รายงานวิจัยฉบับสมบูรณ์

โครงการ โดมิเนชันในไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดและของกึ่งกรุปเชิงเดียวบริบูรณ์

โดย นายสายัญ ปันมา

ม.ค. 2559

สัญญาเลขที่ TRG5680029

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ผู้วิจัย	**สังกัด**
นายสายัญ ปันมา	มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยเชียงใหม่

บทคัดย่อ

รหัสโครงการ : TRG5680029

ชื่อโครงการ : โดมิเนชันในไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ดและของกึ่งกรุปเชิงเดียวบริบูรณ์

ชื่อนักวิจัย : นายสายัญ ปันมา

อีเมลล์ : panmayan@yahoo.com

ระยะเวลาโครงการ : 2 ปี

บทคัดย่อ: ให้ S เป็นกึ่งกรุป และ $A \subseteq S$ และให้ $Cay(S, A)$ เป็นไดกราฟเคย์เลย์ของ S ที่สอดคล้องกับ A ในงานวิจัยนี้เราได้หาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด และของกึ่งกรุปเชิงเดียวบริบูรณ์

คำหลัก: ไดกราฟเคย์เลย์ , กึ่งกรุปคลิฟฟอร์ด, กึ่งกรุปเชิงเดียวบริบูรณ์, จำนวนโดมิเนชัน, จำนวนโททอลโดมิเนชัน

Abstract

Project Code : TRG5680029

Project Title : Dominations in Cayley digraphs of Clifford semigroups and of completely simple semigroups

Investigator : Mr. Sayan Panma

E-mail Address : panmayan@yahoo.com

Project Period : 2 years

Abstract: Let S be a semigroup, $A \subseteq S$ and $Cay(S, A)$ the Cayley digraph of S with respect to A. In this research, we attempt to find the domination number and the total domination number of Cayley digraphs of Clifford semigroups and completely simple semigroups.

Keywords : Cayley digraph, Clifford semigroup, Completely simple semigroup, domination number, total domination number

1. ความสำคัญและที่มาของปัญหา

การศึกษาเกี่ยวกับไดกราฟเคย์เลย์ (Cayley digraph) และกราฟเคย์เลย์(Cayley graph) มีมาแล้วกว่า 130 ปี ผู้ที่ให้นิยามและเริ่มศึกษาเป็นคนแรกคือ Prof. Arthur Cayley ในปี ค.ศ. 1878 โดยเริ่มแรกได้นิยาม ไดกราฟเคย์เลย์ และกราฟเคย์เลย์ มาจากกรุป ดังนี้ ให้ G เป็นกรุป และ $A \subseteq G$ ไดกราฟเคย์เลย์ของ G ที่สอดคล้องกับ A คือ *ไดกราฟ (directed graph)* ที่มี G เป็น *เซตของจุด (vertex set)* และ $E = \{(g, ga) \mid g \in G, a \in A\}$ เป็น *เซตของเส้น (edge set)* จะเขียนแทน ไดกราฟเคย์เลย์ของ G ที่สอดคล้องกับ A ด้วยสัญลักษณ์ $Cay(G, A)$ และจะเรียก $Cay(G, A)$ สั้น ๆ ว่า ไดกราฟเคย์เลย์ของกรุป G และ ถ้า $A = A^{-1} = \{a^{-1} \mid a \in A\}$ แล้วจะเรียก $Cay(G, A)$ ว่า กราฟเคย์เลย์ของกรุป G

จากนั้นได้มีการนำไดกราฟเคย์เลย์ของกรุปและกราฟเคย์เลย์ของกรุป ไปศึกษาอย่างกว้างขวาง เช่น ศึกษาลักษณะเฉพาะของไดกราฟที่เป็นไดกราฟเคย์เลย์ของกรุป ศึกษาลักษณะเฉพาะของกราฟที่เป็นกราฟเคย์เลย์ของกรุป ศึกษาลักษณะของคลาสของ digraph endomorphism ของไดกราฟเคย์เลย์ของกรุปทั้งหมด ศึกษาลักษณะของคลาสของ graph endomorphism ของกราฟเคย์เลย์ของกรุปทั้งหมด ศึกษาลักษณะของคลาสของ digraph automorphism ของไดกราฟเคย์เลย์ของกรุปทั้งหมด และศึกษาลักษณะของคลาสของ graph automorphism ของกราฟเคย์เลย์ของกรุปทั้งหมด เป็นต้น

ต่อมาได้มีผู้ให้ความสนใจอย่างแพร่หลาย และนำกราฟเคย์เลย์ของไปประยุกต์ใช้ในหลายแขนงวิชา เช่น Biology, Chemistry, Physics, Computer science

อาทิเช่น “ Simulations between cellular automata on cayley graphs”

“ Quantum expanders from any classical Cayley graph expander”

“ Quantum walks on Cayley graphs”

เนื่องจากไดกราฟเคย์เลย์ของกรุปมีการศึกษาอย่างแพร่หลายอาทิเช่น [2], [4-7], [10], [18-19] ดังนั้น จึงมีผู้สนใจที่จะขยายการศึกษาไปบนไดกราฟเคย์เลย์ของกึ่งกรุป ซึ่งนิยามของไดกราฟเคย์เลย์ของกึ่งกรุป จะนิยามเช่นเดียวกันกับนิยามของไดกราฟเคย์เลย์ของกรุป เพียงแต่เปลี่ยนพีชคณิตจากกรุป G ไปเป็นกึ่งกรุป S แทน

ไดกราฟเคย์เลย์ของกึ่งกรุปได้ถูกนำไปศึกษาอย่างกว้างขวางดูได้จาก [1],[8],[13-14],[16-17] ซึ่งการศึกษาในเอกสารอ้างอิงเหล่านี้ส่วนใหญ่ได้ขยายผลการศึกษามาจากไดกราฟเคย์เลย์ของกรุป

ด้วยเหตุนี้ผู้วิจัยจึงสนใจที่จะขยายงานวิจัยที่มีผู้ทำไว้บนไดกราฟเคย์เลย์ของกรุป ไปสู่ไดกราฟเคย์เลย์ของกึ่งกรุป ซึ่งเรื่องที่ผู้วิจัยสนใจคือ จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุป ซึ่งมีนิยามดังต่อไปนี้

กำหนดให้ $D = (V, E)$ เป็นไดกราฟ และ $X \subseteq V$ จะเรียกเซต X ว่า *เซตโดมิเนติง(dominating set)* ของ D ถ้าทุกๆ $v \in V \setminus X$ มี $x \in X$ ที่ซึ่ง $(x, v) \in E$ และกำหนด *จำนวนโดมิเนชัน(domination number)* $\gamma(D) = \min\{|X| : X \text{ เป็นเซตโดมิเนติงของ } D\}$

จะเรียกเซต X ว่า *เซตโททอลโดมิเนติง(total dominating set)*ของ D ถ้าทุกๆ $v \in V$ มี $x \in X$ ที่ซึ่ง $(x,v) \in E$ และกำหนด *จำนวนโททอลโดมิเนชัน(total domination number)* $\gamma_t(D) = \min\{|X| : X$ เป็นเซตโททอลโดมิเนติงของ $D\}$

สำหรับนิยามของจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของกราฟ สามารถนิยามได้ทำนองเดียวกันโดยเปลี่ยนเส้น (x,v) เป็น $\{x,v\}$

จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของกราฟได้มีผู้ศึกษากันอย่างกว้างขวาง โดยในปี 1998 [11] Prof. T. Haynes และคณะ ได้รวบรวมทฤษฎีในบทความต่าง ๆ เขียนเป็นหนังสือชื่อว่า Fundamentals of domination in graphs และในปีเดียวกันนี้ [12] Prof. C. Lee ได้ศึกษามิเนชันในไดกราฟโดยได้ให้ขอบเขตของจำนวนโดมิเนชันของไดกราฟ และในปี 1998 นี้เช่นกัน [19] Prof. B. Zelinka ได้ศึกษาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของกราฟเคย์เลย์ของกรุปซึ่งกรุปดังกล่าวอยู่ในรูปผลคูณของกรุปวัฏจักรจำกัด จากนั้นก็ได้มีผู้สนใจศึกษาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของกราฟเคย์เลย์ของกรุปอย่างต่อเนื่องดังในเอกสารอ้างอิง [4-7]

ผู้วิจัยได้ข้อสังเกตว่าการศึกษาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปยังไม่ได้มีการศึกษาดังนั้นผู้วิจัยจึงสนใจที่จะศึกษาเพื่อหาจำนวนดังกล่าว

เนื่องจากกึ่งกรุปสามารถจำแนกได้หลายชนิด และเพื่อที่จะทำให้สามารถเชื่อมโยงกับทฤษฎีบทต่าง ๆ ที่เกี่ยวกับจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของกราฟเคย์เลย์ของกรุป ผู้วิจัยจึงได้สนใจกึ่งกรุปที่นิยามมาจากกรุป กึ่งกรุปดังกล่าวคือ *กึ่งกรุปคลิฟฟอร์ด (Clifford semigroup)* และ *กึ่งกรุปเชิงเดียวบริบูรณ์ (completely simple semigroup)* ซึ่งมีนิยามดังนี้

ให้ Y เป็น *กึ่งแลตทิซ(semilattice)* และ $\{G_\alpha \mid \alpha \in Y\}$ เป็นวงศ์ของกรุป และทุกๆ $\alpha, \beta \in Y$ และ $\alpha \le \beta$ มีฟังก์ชันสาทิสสัณฐาน $f_{\alpha,\beta} : G_\alpha \to G_\beta$ ซึ่ง

(1) สำหรับทุกๆ $\alpha \in Y$ แล้ว $f_{\alpha,\alpha} = id_{G_\alpha}$ (id_{G_α} คือฟังก์ชันเอกลักษณ์บน G_α) และ

(2) สำหรับทุกๆ $\alpha, \beta, \gamma \in Y$ ซึ่ง $\alpha \le \beta \le \gamma$ แล้ว $f_{\beta,\alpha} f_{\gamma,\beta} = f_{\gamma,\alpha}$

กำหนด $S = \bigcup_{\alpha \in Y} G_\alpha$ และการดำเนินการบน S ดังนี้

สำหรับ $x \in G_\alpha$ และ $y \in G_\beta$ แล้ว $xy = f_{\alpha,\alpha\beta}(x) f_{\beta,\alpha\beta}(y)$

สามารถพิสูจน์ได้ไม่ยากว่า S กับการดำเนินการข้างต้นเป็นกึ่งกรุป

จะเรียกกึ่งกรุปนี้ว่า กึ่งกรุปคลิฟฟอร์ด หรือเรียกอีกอย่างหนึ่งว่า *กึ่งแลตทิซอย่างเข้มของกรุป (strong semilattice of groups)* เขียนแทนด้วยสัญลักษณ์ $[Y : G_\alpha, f_{\alpha,\beta}]$

ในปี 2006 ผู้วิจัย Prof. U. Knauer Prof. N. Na Chiangmai และ Prof. Sr. Arworn [16] ได้ให้ลักษณะเฉพาะของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด

<u>ข้อสังเกต 1.</u> ถ้า กึ่งแลตทิซ Y มีสมาชิกเพียงตัวเดียว แล้วกึ่งกรุปคลิฟฟอร์ดจะเป็นกรุป

ให้ G เป็น กรุป, I และ Λ เป็นเซตที่ไม่เป็นเซตว่าง

และ $P=\left[p_{\lambda i} \right]$ เป็น $\Lambda \times I$ เมทริกซ์ ซึ่ง $p_{\lambda i} \in G$ ทุก $i \in I$ และ ทุก $\lambda \in \Lambda$

ให้ $S = G \times I \times \Lambda = \{(g,i,\lambda) \mid g \in G, i \in I, \lambda \in \Lambda\}$

และนิยามการดำเนินการบน S ดังนี้ $(g,i,\lambda)(h,j,\beta) = (gp_{\lambda j}h,i,\beta)$

สามารถพิสูจน์ได้ไม่อยากว่า S เป็นเซมิกรุปภายใต้การดำเนินการข้างต้น

จะเรียกกึ่งกรุปนี้ว่า กึ่งกรุปเชิงเดียวบริบูรณ์ เขียนแทนด้วยสัญลักษณ์ $\mathrm{M}(G,I,\Lambda;P)$

ในปี 2011 ผู้วิจัย J. Meksawang และ Prof. U. Knauer [13] ได้ให้ลักษณะเฉพาะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์

<u>ข้อสังเกต 2.</u> ถ้า I และ Λ ต่างมีสมาชิกเพียงตัวเดียวและ P เป็นเมทริกซ์ที่สมาชิกทุกเป็นสมาชิกเอกลักษณ์ใน G แล้วกึ่งกรุปเชิงเดียวบริบูรณ์จะเป็นกรุป

ให้ S เป็นกึ่งกรุป

1. จะเรียก S ว่า *กึ่งกรุปศูนย์ขวา (right zero semigroup)* ถ้า $xy = y$ ทุก ๆ $x, y \in S$ สำหรับกึ่งกรุปศูนย์ขวาที่มีสมาชิก n ตัว เราจะเขียนแทนด้วย R_n
2. จะเรียก S ว่า *กึ่งกรุปศูนย์ซ้าย (left zero semigroup)* ถ้า $xy = x$ ทุก ๆ $x, y \in S$ สำหรับกึ่งกรุปศูนย์ซ้ายที่มีสมาชิก n ตัว เราจะเขียนแทนด้วย L_n
3. จะเรียก S ว่า *กรุปขวา (right group)* ถ้า S เป็นผลคูณคาร์ทีเซียน (Cartesian product) ของกรุปและกึ่งกรุปศูนย์ขวา
4. จะเรียก S ว่า *กรุปซ้าย (left group)* ถ้า S เป็นผลคูณคาร์ทีเซียนของกรุปและกึ่งกรุปศูนย์ซ้าย
5. จะเรียก S ว่า *กรุปเชิงตั้งฉาก (rectangular group)* ถ้า S เป็นผลคูณคาร์ทีเซียนของกรุปและกึ่งกรุปศูนย์ขวาและกึ่งกรุปศูนย์ซ้าย

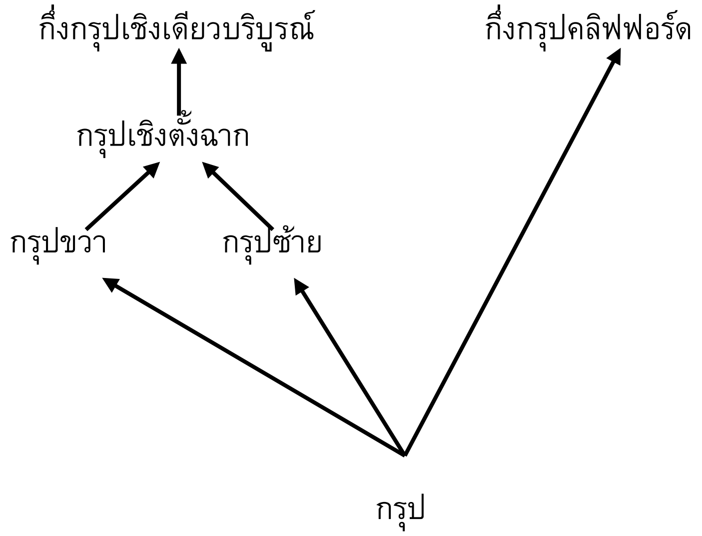
<u>ข้อสังเกต 3.</u> ให้ $S = \mathrm{M}(G,I,\Lambda;P)$ เป็นกึ่งกรุปเชิงเดียวบริบูรณ์

3.1 ถ้า P เป็นเมทริกซ์ที่สมาชิกทุกเป็นสมาชิกเอกลักษณ์ใน G แล้วจะได้ว่า S เป็นกรุปเชิงตั้งฉาก

3.2 ถ้า I มีสมาชิกเพียงตัวเดียวและ P เป็นเมทริกซ์ที่สมาชิกทุกเป็นสมาชิกเอกลักษณ์ใน G แล้วจะได้ว่า S เป็นกรุปขวา

3.3 ถ้า Λ มีสมาชิกเพียงตัวเดียวและ P เป็นเมทริกซ์ที่สมาชิกทุกเป็นสมาชิกเอกลักษณ์ใน G แล้วจะได้ว่า S เป็นกรุปซ้าย

โดยนิยามของกึ่งกรุปคลิฟฟอร์ด และกึ่งกรุปเชิงเดียวบริบูรณ์ จะได้ว่ากึ่งกรุปทั้งสองไม่สามารถเปรียบเทียบกันได้ และยังได้อีกว่ากรุปเป็นทั้งกึ่งกรุปคลิฟฟอร์ดและกึ่งกรุปเชิงเดียวบริบูรณ์ ซึ่งเราสามารถวาดแผนภาพได้ดังนี้



จะเห็นว่ากึ่งกรุปทั้งสองชนิดนิยามมาจากกรุป และจากข้อสังเกต 1 และ 2 เรารู้ว่าเมื่อไหร่ที่ทั้งสองกึ่งกรุปดังกล่าวจะเป็นกรุป ดังนั้นเราจึงสามารถที่จะขยายทฤษฎีที่เกี่ยวกับจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของกราฟเคย์เลย์ของกรุป ไปสู่การหาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุป

ผู้วิจัยจึงสนใจที่จะศึกษา

1. จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด
2. จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์

องค์ความรู้ใหม่ที่ได้จะทำให้ทราบจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด และกึ่งกรุปเชิงเดียวบริบูรณ์ ซึ่งยังไม่มีผู้ไม่มีผู้นำไปศึกษา

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2. วัตถุประสงค์งานวิจัย

2.1. หาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด

2.2. หาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์

3. ระเบียบวิธีวิจัย

ปีที่ 1

1. รวบรวมความรู้พื้นฐานและงานวิจัยที่เกี่ยวข้องเกี่ยวกับไดกราฟเคล์เลย์ และกึ่งกรุปคลิฟฟอร์ด และกึ่งกรุปเชิงเดียวบริบูรณ์
2. หาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด
3. ส่งผลงานให้นักวิจัยที่ปรึกษาตรวจสอบและขอคำแนะนำเพื่อนำมาปรับปรุงงานวิจัย
4. จัดพิมพ์ และส่งงานวิจัยให้วารสารทางคณิตศาสตร์พิจารณาเพื่อตีพิมพ์

ปีที่ 2

1. หาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์
2. ส่งผลงานให้นักวิจัยที่ปรึกษาตรวจสอบและขอคำแนะนำเพื่อนำมาปรับปรุงงานวิจัย
3. จัดพิมพ์ และส่งงานวิจัยให้วารสารทางคณิตศาสตร์พิจารณาเพื่อตีพิมพ์

4. ผลการวิจัย

4.1. จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด

จำนวนโดมิเนชัน $\gamma(Cay(S,A))$ และจำนวนโทเทิลโดมิเนชัน $\gamma_t(Cay(S,A))$ ของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด $S=\bigcup_{\alpha\in Y} G_\alpha$ ซึ่งได้ผลดังนี้ดังนี้

ให้ $S=\bigcup_{\alpha\in Y} G_\alpha$ เป็นกึ่งกรุปคลิฟฟอร์ด และ $A\subseteq S$ และกำหนด

$A_\alpha=\{f_{\gamma,\alpha}(a)\mid a\in A\cap G_\gamma$ ทุก $\gamma\geq\alpha$ ใน $Y\}$

M เป็นเซตของ maximal Chain ทั้งหมดใน Y

C เป็นเซตของสมาชิกมากสุด (maximal element) ใน Y ทั้งหมด

$Y'=\{\alpha\in Y\mid G_\alpha\cap A\neq\varnothing\}$ $\quad$ $A^*=\{\alpha\in\cap M\mid \alpha\wedge\gamma=\alpha$ ทุก $\gamma\in Y'\}$

m^* เป็นสมาชิกมากสุดใน A^* $\quad$ $\overline{M}=\{\alpha\in Y\mid\alpha\wedge\gamma\neq\alpha$ ทุก $\gamma\in Y'\}$

