n+l}w_n\|, \tag{3.49}$$

for all $l \in \{1,2,\dots,N\}$. Since S_l is uniformly continuous, we can show that $\lim_{n\to\infty}\|w_n - S_l w_n\| = 0$. From the closeness of S_l, we get $p = S_l p$. Therefore $p \in \bigcap_{i=1}^N F(S_i)$. Hence $p \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$.

Case 2. Assume that (b) holds. Using the inequalities (2.7) and (3.5), we obtain

$$\begin{aligned}
\phi(u,z_n) &= \phi\Big(u, J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n)\Big)\\
&= \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle\\
&\quad + \big\|\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n\big\|^2\\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_{n-1}\rangle - 2\beta_n\langle u, JT_n x_n\rangle - 2\gamma_n\langle u, JS_n w_n\rangle\\
&\quad + \alpha_n\|Jx_{n-1}\|^2 + \beta_n\|JT_n x_n\|^2 + \gamma_n\|JS_n w_n\|^2 - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|)\\
&\le \alpha_n\phi(u,x_{n-1}) + \beta_n\phi(u,T_n x_n) + \gamma_n\phi(u,S_n w_n) - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|)\\
&\le \alpha_n\phi(u,x_{n-1}) + \beta_n\phi(u,x_n) + \gamma_n\phi(u,w_n) - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|)\\
&\le \alpha_n\phi(u,x_{n-1}) + \beta_n\phi(u,x_n) + \gamma_n\phi(u,x_n)\\
&\quad + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2 - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|)\\
&\le \alpha_n\phi(u,x_{n-1}) + (1-\alpha_n)\phi(u,x_n) + 2r_n\gamma_n\left(\frac{2}{c^2}r_n - \alpha\right)\|Ax_n - Au\|^2\\
&\quad - \beta_n\gamma_n g(\|JT_n x_n - JS_n w_n\|).
\end{aligned} \tag{3.50}$$

This implies that

$$\begin{aligned}\beta_n\gamma_n g(\|JT_nx_n - JS_nw_n\|) &\le \alpha_n\left[\phi(u,x_{n-1}) - \phi(u,x_n)\right] + \phi(u,x_n) - \phi(u,z_n)\\ &\le \alpha_n\Big[\|x_{n-1}\|^2 - \|x_n\|^2 - 2\langle u, Jx_{n-1} - Jx_n\rangle\Big]\\ &\quad + \|x_n\|^2 - \|z_n\|^2 - 2\langle u, Jx_n - Jz_n\rangle\\ &\le \alpha_n\left[\|x_{n-1} - x_n\|(\|x_{n-1}\| + \|x_n\|) + 2\|u\|\|Jx_{n-1} - Jx_n\|\right]\\ &\quad + \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|u\|\|Jx_n - Jz_n\|.\end{aligned} \tag{3.51}$$

It follows from (3.21), (3.24) and the condition $\liminf_{n\to\infty}\beta_n\gamma_n > 0$ that

$$\lim_{n\to\infty} g(\|JT_nx_n - JS_nw_n\|) = 0. \tag{3.52}$$

By the property of function g, we obtain that

$$\lim_{n\to\infty}\|JT_nx_n - JS_nw_n\| = 0. \tag{3.53}$$

Since J^{-1} is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|T_nx_n - S_nw_n\| = \lim_{n\to\infty}\left\|J^{-1}(JT_nx_n) - J^{-1}(JS_nw_n)\right\| = 0. \tag{3.54}$$

On the other hand, we can calculate

$$\begin{aligned}\phi(T_nx_n, z_n) &= \phi\Big(T_nx_n, J^{-1}(\alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n)\Big)\\ &= \|T_nx_n\|^2 - 2\langle T_nx_n, \alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n\rangle\\ &\quad + \left\|\alpha_nJx_{n-1} + \beta_nJT_nx_n + \gamma_nJS_nw_n\right\|^2\\ &\le \|T_nx_n\|^2 - 2\alpha_n\langle T_nx_n, Jx_n\rangle - 2\beta_n\langle T_nx_n, JT_nx_n\rangle - 2\gamma_n\langle T_nx_n, JS_nw_n\rangle\\ &\quad + \alpha_n\|x_n\|^2 + \beta_n\|T_nx_n\|^2 + \gamma_n\|S_nw_n\|^2\\ &\le \alpha_n\phi(T_nx_n, x_n) + \gamma_n\phi(T_nx_n, S_nw_n).\end{aligned} \tag{3.55}$$

Observe that

$$\begin{aligned}\phi(T_nx_n, S_nw_n) &= \|T_nx_n\|^2 - 2\langle T_nx_n, JS_nw_n\rangle + \|S_nw_n\|^2\\ &= \|T_nx_n\|^2 - 2\langle T_nx_n, JT_nx_n\rangle + 2\langle T_nx_n, JT_nx_n - JS_nw_n\rangle + \|S_nw_n\|^2\\ &\le \|S_nw_n\|^2 - \|T_nx_n\|^2 + 2\|T_nx_n\|\|JT_nx_n - JS_nw_n\|\\ &\le \|S_nw_n - T_nx_n\|(\|S_nw_n\| + \|T_nx_n\|) + 2\|T_nx_n\|\|JT_nx_n - JS_nw_n\|.\end{aligned} \tag{3.56}$$

It follows from (3.53) and (3.54) that

$$\lim_{n\to\infty} \phi(T_n x_n, S_n w_n) = 0. \tag{3.57}$$

Applying $\lim_{n\to\infty} \alpha_n = 0$ and (3.57) and the fact that $\{\phi(T_n x_n, x_n)\}$ is bounded to (3.55), we obtain

$$\lim_{n\to\infty} \phi(T_n x_n, z_n) = 0. \tag{3.58}$$

From Lemma 2.2, one obtains

$$\lim_{n\to\infty} \|T_n x_n - z_n\| = 0. \tag{3.59}$$

We observe that

$$\|T_n x_n - x_n\| \leq \|T_n x_n - z_n\| + \|z_n - x_n\|. \tag{3.60}$$

This together with (3.25) and (3.59), we obtain

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{3.61}$$

Noticing that

$$\|x_n - T_{n+l} x_n\| \leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| + \|T_{n+l} x_{n+l} - T_{n+l} x_n\|, \tag{3.62}$$

for all $l \in \{1, 2, \ldots, N\}$. By the uniformly continuity of T_l, (3.16) and (3.61), we obtain

$$\lim_{n\to\infty} \|x_n - T_{n+l} x_n\| = 0, \quad \forall l \in \{1, 2, \ldots, N\}. \tag{3.63}$$

Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in \{1, 2, \ldots, N\}. \tag{3.64}$$

From the closeness of T_i, we get $p = T_i p$. Therefore $p \in \bigcap_{i=1}^{N} F(T_i)$. By the same proof as in Case 1, we obtain that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.65}$$

Hence $w_n \to p$ as $n \to \infty$ for each $i \in I$ and

$$\lim_{n\to\infty} \|J x_n - J w_n\| = 0. \tag{3.66}$$

Combining (3.54), (3.61), and (3.65), we also have

$$\lim_{n\to\infty}\|S_n w_n - w_n\| = 0. \tag{3.67}$$

Moreover

$$\|w_n - w_{n+1}\| \leq \|w_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - w_{n+1}\|. \tag{3.68}$$

By (3.43), (3.15), we have

$$\lim_{n\to\infty}\|w_n - w_{n+1}\| = 0. \tag{3.69}$$

This implies that

$$\lim_{n\to\infty}\|w_n - w_{n+l}\| = 0, \quad \forall l \in \{1,2,\dots,N\}. \tag{3.70}$$

Noticing that

$$\|w_n - S_{n+l}w_n\| \leq \|w_n - w_{n+l}\| + \|w_{n+l} - S_{n+l}w_{n+l}\| + \|S_{n+l}w_{n+l} - S_{n+l}w_n\|, \tag{3.71}$$

for all $l \in \{1,2,\dots,N\}$. Since S_l is uniformly continuous, we can show that $\lim_{n\to\infty}\|w_n - S_l w_n\| = 0$. From the closeness of S_i, we get $p = S_i p$. Therefore $p \in \bigcap_{i=1}^N F(S_i)$. Hence $p \in \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$.

(II) We next show that $p \in VI(C,A)$.
Let $T \subset E \times E^*$ be an operator defined by:

$$Tv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{3.72}$$

By Lemma 2.7, T is maximal monotone and $T^{-1}0 = VI(A,C)$. Let $(v,w) \in G(T)$, since $w \in Tv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $x_n = \Pi_{C_n}x \in C_n \subset C$, we get

$$\langle v - x_n, w - Av\rangle \geq 0. \tag{3.73}$$

Since A is α-inverse-strong monotone, we have

$$\langle v - x_n, w\rangle \geq \langle v - x_n, Av\rangle = \langle v - x_n, Av - Ax_n\rangle + \langle v - x_n, Ax_n\rangle \geq \langle v - x_n, Ax_n\rangle. \tag{3.74}$$

On other hand, from $w_n = \Pi_C J^{-1}(Jx_n - r_n Ax_n)$ and Lemma 2.3, we have $\langle v - w_n, Jw_n - (Jx_n - r_n Ax_n\rangle \geq 0$, and hence

$$\left\langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n\right\rangle \leq 0. \tag{3.75}$$

Because A is $1/\alpha$ constricted, it holds from (3.74) and (3.75) that

$$\begin{aligned}\langle v - x_n, w\rangle &\geq \langle v - x_n, Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{r_n} - Ax_n \right\rangle \\ &= \langle v - w_n, Ax_n\rangle + \langle w_n - x_n, Ax_n\rangle - \langle v - w_n, Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\right\rangle \\ &= \langle w_n - x_n, Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{r_n}\right\rangle \\ &\geq -\|w_n - x_n\| \cdot \|Ax_n\| - \|v - w_n\| \cdot \frac{\|Jx_n - Jw_n\|}{a},\end{aligned} \tag{3.76}$$

for all $n \in \mathbb{N} \cup \{0\}$. By taking the limit as $n \to \infty$ in (3.76) and from (3.43) and (3.44), we have $\langle v - p, w\rangle \geq 0$ as $n \to \infty$. By the maximality of T we obtain $p \in T^{-1}0$ and hence $p \in VI(A, C)$. Hence we conclude that

$$p \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \cap VI(A, C). \tag{3.77}$$

Finally, we show that $p \in \Pi_F x_1$. Indeed, taking the limit as $n \to \infty$ in (3.9), we obtain

$$\langle p - z, Jx_1 - Jp\rangle \geq 0, \quad \forall z \in F \tag{3.78}$$

and hence $p = \Pi_F\, x_1$ by Lemma 2.3. This complete the proof. □

Remark 3.2. Theorem 3.1 improves and extends main results of Iiduka and Takahashi [15], Xu and Ori [19], Qin et al. [21], and Cai and Hu [22] because it can be applied to solving the problem of finding the common element of the set of common fixed points of two families of relatively weak quasi-nonexpansive mappings and the set of solutions of the variational inequality for an inverse-strongly monotone operator.

Strong convergence theorem for approximating a common fixed point of two finite families of closed relatively weak quasi-nonexpansive mappings in Banach spaces may not require that E is 2-uniformly convex. In fact, we have the following theorem.

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed relatively weak quasi-nonexpansive mappings from C into itself with $F \neq \emptyset$, where $F := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i)$. Assume that T_i and S_i are uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algolithm:*

$$
\begin{aligned}
&x_0 = x \in C, \quad \text{chosen arbitrary}, \\
&C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\
&z_n = J^{-1}\left(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n x_n\right), \\
&y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n), \\
&C_{n+1} = \left\{u \in C_n : \phi(u, y_n) \le \delta_n \phi(u, x_1) + (1-\delta_n)\left[\alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)\right]\right\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1,
\end{aligned}
\tag{3.79}
$$

where $T_n = T_{n(\bmod N)}, S_n = S_{n(\bmod N)}$, and J is the normalized duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0,1]$ satisfying the following restrictions:

(C1) $\lim_{n\to\infty} \delta_n = 0$;

(C2) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

(a) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$ *and*

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$, where Π_F is the generalized projection from C onto F.

Proof. Put $A \equiv 40$ in Theorem 3.1. Then, we get that $w_n = x_n$. Thus, the method of the proof of Theorem 3.1 gives the required assertion without the requirement that E is 2-uniformly convex. □

Remark 3.4. Corollary 3.3 improves Theorem 3.1 of Cai and Hu [22] from a finite family of of relatively weak quasi-nonexpansive mappings to two finite families of relatively weak quasi-nonexpansive mappings.

If $E = H$, a Hilbert space, then E is 2-uniformly convex (we can choose $c = 1$) and uniformly smooth real Banach space and closed relatively weak quasi-nonexpansive map reduces to closed weak quasi-nonexpansive map. Furthermore, $J = I$, identity operator on H and $\Pi_C = P_C$, projection mapping from H into C. Thus, the following corollaries hold.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{T_1, T_2, \ldots, T_N\}$ and $\{S_1, S_2, \ldots, S_N\}$ be two finite families of closed weak quasi-nonexpansive mappings from C into itself with $F \ne \emptyset$, where $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap VI(A, C)$ with $\|Ay\| \le \|Ay - Aq\|$ for all $y \in C$ and $q \in F$. Assume that T_i and S_i are uniformly continuous for all $i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{aligned}
&x_0 = x \in C, \quad \text{chosen arbitrary}, \\
&C_1 = C, \quad x_1 = P_{C_1} x_0, \\
&w_n = P_C(x_n - r_n A x_n), \\
&z_n = \left(\alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n S_n w_n\right), \\
&y_n = (\delta_n x_1 + (1-\delta_n) z_n), \\
&C_{n+1} = \left\{u \in C_n : \|u - y_n\|^2 \le \delta_n \|u - x_1\|^2 + (1-\delta_n)\left[\alpha_n \|u - x_{n-1}\|^2 + (1-\alpha_n)\|u - x_n\|^2\right]\right\}, \\
&x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \ge 1,
\end{aligned}
\tag{3.80}
$$

where $T_n = T_{n(\bmod N)}, S_n = S_{n(\bmod N)}$, *and* J *is the normalized duality mapping on* E. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, *and* $\{r_n\}$ *are the sequences in* $[0,1]$ *satisfying the restrictions:*

(C1) $\lim_{n\to\infty} \delta_n = 0$;

(C2) $r_n \subset [a,b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the 2-uniformly convexity constant of* E;

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

(a) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$ *and*

(b) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\liminf_{n\to\infty}\beta_n\gamma_n > 0$.

Then $\{x_n\}$ *converges strongly to* $P_F x_1$, *where* P_F *is the metric projection from* C *onto* F.

Let X be a nonempty closed convex cone in E, and let A be an operator from X into E^*. We define its *polar* in E^* to be the set

$$X^* = \{y^* \in E^* : \langle x, y^*\rangle \geq 0\, \forall x \in X\}. \tag{3.81}$$

Then an element x in X is called a *solution of the complementarity problem* if

$$Ax \in X^*, \qquad \langle x, Ax\rangle = 0. \tag{3.82}$$

The set of all solutions of the complementarity problem is denoted by $CP(A,X)$. Several problem arising in different fields, such as mathematical programming, game theory, mechanics, and geometry, are to find solutions of the complementarity problems.

Theorem 3.6. *Let* X *be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space* E, *let* A *be an* α*-inverse-strongly monotone mapping of* X *into* E^* *with* $\|Ay\| \leq \|Ay - Aq\|$ *for all* $y \in X$ *and* $q \in F$. *Let* $\{T_1, T_2, \ldots, T_N\}$ *and* $\{S_1, S_2, \ldots, S_N\}$ *be two finite families of closed relatively weak quasi-nonexpansive mappings from* X *into itself with* $F \neq \emptyset$, *where* $F := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap CP(A,X)$. *Assume that* T_i *and* S_i *are uniformly continuous for all* $i \in \{1,2,\ldots,N\}$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{aligned}
&x_0 = x \in X, \quad \text{chosen arbitrary},\\
&C_1 = X, \quad x_1 = \Pi_{C_1} x_0,\\
&w_n = \Pi_C J^{-1}(Jx_n - r_n A x_n),\\
&z_n = J^{-1}(\alpha_n J x_{n-1} + \beta_n J T_n x_n + \gamma_n J S_n w_n),\\
&y_n = J^{-1}(\delta_n J x_1 + (1-\delta_n) J z_n),\\
&C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \delta_n \phi(u, x_1) + (1-\delta_n)[\alpha_n \phi(u, x_{n-1}) + (1-\alpha_n)\phi(u, x_n)]\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{aligned} \tag{3.83}$$

where $T_n = T_{n(\text{mod } N)}, S_n = S_{n(\text{mod } N)}$, *and* J *is the normalized duality mapping on* E. *Assume that* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ *and* $\{r_n\}$ *are the sequences in* $[0,1]$ *satisfying the restrictions:*

(C1) $\lim_{n\to\infty} \delta_n = 0$;

(C2) $r_n \subset [a,b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the 2-uniformly convexity constant of* E;

(C3) $\alpha_n + \beta_n + \gamma_n = 1$ *and if one of the following conditions is satisfied*

(a) $\liminf_{n\to\infty}\alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$ *and*

(b) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\liminf_{n\to\infty}\beta_n\gamma_n > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_1$, *where* Π_F *is the generalized projection from* X *onto* F.

Proof. From [25, Lemma 7.1.1], we have $VI(A,X) = CP(A,X)$. From Theorem 3.1, we can obtain the desired conclusion easily. □

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ภาคผนวก 6

Convergence theorems based on the shrinking projection method for variational inequality and equilibrium problems

U. Kamraksa, R. Wangkeeree

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Convergence theorems based on the shrinking projection method for variational inequality and equilibrium problems

Uthai Kamraksa · Rabian Wangkeeree

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Abstract The purpose of this paper is to introduce a hybrid projection algorithm based on the shrinking projection method for two relatively weak nonexpansive mappings. We prove strong convergence theorem which approximate the common element in the fixed point set of two such mappings, the solution set of the variational inequality and the solution set of the equilibrium problem in the framework of Banach spaces. Our results improve and extend previous results.

Keywords Banach space · Fixed point · Projection · Relatively weak nonexpansive mapping · Shrinking projection method · Strong convergence

Mathematics Subject Classification (2000) 47H09 · 47H10

1 Introduction

Let E be a Banach space and let E^* be the dual of E and let C be a closed and convex subset of E. Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x\| = \|x^*\|\}, \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ is the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Some properties of the duality mapping can be found in [9, 33, 39].

U. Kamraksa · R. Wangkeeree (✉)
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
e-mail: rabianw@nu.ac.th

U. Kamraksa
e-mail: uthaikam@hotmail.com

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Let $A : C \to E^*$ be an operator. We consider the following variational inequality:

$$\text{Find } x \in C, \text{ such that } \langle Ax, y - x\rangle \geq 0, \quad \text{for all } y \in C. \tag{1.1}$$

A point $x_0 \in C$ is called a solution of the variational inequality (1.1) if for every $y \in C$, $\langle Ax_0, y - x_0\rangle \geq 0$. The solution set of the variational inequality (1.1) is denoted by $VI(A, C)$.

If C is a nonempty, closed and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive, i.e., $\|P_C x - P_C y\| \leq \|x - y\|$, for all $x, y \in H$. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. Most recently, applying the generalized projection operator in uniformly convex and uniformly smooth Banach spaces, Li [19] established the following Mann type iterative scheme for solving variational inequalities without assuming the monotonicity of A in compact subsets of Banach spaces: For any $x_0 \in C$, define a Mann type iteration scheme as follows

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Pi_C(Jx_n - (Ax_n - \xi)), \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\}$ satisfies conditions $0 \leq \alpha_n \leq 1$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, $A : C \to E^*$ is a continuous mapping on a compact convex subset C of E such that

$$\langle Tx - \xi, J^*(Jx_1 - (Ax - \xi))\rangle \geq 0, \quad \text{for all } x \in C, \xi \in E^*.$$

It is proved in [19] that the variational inequality

$$\langle Ax - \xi, y - x\rangle \geq 0, \quad \forall y \in C$$

has a solution $x^* \in C$ and there exists a subsequence $\{n_i\} \subset \{n\}$ such that $\{x_{n_i}\}$ converges strongly to x^* as $i \to \infty$. Moreover, Fan [13] established some existence results of solutions and the convergence of a Mann type iterative scheme for the variational inequality (1.1) in noncompact subsets of Banach spaces. More precisely, he proved the following theorem:

Theorem Fan (Fan [13], Theorem 3.3) *Let E be a uniformly convex and uniformly smooth Banach space and let C be a closed and convex subset of E. Suppose that there exists a positive number β, such that*

$$\langle Ax, J^*(Jx - \beta Ax)\rangle \geq 0, \quad \textit{for all } x \in C,$$

and $J - \beta A : C \to E^*$ is compact. If

$$\langle Ax, y\rangle \leq 0, \quad \textit{for all } x \in K,\ y \in VI(A, C),$$

then the variational inequality (1.1) has a solution $x^* \in C$ and the sequence $\{x_n\}$ defined by the following iterative scheme:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Pi_C(Jx_n - \beta Ax_n), \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\}$ *satisfies*: $0 < a \leq \alpha_n \leq b < 1$ *for all* $n \in \mathbb{N}$, *for some positive number* $a, b \in (0, 1)$ *satisfying* $a < b$, *converges strongly to* x^*.

Motivated by Li [19] and Fan [13], Liu [25] introduced an iterative sequence for approximating a common element of the set of fixed points of a relatively weak nonexpansive mapping defined by Kohasaka and Takahashi [18] and the solution set of the variational inequality in noncompact subset of Banach spaces without assuming the compactness of the operator $J - \beta A$. More precisely, he proved the following theorem.

Theorem Liu (Liu [25], Theorem 3.1) *Let* E *be a uniformly convex and uniformly smooth Banach space and let* C *be a nonempty, closed and convex subset of* E. *Assume that* A *is a continuous operator of* C *into* E^* *that satisfy conditions* (2.7) *and* (2.8) *and* $S : C \to C$ *is a relatively weak nonexpansive mapping with* $F(S) \cap VI(A, C) \neq \emptyset$. *Then the sequence* $\{x_n\}$ *generated by the following manner*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ y_n = J^*(\delta_n J x_n + (1 - \delta_n) J \Pi_C(J z_n - \beta A z_n)), \\ C_0 = \{u \in C : \phi(u, y_0) \leq \phi(u, x_0)\}, \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(u, y_n) \leq \phi(u, x_n)\}, \\ Q_0 = C, \\ Q_n = \{u \in Q_{n-1} \cap C_{n-1} : \langle J x_0 - J x_n, x_n - u\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} J x_0 \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\delta_n\}$ *satisfy*:

$$0 \leq \delta_n < 1 \quad \text{and} \quad \limsup_{n\to\infty} \delta_n < 1; \qquad 0 < \alpha_n < 1 \quad \text{and} \quad \liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0,$$

converges strongly to $\Pi_{F(S)\cap VI(A,C)} J x_0$.

Let $f : C \times C \to \mathbb{R}$ *be a bifunction. The equilibrium problem for* f *is to find* $\hat{x} \in C$ *such that*

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) *is denoted by* $EP(f)$.

Numerous problems in physics, optimization, and economics can be reduced to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; *see, for instance, Blum and Oettli* [4], *Combettes and Hirstoaga* [10], *and Moudafi* [23]. *On the other hand, there are several methods for approximation fixed points of a nonexpansive mapping*; *see, for instance*, [8, 11, 12, 14, 15, 19–22, 24–29, 33–36, 38]. *Recently, Tada and Takahashi* [31, 32] *and Takahashi and Takahashi* [35] *obtained weak and strong convergence theorems for finding a common elements in the solution set of an equilibrium problem and the set of fixed point of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi* [32] *established a strong convergence theorem for finding a common element of two sets by using the hybrid method introduced in Nakajo and Takahashi* [24]. *They also proved such a strong convergence theorem* *in a uniformly convex and uniformly smooth Banach space. Recently, Takahashi et al.* [38] *introduced a hybrid method which is different from Nakajo and Takahashi's hybrid method. It is called the shrinking projection method. They obtained strong convergence theorem in the frame work of Hilbert spaces. Based on the so-called shrinking projection method of Takahashi et al.* [38], *Takahashi and Zembayashi* [36] *introduced the following iterative scheme*:

$$\begin{cases} x_0 = x \in C, \ C_0 = C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{1.4}$$

where S is a relatively nonexpansive mapping and Π_C is the generalized projection from E onto C. They proved that the sequence $\{x_n\}$ defined by (1.4) *converges strongly to $q = \Pi_{F(T)\cap EP(f)} x_0$ under appropriate conditions imposed on the parameters.*

Motivated and inspired by Li [19], Fan [13], Liu [25] and Takahashi and Zembayashi [36], we introduce a hybrid projection algorithm based on the shrinking projection method for two relatively weak nonexpansive mappings. We prove strong convergence theorem which approximate the common element in the fixed points of two such mappings, the solution set of the variational inequality and the solution set of the equilibrium problem in the framework of Banach spaces.

2 Preliminaries

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [9, 33, 34] for more details.

In [2, 3], Alber introduced the functional $V : E^* \times E \to \mathbb{R}$ defined by

$$V(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2,$$

where $\phi \in E^*$ and $x \in E$.

It is easy to see that

$$V(\phi, x) \geq (\|\phi\| - \|x\|)^2. \tag{2.1}$$

Thus the functional $V : E^* \times E \to \mathbb{R}^+$ is nonnegative.

Now we present several definitions and lemmas which will be used in the proof for the main result in the next section.

Definition 2.1 (Kamimura and Takahashi [14]) If E is a uniformly convex and uniformly smooth Banach space, the generalized projection $\Pi_C : E^* \to C$ is a mapping that assigns an arbitrary point $\phi \in E^*$ to the minimum point of the functional $V(\phi, x)$, i.e., a solution to the minimization problem

$$V(\phi, \Pi_C(\phi)) = \inf_{y \in C} V(\phi, y).$$

Li [20] proved that the generalized projection operator $\Pi_C : E^* \to C$ is continuous, if E is a reflexive, strictly convex and smooth Banach space.

Consider the function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = V(Jy, x), \quad \forall x, y \in E.$$

The following properties of the operator Π_C and V are useful for our paper. (See, for example, [1, 20].)

(i) $V : E^* \times E \to \mathbb{R}$ is continuous.
(ii) $V(\phi, x) = 0$ if and only if $\phi = Jx$.
(iii) $V(J\Pi_C(\phi), x) \leq V(\phi, x)$ for all $\phi \in E^*$ and $x \in E$.
(iv) The operator Π_C is J fixed at each point $x \in C$, i.e., $\Pi_C(Jx) = x$.
(v) If E is smooth, then for any given $\phi \in E^*$, $x \in C$, $x \in \Pi_C(\phi)$ if and only if $\langle \phi - Jx, x - y \rangle \geq 0$, for all $y \in C$.
(vi) The operator $\Pi_C : E^* \to C$ is single valued if and only if E is strictly convex.
(vii) If E is smooth, then for any given point $\phi \in E^*$, $x \in \Pi_C(\phi)$, the following inequality holds

$$V(Jx, y) \leq V(\phi, y) - V(\phi, x) \quad \forall y \in C.$$

(viii) $V(\phi, x)$ is convex with respect to ϕ when x is fixed and with respect to x when ϕ is fixed.
(ix) If E is reflexive, then for any point $\phi \in E^*$, $\Pi_C(\phi)$ is a nonempty, closed, convex and bounded subset of C.

Remark 2.2 If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(y, x) = 0$ then $x = y$. From (1), we have $\|x\| = \|y\|$. This implies $\langle y, Jx \rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$; see [9, 33, 34] for more details.

Using the properties of generalized projection operator Π_C, Alber [1] proved the following theorem.

Lemma 2.3 (Liu [25]) *Let E be a reflexive, strictly convex and smooth Banach space with dual space E^*. Let A be an arbitrary operator from Banach space E to E^* and β an arbitrary fixed positive number. Then $x \in C \subset E$ is a solution of variational inequality* (1.1) *if and only if x is a solution of the operator equation in E*

$$x = \Pi_C(Jx - \beta Ax).$$

Let S be a mapping from C into itself. We denote by $F(S)$ the set of fixed point of S. A point p in C is said to be an asymptotic fixed point of S [30] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed point of S will be denoted by $\hat{F}(S)$. A mapping S from C into itself is called relatively nonexpansive (see e.g., [5]) if $\hat{F}(S) = F(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for all $x \in K$ and $p \in F(S)$. The asymptotic behavior of relatively nonexpansive mappings were studied in [5, 6]. A point p in C is said to be a strong asymptotic fixed point of S if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of strong asymptotic fixed points of S will be denoted by $\tilde{F}(S)$. A mapping S from C into itself is called relatively weak nonexpansive if $\tilde{F}(S) = F(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(S)$. Moreover, if $S : C \to C$ is relatively weak nonexpansive, then using the definition of ϕ (i.e. the same argument as in the proof of [22, p. 260]) one can show that $F(S)$ is closed and convex. It is obvious that relatively nonexpansive mapping is relatively weak nonexpansive mapping. In fact, for any mapping $S : C \to C$ we have $F(S) \subset \tilde{F}(S) \subset \hat{F}(S)$. Therefore, if S is a relatively nonexpansive mapping, then $F(S) = \tilde{F}(S) = \hat{F}(S)$.

Example 2.4 Let E be a smooth strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}0 \neq \emptyset$, then it is proved in [18] that $J_r = (J + rA)^{-1}J$, for $r > 0$ is a relatively weak nonexpansive mapping.

Now we present several useful lemmas for the proof of our main theorem.

Lemma 2.5 (Kamimura and Takahashi [14]) *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, then $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

Lemma 2.6 (Chang [7]) *Let E be a uniformly convex and uniformly smooth Banach space. We have*

$$\|\phi + \Phi\|^2 \le \|\phi\|^2 + 2\langle \Phi, J(\phi + \Phi)\rangle, \quad \forall \phi, \Phi \in E^*.$$

Lemma 2.7 ([8, Lemma 1.4]) *Let X be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|), \qquad (2.2)$$

for all $x, y, z \in B_r(0)$ *and* $\lambda, \mu, \gamma \in [0, 1]$ *with* $\lambda + \mu + \gamma = 1$.

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y); \tag{2.3}$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x, y) = \langle Ax, y - x \rangle, \quad \forall x, y \in C.$$

Then, f satisfies (A1)–(A4).

Lemma 2.8 (Blum and Oettli [4]) *Let* C *be a closed and convex subset of a smooth, strictly convex, and reflexive Banach spaces* E, *let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)–(A4), *and let* $r > 0$ *and* $x \in E$. *Then, there exists* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

Lemma 2.9 (Takahashi and Zembayashi [37]) *Let* C *be a closed and convex subset of a uniformly smooth, strictly convex, and reflexive Banach space* E, *and let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies conditions* (A1)–(A4). *For all* $r > 0$ *and* $x \in E$, *define the mapping*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}. \tag{2.5}$$

Then, the following hold:

(1) T_r *is single-valued*;
(2) T_r *is a firmly nonexpansive-type mapping* [17], *that is, for all* $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle; \tag{2.6}$$

(3) $F(T_r) = \hat{F}(T_r) = EP(f)$;
(4) $EP(f)$ *is closed and convex.*

Lemma 2.10 (Liu [25]) *Let* E *be a uniformly convex and uniformly smooth Banach space, let* C *be a nonempty, closed and convex subset of* E. *Suppose* A *is an operator of* C *into* E^* *and there exists a positive number* β *such that*

$$\langle Ax, J^*(Jx - \beta Ax) \rangle \geq 0, \quad \textit{for all } x \in C, \tag{2.7}$$

and

$$\langle Ax, y\rangle \le 0, \quad \forall x \in C,\ y \in VI(A,C). \tag{2.8}$$

Then $VI(A,C)$ is closed and convex.

Lemma 2.11 (Liu [25]) *If E is a reflexive, strictly convex and smooth Banach space, then $\Pi_C = J^*$.*

3 Main result

Theorem 3.1 *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *Assume that A is a continuous operator of C into E^* satisfying conditions* (2.7) *and* (2.8) *and $S, T : C \to C$ are relatively weak nonexpansive mappings with $F := F(S) \cap F(T) \cap VI(A,C) \cap EP(f) \ne \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \ C_0 = C, \\ z_n = \Pi_C(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ y_n = J^*(\delta_n Jx_n + (1-\delta_n)J\Pi_C(Jz_n - \beta Az_n)), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} Jx \quad \forall n \ge 0. \end{cases} \tag{3.1}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0,1]$ *satisfying the restrictions*:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;
(C2) $0 \le \delta_n < 1, \limsup_{n\to\infty} \delta_n < 1$;
(C3) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$; *and*
(C4) $\liminf_{n\to\infty} \alpha_n\beta_n > 0, \liminf_{n\to\infty} \alpha_n\gamma_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x$.

Proof We divide the proof into five steps.

Step 1. $\Pi_F x$ and $\Pi_{C_{n+1}} x$ are well defined.

From Lemma 2.10, we know that $VI(A,C)$ is closed and convex. By the same argument as in the proof of [22, p. 260], one can show that $F(T) \cap F(S)$ is closed and convex. From Lemma 2.9(4), we also have that $EP(f)$ is closed and convex. Hence F is a nonempty, closed and convex subset of C. Consequently, $\Pi_F x$ is well defined.

Clearly, $C_0 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For all $z \in C_{k+1}$, one obtains that

$$\phi(z, u_n) \le \phi(z, x_n)$$

is equivalent to

$$2(\langle z, Jx_k\rangle - \langle z, Ju_k\rangle) \le \|x_k\|^2 - \|u_k\|^2.$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \ge 0$, C_n is closed and convex. Hence $\Pi_{C_{n+1}}x$ is well defined.

Step 2. $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

We observe that $F \subset C_0 = C$ is obvious. Suppose $F \subset C_k$ for some $k \in \mathbb{N}$. Let $w \in F \subset C_k$, then, from the definitions of ϕ and V, property (iii) of V, Lemma 2.6, conditions (2.7) and (2.8), we have

$$\begin{aligned}
\phi(w, \Pi_C(Jz_n - \beta Az_n)) &= V(J\Pi_C(Jz_n - \beta Az_n), w)\\
&\le V(Jz_n - \beta Az_n, w)\\
&= \|Jz_n - \beta Az_n\|^2 - 2\langle Jz_n - \beta Az_n, w\rangle + \|w\|^2\\
&\le \|Jz_n\|^2 - 2\beta\langle Az_n, J^*(Jz_n - \beta Az_n)\rangle\\
&\quad - 2\langle Jz_n - \beta Az_n, w\rangle + \|w\|^2\\
&\le \|Jz_n\|^2 - 2\langle Jz_n, w\rangle + \|w\|^2 = \phi(w, z_n),
\end{aligned} \tag{3.2}$$

for each $n \in \mathbb{N} \cup \{0\}$. From Lemma 2.9(2), one has that T_{r_n} is a relatively nonexpansive mapping. Therefore, by properties (viii) and (iii) of the operator V and (3.2), we obtain

$$\begin{aligned}
\phi(w, u_k) &= \phi(w, T_{r_k}y_k) \le \phi(w, y_k) = V(Jy_k, w)\\
&\le \delta_k V(Jx_k, w) + (1-\delta_k)V(J\Pi_C(Jz_k - \beta Az_k), w)\\
&= \delta_k\phi(w, x_k) + (1-\delta_k)\phi(w, \Pi_C(Jz_k - \beta Az_k))\\
&\le \delta_k\phi(w, x_k) + (1-\delta_k)\phi(w, z_k)\\
&= \delta_k\phi(w, x_k) + (1-\delta_k)V(Jz_k, w)\\
&\le \delta_k\phi(w, x_k) + (1-\delta_k)V(\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k, w)\\
&= \delta_k\phi(w, x_k) + (1-\delta_k)\phi(w, J^*(\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k))\\
&= \delta_k\phi(w, x_k) + (1-\delta_k)[\|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle - 2\beta_k\langle w, JTx_k\rangle\\
&\quad - 2\gamma_k\langle w, JSx_k\rangle + \|\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k\|^2]\\
&\le \delta_k\phi(w, x_k) + (1-\delta_k)[\|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle - 2\beta_k\langle w, JTx_k\rangle\\
&\quad - 2\gamma_k\langle w, JSx_k\rangle + \alpha_k\|Jx_k\|^2 + \beta_k\|JTx_k\|^2 + \gamma_k\|JSx_k\|^2]\\
&= \delta_k\phi(w, x_k) + (1-\delta_k)[\alpha_k\phi(w, x_k) + \beta_k\phi(w, Tx_k) + \gamma_k\phi(w, Sx_k)]\\
&\le \delta_k\phi(w, x_k) + (1-\delta_k)\phi(w, x_k) = \phi(w, x_k),
\end{aligned} \tag{3.3}$$

which shows that $w \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \ge 0$.

Step 3. $\{x_n\}$ is a convergent sequence in C.

Since $x_n = \Pi_{C_n} Jx$ and $F \subset C_n$, we have $V(Jx, x_n) \le V(Jx, w)$ for each $w \in F$. Therefore, $\{V(Jx, x_n)\}$ is bounded. Moreover, from the definition of V, we have that $\{x_n\}$ is bounded. Since $x_{n+1} = \Pi_{C_{n+1}} Jx \in C_{n+1}$ and $x_n = \Pi_{C_n} Jx$, we have $V(Jx, x_n) \le V(Jx, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Therefore $\{V(Jx, x_n)\}$ is nondecreasing. Hence

$$\lim_{n\to\infty} V(Jx, x_n) \quad \text{exists.}$$

By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m} Jx \in C_n$ for any positive integer $m \ge n$. From property (vii) of the operator Π_C, we have

$$V(Jx_n, x_m) \le V(Jx, x_m) - V(Jx, x_n), \tag{3.4}$$

for each $n \in \mathbb{N} \cup \{0\}$ and any positive integer $m \ge n$. This implies that

$$V(Jx_n, x_m) \to 0 \quad \text{as } n, m \to \infty. \tag{3.5}$$

The definition of ϕ implies that

$$\phi(x_m, x_n) \to 0 \quad \text{as } n, m \to \infty. \tag{3.6}$$

Applying Lemma 2.5, we obtain

$$\|x_m - x_n\| \to 0 \quad \text{as } n, m \to \infty. \tag{3.7}$$

Hence $\{x_n\}$ is a Cauchy sequence. The completeness of a Banach space E and the closeness of C imply that

$$\lim_{n\to\infty} x_n = p, \quad \text{for some } p \in C.$$

Step 4. We show that $p \in F$.