$M^*=\{\alpha\in Y\mid\alpha<m^*\}$ และ $K=Y\setminus(\overline{M}\cup(Y'\cap C)\cup M^*)$ จะได้ว่า

$$|\bigcup_{\alpha\in\overline{M}} G_\alpha|+\sum_{\alpha\in Y'\cap C}\gamma(Cay(G_\alpha,A_\alpha))+\sum_{\alpha\in M^*}\gamma(Cay(G_\alpha,A_\alpha))\leq\gamma(Cay(S,A))$$ และ

$$\gamma(Cay(S,A))\leq |\bigcup_{\alpha\in\overline{M}} G_\alpha|+\sum_{\alpha\in Y'\cap C}\gamma(Cay(G_\alpha,A_\alpha))+\sum_{\alpha\in M^*}\gamma(Cay(G_\alpha,A_\alpha))+\sum_{\alpha\in K}\gamma(Cay(G_\alpha,A_\alpha))$$

$$\sum_{\alpha\in C}\gamma_t(Cay(G_\alpha,A_\alpha))+\sum_{\alpha\in M^*}\gamma_t(Cay(G_\alpha,A_\alpha))\leq\gamma_t(Cay(S,A))$$ และ

$$\gamma_t(Cay(S,A))\leq\sum_{\alpha\in C}\gamma_t(Cay(G_\alpha,A_\alpha))+\sum_{\alpha\in M^*}\gamma_t(Cay(G_\alpha,A_\alpha))+\sum_{\alpha\in K}\gamma_t(Cay(G_\alpha,A_\alpha))$$

ได้ผลงานวิจัยชื่อ Bounds for the Domination and Total Domination Numbers of Cayley Digraphs of Clifford Semigroups

4.2. จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์

ในหัวข้อนี้เราสนใจที่จะหาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ แต่เนื่องจากกึ่งกรุปเชิงเดียวบริบูรณ์มีอยู่หลายชนิด และเป็นการยากที่จะลักษณะของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์ใด ๆ ที่เป็นซีไอกราฟ ดังนั้นผู้วิจัยจึงได้เลือกกรุปซ้ายและกรุปขวาและกรุปเชิงตั้งฉากซึ่งทั้งสามต่างเป็นกึ่งกรุปเชิงเดียวบริบูรณ์

ในระหว่างการวิจัยเนื่องจากกรุปขวาและกรุปเชิงตั้งฉาก มีความซับซ้อนมาก ดังนั้ผู้วิจัยจึงต้องการรู้ลักษณะโครงสร้างของกรึ่งกรุปทั้งสอง เพื่อมาช่วยในการหาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชัน ดังนั้นพิสูจน์ทฤษฎีบทต่าง ๆ ดังนี้

1. ให้ $S = G\times R_n$ เป็นกรุปซ้าย และ $A\subseteq S$

และ $p_1(A)=\{g\in G\,|\,(g,r)\in A$ บาง $r\in R_n\}$ และ $p_2(A)=\{r\in R_n\,|\,(g,r)\in A$ บาง $g\in G\}$

และ $G/\langle p_1(A)\rangle=\{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle,\dots,g_w\langle p_1(A)\rangle\}$ และ

$(g_i\langle p_1(A)\rangle\times p_2(A),E_i)$ เป็นกราฟย่อยของ $Cay(S,A)$ ที่ก่อกำเนิดโดย $g_i\langle p_1(A)\rangle\times p_2(A)$

จะได้ว่า - $Cay(S,A)=(\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle\times p_2(A),E_i))\cup(\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle\times p_2(A),E_i^{'}))$

เมื่อ $E_i^{'}=\{((s,t),(u,v))\,|\,t\notin p_2(A),((s,v),(u,v))\in E_i\}$

- $(g_i\langle p_1(A)\rangle\times p_2(A),E_i)\cong Cay(\langle A\rangle,A)$

2. ให้ $S=G\times L_m\times R_n$ เป็นกรุปสี่เหลี่ยมมุมฉาก และ $A\subseteq S$ และ

และ $p_1(A)=\{g\in G\,|\,(g,l,r)\in A$ บาง $l\in L_m, r\in R_n\}$

และ $G/\langle p_1(A)\rangle=\{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle,\dots,g_w\langle p_1(A)\rangle\}$ และ

$(g_k\langle p_1(A)\rangle\times\{l_i\}\times R_n,E_{ik})$ เป็นกราฟย่อยของ $Cay(S,A)$ ที่ก่อกำเนิดโดย $g_k\langle p_1(A)\rangle\times\{l_i\}\times R_n$

และ $(G\times\{l_i\}\times R_n,E_i)$ เป็นกราฟย่อยของ $Cay(S,A)$ ที่ก่อกำเนิดโดย $G\times\{l_i\}\times R_n$

จะได้ว่า - $(G\times\{l_i\}\times R_n,E_i)=\dot\bigcup_{k=1}^{w}(g_k\langle p_1(A)\rangle\times\{l_i\}\times R_n,E_{ik})$

- $(g_k\langle p_1(A)\rangle\times\{l_i\}\times R_n,E_{ik})\cong Cay(g_k\langle p_1(A)\rangle\times\{l_i\}\times R_n,A^i)$

เมื่อ $A^i=\{(g,l_i,r)\,|\,(g,l,r)\in A\}$

ได้ผลงานวิจัยชื่อ Isomorphism Conditions for Cayley Graphs of Rectangular Groups

จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกรุปซ้ายและกรุปขวา

จำนวนโดมิเนชัน $\gamma(Cay(S,A))$ และจำนวนโทเทิลโดมิเนชัน $\gamma_t(Cay(S,A))$ ของไดกราฟเคย์เลย์ของกรุปขวา $S=G\times R_m$ และกรุปซ้าย $S=G\times L_m$ ซึ่งได้ผลดังนี้

ให้ $S=G\times R_m$ เป็นกรุปขวา และ $A\subseteq S$

และให้ $p_1(A)=\{g\in G \mid (g,r)\in A$ บาง $r\in R_m\}$ และ

$p_2(A)=\{r\in R_m \mid (g,r)\in A$ บาง $g\in G\}$ จะได้ว่า

- ถ้า $|p_2(A)|\neq m$ แล้ว $\gamma(Cay(S,A))=(|R_m|-|p_2(A)|)\times|G|$
- ถ้า $|p_2(A)|=m$ แล้ว $\frac{|S|}{|A|+1}\leq\gamma(Cay(S,A))\leq|G|$
- ถ้า $|A|=|R_m|$ โดยที่ $p_1(A)=G$ และ $p_2(A)=R_m$ แล้ว $\gamma(Cay(S,A))=|G|$
- ถ้า $A=\{a\}\times R_m$ โดยที่ $a\in G$ แล้ว $\gamma(Cay(S,A))=|G|$
- ถ้า $A=K\times R_m$ โดยที่ K เป็นกรุปย่อยของ G แล้ว $\gamma(Cay(S,A))=\frac{|G|}{|K|}$
- ถ้า $p_2(A)=R_m$ แล้ว $\frac{|S|}{|A|}\leq\gamma_t(Cay(S,A))\leq|G|$

ให้ $S=G\times L_m$ เป็นกรุปซ้าย และ $A\subseteq S$

และให้ $p_1(A)=\{g\in G \mid (g,l)\in A$ บาง $l\in L_m\}$ จะได้ว่า

- $\gamma(Cay(S,A))=m\cdot|G/\langle p_1(A)\rangle|\cdot\gamma(Cay(\langle p_1(A)\rangle,p_1(A)))$
- ถ้า $p_1(A)$ บรรจุสมาชิกเอกลักษณ์ของ G และ H เป็นกรุปย่อยของ G ที่มีจำนวนสมาชิกมากที่สุดที่เป็นเซตย่อยของ $p_1(A)$ แล้ว $\frac{|G|}{|p_1(A)|}\leq\frac{\gamma(Cay(S,A))}{|L_m|}\leq[G:H]$ โดยที่ $[G:H]$ คือจำนวนโคเซตที่แตกต่างกันทั้งหมดของ H ใน G
- ถ้า $A=K\times L_m$ โดยที่ K เป็นกรุปย่อยใด ๆ ของ G แล้ว $\gamma(Cay(S,A))=[G:K]|L_m|$
- $\gamma_t(Cay(S,A))=m\cdot|G/\langle p_1(A)\rangle|\cdot\gamma_t(Cay(\langle p_1(A)\rangle,p_1(A)))$

จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกรุปเชิงตั้งฉาก

ให้ $S = G \times L_m \times R_n$ เป็นกรุปสี่เหลี่ยมมุมฉาก และ $A \subseteq S$ และ

ให้ $\overline{A} = \{(g,r) \in G \times R_n \mid (g,l,r) \in A$ บาง $l \in L_m\}$ จะได้ว่า

- $\gamma(Cay(S,A)) = m \cdot \gamma(Cay(G \times R_n, \overline{A}))$
- ถ้า $p_2(\overline{A}) = \{r \in R_n \mid (g,r) \in \overline{A}$ บาง $g \in G\} = R_n$ แล้ว $\gamma_t(Cay(S,A)) = m \cdot \gamma_t(Cay(G \times R_n, \overline{A}))$

ได้ผลงานวิจัยชื่อ Domination in Cayley digraphs of rectangular groups

5. สรุปผลและอภิปรายผล

5.1 จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปคลิฟฟอร์ด

ผลการศึกษาที่ได้ ได้ทฤษฎีบทที่ใช้กำหนดขอบเขตบนและขอบเขตล่างของจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันสำหรับกึ่งกรุปคลิฟฟอร์ดใดๆ อย่าไรก็ตาม เมื่อเรากำหนดเงื่อนไขให้กับกึ่งกรุปคลิฟฟอร์ด เราสามารถใช้ทฤษฎีบทดังกล่าวมากำหนดค่าที่แน่นอนของจำนวนจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันได้

5.2 จำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันของไดกราฟเคย์เลย์ของกึ่งกรุปเชิงเดียวบริบูรณ์

ผลการศึกษาที่ได้ ได้ลักษณะของไดกราฟเคย์เลย์ของกรุปขวาและกรุปเชิงตั้งฉาก ซึ่งนำไปใช้ในการหาจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชัน ทำให้ได้ค่าที่แน่นอนของจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันสำหรับบางกึ่งกรุปเชิงเดียวบริบูรณ์ ซึ่งผู้วิจัยจะต้องหาเครื่องที่จะมาใช้ ในการศึกษาต่อไปเพื่อให้ทราบจำนวนโดมิเนชันและจำนวนโททอลโดมิเนชันสำหรับกึ่งกรุปเชิงเดียวบริบูรณ์ใด ๆ

6. Output ที่ได้จากโครงการ

6.1 ผลงานวิจัยชื่อ Bounds for the Domination and Total Domination Numbers of Cayley Digraphs of Clifford Semigroups

รอการตอบรับการตีพิมพ์

6.2 ผลงานวิจัยชื่อ Isomorphism Conditions for Cayley Graphs of Rectangular Groups

ตีพิมพ์ใน วารสาร Bulletin of the Malaysian Mathematical Sciences Society

6.3 ผลงานวิจัยชื่อ Domination in Cayley digraphs of rectangular groups

รอการตอบรับการตีพิมพ์

7. ภาคผนวก

7.1 Bounds for the Domination and Total Domination Numbers of Cayley Digraphs of Clifford Semigroups

7.2 Isomorphism Conditions for Cayley Graphs of Rectangular Groups

7.3 Domination in Cayley digraphs of rectangular groups

Bounds for the Domination and Total Domination Numbers of Cayley Digraphs of Finite Clifford Semigroups

Krittawit Limkul[1] **and Sayan Panma**[2,*]

[1,2]Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai, 50200, THAILAND
krittawitl@hotmail.com
panmayan@yahoo.com

Abstract : Let $D = (V,E)$ be a digraph with vertex set V and edge set E. A subset $X \subseteq V$ is called *dominating set* of D if every vertex $v \in V \setminus X$ there exists $x \in X$ such that $(x,v) \in E$. The *domination number* $\gamma(D)$ of a digraph D is the minimum cardinality of a dominating set of D. A dominating set X is called *total dominating set* of D if every vertex $v \in V$ there exists $x \in X$ such that $(x,v) \in E$. The *total domination number* $\gamma_t(D)$ of a digraph D is the minimum cardinality of a total dominating set of D. In this paper, we give bounds for the domination numbers and total domination numbers of Cayley digraphs of finite Clifford semigroups.

Keywords : Domination numbers; Dominating sets; Cayley digraphs; Clifford semigroups.
2010 Mathematics Subject Classification : 05C99; 05C69; 05C25; 20M99

1 Introduction

The study of domination have been started by Claude Berge in 1958 and Oystein Ore in 1962. However, it was not until 1977, following an article by E. Cockayne and S. Hedetniemi, that domination become an area of study by many. In 1998, a text devoted to this subject was introduced by T. Haynes, S. Hedetniemi, and P. Slater. Over 2000 articles have been written on domination. While domination in graphs(undirected graphs) have been studied extensively, domination in digraphs(directed graphs) have not yet gained the same amount of attention from researchers. For domination in digraphs, the properties of dominating sets with the smallest cardinality in some special digraphs are introduced. Here we investigate the domination in digraphs which are constructed by algebraic structure, called *Cayley digraphs*.

The Cayley digraph of semigroup S relative to its subset A is a digraph with vertex set S and edge set $\{(s,sa)|s \in S, a \in A\}$, denoted by $Cay(S,A)$.

In 1998 [5], B. Zelinka studied dominating sets and domination numbers of products of circuits. Such graphs are treated algebraically as Cayley graphs of direct products of finite cyclic groups. In 2007 [2], T. T. Chelvam and I. Rani have found the domination number of Cayley graphs of finite cyclic groups relative to its special subsets. In 2011 [1], T. T. Chelvam and S. Mutharasu studied bounds for the domination numbers of Cayley graphs of finite cyclic groups relative to other its special subsets.

*Corresponding author e-mail : panmayan@yahoo.com

In this paper, we attempt to determine the bounds for the domination numbers and the total domination numbers of Cayley digraphs of semigroups S when those semigroups are constructed by groups, called Clifford semigroups.

Let S be a semigroup. The set $C(S)=\{c\in S|cs=sc$ for all $s\in S\}$ is called the *center* of S. An element $e\in S$ is called an *idempotent element* if $e^2=e$, the set of all idempotent elements of S is denoted by $E(S)$. An element $s\in S$ is called a *regular element* if $s=sxs$ for some $x\in S$. A semigroup S is called a *regular semigroup* if all of its elements are regular. A regular semigroup S is called a *Clifford semigroup* if $E(S)\subseteq C(S)$.

Let Y be a partially ordered set and $X\subseteq Y$. An element b of Y is called a *lower bound* of X if $b\leq x$ for every $x\in X$. A lower bound c of X is called *greatest lower bound (meet)* of X if $y\leq c$ for every lower bound y in X. The upper bound and least upper bound (join) are defined dually. For any x,y in partially ordered set Y, denote $x\wedge y$ and $x\vee y$ the greatest lower bound and least upper bound of $\{x,y\}$, respectively. A partially ordered set Y is called a *meet(join) semilattice* if $x\wedge y(x\vee y)\in Y$ for all $x,y\in Y$. A partially ordered set Y is called a *semilattic* if Y is a meet semilattice or a join semilattice. In this paper, we suppose that a semilattice Y is a meet semilattice. For a join semilattice, the results can be proved by the same way.

For any family of nonempty sets $\{X_i|i\in I\}$, we write $\dot{\bigcup\limits_{i\in I}} X_i:=\bigcup_{i\in I}X_i$ if $X_i\cap X_j=\emptyset$ for all $i\neq j$.

Let Y be a semilattice and $\{(G_\alpha,\circ_\alpha)|\alpha\in Y\}$ be a family of groups indexed by Y. Suppose that, for all $\beta\geq\alpha$ in Y, there exists a group homomorphism $f_{\beta,\alpha}:G_\beta\to G_\alpha$ such that

(i) for all $\alpha\in Y$, $f_{\alpha,\alpha}=id_{G_\alpha}$ is the identity mapping on G_α,

(ii) $f_{\beta,\gamma}f_{\alpha,\beta}=f_{\alpha,\gamma}$ for all $\alpha,\beta,\gamma\in Y$ with $\alpha\geq\beta\geq\gamma$,

and the multiplication on $S=\dot{\bigcup\limits_{\alpha\in Y}} G_\alpha$ is defined for $x\in G_\alpha$ and $y\in G_\beta$ by

$$xy=f_{\alpha,\alpha\wedge\beta}(x)\circ_{\alpha\wedge\beta}f_{\beta,\alpha\wedge\beta}(y).$$

It is easy to check that $S=\dot{\bigcup\limits_{\alpha\in Y}} G_\alpha$ under that multiplication is a semigroup, and called a *strong semilattice of groups*. We write $S=[Y;G_\alpha,f_{\alpha,\beta}]$. From [3], we know that S is a Clifford semigroup if and only if S is a strong semilattice of groups. In the sequel, we will use the term strong semilattice of groups instead of Clifford semigroup.

Let $S=[Y;G_\alpha,f_{\alpha,\beta}]$ be a Clifford semigroup, $A\subseteq S$ and $\{C_1,C_2,\ldots,C_n\}$ the set of all maximal chains in Y. Then we put

$Y'=\{\alpha\in Y|G_\alpha\cap A\neq\emptyset\}$,
$A_\alpha=\{f_{\beta,\alpha}(a)|a\in A\cap G_\beta,\beta\geq\alpha\}$,
$M=\{\alpha\in Y|\alpha\wedge\gamma\neq\alpha,\ \forall\gamma\in Y'\}$,
$\bar{A}=C_1\cap C_2\cap\ldots\cap C_n$,
$A^*=\{\alpha\in\bar{A}|\alpha\wedge\gamma=\alpha,\ \forall\gamma\in Y'\}$,
$m^*=max\,A^*$,
$M^*=\{\alpha\in Y\setminus\{m^*\}|\alpha\leq m^*\}$,
$C^*=\{\alpha\in Y|\alpha$ is the maximum of a maximal chain in $Y\}$, and
$K=Y\setminus(M\dot{\bigcup}(Y'\cap C^*)\dot{\bigcup}M^*)$.

It is clear that $Y=M\dot{\bigcup}(Y'\cap C^*)\dot{\bigcup}M^*\dot{\bigcup}K$.

Let $(V_1,E_1),(V_2,E_2),\ldots,(V_n,E_n)$ be digraphs such that $V_i\cap V_j=\emptyset$ for all $i\neq j$. The *disjoint union* of $(V_1,E_1),(V_2,E_2),\ldots,(V_n,E_n)$ is defined as $\dot{\bigcup}_{i=1}^n(V_i,E_i):=(\dot{\bigcup}_{i=1}^nV_i,\dot{\bigcup}_{i=1}^nE_i)$.

It is easy to check that $\gamma(\dot{\bigcup}_{i=1}^{n}(V_i,E_i)) = \sum\limits_{i=1}^{n}\gamma(V_i,E_i)$.

For any nonempty subset A of a group G. Let $\langle A\rangle$ denote the subgroup generated by A in G. If $A=\{a\}\subseteq G$, we write $\langle a\rangle$ in place of $\langle\{a\}\rangle$. By [4], we get the following lemma.

Lemma 1.1. *Let G be a group and $\emptyset\neq A\subseteq G$. Then $Cay(G,A)\cong\dot{\bigcup\limits_{i\in I}}(V_i,E_i)$, where $I=\{1,2,\ldots,\frac{|G|}{|\langle A\rangle|}\}$ and $(V_i,E_i)\cong Cay(\langle A\rangle,A)$ for all $i\in I$.*

If (u,v) is an arc of a digraph D, then v is adjacent from u. The number of vertices from which v is adjacent is the in-degree of v and is denoted by $\overrightarrow{d}(v)$.

The following lemma gives all vertices of in-degree zero in Cayley digraphs of finite Clifford semigroups.

Lemma 1.2. *Let $S=[Y;G_\alpha,f_{\alpha,\beta}]$ be a Clifford semigroup and $A\subseteq S$. Then, in $Cay(S,A)$, $\overrightarrow{d}(x)=0$ if and only if $x\in G_\alpha$ for some $\alpha\in M$.*

Proof. Let $\overrightarrow{d}(x)=0$ where $x\in G_\alpha$ for some $\alpha\in Y$. Assume that $\alpha\notin M$. Then $\alpha\wedge\gamma=\alpha$ for some $\gamma\in Y'$. Let $a\in A\cap G_\gamma$. Then $f_{\gamma,\gamma\wedge\alpha}(a)\in G_{\gamma\wedge\alpha}=G_\alpha$. Since G_α is a group, there exists $y\in G_\alpha$ such that

$$x=y\circ_\alpha f_{\gamma,\alpha}(a)=y\circ_\alpha f_{\gamma,\gamma\wedge\alpha}(a)=ya$$

Thus $\overrightarrow{d}(x)>0$, which is a contradiction.

Conversely, let $x\in G_\beta$ for some $\beta\in M$. Assume that $\overrightarrow{d}(x)>0$, so there exists $x'\in G_\gamma$ for some $\gamma\in Y$ such that (x',x) is an arc in $Cay(S,A)$. Therefore $x=x'a$ for some $a\in A\cap G_\lambda$ and $\lambda\in Y'$. Hence $x=x'a=f_{\gamma,\gamma\wedge\lambda}(x')\circ_{\gamma\wedge\lambda}f_{\lambda,\gamma\wedge\lambda}(a)$. Since $x\in G_\beta$, $\gamma\wedge\lambda=\beta$ by the definition of the multiplication of S. This implies that $\beta\leq\gamma$ and $\beta\leq\lambda$. Thus $\beta\wedge\lambda=\beta$, which is a contradiction since $\beta\in M$. □

Lemma 1.3. *Let $S=[Y;G_\alpha,f_{\alpha,\beta}]$ be a Clifford semigroup and $A\subseteq S$. Then $Cay(S,A)=Cay(\dot{\bigcup\limits_{\alpha\in Y\setminus M^*}}G_\alpha,\dot{\bigcup\limits_{\alpha\in Y\setminus M^*}}G_\alpha\bigcap A)\dot{\bigcup}(\dot{\bigcup\limits_{\alpha\in M^*}}Cay(G_\alpha,A_\alpha))$.*

Proof. Let $D=Cay(S,A)$ and $D'=Cay(\dot{\bigcup\limits_{\alpha\in Y\setminus M^*}}G_\alpha,\dot{\bigcup\limits_{\alpha\in Y\setminus M^*}}G_\alpha\bigcap A)\dot{\bigcup}(\dot{\bigcup\limits_{\alpha\in M^*}}Cay(G_\alpha,A_\alpha))$.
It is easy to see that $V(D)=V(D')=S$. We prove that $E(D)=E(D')$. Let (x,y) be an arc in D. Then $x\in G_\beta$, $y\in G_\gamma$ for some $\beta,\gamma\in Y$. We consider two cases.
Case 1. $\beta\in M^*$. Then $\beta<m^*$. Since (x,y) is an arc of D, $y=xa$ for some $a\in A\cap G_\mu$ for some $\mu\in Y'$ and $m^*\leq\mu$. So $\beta\leq\mu$. This implies that

$$y=f_{\beta,\beta\wedge\mu}(x)\circ_{\beta\wedge\mu}f_{\mu,\beta\wedge\mu}(a)=f_{\beta,\beta}(x)\circ_\beta f_{\mu,\beta}(a)\in G_\beta.$$

Thus $\beta=\gamma$. This means that $(x,y)\in E(Cay(G_\beta,A_\beta))\subseteq E((\dot{\bigcup\limits_{\alpha\in M^*}}Cay(G_\alpha,A_\alpha)))$. Then $(x,y)\in E(D')$.
Case 2. $\beta\in Y\setminus M^*$. Assume that $\gamma\in M^*$. Then $\gamma<m^*$ and by the definition of M^* we have $m^*\wedge\lambda=m^*$ for all $\lambda\in M$. Since $y=xa$ for some $a\in A\cap G_\mu$ for some $\mu\in Y'$, $\gamma=\beta\wedge\mu$ and so $\gamma<\beta$ since $\beta\in Y\setminus M^*$. If $\beta\notin M$, then $\beta\wedge\alpha=\beta$ for some $\alpha\in Y'$. Thus $\beta\leq\alpha$ for some $\alpha\in Y'$. Since $m^*\leq\alpha$ for all $\alpha\in Y'$, $\beta\leq m^*$ or $\beta\geq m^*$, we have the following subcases.
Subcase 2.1 $\beta\leq m^*$. Then $\beta\leq\mu$. This means that $\beta\wedge\mu\geq\beta>\gamma$, it is impossible.
Subcase 2.2 $\beta\geq m^*$. Since $m^*\leq\alpha$ for all $\alpha\in Y'$, $\gamma<m^*\leq\beta\wedge\mu$, it is impossible. Then

$\beta \in M$. Since $m^* \in A^*$, $m^* \wedge \lambda = m^*$ for all $\lambda \in Y'$. Therefore $m^* \wedge \mu = m^*$ because $\mu \in Y'$. Since $\gamma \in M^*$, $\gamma \wedge m^* = \gamma$. Hence

$$\begin{aligned}\beta \wedge m^* &= \beta \wedge (m^* \wedge \mu)\\ &= (\beta \wedge \mu) \wedge m^*\\ &= \gamma \wedge m^*\\ &= \gamma,\end{aligned}$$

which is a contradiction. Therefore $\gamma \notin M^*$. This means that $\gamma \in Y \setminus M^*$. Thus $(x,y) \in E(Cay(\dot{\bigcup}_{\alpha \in Y \setminus M^*} G_\alpha, \dot{\bigcup}_{\alpha \in Y \setminus M^*} G_\alpha \bigcap A))$. Therefore $E(D) \subseteq E(D')$. Next, we will prove that $E(D') \subseteq E(D)$. Let $(x,y) \in E(D')$. We consider two cases.

Case 1. $(x,y) \in E(Cay(\dot{\bigcup}_{\alpha \in Y \setminus M^*} G_\alpha, \dot{\bigcup}_{\alpha \in Y \setminus M^*} G_\alpha \bigcap A))$. Then $y = xa$ for some $a \in \dot{\bigcup}_{\alpha \in Y \setminus M^*} G_\alpha \bigcap A$. Hence $(x,y) \in E(D)$.

Case 2. $(x,y) \in E((\dot{\bigcup}_{\alpha \in M^*} Cay(G_\alpha, A_\alpha)))$. Then $(x,y) \in E(Cay(G_\lambda, A_\lambda))$ for some $\lambda \in M^*$. This implies that $y = x \circ_\lambda a$ for some $a \in A_\lambda$, where $\circ_\lambda$ is the operation of the group G_λ. So $a = f_{\mu,\lambda}(b)$ for some $b \in G_\mu \cap A$, $\mu \geq \lambda$. This means that $y = f_{\lambda,\lambda}(x) \circ_\lambda f_{\mu,\lambda}(b) = f_{\lambda,\lambda\wedge\mu}(x) \circ_{\lambda\wedge\mu} f_{\mu,\lambda\wedge\mu}(b) = xb$. Therefore $(x,y) \in E(D)$. Hence $E(D) = E(D')$, so $D = D'$. □

Lemma 1.4. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, $A \subseteq S$, Y a totally ordered set, and $x_\alpha \in G_\alpha$ for some $\alpha \in Y$. Then, in $Cay(S,A)$, $x_\alpha \notin N(x) := \{y \in S | (x,y) \in E(Cay(S,A))\}$ for all $x \in \dot{\bigcup}_{\beta<\alpha} G_\beta$.*

Proof. Let $x_\alpha \in G_\alpha$ for $\alpha \in Y$. Assume, to the contrary, that $x_\alpha \in N(x_\beta)$ for some $x_\beta \in G_\beta$ and $\beta < \alpha$. Hence $(x_\beta, x_\alpha) \in E(Cay(S,A))$. This implies that $x_\alpha = x_\beta a$ for some $a \in A$. Suppose that $a \in A \cap G_\gamma$ for some $\gamma \in Y$. By the definition of the multiplication of S, we have $\beta \wedge \gamma = \alpha$. So $\beta \geq \alpha$, contrary to $\beta < \alpha$. Therefore $x_\alpha \notin N(x)$ for all $x \in \dot{\bigcup}_{\beta<\alpha} G_\beta$. □

2 Domination Numbers

The following lemma will be used in Theorem 2.2.

Lemma 2.1. *Let $S = [Y; G_\alpha, f_{\alpha,\beta}]$ be a Clifford semigroup, $A \subseteq S$ and X be a dominating set of $Cay(S,A)$ such that $|X| = \gamma(Cay(S,A))$. Then*

i) $\dot{\bigcup}_{\alpha \in M} G_\alpha \subseteq X$,

ii) $Y_\alpha = X \cap G_\alpha$ *is a dominating set of* $Cay(G_\alpha, A_\alpha)$ *for all* $\alpha \in Y' \cap C^*$, *and* $\dot{\bigcup}_{\alpha \in Y' \cap C^*} Y_\alpha \subseteq X$, *and*

iii) $X_\alpha = X \cap G_\alpha$ *is a dominating set of* $Cay(G_\alpha, A_\alpha)$ *such that* $|X_\alpha| = \gamma(Cay(G_\alpha, A_\alpha))$ *for all* $\alpha \in M^*$, *and* $\dot{\bigcup}_{\alpha \in M^*} X_\alpha \subseteq X$.

Proof. *i)* By the definition of the dominating set, the vertices of in-degree zero must belong to any dominating set. Hence $\dot{\bigcup}_{\alpha\in M} G_\alpha \subseteq X$ by Lemma 1.2.

ii) Let $\alpha \in Y'\cap C^*$ and $Y_\alpha = G_\alpha \cap X$. Let $x \in G_\alpha \setminus Y_\alpha$. Thus $x \in S$. Since X is a dominating set of $Cay(S,A)$, there exists $y \in X$ such that $(y,x) \in E(Cay(S,A))$. Therefore $y \in G_\beta$ for some $\beta \in Y$ and $x = ya$ for some $a \in A$. Hence $a \in G_\gamma$ for some $\gamma \in Y$. Thus $x = f_{\beta,\beta\wedge\gamma}(y) \circ_{\beta\wedge\gamma} f_{\gamma,\beta\wedge\gamma}(a)$. Since $x \in G_\alpha$, $\beta\wedge\gamma = \alpha$, so $\beta \geq \alpha$ and $\gamma \geq \alpha$. By the definition of C^*, we obtain that $\alpha = \beta = \gamma$. Then $y \in G_\alpha \cap X$. Therefore Y_α is a dominating set of $Cay(G_\alpha,A_\alpha)$ for all $\alpha \in Y'\cap C^*$. Since $Y_\alpha \subseteq X$ for all $\alpha \in Y'\cap C^*$, $\dot{\bigcup}_{\alpha\in Y'\cap C^*} Y_\alpha \subseteq X$.

iii) By Lemma 1.3, we have

$Cay(S,A) = Cay(\dot{\bigcup}_{\alpha\in Y\setminus M^*} G_\alpha, \dot{\bigcup}_{\alpha\in Y\setminus M^*} G_\alpha \bigcap A) \dot{\bigcup} (\dot{\bigcup}_{\alpha\in M^*} Cay(G_\alpha,A_\alpha))$.

Then $X \cap (\dot{\bigcup}_{\alpha\in M^*} Cay(G_\alpha,A_\alpha))$ is a dominating set of $(\dot{\bigcup}_{\alpha\in M^*} Cay(G_\alpha,A_\alpha))$, and so $X_\alpha = X \cap G_\alpha$ is a dominating set of $Cay(G_\alpha,A_\alpha)$ and $|X_\alpha| = \gamma(Cay(G_\alpha,A_\alpha))$ for all $\alpha \in M^*$. □

Here we give the bounds for the domination numbers of Cayley digraphs of Clifford semigroups.

Theorem 2.2. *Let $S = [Y;G_\alpha,f_{\alpha,\beta}]$ be a Clifford semigroup, $A \subseteq S$. If $D = Cay(S,A)$, then* $|\dot{\bigcup}_{\alpha\in M} G_\alpha| + \sum_{\alpha\in Y'\cap C^*} \gamma(Cay(G_\alpha,A_\alpha)) + \sum_{\alpha\in M^*} \gamma(Cay(G_\alpha,A_\alpha)) \leq \gamma(D)$ *and*

$\gamma(D) \leq |\dot{\bigcup}_{\alpha\in M} G_\alpha| + \sum_{\alpha\in Y'\cap C^*} \gamma(Cay(G_\alpha,A_\alpha)) + \sum_{\alpha\in M^*} \gamma(Cay(G_\alpha,A_\alpha)) + \sum_{\alpha\in K} \gamma(Cay(G_\alpha,A_\alpha))$,

where $K = Y \setminus (M \dot{\bigcup} (Y'\cap C^*) \bigcup M^*)$.

Proof. Let $D = Cay(S,A)$ and $D_\alpha = Cay(G_\alpha,A_\alpha)$. We first show that $|\dot{\bigcup}_{\alpha\in M} G_\alpha| + \sum_{\alpha\in Y'\cap C^*} \gamma(D_\alpha) + \sum_{\alpha\in M^*} \gamma(D_\alpha) \leq \gamma(D)$. Let X be a dominating set of D such that $|X| = \gamma(D)$. Let $Y_\alpha = X \cap G_\alpha$ for all $\alpha \in Y'\cap C^*$ and let $X_\alpha = X \cap G_\alpha$ for all $\alpha \in M^*$. By Lemma 1.5, we have $(\dot{\bigcup}_{\alpha\in M} G_\alpha) \dot{\bigcup} (\dot{\bigcup}_{\alpha\in Y'\cap C^*} Y_\alpha) \dot{\bigcup} (\dot{\bigcup}_{\alpha\in M^*} X_\alpha) \subseteq X$. By the definition, we get M, $Y'\cap C^*$ and M^* are disjoint, so are $\dot{\bigcup}_{\alpha\in M} G_\alpha$, $\dot{\bigcup}_{\alpha\in Y'\cap C^*} Y_\alpha$ and $\dot{\bigcup}_{\alpha\in M^*} X_\alpha$. Hence

$$\begin{aligned}|X| &\geq |(\dot{\bigcup}_{\alpha\in M} G_\alpha) \dot{\bigcup} (\dot{\bigcup}_{\alpha\in Y'\cap C^*} Y_\alpha) \dot{\bigcup} (\dot{\bigcup}_{\alpha\in M^*} X_\alpha)| \\ &= |\dot{\bigcup}_{\alpha\in M} G_\alpha| + \sum_{\alpha\in Y'\cap C^*} |Y_\alpha| + \sum_{\alpha\in M^*} \gamma(D_\alpha).\end{aligned}$$

Since Y_α is a dominating set of D_α for all $\alpha \in Y'\cap C^*$, $|Y_\alpha| \geq \gamma(D_\alpha)$ for all $\alpha \in Y'\cap C^*$. Therefore $|X| \geq |\dot{\bigcup}_{\alpha\in M} G_\alpha| + \sum_{\alpha\in Y'\cap C^*} \gamma(D_\alpha) + \sum_{\alpha\in M^*} \gamma(D_\alpha)$.

Let $K = Y \setminus (M \dot{\bigcup} (Y'\cap C^*) \bigcup M^*)$ and Z_α be a dominating set of D_α such that $|Z_\alpha| = \gamma(D_\alpha)$ for all $\alpha \in K$. It is easy to check that $Y = M \dot{\bigcup} (Y'\cap C^*) \dot{\bigcup} M^* \dot{\bigcup} K$. Since M, $Y' \cap$

C^*, M^* and K are disjoint, so are $\dot{\bigcup}_{\alpha\in M} G_\alpha$, $\dot{\bigcup}_{\alpha\in Y'\cap C^*} X_\alpha$, $\dot{\bigcup}_{\alpha\in M^*} X_\alpha$ and $\dot{\bigcup}_{\alpha\in K} Z_\alpha$. Let $X' = (\dot{\bigcup}_{\alpha\in M} G_\alpha)\dot{\bigcup}(\dot{\bigcup}_{\alpha\in Y\cap C^*} X_\alpha)\dot{\bigcup}(\dot{\bigcup}_{\alpha\in M^*} X_\alpha)$. We will show that $X'\dot{\bigcup}(\dot{\bigcup}_{\alpha\in K} Z_\alpha)$ is a dominating set of D. Let $x\in S\setminus(X'\dot{\bigcup}(\dot{\bigcup}_{\alpha\in K} Z_\alpha))$. Since $\dot{\bigcup}_{\alpha\in M} G_\alpha\subseteq X'$, $x\notin \dot{\bigcup}_{\alpha\in M} G_\alpha$. We need only consider the following three cases.

Case 1. $x\in G_\alpha\setminus X_\alpha$, $\alpha\in Y'\cap C^*$. Since X_α is a dominating set of $Cay(G_\alpha,A_\alpha)$, there exists $x'\in X_\alpha\subseteq \dot{\bigcup}_{\alpha\in Y\cap C^*} X_\alpha$ such that $(x',x)\in E(Cay(S,A))$.

Case 2. $x\in G_\alpha\setminus X_\alpha$, $\alpha\in M^*$. Since X_α is a dominating set of $Cay(G_\alpha,A_\alpha)$, there exists $x'\in X_\alpha\subseteq \dot{\bigcup}_{\alpha\in M^*} X_\alpha$ such that $(x',x)\in E(Cay(S,A))$.

Case 3. $x\in G_\alpha\setminus Z_\alpha$, $\alpha\in K$. Since Z_α is a dominating set of $Cay(G_\alpha,A_\alpha)$, there exists $x'\in Z_\alpha\subseteq \dot{\bigcup}_{\alpha\in K} Z_\alpha$ such that $(x',x)\in E(Cay(S,A))$. Then $X'\dot{\bigcup}(\dot{\bigcup}_{\alpha\in K} Z_\alpha)$ is a dominating set of D. Therefore

$$\begin{aligned}
\gamma(D) &\le |X'\dot{\bigcup}(\dot{\bigcup}_{\alpha\in K} Z_\alpha)|\\
&= |(\dot{\bigcup}_{\alpha\in M} G_\alpha)\dot{\bigcup}(\dot{\bigcup}_{\alpha\in Y\cap C^*} X_\alpha)\dot{\bigcup}(\dot{\bigcup}_{\alpha\in M^*} X_\alpha)| + |(\dot{\bigcup}_{\alpha\in K} Z_\alpha)|\\
&= |\dot{\bigcup}_{\alpha\in M} G_\alpha| + \sum_{\alpha\in Y'\cap C^*}\gamma(D_\alpha) + \sum_{\alpha\in M^*}\gamma(D_\alpha) + \sum_{\alpha\in K}\gamma(D_\alpha).
\end{aligned}$$

□

Let G be a group and $a\in G$. It is clear that $Cay(\langle a\rangle,\{a\})$ is a cycle of length $|\langle a\rangle|$. Therefore $\gamma(Cay(\langle a\rangle,\{a\})) = \lceil\frac{|\langle a\rangle|}{2}\rceil$.