(I) First we show that $p \in F(S) \cap F(T)$.
Take $m = n + 1$ in (3.5), one arrives that

$$\lim_{n\to\infty} V(Jx_n, x_{n+1}) = 0.$$

By the definition of ϕ, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.8}$$

Using Lemma 2.5, we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Note that $x_{n+1} = \Pi_{C_{n+1}} Jx \in C_{n+1}$ then

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n).$$

It follows from (3.8) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Using Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.10}$$

Combining (3.13) with (3.10), one sees that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.11}$$

It follows from $x_n \to p$ as $n \to \infty$ that

$$u_n \to p \text{ as } n \to \infty. \tag{3.12}$$

On the other hand, since J is uniformly norm-to-norm continuous an bounded sets, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.13}$$

Since $\{x_n\}$ is bounded, $\{Jx_n\}$, $\{JTx_n\}$ and $\{JSx_n\}$ are also bounded. Since E is a uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. Let $r = \sup_{n\geq 0}\{\|Jx_n\|, \|JTx_n\|, \|JSx_n\|\}$. Therefore Lemma 2.7 implies that there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$ and inequality (2.2). It follows from the property (iii) of the operator V, (3.2) and the definition of S and T, that

$$\begin{aligned}
\phi(p, z_n) &= V(Jz_n, p) \leq V(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n, p)\\
&= \phi(p, J^*(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n)\\
&= \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, JTx_n\rangle - 2\gamma_n\langle p, JSx_n\rangle\\
&\quad + \|\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n\|^2\\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, JTx_n\rangle - 2\gamma_n\langle p, JSx_n\rangle\\
&\quad + \alpha_n\|Jx_n\|^2 + \beta_n\|JTx_n\|^2 + \gamma_n\|JSx_n\|^2 - \alpha_n\beta_n g(\|JTx_n - Jx_n\|)\\
&= \alpha_n\phi(p, x_n) + \beta_n\phi(p, Tx_n) + \gamma_n\phi(p, Sx_n) - \alpha_n\beta_n g(\|JTx_n - Jx_n\|)\\
&\leq \phi(p, x_n) - \alpha_n\beta_n g(\|JTx_n - Jx_n\|).
\end{aligned} \tag{3.14}$$

From property (viii) of the operator V, (3.2) and (3.14), we obtain

$$\begin{aligned}
\phi(p, u_n) &= \phi(p, T_{r_n} y_n) \leq \phi(p, y_n) = V(Jy_n, p)\\
&\leq \delta_n V(Jx_n, p) + (1 - \delta_n)V(J\Pi_C(Jz_n - \beta Az_n), p)\\
&= \delta_n\phi(p, x_n) + (1 - \delta_n)\phi(p, \Pi_C(Jz_n - \beta Az_n))\\
&\leq \delta_n\phi(p, x_n) + (1 - \delta_n)\phi(p, z_n)\\
&\leq \delta_n\phi(p, x_n) + (1 - \delta_n)[\phi(p, x_n) - \alpha_n\beta_n g(\|JTx_n - Jx_n\|)]
\end{aligned}$$

$$= \phi(p, x_n) - (1-\delta_n)\alpha_n\beta_n g(\|JTx_n - Jx_n\|).$$

Therefore,

$$(1-\delta_n)\alpha_n\beta_n g(\|JTx_n - Jx_n\|) \le \phi(p, x_n) - \phi(p, u_n). \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\phi(p, x_n) - \phi(p, u_n) &= 2\langle Ju_n - Jx_n, p\rangle + \|x_n\|^2 - \|u_n\|^2\\ &= 2\langle Ju_n - Jx_n, p\rangle + (\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|)\\ &\le 2\|Ju_n - Jx_n\|\|w\| + \|x_n - u_n\|(\|x_n\| + \|u_n\|).\end{aligned}$$

It follows from (3.11) and (3.13) that

$$\lim_{n\to\infty}(\phi(p, x_n) - \phi(p, u_n)) = 0. \tag{3.16}$$

From the assumptions $\limsup_{n\to\infty}\delta_n < 1$, $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, (3.15) and (3.20) we have

$$\lim_{n\to\infty} g(\|JTx_n - Jx_n\|) = 0. \tag{3.17}$$

It follows from the property of g that

$$\lim_{n\to\infty}\|JTx_n - Jx_n\| = 0. \tag{3.18}$$

Since J^* is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|J^*Jx_n - J^*JTx_n\| = 0. \tag{3.19}$$

In a similar way, we can apply the condition $\liminf_{n\to\infty}\alpha_n\gamma_n > 0$ to get

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.20}$$

Since $x_n \to p$, we have $p \in \tilde{F}(S)\cap\tilde{F}(T) = F(S)\cap F(T)$. Moreover,

$$Sx_n \to p \quad \text{as } n\to\infty \quad \text{and} \quad Tx_n \to p \quad \text{as } n\to\infty. \tag{3.21}$$

(II) $p \in EP(f)$.

From (3.3), we know that

$$\phi(u, y_n) \le \phi(u, x_n).$$

From $u_n = T_{r_n}y_n$ and Lemma 2.9(2), one has

$$\begin{aligned}\phi(u_n, y_n) = \phi(T_{r_n}y_n, y_n) &\le \phi(w, y_n) - \phi(w, T_{r_n}y_n)\\ &\le \phi(w, x_n) - \phi(w, T_{r_n}y_n) = \phi(w, x_n) - \phi(w, u_n).\end{aligned}$$

It follows from (3.16) that

$$\phi(u_n, y_n) \to 0 \quad \text{as } n \to \infty.$$

Applying Lemma 2.5, we obtain

$$\|u_n - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.22}$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{3.23}$$

From the assumption that $r_n \geq a$, one sees

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.24}$$

Observe that $u_n = T_{r_n} y_n$, one obtains

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy\rangle \geq 0, \quad \forall y \in C. \tag{3.25}$$

From (A2), one arrives that

$$\begin{aligned} \|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -f(u_n, y) \\ &\geq f(y, u_n), \quad \forall y \in C. \end{aligned}$$

Take $n \to \infty$ in the above inequality we get from (A4) and (3.12) that

$$f(y, p) \leq 0, \quad \forall y \in C.$$

For all $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Note that $y, p \in C$, one obtains $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from (A1) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

That is,

$$f(y_t, y) \geq 0. \tag{3.26}$$

Let $t \downarrow 0$, from (A3), we obtain $f(p, y) \geq 0$, for all $y \in C$. This implies that $p \in EP(f)$.

(III) $p \in VI(A, C)$.

From (3.11) and (3.22) we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.27}$$

Since J is uniformly norm-to-norm continuous on bounded set, we have

$$\lim_{n\to\infty} \|Jy_n - Jx_n\| = 0. \tag{3.28}$$

Since $\|Jy_n - Jx_n\| = (1-\delta_n)\|J\Pi_C(Jz_n - \beta Az_n) - Jx_n\|$ and $\limsup_{n\to\infty} \delta_n < 1$, we have

$$\lim_{n\to\infty} \|J\Pi_C(Jz_n - \beta Az_n) - Jx_n\| = 0.$$

Since J^* is also uniformly norm-to-norm continuous on bounded set, we have

$$\|\Pi_C(Jz_n - \beta Az_n) - x_n\| = \lim_{n\to\infty} \|J^*J\Pi_C(Jz_n - \beta Az_n) - J^*Jx_n\| = 0. \tag{3.29}$$

From properties (iii) and (ii) of the operator V, we derive that

$$\begin{aligned}
\phi(x_n, z_n) &= V(Jz_n, x_n) \\
&\le V(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n, x_n) \\
&= \phi(x_n, J^*(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n) \\
&= \|x_n\|^2 - 2\alpha_n\langle x_n, Jx_n\rangle - 2\beta_n\langle x_n, JTx_n\rangle \\
&\quad - 2\gamma_n\langle x_n, JSx_n\rangle + \|\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n\|^2 \\
&\le \|x_n\|^2 - 2\alpha_n\langle x_n, Jx_n\rangle - 2\beta_n\langle x_n, JTx_n\rangle \\
&\quad - 2\gamma_n\langle x_n, JSx_n\rangle + \alpha_n\|Jx_n\|^2 + \beta_n\|JTx_n\|^2 + \gamma_n\|JSx_n\|^2 \\
&= \alpha_n\phi(x_n, x_n) + \beta_n\phi(x_n, Tx_n) + \gamma_n\phi(x_n, Sx_n).
\end{aligned}$$

By the continuity of the function ϕ and (3.21), we have

$$\lim_{n\to\infty} \phi(x_n, z_n) = 0.$$

From Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.30}$$

Using inequalities (3.29) and (3.30) we obtain

$$\|\Pi_C(Jz_n - \beta Az_n) - z_n\| \le \|\Pi_C(Jz_n - \beta Az_n) - x_n\| + \|x_n - z_n\| \to 0. \tag{3.31}$$

Since $x_n \to p$ we get that $z_n \to p$. By the continuity of the operator J, A and Π_C, we have

$$\lim_{n\to\infty} \|\Pi_C(Jz_n - \beta Az_n) - \Pi_C(Jp - \beta Ap)\| = 0. \tag{3.32}$$

Note that

$$\|\Pi_C(Jz_n - \beta Az_n) - p\| \le \|\Pi_C(Jz_n - \beta Az_n) - z_n\| + \|z_n - q\| \to 0, \quad \text{as } n \to \infty. \tag{3.33}$$

Hence, it follows from the uniqueness of the limit that $p = \Pi_C(Jp - \beta Ap)$. From Lemma 2.3, we have $p \in VI(A, C)$. By cases I, II and III, we conclude that $p \in F$.

Step 5. $p = \Pi_F Jx$.

Since $p \in F$, then from property (vii) of the operator Π_C, we have

$$V(J\Pi_F Jx, p) + V(Jx, \Pi_F Jx) \le V(Jx, p). \tag{3.34}$$

On the other hand, since $x_{n+1} = \Pi_{C_{n+1}} Jx$, and $F \subset C_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, then it follows from property (vii) of the operator Π_C that

$$V(Jx_{n+1}, \Pi_F Jx) + V(Jx, x_{n+1}) \le V(Jx, \Pi_F Jx). \tag{3.35}$$

Moreover, by the continuity of the operator V, we get that

$$\lim_{n\to\infty} V(Jx, x_{n+1}) = V(Jx, p). \tag{3.36}$$

Combining (3.34), (3.35) with (3.36), we obtain that $V(Jx, p) = V(Jx, \Pi_F Jx)$. Therefore, it follows from the uniqueness of $\Pi_F Jx$ that $p = \Pi_F Jx$. This completes the proof. □

Remark 3.2 The following sequences of parameters are examples which support our main result:

$$\alpha_n = \frac{1}{3} - \frac{1}{n+1}, \qquad \beta_n = \frac{1}{3} \quad \text{and} \quad \gamma_n = \frac{1}{3} + \frac{1}{n+1}$$
$$r_n = n + 3 \quad \text{and} \quad \delta_n = \frac{n}{2n+1}$$

for all $n \in \mathbb{N}$.

Setting $S = T$ in Theorem 3.1, we obtain the following result.

Corollary 3.3 *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *Assume that A is a continuous operator of C into E^* satisfying conditions* (2.7) *and* (2.8) *and $T : C \to C$ is a relatively weak nonexpansive mapping with $F := F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \ C_0 = C, \\ z_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ y_n = J^*(\delta_n Jx_n + (1-\delta_n)J\Pi_C(Jz_n - \beta Az_n)), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} Jx \quad \forall n \ge 0. \end{cases} \tag{3.37}$$

Assume that $\{\alpha_n\}$ and $\{\delta_n\}$ are the sequences in $[0, 1]$ *satisfying the restrictions*:

(C1) $0 \le \delta_n < 1, \limsup_{n\to\infty} \delta_n < 1$;
(C2) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$; *and*
(C3) $0 < \alpha_n < 1$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_F x$.

Corollary 3.4 *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty, closed and convex subset of E. Assume that A is an continuous operator of C into E^* satisfying conditions (2.7) and (2.8) and $T : C \to C$ is a relatively weak nonexpansive mapping with $F := F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \ C_0 = C, \\ z_n = \Pi_C(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ y_n = J^*(\delta_n J x_n + (1-\delta_n) J \Pi_C (J z_n - \beta A z_n)), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} J x \quad \forall n \ge 0. \end{cases} \tag{3.38}$$

Assume that $\{\alpha_n\}$ *and* $\{\delta_n\}$ *are the sequences in* $[0, 1]$ *satisfying the restrictions*:

(C1) $0 \le \delta_n < 1, \limsup_{n\to\infty} \delta_n < 1$;
(C2) $0 < \alpha_n < 1$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_F x$.

Proof Setting $S = T$, $f(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \ge 0$ in Theorem 3.1, we obtain that $\{x_n\}$ defined by (3.38) converges strongly to $\Pi_F x$. □

Now, we present two examples of mappings which are relatively weak nonexpansive mappings and can be found in Kim and Lee's results [16].

Example 3.5 [16, Example 3.13] Let U denote the unit ball in the space $E = l^p$, where $1 < p < \infty$. Obviously, E is uniformly convex and uniformly smooth. Let $T : E \to E$ be defined by

$$Tx = (0, x_1^2, \lambda_2 x_2, \lambda_3 x_3, \dots)$$

for all $x = (x_1, x_2, x_3, \dots) \in U$, where $\lambda_n = 1 - \frac{1}{n^2}$ for $n \ge 2$ (hence $\prod_{n=2}^{\infty} \lambda_n = \frac{1}{2}$). Therefore,

(1) $F(T) = \{0 = (0, 0, 0, \dots)\}$
(2) T is relatively nonexpansive and hence it is relatively weak nonexpansive.

Next, consider an example where $F(T)$ is not singleton.

Example 3.6 [16, Example 3.14] Let $E = l^p$, where $2 < p < \infty$, and $C = \{x = (x_1, x_2, \dots) \in X; 0 \le x_n \le 1\}$. Then C is a closed convex subset of X. Note that C is not bounded. Let $S : C \to C$ be defined by

$$Sx = (x_1, 0, x_2^2, \lambda_2 x_3, \lambda_2 x_4, \dots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in C$, where $\lambda_n = 1 - \frac{1}{n^2}$ for $n \geq 2$ as in Example 3.5. Then

(1) $F(S) = \{p = (p_1, 0, 0, \ldots) : 0 \leq p_1 \leq 1\}$
(2) S is relatively nonexpansive and hence it is relatively weak nonexpansive.

Remark 3.7 We observe that $0 = (0, 0, \ldots,)$ is a common fixed point of the mapping T in Example 3.5 and the mapping S in Example 3.6.

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ภาคผนวก 7

Strong convergence theorems of viscosity iterative methods for a countable family of strict pseudo-contractions in Banach spaces

R. Wangkeeree, U. Kamraksa

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Research Article

Strong Convergence Theorems of Viscosity Iterative Methods for a Countable Family of Strict Pseudo-contractions in Banach Spaces

Rabian Wangkeeree and Uthai Kamraksa

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Rabian Wangkeeree, rabianw@nu.ac.th

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For a countable family $\{T_n\}_{n=1}^{\infty}$ of strictly pseudo-contractions, a strong convergence of viscosity iteration is shown in order to find a common fixed point of $\{T_n\}_{n=1}^{\infty}$ in either a p-uniformly convex Banach space which admits a weakly continuous duality mapping or a p-uniformly convex Banach space with uniformly Gâteaux differentiable norm. As applications, at the end of the paper we apply our results to the problem of finding a zero of accretive operators. The main result extends various results existing in the current literature.

1. Introduction

Let E be a real Banach space and C a nonempty closed convex subset of E. A mapping $f : C \to C$ is called k-contraction if there exists a constant $0 < k < 1$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$ for all $x, y \in C$. We use $\prod_C$ to denote the collection of all contractions on C. That is, $\prod_C = \{f : f$ is a contraction on $C\}$. A mapping $T : C \to C$ is said to be λ-strictly pseudo-contractive mapping (see, e.g., [1]) if there exists a constant $0 \leq \lambda < 1$, such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda\|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

for all $x, y \in C$. Note that the class of λ-strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mapping T on C such that $\|Tx - Ty\| \leq \|x - y\|$, for all x, $y \in C$. That is, T is nonexpansive if and only if T is a 0-strict pseudo-contraction. A mapping

$T : C \to C$ is said to be λ-strictly pseudo-contractive mapping with respect to p if, for all x, $y \in C$, there exists a constant $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^p \leq \|x - y\|^p + \lambda \|(I - T)x - (I - T)y\|^p. \tag{1.2}$$

A countable family of mapping $\{T_n : C \to C\}_{i=1}^{\infty}$ is called a *family of uniformly λ-strict pseudo-contractions with respect to p*, if there exists a constant $\lambda \in [0, 1)$ such that

$$\|T_n x - T_n y\|^p \leq \|x - y\|^p + \lambda \|(I - T_n)x - (I - T_n)y\|^p, \quad \forall x, y \in C, \ \forall n \geq 1. \tag{1.3}$$

We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

In order to find a fixed point of nonexpansive mapping T, Halpern [2] was the first to introduce the following iteration scheme which was referred to as Halpern iteration in a Hilbert space: u, $x_1 \in C$, $\{\alpha_n\} \subset [0, 1]$,

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \quad n \geq 1. \tag{1.4}$$

He pointed out that the control conditions (C1) $\lim_{n\to\infty} \alpha_n = 0$ and (C2) $\sum_{n=1}^{\infty} = \infty$ are necessary for the convergence of the iteration scheme (1.4) to a fixed point of T. Furthermore, the modified version of Halpern iteration was investigated widely by many mathematicians. Recently, for the sequence of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ with some special conditions, Aoyama et al. [3] introduced a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings $\{T_n : C \to C\}$ satisfying some conditions. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n \tag{1.5}$$

for all $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable, and $\{\alpha_n\}$ is a sequence in $[0, 1]$. They proved that $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $\{T_n\}$. Very recently, Song and Zheng [4] also studied the strong convergence theorem of Halpern iteration (1.5) for a countable family of nonexpansive mappings $\{T_n : C \to C\}$ satisfying some conditions in either a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm or a reflexive Banach space E with a weakly continuous duality mapping. Other investigations of approximating common fixed points for a countable family of nonexpansive mappings can be found in [3, 5–10] and many results not cited here.

On the other hand, in the last twenty years or so, there are many papers in the literature dealing with the iteration approximating fixed points of Lipschitz strongly pseudo-contractive mappings by using the Mann and Ishikawa iteration process. Results which had been known only for Hilbert spaces and Lipschitz mappings have been extended to more general Banach spaces and a more general class of mappings (see, e.g., [1, 11–13] and the references therein).

In 2007, Marino and Xu [12] proved that the Mann iterative sequence converges weakly to a fixed point of λ-strict pseudo-contractions in Hilbert spaces, which extend Reich's theorem [14, Theorem 2] from nonexpansive mappings to λ-strict pseudo-contractions in Hilbert spaces.

Recently, Zhou [13] obtained some weak and strong convergence theorems for λ-strict pseudo-contractions in Hilbert spaces by using Mann iteration and modified Ishikawa iteration which extend Marino and Xu's convergence theorems [12].

More recently, Hu and Wang [11] obtained that the Mann iterative sequence converges weakly to a fixed point of λ-strict pseudo-contractions with respect to p in p-uniformly convex Banach spaces. To be more precise, they obtained the following theorem.

Theorem HW

Let E be a real p-uniformly convex Banach space which satisfies one of the following:

(i) E has a Fréchet differentiable norm;

(ii) E satisfies Opial's property.

Let C a nonempty closed convex subset of E. Let $T : C \to C$ be a λ-strict pseudo-contractions with respect to p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $F(T) \neq \emptyset$. Assume that a real sequence $\{\alpha_n\}$ in $(0,1)$ satisfy the following conditions:

$$0 < \varepsilon \leq \alpha_n \leq 1 - \varepsilon < 1 - \frac{2^{p-2}\lambda}{c_p}, \quad \forall n \geq 1. \tag{1.6}$$

Then Mann iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 1, \end{aligned} \tag{1.7}$$

converges weakly to a fixed point of T.

Very recently, Hu [15] obtained strong convergence theorems on a mixed iteration scheme by the viscosity approximation methods for λ-strict pseudo-contractions in p-uniformly convex Banach spaces with uniformly Gâteaux differentiable norm. To be more precise, Hu [15] obtained the following theorem.

Theorem H. *Let E be a real p-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings. Let $T : C \to C$ be a λ-strict pseudo-contractions with respect to p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $F(T) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$,

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \xi$, *where* $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ *be the sequence generated by the following:*

$$x_1 = x \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n, \quad n \geq 1. \tag{1.8}$$

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

In this paper, motivated by Hu and Wang [11], Hu [15], Aoyama et al. [3] and Song and Zheng [4], we introduce a viscosity iterative approximation method for finding a common fixed point of a countable family of strictly pseudo-contractions which is a unique solution of some variational inequality. We prove the strong convergence theorems of such iterative scheme in either p-uniformly convex Banach space which admits a weakly continuous duality mapping or p-uniformly convex Banach space with uniformly Gâteaux differentiable norm. As applications, at the end of the paper, we apply our results to the problem of finding a zero of an accretive operator. The results presented in this paper improve and extend the corresponding results announced by Hu and Wang [11], Hu [15], Aoyama et al. [3] Song and Zheng [4], and many others.

2. Preliminaries

Throughout this paper, let E be a real Banach space and E^* its dual space. We write $x_n \rightharpoonup x$ (resp., $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (resp., weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence. Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space E. A Banach space E is said to have

(i) *a Gâteaux differentiable norm* (we also say that E is smooth), if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S(E)$,

(ii) *a uniformly Gâteaux differentiable norm*, if for each y in $S(E)$, the limit (2.1) is uniformly attained for $x \in S(E)$,

(iii) *a Fréchet differentiable norm*, if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$,

(iv) *a uniformly Fréchet differentiable norm* (we also say that E is uniformly smooth), if the limit (2.1) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

The modulus of convexity of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = 1, \ \|y\| = 1, \ \|x - y\| \geq \epsilon\right\}, \quad 0 \leq \epsilon \leq 2. \tag{2.2}$$

E is uniformly convex if and only if, for all $0 < \epsilon \leq 2$ such that $\delta_E(\epsilon) > 0$. E is said to be p-uniformly convex, if there exists a constant $a > 0$ such that $\delta_E(\epsilon) \geq a\epsilon^p$.

The following facts are well known which can be found in [16, 17]:

(i) the normalized duality mapping J in a Banach space E with a uniformly Gâteaux differentiable norm is single-valued and strong-weak* uniformly continuous on any bounded subset of E;

(ii) each uniformly convex Banach space E is reflexive and strictly convex and has fixed point property for nonexpansive self-mappings;

(iii) every uniformly smooth Banach space E is a reflexive Banach space with a uniformly Gâteaux differentiable norm and has fixed point property for nonexpansive self-mappings.

Now we collect some useful lemmas for proving the convergence result of this paper.

Lemma 2.1 (see [11]). *Let E be a real p-uniformly convex Banach space and C a nonempty closed convex subset of E. let $T : C \to C$ be a λ-strict pseudo-contraction with respect to p, and $\{\xi_n\}$ a real sequence in $[0,1]$. If $T_n : C \to C$ is defined by $T_n x := (1-\xi_n)x + \xi_n T x$, for all $x \in C$, then for all x, $y \in C$, the inequality holds*

$$\|T_n x - T_n y\|^p \le \|x-y\|^p - (w_p(\xi_n)c_p - \xi_n\lambda)\|(I-T)x - (I-T)y\|^p, \tag{2.3}$$

where c_p is a constant in [18, Theorem 1]. In addition, if $0 \le \lambda < \min\{1, 2^{-(p-2)}c_p\}$, $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$, and $\xi_n \in [0,\xi]$, then $\|T_n x - T_n y\| \le \|x-y\|$, for all x, $y \in C$.

Lemma 2.2 (see [19, 20]). *Let C be a nonempty closed convex subset of a Banach space E which has uniformly Gâteaux differentiable norm, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a k-contraction. Assume that every nonempty closed convex bounded subset of C has the fixed points property for nonexpansive mappings. Then there exists a continuous path: $t \to x_t$, $t \in (0,1)$ satisfying $x_t = tf(x_t) + (1-t)Tx_t$, which converges to a fixed point of T as $t \to 0^+$.*

Lemma 2.3 (see [21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in Banach space E such that*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \quad n \ge 0, \tag{2.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$. Assume

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{2.5}$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Definition 2.4 (see [3]). Let $\{T_n\}$ be a family of mappings from a subset C of a Banach space E into E with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. We say that $\{T_n\}$ satisfies the AKTT-condition if for each bounded subset B of C,

$$\sum_{n=1}^{\infty}\sup_{z\in B}\|T_{n+1}z - T_n z\| < \infty. \tag{2.6}$$

Remark 2.5. The example of the sequence of mappings $\{T_n\}$ satisfying AKTT-condition is supported by Lemma 4.1.

Lemma 2.6 (see [3, Lemma 3.2]). *Suppose that $\{T_n\}$ satisfies AKTT-condition. Then, for each $y \in C$, $\{T_n y\}$ converses strongly to a point in C. Moreover, let the mapping T be defined by*

$$Ty = \lim_{n\to\infty} T_n y, \quad \forall y \in C. \tag{2.7}$$

Then for each bounded subset B of C, $\lim_{n\to\infty} \sup_{z\in B}\|Tz - T_n z\| = 0$.

Lemma 2.7 (see [22]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \tag{2.8}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

By a gauge function φ we mean a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Let E^* be the dual space of E. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated to a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E. \tag{2.9}$$

In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = (\varphi(\|x\|)/\|x\|)J(x)$ for all $x \neq 0$ (see [23]). Browder [23] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Following Browder [23], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \geq 0, \tag{2.10}$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in E, \tag{2.11}$$

where ∂ denotes the subdifferential in the sense of convex analysis (recall that the subdifferential of the convex function $\phi : E \to \mathbb{R}$ at $x \in E$ is the set $\partial\phi(x) = \{x^* \in E^*;\ \phi(y) \geq \phi(x) + \langle x^*, y - x\rangle,\ \text{for all } y \in E\}$).

The following lemma is an immediate consequence of the subdifferential inequality. The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [24].

Lemma 2.8 (see [24]). *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y) \rangle. \tag{2.12}$$

In particular, in a smooth Banach space E, for all $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y) \rangle. \tag{2.13}$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$.*
Then the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y-x\|), \quad \forall x, y \in E. \tag{2.14}$$

3. Main Results

For $T : C \to C$ a nonexpansive mapping, $t \in (0,1)$ and $f \in \prod_C$, $tf + (1-t)T : C \to C$ defines a contraction mapping. Thus, by the Banach contraction mapping principle, there exists a unique fixed point x_t^f satisfying

$$x_t^f = tf(x_t) + (1-t)Tx_t^f. \tag{3.1}$$

For simplicity we will write x_t for x_t^f provided no confusion occurs. Next, we will prove the following lemma.

Lemma 3.1. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ. Let C be a nonempty closed convex subset of E, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \prod_C$. Then the net $\{x_t\}$ defined by* (3.1) *converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality:*

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x} - z) \rangle \leq 0, \quad z \in F(T). \tag{3.2}$$

Proof. We first show that the uniqueness of a solution of the variational inequality (3.2). Suppose both $\tilde{x} \in F(T)$ and $x^* \in F(T)$ are solutions to (3.2), then

$$\begin{aligned} &\langle (I-f)\tilde{x}, J_\varphi(\tilde{x} - x^*) \rangle \leq 0, \\ &\langle (I-f)x^*, J_\varphi(x^* - \tilde{x}) \rangle \leq 0. \end{aligned} \tag{3.3}$$

Adding (3.3), we obtain

$$\langle (I-f)\tilde{x}-(I-f)x^*,\ J_\varphi(\tilde{x}-x^*)\rangle \le 0. \tag{3.4}$$

Noticing that for any $x, y \in E$,

$$\begin{aligned}\langle (I-f)x-(I-f)y, J_\varphi(x-y)\rangle &= \langle x-y, J_\varphi(x-y)\rangle - \langle f(x)-f(y), J_\varphi(x-y)\rangle \\ &\ge \|x-y\|\varphi(\|x-y\|) - \|f(x)-f(y)\|\varphi(\|x-y\|) \\ &\ge \Phi(\|x-y\|) - \alpha\Phi(\|x-y\|) \\ &= (1-\alpha)\Phi(\|x-y\|) \ge 0.\end{aligned} \tag{3.5}$$

From (3.4), we conclude that $\Phi(\|\tilde{x}-x^*\|)=0$. This implies that $\tilde{x}=x^*$ and the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (3.2). Next, we will prove that $\{x_t\}$ is bounded. Take a $p \in F(T)$; then we have

$$\begin{aligned}\|x_t-p\| &= \|tf(x_t)+(1-t)Tx_t-p\| \\ &= \|(1-t)Tx_t-(1-t)p+t(f(x_t)-p)\| \\ &\le (1-t)\|x_t-p\| + t(\alpha\|x_t-p\| + \|f(p)-p\|).\end{aligned} \tag{3.6}$$

It follows that

$$\|x_t-p\| \le \frac{1}{1-\alpha}\|f(p)-p\|. \tag{3.7}$$

Hence $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{T(x_t)\}$. The definition of $\{x_t\}$ implies that

$$\|x_t-Tx_t\| = t\|f(x_t)-Tx_t\| \longrightarrow 0, \quad \text{as } t \longrightarrow 0. \tag{3.8}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_t\}$ that there exists $\{x_{t_n}\}$ which is a subsequence of $\{x_t\}$ converging weakly to $w \in C$ as $n \to \infty$. Since J_φ is weakly sequentially continuous, we have by Lemma 2.8 that

$$\limsup_{n\to\infty}\Phi(\|x_{t_n}-x\|) = \limsup_{n\to\infty}\Phi(\|x_{t_n}-w\|)+\Phi(\|x-w\|), \quad \forall x \in E. \tag{3.9}$$

Let

$$H(x) = \limsup_{n\to\infty}\Phi(\|x_{t_n}-x\|), \quad \forall x \in E. \tag{3.10}$$

It follows that

$$H(x) = H(w)+\Phi(\|x-w\|), \quad \forall x \in E. \tag{3.11}$$

Since

$$\|x_{t_n} - Tx_{t_n}\| = t_n \left\| f(x_{t_n}) - Tx_{t_n} \right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.12}$$

we obtain

$$\begin{aligned} H(Tw) &= \limsup_{n\to\infty} \Phi(\|x_{t_n} - Tw\|) = \limsup_{n\to\infty} \Phi(\|Tx_{t_n} - Tw\|) \\ &\leq \limsup_{n\to\infty} \Phi(\|x_{t_n} - w\|) = H(w). \end{aligned} \tag{3.13}$$

On the other hand, however,

$$H(Tw) = H(w) + \Phi(\|T(w) - w\|). \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\Phi(\|T(w) - w\|) = H(Tw) - H(w) \leq 0. \tag{3.15}$$

This implies that $Tw = w$. Next we show that $x_{t_n} \to w$ as $n \to \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau$, for all $t \geq 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t). \tag{3.16}$$

Following Lemma 2.8, we have

$$\begin{aligned} \Phi(\|x_{t_n} - w\|) &= \Phi\left(\left\|(1-t_n)Tx_{t_n} - (1-t_n)w + t_n\left(f(x_{t_n}) - w\right)\right\|\right) \\ &= \Phi(\|(1-t_n)Tx_{t_n} - (1-t_n)w\|) + t_n\left\langle f(x_{t_n}) - w,\ J(x_{t_n} - w)\right\rangle \\ &\leq \Phi((1-t_n)\|x_{t_n} - w\|) + t_n\left\langle f(x_{t_n}) - f(w),\ J(x_{t_n} - w)\right\rangle \\ &\quad + t_n\left\langle f(w) - w,\ J(x_{t_n} - w)\right\rangle \\ &\leq (1-t_n)\Phi(\|x_{t_n} - w\|) + t_n\left\| f(x_{t_n}) - f(w)\right\| \|J(x_{t_n} - w)\| \\ &\quad + t_n\left\langle f(w) - w,\ J(x_{t_n} - w)\right\rangle \\ &\leq (1-t_n)\Phi(\|x_{t_n} - w\|) + t_n\alpha\|x_{t_n} - w\|\left\|J_\varphi(x_{t_n} - w)\right\| \\ &\quad + t_n\left\langle f(w) - w,\ J(x_{t_n} - w)\right\rangle \\ &= (1-t_n)\Phi(\|x_{t_n} - w\|) + t_n\alpha\Phi(\|x_{t_n} - w\|) \\ &\quad + t_n\left\langle f(w) - w,\ J(x_{t_n} - w)\right\rangle \\ &= (1 - t_n(1-\alpha))\Phi(\|x_{t_n} - w\|) + t_n\left\langle f(w) - w,\ J(x_{t_n} - w)\right\rangle. \end{aligned} \tag{3.17}$$

This implies that

$$\Phi(\|x_{t_n} - w\|) \leq \frac{1}{1-\alpha}\langle f(w) - w,\ J(x_{t_n} - w)\rangle. \tag{3.18}$$

Now observing that $x_{t_n} \rightharpoonup w$ implies $J_\varphi(x_{t_n} - w) \rightharpoonup 0$, we conclude from the last inequality that

$$\Phi(\|x_{t_n} - w\|) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.19}$$

Hence $x_{t_n} \to w$ as $n \to \infty$. Next we prove that w solves the variational inequality (3.2). For any $z \in F(T)$, we observe that

$$\begin{aligned}\langle (I-T)x_t - (I-T)z,\ J_\varphi(x_t - z)\rangle &= \langle x_t - z,\ J_\varphi(x_t - z)\rangle + \langle Tx_t - Tz,\ J_\varphi(x_t - z)\rangle \\ &= \Phi(\|x_t - z\|) - \langle Tz - Tx_t,\ J_\varphi(x_t - z)\rangle \\ &\geq \Phi(\|x_t - z\|) - \|Tz - Tx_t\| \|J_\varphi(x_t - z)\| \\ &\geq \Phi(\|x_t - z\|) - \|z - x_t\| \|J_\varphi(x_t - z)\| \\ &= \Phi(\|x_t - z\|) - \Phi(\|x_t - z\|) = 0.\end{aligned} \tag{3.20}$$

Since

$$x_t = tf(x_t) + (1-t)Tx_t, \tag{3.21}$$

we can derive that

$$(I-f)(x_t) = -\frac{1}{t}(I-T)x_t + (I-T)x_t. \tag{3.22}$$

Thus

$$\begin{aligned}\langle (I-f)(x_t),\ J_\varphi(x_t - z)\rangle &= -\frac{1}{t}\langle (I-T)x_t - (I-T)z,\ J_\varphi(x_t - z)\rangle + \langle (I-T)x_t,\ J_\varphi(x_t - z)\rangle \\ &\leq \langle (I-T)x_t,\ J_\varphi(x_t - z)\rangle.\end{aligned} \tag{3.23}$$

Noticing that

$$x_{t_n} - Tx_{t_n} \longrightarrow w - T(w) = w - w = 0. \tag{3.24}$$

Now replacing t in (3.23) with t_n and letting $n \to \infty$, we have

$$\langle (I-f)w,\ J_\varphi(w - z)\rangle \leq 0. \tag{3.25}$$

So, $w \in F(T)$ is a solution of the variational inequality (3.2), and hence $w = \tilde{x}$ by the uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals $\tilde{x}$. Therefore, $x_t \to \tilde{x}$ as $t \to 0$. This completes the proof. □

Theorem 3.2. *Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_φ, and C a nonempty closed convex subset of E. Let $\{T_n : C \to C\}$ be a family of uniformly λ-strict pseudo-contractions with respect to p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \xi$, *where* $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ be the sequence generated by the following:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \geq 1. \end{aligned} \tag{3.26}$$

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality:

$$\left\langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - z) \right\rangle \leq 0, \quad z \in F(T). \tag{3.27}$$

Proof. Rewrite the iterative sequence (3.26) as follows:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n' x_n + \gamma_n' S_n x_n, \quad n \geq 1, \tag{3.28}$$

where $\beta_n' = \beta_n - (\gamma_n/\xi)(1-\xi)$, $\gamma_n' = \gamma_n/\xi$ and $S_n := (1-\xi)I + \xi T_n$, I is the identity mapping. By Lemma 2.1, S_n is nonexpansive such that $F(S_n) = F(T_n)$ for all $n \in \mathbb{N}$. Taking any $q \in \bigcap_{n=1}^{\infty} F(T_n)$, from (3.28), it implies that

$$\begin{aligned} \|x_{n+1} - q\| &\leq \alpha_n \|f(x_n) - q\| + \beta_n' \|x_n - q\| + \gamma_n' \|S_n x_n - q\| \\ &\leq \alpha_n k \|x_n - q\| + \alpha_n \|f(q) - q\| + (1 - \alpha_n)\|x_n - q\| \\ &= \alpha_n (1-k) \frac{1}{1-k} \|f(q) - q\| + (1 - \alpha_n(1-k))\|x_n - q\| \\ &\leq \max\left\{ \|x_1 - q\|, \frac{1}{1-k}\|f(q) - q\| \right\}. \end{aligned} \tag{3.29}$$

Therefore, the sequence $\{x_n\}$ is bounded, and so are the sequences $\{f(x_n)\}$, $\{S_n x_n\}$. Since $S_n x_n = (1-\xi_n)x_n + \xi_n T_n x_n$ and $\liminf \xi_n > 0$, we know that $\{T_n x_n\}$ is bounded. We note that for any bounded subset B of C,

$$\begin{aligned}\sup_{z\in B}\|S_{n+1}z - S_n z\| &= \sup_{z\in B}[\|((1-\xi_{n+1})z + \xi_{n+1}T_{n+1}z) - ((1-\xi_n)z + \xi_n T_n z)\|] \\ &\le |\xi_{n+1}-\xi_n|\sup_{z\in B}\|z\| + \xi_{n+1}\sup_{z\in B}\|T_{n+1}z - T_n z\| + |\xi_{n+1}-\xi_n|\sup_{z\in B}\|T_n z\| \\ &= |\xi_{n+1}-\xi_n|\sup_{z\in B}(\|z\| + \|Tz\|) + \xi_{n+1}\sup_{z\in B}\|T_{n+1}z - T_n z\|.\end{aligned} \tag{3.30}$$

From $\sum_{n=1}^{\infty}|\xi_{n+1}-\xi_n| < \infty$ and $\{T_n\}$ satisfing AKTT-condition, we obtain that

$$\sum_{n=1}^{\infty}\sup_{z\in B}\|S_{n+1}z - S_n z\| < \infty, \tag{3.31}$$

that is, the sequence $\{S_n\}$ satisfies AKTT-condition. Applying Lemma 2.6, we can take the mapping $S : C \to C$ defined by

$$Sz = \lim_{n\to\infty} S_n z, \quad \forall z \in C. \tag{3.32}$$

Moreover, we have S is nonexpansive and

$$Sz = \lim_{n\to\infty} S_n z = \lim_{n\to\infty}((1-\xi_n)z + \xi_n T_n z) = (1-\xi)z + \xi Tz. \tag{3.33}$$