Corollary 2.3. *Let $S=[Y;G_\alpha,f_{\alpha,\beta}]$ be a Clifford semigroup, Y a totally ordered set and ρ the maximum element Y. Let $D=Cay(S,A)$, where $A=\{a\}$ for some $a\in G_\rho$. Then $\gamma(D)=\frac{|G_\rho|}{|\langle a\rangle|}(\lceil\frac{|\langle a\rangle|}{2}\rceil)+\sum_{\alpha\in M^*}\gamma(Cay(G_\alpha,A_\alpha))$.*

Proof. Let $a\in G_\rho$. Then $Y'=\{\rho\}$ and $M=\emptyset$ since ρ is the maximum. Since $C^*=\{\rho\}$, $Y'\cap C^*=\{\rho\}$. Since Y is the only one maximal chain in Y, $\bar{A}=Y$. Then $A^*=Y$ and $m^*=\rho$. This implies that $M^*=Y\setminus\{\rho\}$. Hence $\gamma(Cay(G_\rho,A_\rho=\{a\}))+\sum_{\alpha\in M^*}\gamma(Cay(G_\alpha,A_\alpha))\le\gamma(D)$ by Theorem 2.1. By Lemma 1.1, we get that $Cay(G_\rho,\{a\}\cong\dot{\bigcup}_{i\in I}(V_i,E_i)$, where $I=\{1,2,\dots,\frac{|G_\rho|}{|\langle a\rangle|}\}$ and $(V_i,E_i)\cong Cay(\langle a\rangle,\{a\})$ for all $i\in I$.

Hence $\frac{|G_\rho|}{|\langle a\rangle|}(\lceil\frac{|\langle a\rangle|}{2}\rceil)+\sum_{\alpha\in M^*}\gamma(Cay(G_\alpha,A_\alpha))\le\gamma(D)$. Since $(Y'\cap C^*)\dot{\bigcup}M^*=Y$, $K=Y\setminus(M\dot{\bigcup}(Y'\cap C^*)\dot{\bigcup}M^*)=\emptyset$. Therefore $\gamma(D)\le\gamma(Cay(G_\rho,A_\rho))+\sum_{\alpha\in M^*}\gamma(Cay(G_\alpha,A_\alpha))=\frac{|G_\rho|}{|\langle a\rangle|}(\lceil\frac{|\langle a\rangle|}{2}\rceil)+\sum_{\alpha\in M^*}\gamma(Cay(G_\alpha,A_\alpha))$ by Theorem 2.1. This implies that $\gamma(D)=\frac{|G_\rho|}{|\langle a\rangle|}(\lceil\frac{|\langle a\rangle|}{2}\rceil)+\sum_{\alpha\in M^*}\gamma(Cay(G_\alpha,A_\alpha))$. □

Corollary 2.4. *Let $S=[Y;G_\alpha,f_{\alpha,\beta}]$ be a Clifford semigroup, Y a totally ordered set and m the minimum element Y. Let $D=Cay(S,A)$, where $A=\{a\}$ for some $a\in G_m$. Then*

$$|\dot{\bigcup_{\alpha\in M}} G_\alpha| \le \gamma(D) \le |\dot{\bigcup_{\alpha\in M}} G_\alpha| + \tfrac{|G_m|}{|<a>|}(\lceil \tfrac{|<a>|}{2}\rceil).$$

Proof. Let $a\in G_m$. We have $Y'=\{m\}$. Since m is the minimum, $Y'\cap C^*=\emptyset$. Since Y is the only one maximal chain in Y, $\bar{A}=Y$. Then $A^*=\{m\}$. Hence $m^*=m$ and so $M^*=\emptyset$. Therefore $|\dot{\bigcup_{\alpha\in M}} G_\alpha|\le\gamma(D)$ by Theorem 2.1. Since $K=Y\setminus(M\dot{\bigcup}(Y'\cap C^*)\dot{\bigcup}M^*)$, $K=\{m\}$. Let X_m be a dominating set of $Cay(G_m,A_m=\{a\})$ such that $|X_m|=\gamma(Cay(G_m,\{a\}))$. By Lemma 1.1, we have $Cay(G_m,\{a\})\cong\dot{\bigcup_{i\in I}}(V_i,E_i)$, where $I=\{1,2,\ldots,\frac{|G_m|}{|\langle a\rangle|}\}$ and $(V_i,E_i)\cong Cay(\langle a\rangle,\{a\})$ for all $i\in I$. Then $|X_m|=\frac{|G_m|}{|<a>|}(\lceil\frac{|<a>|}{2}\rceil)$. By Theorem 2.1, we obtain that $\gamma(D)\le|\dot{\bigcup_{\alpha\in M}} G_\alpha|+\frac{|G_m|}{|<a>|}(\lceil\frac{|<a>|}{2}\rceil)$. □

3 Total Domination Numbers

Recall that a dominating set X is called *total dominating set* of D if every vertex $v\in V$ there exists $x\in X$ such that $(x,v)\in E$. The *total domination number* $\gamma_t(D)$ of a digraph D is the minimum cardinality of a total dominating set of D. So for that definition we get that the total dominating set exists if and only if for every vertices in the digraphs has at least one of its in-degree.

By Lemma 1.2, we get the following observation.

Observation 3.1. *The total dominating set of $Cay(S,A)$ exists if and only if $M=\emptyset$.*

Observation 3.2. *If G is a cycle, then $\gamma_t(G)=|G|$.*

The following lemma shows the property of $Cay(S,A)$ with $M=\emptyset$.

Lemma 3.3. *Let $S=[Y;G_\alpha,f_{\alpha,\beta}]$ be a Clifford semigroup, $A\subseteq S$, and $c_i=maxC_i$ for all $1\le i\le n$. Then $M=\emptyset$ if and only if $c_i\in Y'$ for all $1\le i\le n$.*

Proof. Let $M=\emptyset$. Assume that $c_k\notin Y'$ for some $1\le k\le n$. Then $G_{c_k}\cap A=\emptyset$. Since c_k is the maximal of C_k, $c_k\wedge\gamma\ne c_k$ for all $\gamma\in Y'$. Thus $c_k\in M$, which is a contradiction.
Conversely, suppose that $c_i\in Y'$ for all $1\le i\le n$. Assume to the contrary that $M\ne\emptyset$. Let $\alpha\in M$. Then $\alpha\wedge\gamma\ne\alpha$ for all $\gamma\in Y'$. We consider the following two cases.
Case 1. $\alpha=c_k$ for some $1\le k\le n$. Since $c_i\in Y'$ for all $1\le i\le n$, $c_k\in Y'$ and $c_k\wedge c_k=c_k$, $\alpha=c_k\notin M$, it is impossible.
Case 2. $\alpha\ne c_i\in Y'$ for all $1\le i\le n$. Let $\alpha\in C_k$ for some $1\le k\le n$. Then $c_k>\alpha$. Since $c_k\in Y'$ and $\alpha\wedge c_k=\alpha$, $\alpha\notin M$, it is impossible.
Therefore $M=\emptyset$. □

So, for this section we will assume that $M=\emptyset$. By lemma 3.3 we get the following lemma that will be used in Theorem 3.6.

Lemma 3.4. *Let $S=[Y;G_\alpha,f_{\alpha,\beta}]$ be a Clifford semigroup, $A\subseteq S$ and X be a total dominating set of $Cay(S,A)$ such that $|X|=\gamma_t(Cay(S,A))$. Then* *i)* $Y_\alpha = X \cap G_\alpha$ *is a total dominating set of* $Cay(G_\alpha, A_\alpha)$ *for all* $\alpha \in C^*$*, and* $\dot{\bigcup\limits_{\alpha \in C^*}} Y_\alpha \subseteq X$*, and*

ii) X_α *is a total dominating set of* $Cay(G_\alpha, A_\alpha)$ *such that* $|X_\alpha| = \gamma_t(Cay(G_\alpha, A_\alpha))$ *for all* $\alpha \in M^*$*, and* $\dot{\bigcup\limits_{\alpha \in M^*}} X_\alpha \subseteq X$*,*

Proof. Similarlar to the proof of Lemma 2.1 *ii)* and *iii)*. □

Lemma 3.5. *Let* $S = [Y; G_\alpha, f_{\alpha,\beta}]$ *be a Clifford semigroup,* $A \subseteq S$*, and* $c_i = maxC_i$ *where* $1 \le i \le n$*. If* $f_{\alpha,\beta}$ *is an isomorphism for all* $\alpha, \beta \in Y$ *and* $|A \cap G_\alpha| = 1$ *for all* $\alpha \in Y$*, then* $\gamma_t(Cay(S,A)) = |\dot{\bigcup}_{i=1}^n G_{c_i}|$

Proof. Let $f_{\alpha,\beta}$ is an isomorphism for all $\alpha, \beta \in Y$ and $|A \cap G_\alpha| = 1$ for all $\alpha \in Y$.
First, we will show that $\gamma_t(Cay(S,A)) \ge |\dot{\bigcup}_{i=1}^n G_{c_i}|$. Let X be a total dominating set of $Cay(S,A)$. By Lemma 3.4 *i)*, we have $X \cap G_{c_i}$ is a total dominating set of $Cay(G_{c_i}, A_{c_i})$. Since $|A \cap G_{c_i}| = 1$, $Cay(G_{c_i}, A \cap G_{c_i})$ is a union of disjoint cycle.
Then $\gamma_t(Cay(G_{c_i}, A \cap G_{c_i})) = |G_{c_i}|$. Thus $X \cap G_{c_i} = G_{c_i} \subseteq X$ for all $1 \le i \le n$. Therefore $\gamma_t(Cay(S,A)) \ge |\dot{\bigcup}_{i=1}^n G_{c_i}|$.
Next, we will show that $\gamma_t(Cay(S,A)) \le |\dot{\bigcup}_{i=1}^n G_{c_i}|$. Let $y \in G_\alpha$ for some $\alpha \in Y$. We will show that there exists $x \in \dot{\bigcup}_{i=1}^n G_{c_i}$ such that $(x,y) \in E(Cay(S,A))$. Consider the following two cases
Case 1. $\alpha = c_k$ for some $1 \le k \le n$. Since $|A \cap G_\alpha| = 1$, there exists $x \in G_\alpha$ such that $(x,y) \in E(Cay(S,A))$.
Case 2. $\alpha \ne c_k$ for all $1 \le k \le n$. Let $\alpha \in C_k$ for some $1 \le k \le n$ and $a_\alpha \in G_\alpha \cap A$. Since G_α is a group, there exists $u \in G_\alpha$ such that $y = ua_\alpha$. Since $f_{c_k,\alpha}$ is an isomorphism, there exists $x \in G_{c_k}$ such that $f_{c_k,\alpha}(x) = u$. Then

$$xa_\alpha = f_{c_k, c_k \wedge \alpha}(x) \circ_{c_k \wedge \alpha} f_{\alpha, c_k \wedge \alpha}(a) = f_{c_k,\alpha}(x) \circ_\alpha f_{\alpha,\alpha}(a) = u \circ_\alpha a = y.$$

Hence $(x,y) \in E(Cay(S,A))$, this implies that $\dot{\bigcup}_{i=1}^n G_{c_i}$ is a total dominating set of $Cay(S,A)$. Thus $\gamma_t(Cay(S,A)) \le |\dot{\bigcup}_{i=1}^n G_{c_i}|$.
Therefore $\gamma_t(Cay(S,A)) = |\dot{\bigcup}_{i=1}^n G_{c_i}|$. □

Here we give the bounds for the total domination numbers of Cayley digraphs of Clifford semigroups.

Theorem 3.6. *Let* $S = [Y; G_\alpha, f_{\alpha,\beta}]$ *be a Clifford semigroup,* $A \subseteq S$*. If* $D = Cay(S,A)$*, then* $\sum\limits_{\alpha \in C^*} \gamma_t(Cay(G_\alpha, A_\alpha)) + \sum\limits_{\alpha \in M^*} \gamma_t(Cay(G_\alpha, A_\alpha)) \le \gamma_t(D)$ *and*
$\gamma_t(D) \le \sum\limits_{\alpha \in C^*} \gamma_t(Cay(G_\alpha, A_\alpha)) + \sum\limits_{\alpha \in M^*} \gamma_t(Cay(G_\alpha, A_\alpha)) + \sum\limits_{\alpha \in K} \gamma(Cay(G_\alpha, A_\alpha))$*,*
where $K = Y \setminus (C^* \dot{\bigcup} M^*)$*.*

Proof. Similar to the proof of Theorem 2.2. □

Example 3.7 and 3.8 illustrate the sharpness of the bounds in Theorem 2.2.

Example 3.7. *Let* $Y = \{\alpha, \beta, \gamma, \lambda\}$*, where* $\lambda \le \gamma \le \beta$*,* $\lambda \le \gamma \le \alpha$ *and* $\alpha \wedge \beta = \gamma$*. Let* $G_\alpha = \mathbb{Z}_4 = \{0_\alpha, 1_\alpha, 2_\alpha, 3_\alpha\}$*,* $G_\beta = \mathbb{Z}_4 = \{0_\beta, 1_\beta, 2_\beta, 3_\beta\}$*,* $G_\gamma = \mathbb{Z}_4 = \{0_\gamma, 1_\gamma, 2_\gamma, 3_\gamma\}$*,* $G_\lambda = \mathbb{Z}_2 = \{0_\lambda, 1_\lambda\}$*. Let* $f_{\alpha,\gamma}(0_\alpha) = 0_\gamma$*,* $f_{\alpha,\gamma}(1_\alpha) = 1_\gamma$*,* $f_{\alpha,\gamma}(2_\alpha) = 2_\gamma$*,* $f_{\alpha,\gamma}(3_\alpha) = 3_\gamma$*,* $f_{\beta,\gamma}(0_\beta) = f_{\beta,\gamma}(2_\beta) = 0_\gamma$*,* $f_{\beta,\gamma}(1_\beta) = f_{\beta,\gamma}(3_\beta) = 2_\gamma$*, and* $f_{\gamma,\lambda}(0_\gamma) = f_{\gamma,\lambda}(1_\gamma) = f_{\gamma,\lambda}(2_\gamma) = f_{\gamma,\lambda}(3_\gamma) = 0_\lambda$*.*

Then $S_1 = [Y; G_\alpha, f_{\alpha,\beta}]$ *is a Clifford semigroup (see Fig.1). Consider* $Cay(S_1, A)$*, where* $A = \{2_\beta\}$ *(see Fig.2).*

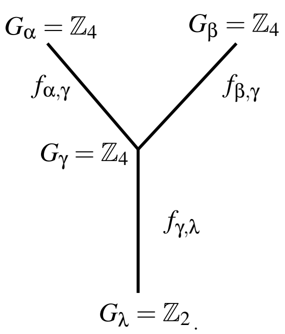

Fig.1. $S_1 = \bigcup_{\alpha \in Y} G_\alpha$

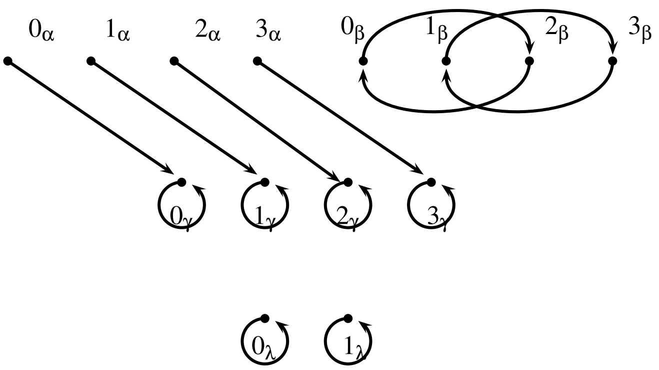

Fig.2. $Cay(S_1, A)$

We see that $Y' = \{\beta\}$*,* $M = \{\alpha\}$*,* $C^* = \{\alpha, \beta\}$*,* $m^* = \gamma$*,* $M^* = \{\lambda\}$*. Then* $Y' \cap C^* = \{\beta\}$*. So* $\gamma(Cay(S, A)) = |G_\alpha| + \gamma(Cay(G_\beta, A_\beta)) + \gamma(Cay(G_\lambda, A_\lambda)) = 4 + 2 + 2 = 8$

Example 3.8. *Let* $Y = \{\alpha, \beta, \gamma, \lambda\}$*, where* $\lambda \leq \gamma \leq \beta$*,* $\lambda \leq \gamma \leq \alpha$ *and* $\alpha \wedge \beta = \gamma$*. Let* $G_\alpha = \mathbb{Z}_4 = \{0_\alpha, 1_\alpha, 2_\alpha, 3_\alpha\}$*,* $G_\beta = \mathbb{Z}_4 = \{0_\beta, 1_\beta, 2_\beta, 3_\beta\}$*,* $G_\gamma = \mathbb{Z}_4 = \{0_\gamma, 1_\gamma, 2_\gamma, 3_\gamma\}$*,* $G_\lambda = \mathbb{Z}_2 = \{0_\lambda, 1_\lambda\}$*. Let* $f_{\alpha,\gamma}$ *and* $f_{\gamma,\lambda}$ *be zero mapping and* $f_{\beta,\gamma}(0_\beta) = 0_\gamma$*,* $f_{\beta,\gamma}(1_\beta) = 1_\gamma$*,* $f_{\beta,\gamma}(2_\beta) = 2_\gamma$*,* $f_{\beta,\gamma}(3_\beta) = 3_\gamma$*. Then* $S_2 = [Y; G_\alpha, f_{\alpha,\beta}]$ *is a Clifford semigroup (see Fig.3). Consider* $Cay(S_2, B)$*, where* $B = \{2_\beta\}$ *(see Fig.4).*

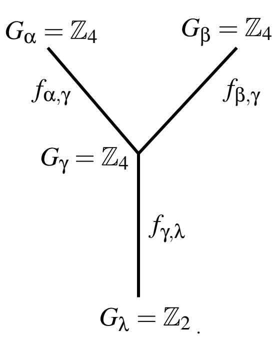

Fig.3. $S_2 = \bigcup_{\alpha \in Y} G_\alpha$

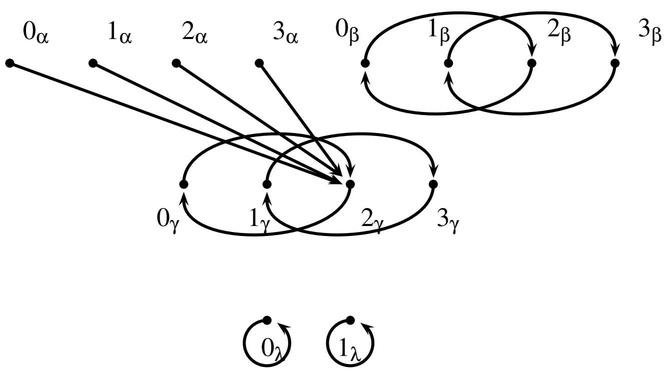

Fig.4. $Cay(S_2, B)$

We see that $Y' = \{\beta\}$*,* $M = \{\alpha\}$*,* $C^* = \{\alpha, \beta\}$*,* $m^* = \gamma$*,* $M^* = \{\lambda\}$*,* $K = \{\gamma\}$*. Then* $Y' \cap C^* = \{\beta\}$*. So* $\gamma(Cay(S_2, B)) = |G_\alpha| + \gamma(Cay(G_\beta, B_\beta)) + \gamma(Cay(G_\lambda, B_\lambda)) + \gamma(Cay(G_\gamma, B_\gamma)) = 4 + 2 + 2 + 2 = 10$.

Example 3.9 and 3.10 illustrate the sharpness of the bounds in Theorem 3.6.

Example 3.9. *Let* $Y = \{\alpha, \beta, \gamma\}$*, where* $\gamma \leq \beta \leq \alpha$*. Let* $G_\alpha = \mathbb{Z}_4 = \{0_\alpha, 1_\alpha, 2_\alpha, 3_\alpha\}$*,* $G_\beta = \mathbb{Z}_4 = \{0_\beta, 1_\beta, 2_\beta, 3_\beta\}$*, and* $G_\gamma = \mathbb{Z}_4 = \{0_\gamma, 1_\gamma, 2_\gamma, 3_\gamma\}$*. Let* $f_{\alpha,\beta}$ *and* $f_{\beta,\gamma}$ *be zero mapping. Then* $S_3 = [Y; G_\alpha, f_{\alpha,\beta}]$ *is a Clifford semigroup (see Fig.5). Consider* $Cay(S_3, A')$*, where* $A' = \{1_\alpha, 2_\alpha, 3_\alpha, 0_\beta, 1_\beta, 2_\beta, 3_\beta\}$ *(see Fig.6).*

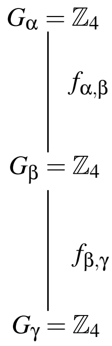

Fig.5.$S_3 = \dot{\bigcup\limits_{\alpha\in Y}} G_\alpha$

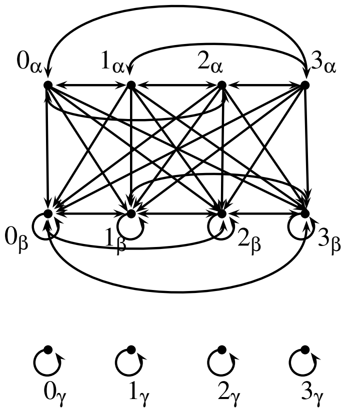

Fig.6. $Cay(S_3, A')$

We see that $Y' = \{\alpha, \beta\}$, $M = \emptyset$, $C^* = \{\alpha\}$, $m^* = \beta$, $M^* = \{\gamma\}$, $K = \{\beta\}$.
Then $\gamma_t(Cay(S_3, A')) = \gamma_t(Cay(G_\alpha, A'_\alpha)) + \gamma_t(Cay(G_\gamma, A'_\gamma)) = 1 + 4 = 5.$

Example 3.10. *Let* $Y = \{\alpha, \beta, \gamma\}$, *where* $\gamma \leq \beta \leq \alpha$. *Let* $G_\alpha = \mathbb{Z}_4 = \{0_\alpha, 1_\alpha, 2_\alpha, 3_\alpha\}$, $G_\beta = \mathbb{Z}_4 = \{0_\beta, 1_\beta, 2_\beta, 3_\beta\}$, *and* $G_\gamma = \mathbb{Z}_4 = \{0_\gamma, 1_\gamma, 2_\gamma, 3_\gamma\}$. *Let* $f_{\alpha,\beta}$ *and* $f_{\beta,\gamma}$ *be zero mapping. Then* $S_3 = [Y; G_\alpha, f_{\alpha,\beta}]$ *is a Clifford semigroup (see Fig.7). Consider* $Cay(S_3, B')$, *where* $B' = \{1_\alpha, 2_\alpha, 3_\alpha, 1_\beta, 2_\beta, 3_\beta\}$ *(see Fig.8).*

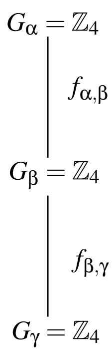

Fig.7.$S_3 = \dot{\bigcup\limits_{\alpha\in Y}} G_\alpha$

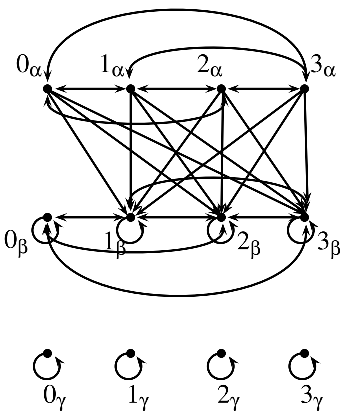

Fig.8.$Cay(S_3, B')$

We see that $Y' = \{\alpha, \beta\}$, $M = \emptyset$, $C^* = \{\alpha\}$, $m^* = \beta$, $M^* = \{\gamma\}$, $K = \{\beta\}$.
Then $\gamma_t(Cay(S_3, B')) = \gamma_t(Cay(G_\alpha, B'_\alpha)) + \gamma_t(Cay(G_\gamma, B'_\gamma)) + \gamma_t(Cay(G_\beta, B'_\beta)) = 1 + 1 + 4 = 6.$

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Isomorphism Conditions for Cayley Graphs of Rectangular Groups

Sayan Panma[1] · **John Meksawang**[1]

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Abstract A rectangular band is defined as a direct product of a left zero semigroup and a right zero semigroup, and a rectangular group is defined as a direct product of a group and a rectangular band. In this paper, we give some equivalent conditions for Cayley graphs of a rectangular group to be isomorphic to each other.

Keywords Cayley graph · Digraph · Rectangular band · Right group · Rectangular group

Mathematics Subject Classification 05C25 · 08B15 · 20M19 · 20M30

1 Introduction

One of the first investigations on Cayley graphs of algebraic structures can be found in Maschke's theorem from 1896 about groups of genus zero, that is, groups G which possess a generating system A such that the Cayley graph $\mathrm{Cay}(G, A)$ is planar, see, for example, [20]. For more results about Cayley graphs of groups, we refer the reader to [3] and [19]. After this, it is natural to investigate Cayley graphs for semigroups which are unions of groups, so-called completely regular semigroups, see, for example, [15]. In [1,16] and [13], Cayley graphs of right(left) groups, rectangular group and

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✉ John Meksawang
john.npu@hotmail.com

Sayan Panma
panmayan@yahoo.com

1 Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand

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finite simple semigroups, respectively, are characterized. Recent studies in a different direction investigate some basic properties of Cayley graphs of ideal in a commutative ring, see [2]. Necessary and sufficient conditions for a Cayley graph of a rectangular group to be isomorphic to a given digraph were given in [16]. In the present paper, we shall give the conditions for two Cayley graphs of a rectangular group to be isomorphic to each other.

All sets in this paper are assumed to be finite. An element z of a semigroup S is a *left(right) zero* of S if $zs = z(sz = z)$ for all $s \in S$. z is a *zero* of S if it is both a left and right zero of S. A semigroup all of whose elements are left(right) zeros is a *left(right) zero semigroup.* A direct product of a group and a left(right) zero semigroup is called a *left(right) group.* A direct product of a left zero and a right zero semigroup is called a *rectangular band.* A *rectangular group* is a direct product of a group and a rectangular band.

The *cardinality* of a set X, denoted by $|X|$, is the number of elements in X. For any nonempty subset A of a semigroups S, let $\langle A \rangle$ denote the *subsemigroup generated by* A in S. Let G be a group and $a \in G$. The *order* of a is the cardinality of the cyclic subsemigroup $\langle \{a\} \rangle$ and is denoted by $ord(a)$.

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *digraph homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v)) \in E_2$, i.e., φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph endomorphism.* If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called a *digraph isomorphism*. If a digraph isomorphism $\varphi : (V_1, E_1) \to (V_2, E_2)$ exists, then the graphs are called *isomorphic* and we write $(V_1, E_1) \cong (V_2, E_2)$. A digraph isomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph automorphism.*

For any family of nonempty set $\{X_i | i \in I\}$, let $\dot{\cup}_{i \in I} X_i$ denote the disjoint union of $X_i, i \in I$.

Let $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ be digraphs such that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The *disjoint union* of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\dot{\bigcup}_{i=1}^{n}(V_i, E_i) = \left(\dot{\cup}_{i=1}^{n} V_i, \dot{\cup}_{i=1}^{n} E_i\right)$. If $V_i = V_j = V$ for all i, j, then the *edge sum* of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\oplus_{i=1}^{n}(V_i, E_i) = (V, \cup_{i=1}^{n} E_n)$.

Let S be a semigroup and $A \subseteq S$. We define the *Cayley graph* $\mathrm{Cay}(S, A)$ of S relative to A as follows: S is the vertex set and $(u, v), u, v \in S$, is an arc in $\mathrm{Cay}(S, A)$ if there exists an element $a \in A$ such that $v = ua$. The set A is called the *connection set* of $\mathrm{Cay}(S, A)$.

A digraph (V, E) is called a *semigroup digraph* or *digraph of a semigroup* if there exist a semigroup S and a connection set $A \subseteq S$ such that (V, E) is isomorphic to the Cayley graph $\mathrm{Cay}(S, A)$. For any $v \in V$, the number of arcs incident to v is the *indegree* of v and is denoted by $\overrightarrow{d}(v)$. The number of arcs incident from v is called the *outdegree* of v and is denoted by $\overleftarrow{d}(v)$.

Throughout the paper, a graph always means a directed graph without multiple edges, but possibly with loops. A subgraph F of a graph D is called a *strong subgraph* of D if and only if whenever u and v are vertices of F and (u, v) is an arc in D, then (u, v) is an arc in F as well. If the vertex set of a strong subgraph F is H, then F is said to be *induced by* H.

2 Cayley Graphs of Rectangular Band

We consider an isomorphism of Cayley graphs of rectangular bands in this section. From now on, G denotes a group, $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup, $L_m = \{l_1, l_2, \ldots, l_m\}$ a left zero semigroup, and p_i the projection into the ith component. For any subgroup K of G, G/K denotes the set of all distinct left cosets of K in G. By the definition of right zero semigroups, we get the following lemma.

Lemma 2.1 *Let $A \subseteq R_n$, and let v be a vertex in $Cay(R_n, A)$. Then*

(1) $\overrightarrow{d}(v) = |R_n|$ if and only if $v \in A$;
(2) $\overrightarrow{d}(v) = 0$ if and only if $v \notin A$.

From the above lemma, we have the following lemma.

Lemma 2.2 *Let $A, B \subseteq R_n$. Then $Cay(R_n, A) \cong Cay(R_n, B)$ if and only if $|A| = |B|$.*

It is known that a rectangular band $S = L_m \times R_n$ is isomorphic to the finite simple semigroup $\mathcal{M}(G, I, \Lambda, P)$, where $G = \{e\}$ is the trivial group, $m = |I|$ and $n = |\Lambda|$. By Theorem 3 in [13], we have the following lemma.

Lemma 2.3 *Let $S = L_m \times R_n$ be a rectangular band and $A \subseteq S$. Then $Cay(S, A)$ is the disjoint union of m isomorphic strong subgraphs $Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A))$ for $i \in \{1, 2, \ldots, m\}$.*

Theorem 2.4 *Let $S = L_m \times R_n$ be a rectangular band and $A, B \subseteq S$. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $|p_2(A)| = |p_2(B)|$.*

Proof ($\Rightarrow$) Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 2.3, we get $\mathrm{Cay}(S, A) \cong \dot{\bigcup}_{i=1}^{m} \mathrm{Cay}(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong \dot{\bigcup}_{i=1}^{m} \mathrm{Cay}(\{l_i\} \times R_n, \{l_i\} \times p_2(B)) \cong \mathrm{Cay}(S, B)$. Then $\mathrm{Cay}(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong \mathrm{Cay}(\{l_i\} \times R_n, \{l_i\} \times p_2(B))$ and thus $\mathrm{Cay}(R_n, p_2(A)) \cong \mathrm{Cay}(R_n, p_2(B))$. By Lemma 2.2, we get $|p_2(A)| = |p_2(B)|$.

($\Leftarrow$) Let $|p_2(A)| = |p_2(B)|$. By Lemma 2.2, we get $\mathrm{Cay}(R_n, p_2(A)) \cong \mathrm{Cay}(R_n, p_2(B))$. Then $\dot{\bigcup}_{i=1}^{m} \mathrm{Cay}(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong \dot{\bigcup}_{i=1}^{m} \mathrm{Cay}(\{l_i\} \times R_n, \{l_i\} \times p_2(B))$. By Lemma 2.3, we get $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. □

3 Cayley Graphs of Right Groups

In this section, we present the conditions for Cayley graphs of a given right group to be isomorphic. By the definition of a right group, we get the two following lemmas.

Lemma 3.1 *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $g, g' \in G$, and $r, r' \in R_n$. Then the following statements are equivalent:*

(1) $((g, r), (g', r'))$ is an arc in $Cay(S, A)$;
(2) There exists $(a, r') \in A$ such that $g' = ga$;
(3) $((g, r'), (g', r'))$ is an arc in $Cay(S, A)$.

Lemma 3.2 *Let $S = G\times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subgraph of $Cay(S, A)$ for $i = 1, 2, \ldots, w$. Then $(g_j\langle p_1(A)\rangle \times p_2(A), E_j)$ and $(g_k\langle p_1(A)\rangle \times p_2(A), E_k)$ are disjoint strong subgraphs of $Cay(S, A)$ for all $j \neq k$.*

Theorem 3.3 *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle\times p_2(A), E_i)$ a strong subgraph of $Cay(S, A)$. Then $Cay(S, A) = \dot{\bigcup}_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle\times R_n, E_i')$, where $E_i' = \{((s, t), (u, v)) \mid t \notin p_2(A), ((s, v), (u, v)) \in E_i\}$.*

Proof Let $D = \dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i')$. It is clear that $S = \dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A)) \bigcup \dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n) = V(D)$. We will prove that $E(\mathrm{Cay}(S, A)) = E(D)$. Let $((g, r), (g', r'))$ be an arc in $\mathrm{Cay}(S, A)$. By Lemma 3.1, there exists $(a, r') \in A$ and $g' = ga$. Hence $g' \in g_{k_1}\langle p_1(A)\rangle$ and $g \in g_{k_2}\langle p_1(A)\rangle$ for some $k_1, k_2 \in \{1, 2, \ldots, w\}$. We have the following cases.

(case 1) If $r \in p_2(A)$, then $(g, r), (g', r') \in \dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A))$. Since $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subgraph of $\mathrm{Cay}(S, A)$, $((g, r), (g', r'))$ is an arc in $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle\times p_2(A), E_i)$. Therefore $((g, r), (g', r'))$ is an arc in D.

(case 2) If $r \notin p_2(A)$, then $((g, r'), (g', r'))$ is also an arc in $\mathrm{Cay}(S, A)$ by Lemma 3.1 and $((g, r), (g', r'))$ is an arc in $\mathrm{Cay}(S, A)$. This implies that $((g, r'), (g', r')) \in E_i$. Then $((g, r), (g', r')) \in E_i'$. Hence $((g, r), (g', r'))$ is an arc in D.

Therefore $E(\mathrm{Cay}(S, A)) \subseteq E(D)$.

To show that $E(D) \subseteq E(\mathrm{Cay}(S, A))$, let $((g, r), (g', r'))$ be an arc in D. We consider two cases.

(case 1) If $((g, r), (g', r'))$ is an arc in $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$, then it is an arc in $\mathrm{Cay}(S, A)$ because $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subgraph of $\mathrm{Cay}(S, A)$.

(case 2) If $((g, r), (g', r'))$ is an arc in $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i')$, then it is an arc in E_k' for some k. We get that $((g, r'), (g', r')) \in E_k$ and this implies that $((g, r'), (g', r'))$ is an arc in $\mathrm{Cay}(S, A)$. By Lemma 3.1, we have $((g, r), (g', r'))$ is also an arc in $\mathrm{Cay}(S, A)$.

Then $E(D) \subseteq E(\mathrm{Cay}(S, A))$. Hence $\mathrm{Cay}(S, A) = D$. □

Lemma 3.4 [14] *Let $S = G \times R_n$ be a right group, and let A be a nonempty subset of S. Then $\langle A\rangle = \langle p_1(A)\rangle \times p_2(A)$.*

Theorem 3.5 *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle\times p_2(A), E_i)$ a strong subgraph of $Cay(S, A)$. Then $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong Cay(\langle A\rangle, A)$ for $i = 1, 2, \ldots, w$.*

Proof We define $f : (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \to \mathrm{Cay}(\langle A\rangle, A)$ by $(g_i a, r) \mapsto (a, r)$ for all $a \in \langle p_1(A)\rangle$ and $r \in p_2(A)$. Clearly, f is a bijection. We will prove that f and f^{-1} are homomorphisms.

For $(g_i a, r), (g_i a', r') \in g_i\langle p_1(A)\rangle \times p_2(A)$, let $((g_i a, r), (g_i a', r'))$ be an arc in $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Since $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subgraph of $\mathrm{Cay}(S, A)$, we get that $((g_i a, r'), (g_i a', r'))$ is an arc in $\mathrm{Cay}(S, A)$. There exists $(a'', r') \in A$ such that $g_i a' = g_i aa''$ so $a' = aa''$. Since $f(g_i a', r') = (a', r') = (aa'', r') = (a, r)(a'', r') = f(g_i a, r)(a'', r')$, we have $(f(g_i a, r), f(g_i a', r'))$ is an arc in $\mathrm{Cay}(\langle A\rangle, A)$. Therefore f is a homomorphism.

Let $(f(g_i a, r), f(g_i a', r'))$ be an arc in $\mathrm{Cay}(\langle A\rangle, A)$. Then there exists $(a'', r'') \in A$ such that $f(g_i a', r') = f(g_i a, r)(a'', r'')$. Therefore $(a', r') = (a, r)(a'', r'') = (aa'', r'')$, $a' = aa''$, and $r' = r''$. Hence $(g_i a', r') = (g_i aa'', r'') = (g_i a, r)\,(a'', r'')$, so $((g_i a, r), (g_i a', r'))$ is an arc in $\mathrm{Cay}(S, A)$. Since $(g_i a, r), (g_i a', r') \in g_i\langle p_1(A)\rangle \times p_2(A)$ and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subgraph of $\mathrm{Cay}(S, A)$, we thus get $((g_i a, r), (g_i a', r'))$ is an arc in $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Therefore f^{-1} is a homomorphism. This means that $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \mathrm{Cay}(\langle A\rangle, A)$. □

Lemma 3.6 *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subgraph of $\mathrm{Cay}(S, A)$. Then for all $v \in V(\mathrm{Cay}(S, A))$, $\overrightarrow{d}(v) \neq 0$ if and only if $v \in \dot{\bigcup}_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$.*

Proof ($\Rightarrow$) Let $v = (h_1, r_1) \in S$ and $\overrightarrow{d}(v) \neq 0$. Then there exists $u = (h_2, r_2) \in S$ such that (u, v) is an arc in $\mathrm{Cay}(S, A)$. Hence there exists $a = (g', r') \in A$ such that $v = ua$. Therefore $(h_1, r_1) = (h_2, r_2)(g', r') = (h_2 g', r')$, which implies that $r_1 = r' \in p_2(A)$. Since $h_1 \in G = \dot{\bigcup}_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big)$, we have $v = (h_1, r_1) \in \dot{\bigcup}_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$.

($\Leftarrow$) Let $v = (h_1, r) \in \dot{\bigcup}_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$. We get that $h_1 \in G$ and $r \in p_2(A)$. We need consider the two cases.

(case 1) If $v \in A$, there exists $(e, r) \in S$, where e is the identity of G. Since $(e, r)(h_1, r) = (eh_1, r) = (h_1, r) = v$, there is an arc from (e, r) to v. Therefore $\overrightarrow{d}(v) \neq 0$.

(case 2) If $v \notin A$, then there exists $(h_2, r) \in A$ for some $h_2 \in G$. Because G is a group and $h_1, h_2 \in G$, this implies that $h_2^{-1} \in G$ and $h_1 h_2^{-1} \in G$. Then we have $(h_1 h_2^{-1}, r) \in S$. Since $(h_1 h_2^{-1}, r)(h_2, r) = (h_1 h_2^{-1} h_2, r) = (h_1, r) = v$, there exists an arc from $(h_1 h_2^{-1}, r)$ to v. Therefore $\overrightarrow{d}(v) \neq 0$.

□

A *path* from a vertex u_0 to some vertex u_n in a graph (V, E) is a sequence of vertices $u_0, u_1, u_2, \ldots, u_n$, where (u_{i-1}, u_i) for all i, is an arc in (V, E). If (u_{i-1}, u_i) or (u_i, u_{i-1}) is an arc in (V, E), then we say that there is *semipath* between u_0 and u_n. A graph (V, E) is *connected* if there is a semipath between any two vertices. The following lemmas will be used in the proof of Theorem 3.14.