It is easy to see that $F(S) = F(T)$. Hence $F(S) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(S_n)$. The iterative sequence (3.28) can be expressed as follows:

$$x_{n+1} = \beta'_n x_n + (1-\beta'_n) y_n, \tag{3.34}$$

where

$$y_n = \frac{\alpha_n}{1-\beta'_n} f(x_n) + \frac{\gamma'_n}{1-\beta'_n} S_n x_n. \tag{3.35}$$

We estimate from (3.35)

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \left\|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}f(x_{n+1})+\frac{\gamma'_{n+1}}{1-\beta'_{n+1}}S_{n+1}x_{n+1}-\frac{\alpha_n}{1-\beta'_n}f(x_n)+\frac{\gamma'_n}{1-\beta'_n}S_nx_n\right\| \\
&\le \frac{\alpha_{n+1}}{1-\beta'_{n+1}}k\|x_{n+1}-x_n\|+\frac{\gamma'_{n+1}}{1-\beta'_{n+1}}\|S_{n+1}x_{n+1}-S_nx_n\| \\
&\quad+\left|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}-\frac{\alpha_n}{1-\beta'_n}\right|\|f(x_n)-S_nx_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta'_{n+1}}k\|x_{n+1}-x_n\|+\frac{\gamma'_{n+1}}{1-\beta'_{n+1}}[\|S_{n+1}x_{n+1}-S_{n+1}x_n\|+\|S_{n+1}x_n-S_nx_n\|] \\
&\quad+\left|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}-\frac{\alpha_n}{1-\beta'_n}\right|\|f(x_n)-S_nx_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta'_{n+1}}k\|x_{n+1}-x_n\|+\frac{\gamma'_{n+1}}{1-\beta'_{n+1}}\left[\|x_{n+1}-x_n\|+\sup_{z\in\{x_n\}}\|S_{n+1}z-S_nz\|\right] \\
&\quad+\left|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}-\frac{\alpha_n}{1-\beta'_n}\right|\|f(x_n)-S_nx_n\|.
\end{aligned} \tag{3.36}$$

Hence

$$\begin{aligned}
\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| &\le \frac{\alpha_{n+1}}{1-\beta'_{n+1}}k\|x_{n+1}-x_n\|+\frac{\gamma'_{n+1}}{1-\beta'_{n+1}}\sup_{z\in\{x_n\}}\|S_{n+1}z-S_nz\| \\
&\quad+\left|\frac{\alpha_{n+1}}{1-\beta'_{n+1}}-\frac{\alpha_n}{1-\beta'_n}\right|\|f(x_n)-S_nx_n\|.
\end{aligned} \tag{3.37}$$

Since $\lim_{n\to\infty}\alpha_n=0$, and $\lim_{n\to\infty}\sup_{z\in\{x_n\}}\|S_{n+1}z-S_nz\|=0$, we have from (3.37) that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.38}$$

Hence, by Lemma 2.3, we obtain

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.39}$$

From (3.35), we get

$$\lim_{n\to\infty}\|y_n-S_nx_n\|=\lim_{n\to\infty}\frac{\alpha_n}{1-\beta'_n}\|f(x_n)-S_nx_n\|=0, \tag{3.40}$$

and so it follows from (3.39) and (3.40) that

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{3.41}$$

It follows from Lemma 2.6 and (3.41), we have

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \\ &\le \|x_n - S_n x_n\| + \sup\{\|S_n z - Sz\| : z \in \{x_n\}\} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.42}$$

Since S is a nonexpansive mapping, we have from Lemma 3.1 that the net $\{x_t\}$ generated by

$$x_t = tf(x_t) + (1-t)Sx \tag{3.43}$$

converges strongly to $\tilde{x} \in F(S)$, as $t \to 0^+$. Next, we prove that

$$\limsup_{n\to\infty} \langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \le 0. \tag{3.44}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty} \langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle = \limsup_{n\to\infty} \langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_n - \tilde{x})\rangle. \tag{3.45}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in C$ as $i \to \infty$. Since J_φ is weakly continuous, we have by Lemma 2.8 that

$$\limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - x\right\|\right) = \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - w\right\|\right) + \Phi(\|x - w\|), \quad \forall x \in E. \tag{3.46}$$

Let

$$H(x) = \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - x\right\|\right), \quad \forall x \in E. \tag{3.47}$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \quad \forall x \in E. \tag{3.48}$$

From (3.42), we obtain

$$\begin{aligned}H(Sw) &= \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - Sw\right\|\right) = \limsup_{i\to\infty} \Phi\left(\left\|Sx_{n_{k_i}} - Sw\right\|\right) \\ &\le \limsup_{i\to\infty} \Phi\left(\left\|x_{n_{k_i}} - w\right\|\right) = H(w).\end{aligned} \tag{3.49}$$

On the other hand, however,

$$H(Sw) = H(w) + \Phi(\|S(w) - w\|). \tag{3.50}$$

It follows from (3.49) and (3.50) that

$$\Phi(\|S(w) - w\|) = H(Sw) - H(w) \leq 0. \tag{3.51}$$

This implies that $Sw = w$, that is, $w \in F(S) = F(T)$. Since the duality map J_φ is single-valued and weakly continuous, we get that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_n - \tilde{x})\rangle &= \lim_{k\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle \\
&= \lim_{i\to\infty}\left\langle f(\tilde{x}) - \tilde{x}, J_\varphi\left(x_{n_{k_i}} - \tilde{x}\right)\right\rangle \\
&= \langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - w)\rangle \leq 0
\end{aligned} \tag{3.52}$$

as required. Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$.

$$\begin{aligned}
\Phi(\|x_{n+1} - \tilde{x}\|) &= \Phi\left(\left\|\alpha_n\left(f(x_n) - f(\tilde{x})\right) + \beta_n'(x_n - \tilde{x}) + \gamma_n'(S_n x_n - \tilde{x}) + \alpha_n\left(f(\tilde{x}) - \tilde{x}\right)\right\|\right) \\
&\leq \Phi\left(\left\|\alpha_n\left(f(x_n) - f(\tilde{x})\right) + \beta_n'(x_n - \tilde{x}) + \gamma_n'(S_n x_n - \tilde{x})\right\|\right) \\
&\quad + \alpha_n\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\leq \Phi\left(\alpha_n k\|x_n - \tilde{x}\| + \beta_n'\|x_n - \tilde{x}\| + \gamma_n'\|x_n - \tilde{x}\|\right) \\
&\quad + \alpha_n\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&= \Phi((1 - \alpha_n(1-k))\|x_n - \tilde{x}\|) + \alpha_n\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\leq (1 - \alpha_n(1-k))\Phi(\|x_n - \tilde{x}\|) + \alpha_n\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle.
\end{aligned} \tag{3.53}$$

It follows that from condition (i) and (3.44) that

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=1}^{\infty}\alpha_n = \infty, \qquad \limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \leq 0. \tag{3.54}$$

Apply Lemma 2.7 to (3.53) to conclude $\Phi(\|x_{n+1} - \tilde{x}\|) \to 0$ as $n \to \infty$; that is, $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

If $\{T_n : C \to C\}$ is a family of nonexpansive mappings, then we obtain the following results.

Corollary 3.3. *Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_φ, and C a nonempty closed convex subset of E. Let $\{T_n : C \to C\}$ be a family of* *nonexpansive mappings such that* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$*. Let* $f : C \to C$ *be a* k*-contraction with* $k \in (0,1)$*. Assume that real sequences* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$*;*

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Let $\{x_n\}$ *be the sequence generated by the following:*

$$x_1 = x \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \geq 1. \tag{3.55}$$

Suppose that $\{T_n\}$ *satisfies the AKTT-condition. Let* T *be a mapping of* C *into itself defined by* $Tz = \lim_{n\to\infty} T_n z$ *for all* $z \in C$ *and suppose that* $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$*. Then the sequence* $\{x_n\}$ *converges strongly* $\tilde{x}$ *which solves the variational inequality:*

$$\langle (I - f)\tilde{x}, J_{\varphi}(\tilde{x} - z) \rangle \leq 0, \quad z \in F(T). \tag{3.56}$$

Corollary 3.4. *Let* E *be a real* p*-uniformly convex Banach space with a weakly continuous duality mapping* J_{φ}*, and* C *a nonempty closed convex subset of* E*. Let* $T : C \to C$ *be a* λ*-strict pseudo-contraction with respect to* p*,* $\lambda \in [0, \min\{1, 2^{-(p-2)} c_p\})$ *and* $F(T) \neq \emptyset$*. Let* $f : C \to C$ *be a* k*-contraction with* $k \in (0,1)$*. Assume that real sequences* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$*;*

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \xi$*, where* $\xi = 1 - 2^{p-2} \lambda c_p^{-1}$.

Let $\{x_n\}$ *be the sequence generated by the following*

$$x_1 = x \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n, \quad n \geq 1. \tag{3.57}$$

Then the sequence $\{x_n\}$ *converges strongly to* $\tilde{x}$ *which solves the following variational inequality:*

$$\langle (I - f)\tilde{x}, J_{\varphi}(\tilde{x} - z) \rangle \leq 0, \quad z \in F(T). \tag{3.58}$$

Theorem 3.5. *Let* E *be a real* p*-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and* C *a nonempty closed convex subset of* E *which has the fixed point property for nonexpansive mappings. Let* $\{T_n : C \to C\}$ *be a family of uniformly*λ*-strict pseudo-contractions with respect to*

p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \xi$, *where* $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ be the sequence generated by the following:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \geq 1. \end{aligned} \tag{3.59}$$

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point $\widetilde{x}$ of $\{T_n\}$.

Proof. It follows from the same argumentation as Theorem 3.2 that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$, where S is a nonexpansive mapping defined by (3.32). From Lemma 2.2 that the net $\{x_t\}$ generated by $x_t = tf(x_t) + (1-t)Sx_t$ converges strongly to $\widetilde{x} \in F(S) = F(T)$, as $t \to 0^+$. Obviously,

$$x_t - x_n = (1-t)(Sx_t - x_n) + t\big(f(x_t) - x_n\big). \tag{3.60}$$

In view of Lemma 2.8, we calculate

$$\begin{aligned} \|x_t - x_n\|^2 &\leq (1-t)^2\|Sx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\ &\leq \left(1 - 2t + t^2\right)(\|x_t - x_n\| + \|Sx_n - x_n\|)^2 \\ &\quad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2 \end{aligned} \tag{3.61}$$

and therefore

$$\langle f(x_t) - x_t, J(x_n - x_t)\rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{(1+t)^2\|x_n - Sx_n\|}{2t}(2\|x_t - x_n\| + \|x_n - Sx_n\|). \tag{3.62}$$

Since $\{x_n\}$, $\{x_t\}$ and $\{Sx_n\}$ are bounded and $\lim_{n\to\infty}(\|x_n - Sx_n\|/2t) = 0$, we obtain

$$\limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle \leq \frac{t}{2}M, \tag{3.63}$$

where $M = \sup_{n\geq 1, t\in(0,1)}\{\|x_t - x_n\|^2\}$. We also know that

$$\begin{aligned} \langle f(\widetilde{x}) - \widetilde{x}, J(x_n - \widetilde{x})\rangle &= \langle f(x_t) - x_t, J(x_n - x_t)\rangle + \langle f(\widetilde{x}) - f(x_t) + x_t - \widetilde{x}, J(x_n - x_t)\rangle \\ &\quad + \langle f(\widetilde{x}) - \widetilde{x}, J(x_n - \widetilde{x}) - J(x_n - x_t)\rangle. \end{aligned} \tag{3.64}$$

From the fact that $x_t \to \tilde{x} \in F(T)$, as $t \to 0$, $\{x_n\}$ is bounded and the duality mapping J is norm-to-weak* uniformly continuous on bounded subset of E, it follows that as $t \to 0$,

$$\begin{aligned} &\langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x}) - J(x_n - x_t)\rangle \longrightarrow 0, \quad \forall n \in \mathbb{N}, \\ &\langle f(\tilde{x}) - f(x_t) + x_t - \tilde{x}, J(x_n - x_t)\rangle \longrightarrow 0, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.65}$$

Combining (3.63), (3.64) and two results mentioned above, we get

$$\limsup_{n\to\infty} \langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x})\rangle \le 0. \tag{3.66}$$

From (3.28) and Lemma 2.8, we get

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &\le \left\|\alpha_n(f(x_n) - f(\tilde{x})) + \beta_n'(x_n - \tilde{x}) + \gamma_n'(S_n x_n - \tilde{x})\right\|^2 \\ &\quad + 2\alpha_n\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\ &\le (1 - \alpha_n(1-k))\|x_n - \tilde{x}\|^2 + 2\alpha_n\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle. \end{aligned} \tag{3.67}$$

Hence applying in Lemma 2.7 to (3.67), we conclude that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. □

Corollary 3.6. *Let E be a real p-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings. Let $\{T_n : C \to C\}$ be a family of nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = +\infty$;

(iii) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$.

Let $\{x_n\}$ be the sequence generated by the following:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n, \quad n \ge 1. \end{aligned} \tag{3.68}$$

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point $\tilde{x}$ of $\{T_n\}$.

Corollary 3.7. *Let E be a real p-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of E which has the fixed point property for nonexpansive mappings. Let $T : C \to C$ be a λ-strict pseudo-contractions with respect to*

p, $\lambda \in [0, \min\{1, 2^{-(p-2)}c_p\})$ and $F(T) \neq \emptyset$. Let $f : C \to C$ be a k-contraction with $k \in (0,1)$. Assume that real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \xi$, *where* $\xi = 1 - 2^{p-2}\lambda c_p^{-1}$.

Let $\{x_n\}$ be the sequence generated by the following:

$$x_1 = x \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T x_n, \quad n \geq 1. \tag{3.69}$$

Then the sequence $\{x_n\}$ converges strongly to a common fixed point $\tilde{x}$ of $\{T_n\}$.

4. Some Applications for Accretive Operators

We consider the problem of finding a zero of an accretive operator. An operator $\Psi \subset E \times E$ is said to be accretive if for each (x_1, y_1) and $(x_2, y_2) \in \Psi$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. An accretive operator Ψ is said to satisfy the range condition if $\overline{D(\Psi)} \subset R(I + \lambda\Psi)$ for all $\lambda > 0$, where $D(\Psi)$ is the domain of Ψ, I is the identity mapping on E, $R(I + \lambda\Psi)$ is the range of $I + \lambda\Psi$, and $\overline{D(\Psi)}$ is the closure of $D(\Psi)$. If Ψ is an accretive operator which satisfies the range condition, then we can define, for each $\lambda > 0$, a mapping $J_\lambda : R(I + \lambda\Psi) \to D(\Psi)$ by $J_\lambda = (I + \lambda\Psi)^{-1}$, which is called the resolvent of Ψ. We know that J_λ is nonexpansive and $F(J_\lambda) = \Psi^{-1}(0)$ for all $\lambda > 0$. We also know the following [25]: For each λ, $\mu > 0$ and $x \in R(I + \lambda\Psi) \cap R(I + \mu\Psi)$, it holds that

$$\left\| J_\lambda x - J_\mu x \right\| \leq \frac{|\lambda - \mu|}{\lambda} \|x - J_\lambda x\|. \tag{4.1}$$

By the proof of Theorem 4.3 in [3], we have the following lemma.

Lemma 4.1. *Let E be a Banach space and C a nonempty closed convex subset of E. Let $\Psi \subseteq E \times E$ be an accretive operator such that $\Psi^{-1}0 \neq \emptyset$ and $\overline{D(\Psi)} \subset C \subset \bigcap_{\lambda>0} R(I + \lambda\Psi)$. Suppose that $\{\lambda_n\}$ is a sequence of $(0,\infty)$ such that $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then*

(i) *The sequence $\{J_{\lambda_n}\}$ satisfies the AKTT-condition.*

(ii) $\lim_{n\to\infty} J_{\lambda_n} z = J_\lambda z$ *for all* $z \in C$ *and* $F(J_\lambda) = \bigcap_{n=1}^{\infty} F(J_{\lambda_n})$ *where* $\lambda_n \to \lambda$ *as* $n \to \infty$.

By Corollary 3.3, we obtain the following result.

Theorem 4.2. *Let E be a real p-uniformly convex Banach space with a weakly continuous duality mapping J_φ, and C a nonempty closed convex subset of E. Let Ψ is an m-accretive operator in E such* *that* $\Psi^{-1}0 \neq \emptyset$. *Let* $f : C \to C$ *be a* k*-contraction with* $k \in (0,1)$. *Assume that real sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;

(iv) $\{\lambda_n\}$ *is a sequence of* $(0,\infty)$ *such that* $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Let $\{x_n\}$ *be the sequence generated by the following:*

$$x_1 = x \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n} x_n, \quad n \geq 1. \tag{4.2}$$

Then the sequence $\{x_n\}$ *converges strongly* $\tilde{x}$ *which solves the following variational inequality:*

$$\left\langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - z) \right\rangle \leq 0, \quad z \in F(J_\lambda). \tag{4.3}$$

By Corollary 3.6, we obtain the following result.

Theorem 4.3. *Let E be a real* p*-uniformly convex Banach space with uniformly Gâteaux differentiable norm, and C a nonempty closed convex subset of E. Let* Ψ *is an* m*-accretive operator in E such that* $\Psi^{-1}0 \neq \emptyset$. *Let* $f : C \to C$ *be a* k*-contraction with* $k \in (0,1)$. *Assume that real sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;

(iv) $\{\lambda_n\}$ *is a sequence of* $(0,\infty)$ *such that* $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Let $\{x_n\}$ *be the sequence generated by the following:*

$$x_1 = x \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n} x_n, \quad n \geq 1. \tag{4.4}$$

Then the sequence $\{x_n\}$ *converges strongly* $\tilde{x}$ *in* $\Psi^{-1}0$.

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The general iterative methods for nonexpansive mappings in Banach Spaces

R. Wangkeeree, N. Petrot, R. Wangkeeree

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The general iterative methods for nonexpansive mappings in Banach spaces

Rattanaporn Wangkeeree · Narin Petrot · Rabian Wangkeeree

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Abstract In this paper, we introduce a general iterative approximation method for finding a common fixed point of a countable family of nonexpansive mappings which is a unique solution of some variational inequality. We prove the strong convergence theorems of such iterative scheme in a reflexive Banach space which admits a weakly continuous duality mapping. As applications, at the end of the paper, we apply our results to the problem of finding a zero of an accretive operator. The main result extends various results existing in the current literature.

Keywords Iterative approximation method · Nonexpansive mapping · Strong convergence theorem · Banach space · Common fixed point

Mathematics Subject Classification (2000) 47H05 · 47H09 · 47J25 · 65J15

1 Introduction

In recent years, the existence of common fixed points for a finite family of nonexpansive mappings has been considered by many authors (see [1,2,24,25] and the references therein). The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [3,7]). The problem of finding an optimal point that minimizes a given cost function over the common set of fixed

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R. Wangkeeree · N. Petrot · R. Wangkeeree (✉)
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
e-mail: rabianw@nu.ac.th

R. Wangkeeree
e-mail: rattanapornw@nu.ac.th

N. Petrot
e-mail: narinp@nu.ac.th

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points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance (see [2,6,19,33]). A simple algorithmic solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation (see [11,33]).

Let E be a normed linear space. Recall that a mapping $T : E \to E$ is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \ \ \forall x, y \in E. \tag{1.1}$$

We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in E : Tx = x\}$. A self mapping $f : E \to E$ is a contraction on E if there exists a constant $\alpha \in (0, 1)$ and $x, y \in E$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|. \tag{1.2}$$

We use Π_E to denote the collection of all contractions on E. That is, $\Pi_E = \{f : f \text{ is a contraction on } E\}$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping ([4,23,32]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : E \to E$ by

$$T_t x = tu + (1 - t)Tx, \ \ \forall x \in E, \tag{1.3}$$

where $u \in E$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in E. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [4] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [23] extended Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from E onto $F(T)$. Xu [32] proved Reich's results hold in reflexive Banach spaces which have a weakly continuous duality mapping.

In the last ten years or so, the iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [8,29,30] and the references therein. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \ge \bar{\gamma}\|x\|^2 \ \text{ for all } x \in H. \tag{1.4}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. In 2003, Xu ([30]) proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \ \ n \ge 0, \tag{1.6}$$

converges strongly to the unique solution of the minimization problem (1.5) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [16] introduced the following iterative iterative process for nonexpansive mappings (see [20,31] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \ \ n \geq 0, \tag{1.7}$$

where $\{\sigma_n\}$ is a sequence in (0, 1). It is proved [16,31] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.7) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, x \in H. \tag{1.8}$$

Recently, Marino and Xu [15] mixed the iterative method (1.6) and the viscosity approximation method (1.7) and considered the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.9}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the following conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and
(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

then the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution x^* in H of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, x \in H \tag{1.10}$$

which is the optimality condition for the minimization problem: $\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where h is a potential function for $\gamma f (i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

On the other hand, in order to finding a fixed point of nonexpansive mapping T, Halpern [10] was the first who introduced the following iteration scheme which was referred to as Halpern iteration in a Hilbert space : $u, x_0 \in C$, $\{\alpha_n\} \subset [0, 1]$,

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, n \geq 0. \tag{1.11}$$

He pointed out that the control conditions (C1) $\lim_{n\to\infty} \alpha_n = 0$ and (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the convergence of the iteration scheme (1.11) to a fixed point of T. Furthermore, the modified version of Halpern iteration was investigated widely by many mathematicians. Recently, for the sequence of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ with some special conditions, Aoyama et al. [1] introduced a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings $\{T_n : C \to C\}$ satisfying some conditions. Let $x_0 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T_n x_n, \ \ n \geq 0, \tag{1.12}$$

where C is a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gáteaux differentiable, $\{\alpha_n\}$ is a sequence in [0, 1]. They proved that $\{x_n\}$ defined by (1.12) converges strongly to a common fixed point of $\{T_n\}$ provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Very recently, Song and Zheng [26] also studied the strong convergence theorem of Halpern iteration (1.12) for a countable family of nonexpansive mappings $\{T_n : C \to C\}$ satisfying some conditions in either a reflexive and strictly convex Banach space with a uniformly Gáteaux differentiable norm or a reflexive Banach space E with a weakly continuous duality mapping. Other investigations of approximating common fixed points for a countable family of nonexpansive mappings can be found in Refs. [1,12,14,17,18] and many results not cited here.

All of the above bring us the following conjectures?

Question 1 Can Theorem of Marino and Xu [15] be extended from Hilbert space to a general Banach space? such as reflexive Banach space.

Question 2 Can we extend the iterative method of algorithm (1.9) to a general algorithm defined by a countable family of nonexpansive mappings?.

Question 3 Could we weaken or remove the control condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on parameter $\{\alpha_n\}$ in [15, Theorem 3.4]?.

In this paper, motivated by Marino and Xu [15], Aoyama et al. [1] and Song and Zheng [26], we introduce a general iterative approximation method for finding a common fixed point of a countable family of nonexpansive mappings which is a unique solution of some variational inequality. We prove the strong convergence theorems of such iterative scheme in a reflexive Banach space which admits a weakly continuous duality mapping. As applications, at the end of the paper, we apply our results to the problem of finding a zero of an accretive operator. The results presented in this paper improve and extend the corresponding results announced by Marino and Xu [15], Aoyama et al. [1] and many others.

2 Preliminaries

Throughout this paper, let E be a real Banach space and E^* be its dual space. We write $x_n \rightharpoonup x$ (respectively $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (respectively weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence. Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [28]). A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$.

By a gauge function φ we mean a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Let E^* be the dual space of E. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated to a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|\varphi(\|x\|),\ \ \|f^*\| = \varphi(\|x\|)\},\ \ \forall x \in E.$$

In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ for all $x \neq 0$ (see [5]). Browder [5] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Following Browder [5], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that l^p has a weakly continuous duality mapping with a gauge function

$\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \forall t \geq 0,$$

then

$$J_\varphi(x) = \partial \Phi(\|x\|), \forall x \in E,$$

where ∂ denotes the sub-differential in the sense of convex analysis.

Now we collect some useful lemmas for proving the convergence result of this paper.

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [13].

Lemma 2.1 ([13]) *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) For all $x, y \in E$, the following inequality holds:

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y) \rangle.$$

In particular, for all $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle.$$

(ii) Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E.$$

Lemma 2.2 ([29]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n) a_n + b_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\limsup_{n\to\infty} b_n/\alpha_n \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([27]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that*

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) y_n, n \geq 0$$

where $\{\lambda_n\}$ is a sequence in $[0, 1]$ such that

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1.$$

Assume

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Definition 2.4 Let $\{T_n\}$ be a family of mappings from a subset C of a Banach space E into E with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. We say that

(i) $\{T_n\}$ satisfies the AKTT-condition [1] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup_{z \in B} \|T_{n+1}z - T_n z\| < \infty. \tag{2.1}$$

(ii) $\{T_n\}$ satisfies the PU-condition [21] if for each bounded subset D of C, there exists a continuous increasing and convex function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$h(0) = 0 \text{ and } \lim_{k,l \to \infty} \sup_{z \in D} h(\|T_k z - T_l z\|) = 0. \tag{2.2}$$

Remark 2.5 (i) The example of the sequence of mappings $\{T_n\}$ satisfying AKTT-condition and PU-condition is supported by Lemma 4.1, respectively.

(ii) If $\{T_n\}$ satisfies the AKTT-condition, then $\{T_n\}$ satisfies the PU-condition by Remak 3.2 in [21].

Lemma 2.6 [21, Lemma 3.1] *Suppose that there exists a continuous increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (2.2). Then*

(i) For each $x \in C$, $\{T_n x\}$ is a convergent sequence in C.
(ii) Let the mapping $T : C \to C$ be defined by

$$Tx = \lim_{n \to \infty} T_n x, \text{ for all } x \in C. \tag{2.3}$$

Then $\lim_{n\to\infty} \sup_{z \in D} h(\|Tz - T_n z\|) = 0$ for each bounded subset D of C. Moreover, the properties of h imply that $\lim_{n\to\infty} \sup_{z\in D} \|Tz - T_n z\| = 0$.

3 Main results

In a Banach space E having a weakly continuous duality mapping J_φ with a gauge function φ, an operator A is said to be *strongly positive* if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x)\rangle \geq \bar{\gamma}\|x\|\varphi(\|x\|) \tag{3.1}$$

and

$$\|\alpha I - \beta A\| = \sup_{\|x\| \leq 1} |\langle(\alpha I - \beta A)x, J_\varphi(x)\rangle|, \quad \alpha \in [0, 1], \beta \in [-1, 1], \tag{3.2}$$

where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (3.1) reduce to (1.4). The next valuable lemma is proved for applying our main results.

Lemma 3.1 *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ. Let A be a strong positive linear bounded operator on E with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\bar{\gamma})$.*

Proof From (3.2), we obtain that $\|A\| = \sup_{\|x\|\leq 1} |\langle Ax, J_\varphi(x)\rangle|$. Now for any $x \in E$ with $\|x\| = 1$, we see that

$$\langle(I - \rho A)x, J_\varphi(x)\rangle = \varphi(1) - \rho\langle Ax, J_\varphi(x)\rangle \geq \varphi(1) - \rho\|A\| \geq 0.$$

That is $I-\rho A$ is positive. It follows that

$$\begin{aligned}\|I-\rho A\| &= \sup\{\langle (I-\rho A)x, J_\varphi(x)\rangle : x\in E, \|x\|=1\}\\ &= \sup\{\varphi(1)-\rho\langle Ax, J_\varphi(x)\rangle : x\in E, \|x\|=1\}\\ &\le \varphi(1)-\rho\bar{\gamma}\varphi(1)=\varphi(1)(1-\rho\bar{\gamma}).\end{aligned}$$

□

Let E be a Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on [0, 1] i.e. $T([0,1])\subset[0,1]$. For any nonexpansive mapping $T:E\to E$, $t\in(0,1)$, $f\in\Pi_E$ and A is a strongly positive bounded linear operator with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\bar{\gamma}\varphi(1)}{\alpha}$, the mapping $S_t:E\to E$ defined by

$$S_t(x)=t\gamma f(x)+(I-tA)Tx, \forall x\in E$$

is a contraction mapping. Indeed, for any $x,y\in E$,

$$\begin{aligned}\|S_t(x)-S_t(y)\| &= \|t\gamma(f(x)-f(y))+(I-tA)(Tx-Ty)\|\\ &\le t\gamma\|f(x)-f(y)\|+\|I-tA\|\|Tx-Ty\|\\ &\le t\gamma\alpha\|x-y\|+\varphi(1)(1-t\bar{\gamma})\|x-y\|\\ &\le [1-t(\varphi(1)\bar{\gamma}-\gamma\alpha)]\|x-y\|.\end{aligned}\tag{3.3}$$

Thus, by Banach contraction mapping principle, there exists a unique fixed point x_t in E that is

$$x_t=t\gamma f(x_t)+(I-tA)Tx_t.\tag{3.4}$$

Remark 3.2 We note that l^p space has a weakly continuous duality mapping with a gauge function $\varphi(t)=t^{p-1}$ for all $1<p<\infty$. This show that φ is invariant on [0, 1].

Lemma 3.3 *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on* [0, 1]. *Let $T:E\to E$ be a nonexpansive mapping with $F(T)\neq\emptyset$ and $f\in\Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\bar{\gamma}\varphi(1)}{\alpha}$. Then the net $\{x_t\}$ defined by (3.4) converges strongly as $t\to 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality:*

$$\langle (A-\gamma f)\tilde{x}, J_\varphi(\tilde{x}-z)\rangle\le 0, z\in F(T).\tag{3.5}$$

Proof We first show that the uniqueness of a solution of the variational inequality (3.5). Suppose both $\tilde{x}\in F(T)$ and $x^*\in F(T)$ are solutions to (3.5), then

$$\langle (A-\gamma f)\tilde{x}, J_\varphi(\tilde{x}-x^*)\rangle\le 0\tag{3.6}$$

and

$$\langle (A-\gamma f)x^*, J_\varphi(x^*-\tilde{x})\rangle\le 0.\tag{3.7}$$

Adding (3.6) and (3.7), we obtain

$$\langle (A-\gamma f)\tilde{x}-(A-\gamma f)x^*, J_\varphi(\tilde{x}-x^*)\rangle\le 0.\tag{3.8}$$

Noticing that for any $x, y \in E$,

$$
\begin{aligned}
&\langle (A-\gamma f)x-(A-\gamma f)y, J_\varphi(x-y)\rangle \\
&\quad = \langle A(x-y), J_\varphi(x-y)\rangle - \gamma\langle f(x)-f(y), J_\varphi(x-y)\rangle \\
&\quad \geq \bar{\gamma}\|x-y\|\varphi(\|x-y\|) - \gamma\|f(x)-f(y)\|\|J_\varphi(x-y)\| \\
&\quad \geq \bar{\gamma}\Phi(\|x-y\|) - \gamma\alpha\Phi(\|x-y\|) \\
&\quad = (\bar{\gamma}-\gamma\alpha)\Phi(\|x-y\|) \\
&\quad \geq (\bar{\gamma}\varphi(1)-\gamma\alpha)\Phi(\|x-y\|) \geq 0. \qquad (3.9)
\end{aligned}
$$

Therefore $\tilde{x} = x^*$ and the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (3.5). Next, we will prove that $\{x_t\}$ is bounded. Take a $p \in F(T)$, then we have

$$
\begin{aligned}
\|x_t - p\| &= \|t\gamma f(x_t) + (I - tA)Tx_t - p\| \\
&= \|(I-tA)Tx_t - (I-tA)p + t(\gamma f(x_t) - A(p))\| \\
&\leq \varphi(1)(1-t\bar{\gamma})\|x_t-p\| + t(\gamma\alpha\|x_t-p\| + \|\gamma f(p) - A(p)\|).
\end{aligned}
$$

It follows that

$$
\|x_t - p\| \leq \frac{1}{\bar{\gamma}\varphi(1)-\gamma\alpha}\|\gamma f(p) - A(p)\|.
$$

Hence $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{AT(x_t)\}$. The definition of $\{x_t\}$ implies that

$$
\|x_t - Tx_t\| = t\|\gamma f(x_t) - A(Tx_t)\| \to 0 \text{ as } t \to 0. \qquad (3.10)
$$

If follows from reflexivity of E and the boundedness of sequence $\{x_t\}$ that there exists $\{x_{t_n}\}$ which is a subsequence of $\{x_t\}$ converging weakly to $w \in E$ as $n \to \infty$. Since J_φ is weakly sequentially continuous, we have by Lemma 2.1 that

$$
\limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - w\|) + \Phi(\|x - w\|), \text{ for all } x \in E.
$$

Let

$$
H(x) = \limsup_{n\to\infty} \Phi(\|x_{t_n} - x\|), \text{ for all } x \in E.
$$

It follows that

$$
H(x) = H(w) + \Phi(\|x - w\|), \text{ for all } x \in E.
$$

Since

$$
\|x_{t_n} - Tx_{t_n}\| = t_n\|\gamma f(x_{t_n}) - A(Tx_{t_n})\| \to 0 \text{ as } n \to \infty.
$$

We obtain

$$
\begin{aligned}
H(Tw) &= \limsup_{n\to\infty} \Phi(\|x_{t_n} - Tw\|) = \limsup_{n\to\infty} \Phi(\|Tx_{t_n} - Tw\|) \\
&\leq \limsup_{n\to\infty} \Phi(\|x_{t_n} - w\|) = H(w). \qquad (3.11)
\end{aligned}
$$

On the other hand, however,

$$
H(Tw) = H(w) + \Phi(\|T(w) - w\|). \qquad (3.12)
$$

It follows from (3.11) and (3.12) that

$$
\Phi(\|T(w) - w\|) = H(Tw) - H(w) \leq 0.
$$

This implies that $Tw = w$. Next we show that $x_{t_n} \to w$ as $n \to \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau, \forall t \geq 0$, and $\varphi : [0, \infty) \to [0, \infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k \int_0^t \varphi(kx)dx \leq k \int_0^t \varphi(x)dx = k\Phi(t).$$

Following Lemma 2.1, we have

$$\begin{aligned}
\Phi(\|x_{t_n} - w\|) &= \Phi((I - t_nA)Tx_{t_n} - (I - t_nA)w + t_n(\gamma f(x_{t_n}) - A(w))) \\
&= \Phi(\|(I - t_nA)Tx_{t_n} - (I - t_nA)w\|) + t_n\langle \gamma f(x_{t_n}) - A(w), J_\varphi(x_{t_n} - w)\rangle \\
&\leq \Phi(\varphi(1)(1 - t_n\bar{\gamma})\|x_{t_n} - w\|) + t_n\gamma\langle f(x_{t_n}) - f(w), J_\varphi(x_{t_n} - w)\rangle \\
&\quad + t_n\langle \gamma f(w) - A(w), J_\varphi(x_{t_n} - w)\rangle \\
&\leq \varphi(1)(1 - t_n\bar{\gamma})\Phi(\|x_{t_n} - w\|) + t_n\gamma\|f(x_{t_n}) - f(w)\|\|J_\varphi(x_{t_n} - w)\| \\
&\quad + t_n\langle \gamma f(w) - A(w), J_\varphi(x_{t_n} - w)\rangle \\
&\leq \varphi(1)(1 - t_n\bar{\gamma})\Phi(\|x_{t_n} - w\|) + t_n\gamma\alpha\|x_{t_n} - w\|\|J_\varphi(x_{t_n} - w)\| \\
&\quad + t_n\langle \gamma f(w) - A(w), J_\varphi(x_{t_n} - w)\rangle \\
&= \varphi(1)(1 - t_n\bar{\gamma})\Phi(\|x_{t_n} - w\|) + t_n\gamma\alpha\Phi(\|x_{t_n} - w\|) \\
&\quad + t_n\langle \gamma f(w) - A(w), J_\varphi(x_{t_n} - w)\rangle \\
&= (1 - t_n(\bar{\gamma}\varphi(1) - \gamma\alpha))\Phi(\|x_{t_n} - w\|) + t_n\langle \gamma f(w) - A(w), J_\varphi(x_{t_n} - w)\rangle.
\end{aligned} \tag{3.13}$$

This implies that

$$\Phi(\|x_{t_n} - w\|) \leq \frac{1}{\bar{\gamma}\varphi(1) - \gamma\alpha}\langle \gamma f(w) - A(w), J_\varphi(x_{t_n} - w)\rangle.$$

Now observing that $x_{t_n} \rightharpoonup w$ implies $J_\varphi(x_{t_n} - w) \rightharpoonup 0$, we conclude from the last inequality that

$$\Phi(\|x_{t_n} - w\|) \to 0 \text{ as } n \to \infty.$$

Hence $x_{t_n} \to w$ as $n \to \infty$. Next we prove that w solves the variational inequality (3.5). For any $z \in F(T)$, we observe that

$$\begin{aligned}
\langle (I - T)x_t - (I - T)z, J_\varphi(x_t - z)\rangle &= \langle x_t - z, J_\varphi(x_t - z)\rangle + \langle Tx_t - Tz, J_\varphi(x_t - z)\rangle \\
&= \Phi(\|x_t - z\|) - \langle Tz - Tx_t, J_\varphi(x_t - z)\rangle \\
&\geq \Phi(\|x_t - z\|) - \|Tz - Tx_t\|\|J_\varphi(x_t - z)\| \\
&\geq \Phi(\|x_t - z\|) - \|z - x_t\|\|J_\varphi(x_t - z)\| \\
&= \Phi(\|x_t - z\|) - \Phi(\|x_t - z\|) = 0.
\end{aligned} \tag{3.14}$$

Since

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t,$$

we can derive that

$$(A - \gamma f)(x_t) = -\frac{1}{t}(I - T)x_t + (A(I - T)x_t.$$

Thus

$$\langle (A-\gamma f)(x_t), J_\varphi(x_t - z)\rangle = -\frac{1}{t}\langle (I-T)x_t - (I-T)z, J_\varphi(x_t - z)\rangle \\ +\langle A(I-T)x_t, J_\varphi(x_t - z)\rangle \le \langle A(I-T)x_t, J_\varphi(x_t - z)\rangle. \tag{3.15}$$

Noticing that

$$x_{t_n} - Tx_{t_n} \to w - T(w) = w - w = 0.$$

Now replacing t in (3.15) with t_n and letting $n \to \infty$, we have

$$\langle (A-\gamma f)w, J_\varphi(w - z)\rangle \le 0.$$

So, $w \in F(T)$ is a solution of the variational inequality (3.5), and hence $w = \tilde{x}$ by the uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$(at $t \to 0$) equals $\tilde{x}$. Therefore, $x_t \to \tilde{x}$ as $t \to 0$. This completes the proof. □

Theorem 3.4 *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on* $[0, 1]$. *Let $\{T_n : E \to E\}_{i=1}^\infty$ be a countable family of nonexpansive mappings with $\cap_{n=1}^\infty F(T_n) \neq \emptyset$ and $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following:*

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1-\beta_n)T_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n,\ n \ge 0 \end{cases} \tag{3.16}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset [0, 1]$ are real sequences satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(C2) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

Suppose that $\{T_n\}$ satisfies the AKTT-condition. Let the mapping $T : E \to E$ be defined by (2.3) and suppose that $F(T) = \cap_{n=1}^\infty F(T_n)$. Then $\{x_n\}$ converges strongly to $\tilde{x}$ that is obtained in Lemma 3.3.

Proof Since $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n < \varphi(1)\|A\|^{-1}$ for all n. By Lemma 3.1, we have $\|I - \alpha_n A\| \le \varphi(1)(1 - \alpha_n\bar{\gamma})$. We first observe that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(T)$ to obtain

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)Tx_n - p\| \\ &= \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - Tp)\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\|, \end{aligned} \tag{3.17}$$

and so

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - A(p)) + (I - \alpha_n A)y_n - (I - \alpha_n A)p\| \\ &\le (1 - \alpha_n(\bar{\gamma}\varphi(1) - \gamma\alpha))\|y_n - p\| + \alpha_n\|\gamma f(x_n) - A(p)\| \\ &= (1 - \alpha_n(\bar{\gamma}\varphi(1) - \gamma\alpha))\|x_n - p\| + \alpha_n(\bar{\gamma} - \gamma\alpha)\frac{\|\gamma f(x_n) - A(p)\|}{\tau - \gamma\alpha}. \end{aligned}$$

It follows from induction that

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma f(x_n) - A(p)\|}{\bar{\gamma}\varphi(1) - \gamma\alpha}\right\}, n \ge 0. \tag{3.18}$$

The boundedness of $\{x_n\}$ implies that $\{y_n\}$, $\{Tx_n\}$ and $\{f(x_n)\}$ are bounded. Now we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From the definition of $\{x_n\}$, it is easily seen that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)y_{n+1} - \alpha_n\gamma f(x_n) - (I - \alpha_n A)y_n\| \\
&= \|\alpha_{n+1} f(x_{n+1}) + (I - \alpha_{n+1}A)y_{n+1} - (I - \alpha_{n+1}A)y_n + (1 - \alpha_{n+1}A)y_n \\
&\quad -\alpha_n\gamma f(x_n) - (I - \alpha_n A)y_n - \alpha_{n+1}\gamma f(x_n) + \alpha_{n+1}\gamma f(x_n)\| \\
&= \|(I - \alpha_{n+1}A)(y_{n+1} - y_n) + (\alpha_n - \alpha_{n+1})(Ay_n - \gamma f(x_n)) \\
&\quad +\alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n))\| \\
&\le \varphi(1)(1 - \alpha_{n+1}\bar{\gamma})\|y_{n+1} - y_n\| + |\alpha_n - \alpha_{n+1}|\|Ay_n - \gamma f(x_n)\| \\
&\quad +\alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\|
\end{aligned} \tag{3.19}$$

for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\beta_{n+1}x_{n+1} + (1 - \beta_{n+1})T_{n+1}x_{n+1} - \beta_n x_n - (1 - \beta_n)T_n x_n\| \\
&= \|\beta_{n+1}x_{n+1} + (1 - \beta_{n+1})T_{n+1}x_{n+1} - (1 - \beta_{n+1})T_{n+1}x_n \\
&\quad +(1 - \beta_{n+1})T_{n+1}x_n - \beta_n x_n - (1 - \beta_n)T_n x_n - (1 - \beta_n)T_{n+1}x_n \\
&\quad +(1 - \beta_n)T_{n+1}x_n - \beta_{n+1}x_n + \beta_{n+1}x_n\| \\
&= \|(1 - \beta_{n+1})(T_{n+1}x_{n+1} - T_{n+1}x_n) + (\beta_n - \beta_{n+1})(T_{n+1}x_n - x_n) \\
&\quad +(1 - \beta_n)(T_{n+1}x_n - T_n x_n) + \beta_{n+1}(x_{n+1} - x_n)\| \\
&\le (1 - \beta_{n+1})\|x_{n+1} - x_n\| + |\beta_n - \beta_{n+1}|\|T_{n+1}x_n - x_n\| \\
&\quad +(1 - \beta_n)\|T_{n+1}x_n - T_n x_n\| + \beta_{n+1}\|x_{n+1} - x_n\| \\
&\le \|x_{n+1} - x_n\| + |\beta_n - \beta_{n+1}|\|T_{n+1}x_n - x_n\| + \|T_{n+1}x_n - T_n x_n\|
\end{aligned} \tag{3.20}$$

for all $n \in \mathbb{N}$. Substitution (3.20) in (3.19), we obtain

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\le \varphi(1)(1 - \alpha_{n+1}\bar{\gamma})\{\|x_{n+1} - x_n\| + |\beta_n - \beta_{n+1}|\|T_{n+1}x_n - x_n\| \\
&\quad +\|T_{n+1}x_n - T_n x_n\|\} + |\alpha_n - \alpha_{n+1}|\|Ay_n - \gamma f(x_n)\| \\
&\quad +\alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| \\
&\le \varphi(1)(1 - \alpha_{n+1}\bar{\gamma})\{\|x_{n+1} - x_n\| + |\beta_n - \beta_{n+1}|\|T_{n+1}x_n - x_n\| \\
&\quad +\|T_{n+1}x_n - T_n x_n\|\} + |\alpha_n - \alpha_{n+1}|\|Ay_n - \gamma f(x_n)\| \\
&\quad +\alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| \\
&= (1 - \alpha_{n+1}(\bar{\gamma}\varphi(1) - \gamma\alpha))\|x_{n+1} - x_n\| + (1 - \alpha_{n+1}\bar{\gamma})|\beta_n - \beta_{n+1}| \\
&\quad \times\|T_{n+1}x_n - x_n\| + (1 - \alpha_{n+1}\bar{\gamma})\|T_{n+1}x_n - T_n x_n\| \\
&\quad +|\alpha_n - \alpha_{n+1}|\|Ay_n - \gamma f(x_n)\| \\
&\le (1 - \alpha_{n+1}(\bar{\gamma}\varphi(1) - \alpha\gamma))\|x_{n+1} - x_n\| + |\beta_n - \beta_{n+1}|\|T_{n+1}x_n - x_n\| \\
&\quad +\|T_{n+1}x_n - T_n x_n\| + |\alpha_n - \alpha_{n+1}|\|Ay_n - \gamma f(x_n)\|
\end{aligned}$$

$$\leq (1-\alpha_{n+1}(\bar{\gamma}\varphi(1)-\alpha\gamma))\|x_{n+1}-x_n\| + (|\beta_n-\beta_{n+1}| + |\alpha_n-\alpha_{n+1}|)M + \|T_{n+1}x_n - T_n x_n\| \tag{3.21}$$

for each $n \in \mathbb{N}$, where $M = \sup_{n\geq 1}\{\|T_{n+1}x_n - x_n\|, \|Ay_n - \gamma f(x_n)\|\}$. Putting $\mu_n = (|\beta_n - \beta_{n+1}| + |\alpha_n - \alpha_{n+1}|)M + \|T_{n+1}x_n - T_n x_n\|$,
we have

$$\sum_{n=1}^{\infty} \mu_n = \sum_{n=1}^{\infty}((|\beta_n - \beta_{n+1}| + |\alpha_n - \alpha_{n+1}|)M + \|T_{n+1}x_n - T_n x_n\|)$$
$$\leq M\sum_{n=1}^{\infty}(|\beta_n - \beta_{n+1}| + |\alpha_n - \alpha_{n+1}|) + \sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_n z\| : z \in \{x_n\}\} < \infty.$$

Therefore it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (3.16), we have

$$\|x_{n+1} - y_n\| = \alpha_n\|\gamma f(x_n) - Ay_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.22}$$

Next, we show that

$$\|x_n - Tx_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.23}$$

Since $\lim_{n\to\infty}\beta_n = 0$ holds, we obtain

$$\|T_n x_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T_n x_n\|$$
$$\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - T_n x_n\|.$$

This implies that

$$(1 - \beta_n)\|T_n x_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

It follow from (3.22) that

$$\|T_n x_n - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.24}$$

Therefore, we have

$$\|Tx_n - x_n\| \leq \|Tx_n - T_n x_n\| + \|T_n x_n - x_n\|$$
$$\leq \sup\{\|Tz - T_n z\| : z \in \{x_n\}\} + \|T_n x_n - x_n\| \tag{3.25}$$

Hence, by (3.24) and Lemma 2.6, we get $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.
Next, we prove that

$$\limsup_{n\to\infty}\langle\gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \leq 0, \tag{3.26}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle\gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle = \limsup_{n\to\infty}\langle\gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle. \tag{3.27}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in E$ as $i \to \infty$. Since J_φ is weakly continuous, we have by Lemma 2.1 that

$$\limsup_{n\to\infty}\Phi(\|x_{n_{k_i}} - x\|) = \limsup_{n\to\infty}\Phi(\|x_{n_{k_i}} - w\|) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

Let

$$H(x) = \limsup_{n\to\infty}\Phi(\|x_{n_{k_i}} - x\|), \text{ for all } x \in E.$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

From (3.39), we obtain

$$\begin{aligned} H(Tw) &= \limsup_{i\to\infty} \Phi(\|x_{n_{k_i}} - Tw\|) = \limsup_{i\to\infty} \Phi(\|Tx_{n_{k_i}} - Tw\|) \\ &\le \limsup_{i\to\infty} \Phi(\|x_{n_{k_i}} - w\|) = H(w) \end{aligned} \tag{3.28}$$

On the other hand, however,

$$H(Tw) = H(w) + \Phi(\|T(w) - w\|) \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$\Phi(\|T(w) - w\|) = H(Tw) - H(w) \le 0.$$

This implies that $Tw = w$. Since the duality map J_φ is single-valued and weakly continuous, we get that

$$\begin{aligned} \limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle &= \lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle \\ &= \lim_{i\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_{k_i}} - \tilde{x})\rangle \\ &= \langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - w)\rangle \le 0 \end{aligned}$$

as required. Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$.

$$\begin{aligned} \Phi(\|x_{n+1} - \tilde{x}\|) &= \Phi(\|\alpha_n(\gamma f(x_n) - A\tilde{x}) + (I - \alpha_n A)y_n - (I - \alpha_n A)\tilde{x}\|) \\ &\le \Phi(\|(I - \alpha_n A)y_n - (I - \alpha_n A)\tilde{x}\|) + \alpha_n\langle \gamma f(x_n) - A\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\ &\le \varphi(1)(1 - \alpha_n\bar{\gamma})\Phi(\|y_n - \tilde{x}\|) + \alpha_n\langle \gamma f(x_n) - A\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\ &\le (1 - \alpha_n\bar{\gamma})\Phi(\|x_n - \tilde{x}\|) + \alpha_n\langle \gamma f(x_n) - A\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle. \end{aligned} \tag{3.30}$$

It follows that from condition (C1) and (3.26) that

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \limsup_{n\to\infty}\langle \gamma f(x_n) - A\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \le 0.$$

Apply Lemma 2.2 to (3.30) to conclude $\Phi(\|x_{n+1} - \tilde{x}\|) \to 0$ as $n \to \infty$; that is, $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

By Theorem 3.4, we can obtain some new and interesting strong convergence theorems. Now we give some examples as follows:

Setting $\beta_n = 0$ and $T_n := T$ a nonexpansive mapping in Theorem 3.4, we have the following result.

Corollary 3.5 *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0, 1]$. Let T be a nonexpansive mappings with $F(T) = \emptyset$ and $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following:*

$$\begin{cases} x_0 = x \in E, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Tx_n, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $\tilde{x}$ *that is obtained in Lemma 3.3.*

Setting $E := H$ a real Hilbert space in Corollary 3.5, we have the following result.

Corollary 3.6 [15, Theorem 3.3] *Let* H *be a real Hilbert space. Let* T *be a nonexpansive mappings with* $F(T) = \emptyset$ *and* $f \in \Pi_H$, *let* A *be a strongly positive bounded linear operator with coefficient* $\bar{\gamma} > 0$ *and* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. *Let the sequence* $\{x_n\}$ *be generated by the following:*

$$\begin{cases} x_0 = x \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 0 \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ *is a real sequence satisfying the following conditions:*
$\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
Then $\{x_n\}$ *converges strongly to* $\tilde{x}$ *which solves the variational inequality:*

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z \rangle \leq 0, z \in F(T).$$

Theorem 3.7 *Let* E *be a reflexive Banach space which admits a weakly continuous duality mapping* J_φ *with gauge* φ *such that* φ *is invariant on* $[0, 1]$. *Let* $\{T_n : E \to E\}_{i=1}^{\infty}$ *be a countable family of nonexpansive mappings with* $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *and* $f \in \Pi_E$, *let* A *be a strongly positive bounded linear operator with coefficient* $\bar{\gamma} > 0$ *and* $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. *Let the sequence* $\{x_n\}$ *be generated by the following:*

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1 - \beta_n) T_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \ n \geq 0 \end{cases} \tag{3.31}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*
(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Suppose that $\{T_n\}$ *satisfies the* PU*-condition. Let the mapping* $T : E \to E$ *be defined by (2.3) and suppose that* $F(T) = \cap_{n=1}^{\infty} F(T_n)$. *Then* $\{x_n\}$ *converges strongly to* $\tilde{x}$ *that is obtained in Lemma 3.3.*

Proof By using the same arguments and techniques as those of Theorem 3.4, we note that $\{x_n\}$ is bounded. This implies that $\{y_n\}$, $\{Tx_n\}$ and $\{f(x_n)\}$ are bounded. Now we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

We rewrite x_{n+1} in the form

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n z_n, \tag{3.32}$$

where

$$\lambda_n = 1 - (1 - \alpha_n)\beta_n$$

and

$$z_n = \frac{\alpha_n \beta_n}{\lambda_n}(I - A)x_n + \frac{(1 - \beta_n)}{\lambda_n}(I - \alpha_n A)T_n x_n + \frac{\alpha_n}{\lambda_n}\gamma f(x_n). \tag{3.33}$$

Since $\alpha_n \to 0$ and $0 < a \le \beta_n \le b < 1$, then

$$0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1.$$

Next some manipulations give us that

$$\begin{aligned}
z_{n+1}-z_n &= \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}(I-A)x_{n+1} + \frac{(1-\beta_{n+1})}{\lambda_{n+1}}(I-\alpha_{n+1}A)T_{n+1}x_{n+1} + \frac{\alpha_{n+1}}{\lambda_{n+1}}\gamma f(x_{n+1}) \\
&\quad -\frac{\alpha_n\beta_n}{\lambda_n}(I-A)x_n - \frac{(1-\beta_n)}{\lambda_n}(I-\alpha_n A)T_n x_n - \frac{\alpha_n}{\lambda_n}\gamma f(x_n) \\
&= \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}(I-A)x_{n+1} - \frac{\alpha_n\beta_n}{\lambda_n}(I-A)x_n + \frac{(1-\beta_{n+1})}{\lambda_{n+1}}T_{n+1}x_{n+1} \\
&\quad -\frac{(1-\beta_{n+1})}{\lambda_{n+1}}T_n x_n + \frac{(1-\beta_{n+1})}{\lambda_{n+1}}T_n x_n \\
&\quad -\alpha_{n+1}\frac{(1-\beta_{n+1})}{\lambda_{n+1}}A(T_{n+1}x_{n+1}) - \frac{(1-\beta_n)}{\lambda_n}T_n x_n \\
&\quad -\alpha_n\frac{(1-\beta_n)}{\lambda_n}A(T_n x_n) + \frac{\alpha_{n+1}}{\lambda_{n+1}}(\gamma f(x_{n+1})-\gamma f(x_n)) + \left(\frac{\alpha_{n+1}}{\lambda_{n+1}}-\frac{\alpha_n}{\lambda_n}\right)\gamma f(x_n) \\
&= \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}(I-A)x_{n+1} - \frac{\alpha_n\beta_n}{\lambda_n}(I-A)x_n + \frac{(1-\beta_{n+1})}{\lambda_{n+1}}(T_{n+1}x_{n+1} - T_n x_n) \\
&\quad + \left(\frac{1-\beta_{n+1}}{\lambda_{n+1}} - \frac{1-\beta_n}{\lambda_n}\right)T_n x_n - \alpha_{n+1}\frac{(1-\beta_{n+1})}{\lambda_{n+1}}A(Tx_{n+1}) \\
&\quad +\alpha_n\frac{(1-\beta_n)}{\lambda_n}A(Tx_n) + \frac{\alpha_{n+1}}{\lambda_{n+1}}(\gamma f(x_{n+1})-\gamma f(x_n)) + \left(\frac{\alpha_{n+1}}{\lambda_{n+1}}-\frac{\alpha_n}{\lambda_n}\right)\gamma f(x_n)
\end{aligned}$$

Hence

$$\begin{aligned}
\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| &\le \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}\|(I-A)x_{n+1}\| + \frac{\alpha_n\beta_n}{\lambda_n}\|(I-A)x_n\| \\
&\quad + \left(\frac{1-\beta_{n+1}}{\lambda_{n+1}} - 1\right)\|x_{n+1}-x_n\| \\
&\quad + \frac{1-\beta_{n+1}}{\lambda_{n+1}}\|T_{n+1}x_n - T_n x_n\| \\
&\quad + \left(\frac{1-\beta_{n+1}}{\lambda_{n+1}} - \frac{1-\beta_n}{\lambda_n}\right)\|Tx_n\| \\
&\quad +\alpha_{n+1}\frac{(1-\beta_{n+1})}{\lambda_{n+1}}\|A(Tx_{n+1})\| + \alpha_n\frac{(1-\beta_n)}{\lambda_n}\|A(Tx_n)\| \\
&\quad + \frac{\alpha_{n+1}}{\lambda_{n+1}}\|(\gamma f(x_{n+1})-\gamma f(x_n))\| + \left|\frac{\alpha_{n+1}}{\lambda_{n+1}}-\frac{\alpha_n}{\lambda_n}\right|\|\gamma f(x_n)\|.
\end{aligned} \tag{3.34}$$

Since $\lambda_n = 1-(1-\alpha_n)\beta_n$ and $\alpha_n \to 0$, then

$$\lim_{n\to\infty}\frac{1-\beta_n}{\lambda_n} = \lim_{n\to\infty}\left(1 - \frac{\alpha_n\beta_n}{\lambda_n}\right) = 1. \tag{3.35}$$

Next, we will prove that $\lim_{n\to\infty}\omega_n = 0$, where $\omega_n := \|T_{n+1}x_n - T_n x_n\|$. Indeed, Since $\{x_n\}$ is bounded, there exists a bounded subset D of E such that $\{x_n\} \subset D$. We observe that

$$\frac{1}{2}\omega_n = \frac{1}{2}\|T_{n+1}x_n - T_n x_n\| \leq \frac{1}{2}\|T_{n+1}x_n - Tx_n\| + \frac{1}{2}\|Tx_n - T_n x_n\|.$$

Since $\{T_n\}$ satisfies PU-condition, then there exists an increasing, continuous and convex function h from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that (2.2) holds. Then

$$\begin{aligned} h\left(\frac{1}{2}\omega_n\right) &\leq \frac{1}{2}h(\|T_{n+1}x_n - Tx_n\|) + \frac{1}{2}h(\|Tx_n - T_n x_n\|) \\ &= \frac{1}{2}\sup_{z\in D} h(\|T_{n+1}z - Tz\|) + \frac{1}{2}\sup_{z\in D} h(\|T_n z - Tz\|). \end{aligned} \tag{3.36}$$

Applying Lemma 2.6 to the above inequality, we obtain that

$$\lim_{n\to\infty} h\left(\frac{1}{2}\omega_n\right) = 0$$

The properties of the function h implies that

$$\lim_{n\to\infty} \omega_n = 0. \tag{3.37}$$

From (3.35), (3.37) and (3.34), we have

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Using Lemma 2.3, we obtain that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \lambda_n)\|z_n - x_n\| = 0. \tag{3.38}$$

Next we show that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.39}$$

Indeed, we observe that

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n - T_n x_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n(\gamma f(x_n) - A(T_n x_n)) + (I - \alpha_n A)y_n - (I - \alpha_n A)T_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - A(T_n x_n)\| + \|(I - \alpha_n A)y_n - (I - \alpha_n A)T_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - A(T_n x_n)\| + \varphi(1)(1 - \alpha_n\bar{\gamma})\|y_n - T_n x_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - A(T_n x_n)\| + (1 - \alpha_n\bar{\gamma})\beta_n\|x_n - T_n x_n\|. \end{aligned} \tag{3.40}$$

Hence

$$\|x_n - T_n x_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - A(T_n x_n)\| - \frac{\alpha_n\bar{\gamma}\beta_n}{1-\beta_n}\|x_n - T_n x_n\|.$$

Using our assumptions and (3.38), we conclude that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.41}$$

Using Lemma 2.6, we have

$$\lim_{n\to\infty} \sup_{y\in D} h(\|Ty - T_n y\|) = 0. \tag{3.42}$$

Next, we observe that

$$\begin{aligned} h\left(\frac{1}{2}\|Tx_n - x_n\|\right) &\leq \frac{1}{2}h(\|Tx_n - T_n x_n\|) + \frac{1}{2}h(\|T_n x_n - x_n\|) \\ &\leq \frac{1}{2}\sup_{y\in D} h(\|Ty - T_n y\|) + \frac{1}{2}h(\|T_n x_n - x_n\|). \end{aligned}$$

Applying (3.41) and (3.42) to the last inequality, we have

$$\lim_{n\to\infty} h\left(\frac{1}{2}\|Tx_n - x_n\|\right) = 0. \tag{3.43}$$

It follows from the properties of h that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.44}$$

It follows from the same arguments and techniques as those of Theorem 3.4 that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Remark 3.8 (i) In contrast to results in [15, Theorem 3.4], the restriction $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on parameter $\{\alpha_n\}$ is removed.

(ii) In contrast to results in [15, Theorem 3.4], these results with respect to a nonexpansive mapping are extended to a countable family of nonexpansive mappings.

Setting $E := H$ a real Hilbert space, we obtain the following result.

Corollary 3.9 *Let H be a real Hilbert space. Let $\{T_n : H \to H\}_{i=1}^{\infty}$ be a countable family of nonexpansive mappings with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $f \in \Pi_H$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following:*

$$\begin{cases} x_0 = x \in H, \\ y_n = \beta_n x_n + (1 - \beta_n) T_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \ n \geq 0 \end{cases} \tag{3.45}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*
(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Suppose that $\{T_n\}$ satisfies the PU-condition. Let the mapping $T : H \to H$ be defined by (2.3) and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z \rangle \leq 0, z \in F(T).$$

4 Some applications

We consider the problem of finding a zero of an accretive operator. An operator $\Psi \subset E \times E$ is said to be accretive if for each (x_1, y_1) and $(x_2, y_2) \in \Psi$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. An accretive operator Ψ is said to satisfy the range condition if $\overline{D(\Psi)} \subset R(I + \lambda\Psi)$ for all $\lambda > 0$, where $D(\Psi)$ is the domain of Ψ, I is the identity mapping on E, $R(I + \lambda\Psi)$ is the range of $I + \lambda\Psi$, and $\overline{D(\Psi)}$ is the closure of $D(\Psi)$. If Ψ is an accretive operator which satisfies the range condition, then we can define, for each $\lambda > 0$, a mapping $J_\lambda : R(I + \lambda\Psi) \to D(\Psi)$ by $J_\lambda = (I - \lambda\Psi)^{-1}$, which is called the resolvent of Ψ. We know that J_λ is nonexpansive and $F(J_\lambda) = \Psi^{-1}(0)$ for all $\lambda > 0$. We also know the following [9]: For each $\lambda, \mu > 0$ and $x \in R(I + \lambda\Psi) \cap R(I + \mu\Psi)$, it holds that

$$\|J_\lambda x - J_\mu x\| \leq \frac{|\lambda - \mu|}{\lambda} \|x - J_\lambda x\|. \tag{4.1}$$

From the Resolvent identity, we have the following lemma.

Lemma 4.1 *Let E be a Banach space and C a nonempty closed convex subset of E. Let $\Psi \subseteq E \times E$ be an accretive operator such that $\Psi^{-1}0 \neq \emptyset$ and $\overline{D(\Psi)} \subset C \subset \cap_{\lambda>0} R(I+\lambda\Psi)$. Suppose that $\{\lambda_n\}$ is a sequence of $(0, \infty)$ such that $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$. Then*

- (i) *The sequence $\{J_{\lambda_n}\}$ satisfies the PU-condition and hence it satisfies AKTT-condition.*
- (ii) *$\lim_{n\to\infty} J_{\lambda_n} z = J_\lambda z$ for all $z \in C$ and $F(J_\lambda) = \cap_{n=1}^{\infty} F(J_{\lambda_n})$ where $\lambda_n \to \lambda$ as $n \to \infty$.*

Proof (i) By the proof of Theorem 4.3 in [1], we obtain that

$$\sum_{n=1}^{\infty} \sup\{\|J_{\lambda_{n+1}} z - J_{\lambda_n} z\| : z \in D\} < \infty \tag{4.2}$$

for every bounded subset D of C. If $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, increasing function such that $h(0) = 0$, we have for each $k > l$,

$$\begin{aligned} \|J_{\lambda_k} z - J_{\lambda_l} z\| &\leq \sum_{n=l}^{k-1} \|J_{\lambda_{n+1}} z - J_{\lambda_n} z\| \leq \sum_{n=l}^{k-1} \sup_{z\in D} \|J_{\lambda_{n+1}} z - J_{\lambda_n} z\| \\ &\leq \sum_{n=l}^{\infty} \sup_{z\in D} \|J_{\lambda_{n+1}} z - J_{\lambda_n} z\| \end{aligned} \tag{4.3}$$

for every bounded subset D of C. Since h is increasing, we obtain that

$$h(\|J_{\lambda_k} z - J_{\lambda_l} z\|) \leq h\left(\sum_{n=l}^{\infty} \sup_{z\in D} \|J_{\lambda_{n+1}} z - J_{\lambda_n} z\|\right) = h(V_l)$$

for every $z \in D$, where $V_l = \sum_{n=l}^{\infty} \sup_{z\in D} \|J_{\lambda_{n+1}} z - J_{\lambda_n} z\|$. Then

$$\sup_{z\in D} h(\|J_{\lambda_k} z - J_{\lambda_l} z\|) \leq h(V_l).$$

It follows from the continuity of h and (4.2) that $h(V_l) \to 0$ as $l \to \infty$. This implies that

$$\lim_{l\to\infty} \sup_{z\in D} h(\|J_{\lambda_k} z - J_{\lambda_l} z\|) = 0.$$

Hence $\{J_{\lambda_n}\}$ satisfies the PU-condition.

(ii) By the proof of Theorem 4.3 in [1].

□

From Lemma 4.1 and Theorem 3.4, we obtain the following result.

Theorem 4.2 *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0, 1]$. Let $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let Ψ is an m-accretive operator in E such that $\Omega := \Psi^{-1}0 \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by the following:*

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1 - \beta_n) J_{\lambda_n} x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \ n \geq 0 \end{cases} \tag{4.4}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset [0, 1]$ are real sequences satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(C3) $\{\lambda_n\}$ *is a sequence of* $(0, \infty)$ *such that* $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in \cap_{n=1}^{\infty} F(J_{\lambda_n}) = \Psi^{-1}0.$$

From Lemma 4.1 and Theorem 3.7, we obtain the following result.

Theorem 4.3 *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0, 1]$. Let $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let Ψ is an m-accretive operator in E such that $\Omega := \Psi^{-1}0 \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by the following:*

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1 - \beta_n) J_{\lambda_n} x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad n \geq 0 \end{cases} \tag{4.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\{\lambda_n\}$ *is a sequence of* $(0, \infty)$ *such that* $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in \cap_{n=1}^{\infty} F(J_{\lambda_n}) = \Psi^{-1}0.$$

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Generalized Equilibrium Problems and fixed point problems for nonexpansive semigroups in Hilbert spaces

U. Kamraksa and R. Wangkeeree

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Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces

Uthai Kamraksa · Rabian Wangkeeree

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Abstract In this paper, we introduce two iterative schemes (one implicit and one explicit) for finding a common element of the set of solutions of the generalized equilibrium problems and the set of all common fixed points of a nonexpansive semigroup in the framework of a real Hilbert space. We prove that both approaches converge strongly to a common element of such two sets. Such common element is the unique solution of a variational inequality, which is the optimality condition for a minimization problem. Furthermore, we utilize the main results to obtain two mean ergodic theorems for nonexpansive mappings in a Hilbert space. The results of this paper extend and improve the results of Li et al. (J Nonlinear Anal 70:3065–3071, 2009), Cianciaruso et al. (J Optim Theory Appl 146:491–509, 2010) and many others.

Keywords Generalized equilibrium problem · Nonexpansive semigroup · Minimization problem · Fixed point · Hilbert space

1 Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $G : H \times H \to \mathbb{R}$ be an equilibrium bifunction, i.e., $G(u, u) = 0$ for each $u \in H$ and $\Psi : H \to H$ is a mapping. Then, we consider the following generalized equilibrium problem (for short, GEP):

$$\text{Finding } x^* \in H \text{ such that } G(x^*, y) + \langle \Psi x^*, y - x^* \rangle \geq 0, \quad \forall y \in H. \tag{1.1}$$

The problem (1.1) was studied by Moudafi [11]. The set of solutions for the problem GEP (1.1) is denoted by $GEP(G, \Psi)$.

U. Kamraksa · R. Wangkeeree (✉)
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
e-mail: rabianw@nu.ac.th

U. Kamraksa
e-mail: uthaikam@hotmail.com

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Special cases.

1. If $\Psi \equiv 0$, then GEP (1.1) reduces to the following classical equilibrium problem (for short, EP):
$$\text{Finding } x^* \in H \text{ such that } G(x^*, y) \geq 0, \quad \forall y \in H. \tag{1.2}$$
The set of solutions for the problem EP (1.2) is denoted by $EP(G)$.
2. If $G \equiv 0$, then GEP (1.1) reduces to the following classical variational inequality problem (for short VIP):
$$\text{Finding } x^* \in H \text{ such that } \langle \Psi x^*, y - x^* \rangle \geq 0, \quad \forall y \in H. \tag{1.3}$$
The set of solutions for the problem VIP (1.3) is denoted by $VI(\Psi, H)$.

The problem (1.1) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, the classical equilibrium problems, and others; see e.g., [2,5,8,10,22,23,29].

Recall that a mapping $\Psi : H \to H$ is said to be α-inverse-strongly monotone [4,16] if there exists a positive real number α such that

$$\langle \Psi x - \Psi y, x - y \rangle \geq \alpha \|\Psi x - \Psi y\|^2, \quad \forall x, y \in H.$$

A mapping T of H into itself is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in H$. We denote by $F(T)$ the set of fixed points of T. A family $\mathcal{S} := \{T(s) : 0 \leq s < \infty\}$ of mappings of H into itself is called a nonexpansive semigroup on H if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in H$;
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in H$ and $s \geq 0$;
(iv) for all $x \in H$, $s \mapsto T(s)x$ is continuous.

We denote by $F(T(s)) = \{x \in C : T(s)x = x\}$ the set of fixed points of $T(s)$ and by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, i.e. $F(\mathcal{S}) = \cap_{s \geq 0} F(T(s))$. It is known that $F(\mathcal{S})$ is closed and convex.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see e.g., [7,24–26] and the references therein. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2 \quad \text{for all} \quad x \in H. \tag{1.4}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.5}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. In 2003, Xu [27] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0, \tag{1.6}$$

converges strongly to the unique solution of the minimization problem (1.5) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Using the viscosity approximation method, Moudafi [15] introduced the following iterative iterative process for nonexpansive mappings (see [28] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \quad n \geq 0, \tag{1.7}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [15,28] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.7) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in H. \tag{1.8}$$

Recently, Marino and Xu [14] was combine the iterative method (1.6) with the viscosity approximation method (1.7) and consider the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.9}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in H \tag{1.10}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for $\gamma f (i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

In [12] motivated and inspired by Marino and Xu [14], Li, Li and Su introduced the following two iterative methods (see (1.11) and (1.12)) for the approximation of common fixed points of a one-parameter nonexpansive semigroup $\{T(s) : 0 \leq s < \infty\}$ on a nonempty closed convex subset C in a Hilbert space:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s) x_n ds, \tag{1.11}$$

$$y_{n+1} = \alpha_n \gamma f(y_n) + (I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s) y_n ds, \tag{1.12}$$

where $A : C \to H$ is a linear bounded strongly positive operator and $f : H \to H$ is an α-contraction, $\{\alpha_n\}$ and $\{s_n\}$ are sequences in $[0, 1]$ and $(0, \infty)$, respectively. They obtained some convergence theorems under some appropriate control conditions on parameter $\{\alpha_n\}$ and $\{s_n\}$.

Very recently, improving Plubtieng and Punpaeng [17], Cianciaruso et al. [6] introduced the following iterative method, that include equilibrium problems and fixed points problems for nonexpansive semigroups $\mathcal{S} = \{T(s)\}_{s \geq 0}$ on a Hilbert space H,

$$\begin{cases} x_1 \in H \text{ chosen arbitrary}, \\ G(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds, \quad \forall n \geq 1, \end{cases} \tag{1.13}$$

where $A : C \to H$ is a linear bounded strongly positive operator and $f : H \to H$ is an α-contraction. They proved that the iterative scheme $\{x_n\}$ defined by (1.13) converges strongly to a common element of $z \in F(\mathcal{S}) \cap EP(F)$ solving the variational inequality $\langle(\gamma f - A)z, p - z\rangle \leq 0, \forall p \in F(\mathcal{S}) \cap EP(G)$ provided $\{\alpha_n\}$, $\{r_n\}$ and $\{s_n\}$ are real sequences satisfying the following control conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(C2) $\lim_{n\to+\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(C3) $\lim_{n\to+\infty} s_n = +\infty$ and $\lim_{n\to+\infty} \frac{|s_n - s_{n+1}|}{s_n} \frac{1}{\alpha_n} = 0$.

All of the above bring us the following conjectures?

Question 1.1 (i) Could we weaken or remove the control condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on parameter $\{\alpha_n\}$ in (C1)?
(ii) Could we weaken the control condition $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ on parameter $\{r_n\}$ in (C2)?
(iii) Could we weaken the control condition $\lim_{n\to+\infty} \frac{|s_n - s_{n+1}|}{s_n} \frac{1}{\alpha_n} = 0$ in (C3)?
(iv) Could we construct an iterative algorithm to approximate a common element of the set of solutions of the generalized equilibrium problem and the set of all common fixed points of a nonexpansive semigroup?

In this paper, motivated by Li et al. [12], Plubtieng and Punpaeng [17] and Cianciaruso et al. [6], we suggest and analyze an iterative scheme for finding a common element of the set of solutions of the generalized equilibrium problem and the set of all common fixed points of a nonexpansive semigroup in the framework of a real Hilbert space under weak conditions imposed on the parameters. Furthermore, by using these results, we obtain two mean ergodic theorems for a nonexpansive mapping in a real Hilbert spaces. The results in this paper generalize and improve some well-known results in Li et al. [6,12,17] and many others.

2 Preliminaries

This section collects some results that will be used in the proofs of our main results.

Lemma 2.1 [20] *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $(T(s))_{s\geq 0}$ be a nonexpansive semigroup on C. Then, for every $h \geq 0$,*

$$\lim_{t\to+\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x ds - T(h)\frac{1}{t}\int_0^t T(s)x ds \right\| = 0.$$

Lemma 2.2 [3] (Demiclosedness Principle) *Let X be a uniformly convex Banach space, let C be a nonempty closed convex subset of X and let $T : C \to X$ be a nonexpansive mapping. Then, the mapping $(I - T)$ is demiclosed on C, i.e., if $\{x_n\}$ is weakly convergent to x and $\{(I - T)(x_n)\}$ is strongly convergent to y, then $(I - T)x = y$.*

Lemma 2.3 *For all $x, y \in H$, the inequality $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ holds.*

Lemma 2.4 [21] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that*

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)y_n, \quad n \geq 0$$

where $\{\lambda_n\}$ *is a sequence in* $[0, 1]$ *such that*

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1.$$

Assume

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

We recall that, if C is a closed convex subset of real Hilbert space H, the metric projection $P_C : H \to C$ is the mapping defined as follows: for each $x \in H$, $P_C x$ is the only point in C with the property that $\|x - P_C x\| = \inf_{y\in C} \|x - y\|$.

Lemma 2.5 [3] *Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C. Given $x \in H$ and $z \in C$, $z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0$ for all $y \in C$.*

Lemma 2.6 [14] *Let H be a Hilbert space and let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. If $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.7 [14] *Let C be a nonempty closed convex subset of a real Hilbert space H, let $f : H \to H$ be an α-contraction $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$. Then, for every $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, $(A - \gamma f)$ is a strongly monotone with coefficient $(\bar{\gamma} - \alpha\gamma)$, i.e.*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in H.$$

Lemma 2.8 [27] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \sigma_n + \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\sigma_n\}$, $\{\delta_n\}$ are sequences of real numbers such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$;
(iii) $\delta_n \geq 0$ *and* $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

In order to solve the equilibrium problem for a function $G : H \times H \to \mathbb{R}$, we assume that:

(E1) $G(x, x) = 0$, for all $x \in H$;
(E2) $G(x, y) + G(y, x) \leq 0$, for all $(x, y) \in H \times H$(i.e. G is monotone);
(E3) for each $x, y, z \in H$, $\limsup_{t\to 0} G(tz + (1 - t)x, y) \leq G(x, y)$;
(E4) the function $y \mapsto G(x, y)$ is convex and lower semicontinuous for each $x \in H$.

Lemma 2.9 [2] *Let C be a nonempty closed and convex subset of a real Hilbert space H and $G : C \times C \to \mathbb{R}$ a function satisfying the conditions* (E1)–(E4). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.10 [13] *Let C be a nonempty closed and convex subset of a real Hilbert space H and $G : C \times C \to \mathbb{R}$ a function satisfying the condition* (E1)–(E4). *For $r > 0$ and $x \in H$, let $S_r : H \to C$ be a r-resolvent defined by*

$$T_r(x) = \left\{ z \in C : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}, \quad x \in H.$$

Then:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, i.e.*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H;$$

(iii) *$F(T_r) = EP(G)$;*
(iv) *EP(G) is closed and convex.*

Remark 2.11 For any $x \in H$ and $r > 0$, by Lemma 2.10 (i), there exists $u \in H$ such that

$$G(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \quad \forall\, y \in H. \tag{2.1}$$

Replacing x with $x - r\Psi x \in H$ in (2.1), we have

$$G(u, y) + \langle \Psi x, y - u\rangle + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \quad \forall\, y \in H, \tag{2.2}$$

where $\Psi : H \to H$ is an inverse-strongly monotone mapping.