Lemma 3.7 *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subgraph of $\mathrm{Cay}(S, A)$. Then for any $i \in \{1, 2, \ldots, w\}$, $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is connected.*

Proof Let $(g_i x, \beta), (g_i y, \gamma) \in \big(g_i\langle p_1(A)\rangle \times p_2(A)\big)$. Then $(x, \beta), (y, \gamma) \in \langle p_1(A)\rangle \times p_2(A) = \langle A\rangle$. There are $a_1, a_2, \ldots, a_q \in A$ such that $(y, \gamma) = (x, \beta)a_1a_2\ldots a_q$ for some $q \le |A|$. Hence $(g_i y, \gamma) = (g_i x, \beta)a_1a_2\ldots a_q$. This means that there is an arc from $(g_i x, \beta)a_1a_2\ldots a_{q-1}$ to $(g_i y, \gamma)$. Since $\big((g_i x, \beta), (g_i x, \beta)a_1\big)$, $\big((g_i x, \beta)a_1, (g_i x, \beta)a_1a_2\big), \ldots, \big((g_i x, \beta)a_1a_2\ldots a_{q-2}, (g_i x, \beta)a_1a_2\ldots a_{q-1}\big)$ are arcs in $Cay(S, A)$, there is a path $(g_i x, \beta)$, $(g_i x, \beta)a_1$, $(g_i x, \beta)a_1a_2, \ldots, (g_i x, \beta)a_1a_2\ldots a_{q-1}$, $(g_i y, \gamma)$ in $Cay(S, A)$. We conclude that $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is connected. $\square$

Since a strong subgraph $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is connected, we have $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (g_i\langle p_1(A)\rangle \times R_n, E_i')$ is also connected for any $i \in \{1, 2, \ldots, w\}$, where E' is defined as in Theorem 3.3.

Lemma 3.8 *Let $S = G \times R_n$ be a right group, A and B be nonempty subsets of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $G/\langle p_1(B)\rangle = \{h_1\langle p_1(B)\rangle, h_2\langle p_1(B)\rangle, \ldots, h_z\langle p_1(B)\rangle\}$. If $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i') \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E_j')$, then $w = z$ and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$ for all i, j.*

Proof Let $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i') \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E_j')$. Then there exists an isomorphism $f : \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A)) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n) \to \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B)) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n)$. By Lemma 3.6, we get that $|\dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)| = |\dot\bigcup_{j=1}^{z}\big(h_j\langle p_1(B)\rangle\big) \times p_2(B)|$ and we have $f(\dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)) = \dot\bigcup_{j=1}^{z}\big(h_j\langle p_1(B)\rangle\big) \times p_2(B)$. Since f is an isomorphism, we thus get the restriction of f to $\dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$ is an isomorphism from $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ to $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Therefore $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. In view of Theorem 3.5 and Lemma 3.7, we get that $w = z$ and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. $\square$

Lemma 3.9 *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. If $Cay(S, A) \cong Cay(S, B)$, then $|p_2(A)| = |p_2(B)|$.*

Proof Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$ and $G/\langle p_1(B)\rangle = \{h_1\langle p_1(B)\rangle, h_2\langle p_1(B)\rangle, \ldots, h_z\langle p_1(B)\rangle\}$. Assume that $Cay(S, A) \cong Cay(S, B)$. By Theorem 3.3 and Lemma 3.8, we get that $\left|\dot\bigcup_{i=1}^{w} g_i\langle p_1(A)\rangle \times p_2(A)\right| = \left|\dot\bigcup_{j=1}^{w} h_j\langle p_1(B)\rangle \times p_2(B)\right|$ for all $g_i, h_j \in G$. Since $\dot\bigcup_{i=1}^{w} g_i\langle p_1(A)\rangle = G = \dot\bigcup_{j=1}^{w} h_j\langle p_1(B)\rangle$, we have $|G \times p_2(A)| = |G \times p_2(B)|$. Therefore $|G| \times |p_2(A)| = |G| \times |p_2(B)|$. Hence $|p_2(A)| = |p_2(B)|$. $\square$

By Theorem 4 in [13], we have the next lemma.

Lemma 3.10 *Let $S = G \times R_n$ be right group, and let $(g, \lambda), (h, \beta) \in S$, where $g, h \in G$ and $\lambda, \beta \in R_n$. Then $Cay(S, \{(g, \lambda)\}) \cong Cay(S, \{(h, \beta)\})$ if and only if $ord(g) = ord(h)$.*

Theorem 3.11 *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. Let $A_r = \langle p_1(A)\rangle \times \{r\}$, $\hat{A}_r = A \cap A_r$ and $\hat{A} = \{\hat{A}_r \big| r \in p_2(A)\}$. B_r, $\hat{B}_r$ and $\hat{B}$ are defined similarly. If $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$, then $\big|\hat{A}\big| = \big|\hat{B}\big|$ and $\big|\langle p_1(A)\rangle\big| = \big|\langle p_1(B)\rangle\big|$.*

Proof Let $\text{Cay}(\langle A\rangle, A) \cong \text{Cay}(\langle B\rangle, B)$. By Lemma 3.9, $\big|p_2(A)\big| = \big|p_2(B)\big|$ and then $\big|\hat{A}\big| = \big|\hat{B}\big|$. Since $\text{Cay}(\langle A\rangle, A) \cong \text{Cay}(\langle B\rangle, B)$, we get that $\big|\langle A\rangle\big| = \big|\langle B\rangle\big|$. By Lemma 3.4,

$$\begin{gathered}
\big|\langle p_1(A)\rangle \times p_2(A)\big| = \big|\langle p_1(B)\rangle \times p_2(B)\big|;\\
\big|\langle p_1(A)\rangle\big| \times \big|p_2(A)\big| = \big|\langle p_1(B)\rangle\big| \times \big|p_2(B)\big|;\\
\big|\langle p_1(A)\rangle\big| = \big|\langle p_1(B)\rangle\big|.
\end{gathered}$$

□

Theorem 3.12 *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. Let $A_r = \langle p_1(A)\rangle \times \{r\}$, $\hat{A}_r = A \cap A_r$ and $\hat{A} = \{\hat{A}_r \big| r \in p_2(A)\}$. B_r, $\hat{B}_r$ and $\hat{B}$ are defined similarly. Then $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$ if the following conditions hold:*

(1) *$\big|\hat{A}\big| = \big|\hat{B}\big|$ and $\big|\langle p_1(A)\rangle\big| = \big|\langle p_1(B)\rangle\big|$;*
(2) *There exists a bijection $f : \hat{A} \to \hat{B}$ such that $\big|\hat{A}_r\big| = \big|f(\hat{A}_r)\big|$ for all $\hat{A}_r \in \hat{A}$;*
(3) *For each $\hat{A}_r \in \hat{A}$, there exists a bijection $\varphi_r : \hat{A}_r \to f(\hat{A}_r)$ such that $ord(p_1(a)) = ord(p_1(\varphi_r(a)))$ for all $a \in \hat{A}_r$.*

Proof By (1), $\big|\langle A\rangle\big| = \big|\langle B\rangle\big|$. By Lemma 3.10 and (3), we get that $\text{Cay}(\langle A\rangle, \{a\}) \cong \text{Cay}(\langle B\rangle, \{\varphi_r(a)\})$ for all $a \in \hat{A}_r$. Then $\text{Cay}(\langle A\rangle, \hat{A}_r) = \oplus_{a\in\hat{A}_r}\text{Cay}(\langle A\rangle, \{a\}) \cong \oplus_{a\in\hat{A}_r}\text{Cay}(\langle B\rangle, \{\varphi_r(a)\}) = \text{Cay}(\langle B\rangle, \varphi_r(\hat{A}_r))$.

By (2), $\text{Cay}(\langle A\rangle, \hat{A}_r) \cong \text{Cay}(\langle B\rangle, f(\hat{A}_r))$ for all $\hat{A}_r \in \hat{A}$. Then

$$\begin{gathered}
\oplus_{\hat{A}_r\in\hat{A}}Cay\big(\langle A\rangle, \hat{A}_r\big) \cong \oplus_{\hat{A}_r\in\hat{A}}Cay\big(\langle B\rangle, f(\hat{A}_r)\big);\\
Cay\big(\langle A\rangle, \cup_{\hat{A}_r\in\hat{A}}\hat{A}_r\big) \cong Cay\big(\langle B\rangle, \cup_{\hat{A}_r\in\hat{A}}f(\hat{A}_r)\big);\\
Cay\big(\langle A\rangle, A\big) \cong Cay\big(\langle B\rangle, B\big).
\end{gathered}$$

□

Lemma 3.13 *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subgraph of $Cay(S, A)$. Then for every $i \in \{1, 2, \ldots, w\}$, $\dot{\cup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot{\cup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i') = \dot{\cup}_{i=1}^{w}\big((g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (g_i\langle p_1(A)\rangle \times R_n, E_i')\big)$, where E' is defined as in Theorem 3.3.*

Theorem 3.14 *Let $S = G \times R_n$ be a right group, A and B be nonempty subsets of S. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$.*

Proof Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$ and $G/\langle p_1(B)\rangle = \{h_1\langle p_1(B)\rangle, h_2\langle p_1(B)\rangle, \ldots, h_z\langle p_1(B)\rangle\}$.

($\Rightarrow$) Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Then there exists an isomorphism $f : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$. By Theorem 3.3, we get that $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times R_n, E_i') \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times R_n, E_j')$. In view of Lemma 3.8, $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. By Theorem 3.5, we get $\mathrm{Cay}(\langle A\rangle, A) \cong \mathrm{Cay}(\langle B\rangle, B)$.

($\Leftarrow$) Let $\mathrm{Cay}(\langle A\rangle, A) \cong \mathrm{Cay}(\langle B\rangle, B)$. By Lemma 3.12, $\left|\langle p_1(A)\rangle\right| = \left|\langle p_1(B)\rangle\right|$ and thus $w = |G|/|\langle p_1(A)\rangle| = |G|/|\langle p_1(B)\rangle| = z$. By Theorem 3.5, we get $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$ for all $i, j \in \{1, 2, \ldots, w\}$. It follows that $\dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. There exists an isomorphism $f : \dot\bigcup_{i=1}^{w}(g_i \langle p_1(A)\rangle \times p_2(A), E_i) \to \dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Therefore $|\dot\bigcup_{i=1}^{w}(g_i \langle p_1(A)\rangle \times p_2(A)| = |\dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times p_2(B)|$. Since $\dot\bigcup_{i=1}^{w} g_i\langle p_1(A)\rangle = G = \dot\bigcup_{j=1}^{z} h_j\langle p_1(B)\rangle$, then $\left|G \times p_2(A)\right| = \left|G \times p_2(B)\right|$. Hence $\left|G\right| \times \left|p_2(A)\right| = \left|G\right| \times \left|p_2(B)\right|$ and thus $\left|p_2(A)\right| = \left|p_2(B)\right|$. Suppose that $R_n \backslash p_2(A) = \{q_1, q_2, \ldots, q_m\}$ and $R_n \backslash p_2(B) = \{q_1', q_2', \ldots, q_m'\}$. Let $r \in p_2(A)$. Define $T : \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i') \to \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E_j')$ by

$$T(s, r_l) = \begin{cases} f(s, r_l) & \text{if } r_l \in p_2(A) \\ (p_1(f(s, r)), q_k') & \text{if } r_l = q_k \text{ for some } q_k \in R_n \backslash p_2(A). \end{cases}$$

Clearly, T is well defined and is surjective. To show T is injective, let $x_1 = (u_1, \lambda_1), x_2 = (u_2, \lambda_2) \in S$ and $T(x_1) = T(x_2)$. We need to consider the following two cases:

(case 1) If $\lambda_1, \lambda_2 \in p_2(A)$, then $T(x_1) = f(x_1)$ and $T(x_2) = f(x_2)$. Since $T(x_1) = T(x_2)$, we have $f(x_1) = f(x_2)$ and $x_1 = x_2$ because f is a graph isomorphism.

(case 2) If $\lambda_1, \lambda_2 \notin p_2(A)$, assume that $\lambda_1 = q_l$ and $\lambda_2 = q_k$. Thus $T(x_1) = (p_1(f(u_1, r)), q_l')$ and $T(x_2) = (p_1(f(u_2, r)), q_k')$. Since $T(x_1) = T(x_2)$, we have $(p_1(f(u_1, r)), q_l') = (p_1(f(u_2, r)), q_k')$. It follows that $q_l' = q_k'$ and $p_1(f(u_1, r)) = p_1(f(u_2, r))$. Hence by the definition of T, $\lambda_1 = \lambda_2$. Since f is a graph isomorphism, we have $u_1 = u_2$. Therefore $x_1 = x_2$.

By the above two cases, we conclude that T is an injection. We will prove that T and T^{-1} are homomorphisms.

Assume that $((x, r_c), (y, r_d))$ is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times R_n, E_i')$. Thus $(y, r_d) = (x, r_c)(a, r_t)$ for some $(a, r_t) \in A$. Hence $(y, r_d) = (xa, r_t)$ and thus $r_d = r_t \in p_2(A)$ and $y = xa$. We have the following two cases.

(case 1) $r_c \in p_2(A)$. Then $\big(T(x, r_c), T(y, r_d)\big) = \big(f(x, r_c), f(y, r_d)\big)$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E_j')$ since f is an isomorphism.

(case 2) $r_c \in R_n \backslash p_2(A)$. Then $r_c = q_k$ for some $k \in \{1, 2, \ldots, m\}$. Hence $((x, r_c), (y, r_d))$ is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i')$. Then $((x, r_d), (y, r_d))$ is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. By Lemma 3.1, $((x, r), (y, r_d))$ is also an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. It follows that $(f(x, r), f(y, r_d))$

is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Let $f(x,r) = (x', r')$ and $f(y, r_d) = (y', r'_d)$. Therefore $\big((x', r'), (y', r'_d)\big)$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$ and thus $\big((x', r'_d), (y', r'_d)\big)$ is also an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Therefore $((x', q'_k), (y', r'_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. This means that $((x', q'_k), (y', r'_d)) = \big((p_1(f(x,r)), q'_k), (y', r'_d)\big) = (T(x, r_c), T(y, r_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. Hence $(T(x, r_c), T(y, r_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$.

Thus T is a homomorphism.

Assume that $(T(x, r_c), T(y, r_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. We have the following two cases.

(case 1) If $(T(x, r_c), T(y, r_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$, then we get that $T(x, r_c) = f(x, r_c)$ and $T(y, r_d) = f(y, r_d)$. Therefore $(f(x, r_c), f(y, r_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Since f is a graph isomorphism from $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ to $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$, we get that $((x, r_c), (y, r_d))$ is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ and it is also an arc in $\dot\bigcup_{i=1}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$.

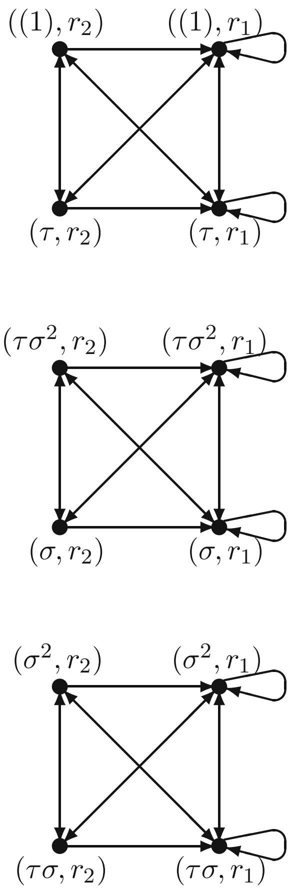
(case 2) Suppose that $(T(x, r_c), T(y, r_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. Then $r_c = q_k$ for some $k \in \{1, 2, \ldots, m\}$. Let $T(y, r_d) = f(y, r_d) = (y', r'_d)$. Then $((p_1(f(x,r)), q'_k), (y', r'_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$ and so $((p_1(f(x,r)), r'_d), (y', r'_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Hence there exists $(b, r'_d) \in B$ such that $(y', r'_d) = (p_1(f(x,r)), r'_d)(b, r'_d)$. Let $f(x,r) = f(x', r')$. Then $f(y, r_d) = (x', r'_d)(b, r'_d) = (x'b, r'_d) = (x', r')(b, r'_d) = f(x,r)\,(b, r'_d)$. This means that $(f(x,r), f(y, r_d))$ is an arc in $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Then $((x,r), (y, r_d))$ is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Therefore $((x, r_c), (y, r_d))$ is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$ and it is also an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$.

Thus T^{-1} is a homomorphism. Hence $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_j\langle p_1(A)\rangle \times R_n, E'_i) \cong \dot\bigcup_{j\in I}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. By Theorem 3.3, we have $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. □

Example 1 Let $S = S_3 \times R_2$ be a right group, where $S_3 = \{(1), \sigma, \sigma^2, \tau, \tau\sigma^2, \tau\sigma\}$ is the symmetric group with (1) an identity, $\sigma = (123), \sigma^2 = (132), \tau = (12), \tau\sigma^2 = (13), \tau\sigma = (23)$. Let $A = \{((1), r_1), (\tau, r_1), (\tau, r_2)\}$ and $B = \{(\tau\sigma, r_1), ((1), r_2), (\tau\sigma, r_2)\}$. It is easily seen that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ (see Figs. 1 and 2).

We have $\hat{A}_{r_1} = \{((1), r_1), (\tau, r_1)\}$, $\hat{A}_{r_2} = \{(\tau, r_2)\}$, $\hat{B}_{r_1} = \{(\tau\sigma, r_1)\}$, and $\hat{B}_{r_2} = \{((1), r_2), (\tau\sigma, r_2)\}$. Therefore $\hat{A} = \{\hat{A}_{r_1}, \hat{A}_{r_2}\}$, $\hat{B} = \{\hat{B}_{r_1}, \hat{B}_{r_2}\}$, and thus $|\hat{A}| = |\hat{B}|$. Since $\langle p_1(A)\rangle = \{e, \tau\}$ and $\langle p_1(B)\rangle = \{e, \tau\sigma\}$, then $|\langle p_1(A)\rangle| = |\langle p_1(B)\rangle|$.

Fig. 1 Cay(S, A)



We thus get $|\hat{A}_{r_1}| = 2 = |\hat{B}_{r_2}|$ and $|\hat{A}_{r_2}| = 1 = |\hat{B}_{r_1}|$. There exists a bijective function f from $\hat{A}$ to $\hat{B}$ such that $f(\hat{A}_{r_1}) = \hat{B}_{r_2}$ and $f(\hat{A}_{r_2}) = \hat{B}_{r_1}$.

Moreover, there are bijective functions

$$\begin{aligned} \varphi_{r_1} : \hat{A}_{r_1} \to \hat{B}_{r_2} \quad \text{such that} \quad & \varphi_{r_1}((1), r_1) = ((1), r_2) \\ & \varphi_{r_1}(\tau, r_1) = (\tau\sigma, r_2) \\ \text{and} \quad \varphi_{r_2} : \hat{A}_{r_2} \to \hat{B}_{r_1} \quad \text{such that} \quad & \varphi_{r_2}(\tau, r_2) = (\tau\sigma, r_1) \end{aligned}$$

such that $ord(p_1(a)) = ord(p_1(\varphi_{r_1}(a)))$ and $ord(p_1(b)) = ord(p_1(\varphi_{r_2}(b)))$ for all $a \in \hat{A}_{r_1}$ and $b \in \hat{A}_{r_2}$. According to Theorem 3.12, $\text{Cay}(\langle A \rangle, A) \cong \text{Cay}(\langle B \rangle, B)$.

4 Cayley Graphs of Rectangular Groups

By Theorem 4 in [13], we have the following theorem, which presents an equivalent condition for two Cayley graphs of a given rectangular group relative to one-element subsets are isomorphic to each other.