3 Implicit iterative approximation methods

In this section, for finding a common element of the set of solutions of the generalized equilibrium problem and the set of all common fixed points of a nonexpansive semigroup, we prove a strong convergence theorem of an implicit iterative sequence.

Theorem 3.1 *Let $\mathcal{S} = (T(s))_{s\geq 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \to H$ be an α-contraction, $A : H \to H$ a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$, $G : H \times H \to \mathbb{R}$ a mapping satisfying hypotheses* (E1)–(E4) *and $\Psi : H \to H$ an inverse-strongly monotone mappings with coefficients δ such that $F(\mathcal{S}) \cap GEP(G, \Psi) \neq \emptyset$. Let $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, 2\delta)$ and $\{s_n\} \subset (0, \infty)$ be the real sequences. Then the following hold.*

(i) *For any $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, there exists a unique sequence $\{x_n\} \subset H$ such that*

$$\begin{cases} G(u_n, y) + \langle \Psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, & \forall n \geq 1. \end{cases} \tag{3.1}$$

(ii) *If $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty} r_n > 0$, and $\lim_{n\to\infty} s_n = +\infty$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to z, which is a unique solution in $F(\mathcal{S}) \cap GEP(G, \Psi)$ of the variational inequality*

$$\langle (\gamma f - A)z, p - z\rangle \leq 0, \quad \forall p \in F(\mathcal{S}) \cap GEP(G, \Psi). \tag{3.2}$$

Proof We first show that $\{x_n\}$ is well defined. For each $n \geq 1$, we consider the mapping $S_n : H \to H$ defined by

$$S_n x := \alpha_n \gamma f(x) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)x ds. \tag{3.3}$$

for all $x \in H$. We claim that S_n is contractive with the coefficient $(1 - \alpha_n(\bar{\gamma} - \gamma\alpha))$. We observe that $T_{r_n}(I - r_n\Psi)$ is nonexpansive for all $n \geq 1$. Indeed, for any $x, y \in H$,

$$\begin{aligned}
\|T_{r_n}(I - r_n\Psi)x - T_{r_n}(I - r_n\Psi)y\|^2 &\leq \|(I - r_n\Psi)x - (I - r_n\Psi)y\|^2 \\
&= \|(x - y) - r_n(\Psi x - \Psi y)\|^2 \\
&= \|x-y\|^2 - 2r_n\langle x-y, \Psi x-\Psi y\rangle + r_n^2\|\Psi x-\Psi y\|^2 \\
&\leq \|x - y\|^2 - 2r_n\delta\|\Psi x - \Psi y\|^2 + r_n^2\|\Psi x - \Psi y\|^2 \\
&= \|x - y\|^2 + r_n(r_n - 2\delta)\|\Psi x - \Psi y\|^2 \\
&\leq \|x - y\|^2.
\end{aligned} \tag{3.4}$$

It follows from Lemma 2.6 and (3.4) that

$$\begin{aligned}
\|S_n x - S_n y\| &\leq \Bigg\|\alpha_n\gamma f(x) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)x ds \\
&\quad - \alpha_n\gamma f(y) - (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)y ds\Bigg\| \\
&\leq \alpha_n\gamma\|f(x) - f(y)\| + (1 - \alpha_n\bar{\gamma})\Bigg\|\frac{1}{s_n}\int_0^{s_n}[T(s)T_{r_n}(I - r_n\Psi)x \\
&\quad - T(s)T_{r_n}(I - r_n\Psi)y]ds\Bigg\| \\
&\leq \alpha_n\gamma\alpha\|x - y\| + (1 - \alpha_n\bar{\gamma})\|T_{r_n}(I - r_n\Psi)x - T_{r_n}(I - r_n\Psi)y\| \\
&\leq \alpha_n\gamma\alpha\|x - y\| + (1 - \alpha_n\bar{\gamma})\|x - y\| \\
&= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x - y\|.
\end{aligned}$$

Since $0 < 1 - \alpha_n(\bar{\gamma} - \gamma\alpha) < 1$, it follows that S_n is a contraction. Therefore by Banach contraction principle, S_n has a unique fixed point $x_n \in H$ such that

$$x_n = \alpha_n\gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)x_n ds.$$

Next, we will show that $\{x_n\}$ is bounded. Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, with no loss of generality, that $\alpha_n < \|A\|^{-1}$, for all $n \geq 1$. Note that u_n can be written as $u_n = T_{r_n}(x_n - r_n\Psi x_n)$ for all $n \geq 1$. Take $p \in F(\mathcal{S}) \cap GEP(G, \Psi)$. Applying $p = T_{r_n}(p - r_n\Psi p)$ and (3.4), we obtain the following

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|T_{r_n}(x_n - r_n\Psi x_n) - T_{r_n}(p - r_n\Psi p)\|^2 \\
&\leq \|x_n - p\|^2 + r_n(r_n - 2\delta)\|\Psi x_n - \Psi p\|^2 \leq \|x_n - p\|^2.
\end{aligned} \tag{3.5}$$

This implies that

$$\begin{aligned}\|x_n - p\| &\le \alpha_n\|\gamma f(x_n) - Ap\| + (1-\alpha_n\bar{\gamma})\frac{1}{s_n}\int_0^{s_n}\|T(s)u_n - T(s)p\|ds\\ &\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + (1-\alpha_n\bar{\gamma})\|x_n - p\|.\end{aligned}$$

Hence,

$$\|x_n - p\| \le \frac{1}{\bar{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\|,$$

i.e., $\{x_n\}$ is bounded and so is $\{u_n\}$. Now, we will prove that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0.$$

From Lemma 2.3 and (3.5), we have

$$\begin{aligned}\|x_n - p\|^2 &\le (1-\alpha_n\bar{\gamma})^2\left\|\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds - p\right\|^2\\ &\quad + 2\alpha_n\langle\gamma f(x_n) - Ap + \gamma f(p) - \gamma f(p), x_n - p\rangle\\ &\le (1+\alpha_n^2\bar{\gamma}^2 - 2\alpha_n\bar{\gamma})\|u_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2\\ &\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle\\ &\le (1+\alpha_n^2\bar{\gamma}^2)\|u_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle\\ &\le (1+\alpha_n^2\bar{\gamma}^2)(\|x_n - p\|^2 + r_n(r_n - 2\delta)\|\Psi x_n - \Psi p\|^2)\\ &\quad - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle\\ &= (1+\alpha_n^2\bar{\gamma}^2)\|x_n - p\|^2 + (1+\alpha_n^2\bar{\gamma}^2)r_n(r_n - 2\delta)\|\Psi x_n - \Psi p\|^2\\ &\quad - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle\end{aligned}$$

and hence

$$\begin{aligned}(1+\alpha_n^2\bar{\gamma}^2)r_n(2\delta - r_n)\|\Psi x_n - \Psi p\|^2 &\le \alpha_n(\alpha_n\bar{\gamma}^2 - 2\gamma\alpha)\|x_n - p\|^2\\ &\quad + 2\alpha_n\langle\gamma f(p) - Ap, x_n - p\rangle.\end{aligned}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and $\liminf_{n\to\infty} r_n > 0$, we have

$$\|\Psi x_n - \Psi p\| \to 0, \text{ as } n\to\infty. \tag{3.6}$$

On the other hand, using Lemma 2.10 and (3.5), we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n\Psi x_n) - T_{r_n}(p - r_n\Psi p)\|^2\\ &\le \langle x_n - r_n\Psi x_n - (p - r_n\Psi p), u_n - p\rangle\\ &= \frac{1}{2}(\|(x_n - r_n\Psi x_n) - (p - r_n\Psi p)\|^2 + \|u_n - p\|^2\\ &\quad - \|(x_n - r_n\Psi x_n) - (p - r_n\Psi p) - (u_n - p)\|^2\\ &\le \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|(x_n - u_n) - r_n(\Psi x_n - \Psi p)\|^2)\\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2\\ &\quad + 2r_n\langle x_n - u_n, \Psi x_n - \Psi p\rangle - r_n^2\|\Psi x_n - \Psi p\|^2).\end{aligned}$$

So, we have

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, \Psi x_n - \Psi p\rangle - r_n^2\|\Psi x_n - \Psi p\|^2. \quad (3.7)$$

It follows from Lemma 2.3 and (3.7), for any $p \in F(\mathcal{S}) \cap GEP(G, \Psi)$,

$$\begin{aligned}
\|x_n - p\|^2 &\le (1 - \alpha_n\bar{\gamma})^2 \left\| \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p \right\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ap + \gamma f(p) - \gamma f(p), x_n - p\rangle \\
&\le (1 + \alpha_n^2\bar{\gamma}^2 - 2\alpha_n\bar{\gamma})\|u_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle \\
&\le (1 + \alpha_n^2\bar{\gamma}^2)\|u_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle \\
&\le \|u_n - p\|^2 + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, \Psi x_n - \Psi p\rangle - r_n^2\|\Psi x_n - \Psi p\|^2 \\
&\quad + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|\Psi x_n - \Psi p\| - r_n^2\|\Psi x_n - \Psi p\|^2 \\
&\quad + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 - 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\langle \gamma f(p) - Ap, x_n - p\rangle. \quad (3.8)
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_n - u_n\|^2 &\le \|\Psi x_n - \Psi p\|[2r_n\|x_n - u_n\| - r_n^2\|\Psi x_n - \Psi p\|] \\
&\quad + \alpha_n[\bar{\gamma}^2\|x_n - p\|^2 + 2\gamma\alpha\|x_n - p\|^2 + 2\langle \gamma f(p) - Ap, x_n - p\rangle]. \quad (3.9)
\end{aligned}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and (3.6), we can conclude that

$$\|x_n - u_n\| \to 0, \quad \text{as } n \to \infty. \quad (3.10)$$

On the other hand, let $z_1 = P_{F(\mathcal{S})}x_1$ and $D = \{z \in H : \|z - z_1\| \le \frac{1}{\bar{\gamma} - \gamma\alpha}\|\gamma f(z_1) - Az_1\|\}$. Then D is a nonempty closed bounded convex subset of H which is $T(s)$-invariant for each $s \in [0, \infty)$ and contains $\{x_n\}$ and $\{u_n\}$. We may assume, without loss of generality, that $\mathcal{S} = (T(s))_{s\ge 0}$ is a nonexpansive semigroup on D. In view of Lemma 2.1, we can obtain that

$$\begin{aligned}
&\lim_{n\to\infty}\left\| \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - T(s)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right)\right\| \\
&\le \lim_{n\to\infty}\sup_{z\in D}\left\| \frac{1}{s_n}\int_0^{s_n} T(s)z ds - T(s)\left(\frac{1}{s_n}\int_0^{s_n} T(s)z ds\right)\right\| = 0 \quad (3.11)
\end{aligned}$$

for every $s \in [0, \infty)$. We observe that, for any $0 \leq s < \infty$,

$$\begin{aligned}
\|T(s)x_n - x_n\| &\leq \left\| T(s)x_n - T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \right\| \\
&\quad + \left\| T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \right\| \\
&\quad + \left\| \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - x_n \right\| \\
&\leq 2\left\| x_n - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \right\| \\
&\quad + \left\| T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \right\| \\
&= 2\alpha_n \left\| \gamma f(x_n) - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds \right\| \\
&\quad + \left\| T(s)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \right\|.
\end{aligned}$$

Applying (3.11), Lemma 2.1 and the boundedness of $\{x_n\}$, $\{u_n\}$, we obtain that

$$\|T(s)x_n - x_n\| \to 0, \text{ as } n \to \infty \quad \text{for all } 0 \leq s < \infty. \tag{3.12}$$

Consider a subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $z \in H$. Next, we show that $z \in F(\mathcal{S}) \cap GEP(G, \Psi)$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup z$ as $i \to \infty$. From $\|T(s)x_{n_i} - x_{n_i}\| \to 0$ and the demiclosedness principle of $I - T(s)$ for all $0 \leq s < \infty$, one sees that

$$T(s)z = z \quad \text{for all } 0 \leq s < \infty \text{ that is } z \in F(\mathcal{S}).$$

Next, we show that $z \in GEP(G, \Psi)$. From $\|x_{n_i} - u_{n_i}\| \to 0$, one sees that

$$u_{n_i} \rightharpoonup z \quad \text{and} \quad T(s)x_{n_i} \rightharpoonup z, \quad \text{as } i \to \infty \text{ for all } 0 \leq s < \infty.$$

Putting $\{x_i\} := \{x_{n_i}\}$, $\{u_i\} := \{u_{n_i}\}$ and $\{r_i\} := \{r_{n_i}\}$. Since $u_n = T_{r_n}(x_n - r_n\Psi x_n)$, for any $y \in H$ we have

$$G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0.$$

From (E2), we have

$$\langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq G(y, u_n).$$

Replacing n by n_i, we have

$$\langle \Psi x_i, y - u_i \rangle + \frac{1}{r_i}\langle y - u_i, u_k - x_i \rangle \geq G(y, u_i), \quad \text{for all } y \in H. \tag{3.13}$$

Put $u_t = ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in H$. Then, we have $u_t \in H$. So from (3.13) we have

$$\langle u_t - u_i, \Psi u_t\rangle \geq \langle u_t - u_i, \Psi u_t\rangle - \langle u_t - u_i, \Psi x_i\rangle - \left\langle u_t - u_i, \frac{u_i - x_i}{r_i}\right\rangle + G(u_t, u_i)$$
$$= \langle u_t - u_i, \Psi u_t - \Psi u_i\rangle + \langle u_t - u_i, \Psi u_i - \Psi x_i\rangle - \left\langle u_t - u_i, \frac{u_i - x_i}{r_i}\right\rangle + G(u_t, u_i).$$

Since $\|u_i - x_i\| \to 0$, we have $\|\Psi u_i - \Psi x_i\| \to 0$. Further, from monotonicity of Ψ, we have $\langle u_t - u_i, \Psi u_t - \Psi u_i\rangle \geq 0$. So, from (E4) we have

$$\langle u_t - z, \Psi u_t\rangle \geq G(u_t, z), \tag{3.14}$$

as $i \to \infty$. From (E1) and (E4) and (3.14), we also have

$$\begin{aligned} 0 &= G(u_t, u_t) \leq tG(u_t, y) + (1-t)G(u_t, z) \\ &\leq tG(u_t, y) + (1-t)\langle u_t - z, \Psi u_t\rangle \\ &= tG(u_t, y) + (1-t)\langle y - z, \Psi u_t\rangle \end{aligned}$$

and hence

$$0 \leq G(u_t, y) + (1-t)\langle y - z, \Psi u_t\rangle.$$

Letting $t \to \infty$, we have, for each $y \in C$,

$$0 \leq G(z, y) + \langle y - z, \Psi z\rangle.$$

This implies $z \in GEP(G, \Psi)$. Hence $z \in F(\mathcal{S}) \cap GEP(G, \Psi)$ is proved. Next, we show that z solves the variational inequality (3.2). We observe that

$$\begin{aligned} \|x_n - z\|^2 &= \alpha_n\langle \gamma f(x_n) - Az, x_n - z\rangle \\ &\quad + \left\langle (I - \alpha_n A)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - z\right), x_n - z\right\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})\|x_n - z\|^2 + \alpha_n\langle \gamma f(x_n) - Az, x_n - z\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})\|x_n - z\|^2 + \alpha_n\gamma\alpha\|x_n - z\|^2 + \alpha_n\langle \alpha f(z) - Az, x_n - z\rangle. \end{aligned}$$

This implies that

$$\|x_n - z\|^2 \leq \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(z) - Az, x_n - z\rangle.$$

In particular, we have

$$\|x_i - z\|^2 \leq \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(z) - Az, x_i - z\rangle. \tag{3.15}$$

Since $x_i \rightharpoonup z$, it follows from (3.15) that $x_i \to z$ as $i \to \infty$. We rewrite $(A - \gamma f)x_n$ as

$$(A - \gamma f)x_n = -\frac{1}{\alpha_n}(I - \alpha_n A)\left[x_n - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I - r_n\Psi)x_n ds\right]$$

and utilize the fact that $(I - T)$ is monotone if T is nonexpansive. Hence, for any $p \in F(\mathcal{S}) \cap GEP(G, \Psi)$, we have

$$\begin{aligned}
\langle (A-\gamma f)x_n, x_n-p\rangle &= -\frac{1}{\alpha_n}\left\langle (I-\alpha_n A)\left[x_n - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)x_n ds\right], x_n-p\right\rangle \\
&= -\frac{1}{\alpha_n}\left[\left\langle\left(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\right)x_n\right.\right. \\
&\quad \left.\left. -\left(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\right)p, x_n-p\right\rangle\right] \\
&\quad +\frac{1}{s_n}\left\langle A\int_0^{s_n}[x_n - T(s)u_n]ds, x_n-p\right\rangle. \\
&= -\frac{1}{\alpha_n}\left[\left\langle\left(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\right)x_n\right.\right. \\
&\quad \left.\left. -\left(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\right)p, x_n-p\right\rangle ds\right] \\
&\quad +\frac{1}{s_n}\left\langle A\int_0^{s_n}[x_n - T(s)u_n]ds, x_n-p\right\rangle. \qquad (3.16)
\end{aligned}$$

Since the map $\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds$ is nonexpansive, $I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds$ is monotone. This implies that

$$\left\langle\left(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\right)x_n - \left(I - \frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}(I-r_n\Psi)ds\right)p, x_n-p\right\rangle \geq 0.$$

This together with (3.16), we obtain that

$$\langle (A-\gamma f)x_n, x_n-p\rangle \leq \left\langle Ax_n - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds, x_n-p\right\rangle.$$

By the definition of x_n, we obtain that

$$Ax_n - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds = \alpha_n A\left(\gamma f(x_n) - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds\right).$$

Then,

$$\langle (A-\gamma f)x_n, x_n-p\rangle \leq \alpha_n\left\langle A\left(\gamma f(x_n) - \frac{A}{s_n}\int_0^{s_n} T(s)u_n ds\right), x_n-p\right\rangle. \qquad (3.17)$$

In particular, we have

$$\langle (A-\gamma f)x_i, x_i - p\rangle \le \alpha_i \left\langle A\left(\gamma f(x_i) - \frac{A}{s_i}\int_0^{s_i} T(s)u_i ds\right), x_i - p\right\rangle. \tag{3.18}$$

where $\alpha_i := \alpha_{n_i}$. Passing to the limit $i \to \infty$, by the boundedness of x_i and u_i we obtain

$$\langle (A-\gamma f)z, z-p\rangle = \lim_{i\to\infty}\langle (A-\gamma f)x_i, x_i - p\rangle \le 0, \quad \forall p \in F(\mathcal{S}) \cap GEP(G,\Psi). \tag{3.19}$$

That is, $z \in F(\mathcal{S}) \cap GEP(G,\Psi)$ is a solution of the variational inequality (3.2). Finally, we will show that the sequence $\{x_n\}$ converges strongly to z. Assume that there exits a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^*$ as $k \to \infty$. By the same methods as in the above proof, we obtain $x^* \in F(\mathcal{S}) \cap GEP(G,\Psi)$. If follows from the inequality (3.19) that

$$\langle (A-\gamma f)z, z - x^*\rangle \le 0. \tag{3.20}$$

Interchange z and x^* to obtain

$$\langle (A-\gamma f)x^*, x^* - z\rangle \le 0. \tag{3.21}$$

Adding the inequalities (3.20) and (3.21), yields

$$(\bar{\gamma} - \gamma\alpha)\|z - x^*\|^2 \le \langle z - x^*, (A-\gamma f)z - (A-\gamma f)x^*\rangle \le 0$$

by Lemma 2.7. Hence $z = x^*$ and therefore $x_n \to z$ as $n \to \infty$. This completes the proof. □

Setting $G \equiv 0$, $\Psi \equiv 0$, $r_n \equiv 1$ in Theorem 3.1, we have the following result.

Corollary 3.2 [12, Theorem 3.1] *Let C be nonempty closed convex subset of a real Hilbert space H. Suppose that $f : C \to C$ is a fixed contractive mapping with coefficient $0 < \alpha < 1$, and $\mathcal{S} = \{T(s) : s \ge 0\}$ is a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty, and A a strong positive linear bounded operator with coefficient $\bar{\gamma} > 0$, $\{\alpha_n\} \subset (0,1)$, $\{s_n\} \subset (0,\infty)$ are real sequences such that $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty} s_n = \infty$, then for any $0 < \gamma < \bar{\gamma}/\alpha$, there is a unique sequence $\{x_n\} \subset C$ such that*

$$x_n = (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds + \alpha_n \gamma f(x_n)$$

and the sequence $\{x_n\}$ converges strongly to the unique solution $z \in F(\mathcal{S})$ of the variational inequality $\langle (\gamma f - A)z, p - z\rangle \le 0, \forall p \in F(\mathcal{S})$.

4 Explicit iterative approximation methods

Theorem 4.1 *Let $\mathcal{S} = (T(s))_{s\ge 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f : H \to H$ be an α-contraction, $A : H \to H$ a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma}$ and let γ be a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $G : H \times H \to \mathbb{R}$ be a mapping satisfying hypotheses* (E1)–(E4) *and $\Psi : H \to H$ an* *inverse-strongly monotone mapping with coefficients* δ *such that* $F(\mathcal{S}) \cap GEP(G, \Psi) \neq \emptyset$. *Let the sequences* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *be generated by*

$$\begin{cases} x_1 \in H \text{ chosen arbitrary}, \\ G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ y_n = \beta_n x_n + (1 - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where the real sequences $\{r_n\} \subset (0, 2\delta)$, $\{s_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *in* $(0, 1)$ *satisfy the following conditions:*

(D1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$,
(D2) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,
(D3) $\lim_{n\to\infty} s_n = +\infty$, $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$, *and*
(D4) $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0$.

Then the sequences $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converge strongly to* z *which is a unique solution in* $F(\mathcal{S}) \cap GEP(G, \Psi)$ *of the variational inequality* (3.2).

Proof We divide the proof of Theorem 4.1 into five steps:

Step 1 Firstly, we show that $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are bounded.
Note that u_n can be written as $u_n = T_{r_n}(x_n - r_n \Psi x_n)$ for all $n \geq 1$. Take $p \in F(\mathcal{S}) \cap GEP(G, \Psi)$. Since $p = T_{r_n}(p - r_n \Psi p)$ and $\Psi : H \to H$ is an inverse-strongly monotone mapping with coefficients δ satisfying $0 \leq r_n \leq 2\delta$, we obtain the following

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n \Psi x_n) - T_{r_n}(p - r_n \Psi p)\|^2 \\ &\leq \|(x_n - r_n \Psi x_n) - (p - r_n \Psi p)\|^2 \\ &= \|(x_n - p) - r_n(\Psi x_n - \Psi p)\|^2 \\ &= \|x_n - p\|^2 - 2r_n \langle x_n - p, \Psi x_n - \Psi p \rangle + r_n^2 \|\Psi x_n - \Psi p\|^2 \\ &\leq \|x_n - p\|^2 - 2r_n \delta \|\Psi x_n - \Psi p\|^2 + r_n^2 \|\Psi x_n - \Psi p\|^2 \\ &= \|x_n - p\|^2 + r_n(r_n - 2\delta) \|\Psi x_n - \Psi p\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{4.2}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ we may assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ for all $n \in \mathbb{N}$. Applying Lemma 2.6 and (4.2), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| \alpha_n \gamma f(x_n) + (I - \alpha_n A)(\beta_n x_n + (1 - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds) - p \right\| \\ &= \left\| \alpha_n \gamma f(x_n) - \alpha_n \gamma f(p) + \alpha_n \gamma f(p) - \alpha_n A p \right. \\ &\quad \left. + (I - \alpha_n A)(\beta_n x_n + (1 - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - p) \right\| \\ &\leq \|\alpha_n \gamma (f(x_n) - f(p))\| + \|\alpha_n(\gamma f(p) - Ap)\| \\ &\quad + (1 - \alpha_n \bar{\gamma}) \left\| \beta_n x_n + (1 - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s) u_n ds - p \right\| \end{aligned}$$

$$
\begin{aligned}
&= \|\alpha_n\gamma(f(x_n)-f(p))\| + \|\alpha_n(\gamma f(p)-Ap)\| \\
&\quad + (1-\alpha_n\bar{\gamma})\left\|\beta_n(x_n-p)+(1-\beta_n)\frac{1}{s_n}\int_0^{s_n}[T(s)u_n-T(s)p]ds\right\| \\
&\leq \alpha_n\gamma\|f(x_n)-f(p)\| + \alpha_n\|\gamma f(p)-Ap\| \\
&\quad + (1-\alpha_n\bar{\gamma})(\beta_n\|x_n-p\|+(1-\beta_n)\|u_n-p\|) \\
&\leq \alpha_n\gamma\|f(x_n)-f(p)\| + \alpha_n\|\gamma f(p)-Ap\| \\
&\quad + (1-\alpha_n\bar{\gamma})(\beta_n\|x_n-p\|+(1-\beta_n)\|x_n-p\|) \\
&\leq (1-\alpha_n(\bar{\gamma}-\gamma\alpha))\|x_n-p\| + \alpha_n\|\gamma f(p)-Ap\|. \qquad (4.3)
\end{aligned}
$$

From a simple inductive process, it follows that

$$
\|x_n-p\| \leq \max\left\{\|x_1-p\|, \frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\right\}, \quad n \geq 1,
$$

which yields that $\{x_n\}$ is bounded, so is $\{u_n\}$. Moreover, since

$$
\begin{aligned}
\|y_n-p\| &= \left\|\beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&= \left\|\beta_n x_n - \beta_n p + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - (1-\beta_n)p\right\| \\
&= \left\|\beta_n(x_n-p) + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n}[T(s)u_n-T(s)p]ds\right\| \\
&\leq \beta_n\|x_n-p\| + (1-\beta_n)\|u_n-p\| \\
&\leq \beta_n\|x_n-p\| + (1-\beta_n)\|x_n-p\| \\
&= \|x_n-p\|, \qquad (4.4)
\end{aligned}
$$

$\{y_n\}$ is also bounded.

Step 2 Now we show that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0, \lim_{n\to\infty}\|y_{n+1}-y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|u_{n+1}-u_n\| = 0.
$$

We rewrite x_{n+1} in the form:

$$
x_{n+1} = (1-\lambda_n)x_n + \lambda_n z_n, \qquad (4.5)
$$

where

$$
\lambda_n = 1-(1-\alpha_n)\beta_n
$$

and

$$
z_n = \frac{\alpha_n\beta_n}{\lambda_n}(I-A)x_n + \frac{(1-\beta_n)}{\lambda_n}(I-\alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds + \frac{\alpha_n}{\lambda_n}\gamma f(x_n). \qquad (4.6)
$$

Since $\alpha_n \to 0$ and $0 < a \leq \beta_n \leq b < 1$, we have

$$
0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 1.
$$

Next some manipulations give us that

$$\begin{aligned}
z_{n+1} - z_n = & \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}}(I - A)x_{n+1} - \frac{\beta_n \alpha_n}{\lambda_n}(I - A)x_n \\
& + \frac{1 - \beta_{n+1}}{\lambda_{n+1}} \left(\frac{1}{s_{n+1}} \int_0^{s_{n+1}} T(s)u_{n+1}ds - \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \right) \\
& - \frac{(1 - \beta_{n+1})\alpha_{n+1}}{\lambda_{n+1}} A \left(\frac{1}{s_{n+1}} \int_0^{s_{n+1}} T(s)u_{n+1}ds - \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \right) \\
& + \left(\frac{1 - \beta_{n+1}}{\lambda_{n+1}} - \frac{1 - \beta_n}{\lambda_n} \right) \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \\
& - \left(\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n} \right) (1 - \beta_n) A \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \\
& + \frac{\alpha_{n+1}}{\lambda_{n+1}} (\beta_n - \beta_{n+1}) A \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \\
& + \frac{\alpha_{n+1}}{\lambda_{n+1}} (\gamma f(x_{n+1}) - \gamma f(x_n)) + \left(\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n} \right) \gamma f(x_n).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
& \leq \frac{\beta_{n+1}\alpha_{n+1}}{\lambda_{n+1}} \|(I - A)x_{n+1}\| + \frac{\beta_n \alpha_n}{\lambda_n} \|(I - A)x_n\| + \left| \frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n} \right| \|\gamma f(x_n)\| \\
& \quad + \left(\frac{1 - \beta_{n+1}}{\lambda_{n+1}} - 1 \right) \left\| \frac{1}{s_{n+1}} \int_0^{s_{n+1}} T(s)u_{n+1}ds - Tx_{n+1} + \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \right\| \\
& \quad + \frac{(1 - \beta_{n+1})\alpha_{n+1}}{\lambda_{n+1}} \|A\| \left\| \frac{1}{s_{n+1}} \int_0^{s_{n+1}} T(s)u_{n+1}ds - Tx_{n+1} + \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \right\| \\
& \quad + \left| \frac{1 - \beta_{n+1}}{\lambda_{n+1}} - \frac{1 - \beta_n}{\lambda_n} \right| \left\| \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \right\| \\
& \quad + \left| \frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n} \right| \left\| (1 - \beta_n) A \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \right\| \\
& \quad + \frac{\alpha_{n+1}}{\lambda_{n+1}} |\beta_n - \beta_{n+1}| \left\| A \frac{1}{s_n} \int_0^{s_n} T(s)u_n ds \right\| \\
& \quad + \frac{\alpha_{n+1}}{\lambda_{n+1}} \|(\gamma f(x_{n+1}) - \gamma f(x_n))\| + \left| \frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n} \right| \|\gamma f(x_n)\|.
\end{aligned}$$

Since $\lambda_n = 1 - (1 - \alpha_n)\beta_n$ and $\alpha_n \to 0$, then

$$\lim_{n\to\infty} \frac{1-\beta_n}{\lambda_n} = \lim_{n\to\infty}\left(1 - \frac{\alpha_n\beta_n}{\lambda_n}\right) = 1.$$

Then last inequality implies

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$$

and so an application of Lemma 2.4 asserts that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{4.7}$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\lambda_n)\|z_n - x_n\| = 0. \tag{4.8}$$

From the fact that

$$\left(\frac{1}{a} - \frac{1}{b}\right) b = -\frac{a-b}{a},$$

for all nonzero real numbers a, b, we obtain that, for any $p \in F(\mathcal{S})$,

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \Bigg\| \beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \\
&\quad - \beta_{n-1}x_{n-1} - (1-\beta_{n-1})\frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds \Bigg\| \\
&= \Bigg\| \beta_n x_n - \beta_{n-1}x_{n-1} + \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds \\
&\quad - \frac{\beta_n}{s_n}\int_0^{s_n} T(s)u_n ds + \frac{\beta_{n-1}}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds \Bigg\| \\
&\le \Bigg\| \beta_n(x_n - x_{n-1}) + (\beta_n - \beta_{n-1})x_{n-1} + (1-\beta_n)\Bigg(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \\
&\quad - \frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\Bigg) - (\beta_n - \beta_{n-1})\frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds \Bigg\| \\
&= \Bigg\| \beta_n(x_n - x_{n-1}) + (\beta_n - \beta_{n-1})x_{n-1} \\
&\quad + (1-\beta_n)\Bigg\{\frac{1}{s_n}\int_0^{s_n}[T(s)u_n - T(s)u_{n-1}]ds + \left(\frac{1}{s_n} - \frac{1}{s_{n-1}}\right)
\end{aligned}$$

$$
\times \int_0^{s_{n-1}} [T(s)u_{n-1} - T(s)p]ds + \frac{1}{s_n} \int_{s_{n-1}}^{s_n} [T(s)u_{n-1} - T(s)p]ds \Bigg\}
$$

$$
- (\beta_n - \beta_{n-1}) \frac{1}{s_{n-1}} \int_0^{s_{n-1}} T(s)u_{n-1}ds \Bigg\|
$$

$$
\leq \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) \{\|u_n - u_{n-1}\|
$$

$$
+ \left(\frac{2|s_n - s_{n-1}|}{s_n}\right) \|u_{n-1} - p\| \Big\} + |\beta_n - \beta_{n-1}| \left\| \frac{1}{s_{n-1}} \int_0^{s_{n-1}} T(s)u_{n-1}ds \right\|. \tag{4.9}
$$

On the other hand, we observe that

$$
u_n = T_{r_n}(x_n - r_n \Psi x_n) \quad \text{and} \quad u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1} \Psi x_{n+1})
$$

we have

$$
G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \text{for all } y \in H \tag{4.10}
$$

and

$$
G(u_{n+1}, y) + \langle \Psi x_{n+1}, y - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0 \quad \text{for all } y \in H. \tag{4.11}
$$

Putting $y = u_{n+1}$ in (4.10) and $y = u_n$ in (4.11), we have

$$
G(u_n, u_{n+1}) + \langle \Psi x_n, u_{n+1} - u_n \rangle + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0
$$

and

$$
G(u_{n+1}, u_n) + \langle \Psi x_{n+1}, u_n - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.
$$

Adding the above two inequalities, the monotonicity of F implies that

$$
\langle \Psi x_{n+1} - \Psi x_n, u_n - u_{n+1} \rangle + \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0.
$$

Hence

$$
\begin{aligned}
0 &\leq \langle u_n - u_{n+1}, r_n(\Psi x_{n+1} - \Psi x_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n) \rangle \\
&= \Big\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right) u_{n+1} + (x_{n+1} - r_n \Psi x_{n+1}) \\
&\quad - (x_n - r_n \Psi x_n) - x_{n+1} + \frac{r_n}{r_{n+1}} x_{n+1} \Big\rangle \\
&= \Big\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right) (u_{n+1} - x_{n+1}) + (x_{n+1} - r_n \Psi x_{n+1}) \\
&\quad - (x_n - r_n \Psi x_n) \Big\rangle.
\end{aligned}
$$

It follows that

$$\|u_{n+1}-u_n\|^2 \leq \|u_{n+1}-u_n\| \left\{ \left|1-\frac{r_n}{r_{n+1}}\right| \|u_{n+1}-x_{n+1}\| + \|x_{n+1}-x_n\| \right\}$$

and hence

$$\|u_{n+1}-u_n\| \leq \left|1-\frac{r_n}{r_{n+1}}\right| \|u_{n+1}-x_{n+1}\| + \|x_{n+1}-x_n\|. \tag{4.12}$$

Since $\liminf_{n\to\infty} r_n$ is strictly positive, there exists $b > 0$ such that $r_n > b$ for large $n \in \mathbb{N}$. Then,

$$\|u_{n+1}-u_n\| \leq \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{b}\|u_{n+1}-x_{n+1}\|. \tag{4.13}$$

Using (4.9) and (4.13), we can obtain

$$\begin{aligned}
\|y_n-y_{n-1}\| \leq{}& \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| + (1-\beta_n)\Bigg\{\|x_n-x_{n-1}\| \\
&+ \frac{|r_n-r_{n-1}|}{b}\|u_n-x_n\| + \left(\frac{2|s_n-s_{n-1}|}{s_n}\right)\|u_{n-1}-p\|\Bigg\} \\
&+ |\beta_n-\beta_{n-1}| \left\| \frac{1}{s_{n-1}} \int_0^{s_{n-1}} T(s)u_{n-1}ds \right\| \\
={}& \|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| + (1-\beta_n)\left\{\frac{|r_n-r_{n-1}|}{b}\|u_n-x_n\|\right. \\
&\left. + \left(\frac{2|s_n-s_{n-1}|}{s_n}\right)\|u_{n-1}-p\|\right\} + |\beta_n-\beta_{n-1}| \left\| \frac{1}{s_{n-1}} \int_0^{s_{n-1}} T(s)u_{n-1}ds \right\|.
\end{aligned} \tag{4.14}$$

From (4.8) and (D2)–(D4), it follows that also

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = \lim_{n\to\infty}\|y_{n+1}-y_n\| = 0. \tag{4.15}$$

Step 3 Now we will prove that

$$\lim_{n\to\infty}\|x_n-y_n\| = \lim_{n\to\infty}\|u_n-y_n\| = \lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{4.16}$$

In fact, since

$$\|x_n-y_n\| \leq \|y_n-y_{n-1}\| + \alpha_{n-1}\|\gamma f(x_{n-1}) - Ay_{n-1}\|,$$

and from the boundedness of $\{f(x_{n-1})\}$, $\{A(y_{n-1})\}$ and $\lim_{n\to\infty}\|y_n-y_{n-1}\| = 0$, it follows that

$$\lim_{n\to\infty}\|x_n-y_n\| = 0.$$

From Lemma 2.3, it follows that

$$\|x_{n+1}-p\|^2 \leq \|(I-\alpha_n A)(y_n-p)\|^2 + 2\alpha_n\langle\gamma f(x_n) - Ap, x_{n+1}-p\rangle. \tag{4.17}$$

i.e.,

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n\bar{\gamma})^2\|y_n-p\|^2 + 2\alpha_n\langle\gamma f(x_n) - Ap, x_{n+1}-p\rangle. \tag{4.18}$$

Using (4.4) and (3.5), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n\bar{\gamma})^2(\beta_n\|x_n-p\|+(1-\beta_n)\|u_n-p\|)^2\\ &\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\leq (1-2\alpha_n\bar{\gamma}+\alpha_n^2\bar{\gamma}^2)(\beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-p\|^2)\\ &\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\leq (1+\alpha_n^2\bar{\gamma}^2)(\beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-p\|^2)\\ &\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &= (1+\alpha_n^2\bar{\gamma}^2)\beta_n\|x_n-p\|^2+(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)\|u_n-p\|^2\\ &\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\leq (1+\alpha_n^2\bar{\gamma}^2)\beta_n\|x_n-p\|^2+(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)(\|x_n-p\|^2\\ &\quad+r_n(r_n-2\delta)\|\Psi x_n-\Psi p\|^2)\\ &\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &= (1+\alpha_n^2\bar{\gamma}^2)\|x_n-p\|^2+(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)r_n(r_n-2\delta)\|\Psi x_n-\Psi p\|^2\\ &\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\leq \|x_n-p\|^2+\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\quad+(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)r_n(r_n-2\delta)\|\Psi x_n-\Psi p\|^2 \end{aligned} \tag{4.19}$$

and hence

$$\begin{aligned}(1+\alpha_n^2\bar{\gamma}^2)(1-\beta_n)r_n(2\delta-r_n)\|\Psi x_n-\Psi p\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\quad+\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2\\ &\leq \|x_n-x_{n+1}\|(\|x_{n+1}-p\|+\|x_n-p\|)\\ &\quad+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\quad+\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2.\end{aligned}$$