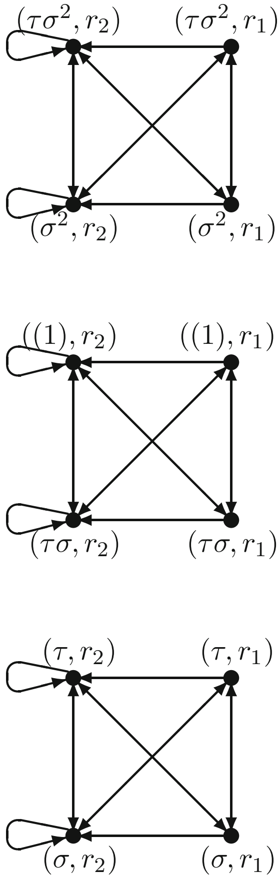


Fig. 2 Cay(S, B)

Theorem 4.1 *Let $S = G \times L_n \times R_m$ be a rectangular group, $a = (g_1, l_1, r_1)$, $b = (g_2, l_2, r_2) \in S$. Then $Cay(S, \{a\}) \cong Cay(S, \{b\})$ if and only if $ord(g_1) = ord(g_2)$.*

Lemma 4.2 *Let $S = G \times L_m \times R_n$ be a rectangular group, A a nonempty subset of S, and $(g_1, l_1, r_1), (g_2, l_2, r_2) \in S$. Then $\big((g_1, l_1, r_1), (g_2, l_2, r_2)\big)$ is an arc in $Cay(S, A)$ if and only if there exists $(a, l, r_2) \in A$ such that $g_2 = g_1 a$ and $l_1 = l_2$.*

As a direct consequence of Lemma 4.2, we have the following lemma.

Lemma 4.3 *Let $S = G \times L_m \times R_n$ be a rectangular group, A a nonempty subset of S. Then $Cay(S, A)$ is the disjoint union of m isomorphic strong subgraphs $(G \times \{l_i\} \times R_n, E_i)$ for $i = 1, 2, \ldots, m$.*

Lemma 4.4 *Let $S = G \times L_m \times R_n$ be a rectangular group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik})$ a strong subgraph of $Cay(S, A)$. Then the following conditions hold:*

(1) $(G \times \{l_i\} \times R_n, E_i) = \dot{\bigcup}_{k=1}^{w}(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik})$;
(2) $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik}) = Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, A^i)$ *where* $A^i = \{(g, l_i, r) | (g, l, r) \in A$ *for all* $l \in L_m\}$.

Proof (1) Note that $G = \dot{\bigcup}_{k=1}^{w} g_k\langle p_1(A)\rangle$, then $G \times \{l_i\} \times R_n = \dot{\bigcup}_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n)$. Let $\big((g, l_i, r), (g', l_i, r')\big) \in E_i$. By Lemma 4.2, there exists $(a, l, r') \in A$ such that $g' = ga$. Hence $g \in g_p\langle p_1(A)\rangle$, $g' \in g_q\langle p_1(A)\rangle$ for some $p, q \in \{1, 2, \ldots, w\}$. A simple computation shows that $p = q$ and then $(g, l_i, r), (g', l_i, r') \in (g_p\langle p_1(A)\rangle \times \{l_i\} \times R_n)$. Because $(g_p\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ip})$ is the strong subgraph of $\mathrm{Cay}(S, A)$, therefore $\big((g, l_i, r), (g', l_i, r')\big) \in \dot{\bigcup}_{k=1}^{w} E_{ik}$. Hence $E_i \subseteq \dot{\bigcup}_{k=1}^{w} E_{ik}$. Similarly, we can prove that $\dot{\bigcup}_{k=1}^{w} E_{ik} \subseteq E_i$, and then $E_i = \dot{\bigcup}_{k=1}^{w} E_{ik}$. We conclude that $(G \times \{l_i\} \times R_n, E_i) = \dot{\bigcup}_{k=1}^{w}(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik})$.

(2) Let $D = \mathrm{Cay}(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, A^i)$, we will prove that $E_{ik} = E(D)$. Let $\big((g, l_i, r), (g', l_i, r')\big) \in E_{ik}$. By Lemma 4.2, there exists $(a, l, r') \in A$ such that $g' = ga$. By Lemma 4.2 again, we get that $\big((g, l_i, r), (g', l_i, r')\big) \in E(D)$. This shows that $E_{ik} \subseteq E(D)$. Let $\big((g, l_i, r), (g', l_i, r')\big) \in E(D)$. By Lemma 4.2, there exists $(a, l_i, r') \in A^i$ such that $g' = ga$. We get that $(a, j, r') \in A$ for some $j \in L_m$. Then by Lemma 4.2 again, $((g, l_i, r), (g', l_i, r')) \in E_{ik}$. This shows that $E(D) \subseteq E_{ik}$. Therefore $E_{ik} = E(D)$. We conclude that $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik}) = \mathrm{Cay}(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, A^i)$.

□

Theorem 4.5 *Let* $S = G \times L_m \times R_n$ *be a rectangular group,* A, B *nonempty subsets of* S. *Let* $S' = G \times R_n$. *Then* $Cay(S, A) \cong Cay(S, B)$ *if and only if* $Cay(S', A') \cong Cay(S', B')$, *where* $A' = \{(g, r) \mid (g, l, r) \in A$ *for some* $l \in L_m\}$ *and* $B' = \{(g, r) \mid (g, l, r) \in B$ *for some* $l \in L_m\}$.

Proof Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, $G/\langle p_1(B)\rangle = \{h_1\langle p_1(B)\rangle, h_2\langle p_1(B)\rangle, \ldots, h_z\langle p_1(B)\rangle\}$. We let $(G \times \{l_i\} \times R_n, E_i^A)$, $A_i^k = (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik})$ be a strong subgraph of $\mathrm{Cay}(S, A)$, and let $(G \times \{l_j\} \times R_n, E_j^B)$, $B_j^t = (h_t\langle p_1(B)\rangle \times \{l_j\} \times R_n, E_{jt})$ be a strong subgraph of $\mathrm{Cay}(S, B)$. By Lemma 4.3 and Lemma 4.4(1), we have $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$

$$\begin{aligned}
&\Leftrightarrow \mathrm{Cay}(G \times L_m \times R_n, A) \cong \mathrm{Cay}(G \times L_m \times R_n, B)\\
&\Leftrightarrow \dot{\bigcup}_{i=1}^{m}(G \times \{l_i\} \times R_n, E_i^A) \cong \dot{\bigcup}_{j=1}^{m}(G \times \{l_j\} \times R_n, E_j^B)\\
&\Leftrightarrow \dot{\bigcup}_{i=1}^{m}\dot{\bigcup}_{k=1}^{w} A_i^k \cong \dot{\bigcup}_{j=1}^{m}\dot{\bigcup}_{t=1}^{z} B_j^t.
\end{aligned}$$

Since A_i^k and B_j^t are connected subgraphs, we get that $w = z$. Then for each i, k, there exist j, t such that $A_i^k \cong B_j^t$. Let $D_k^A = (g_k\langle p_1(A)\rangle \times p_2(A'), E_k)$ and $D_t^B = (h_t\langle p_1(B)\rangle \times p_2(B'), E_t)$ be strong subgraphs of $\mathrm{Cay}(g_k\langle p_1(A)\rangle \times R_n, A')$ and $\mathrm{Cay}(h_t\langle p_1(B)\rangle \times R_n, B')$, respectively. Let $A^i = \{(g, l_i, r) | (g, l, r) \in A\}$, $B^j = \{(h, l_j, r) | (h, l, r) \in B\}$. By Lemma 4.4(2) and Theorem 3.3, we have

$$\mathrm{Cay}(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, A^i) \cong \mathrm{Cay}(h_t\langle p_1(B)\rangle \times \{l_j\} \times R_n, B^j)$$

$$\Leftrightarrow \operatorname{Cay}(g_k\langle p_1(A)\rangle \times R_n, A') \cong \operatorname{Cay}(h_t\langle p_1(B)\rangle \times R_n, B')$$
$$\Leftrightarrow \dot{\bigcup}_{k=1}^{w} D_k^A \cup (g_k\langle p_1(A)\rangle \times R_n, E_{A'}) \cong \dot{\bigcup}_{t=1}^{z} D_t^B \cup (h_t\langle p_1(B)\rangle \times R_n, E_{B'}).$$

By Lemma 3.8 and Theorem 3.5, we have $\dot{\bigcup}_{k=1}^{w} D_k^A \cong \dot{\bigcup}_{t=1}^{z} D_t^B \Leftrightarrow D_k^A \cong D_t^B \Leftrightarrow \operatorname{Cay}(\langle A'\rangle, A') \cong \operatorname{Cay}(\langle B'\rangle, B') \Leftrightarrow \operatorname{Cay}(S', A') \cong \operatorname{Cay}(S', B')$. □

Example 2 Let $S = S_3 \times L_2 \times R_2$ be a rectangular group, where $S_3 = \{(1), \sigma, \sigma^2, \tau, \tau\sigma^2, \tau\sigma\}$ is the symmetric group, and let $A = \{((1), l_1, r_1), (\tau, l_1, r_1), (\tau, l_1, r_2)\}$, $B = \{(\tau\sigma, l_2, r_1), ((1), l_2, r_2), (\tau\sigma, l_2, r_2)\}$ be subsets of S.

We have $A' = \{((1), r_1), (\tau, r_1), (\tau, r_2)\}$ and $B' = \{(\tau\sigma, r_1), ((1), r_2), (\tau\sigma, r_2)\}$. By Example 1, $\operatorname{Cay}(S', A') \cong \operatorname{Cay}(S', B')$. By Theorem 4.5, $\operatorname{Cay}(S, A) \cong \operatorname{Cay}(S, B)$.

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Domination in Cayley Digraphs of Right and Left Groups

Nuttawoot Nupo and Sayan Panma

Abstract

Let $Cay(S, A)$ denote a Cayley digraph of a semigroup S with a connection set A. A semigroup S is said to be a right group if it is isomorphic to the direct product of a group and a right zero semigroup and S is called a left group if it is isomorphic to the direct product of a group and a left zero semigroup. In this paper, we attempt to find the value or bounds for the domination number of Cayley digraphs of right groups and left groups. Some examples which give sharpness of those bounds are also shown. Moreover, we consider the total domination number and give the necessary and sufficient conditions for the existence of the total dominating set in Cayley digraphs of right groups and left groups.

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Keywords: Cayley digraph, Right group, Left group, Domination number, Total domination number

1 Introduction

Let S be a semigroup and A a subset of S. Recall that the Cayley digraph $Cay(S, A)$ of S with the connection set A is defined as the digraph with vertex set S and arc set $E(Cay(S, A)) = \{(x, xa)|x \in S, a \in A\}$ (see [7]). Clearly, if A is an empty set, then $Cay(S, A)$ is an empty graph.

Arthur Cayley (1821-1895) introduced Cayley graphs of groups in 1878. Cayley graphs of groups have been extensively studied and many interesting results have been obtained (see for examples, [1], [12], [13], and [14]). Also, the Cayley graphs of semigroups have been considered by many authors. Many new interesting results on Cayley graphs of semigroups have recently appeared in various journals (see for examples, [4], [8], [9], [10], [12], [15], [16], [17], [18], and [19]). Furthermore, some properties of the Cayley digraphs of left groups and right groups are obtained by some authors (see for examples, [11], [16], [17], [18], and [21]).

The concept of domination for Cayley graphs has been studied by various authors (see for examples, [3], [5], [22], [24], and [25]). The total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi (see [3]) and is now well studied in graph theory. Tamizh Chelvam and Rani (see for examples, [24], [25], and [26]) have obtained bounds for various domination parameters for a class of Circulant graphs. Here we shall study some domination parameters of Cayley digraphs of right groups and left groups. All graphs considered in this paper are finite directed graphs. The terminologies and notations which related to our paper will be defined in the next section.

2 Preliminaries and Notations

In this section, we give some preliminaries needed in what follows on digraphs, semigroups, Cayley graphs of semigroups, domination number, and total domination number. For more information on digraphs, we refer to [2], and for semigroups see [6]. All sets in this paper are assumed to be finite.

A semigroup S is said to be a *right (left) zero semigroup* if $xy = y(xy = x)$ for all $x, y \in S$. A semigroup S is called a *right (left) group* if it is isomorphic to the direct product $G \times R_m(G \times L_m)$ of a group G and an $m-$ element right (left) zero semigroup $R_m(L_m)$. If $m = 1$, then we can consider a Cayley digraph of a right (left) group $G \times R_m(G \times L_m)$ as a Cayley digraph of a group G. So in this paper, we will consider in the case where $m \geq 2$.

Let $D = (V, E)$ be a digraph. A set $X \subseteq V$ of vertices in a digraph D is called a *dominating set* if every vertex $v \in V \setminus X$, there exists $x \in X$ such that $(x, v) \in E$ and we call that x dominates v or v is dominated by x. The *domination number* $\gamma(D)$ of a digraph D is the minimum cardinality of a dominating set in D and the corresponding dominating set is called a *$\gamma-$set*. A set $X \subseteq V$ is called a *total dominating set* if every vertex $v \in V$, there exists $x \in X$ such that $(x, v) \in E$. The *total domination number* $\gamma_t(D)$ equals the minimum cardinality among all the total dominating sets in D and the corresponding total dominating set is called a *γ_t-set*.

For any family of nonempty sets $\{X_i | i \in I\}$, we write $\dot{\bigcup\limits_{i\in I}} X_i := \bigcup\limits_{i\in I} X_i$ if $X_i \cap X_j = \emptyset$ for all $i \neq j$. Let $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ be digraphs such that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The *disjoint union* of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\dot{\bigcup}_{i=1}^{n}(V_i, E_i) := (\dot{\bigcup}_{i=1}^{n} V_i, \dot{\bigcup}_{i=1}^{n} E_i)$. It is easy to verify that $\gamma(\dot{\bigcup}_{i=1}^{n}(V_i, E_i)) = \sum\limits_{i=1}^{n} \gamma(V_i, E_i)$, $\gamma_t(\dot{\bigcup}_{i=1}^{n}(V_i, E_i)) = \sum\limits_{i=1}^{n} \gamma_t(V_i, E_i)$, and if for each $i, j \in I$ with $(V_i, E_i) \cong (V_j, E_j)$, then $\gamma(V_i, E_i) = \gamma(V_j, E_j)$ and $\gamma_t(V_i, E_i) = \gamma_t(V_j, E_j)$. From now on, $|A|$ denotes the cardinality of A where A is any finite set and p_i denotes the projection map on the i^{th} coordinate of a triple where $i \in \{1, 2, 3\}$. A subdigraph F of a digraph G is called a strong subdigraph of G if and only if whenever u and v are vertices of F and (u, v) is an arc in G, then (u, v) is an arc in F as well. Moreover, we denote by G a finite group and G_k a group of order k, and let $R_m(L_m)$ denote a right (left) zero semigroup with m elements. Now, we recall some results which are needed in the sequel as below for further references.

Lemma 2.1 ([20]). *Let $S = G \times R_m$ be a right group, A a nonempty subset of S such that $p_2(A) = R_m, G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$, and let $I = \{1, 2, \ldots, k\}$. Then*

1. *$S/\langle A\rangle = \{g_i\langle p_1(A)\rangle \times R_m | i \in I\}$ and $S = \dot{\bigcup\limits_{i\in I}} (g_i\langle p_1(A)\rangle \times R_m)$,*

2. *$Cay(S, A) = \dot{\bigcup\limits_{i\in I}} ((g_i\langle p_1(A)\rangle \times R_m), E_i)$ where $((g_i\langle p_1(A)\rangle \times R_m), E_i)$ is a strong subdigraph of $Cay(S, A)$ with $((g_i\langle p_1(A)\rangle \times R_m), E_i) \cong Cay(\langle A\rangle, A)$ for all $i \in I$.*

Lemma 2.2 ([20]). *Let $S = G \times R_m$ be a right group and A a nonempty subset of S. Then $\langle A\rangle = \langle p_1(A)\rangle \times p_2(A)$ is a right group contained in S.*

Lemma 2.3 ([16]). *Let $S = G \times L_m$ be a left group and A a nonempty subset of S. Then the following conditions hold :*

1. *for each $l \in L_m$, $Cay(G \times \{l\}, p_1(A) \times \{l\}) \cong Cay(G, p_1(A))$,*

2. *$Cay(S, A) = \dot{\bigcup\limits_{l\in L_m}} Cay(G \times \{l\}, p_1(A) \times \{l\})$.*

Lemma 2.4 ([20]). *Let $S = G \times L_m$ be a left group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$, and let $I = \{1, 2, \ldots, k\}$. Then*

1. *$S/\langle A\rangle = \{g_i\langle p_1(A)\rangle \times \{l\} | i \in I, l \in L_m\}$ and $S = \dot{\bigcup\limits_{i\in I, l\in L_m}} (g_i\langle p_1(A)\rangle \times \{l\})$,*

2. *$Cay(S, A) = \dot{\bigcup\limits_{i\in I, l\in L_m}} ((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ where $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ is a strong subdigraph of $Cay(S, A)$ with $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il}) \cong Cay(\langle p_1(A)\rangle, p_1(A))$ for all $i \in I, l \in L_m$.*

The following lemmas give the results for the domination number and the total domination number of Cayley graphs of the group $\mathbb{Z}_n$ relative to the specific connection sets.

Lemma 2.5 ([23]). *Let $n \geq 3$ be an odd integer, $m = \frac{n-1}{2}$ and $A = \{m, n - m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\} \subseteq \mathbb{Z}_n$ where $1 \leq k \leq m$. Then $\gamma_t(Cay(\mathbb{Z}_n, A)) = \lceil \frac{n}{2k} \rceil$.*

Lemma 2.6 ([23]). *Let $n \geq 3$ be an even integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and $A = \{\frac{n}{2}, m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\} \subseteq \mathbb{Z}_n$ where $1 \leq k \leq m$. Then $\gamma_t(Cay(\mathbb{Z}_n, A)) = \lceil \frac{n}{2k+1} \rceil$.*

3 Main results

In this section, we give some results for the domination parameters of Cayley digraphs of right groups and left groups related to the according connection sets such as the domination number and total domination number. We divide this section into two parts. The first part gives some results for the domination in Cayley digraphs of right groups and the second part for left groups. Hereafter, we will denote by D the Cayley digraph $Cay(S, A)$ of a semigroup S with a connection set A.

3.1 The domination parameters of Cayley digraphs of right groups

In this part, we study the domination parameters of Cayley digraphs of right groups relative to the appropriate connection sets. We start with the theorem which describes the domination number in Cayley digraphs of right groups with any connection set A where $|p_2(A)| \neq |R_m|$.

Theorem 3.1. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S. If $|p_2(A)| \neq |R_m|$, then $\gamma(D) = (|R_m| - |p_2(A)|) \times |G|$.*

Proof. Suppose that $|p_2(A)| \neq |R_m|$, we have $|p_2(A)| < |R_m|$. Let $Y = \{(x, a) \in S | a \notin p_2(A)\}$. We will show that Y is a dominating set of D. Let $(b, c) \in S \setminus Y$. Then $b \in G$ and $c \in p_2(A)$, i.e., there exists $d \in p_1(A) \subseteq G$ such that $(d, c) \in A$. Since G is a group, there exists $y \in G$ such that $b = yd$. From $|R_m| > |p_2(A)|$, we get that there exists $r \in R_m \setminus p_2(A)$ which leads to $(y, r) \in Y$. Thus $(b, c) = (yd, c) = (y, r)(d, c)$. Therefore, Y is a dominating set of D. Hence $\gamma(D) \leq |Y| = (|R_m| - |p_2(A)|) \times |G|$. Now, we assume in the contrary that

$\gamma(D) < (|R_m| - |p_2(A)|) \times |G|$. Let $X \subseteq S$ be a dominating set of D such that X is a $\gamma-$set, i.e., $|X| = \gamma(D) < (|R_m| - |p_2(A)|) \times |G|$. We have

$$\begin{aligned}
|S \setminus X| &= |S| - |X| \\
&> nm - [(|R_m| - |p_2(A)|) \times |G|] \\
&= nm - [(m - |p_2(A)|) \times n] \\
&= nm - nm + n(|p_2(A)|) \\
&= n(|p_2(A)|) \\
&= |G \times p_2(A)|.
\end{aligned}$$

Thus there exists at least one element $(a, b) \in (S \setminus X) \setminus (G \times p_2(A))$, i.e., $(a, b) \in S \setminus X$ and $(a, b) \notin G \times p_2(A)$. Since $a \in G$, we obtain that $b \notin p_2(A)$. From $(a, b) \in S \setminus X$ and X is the dominating set of D, there exists $(x, y) \in X$ such that $((x, y), (a, b)) \in E(D)$. Thus $(a, b) = (x, y)(c, d) = (xc, yd) = (xc, d)$ for some $(c, d) \in A$. We conclude that $b = d \in p_2(A)$ which is a contradiction. Therefore, $\gamma(D) \not< (|R_m| - |p_2(A)|) \times |G|$, i.e., $\gamma(D) = (|R_m| - |p_2(A)|) \times |G|$, as required. $\square$

The next theorem gives the bounds of the domination number in Cayley digraphs of right groups with any connection set A where $|p_2(A)| = |R_m|$.