From $\lim_{n\to\infty}\alpha_n=0$, $\liminf_{n\to\infty}r_n>0$ and (4.8), we have

$$\|\Psi x_n-\Psi p\|\to 0,\ \text{ as } n\to\infty. \tag{4.20}$$

Moreover, for $p\in F(\mathcal{S})\cap GEP(G,\Psi)$, we have that,

$$\begin{aligned}\|u_n-p\|^2 &\leq \|x_n-p\|^2-\|x_n-u_n\|^2\\ &\quad+2r_n\langle x_n-u_n,\Psi x_n-\Psi p\rangle-r_n^2\|\Psi x_n-\Psi p\|^2.\end{aligned} \tag{4.21}$$

From (4.4), we obtain

$$\begin{aligned}\|y_n-p\|^2 &\leq \left\|\beta_n(x_n-p)+(1-\beta_n)\frac{1}{s_n}\int_0^{s_n}[T(s)u_n-T(s)p]ds\right\|^2\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-p\|^2.\end{aligned} \tag{4.22}$$

From (4.18), (4.21) and (4.22), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n\bar{\gamma})^2\beta_n\|x_n-p\|^2+(1-\alpha_n\bar{\gamma})^2(1-\beta_n)\|u_n-p\|^2\\ &\quad +2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\alpha_n\langle\gamma f(p)-Ap,x_{n+1}-p\rangle\\ &\le (1-\alpha_n\bar{\gamma})^2\beta_n\|x_n-p\|^2+(1-\alpha_n\bar{\gamma})^2(1-\beta_n)(\|x_n-p\|^2-\|x_n-u_n\|^2\\ &\quad +2r_n\langle x_n-u_n,\Psi x_n-\Psi p\rangle-r_n^2\|\Psi x_n-\Psi p\|^2)\\ &\quad +2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\alpha_n\langle\gamma f(p)-Ap,x_{n+1}-p\rangle\\ &\le (1-\alpha_n\bar{\gamma})^2\|x_n-p\|^2-(1-\alpha_n\bar{\gamma})^2(1-\beta_n)\|x_n-u_n\|^2\\ &\quad +(1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n-u_n\|\|\Psi x_n-\Psi p\|\\ &\quad -(1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n-\Psi p\|^2\\ &\quad +2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\alpha_n\langle\gamma f(p)-Ap,x_{n+1}-p\rangle\end{aligned}\tag{4.23}$$

and hence,

$$\begin{aligned}&(1-\alpha_n\bar{\gamma})^2(1-b)\|x_n-u_n\|^2\\ &\le (1-\alpha_n\bar{\gamma})^2\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +(1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n-u_n\|\|\Psi x_n-\Psi p\|\\ &\quad -(1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n-\Psi p\|^2+2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle\\ &\quad +2\alpha_n\langle\gamma f(p)-Ap,x_{n+1}-p\rangle\\ &\le (1-2\alpha_n\bar{\gamma}+\alpha_n^2\bar{\gamma}^2)\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +(1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n-u_n\|\|\Psi x_n-\Psi p\|\\ &\quad -(1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n-\Psi p\|^2+2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle\\ &\quad +2\alpha_n\langle\gamma f(p)-Ap,x_{n+1}-p\rangle\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2-2\alpha_n\bar{\gamma}\|x_n-p\|^2+\alpha_n^2\bar{\gamma}^2\|x_n-p\|^2\\ &\quad +(1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n-u_n\|\|\Psi x_n-\Psi p\|\\ &\quad -(1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n-\Psi p\|^2\\ &\quad +2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\alpha_n\langle\gamma f(p)-Ap,x_{n+1}-p\rangle\\ &\le \|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)\\ &\quad +\alpha_n[\alpha_n\bar{\gamma}^2\|x_n-p\|^2-2\bar{\gamma}\|x_n-p\|^2+2\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle\\ &\quad +2\langle\gamma f(p)-Ap,x_{n+1}-p\rangle]\\ &\quad +\|\Psi x_n-\Psi p\|[(1-\alpha_n\bar{\gamma})^2(1-\beta_n)2r_n\|x_n-u_n\|\\ &\quad -(1-\alpha_n\bar{\gamma})^2(1-\beta_n)r_n^2\|\Psi x_n-\Psi p\|].\end{aligned}$$

From (3.6), $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, the boundedness of $\{x_n\}$ and hypothesis (D1), it follows that

$$\lim_{n\to\infty}\|x_n-u_n\|=0,\tag{4.24}$$

and consequently

$$\lim_{n\to\infty}\|y_n-u_n\|=0.\tag{4.25}$$

From (4.24) and (4.25), we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0.\tag{4.26}$$

Putting $t_n = \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds$, since

$$\begin{aligned}\|x_n - t_n\| &\le \|x_n - y_n\| + \|y_n - t_n\| \\ &\le \|x_n - y_n\| + \|\beta_n x_n + (1-\beta_n)t_n - t_n\| \\ &\le \|x_n - y_n\| + \beta_n\|x_n - t_n\|,\end{aligned}$$

we have

$$(1-\beta_n)\|x_n - t_n\| \le \|x_n - y_n\|.$$

From (C4) and (4.26), it follows that

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{4.27}$$

By (4.24) and (4.27), we have

$$\lim_{n\to\infty}\|t_n - u_n\| = 0. \tag{4.28}$$

Step 4 Next, we show that

$$\limsup_{n\to\infty}\langle (A-\gamma f)z, z - x_n\rangle \le 0, \tag{4.29}$$

where $z = P_{F(\mathcal{S})\cap GEP(G,\Psi)}(I - A + \gamma f)(z)$ is a unique solution of the variational inequality (3.2). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle (A-\gamma f)z, z - x_{n_i}\rangle = \limsup_{n\to\infty}\langle (A-\gamma f)z, z - x_n\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. From $\lim_{n\to\infty}\|x_n - t_n\| = 0$, we obtain $t_{n_i} \rightharpoonup w$. Let $z_1 = P_{F(\mathcal{S})}x_1$ and $D = \{z \in H : \|z - z_1\| \le \|x_1 - z_1\| + \frac{1}{\bar{\gamma}-\gamma\alpha}\|\gamma f(z_1) - Az_1\|\}$. Then D is a nonempty closed bounded convex subset of H which is $T(s)$-invariant for each $s \in [0,\infty)$ and contains $\{x_n\}$, $\{u_n\}$. We may assume, without loss of generality, that $\mathcal{S} = (T(s))_{s\ge 0}$ is a nonexpansive semigroup on D. In view of Lemma 2.1, we can obtain that, for every $s \ge 0$,

$$\lim_{n\to\infty}\|t_n - T(s)t_n\| = 0.$$

By the same argument as in the proof of Theorem 3.1, we conclude that $w \in F(\mathcal{S}) \cap GEP(G,\Psi)$. This implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle (A-\gamma f)z, z - x_n\rangle &= \lim_{i\to\infty}\langle (A-\gamma f)z, z - x_{n_i}\rangle \\ &= \langle (A-\gamma f)z, z - w\rangle \le 0.\end{aligned} \tag{4.30}$$

Step 5 Finally, we prove that $x_n \to z$ and $u_n \to z$ as $n \to \infty$. From (4.18), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le (1-\alpha_n\bar{\gamma})^2\|x_n - z\|^2 + 2\alpha_n\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle \\ &\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\ &\le (1-\alpha_n\bar{\gamma})^2\|x_n - z\|^2 + 2\alpha_n\gamma\alpha\|x_n - z\|\|x_{n+1} - z\| \\ &\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\ &\le (1-\alpha_n\bar{\gamma})^2\|x_n - z\|^2 + 2\alpha_n\gamma\alpha(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle \\ &\le ((1-\alpha_n\bar{\gamma})^2 + \alpha_n\gamma\alpha)\|x_n - z\|^2 + \alpha_n\gamma\alpha\|x_{n+1} - z\|^2 \\ &\quad + 2\alpha_n\langle \gamma f(z) - Az, x_{n+1} - z\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \frac{1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-z\|^2\\ &\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(z)-Az, x_{n+1}-z\rangle,\\ &=\left(1-\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right)\|x_n-z\|^2+\frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\alpha}\|x_n-z\|^2\\ &\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(z)-Az, x_{n+1}-z\rangle.\end{aligned}$$

Setting

$$M := \sup_{n\in\mathbb{N}}\|x_n-z\|^2, \tag{4.31}$$

we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \left(1-\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right)\|x_n-z\|^2+\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\\ &\quad\times\left(\frac{\alpha_n\bar{\gamma}^2M}{2(\bar{\gamma}-\gamma\alpha)}+\frac{1}{(\bar{\gamma}-\gamma\alpha)}\langle\gamma f(z)-Az, x_{n+1}-z\rangle\right).\end{aligned} \tag{4.32}$$

Setting $\gamma_n = \frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}$ and $\beta_n := \frac{(\alpha_n\bar{\gamma}^2)M}{2(\bar{\gamma}-\gamma\alpha)}+\frac{1}{\bar{\gamma}-\gamma\alpha}\langle\gamma f(z)-Az, x_{n+1}-z\rangle$. It is easily to see that $\sum_{n=1}^{\infty}\gamma_n=\infty$ and $\limsup_{n\to\infty}\beta_n\le 0$ by (4.29). Hence, by Lemma 2.8, the sequence $\{x_n\}$ converges strongly to z. From $\lim_{n\to\infty}\|x_n-y_n\|=0$ and $\lim_{n\to\infty}\|x_n-u_n\|=0$, we conclude that $\{y_n\}$ and $\{u_n\}$ also converge strongly to z as $n\to\infty$. This completes the proof of Theorem 4.1. □

Setting $\Psi\equiv 0$ in Theorem 4.1, we obtain the following results.

Corollary 4.2 *Let $\mathcal{S}=(T(s))_{s\ge 0}$ be a nonexpansive semigroup on a real Hilbert space H. Let $f: H\to H$ be an α-contraction, $A: H\to H$ a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$ and let γ be a real number such that $0<\gamma<\frac{\bar{\gamma}}{\alpha}$. Let $G: H\times H\to\mathbb{R}$ be a mapping satisfying hypotheses (E1)-(E4). Assume that $F(\mathcal{S})\cap EP(G)\neq\emptyset$ and the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1\in H \text{ chosen arbitrary},\\ G(u_n,y)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle\ge 0, \quad \forall y\in H,\\ y_n=\beta_n x_n+(1-\beta_n)\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds,\\ x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_n A)y_n, \quad \forall n\ge 1,\end{cases}$$

where the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{s_n\}$, $\{r_n\}$ satisfy the following conditions:

(D1) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=+\infty$,
(D2) $\liminf_{n\to\infty}r_n\ge 0$, $\lim_{n\to\infty}|r_{n+1}-r_n|=0$,
(D3) $\lim_{n\to\infty}s_n=+\infty$, $\lim_{n\to\infty}\frac{|s_n-s_{n-1}|}{s_n}=0$, *and*
(D4) $0<a\le\beta_n\le b<1$, $\lim_{n\to\infty}|\beta_n-\beta_{n-1}|=0$.

Then the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to z which is a unique solution in $F(\mathcal{S})\cap EP(G)$ of the variational inequality $\langle(\gamma f-A)z, p-z\rangle\le 0, \forall p\in F(\mathcal{S})\cap EP(G)$.

Remark 4.3 Theorem 4.1 and Corollary 4.2 generalize and improve [6, Theorem 4.1]. In fact,

(i) The conditions (C1) and (C2) can be replaced by the weaker conditions (D1) and (D2) respectively.
(ii) The control condition $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} \frac{1}{\alpha_n} = 0$ on (C3) is placed by the *strictly weaker* condition: $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$ in (D3) as shown in the next example.

Example 4.4 (a) If $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} \frac{1}{\alpha_n} = 0$, then $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = 0$.
(b) The converse of (a) is not true.

Proof Since $\{s_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$, we obtain

$$\frac{|s_n - s_{n-1}|}{s_n} \le \frac{|s_n - s_{n-1}|}{s_n} \frac{1}{\alpha_n}.$$

Then it is easy to see that (a) is true. Let $s_n = n$ and $\alpha_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} = \lim_{n\to\infty} \frac{1}{n} = 0$ but $\lim_{n\to\infty} \frac{|s_n - s_{n-1}|}{s_n} \frac{1}{\alpha_n} = 1$. Then converse of (a) is not true. Hence (b) is proved. □

A strong mean convergence theorem for nonexpansive mappings was first established for odd mappings by Baillon [1], and it was generalized to that for nonlinear semigroups by Reich [9,18,19]. It follows from the above proof that Theorems 3.1 and 4.1 are valid for nonexpansive mappings. Thus, we have the following mean ergodic theorems of implicit and explicit iterative methods for nonexpansive mappings in a Hilbert space.

Corollary 4.5 *Let H be a a real Hilbert space and $f : H \to H$ be an α-contraction, $A : H \to H$ a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$, $T : H \to H$ a nonexpansive mapping and $G : H \times H \to \mathbb{R}$ be a mapping satisfying hypotheses* (E1)–(E4) *and $\Psi : H \to H$ an inverse-strongly monotone mapping with coefficients δ. If $F(T) \cap GEP(G, \Psi) \neq \emptyset$, then for any $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, there exists a unique sequence $\{x_n\} \subset H$ such that*

$$\begin{cases} G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in H, \\ x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{n+1} \sum_{j=0}^{n} T^j u_n, \quad \forall n \ge 1, \end{cases} \tag{4.33}$$

where $\{\alpha_n\}$ and $\{r_n\}$ are real sequences in $(0, 1)$ and $(0, 2\delta)$ respectively. Furthermore, if $\lim_{n\to\infty} \alpha_n = 0$, and $\liminf_{n\to\infty} r_n > 0$, then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to z which is a unique solution in $F(T) \cap GEP(G, \Psi)$ of the variational inequality $\langle (\gamma f - A)z, p - z \rangle \le 0, \forall p \in F(T) \cap GEP(G, \Psi)$.

Corollary 4.6 *Let H be a a real Hilbert space, $f : H \to H$ an α-contraction, $A : H \to H$ a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma}$, γ a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $T : H \to H$ be a nonexpansive mapping, $G : H \times H \to \mathbb{R}$ a mapping satisfying hypotheses (E1)-(E4), and $\Psi : H \to H$ an inverse-strongly monotone mapping with coefficients δ. Assume that $F(T) \cap GEP(G, \Psi) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} x_1 \in H \text{ chosen arbitrary}, \\ G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in H, \\ y_n = \beta_n x_n + (1 - \beta_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j u_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad \forall n \ge 1, \end{cases} \tag{4.34}$$

where the real sequences $\{\beta_n\}, \{\alpha_n\} \subset (0, 1)$ *and* $\{r_n\} \subset (0, 2\delta)$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n\geq 0} \alpha_n = +\infty,$
(ii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0,$
(iii) $0 < a \leq \beta_n \leq b < 1$ *and* $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0.$

Then the sequences $\{x_n\}, \{u_n\}$ *and* $\{y_n\}$ *converge strongly to* z *which is a unique solution in* $F(T) \cap GEP(G, \Psi)$ *of the variational inequality* $\langle(\gamma f - A)z, p - z\rangle \leq 0, \forall p \in F(T) \cap GEP(G, \Psi)$.

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The general hybrid approximation methods for

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THE GENERAL HYBRID APPROXIMATION METHODS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

RABIAN WANGKEEREE

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.

Abstract. In this paper, we introduce two general hybrid iterative approximation methods (one implicit and one explicit) for finding a fixed point of a nonexpansive mapping which solving the variational inequality generated by two strongly positive bounded linear operators. Strong convergence theorems of the proposed iterative methods are obtained in a reflexive Banach space which admits a weakly continuous duality mapping. The results presented in this paper improve and extend the corresponding results announced by Marino and Xu [G. Marino, H.K. Xu, A general iterative method for nonexpansive mapping in Hilbert spaces, J. Math. Anal. Appl. 318(2006) 43-52], Wangkeeree, Petrot and Wangkeeree [R. Wangkeeree, N. Petrot, and R. Wangkeeree, The general iterative methods for nonexpansive mappings in Banach spaces, Journal of Global Optimization, DOI 10.1007/s10898-010-9617-6], and Ceng, Guu and Yao [L. C. Ceng, S.M. Guu and J.C. Yao, Hybrid viscosity-like approximation methods for nonexpansive mappings in Hilbert spaces, Computers & Mathematics with Applications, 58,3,(2009),605-617].

Keywords: Nonexpansive mapping, strong convergence, weakly continuous duality mapping, strongly positive bounded linear operator, Banach space, fixed point.

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1. Introduction

Let C be a nonempty subset of a normed linear space E. Recall that a mapping $T : C \longrightarrow C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in E. \tag{1.1}$$

We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in E : Tx = x\}$. A self mapping $f : E \to E$ is a contraction on E if there exists a constant $\alpha \in (0, 1)$ and $x, y \in E$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|. \tag{1.2}$$

One classical way to study nonexpansive mappings is to use contractions to approximate a non-expansive mapping ([2, 9, 15]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : E \to E$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in E, \tag{1.3}$$

where $u \in E$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in E. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [2] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [9] extended Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from E onto $F(T)$. Xu [15] proved Reich's results hold in reflexive Banach spaces which have a weakly continuous duality mapping.

The iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [4, 11, 13, 14] and the references therein. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2 \quad \text{for all } x \in H. \tag{1.4}$$

Email address: rabianw@nu.ac.th.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H :

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where T is a nonexpansive mapping on H and b is a given point in H. In 2003, Xu ([13]) proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily:

$$x_{n+1} = (I - \lambda_n A)Tx_n + \lambda_n u,\ n \geq 0, \tag{1.6}$$

converges strongly to the unique solution of the minimization problem (1.5) provided the sequence $\{\lambda_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [7] introduced the following iterative iterative process for nonexpansive mappings (see [8, 16] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \lambda_n)Tx_n + \lambda_n f(x_n),\ n \geq 0, \tag{1.7}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$. It is proved [7, 16] that under certain appropriate conditions imposed on $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (1.7) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, x \in H. \tag{1.8}$$

Recently, Marino and Xu [6] mixed the iterative method (1.6) and the viscosity approximation method (1.7) and considered the following general iterative method:

$$x_{n+1} = (I - \lambda_n A)Tx_n + \lambda_n \gamma f(x_n), n \geq 0, \tag{1.9}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\lambda_n\}$ of parameters satisfies the following appropriate conditions : $\lim_{n\longrightarrow\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and either $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$, then the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution x^* in H of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, x \in H \tag{1.10}$$

which is the optimality condition for the minimization problem: $\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where h is a potential function for $\gamma f (i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

Very recently, Wangkeeree, Petrot and Wangkeeree [12] extended Marino and Xu's result to the setting of Banach spaces and obtained the strong convergence theorems in a reflexive Banach space which admits a weakly continuous duality mapping. Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0, 1]$. Let $T : E \longrightarrow E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, f a contraction with coefficient $0 < \alpha < 1$ and A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Define the net $\{x_t\}$ by

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t. \tag{1.11}$$

It is proved in [12] that $\{x_t\}$ converges strongly as $t \longrightarrow 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality :

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T). \tag{1.12}$$

On the other hand, Ceng, Guu and Yao [3], introduced the iterative approximation method for solving the variational inequality generated by two strongly positive bounded linear operators on a real Hilbert space H. Let $f : H \longrightarrow H$ be a contraction with coefficient $0 < \alpha < 1$ and let $A, B : H \longrightarrow H$ be two strongly positive bounded linear operators with coefficient $\bar{\gamma} \in (0, 1)$ and $\beta > 0$, respectively. Assume that $0 < \gamma\alpha < \beta$, $\{\lambda_n\}$ is a sequence in $(0, 1)$, $\{\mu_n\}$ is a sequence in $(0, \min\{1, \|B\|^{-1}\})$. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \lambda_n A)Tx_n + \lambda_{n+1}[Tx_n - \mu_{n+1}(BTx_n - \gamma f(x_n))], n \geq 0. \tag{1.13}$$

It is proved in [3, Theorem 3.1] that if the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the following conditions :
(C1) $\lim_{n\longrightarrow\infty} \lambda_n = 0$;
(C2) $\sum_{n=1}^{\infty} \lambda_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;
(C4) $\frac{1-\bar{\gamma}}{\beta-\gamma\alpha} < \lim_{n\to\infty} \mu_n = \mu < \frac{2-\bar{\gamma}}{\beta-\gamma\alpha}$;

then the sequence $\{x_n\}$ generated by (1.13) converges strongly to the unique solution $\tilde{x}$ in H of the variational inequality

$$\langle (A - I + \mu(B - \gamma f))\tilde{x}, \tilde{x} - z\rangle \leq 0, z \in F(T). \tag{1.14}$$

Observe that if $B = I$ and $\mu_n = 1$ for all $n \geq 1$, then algorithm (1.13) reduces to (1.9). Moreover, the variational inequality (1.14) reduces to (1.10). Furthermore, the applications of these results to constrained generalized pseudoinverse are studied.

In this paper, motivated by Marino and Xu [6], Wangkeeree, Petrot and Wangkeeree [12] and Ceng Guu and Yao [3], we introduce two general iterative approximation methods (one implicit and one explicit) for finding a fixed point of a nonexpansive mapping which solving the variational inequality generated by two strongly positive bounded linear operators. Strong convergence theorems of the proposed iterative methods are obtained in a reflexive Banach space which admits a weakly continuous duality mapping. The results presented in this paper improve and extend the corresponding results announced by Marino and Xu [6], Wangkeeree, Petrot and Wangkeeree [12] and Ceng, Guu and Yao [3] and many others.

2. Preliminaries

Throughout this paper, let E be a real Banach space and E^* be its dual space. We write $x_n \rightharpoonup x$ (respectively $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (respectively weak*) converges to x; as usual $x_n \longrightarrow x$ will symbolize strong convergence. Let $U_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to *uniformly convex* if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U_E$, $\|x - y\| \geq \varepsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [10]). A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U_E$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U_E$.

By a gauge function φ we mean a continuous strictly increasing function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Let E^* be the dual space of E. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated to a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \quad \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E.$$

In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ for all $x \neq 0$ (see [1]). Browder [1] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Following Browder [1], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \forall t \geq 0,$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \forall x \in E,$$

where ∂ denotes the sub-differential in the sense of convex analysis.
Now we collect some useful lemmas for proving the convergence result of this paper.

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [5].

Lemma 2.1. ([5]) *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle.$$

In particular, for all $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle.$$

(ii) *Assume that a sequence* $\{x_n\}$ *in* E *converges weakly to a point* $x \in E$. *Then the following identity holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E.$$

Lemma 2.2. ([14]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \alpha_n)a_n + b_n,$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{b_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup_{n\to\infty} b_n/\alpha_n \le 0$ *or* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

In a Banach space E having a weakly continuous duality mapping J_φ with a gauge function φ, an operator A is said to be *strongly positive* [12] if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x)\rangle \ge \bar{\gamma}\|x\|\varphi(\|x\|) \tag{2.1}$$

and

$$\|\alpha I - \beta A\| = \sup_{\|x\|\le 1} |\langle(\alpha I - \beta A)x, J_\varphi(x)\rangle|, \quad \alpha \in [0, 1], \beta \in [-1, 1], \tag{2.2}$$

where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (2.1) reduce to (1.4). The next valuable lemma can be found in [12].

Lemma 2.3. [12, Lemma 3.1] *Assume that a Banach space* E *has a weakly continuous duality mapping* J_φ *with gauge* φ. *Let* A *be a strongly positive bounded linear operator on* E *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \le \varphi(1)\|A\|^{-1}$. *Then* $\|I - \rho A\| \le \varphi(1)(1 - \rho\bar{\gamma})$.

3. Main Results

Now, we are a position to state and prove our main results.

Lemma 3.1. *Let* E *be a Banach space which admits a weakly continuous duality mapping* J_φ *with gauge* φ *such that* φ *is invariant on* $[0, 1]$ *i.e.* $T([0, 1]) \subset [0, 1]$. *Let* $T : E \longrightarrow E$ *be a nonexpansive mapping,* $f : E \longrightarrow E$ *a contraction with coefficient* $\alpha \in (0, 1)$. *Let* A *and* B *be two strongly positive bounded linear operators with coefficients* $\bar{\gamma} > 0$ *and* $\beta > 0$, *respectively. Let* γ *and* μ *be two constants satisfying the condition* (C^*) :

$$(C^*) : 0 < \gamma < \frac{\beta\varphi(1)}{\alpha} \text{ and } \frac{\varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha} < \mu \le \min\left\{1, \varphi(1)\|B\|^{-1}, \frac{1 + \varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha}\right\}.$$

Then for any $\lambda \in (0, \min\{1, \varphi(1)\|A\|^{-1}\})$, *the mapping* $S_\lambda : E \longrightarrow E$ *defined by*

$$S_\lambda(x) = (I - \lambda A)Tx + \lambda[Tx - \mu(BTx - \gamma f(x))], \forall x \in E. \tag{3.1}$$

is a contraction with coefficient $1 - \lambda\tau$, *where* $\tau := \varphi(1)\bar{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha)$.

Proof. Observe that

$$\begin{aligned} \mu \le \frac{1 + \varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha} \quad &\Leftrightarrow \quad \mu(\varphi(1)\beta - \gamma\alpha) \le 1 + \varphi(1) - \varphi(1)\bar{\gamma} \\ &\Leftrightarrow \quad \varphi(1)\bar{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha) \le 1 \end{aligned}$$

and

$$\begin{aligned} \frac{\varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha} < \mu \quad &\Leftrightarrow \quad \varphi(1) - \varphi(1)\bar{\gamma} < \mu(\varphi(1)\beta - \gamma\alpha) \\ &\Leftrightarrow \quad 0 < \varphi(1)\bar{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha). \end{aligned}$$

This show that $\tau := \varphi(1)\bar{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha) \in (0, 1]$. Using Lemma 2.3, we obtain

$$\begin{aligned} \|S_\lambda(x) - S_\lambda(y)\| &= \|(I - \lambda A)Tx + \lambda[Tx - \mu(BTx - \gamma f(x))] - (I - \lambda A)Ty - \lambda[Ty - \mu(BTy - \gamma f(y))]\| \\ &\le \|(I - \lambda A)Tx - (I - \lambda A)Ty\| + \lambda\|Tx - \mu(BTx - \gamma f(x)) - [Ty - \mu(BTy - \gamma f(y))]\| \\ &\le \|I - \lambda A\|\|Tx - Ty\| + \lambda[\|(I - \mu B)Tx - (I - \mu B)Ty\| + \gamma\mu\|f(x) - f(y)\|] \\ &\le \|I - \lambda A\|\|Tx - Ty\| + \lambda[\|I - \mu B\|\|Tx - Ty\| + \gamma\mu\|f(x) - f(y)\|] \end{aligned}$$

$$
\begin{aligned}
&\le \varphi(1)(1-\lambda\bar{\gamma})\|x-y\| + \lambda[\varphi(1)(1-\mu\beta)\|x-y\| + \gamma\mu\alpha\|x-y\|] \\
&= [\varphi(1)(1-\lambda\bar{\gamma}) + \lambda[\varphi(1)(1-\mu\beta) + \gamma\mu\alpha]]\|x-y\| \\
&= [\varphi(1)(1-\lambda\bar{\gamma}) + \lambda[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]]\|x-y\| \\
&= [\varphi(1) - \lambda[\varphi(1)\bar{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha)]]\|x-y\| \\
&= (\varphi(1) - \lambda\tau)\|x-y\| \\
&\le (1-\lambda\tau)\|x-y\|. \qquad (3.2)
\end{aligned}
$$

Hence S_λ is a contraction with coefficient $1-\lambda\tau$. □

Applying the Banach contraction principle to Lemma 3.1, there exists a unique fixed point x_λ of S_λ in E, that is

$$x_\lambda = (I-\lambda A)Tx_\lambda + \lambda[Tx_\lambda - \mu(BTx_\lambda - \gamma f(x_\lambda))], \text{ for all } \lambda \in (0,1). \qquad (3.3)$$

Remark 3.2. For each $1 < p < \infty$, l^p space has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ which is invariant on $[0,1]$.

Theorem 3.3. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \longrightarrow E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \longrightarrow E$ a contraction with coefficient $\alpha \in (0,1)$, and A, B two strongly positive bounded linear operators with coefficients $\bar{\gamma} > 0$ and $\beta > 0$, respectively. Let γ and μ be two constants satisfying the condition (C^*). Then the net $\{x_\lambda\}$ defined by (3.3) converges strongly as $\lambda \longrightarrow 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality :*

$$\langle (A - I + \mu(B-\gamma f))\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \le 0, z \in F(T). \qquad (3.4)$$

Proof. We first show that the uniqueness of a solution of the variational inequality (3.4). Suppose both $\tilde{x} \in F(T)$ and $x^* \in F(T)$ are solutions to (3.4), then

$$\langle (A - I + \mu(B-\gamma f))\tilde{x}, J_\varphi(\tilde{x} - x^*)\rangle \le 0, \qquad (3.5)$$

and

$$\langle (A - I + \mu(B-\gamma f))x^*, J_\varphi(x^* - \tilde{x})\rangle \le 0. \qquad (3.6)$$

Adding (3.5) and (3.6), we obtain

$$\langle (A - I + \mu(B-\gamma f))\tilde{x} - (A - I + \mu(B-\gamma f))x^*, J_\varphi(\tilde{x} - x^*)\rangle \le 0. \qquad (3.7)$$

On the other hand, we observe that

$$
\begin{aligned}
\frac{\varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha} < \mu \quad &\Leftrightarrow \quad \varphi(1) - \varphi(1)\bar{\gamma} < \mu(\varphi(1)\beta - \gamma\alpha) \\
&\Leftrightarrow \quad 1 - \bar{\gamma} < \mu\left(\beta - \frac{\gamma\alpha}{\varphi(1)}\right) \\
&\Leftrightarrow \quad 0 < \bar{\gamma} - 1 + \mu\left(\beta - \frac{\gamma\alpha}{\varphi(1)}\right). \qquad (3.8)
\end{aligned}
$$

It then follows that, for any $x, y \in E$,

$$
\begin{aligned}
&\langle (A - I + \mu(B-\gamma f))x - (A - I + \mu(B-\gamma f))y, J_\varphi(x-y)\rangle \\
&= \langle A(x-y) - (x-y) + \mu[(B-\gamma f)x - (B-\gamma f)y], J_\varphi(x-y)\rangle \\
&= \langle A(x-y), J_\varphi(x-y)\rangle - \langle x-y, J_\varphi(x-y)\rangle \\
&\quad + \mu\langle (B-\gamma f)x - (B-\gamma f)y, J_\varphi(x-y)\rangle \\
&\ge \bar{\gamma}\|x-y\|\varphi(\|x-y\|) - \|x-y\|\varphi(\|x-y\|) + \mu\langle B(x-y), J_\varphi(x-y)\rangle \\
&\quad -\mu\gamma\langle f(x) - f(y), J_\varphi(x-y)\rangle \\
&\ge \bar{\gamma}\|x-y\|\varphi(\|x-y\|) - \|x-y\|\varphi(\|x-y\|) + \mu\beta\|x-y\|\varphi(\|x-y\|) - \mu\gamma\|f(x)-f(y)\|\|J_\varphi(x-y)\| \\
&\ge \bar{\gamma}\Phi(\|x-y\|) - \Phi(\|x-y\|) + \mu\beta\Phi(\|x-y\|) - \mu\gamma\alpha\Phi(\|x-y\|) \\
&= (\bar{\gamma} - 1 + \mu\beta - \mu\gamma\alpha)\bar{\gamma}\Phi(\|x-y\|) \\
&= (\bar{\gamma} - 1 + \mu(\beta - \gamma\alpha))\bar{\gamma}\Phi(\|x-y\|) \\
&\ge \left(\bar{\gamma} - 1 + \mu\left(\beta - \frac{\gamma\alpha}{\varphi(1)}\right)\right)\bar{\gamma}\Phi(\|x-y\|) \ge 0. \qquad (3.9)
\end{aligned}
$$

Applying (3.9) to (3.7), we obtain that $\tilde{x} = x^*$ and the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (3.4). Next, we will prove that $\{x_\lambda\}$ is bounded. Take a $p \in F(T)$, and denote the mapping S_λ by

$$S_\lambda := (I - \lambda A)T + \lambda[T - \mu(BT - \gamma f)], \text{ for all } \lambda \in (0,1).$$

From Lemma 3.1, we have

$$\begin{aligned}
\|x_\lambda - p\| &\leq \|S_\lambda x_\lambda - S_\lambda p\| + \|S_\lambda p - p\| \\
&\leq (1-\lambda\tau)\|x_\lambda - p\| + \|(I-\lambda A)Tp + \lambda[Tp - \mu(BTp - \gamma fp)] - p\| \\
&= (1-\lambda\tau)\|x_\lambda - p\| + \lambda\| - Ap + p - \mu(Bp - \gamma fp)\| \\
&\leq (1-\lambda\tau)\|x_\lambda - p\| + \lambda[\|I - A\|\|p\| + \mu\|Bp - \gamma fp\|],
\end{aligned}$$

where $\tau := \varphi(1)\bar{\gamma} - \varphi(1) + \mu(\varphi(1)\beta - \gamma\alpha) \in (0,1]$. It follows that

$$\|x_\lambda - p\| \leq \frac{1}{\tau}[\|I - A\|\|p\| + \mu\|Bp - \gamma fp\|].$$

Hence $\{x_\lambda\}$ is bounded, so are $\{f(x_\lambda)\}$, $\{AT(x_\lambda)\}$ and $\{BT(x_\lambda)\}$. The definition of $\{x_\lambda\}$ implies that

$$\|x_\lambda - Tx_\lambda\| = \lambda\|Tx_\lambda - \mu(BTx_\lambda - \gamma f(x_\lambda)) - ATx_\lambda\| \longrightarrow 0 \text{ as } \lambda \longrightarrow 0. \tag{3.10}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_\lambda\}$ that there exists $\{x_{\lambda_n}\}$ which is a subsequence of $\{x_\lambda\}$ converging weakly to $w \in E$ as $n \longrightarrow \infty$. Since J_φ is weakly sequentially continuous, we have by Lemma 2.1 that

$$\limsup_{n\longrightarrow\infty} \Phi(\|x_{\lambda_n} - x\|) = \limsup_{n\longrightarrow\infty} \Phi(\|x_{\lambda_n} - w\|) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

Let

$$H(x) = \limsup_{n\longrightarrow\infty} \Phi(\|x_{\lambda_n} - x\|), \text{ for all } x \in E.$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

Since

$$\|x_{\lambda_n} - Tx_{\lambda_n}\| = \lambda_n\|Tx_{\lambda_n} - \mu(BTx_{\lambda_n} - \gamma f(x_{\lambda_n})) - ATx_{\lambda_n}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

We obtain

$$\begin{aligned}
H(Tw) &= \limsup_{n\longrightarrow\infty} \Phi(\|x_{\lambda_n} - Tw\|) = \limsup_{n\longrightarrow\infty} \Phi(\|Tx_{\lambda_n} - Tw\|) \\
&\leq \limsup_{n\longrightarrow\infty} \Phi(\|x_{\lambda_n} - w\|) = H(w).
\end{aligned} \tag{3.11}$$

On the other hand, however,

$$H(Tw) = H(w) + \Phi(\|T(w) - w\|). \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\Phi(\|T(w) - w\|) = H(Tw) - H(w) \leq 0$$

which gives us, $Tw = w$. Next we show that $x_{\lambda_n} \longrightarrow w$ as $n \longrightarrow \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau, \forall t \geq 0$, and $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t).$$

Following Lemma 2.1, we have

$$\begin{aligned}
\Phi(\|x_{\lambda_n} - w\|) &= \Phi(\|(I - \lambda_n A)Tx_{\lambda_n} + \lambda_n[Tx_{\lambda_n} - \mu(BTx_{\lambda_n} - \gamma f(x_{\lambda_n}))] - (I - \lambda_n A)w - \lambda_n Aw\|) \\
&\leq \Phi(\|(I - \lambda_n A)Tx_{\lambda_n} - (I - \lambda_n A)w\|) \\
&\quad + \lambda_n\langle Tx_{\lambda_n} - \mu(BTx_{\lambda_n} - \gamma f(x_{\lambda_n})) - Aw, J_\varphi(x_{\lambda_n} - w)\rangle \\
&\leq \Phi(\varphi(1)(1 - \lambda_n\bar{\gamma})\|x_{\lambda_n} - w\|) \\
&\quad + \lambda_n\langle (I - \mu B)Tx_{\lambda_n} + \mu\gamma f(x_{\lambda_n}) - Aw, J_\varphi(x_{\lambda_n} - w)\rangle \\
&\leq \varphi(1)(1 - \lambda_n\bar{\gamma})\Phi(\|x_{\lambda_n} - w\|) \\
&\quad + \lambda_n\langle (I - \mu B)Tx_{\lambda_n} - (I - \mu B)w + \mu\gamma f(x_{\lambda_n}) - \mu\gamma f(w), J_\varphi(x_{\lambda_n} - w)\rangle \\
&\quad + \lambda_n\langle (I - \mu B)w + \mu\gamma f(w) - Aw, J_\varphi(x_{\lambda_n} - w)\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \varphi(1)(1-\lambda_n\bar{\gamma})\Phi(\|x_{\lambda_n}-w\|)+\lambda_n\langle (I-\mu B)Tx_{\lambda_n}-(I-\mu B)w, J_\varphi(x_{\lambda_n}-w)\rangle\\
&\quad +\lambda_n\mu\gamma\langle f(x_{\lambda_n})-f(w), J_\varphi(x_{\lambda_n}-w)\rangle+\lambda_n\langle (I-\mu B)w+\mu\gamma f(w)-Aw, J_\varphi(x_{\lambda_n}-w)\rangle\\
&\le \varphi(1)(1-\lambda_n\bar{\gamma})\Phi(\|x_{\lambda_n}-w\|)+\lambda_n\|(I-\mu B)Tx_{\lambda_n}-(I-\mu B)w\|\|J_\varphi(x_{\lambda_n}-w)\|\\
&\quad +\lambda_n\mu\gamma\|f(x_{\lambda_n})-f(w)\|\|J_\varphi(x_{\lambda_n}-w)\|+\lambda_n\langle (I-\mu B)w+\mu\gamma f(w)-Aw, J_\varphi(x_{\lambda_n}-w)\rangle\\
&\le \varphi(1)(1-\lambda_n\bar{\gamma})\Phi(\|x_{\lambda_n}-w\|)+\lambda_n\varphi(1)(1-\mu\beta)\|x_{\lambda_n}-w\|\|J_\varphi(x_{\lambda_n}-w)\|\\
&\quad +\lambda_n\mu\gamma\alpha\|x_{\lambda_n}-w\|\|J_\varphi(x_{\lambda_n}-w)\|+\lambda_n\langle (I-\mu B)w+\mu\gamma f(w)-Aw, J_\varphi(x_{\lambda_n}-w)\rangle\\
&= [\varphi(1)(1-\lambda_n\bar{\gamma})+\lambda_n(\varphi(1)(1-\mu\beta)+\mu\gamma\alpha)]\Phi(\|x_{\lambda_n}-w\|)\\
&\quad +\lambda_n\langle (I-\mu B)w+\mu\gamma f(w)-Aw, J_\varphi(x_{\lambda_n}-w)\rangle\\
&= [\varphi(1)-\lambda_n(\varphi(1)\bar{\gamma}-\varphi(1)+\mu(\varphi(1)\beta-\gamma\alpha))]\Phi(\|x_{\lambda_n}-w\|)\\
&\quad +\lambda_n\langle (I-\mu B)w+\mu\gamma f(w)-Aw, J_\varphi(x_{\lambda_n}-w)\rangle\\
&\le [1-\lambda_n(\varphi(1)\bar{\gamma}-\varphi(1)+\mu(\varphi(1)\beta-\gamma\alpha))]\Phi(\|x_{\lambda_n}-w\|)\\
&\quad +\lambda_n\langle (I-\mu B)w+\mu\gamma f(w)-Aw, J_\varphi(x_{\lambda_n}-w)\rangle. \qquad (3.13)
\end{aligned}$$

Thus,

$$\Phi(\|x_{\lambda_n}-w\|)\le \frac{1}{\varphi(1)\bar{\gamma}-\varphi(1)+\mu(\varphi(1)\beta-\gamma\alpha)}\langle (I-\mu B)w+\mu\gamma f(w)-Aw, J_\varphi(x_{\lambda_n}-w)\rangle.$$

Now observing that $x_{\lambda_n}\rightharpoonup w$ implies $J_\varphi(x_{\lambda_n}-w)\rightharpoonup 0$, we conclude from the last inequality that

$$\Phi(\|x_{\lambda_n}-w\|)\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Hence $x_{\lambda_n}\longrightarrow w$ as $n\longrightarrow\infty$. Next we prove that w solves the variational inequality (3.4). For any $z\in F(T)$, we observe that

$$\begin{aligned}
\langle (I-T)x_\lambda-(I-T)z, J_\varphi(x_\lambda-z)\rangle &= \langle x_t-z, J_\varphi(x_\lambda-z)\rangle+\langle Tx_t-Tz, J_\varphi(x_\lambda-z)\rangle\\
&= \Phi(\|x_\lambda-z\|)-\langle Tz-Tx_t, J_\varphi(x_\lambda-z)\rangle\\
&\ge \Phi(\|x_\lambda-z\|)-\|Tz-Tx_t\|\|J_\varphi(x_\lambda-z)\|\\
&\ge \Phi(\|x_\lambda-z\|)-\|z-x_t\|\|J_\varphi(x_\lambda-z)\|\\
&= \Phi(\|x_\lambda-z\|)-\Phi(\|x_\lambda-z\|)=0. \qquad (3.14)
\end{aligned}$$

Since

$$x_\lambda=(I-\lambda_n A)Tx_{\lambda_n}+\lambda_n[Tx_{\lambda_n}-\mu(BTx_{\lambda_n}-\gamma f(x_{\lambda_n}))],$$

we can derive that

$$\lambda_n[Ax_{\lambda_n}-(I-\mu B)x_{\lambda_n}]=(I-\lambda_n A)Tx_{\lambda_n}-(I-\lambda_n A)x_{\lambda_n}+\lambda_n(I-\mu B)Tx_{\lambda_n}-\lambda_n(I-\mu B)x_{\lambda_n}+\lambda_n\gamma f(x_{\lambda_n}).$$

That is

$$[A-I+\mu(B-\gamma f)]x_{\lambda_n}=-\frac{1}{\lambda_n}[(I-\lambda_n A)(I-T)x_{\lambda_n}+\lambda_n(I-\mu B)(I-T)x_{\lambda_n}].$$

Using (3.14), for each $p\in F(T)$, we have

$$\begin{aligned}
&\langle [A-I+\mu(B-\gamma f)]x_{\lambda_n}, J_\varphi(x_{\lambda_n}-p)\rangle\\
&= -\frac{1}{\lambda_n}\left[\langle (I-\lambda_n A)(I-T)x_{\lambda_n}, J_\varphi(x_{\lambda_n}-p)\rangle+\lambda_n\langle (I-\mu B)(I-T)x_{\lambda_n}, J_\varphi(x_{\lambda_n}-p)\rangle\right]\\
&= -\frac{1}{\lambda_n}\langle (I-T)x_{\lambda_n}-(I-T)p, J_\varphi(x_{\lambda_n}-p)\rangle+\langle A(I-T)x_{\lambda_n}, J_\varphi(x_{\lambda_n}-p)\rangle\\
&\quad -\langle (I-T)x_{\lambda_n}-(I-T)p, J_\varphi(x_{\lambda_n}-p)\rangle+\mu\langle B(I-T)x_{\lambda_n}, J_\varphi(x_{\lambda_n}-p)\rangle\\
&\le \langle A(I-T)x_{\lambda_n}, J_\varphi(x_{\lambda_n}-p)\rangle+\langle B(I-T)x_{\lambda_n}, J_\varphi(x_{\lambda_n}-p)\rangle\\
&\le \|A\|\|x_{\lambda_n}-Tx_{\lambda_n}\|\|J_\varphi(x_{\lambda_n}-p)\|+\mu\|B\|\|x_{\lambda_n}-Tx_{\lambda_n}\|\|J_\varphi(x_{\lambda_n}-p)\|\\
&\le \|x_{\lambda_n}-Tx_{\lambda_n}\|M, \qquad (3.15)
\end{aligned}$$

where M is a constant satisfying $M\ge\sup_{n\ge1}\{\|A\|\|J_\varphi(x_{\lambda_n}-p)\|, \mu\|B\|\|J_\varphi(x_{\lambda_n}-p)\|\}$. Noticing that

$$x_{\lambda_n}-Tx_{\lambda_n}\longrightarrow w-T(w)=w-w=0.$$

It follows from (3.15) that

$$\langle (A-I+\mu(B-\gamma f))w, J_\varphi(w-p)\rangle\le 0.$$

So, $w \in F(T)$ is a solution of the variational inequality (3.4), and hence $w = \tilde{x}$ by the uniqueness. In a summary, we have shown that each cluster point of $\{x_\lambda\}$(at $\lambda \longrightarrow 0$) equals $\tilde{x}$. Therefore, $x_\lambda \longrightarrow \tilde{x}$ as $\lambda \longrightarrow 0$. This completes the proof.

□

According to the definition of strongly positive operator A in a Banach space E having a weakly continuous duality mapping J_φ with a gauge function φ, an operator A is said to be *strongly positive* [12] if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x)\rangle \geq \bar{\gamma}\|x\|\varphi(\|x\|)$$

and

$$\|\alpha I - \beta A\| = \sup_{\|x\|\leq 1} |\langle(\alpha I - \beta A)x, J_\varphi(x)\rangle|, \quad \alpha \in [0,1], \beta \in [-1,1],$$

where I is the identity mapping. We may assume, without loss of generality, that $\bar{\gamma} < 1$. Therefore, if $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$, then we have the Corollary 3.4 immediately. Indeed, putting $B = I$ and $\beta = 1$, we have

$$\frac{\varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha} = \frac{\varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1) - \gamma\alpha} < 1 < \frac{1 + \varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1) - \gamma\alpha} = \frac{1 + \varphi(1) - \varphi(1)\bar{\gamma}}{\varphi(1)\beta - \gamma\alpha}.$$

Taking $\mu \equiv 1$ in Theorem 3.3, we obtain the following result.

Corollary 3.4. [12, Lemma 3.3] *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \longrightarrow E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \longrightarrow E$ a contraction with coefficient $\alpha \in (0,1)$, and A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Then the net $\{x_\lambda\}$ defined by*

$$x_\lambda = (I - \lambda A)Tx_\lambda + \lambda\gamma f(x_\lambda),$$

converges strongly as $\lambda \longrightarrow 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality :

$$\langle(A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(T).$$

Corollary 3.5. [6, Theorem 3.2] *Let H be a real Hilbert space. Let $T : H \longrightarrow H$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : H \longrightarrow H$ a contraction with coefficient $\alpha \in (0,1)$, and A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Then the net $\{x_\lambda\}$ defined by*

$$x_\lambda = (I - \lambda A)Tx_\lambda + \lambda\gamma f(x_\lambda),$$

converges strongly as $\lambda \longrightarrow 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality :

$$\langle(A - \gamma f)\tilde{x}, \tilde{x} - z\rangle \leq 0, z \in F(T).$$

Theorem 3.6. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T : E \longrightarrow E$ be a nonexpansive mapping with $F(T) \neq \emptyset$, $f : E \longrightarrow E$ a contraction with coefficient $\alpha \in (0,1)$, and A and B two strongly positive bounded linear operators with coefficients $\bar{\gamma} > 0$ and $\beta > 0$, respectively. Let $x_0 \in E$ be arbitrary and the sequence $\{x_n\}$ be generated by the following iterative scheme :*

$$x_{n+1} = (I - \lambda_n A)Tx_n + \lambda_n[Tx_n - \mu(BTx_n - \gamma f(x_n))], \text{ for all } n \geq 0, \tag{3.16}$$

where γ and μ are two constants satisfying the condition (C^*) and $\{\lambda_n\}$ is a real sequence in $(0,1)$ satisfying the following conditions :

(C1) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$
(C2) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Then the sequence $\{x_n\}$ defined by (3.16) converges strongly to a fixed point $\tilde{x}$ of T that is obtained by Theorem 3.3.

Proof. We first prove that $\{x_n\}$ is bounded. Take a $p \in F(T)$, and denote

$$S_{\lambda_n} := (I - \lambda_n A)T + \lambda_n[T - \mu(BT - \gamma f)].$$

Using Lemma 3.1, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \|S_{\lambda_n}x_n - S_{\lambda_n}p\| + \|S_{\lambda_n}p - p\| \\ &\leq (1 - \lambda_n\tau)\|x_n - p\| + \|(I - \lambda_n A)Tp + \lambda_n[Tp - \mu(BTp - \gamma fp)] - p\| \\ &= (1 - \lambda_n\tau)\|x_n - p\| + \lambda_n\| - Ap + p - \mu(Bp - \gamma fp)\| \end{aligned}$$

$$\begin{aligned} &\leq (1-\lambda_n\tau)\|x_n-p\|+\lambda_n[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|] \\ &= (1-\lambda_n\tau)\|x_n-p\|+\lambda_n\tau\frac{[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|]}{\tau} \\ &\leq \max\left\{\|x_n-p\|, \frac{\tau[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|]}{\tau}\right\} \end{aligned}$$

where $\tau := \varphi(1)\bar{\gamma}-\varphi(1)+\mu(\varphi(1)\beta-\gamma\alpha)\in(0,1]$. By induction, it is easy to see that

$$\|x_n-p\|\leq\max\left\{\|x_0-p\|, \frac{\tau[\|I-A\|\|p\|+\mu\|Bp-\gamma fp\|]}{\tau}\right\}, \text{ for all } n\geq 0.$$

Thus, $\{x_n\}$ is bounded, and hence so are $\{y_n\},\{ATx_n\},\{BTx_n\}$ and $\{f(x_n)\}$. Now we show that

$$\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0.$$

From the definition of $\{x_n\}$, it is easily seen that

$$\begin{aligned} S_{\lambda_{n+1}}x_n-S_{\lambda_n}x_n &= (I-\lambda_{n+1}A)Tx_n+\lambda_{n+1}[Tx_n-\mu(BTx_n-\gamma f(x_n))] \\ &\quad -(I-\lambda_nA)Tx_n-\lambda_n[Tx_n-\mu(BTx_n-\gamma f(x_n))] \\ &= (\lambda_n-\lambda_{n+1})ATx_n+(\lambda_{n+1}-\lambda_n)Tx_n+\mu(\lambda_n-\lambda_{n+1})(BTx_n-\gamma f(x_n)) \\ &= (\lambda_{n+1}-\lambda_n)(I-A)Tx_n+\mu(\lambda_n-\lambda_{n+1})(BTx_n-\gamma f(x_n)). \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+2}-x_{n+1}\| &= \|S_{\lambda_{n+1}}x_{n+1}-S_{\lambda_n}x_n\| \\ &\leq \|S_{\lambda_{n+1}}x_{n+1}-S_{\lambda_{n+1}}x_n\|+\|S_{\lambda_{n+1}}x_n-S_{\lambda_n}x_n\| \\ &\leq (1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|(I-A)Tx_n\| \\ &\quad +\mu|\lambda_n-\lambda_{n+1}|\|BTx_n-\gamma f(x_n)\| \\ &\leq (1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\|+(1+\mu)|\lambda_{n+1}-\lambda_n|M \\ &= (1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\|+(1+\mu)\lambda_{n+1}\tau\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}\tau}M, \end{aligned}$$

where M is a constant satisfying $M\geq\sup\{\|(I-A)Tx_n\|,\|BTx_n-\gamma f(x_n)\|\}$. From condition (C2) we deduce that either $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|M<\infty$ or $\lim_{n\to\infty}\frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}\tau}M=0$. Therefore it follows from Lemma 2.2 that $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$. It then follows that

$$\begin{aligned} \|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\| \\ &= \|x_n-x_{n+1}\|+\lambda_n\|Tx_n-\mu(BTx_n-\gamma f(x_n))-ATx_n\|\longrightarrow 0. \end{aligned} \tag{3.17}$$

Next, we prove that

$$\limsup_{n\longrightarrow\infty}\langle-(A-I+\mu(B-\gamma f))\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle\leq 0. \tag{3.18}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\longrightarrow\infty}\langle-(A-I+\mu(B-\gamma f))\tilde{x}, J_\varphi(x_{n_k}-\tilde{x})\rangle=\limsup_{n\longrightarrow\infty}\langle-(A-I+\mu(B-\gamma f))\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle. \tag{3.19}$$

If follows from reflexivity of E and the boundedness of a sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w\in E$ as $i\longrightarrow\infty$. Since J_φ is weakly continuous, we have by Lemma 2.1 that

$$\limsup_{n\longrightarrow\infty}\Phi(\|x_{n_{k_i}}-x\|)=\limsup_{n\longrightarrow\infty}\Phi(\|x_{n_{k_i}}-w\|)+\Phi(\|x-w\|), \text{ for all } x\in E.$$

Let

$$H(x)=\limsup_{n\longrightarrow\infty}\Phi(\|x_{n_{k_i}}-x\|), \text{ for all } x\in E.$$

It follows that

$$H(x)=H(w)+\Phi(\|x-w\|), \text{ for all } x\in E.$$

From (3.17), we obtain

$$\begin{aligned} H(Tw) &= \limsup_{i\longrightarrow\infty}\Phi(\|x_{n_{k_i}}-Tw\|)=\limsup_{i\longrightarrow\infty}\Phi(\|Tx_{n_{k_i}}-Tw\|) \\ &\leq \limsup_{i\longrightarrow\infty}\Phi(\|x_{n_{k_i}}-w\|)=H(w). \end{aligned} \tag{3.20}$$

On the other hand, however,

$$H(Tw) = H(w) + \Phi(\|T(w) - w\|). \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\Phi(\|T(w) - w\|) = H(Tw) - H(w) \leq 0.$$

This implies that $Tw = w$. Since the duality map J_φ is single-valued and weakly continuous, we get that

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle &= \lim_{k\longrightarrow\infty}\langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle \\
&= \lim_{i\longrightarrow\infty}\langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(x_{n_{k_i}} - \tilde{x})\rangle \\
&= \langle -(A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(w - \tilde{x})\rangle \\
&= \langle (A - I + \mu(B - \gamma f))\tilde{x}, J_\varphi(\tilde{x} - w)\rangle \leq 0
\end{aligned}$$

as required. Finally, we show that $x_n \longrightarrow \tilde{x}$ as $n \longrightarrow \infty$.

$$\begin{aligned}
\Phi(\|x_{n+1} - \tilde{x}\|) &= \Phi(\|(I - \lambda_n A)Tx_n + \lambda_n[Tx_n - \mu(BTx_n - \gamma f(x_n))] - (I - \lambda_n A)\tilde{x} - \lambda_n A\tilde{x}\|) \\
&\leq \Phi(\|(I - \lambda_n A)Tx_n - (I - \lambda_n A)\tilde{x}\|) \\
&\quad + \lambda_n\langle Tx_n - \mu(BTx_n - \gamma f(x_n)) - A\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\leq \varphi(1)(1 - \lambda_n\bar{\gamma})\Phi(\|x_n - \tilde{x}\|) + \lambda_n\langle (I - \mu B)Tx_n + \gamma\mu f(x_n)) - A\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&= \varphi(1)(1 - \lambda_n\bar{\gamma})\Phi(\|x_n - \tilde{x}\|) \\
&\quad + \lambda_n\Big[\langle (I - \mu B)Tx_n + \gamma\mu f(x_n)) - (I - \mu B)Tx_{n+1} - \gamma\mu f(x_{n+1}), J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\quad + \langle (I - \mu B)Tx_{n+1} + \gamma\mu f(x_{n+1}) - (I - \mu B)\tilde{x} - \gamma\mu f(\tilde{x}), J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\quad + \langle (I - \mu B)\tilde{x} + \gamma\mu f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_{n+1} - \tilde{x})\rangle\Big] \\
&= \varphi(1)(1 - \lambda_n\bar{\gamma})\Phi(\|x_n - \tilde{x}\|) \\
&\quad + \lambda_n\Big[\langle (I - \mu B)(Tx_n - Tx_{n+1}), J_\varphi(x_{n+1} - \tilde{x})\rangle + \gamma\mu\langle f(x_n) - f(x_{n+1}), J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\quad + \langle (I - \mu B)(Tx_{n+1} - \tilde{x}), J_\varphi(x_{n+1} - \tilde{x})\rangle + \gamma\mu\langle f(x_{n+1}) - f(\tilde{x}), J_\varphi(x_{n+1} - \tilde{x})\rangle \\
&\quad + \langle (I - A - \mu(B - \gamma f)\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle\Big] \\
&= \varphi(1)(1 - \lambda_n\bar{\gamma})\Phi(\|x_n - \tilde{x}\|) \\
&\quad + \lambda_n\Big[\varphi(1)(1 - \mu\beta)\|x_n - x_{n+1}\|\|J_\varphi(x_{n+1} - \tilde{x})\| + \gamma\mu\alpha\|x_n - x_{n+1}\|\|J_\varphi(x_{n+1} - \tilde{x})\| \\
&\quad + \varphi(1)(1 - \mu\beta)\|x_{n+1} - \tilde{x}\|\|J_\varphi(x_{n+1} - \tilde{x})\| + \gamma\mu\alpha\|x_{n+1} - \tilde{x}\|\|J_\varphi(x_{n+1} - \tilde{x})\| \\
&\quad + \langle (I - A - \mu(B - \gamma f)\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle\Big] \\
&\leq \varphi(1)(1 - \lambda_n\bar{\gamma})\Phi(\|x_n - \tilde{x}\|) \\
&\quad + \lambda_n\Big[\varphi(1)(1 - \mu\beta)\|x_n - x_{n+1}\|M' + \gamma\mu\alpha\|x_n - x_{n+1}\|M' \\
&\quad + \langle (I - A - \mu(B - \gamma f)\tilde{x}, J_\varphi(x_{n+1} - \tilde{x})\rangle\Big] \\
&\quad + \lambda_n[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]\Phi(\|x_{n+1} - \tilde{x}\|),
\end{aligned} \tag{3.22}$$

where M' is a constant satisfying $M' \geq \sup_{n\geq 0}\|J_\varphi(x_{n+1} - \tilde{x})\|$. It then follows that

$$\Phi(\|x_{n+1} - \tilde{x}\|) \leq \frac{\varphi(1)(1 - \lambda_n\bar{\gamma})}{1 - \lambda_n[\varphi(1) - \mu(\varphi(1)\beta - \gamma\alpha)]}\Phi(\|x_n - \tilde{x}\|)$$

$$
\begin{aligned}
&+\lambda_n\left[\frac{\varphi(1)(1-\mu\beta)}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M'\right.\\
&+\frac{\gamma\mu\alpha}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M'\\
&\left.+\frac{1}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\langle(I-A-\mu(B-\gamma f)\tilde{x},J_\varphi(x_{n+1}-\tilde{x})\rangle\right]\\
=&\left(1-\lambda_n\frac{[\varphi(1)\bar{\gamma}-(\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha))]}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\right)\Phi(\|x_n-\tilde{x}\|)\\
&+\lambda_n\left[\frac{\varphi(1)(1-\mu\beta)}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M'\right.\\
&+\frac{\gamma\mu\alpha}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M'\\
&\left.+\frac{1}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\langle-(A-I+\mu(B-\gamma f)\tilde{x},J_\varphi(x_{n+1}-\tilde{x})\rangle\right].
\end{aligned}
\tag{3.23}
$$

Put

$$
\gamma_n=\lambda_n\frac{[\varphi(1)\bar{\gamma}-(\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha))]}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}
$$

and

$$
\begin{aligned}
\delta_n\ =\ &\frac{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}{[\varphi(1)\bar{\gamma}-(\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha))]}\left[\frac{\varphi(1)(1-\mu\beta)}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M'\right.\\
&+\frac{\gamma\mu\alpha}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\|x_n-x_{n+1}\|M'\\
&\left.+\frac{1}{1-\lambda_n[\varphi(1)-\mu(\varphi(1)\beta-\gamma\alpha)]}\langle-(A-I+\mu(B-\gamma f)\tilde{x},J_\varphi(x_{n+1}-\tilde{x})\rangle\right].
\end{aligned}
$$

It follows that from condition (C1), $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$ and (3.18) that

$$
\lim_{n\longrightarrow\infty}\gamma_n=0,\qquad \sum_{n=1}^{\infty}\gamma_n=\infty \text{ and } \limsup_{n\longrightarrow\infty}\delta_n\le 0.
$$

The inequality (3.23) reduces to the following :

$$
\Phi(\|x_{n+1}-\tilde{x}\|)\le(1-\gamma_n)\Phi(\|x_n-\tilde{x}\|)+\gamma_n\delta_n
$$

Applying Lemma 2.2, we conclude that $\Phi(\|x_{n+1}-\tilde{x}\|)\longrightarrow 0$ as $n\longrightarrow\infty$; that is, $x_n\longrightarrow\tilde{x}$ as $n\longrightarrow\infty$. This completes the proof.

□

Remark 3.7. In comparison to the results in [3, Theorem 3.1], the strong convergence in a real Hilbert space is extended to the strong convergence in a reflexive Banach space which admits a weakly continuous duality mapping.

Setting $B\equiv I$, and $\mu\equiv 1$ in Theorem 3.6, we obtain the following result.

Corollary 3.8. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $T:E\longrightarrow E$ be a nonexpansive mapping with $F(T)\neq\emptyset$, $f:E\longrightarrow E$ a contraction with coefficient $\alpha\in(0,1)$, and A a strongly positive bounded linear operator with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let $x_0\in E$ be arbitrary and the sequence $\{x_n\}$ be generated by the following iterative scheme :*

$$
x_{n+1}=(I-\lambda_nA)Tx_n+\lambda_n\gamma f(x_n),\ \textit{for all } n\ge 0,
$$

where $\{\lambda_n\}$ is a real sequence in $(0,1)$ satisfying the following conditions :

(C1) $\lim_{n\to\infty}\lambda_n=0$ *and* $\sum_{n=1}^{\infty}\lambda_n=\infty$

(C2) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Then the sequence $\{x_n\}$ *converges strongly to a fixed point* $\tilde{x}$ *of* T *which solves the variational inequality :*

$$\langle (A - \gamma f)\tilde{x}, J_{\varphi}(\tilde{x} - z)\rangle \leq 0, z \in F(T).$$

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ภาคผนวก 11

Iterative Algorithms for solving Mixed Equilibrium Problem and variational inequality problem of a finite family of asymptotically strict pseudo-contractions

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ITERATIVE ALGORITHMS FOR SOLVING MIXED EQUILIBRIUM PROBLEM AND VARIATIONAL INEQUALITY PROBLEM OF A FINITE FAMILY OF ASYMPTOTICALLY STRICT PSEUDO-CONTRACTIONS*

UTHAI KAMRAKSA AND RABIAN WANGKEEREE*

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.

Abstract. In this paper, we introduce the iterative schemes for finding a common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of a finite family of asymptotically k-strict pseudo-contractions, and the set of the solutions of a variational inequality for a monotone, Lipschitz continuous mapping in the framework of Hilbert spaces. Both weak and strong convergence theorems are obtained. Our results extend the corresponding recent results of Peng [J. W. Peng, Iterative Algorithms for Mixed Equilibrium Problems, Strict Pseudocontractions and Monotone Mappings, J Optim. Theory Appl. DOI 10.1007/s10957-009-9585-5.] and Qin, Cho, Kang and Shang [X. Qin, Y. J. Cho, S. M. Kang, and M. Shang, A hybrid iterative scheme for asymptotically k-strict pseudo-contractions in Hilbert spaces, Nonlinear Analysis, 70, 5,(2009), 1902-1911].

Keywords: Mixed equilibrium problem, Variational inequality problem, Asymptotically k-strict pseudo-contraction, Fixed point, Hilbert space.

AMS Subject Classification: 47H09, 47H10, 47H17.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, C is a nonempty closed convex subset of H. Let $\varphi : C \longrightarrow \mathbb{R}$ be a real-valued function and $F : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction, i.e., $F(u,u) = 0$ for each $u \in C$. The mixed equilibrium problem (for short, MEP) is to find $x^* \in C$ such that

$$MEP : F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions for the problem MEP (1.1) is denoted by $MEP(F, \varphi)$.

Special cases.

(1) If $\varphi \equiv 0$, then MEP (1.1) reduces to the following classical equilibrium problem (for short, EP):

$$\text{Finding } x^* \in C \text{ such that } F(x^*, y) \geq 0, \forall y \in C. \tag{1.2}$$

The set of solutions for the problem EP (1.2) is denoted by $EP(F)$.

(2) If $\varphi \equiv 0$ and $F(x,y) = \langle Ax, y - x \rangle$ for all $x, y \in C$, where A is a mapping from C into H, then MEP (1.1) reduces to the following classical variational inequality problem (for short VIP):

$$\text{Finding } x^* \in C \text{ such that } \langle Ax^*, y - x^* \rangle \geq 0, \forall y \in C. \tag{1.3}$$

The set of solutions for the problem VIP (1.3) is denoted by $VI(C, A)$.

(3) If $F \equiv 0$, then MEP (1.1) becomes the following minimize problem:

$$\text{Finding } x^* \in C \text{ such that } \varphi(y) - \varphi(x^*) \geq 0, \forall y \in C. \tag{1.4}$$

The set of solutions for the problem (1.4) is denoted by $Argmin(\varphi)$.

The problem (1.1) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, the equilibrium problems and others; see, e.g., [2, 6, 10, 29] and the reference therein. First we recall some relevant important results as follows.

In 1997, Combettes and Hirstoaga [7] introduced an iterative method of finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, Takahashi and Takahashi [21] introduced another iterative scheme for finding a common element of the set of solutions of EP and the set of fixed point points of a nonexpansive mapping. Using the idea of Takahashi and

*Corresponding author:
Email address: uthaikam@hotmail.com(Uthai Kamraksa) and rabianw@nu.ac.th(Rabian Wangkeeree).

Takahashi [21], Plubtieng and Punpaeng [15] introduced an the general iterative method for finding a common element of the set of solutions of EP and the set of fixed points of a nonexpansive mapping which is the optimality condition for the minimization problem in a Hilbert space. Furthermore, Yao, Liou and Yao [27] introduced some new iterative schemes for finding a common element of the set of solutions of EP and the set of common fixed points of finitely (infinitely) nonexpansive mappings. Very recently, Ceng and Yao [4] considered a new iterative scheme for finding a common element of the set of solutions of MEP and the set of common fixed points of finitely many nonexpansive mappings. Their results extend and improve the corresponding results in [7, 21, 27].

Recall that a mapping $T : C \longrightarrow C$ is said to be asymptotically k-strictly pseudo-contractive (The class of asymptotically k-strict pseudo-contractive maps was first introduced in Hilbert spaces by Qihou [16]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} k_n = 1$ such that there exists $k \in [0, 1)$ such that

$$\|T^n x - T^n y\|^2 \leq k_n^2 \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \tag{1.5}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Note that the class of asymptotically k-strict pseudo-contractions strictly includes the class of asymptotically nonexpansive mappings [8] which are mappings T on C such that

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|, \forall x, y \in C, \tag{1.6}$$

where the sequence $\{k_n\} \subset [1, \infty)$ is such that $\lim_{n\longrightarrow\infty} k_n = 1$. That is, T is asymptotically nonexpansive if and only if T is asymptotically 0-strict pseudo-contractive.

Recall that a mapping $T : C \longrightarrow C$ is called a k-strict pseudo-contraction mapping if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.7}$$

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. That is, T is nonexpansive if and only if T is 0-strict pseudo-contractive. Note that the class of strict pseudo-contraction mappings strictly includes the class of nonexpansive mappings. Clearly, T is nonexpansive if and only if T is a 0-strict pseudo-contraction.

For solving the mixed equilibrium problem for an equilibrium bifunction $F : C \times C \longrightarrow \mathbb{R}$, let us assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
- (A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;
- (A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex;
- (A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;
- (B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that, for any $z \in C \backslash D_x$

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

- (B2) C is a bounded set.

Construction of fixed points of nonexpansive mappings via Manns algorithm [11] has extensively been investigated in the literature; See, for example [3, 11, 23, 25, 26, 28] and references therein. If T is a nonexpansive self-mapping of C, then Mann's algorithm generates, initializing with an arbitrary $x_1 \in C$, a sequence according to the recursive manner

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \forall n \geq 1, \tag{1.8}$$

where $\{\alpha_n\}$ is a real control sequence in the interval $(0, 1)$.

If $T : C \longrightarrow C$ is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Manns algorithm converges weakly to a fixed point of T. Reich [19] showed that the conclusion also holds good in the setting of uniformly convex Banach spaces with a Fréhet differentiable norm. It is well known that Reich's result is one of the fundamental convergence results. Recently, Marino and Xu [12] extended Reich's result

[19] to strict pseudo-contraction mappings in the setting of Hilbert spaces. Very recently, Ceng, AI-Homidan, Ansari and Yao [5] introduced an iterative algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a strict pseudocontraction mapping and obtained a weak convergence theorem.

Very recently, inspired and motivated by the above ideas, Peng [14] introduced some iterative algorithms based on the extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a strict pseudocontraction and the set of the solution sets of a variational inequality for a monotone, Lipschitz continuous mapping and obtained both weak convergence theorem and strong convergence theorem for the sequences generated by these processes. More precisely, Peng [14] proved the following strong and weak convergence theorems.

Theorem PS. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let A be a monotone and δ-Lipschitz continuous mapping of C into H. Let $T : C \longrightarrow C$ be an k-strict pseudo-contractive mapping for some $0 \le k < 1$ such that $\Omega := F(T) \cap VI(C,A) \cap MEP(F,\varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in H, \ \text{chosen arbitrary}, \\ F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - x_{n-1} \rangle \ge 0, \quad \forall y \in C, \\ y_{n-1} = P_C(u_{n-1} - \lambda_{n-1} A u_{n-1}), \\ t_{n-1} = P_C(u_{n-1} - \lambda_{n-1} A y_{n-1}), \\ z_{n-1} = \alpha_{n-1} t_{n-1} + (1 - \alpha_{n-1}) T t_{n-1}, \\ C_{n-1} = \{ z \in C : \|z_{n-1} - z\|^2 \le \|x_{n-1} - z\|^2 - (1 - \alpha_{n-1})(\alpha_{n-1} - k) \|t_{n-1} - T t_{n-1}\|^2 \}, \\ Q_{n-1} = \{ z \in H : \langle x_{n-1} - z, x - x_{n-1} \rangle \ge 0 \}, \\ x_n = P_{C_{n-1} \cap Q_{n-1}} x, \quad \forall n \ge 1. \end{cases} \tag{1.9}$$

Assume that $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{\delta})$, $\{\alpha_n\} \subset [c,d]$ for some c, $d \in (k,1)$ and $\{r_n\} \subset (0,\infty)$ such that $\liminf_{n \longrightarrow \infty} r_n > 0$. Then $\{x_n\}$,$\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ converge strongly to $w = P_\Omega(x)$.

Theorem PW. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let A be a monotone and δ-Lipschitz continuous mapping of C into H. Let $T : C \longrightarrow C$ be an k-strict pseudo-contractive mapping for some $0 \le k < 1$ such that $\Omega := F(T) \cap VI(C,A) \cap MEP(F,\varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in H, \ \text{chosen arbitrary}, \\ F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - x_{n-1} \rangle \ge 0, \quad \forall y \in C, \\ y_{n-1} = P_C(u_{n-1} - \lambda_{n-1} A u_{n-1}), \\ t_{n-1} = P_C(u_{n-1} - \lambda_{n-1} A y_{n-1}), \\ x_n = \alpha_{n-1} t_{n-1} + (1 - \alpha_{n-1}) T t_{n-1}. \end{cases} \tag{1.10}$$

Assume that $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{\delta})$, $\{\alpha_n\} \subset [c,d]$ for some c, $d \in (k,1)$ and $\{r_n\} \subset (0,\infty)$ such that $\liminf_{n \longrightarrow \infty} r_n > 0$. Then $\{x_n\}$,$\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ converge weakly to $w \in \Omega$, where $w = \lim_{n \longrightarrow \infty} P_\Omega x_n$.

On the other hand, very recently, Qin, Cho, Kang and Shang [18] introduced the following algorithm for a finite family of asymptotically k-strict pseudo-contractions. Let $x_0 \in C$ and $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(0,1)$. The sequence $\{x_n\}$ generated by the following way:

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1 - \alpha_0) T_1 x_0, \\ x_2 &= \alpha_1 x_1 + (1 - \alpha_1) T_2 x_1 \\ \cdots & \\ x_N &= \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1}) T_N x_{N-1} \end{aligned}$$

$$\begin{aligned} x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_1^2 x_N \\ &\cdots \\ x_{2N} &= \alpha_{2N-1}x_{2N-1} + (1-\alpha_{2N-1})T_N^2 x_{2N-1} \\ x_{2N+1} &= \alpha_{2N}x_{2N} + (1-\alpha_{2N})T_1^3 x_{2N} \\ &\cdots \end{aligned}$$

is called the explicit iterative sequence of a finite family of asymptotically k-strict pseudo-contractions $\{T_1, T_2, \ldots, T_N\}$. Since, for each $n \geq 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. Hence the above table can be written in the following form:

$$x_n = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)} x_{n-1}, \forall n \geq 1. \tag{1.11}$$

Under appropriate conditions on the parameters, they obtained some strong and weak convergence theorems for a finite family of asymptotically k-strict pseudo-contractions in the framework of Hilbert spaces.

All of the above bring us the following conjectures?.

Question.

(i) *Could both Theorem PS and Theorem PW be extended to more general class of asymptotically strict pseudo-contractive mappings?.*

(ii) *Could we construct the iterative algorithms generalized algorithms* (1.9)*,* (1.10) *and* (1.11) *to approximate a common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of asymptotically strict pseudo-contractions and the set of the solution sets of a variational inequality for a monotone, Lipschitz continuous mapping?.*

Inspired and motivated by the above researchs, we suggest and analyze the iterative schemes for finding a common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of a finite family of asymptotically k-strict pseudo-contractions, and the set of the solution sets of a variational inequality for a monotone, Lipschitz continuous mapping in the framework of Hilbert spaces. Both strong and weak convergence theorems are obtained. Our results extend the corresponding recent results of Peng [14] and Qin, Cho, Kang and Shang [18] and many others.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let symbols $\longrightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In Hilbert space H, it is well known that

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \quad \forall x, y \in H, \tag{2.1}$$

and

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad \forall t \in [0,1], \forall x, y \in H. \tag{2.2}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$. For more details see [24]. It is easy to see that the following is true:

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda A u), \lambda > 0. \tag{2.6}$$

A set-valued mapping $S : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H$, $f \in Sx$ and $g \in Sy$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $S : H \longrightarrow 2^H$ is maximal if the graph of $G(S)$ of S is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping S is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(S)$ implies $f \in Sx$. Let A be a monotone map of C into H, L-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle u - v, w\rangle \geq 0, \forall u \in C\}$. Define

$$Sv = \begin{cases} Av + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{2.7}$$

Then S is the maximal monotone and $0 \in Sv$ if and only if $v \in VI(A, C)$; see [20].

Lemma 2.1. [13] *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and point $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$ the set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\},$$

is convex and closed.

Lemma 2.2. ([12, Lemma 1.3])*Let C be a closed convex subset of H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \geq 0, \forall y \in C.$$

Lemma 2.3 (Kim and Xu [9])**.** *Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H and $T : C \longrightarrow C$ be an asymptotically k-strict pseudo-contractive mapping for some $0 \leq k < 1$ with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and the fixed point set of T is nonempty. Then $(I - T)$ is demiclosed at zero.*

Lemma 2.4. ([17, Lemma 2]) *Let the sequence of numbers $\{a_n\}$ and $\{b_n\}$ be satisfy that*

$$a_{n+1} \leq (1 + b_n)a_n, a_n \geq 0, b_n \geq 0, \text{ and } \sum_{n=1}^{\infty} b_n < \infty, \forall n \geq 1.$$

If $\liminf_{n\longrightarrow\infty} a_n = 0$, then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.5. ([23]) *Let $\{r_n\}, \{s_n\}$ and $\{t_n\}$ be the three nonnegative sequences satisfying the following condition:*

$$t_{n+1} \leq (1 + s_n)r_n + t_n, \quad \forall n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\longrightarrow\infty} r_n$ exists.

Lemma 2.6 (Kim and Xu [9])**.** *Let H be a real Hilbert space, C a nonempty subset of H and $T : C \longrightarrow C$ be an asymptotically k-strict pseudo-contractive mapping. Then T is uniformly L-Lipschitzian.*

Lemma 2.7 (Qin, Cho, Kang, and Shang [18])**.** *Let H be a real Hilbert space, C a nonempty subset of H and $T : C \longrightarrow C$ be an asymptotically k-strict pseudo-contractive mapping. Then the fixed point set $F(T)$ is closed and convex.*

Lemma 2.8. [7] *Let C be a nonempty closed convex subset of H. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\},$$

for all $x \in H$. Assume that either (B1) or (B2) holds. Then, the following conclusions hold:

(i) *For each $x \in H, T_r(x) \neq \emptyset$.*
(ii) *T_r is single-valued;*
(iii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
(iv) *$F(T_r) = MEP(F, \varphi)$.*
(v) *$MEP(F, \varphi)$ is closed and convex.*

3. Main Results

We are now in a position to prove the main result of this paper.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let A be a monotone and δ-Lipschitz continuous mapping of C into H. Let, for each $1 \leq i \leq N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strict pseudo-contractive mapping for some $0 \leq k_i < 1$ and a sequence $\{k_{n,i}\}$. Let $k = \max\{k_i : 1 \leq i \leq N\}$ and $k_n = \max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that $\Omega := \cap_{i=1}^N F(T_i) \cap VI(C, A) \cap MEP(F, \varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in H, \text{ chosen arbitrary,} \\ F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C, \\ y_{n-1} = P_C(u_{n-1} - \lambda_{n-1} A u_{n-1}), \\ t_{n-1} = P_C(u_{n-1} - \lambda_{n-1} A y_{n-1}), \\ z_{n-1} = \alpha_{n-1} t_{n-1} + (1 - \alpha_{n-1}) T_{i(n)}^{h(n)} t_{n-1}, \\ C_{n-1} = \{z \in C : \|z_{n-1} - z\|^2 \leq \|x_{n-1} - z\|^2 + \theta_{n-1}\}, \\ Q_{n-1} = \{z \in H : \langle x_{n-1} - z, x - x_{n-1}\rangle \geq 0\}, \\ x_n = P_{C_{n-1} \cap Q_{n-1}} x, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\theta_{n-1} = (k_{h(n)}^2 - 1)(1 - \alpha_{n-1})\rho_{n-1}^2 \longrightarrow 0$ as $n \longrightarrow \infty$, where $\rho_{n-1} = \sup\{\|x_{n-1} - z\| : z \in \Omega\} < \infty$. Assume that $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\delta})$, $\{\alpha_n\} \subset [c, d]$ for some c, $d \in (k, 1)$ and $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n \longrightarrow \infty} r_n > 0$. Then $\{x_n\}$,$\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ converge strongly to $w = P_\Omega(x)$.

Proof. It is obvious that $VI(C, A)$ is closed and convex. By Lemma 2.7 and Lemma 2.8, we have $\cap_{i=1}^N F(T_i)$ and $MEP(F, \varphi)$ are closed and convex. Hence $\Omega := \cap_{i=1}^N F(T_i) \cap VI(C, A) \cap MEP(F, \varphi)$ is closed and convex. This implies that $P_\Omega(x)$ is well defined.

Next, we prove that the sequence $\{x_n\}$ is well defined. From the definition of C_{n-1} and Q_{n-1}, it is obvious that C_{n-1} is closed and Q_{n-1} is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. We prove that C_{n-1} is convex. For any z_1, $z_2 \in C_{n-1}$ and $t \in (0, 1)$, put $z = tz_1 + (1 - t)z_2$. It is sufficient to show that $z \in C_{n-1}$. Since the inequality

$$\|z_{n-1} - z\|^2 \leq \|x_{n-1} - z\|^2 + \theta_{n-1}$$

is equivalent to

$$2\langle x_{n-1} - z_{n-1}, z\rangle \leq \|x_{n-1}\|^2 - \|z_{n-1}\|^2 + \theta_{n-1},$$

we have $z \in C_{n-1}$. Therefore C_{n-1} is convex and hence $C_{n-1} \cap Q_{n-1}$ is a closed and convex subset of H for any $n \in \mathbb{N}$.

Next, we show that $\Omega \subset C_{n-1}$ for any $n \in \mathbb{N}$. Indeed, let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.8. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u)$. From $u_{n-1} = T_{r_{n-1}} x_{n-1} \in C$, we have

$$\|u_{n-1} - u\| = \|T_{r_{n-1}}(x_{n-1}) - T_{r_{n-1}}(u_{n-1})\| \leq \|x_{n-1} - u\|. \tag{3.2}$$

From (2.5), the monotonicity of A, and $u \in VI(C, A)$, we have

$$\begin{aligned} \|t_{n-1} - u\|^2 &\leq \|u_{n-1} - \lambda_{n-1} A y_{n-1} - u\|^2 - \|u_{n-1} - \lambda_{n-1} A y_{n-1} - t_{n-1}\|^2 \\ &= \|u_{n-1} - u\|^2 - \|u_{n-1} - t_{n-1}\|^2 + 2\lambda_{n-1}\langle A y_{n-1}, u - t_{n-1}\rangle \\ &= \|u_{n-1} - u\|^2 - \|u_{n-1} - t_{n-1}\|^2 + 2\lambda_{n-1}(\langle A y_{n-1} - Au, u - y_{n-1}\rangle \\ &\quad + \langle Au, u - y_{n-1}\rangle + \langle A y_{n-1}, y_{n-1} - t_{n-1}\rangle) \\ &\leq \|u_{n-1} - u\|^2 - \|u_{n-1} - t_{n-1}\|^2 + 2\lambda_{n-1}\langle A y_{n-1}, y_{n-1} - t_{n-1}\rangle \\ &= \|u_{n-1} - u\|^2 - \|u_{n-1} - y_{n-1}\|^2 - 2\langle u_{n-1} - y_{n-1}, y_{n-1} - t_{n-1}\rangle \\ &\quad - \|y_{n-1} - t_{n-1}\|^2 + 2\lambda_{n-1}\langle A y_{n-1}, y_{n-1} - t_{n-1}\rangle \\ &= \|u_{n-1} - u\|^2 - \|u_{n-1} - y_{n-1}\|^2 - \|y_{n-1} - t_{n-1}\|^2 \\ &\quad + 2\langle u_{n-1} - \lambda_{n-1} A y_{n-1} - y_{n-1}, t_{n-1} - y_{n-1}\rangle. \end{aligned}$$

Further, Since $y_{n-1} = P_C(u_{n-1} - \lambda_{n-1}Au_{n-1})$ and A is δ-Lipschitz continuous, we have

$$\begin{aligned}\langle u_{n-1} - \lambda_n Ay_{n-1} - y_{n-1}, t_{n-1} - y_{n-1}\rangle &= \langle u_{n-1} - \lambda_{n-1}Au_{n-1} - y_{n-1}, t_{n-1} - y_{n-1}\rangle \\ &\quad + \langle \lambda_{n-1}Au_{n-1} - \lambda_{n-1}Ay_{n-1}, t_{n-1} - y_{n-1}\rangle \\ &\le \langle \lambda_{n-1}Au_{n-1} - \lambda_{n-1}Ay_{n-1}, t_{n-1} - y_{n-1}\rangle \\ &\le \lambda_{n-1}\delta\|u_{n-1} - y_{n-1}\|\|t_{n-1} - y_{n-1}\|.\end{aligned}$$

So, we have

$$\begin{aligned}\|t_{n-1} - u\|^2 &\le \|u_{n-1} - u\|^2 - \|u_{n-1} - y_{n-1}\|^2 - \|y_{n-1} - t_{n-1}\|^2 \\ &\quad + 2\lambda_{n-1}\delta\|u_{n-1} - y_{n-1}\|\|t_{n-1} - y_{n-1}\| \\ &\le \|u_{n-1} - u\|^2 - \|u_{n-1} - y_{n-1}\|^2 - \|y_{n-1} - t_{n-1}\|^2 + \lambda_{n-1}^2\delta^2\|u_{n-1} - y_{n-1}\|^2 \\ &\quad + \|t_{n-1} - y_{n-1}\|^2 \\ &= \|u_{n-1} - u\|^2 + (\lambda_{n-1}^2\delta^2 - 1)\|u_{n-1} - y_{n-1}\|^2 \\ &\le \|u_{n-1} - u\|^2. \end{aligned} \tag{3.3}$$

It follows from (3.2), (3.3), $z_{n-1} = \alpha_{n-1}t_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)}t_{n-1}$ and $u = T_{i(n)}^{h(n)}u$ that

$$\begin{aligned}\|z_{n-1} - u\|^2 &= \|\alpha_{n-1}(t_{n-1} - u) + (1 - \alpha_{n-1})(T_{i(n)}^{h(n)}t_{n-1} - u)\|^2 \\ &= \alpha_{n-1}\|t_{n-1} - u\|^2 + (1 - \alpha_{n-1})\|T_{i(n)}^{h(n)}t_{n-1} - u\|^2 \\ &\quad - \alpha_{n-1}(1 - \alpha_{n-1})\|T_{i(n)}^{h(n)}t_{n-1} - t_{n-1}\| \\ &\le \alpha_{n-1}\|t_{n-1} - u\|^2 - \alpha_{n-1}(1 - \alpha_{n-1})\|T_{i(n)}^{h(n)}t_{n-1} - t_{n-1}\|^2 \\ &\quad + (1 - \alpha_{n-1})[k_{h(n)}^2\|t_{n-1} - u\|^2 + k\|T_{i(n)}^{h(n)}t_{n-1} - t_{n-1}\|] \\ &\le \|t_{n-1} - u\|^2 + (k_{h(n)}^2 - 1)(1 - \alpha_{n-1})\rho_{n-1}^2 \\ &\quad - (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}t_{n-1} - t_{n-1}\|^2 \\ &\le \|u_{n-1} - u\|^2 + \theta_{n-1} \\ &\quad - (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)}t_{n-1} - t_{n-1}\|^2 \\ &\le \|x_{n-1} - u\|^2 + \theta_{n-1}. \end{aligned} \tag{3.4}$$

Therefore, $u \in C_{n-1}$ for all $n \ge 1$.

Next, we show that

$$\Omega \subset Q_{n-1}, \qquad \forall n \ge 1. \tag{3.5}$$

We prove this by induction. For $n = 1$, we have $\Omega \subset C = Q_0$. Assume that $\Omega \subset Q_{n-1}$. Since x_n is the projection of x_0 onto $C_{n-1} \cap Q_{n-1}$, by Lemma 2.2, we have

$$\langle x_0 - x_n, x_n - z\rangle \ge 0, \qquad \forall z \in C_{n-1} \cap Q_{n-1}.$$

In particular, we have

$$\langle x_0 - x_n, x_n - z\rangle \ge 0$$

for each $u \in \Omega$ and hence $u \in Q_n$. Hence (3.5) holds for all $n \ge 1$. Therefore, we obtain that

$$\Omega \subset C_{n-1} \cap Q_{n-1}, \qquad \forall n \ge 1.$$

Next, we show that $\lim_{n\longrightarrow\infty} \|x_{n-1} - x\|$ exists. Let $l_0 = P_\Omega x$. From $x_n = P_{C_{n-1}\cap Q_{n-1}}x$ and $l_0 \in \Omega \subset C_{n-1} \cap Q_{n-1}$, we have

$$\|x_n - x\| \le \|l_0 - x\|, \tag{3.6}$$

for every $n = 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. From (3.2)-(3.4), we also obtain that $\{t_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded. Since $x_n \in C_{n-1} \cap Q_{n-1} \subset C_{n-1}$ and $x_{n-1} = P_{Q_{n-1}}(x)$, we have

$$\|x_{n-1} - x\| \le \|x_n - x\|,$$

for every $n = 1, 2, \ldots$. Therefore, $\lim_{n\longrightarrow\infty} \|x_{n-1} - x\|$ exists.

Since $x_{n-1} = P_{Q_{n-1}}(x)$ and $x_n \in Q_{n-1}$, using (2.1) we have

$$\begin{aligned} \|x_n - x_{n-1}\|^2 &= \|(x_n - x) - (x_{n-1} - x)\|^2 \\ &= \|x_n - x\|^2 - \|x_{n-1} - x\|^2 - 2\langle x_n - x_{n-1}, x_{n-1} - x\rangle \\ &\leq \|x_n - x\|^2 - \|x_{n-1} - x\|^2 \end{aligned} \tag{3.7}$$

for every $n = 1, 2, \ldots$. It follows from the existence of $\lim_{n\longrightarrow\infty} \|x_n - x\|$ that

$$\lim_{n\longrightarrow\infty} \|x_n - x_{n-1}\| = 0. \tag{3.8}$$

Since $x_n \in C_{n-1}$, we have

$$\|z_{n-1} - x_n\|^2 \leq \|x_{n-1} - x_n\|^2 + \theta_{n-1}.$$

So we have

$$\lim_{n\longrightarrow\infty} \|z_{n-1} - x_n\| = 0.$$

It follows from (3.8) and the last inequality that

$$\|x_{n-1} - z_{n-1}\| \leq \|x_{n-1} - x_n\| + \|x_n - z_{n-1}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.9}$$

for every $n = 1, 2, \ldots$. For $u \in \Omega$, from (3.4) we obtain

$$\|z_{n-1} - u\|^2 \leq \|x_{n-1} - u\|^2 + (\lambda_{n-1}^2\delta^2 - 1)\|u_{n-1} - y_{n-1}\|^2 + \theta_{n-1} \tag{3.10}$$

Thus, we have

$$\begin{aligned} \|u_{n-1} - y_{n-1}\|^2 &\leq \frac{1}{1 - \lambda_{n-1}^2\delta^2}(\|x_{n-1} - u\|^2 - \|z_{n-1} - u\|^2 + \theta_{n-1}) \\ &\leq \frac{1}{1 - b^2\delta^2}[(\|x_{n-1} - u\| + \|z_{n-1} - u\|)\|x_{n-1} - z_{n-1}\| + \theta_{n-1}]. \end{aligned}$$

It follows from (3.9), $\lim_{n\longrightarrow\infty} \theta_{n-1} = 0$ and the boundedness of the sequences $\{x_n\}$ and $\{z_n\}$ that

$$\|u_{n-1} - y_{n-1}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.11}$$

From the definition of t_{n-1} and y_{n-1}, we have

$$\begin{aligned} \|t_{n-1} - y_{n-1}\| &= \|P_C(u_{n-1} - \lambda_{n-1}Ay_{n-1}) - P_C(u_{n-1} - \lambda_{n-1}Au_{n-1})\| \\ &\leq \|(u_{n-1} - \lambda_{n-1}Ay_{n-1}) - (u_{n-1} - \lambda_{n-1}Au_{n-1})\| \\ &\leq \lambda_{n-1}\delta\|y_{n-1} - u_{n-1}\|. \end{aligned} \tag{3.12}$$

Using (3.11), we obtain that $\lim_{n\longrightarrow\infty} \|t_{n-1} - y_{n-1}\| = 0$. From

$$\|u_n - t_n\| \leq \|u_n - y_n\| + \|y_n - t_n\|.$$

Hence

$$\|u_n - t_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.13}$$

Since A is δ-Lipschitz continuous, we have $\|Ay_{n-1} - At_{n-1}\| \longrightarrow 0$.

From the fact that $k < c \leq \alpha_{n-1} \leq d < 1$ and (3.4), we have

$$\begin{aligned} (1-d)(c-k)\|t_{n-1} - T_{i(n)}^{h(n)}t_{n-1}\|^2 &\leq (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|t_{n-1} - T_{i(n)}^{h(n)}t_{n-1}\|^2 \\ &\leq \|x_{n-1} - u\|^2 - \|z_{n-1} - u\|^2 + \theta_{n-1} \\ &\leq (\|x_{n-1} - u\| + \|z_{n-1} - u\|)\|x_{n-1} - z_{n-1}\| + \theta_{n-1}. \end{aligned}$$

It follows from (3.9), $\lim_{n\longrightarrow\infty} \theta_{n-1} = 0$ that

$$\lim_{n\longrightarrow\infty} \|t_{n-1} - T_{i(n)}^{h(n)}t_{n-1}\| = 0. \tag{3.14}$$

For $u \in \Omega$, we have from Lemma 2.8,

$$\begin{aligned} \|u_{n-1} - u\|^2 &= \|T_{r_{n-1}}x_{n-1} - T_{r_{n-1}}u\|^2 \\ &\leq \langle T_{r_{n-1}}x_{n-1} - T_{r_{n-1}}u, x_{n-1} - u\rangle \\ &= \frac{1}{2}\{\|u_{n-1} - u\|^2 + \|x_{n-1} - u\|^2 - \|x_{n-1} - u_{n-1}\|^2\}. \end{aligned}$$

Hence,

$$\|u_{n-1} - u\|^2 \leq \|x_{n-1} - u\|^2 - \|x_{n-1} - u_{n-1}\|^2. \tag{3.15}$$

It follows from (3.4) and (3.15) that

$$\begin{aligned}\|z_{n-1}-u\|^2 &\leq \|u_{n-1}-u\|^2+\theta_{n-1}\\ &\leq \|x_{n-1}-u\|^2-\|x_{n-1}-u_{n-1}\|^2+\theta_{n-1}. \end{aligned} \tag{3.16}$$

Hence

$$\begin{aligned}\|x_{n-1}-u_{n-1}\|^2 &\leq \|x_{n-1}-u\|^2-\|z_{n-1}-u\|^2+\theta_{n-1}\\ &\leq (\|x_{n-1}-u\|+\|z_{n-1}-u\|)\|x_{n-1}-z_{n-1}\|+\theta_{n-1}. \end{aligned} \tag{3.17}$$

Since $\|x_{n-1}-z_{n-1}\| \longrightarrow 0$, $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain

$$\lim_{n\longrightarrow\infty}\|x_{n-1}-u_{n-1}\|=0. \tag{3.18}$$

Observe that

$$\|t_{n-1}-x_{n-1}\|\leq\|t_{n-1}-u_{n-1}\|+\|x_{n-1}-u_{n-1}\|.$$

Using the inequality (3.13) and (3.18), we get

$$\lim_{n\longrightarrow\infty}\|t_{n-1}-x_{n-1}\|=0. \tag{3.19}$$

Moreover, we have

$$\|t_n-t_{n-1}\|\leq\|t_n-x_n\|+\|x_n-x_{n-1}\|+\|x_{n-1}-t_{n-1}\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{3.20}$$

It follows that

$$\lim_{n\longrightarrow\infty}\|t_n-t_{n+j}\|=0, \forall j=1,2,\dots,N. \tag{3.21}$$

We claim that

$$\lim_{n\longrightarrow\infty}\|t_n-T_lt_n\|=0, \forall l=1,2,\dots,N.$$

Since, for any positive integer $n>N$, it can be written as $n=(k(n)-1)N+i(n)$, where $i(n)\in\{1,2,\dots,N\}$. Observe that

$$\begin{aligned}\|t_{n-1}-T_nt_{n-1}\| &\leq \|t_{n-1}-T_{i(n)}^{h(n)}t_{n-1}\|+\|T_{i(n)}^{h(n)}t_{n-1}-T_nt_{n-1}\|\\ &= \|t_{n-1}-T_{i(n)}^{h(n)}t_{n-1}\|+\|T_{i(n)}^{h(n)}t_{n-1}-T_{i(n)}t_{n-1}\|\\ &\leq \|t_{n-1}-T_{i(n)}^{h(n)}t_{n-1}\|+L\|T_{i(n)}^{h(n)-1}t_{n-1}-t_{n-1}\|\\ &\leq \|t_{n-1}-T_{i(n)}^{h(n)}t_{n-1}\|+L[\|T_{i(n)}^{h(n)-1}t_{n-1}-T_{i(n-N)}^{h(n)-1}t_{n-N}\|\\ &\quad+\|T_{i(n-N)}^{h(n)-1}t_{n-N}-t_{(n-N)-1}\|+\|t_{(n-N)-1}-t_{n-1}\|]. \end{aligned} \tag{3.22}$$

Since, for each $n>N$, $n=(n-N)(\mod N)$ and $n=(k(n)-1)N+i(n)$, we have

$$n-N=(k(n)-1)N+i(n)=(k(n-N)-1)N+i(n-N).$$

That is

$$k(n-N)=k(n)-1,\ i(n-N)=i(n).$$

Observe that

$$\|T_{i(n)}^{h(n)-1}t_{n-1}-T_{i(n-N)}^{h(n)-1}t_{n-N}\|\leq L\|t_{n-1}-t_{n-N}\| \tag{3.23}$$

and

$$\begin{aligned}\|T_{i(n-N)}^{h(n)-1}t_{n-N}-t_{(n-N)-1}\| &= \|T_{i(n-N)}^{h(n-N)}t_{n-N}-t_{(n-N)-1}\|\\ &\leq \|T_{i(n-N)}^{h(n-N)}t_{n-N}-T_{i(n-N)}^{h(n)-N}t_{(n-N)-1}\|+\|T_{i(n-N)}^{h(n-N)}t_{(n-N)-1}-u_{n-N-1}\|\\ &\leq L\|t_{(n-N)-1}-t_{n-N}\|+\|T_{i(n-N)}^{h(n-N)}t_{(n-N)-1}-t_{n-N-1}\|. \end{aligned} \tag{3.24}$$

It follows from (3.22), (3.23) and (3.24) that

$$\begin{aligned}\|t_{n-1}-T_nt_{n-1}\| &\leq \|t_{n-1}-T_{i(n)}^{h(n)}t_{n-1}\|+L(L\|t_{n-1}-t_{n-N}\|+L\|t_{(n-N)-1}-t_{n-N}\|\\ &\quad+\|T_{i(n-N)}^{h(n-N)}t_{(n-N)-1}-u_{n-N-1}\|+\|t_{(n-N)-1}-t_{n-1}\|). \end{aligned} \tag{3.25}$$

Applying (3.14) and (3.21) to (3.25), we obtain

$$\lim_{n\longrightarrow\infty}\|t_{n-1}-T_nt_{n-1}\|=0. \tag{3.26}$$

Observing that

$$\begin{aligned}\|t_n - T_n t_n\| &\leq \|t_n - t_{n-1}\| + \|t_{n-1} - T_n t_{n-1}\| + \|T_n t_{n-1} - T_n t_n\|.\\ &\leq (1+L)\|t_n - t_{n-1}\| + \|t_{n-1} - T_n t_{n-1}\|.\end{aligned}$$

From (3.19) and (3.26), one can easily see that

$$\lim_{n\longrightarrow\infty} \|t_n - T_n t_n\| = 0.$$

We also have

$$\begin{aligned}\|t_n - T_{n+j} t_n\| &\leq \|t_n - t_{n+j}\| + \|t_{n+j} - T_{n+j} t_{n+j}\| + \|T_{n+j} t_{n+j} - T_{n+j} t_n\|.\\ &\leq (1+L)\|t_n - t_{n+j}\| + \|t_{n+j} - T_{n+j} t_{n+j}\|\end{aligned}$$

for any $j = 1, 2, ..., N$, which give that

$$\lim_{n\longrightarrow\infty} \|t_n - T_l t_n\| = 0;\ \forall l = 1, 2, \ldots, N. \tag{3.27}$$

Moreover, for each $l \in \{1, 2, \ldots, N\}$, we obtain that

$$\begin{aligned}\|x_n - T_l x_n\| &\leq \|x_n - t_n\| + \|t_n - T_l t_n\| + \|T_l t_n - T_l x_n\|\\ &\leq (1+L)\|x_n - t_n\| + \|t_n - T_l t_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (3.28)\end{aligned}$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From the fact that

$$\|x_{n-1} - t_{n-1}\| \longrightarrow 0, \|t_{n-1} - x_{n-1}\| \longrightarrow 0, \|y_{n-1} - u_{n-1}\| \longrightarrow 0,$$

we obtain that

$$u_{n_i} \rightharpoonup w, t_{n_i} \rightharpoonup w \text{ and } y_{n_i} \rightharpoonup w, \text{ as } i \longrightarrow \infty.$$

Since $\{x_{n_i}\}, \subset C$ and C is closed and convex, we obtain $w \in C$.
Next, we prove that $w \in \Omega := \cap_{i=1}^N F(T_i) \cap VI(C, A) \cap MEP(F, \varphi)$.

(a) First, we show that $w \in \cap_{i=1}^N F(T_i)$.

From (3.27), we get

$$\|t_{n_i} - T_l t_{n_i}\| \longrightarrow 0 \text{ as } i \longrightarrow \infty,\ \forall l = 1, 2, \ldots, N.$$

By Lemma 2.3, we know that $I - T_l$ is demiclosed at zero and hence $w \in F(T_l), \forall l = 1, 2, \ldots, N$. That is

$$w \in \cap_{i=1}^N F(T_i).$$

(b) Next we prove that $w \in MEP(F, \varphi)$.

By $u_n = T_{r_n} x_n$, we know that

$$F(u_n, y) + \varphi(y) + \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C.$$

It follows from (A2) that

$$\varphi(y) + \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \qquad \forall y \in C.$$

Hence,

$$\varphi(y) + \varphi(u_{n_i}) + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}), \qquad \forall y \in C.$$

It follows from (A4), (A5), and the weakly lower semicontinuity of φ, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$ and $u_{n_i} \rightharpoonup w$ that

$$F(y, w) + \varphi(w) - \varphi(y) \leq 0, \qquad \forall y \in C.$$

For $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $F(y_t, w) + \varphi(w) + \varphi(y_t) \leq 0$. So by (A4) and the convexity of φ, we have

$$\begin{aligned}0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t)\\ &\leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t)\\ &\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)].\end{aligned}$$

Dividing by t, we get

$$F(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0.$$

Letting $t \longrightarrow 0$, it follows from (A3) and the weakly lower semicontinuity of φ that

$$F(w, y) + \varphi(y) - \varphi(w) \geq 0,$$

for all $y \in C$ and hence $w \in MEP(F, \varphi)$.

(c) Next, we prove that $w \in VI(A, C)$.

For this purpose, let S be the maximal monotone mapping defined by (2.7):

$$Sv = \begin{cases} Av + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

For any given $(v, q) \in G(S)$, hence $q - Av \in N_C(v)$. Since $v_n \in C$, we have

$$\langle v - v_n, q - Av \rangle \geq 0.$$

On the other hand, from $y_n = P_C(u_n - \lambda_n Au_n)$, we have

$$\langle v - y_n, y_n - (u_n - \lambda_n Au_n) \rangle \geq 0 \tag{3.29}$$

that is,

$$\langle v - y_n, \frac{y_n - u_n}{\lambda_n} + Au_n \rangle \geq 0. \tag{3.30}$$

Therefore, we obtain

$$\begin{aligned} \langle v - y_{n_i}, w \rangle &\geq \langle v - y_{n_i}, Av \rangle \geq \langle v - y_{n_i}, Av \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \rangle \\ &= \langle v - y_{n_i}, Av - Au_{n_i} - \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &= \langle v - y_{n_i}, Av - Ay_{n_i} \rangle + \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i} \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\ &\geq \langle v - y_{n_i}, Ay_{n_i} \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i} \rangle \\ &= \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i} \rangle - \langle v - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle. \end{aligned} \tag{3.31}$$

Noting that $\|y_{n_i} - u_{n_i}\| \longrightarrow 0$ as $i \longrightarrow \infty$ and A is Lipschitz continuous, hence from (3.31), we obtain

$$\langle v - w, q \rangle \geq 0.$$

Since S is maximal monotone, we have $w \in S^{-1}0$, and hence $w \in VI(A, C)$. From (a), (b) and (c), we conclude that $w \in \Omega$.

From $l_0 = P_\Omega(x)$, $w \in \Omega$ and (3.6), we have

$$\|l_0 - x\| \leq \|w - x\| \leq \liminf_{i \longrightarrow \infty} \|x_{n_i} - x\| \leq \limsup_{i \longrightarrow \infty} \|x_{n_i} - x\| \leq \|l_0 - x\|.$$

So, we obtain

$$\lim_{i \longrightarrow \infty} \|x_{n_i} - x\| = \|w - x\|.$$

From $x_{n_i} - x \rightharpoonup w - x$, we have $x_{n_i} - x \longrightarrow w - x$ and hence $x_{n_i} \longrightarrow w$. Since $x_n = P_{Q_n}(x)$ and $l_0 \in \Omega \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|l_0 - x_{n_i}\|^2 = \langle l_0 - x_{n_i}, x_{n_i} - x \rangle + \langle l_0 - x_{n_i}, x - l_0 \rangle \geq \langle l_0 - x_{n_i}, x - l_0 \rangle.$$

As $i \longrightarrow \infty$, we obtain $-\|l_0 - w\|^2 \geq \langle l_0 - w, x - l_0 \rangle \geq 0$ by $l_0 = P_\Omega(x)$ and $w \in \Omega$. Hence we have $w = l_0$. This implies that $x_n \longrightarrow l_0$. It is easy to see $u_n \longrightarrow l_0$, $y_n \longrightarrow l_0$, $t_n \longrightarrow l_0$ and $z_n \longrightarrow l_0$. The proof is completed. □

Remark 3.2. Theorem 3.1 generalizes and improves Theorem PS in [14] from a k-strict pseudo-contractive mapping to more general class of asymptotically k-strict pseudo-contractive mappings?.

Theorem 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \longrightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let A be a monotone and δ-Lipschitz continuous mapping of C into H. Let, for each $1 \le i \le N$, $T_i : C \longrightarrow C$ be an asymptotically k_i-strict pseudo-contractive mapping for some $0 \le k_i < 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = \max\{k_i : 1 \le i \le N\}$ and $k_n = \max\{k_{n,i} : 1 \le i \le N\}$. Assume that $F := \cap_{i=1}^{N} F(T_i) \cap VI(C,A) \cap MEP(F,\varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{t_n\}$, $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in H, \text{ chosen arbitrary,} \\ F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C, \\ y_{n-1} = P_C(u_{n-1} - \lambda_{n-1}Au_{n-1}), \\ t_{n-1} = P_C(u_{n-1} - \lambda_{n-1}Ay_{n-1}), \\ x_n = \alpha_{n-1}t_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)} t_{n-1}, \quad \forall n \ge 1, \end{cases} \tag{3.32}$$

where $\theta_{n-1} = (k_{h(n)}^2 - 1)(1 - \alpha_{n-1})\rho_{n-1}^2 \longrightarrow 0$ as $n \longrightarrow \infty$, where $\rho_{n-1} = \sup\{\|x_{n-1} - z\| : z \in F\} < \infty$. Assume that $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c,d]$ for some c, $d \in (k,1)$ and $\{r_n\} \subset (0,\infty)$ such that $\liminf_{n\longrightarrow\infty} r_n > 0$. Then $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ converge weakly to $w \in \Omega$, where $w = \lim_{n\longrightarrow\infty} P_\Omega x_n$.

Proof. Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.8. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u)$. As in the proof of Theorem 3.1, we know that (3.2), (3.3), (3.12) and (3.15) still hold. It follows from (3.2), (3.3) and $x_n = \alpha_{n-1}t_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)} t_{n-1}$ that

$$\begin{aligned} \|x_n - u\|^2 &= \|\alpha_{n-1}(t_{n-1} - u) + (1 - \alpha_{n-1})(T_{i(n)}^{h(n)} t_{n-1} - u)\|^2 \\ &= \alpha_{n-1}\|t_{n-1} - u\|^2 + (1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} t_{n+1} - u\|^2 \\ &\quad - \alpha_{n-1}(1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} t_{n-1} - u_{n-1}\| \\ &\le \alpha_{n-1}\|t_{n-1} - u\|^2 - \alpha_{n-1}(1 - \alpha_{n-1})\|T_{i(n)}^{h(n)} t_{n-1} - t_{n-1}\| \\ &\quad + (1 - \alpha_{n-1})[k_{h(n)}^2\|t_{n-1} - u\|^2 + k\|T_{i(n)}^{h(n)} t_{n-1} - u_{n-1}\|] \\ &\le k_{h(n)}^2\|t_{n-1} - u\|^2 - (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)} t_{n-1} - t_{n-1}\|^2 \\ &\le k_{h(n)}^2\|u_{n-1} - u\|^2 + k_{h(n)}^2(\lambda_n^2 k^2 - 1)\|u_{n-1} - y_{n-1}\|^2 \\ &\quad - (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)} t_{n-1} - t_{n-1}\|^2 \\ &\le k_{h(n)}^2\|x_{n-1} - u\|^2 + k_{h(n)}^2(\lambda_n^2 k^2 - 1)\|u_{n-1} - y_{n-1}\|^2 \\ &\quad - (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)} t_{n-1} - t_{n-1}\|^2 \\ &\le (1 + (k_{h(n)}^2 - 1))\|x_{n-1} - q\|^2. \end{aligned} \tag{3.33}$$

It follows from Lemma 2.5 that $\lim_{n\longrightarrow\infty} \|x_n - u\|$ exists and hence $\{x_n\}$ is bounded. From (3.2) and (3.3), we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded. By (3.33), we have

$$k_{h(n)}^2(1 - \lambda_n^2 k^2)\|u_{n-1} - y_{n-1}\|^2 \le k_{h(n)}^2\|x_{n-1} - u\|^2 - \|x_n - u\|^2.$$

and hence

$$\|u_{n-1} - y_{n-1}\|^2 \le \frac{1}{k_{h(n)}^2(1 - \lambda_n^2 k^2)}(k_{h(n)}^2\|x_{n-1} - u\|^2 - \|x_n - u\|^2).$$

The existence of $\lim_{n\longrightarrow\infty} \|x_n - u\|$ and $\lim_{n\longrightarrow\infty} k_{h(n)} = 1$ imply that

$$\|u_{n-1} - y_{n-1}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

It follows from (3.12) that $\lim_{n\longrightarrow\infty} \|t_n - y_n\| = 0$. From $\|u_n - t_n\| \le \|u_n - y_n\| + \|y_n - t_n\|$ we also have

$$\|u_n - t_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

As A is δ-Lipschitz continuous, we have $\|Ay_{n-1} - At_{n-1}\| \longrightarrow 0$.
From (3.33) and (3.2), we also have

$$\|x_n - u\|^2 \leq k_{h(n)}^2 \|x_{n-1} - u\|^2 - (1 - \alpha_{n-1})(\alpha_{n-1} - k)\|T_{i(n)}^{h(n)} t_{n-1} - t_{n-1}\|^2, \tag{3.34}$$

for every $n = 1, 2, \ldots$. From $k < c \leq \alpha_n \leq d < 1$ and (3.34), we have

$$\begin{aligned}(1-d)(c-k)\|t_{n-1} - T_{i(n)}^{h(n)} t_{n-1}\|^2 &\leq (1 - \alpha_{n-1})(k - \alpha_{n-1})\|t_{n-1} - T_{i(n)}^{h(n)} t_{n-1}\|^2 \\ &\leq k_{h(n)}^2 \|x_{n-1} - u\|^2 - \|x_n - u\|^2.\end{aligned}$$

This implies that

$$\lim_{n \longrightarrow \infty} \|t_{n-1} - T_{i(n)}^{h(n)} t_{n-1}\| = 0. \tag{3.35}$$

Then, by (3.33) and (3.15), we have

$$\begin{aligned}\|x_n - u\|^2 &\leq k_{h(n)}^2 \|u_{n-1} - u\|^2 \\ &\leq k_{h(n)}^2 \|x_{n-1} - u\|^2 - k_{h(n)}^2 \|x_{n-1} - u_{n-1}\|^2.\end{aligned}$$

Hence,

$$k_{h(n)}^2 \|x_{n-1} - u_{n-1}\|^2 \leq k_{h(n)}^2 \|x_{n-1} - u\|^2 - \|x_n - u\|^2. \tag{3.36}$$

Thus, we obtain

$$\|x_{n-1} - u_{n-1}\| \longrightarrow 0.$$

From $\|t_{n-1} - x_{n-1}\| \leq \|t_{n-1} - u_{n-1}\| + \|x_{n-1} - u_{n-1}\|$, we also have

$$\|t_{n-1} - x_{n-1}\| \longrightarrow 0.$$

The same methods as in the proof of Theorem 3.1 imply that for each $l \in \{1, 2, \ldots, N\}$,

$$\lim_{n \longrightarrow \infty} \|x_n - T_l x_n\| = 0. \tag{3.37}$$

Since $\{x_n\}$ is bounded and H is reflexive, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From $\|x_{n-1} - u_{n-1}\| \longrightarrow 0$ and $\|t_{n-1} - x_{n-1}\| \longrightarrow 0$, we obtain that $u_{n_i} \rightharpoonup w$ and $t_{n_i} \rightharpoonup w$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we obtain $w \in C$. Exactly as in the proof of Theorem 3.1, we can obtain that $w \in \Omega$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$. Then $z \in \Omega$. Let us show $w = z$. Assume that $w \neq z$. From the Opial condition, we have

$$\begin{aligned}\lim_{n \longrightarrow \infty} \|x_n - w\| &= \liminf_{i \longrightarrow \infty} \|x_{n_i} - w\| \\ &< \lim_{i \longrightarrow \infty} \|x_{n_i} - z\| = \lim_{n \longrightarrow \infty} \|x_n - z\| \\ &= \liminf_{j \longrightarrow \infty} \|x_{n_j} - z\| \\ &< \liminf_{j \longrightarrow \infty} \|x_{n_j} - w\| = \lim_{n \longrightarrow \infty} \|x_n - w\|.\end{aligned} \tag{3.38}$$

This is a contradiction. Thus, we have $w = z$. This implies that $x_n \rightharpoonup w \in \Omega$. Since $\|x_{n-1} - u_{n-1}\| \longrightarrow 0$, we have $u_n \rightharpoonup w \in \Omega$. Since $\|y_{n-1} - u_{n-1}\| \longrightarrow 0$, we have also $y_n \rightharpoonup w \in \Omega$.

Now we put $w_n = P_\Omega(x_n)$. We show that $w = \lim_{n \longrightarrow \infty} w_n$. From $w_n = P_\Omega(x_n)$ and $w \in \Omega$, we have

$$\langle w - w_n, w_n - x_n \rangle \geq 0. \tag{3.39}$$

From (3.33) and Lemma 3.2 in [22], we know that $\{w_n\}$ converges strongly to some $w_0 \in \Omega$. Then, we have

$$\langle w - w_0, w_0 - w \rangle \geq 0$$

and hence $w = w_0$. The proof is now complete. □

Remark 3.4. (i) Theorem 3.3 generalizes and improves Theorem PW in [14] from a k-strict pseudo-contractive mapping to more general class of asymptotically k-strict pseudo-contractive mappings.

(ii) Setting $A = 0$ and $\varphi = 0$ in Theorem 3.3, we obtain the result which generalizes and improves Theorem 3.1 in Ceng, AI-Homidan, Ansari and Yao [5] from a k-strict pseudo-contractive mapping to a finite family of asymptotically k-strict pseudo-contractive mappings?.

A direct consequences of Theorem 3.3, we derive the following theorem of Qin, Cho, Kang and Shang [18].

Theorem 3.5. ([18, Theorem 2.1]) *Let C be nonempty closed convex subset of a real Hilbert space H and $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N, T_i : C \longrightarrow C$ be an asymptotically k_i-strict pseudo-contractive mapping for some $0 \leq k_i \leq 1$ and a sequence $\{k_{n,i}\}$ such that $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $k = max\{k_i : 1 \leq i \leq N\}$ and $\{k_n\} = max\{k_{n,i} : 1 \leq i \leq N\}$. Assume that $\cap_{i=1}^{N} F(T_i)$ is nonempty. For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated initially by arbitrary element $x_0 \in C$ and then by*

$$x_n = \alpha_{n-1}x_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)}x_{n-1}; \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$. Then, the sequences $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$, $r_n = 1$ for all $n \geq 0$ and $A = 0$ in Theorem 3.3. Thus, we have $u_n = x_n$. Then the sequence $\{x_n\}$ generated in Theorem 3.5 converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^{N}$.

□

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