Theorem 3.2. *Let $S = G \times R_m$ be a right group A a nonempty subset of S. If $|p_2(A)| = |R_m|$, then $\frac{|S|}{|A|+1} \leq \gamma(D) \leq |G|$.*

Proof. Assume that $|p_2(A)| = |R_m|$. We first prove the right inequality, i.e., $\gamma(D) \leq |G|$. For each $r \in R_m$, let $Y = \{(x, r) | x \in G\} = G \times \{r\}$. We will show that Y is a dominating set of D. Let $(a, b) \in S \setminus Y$. Then $a \in G$ and $b \in R_m$ such that $b \neq r$. Since $|p_2(A)| = |R_m|$ and $p_2(A) \subseteq R_m$, we get that $R_m = p_2(A)$ and then $b \in p_2(A)$. Thus there exists $c \in p_1(A) \subseteq G$ such that $(c, b) \in A$. Since G is a group, there exists $g \in G$ such that $a = gc$. We obtain that $(a, b) = (gc, b) = (g, r)(c, b)$ where $(g, r) \in Y$. Hence Y is the dominating set of D. Therefore, $\gamma(D) \leq |Y| = |G \times \{r\}| = |G|$.
Now, we will prove the left inequality. Let X be the dominating set of D such that X is a $\gamma-$set, i.e., $|X| = \gamma(D)$. Then for each $(a, b) \in S \setminus X$, we get that $(a, b) = (x, y)(s, t)$ for some $(x, y) \in X$ and $(s, t) \in A$ which implies that $S \setminus X \subseteq XA$. Hence $|S \setminus X| \leq |XA|$. Since every element of X has the same out-degree $|A|$, we obtain that

$$\gamma(D)|A| = |X||A| \geq |XA| \geq |S \setminus X| = |S| - |X| = |S| - \gamma(D).$$

Then $\gamma(D)|A| \geq |S| - \gamma(D)$ which leads to $|S| \leq \gamma(D)|A| + \gamma(D) = \gamma(D)(|A|+1)$. Hence $\gamma(D) \geq \frac{|S|}{|A|+1}$. $\square$

In the following example, we present the sharpness of the bounds given in Theorem 3.2.

Example 3.3. *Let $\mathbb{Z}_3 \times R_2$ be a right group where $\mathbb{Z}_3$ is a group of integers modulo 3 under the addition and $R_2 = \{r_1, r_2\}$ is a right zero semigroup.*

(1). *Consider the Cayley digraph* $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})$.

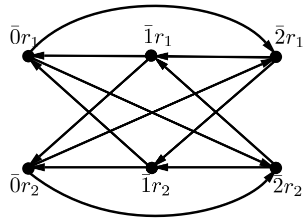

Figure 1: $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})$.

We have $X = \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1)\}$ *is a* $\gamma-$*set of* $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})$ *and* $\gamma(Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})) = |X| = 3 = |\mathbb{Z}_3|$. *Similarly,* $\gamma(Cay(\mathbb{Z}_n \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})) = |\mathbb{Z}_n|$ *where* $n \in \mathbb{N}$.

(2). *Consider the Cayley digraph* $Cay(\mathbb{Z}_4 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})$.

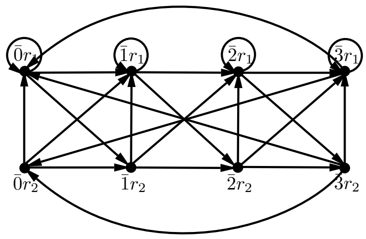

Figure 2: $Cay(\mathbb{Z}_4 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})$.

We have $Y = \{(\bar{0}, r_2), (\bar{2}, r_2)\}$ *is a* $\gamma-$*set of* $Cay(\mathbb{Z}_4 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})$ *and* $\gamma(Cay(\mathbb{Z}_4 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})) = |Y| = 2 = \frac{|\mathbb{Z}_4 \times R_2|}{|\{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\}|+1}$. *Similarly, we also obtain* $\gamma(Cay(\mathbb{Z}_{2k} \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})) = k = \frac{4k}{4} = \frac{|\mathbb{Z}_{2k} \times R_2|}{|\{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\}|+1}$ *with a* $\gamma-$*set* $\{(\bar{0}, r_2), (\bar{2}, r_2), (\bar{4}, r_2), \dots, (\overline{2k-2}, r_2)\}$ *where* $k \in \mathbb{N}$.

The following theorems show the values for the domination number in Cayley digraphs of right groups according to the specific connection sets. We begin with two lemmas which are refered in the proof of theorems.

Lemma 3.4. *Let* $S = G \times R_m$ *be a right group and* A *a nonempty subset of* S *such that* $p_1(A) = G$, $p_2(A) = R_m$, *and* $|A| = |R_m|$. *For each* $(x_1, r_1), (x_2, r_2) \in S$, *if* $(x_1, r_1)(y_1, s_1) = (x_2, r_2)(y_2, s_2)$ *for some* $(y_1, s_1), (y_2, s_2) \in A$, *then* $x_1 = x_2$.

Proof. Let $(x_1, r_1), (x_2, r_2) \in S$ be such that $(x_1, r_1)(y_1, s_1) = (x_2, r_2)(y_2, s_2)$ for some $(y_1, s_1), (y_2, s_2) \in A$. Thus $(x_1y_1, r_1s_1) = (x_2y_2, r_2s_2)$, that is, $(x_1y_1, s_1) = (x_2y_2, s_2)$. Then $x_1y_1 = x_2y_2$ and $s_1 = s_2$. Since we know that $p_1(A) = G$, $p_2(A) = R_m$, and $|A| = |R_m|$, these imply $y_1 = y_2$. From $x_1y_1 = x_2y_2$

where x_1, x_2, y_1, y_2 are elements of a group G and $y_1 = y_2$, we can conclude that $x_1 = x_2$ by the cancellation law. $\square$

Lemma 3.5. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $A = \{a\} \times R_m$ where $a \in G$. Let Y be a dominating set of D. If there exists $x \in G$ such that $x \notin p_1(Y)$, then $(xa, r) \in Y$ for all $r \in R_m$.*

Proof. Let Y be a dominating set of D. Suppose that there exists $x \in G$ such that $x \notin p_1(Y)$ and assume in the contrary that there exists $r \in R_m$ such that $(xa, r) \notin Y$. Since Y is a dominating set of D, there exists $(y, r') \in Y$ such that $((y, r'), (xa, r)) \in E(D)$, i.e., $(xa, r) = (y, r')(a, r)$ where $(a, r) \in A$. Hence $xa = ya$ which implies that $x = y \in p_1(Y)$ which contradicts our supposition. Therefore, $(xa, r) \in Y$ for all $r \in R_m$. $\square$

Theorem 3.6. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $p_1(A) = G$, $p_2(A) = R_m$, and $|A| = |R_m|$. Then $\gamma(D) = |G|$.*

Proof. Assume that the conditions hold. Since $|p_2(A)| = |R_m|$, we obtain that $\gamma(D) \leq |G|$ by Theorem 3.2. Now, suppose that there exists a dominating set Y such that $|Y| < |G|$. Then there exists $g \in G$ such that $g \notin p_1(Y)$. We first prove that for each $r \in R_m$, $(g, r)A \subseteq Y$. Let $r \in R_m$ and $(x, y) \in (g, r)A$. Then $(x, y) = (g, r)(g_1, r_1)$ for some $(g_1, r_1) \in A$. If $(x, y) \notin Y$, then there exists $(g', r') \in Y$ such that $(x, y) = (g', r')(g_2, r_2)$ for some $(g_2, r_2) \in A$ since Y is a dominating set of D. Thus $(g, r)(g_1, r_1) = (x, y) = (g', r')(g_2, r_2)$ where $(g_1, r_1), (g_2, r_2) \in A$. By Lemma 3.4, we can conclude that $g = g' \in p_1(Y)$ which is a contradiction. Hence $(x, y) \in Y$ which leads to $(g, r)A \subseteq Y$. Since $p_1(A) = G$, we obtain that the identity element e of G lies in $p_1(A)$. Then there exists $s \in p_2(A)$ such that $(e, s) \in A$ and $(g, s) = (g, r)(e, s) \in (g, r)A \subseteq Y$. Whence $g \in p_1(Y)$ which contradicts the above supposition. Therefore, $\gamma(D) = |G|$. $\square$

Theorem 3.7. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $A = \{a\} \times R_m$ where $a \in G$. Then $\gamma(D) = |G|$.*

Proof. Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $A = \{a\} \times R_m$ where $a \in G$. Then $|p_2(A)| = |R_m|$. By Theorem 3.2, we obtain that $\gamma(D) \leq |G|$. Assume that there exists a dominating set Y of D such that $|Y| < |G|$. Then there exists $x \in G$ such that $x \notin p_1(Y)$. Let $U = \{u \in G | u \notin p_1(Y)\}$. Assume that $|U| = k$ where $1 \leq k \leq n - 1$. For each $u \in U$, we obtain by Lemma 3.5 that $(ua, r) \in Y$ for all $r \in R_m$. Hence there exists at least one element $q \in p_1(Y)$ such that $(q, r) \in Y$ for all $r \in R_m$. Let $V = \{v \in p_1(Y) | (v, r) \in Y \text{ for all } r \in R_m\}$. Assume that $|V| = l$ where $1 \leq l \leq n-k$. By Lemma 3.5 again, we get that $|Y| \geq ml + [(n-k)-l] + (k-l) = ml + n - 2l = n + (m-2)l$. Since $m \geq 2$, we obtain that $|Y| \geq n + (m-2)l \geq n$, a contradiction. Therefore, $\gamma(D) = |G|$, as required. $\square$

Theorem 3.8. *Let $S = G \times R_m$ be a right group and $A = K \times R_m$ a nonempty subset of S where K is any subgroup of G. Then $\gamma(D) = \frac{|G|}{|K|}$.*

Proof. Let $S = G \times R_m$ be a right group and $A = K \times R_m$ a nonempty subset of S where K is a subgroup of a group G. Consider the set of all left cosets of K in G, $G/K = \{g_1K, g_2K, \ldots, g_tK\}$, we obtain that the index of K in G equals t, i.e., $[G : K] = t$. Let $I = \{1, 2, \ldots, t\}$ be an index set. By Lemma

2.1, we have $S/\langle A\rangle = \{g_iK \times R_m | i \in I\}$ such that $S = \dot{\bigcup\limits_{i\in I}}(g_iK \times R_m)$ and $Cay(S, A) = \dot{\bigcup\limits_{i\in I}}((g_iK \times R_m), E_i)$ where $((g_iK \times R_m), E_i)$ is a strong subdigraph of $Cay(S, A)$ with $((g_iK \times R_m), E_i) \cong Cay(\langle A\rangle, A)$ for all $i \in I$. Thus

$$\begin{aligned}
\gamma(D) &= \gamma(Cay(S, A)) \\
&= \gamma(\dot{\bigcup_{i\in I}}((g_iK \times R_m), E_i)) \\
&= \sum_{i\in I} \gamma((g_iK \times R_m), E_i) \\
&= |I|[\gamma(Cay(\langle A\rangle, A))] \\
&= t[\gamma(Cay(\langle A\rangle, A))].
\end{aligned}$$

By Lemma 2.2, we can conclude that $\langle A\rangle = \langle p_1(A)\rangle \times p_2(A) = \langle K\rangle \times R_m = K \times R_m = A$. Furthermore, we can prove that $\gamma(Cay(\langle A\rangle, A)) = 1$ which implies that $\gamma(D) = t = [G : K] = \frac{|G|}{|K|}$. □

The next theorem gives the necessary and sufficient conditions for the existence of the total dominating set in Cayley digraphs of right groups with connection sets.

Theorem 3.9. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S. Then the total dominating set of D exists if and only if $p_2(A) = R_m$.*

Proof. We first prove the necessary condition by assume that the total dominating set of D exists, say T. We will show that $p_2(A) = R_m$. By the definition of the connection set A, we know that $p_2(A) \subseteq R_m$. Let $r \in R_m$. Then for each $a \in G$, we get that (a, r) is dominated by a vertex (x, y) for some $(x, y) \in T$ since T is the total dominating set of D. Thus there exists $(a', r') \in A$ such that $(a, r) = (x, y)(a', r') = (xa', yr') = (xa', r')$ which implies that $r = r'$, i.e., $r \in p_2(A)$. Therefore, $p_2(A) = R_m$.
Conversely, we prove the sufficient condition by suppose that $p_2(A) = R_m$. We will prove that every vertex has an in-degree in D. Let $(g, r) \in G \times R_m$. Then $r \in R_m = p_2(A)$. Thus there exists $a \in p_1(A)$ such that $(a, r) \in A$. We obtain that $((ga^{-1}, r'), (g, r)) = ((ga^{-1}, r'), (ga^{-1}, r')(a, r)) \in E(D)$, i.e., (g, r) is dominated by (ga^{-1}, r'). So we can conclude that every vertex of D always has an in-degree in D. If we take $T = V(D) = S$, then we can see that T is a total dominating set of D since for each $(x, y) \in S$, (x, y) is dominated by some vertices in T. Hence the total dominating set of D always exists if $p_2(A) = R_m$. □

Theorem 3.10. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $p_2(A) = R_m$. Then $\frac{|S|}{|A|} \leq \gamma_t(D) \leq |G|$.*

Proof. Let A be a connection set of D such that $p_2(A) = R_m$. We know that the total dominating set of D exists by Theorem 3.9. For each $r \in R_m$, let $T = \{(g, r) | g \in G\} = G \times \{r\}$. We will show that T is a total dominating set of D. Let $(x, y) \in S = G \times R_m$. Since $p_2(A) = R_m$, we get that $y \in p_2(A)$ which implies that there exists $z \in p_1(A)$ such that $(z, y) \in A$. Since G is a group and $x, z \in G$, we obtain that $x = hz$ for some $h \in G$. Thus there exists

$(h, r) \in T$ such that $(x, y) = (hz, y) = (h, r)(z, y)$. Hence (x, y) is dominated by the vertex (h, r) in T. We can conclude that T is the total dominating set of D which leads to

$$\gamma_t(D) \leq |T| = |G \times \{r\}| = |G|.$$

Next, we will show that $\gamma_t(D) \geq \frac{|S|}{|A|}$. Assume in the contrary that there exists a total dominating set T' such that $|T'| < \frac{|S|}{|A|}$. Thus $|T'A| \leq |T'||A| < |S|$ which implies that there exists at least one element $(p, q) \in S$ but $(p, q) \notin T'A$. Hence there is no an element in T' which dominates (p, q), this contradicts the property of the total dominating set T'. Consequently, $\gamma_t(D) \geq \frac{|S|}{|A|}$, as required. $\square$

In the following example, we give the sharpness of the bounds given in Theorem 3.10.

Example 3.11. *Let $\mathbb{Z}_3 \times R_2$ be a right group where $\mathbb{Z}_3$ is a group of integers modulo 3 under the addition and $R_2 = \{r_1, r_2\}$ is a right zero semigroup.*

(1). *Consider the Cayley digraph $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})$.*

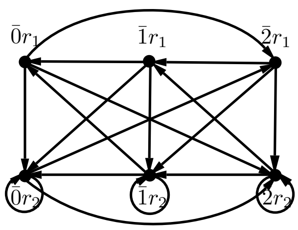

Figure 3: $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})$.

We have $X = \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1)\}$ is a γ_t−set of $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})$ and $\gamma_t(Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})) = |X| = 3 = |\mathbb{Z}_3|$. Similarly, we can get that $\gamma_t(Cay(\mathbb{Z}_n \times R_2, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})) = |\mathbb{Z}_n|$ with a γ_t−set $\mathbb{Z}_n \times \{r_1\}$ where $n \in \mathbb{N}$.

(2). *Consider $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})$.*

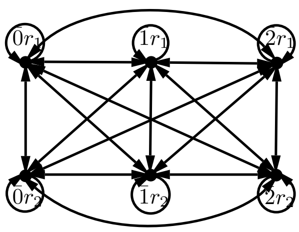

Figure 4: $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})$.

We obtain that $Y = \{(\bar{0}, r_1)\}$ *is a* γ_t-*set of* $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})$ *and* $\gamma_t(Cay(\mathbb{Z}_3 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})) = |Y| = 1 = \frac{|\mathbb{Z}_3 \times R_2|}{|\{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\}|}$.
Similarly, $\gamma_t(Cay(\mathbb{Z}_{3k} \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})) = \frac{|\mathbb{Z}_{3k} \times R_2|}{|\{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\}|}$ *with the* γ_t-*set* $\{(\bar{0}, r_1), (\bar{3}, r_1), (\bar{6}, r_1), \ldots, (\overline{3k-3}, r_1)\}$ *where* $k \in \mathbb{N}$.

3.2 The domination parameters of Cayley digraphs of left groups

The following result gives us the domination number of a Cayley digraph of a left group $G \times L_m$ with a connection set A in the term of a domination number of a Cayley digraph of the subgroup $\langle p_1(A) \rangle$ of G.

Theorem 3.12. *Let* $S = G \times L_m$ *be a left group,* A *a nonempty subset of* S*, and* $G/\langle p_1(A) \rangle = \{g_1\langle p_1(A) \rangle, g_2\langle p_1(A) \rangle, \ldots, g_k\langle p_1(A) \rangle\}$*. Then* $\gamma(D) = m \cdot k \cdot \gamma(Cay(\langle p_1(A) \rangle, p_1(A)))$.

Proof. Let $I = \{1, 2, \ldots, k\}$. By Lemma 2.4, we have $D = \dot{\bigcup}_{i \in I, l \in L_m} ((g_i\langle p_1(A) \rangle \times \{l\}), E_{il})$ such that a digraph $((g_i\langle p_1(A) \rangle \times \{l\}), E_{il})$ is the strong subdigraph of D with $((g_i\langle p_1(A) \rangle \times \{l\}), E_{il}) \cong Cay(\langle p_1(A) \rangle, p_1(A))$ for all $i \in I, l \in L_m$. Therefore, $\gamma(D) = \sum_{i=1}^{k} \sum_{l=1}^{m} \gamma((g_i\langle p_1(A) \rangle \times \{l\}), E_{il})$ and we can conclude that $\gamma(D) = m \cdot k \cdot \gamma(Cay(\langle p_1(A) \rangle, p_1(A)))$. □

Sometimes, it is not easy to find $\gamma(Cay(\langle p_1(A) \rangle, p_1(A)))$ so we can not find $\gamma(D)$ actually. However, we can know the bound of $\gamma(D)$ which is not depend on $\gamma(Cay(\langle p_1(A) \rangle, p_1(A)))$. The next theorem gives the bounds of the domination number in Cayley digraphs of left groups with the according connection sets.

Theorem 3.13. *Let* $S = G \times L_m$ *be a left group and* A *a nonempty subset of* S *such that the identity of* G *lies in* $p_1(A)$*. If* H *is a subgroup of* G *with a maximum cardinality and contained in* $p_1(A)$*, then* $\frac{|G|}{|p_1(A)|} \leq \frac{\gamma(D)}{|L_m|} \leq [G : H]$ *where* $[G : H]$ *is the index of* H *in* G.

Proof. Suppose that H is the subgroup of G with a maximum cardinality such that $H \subseteq p_1(A)$. We will show that $\gamma(D) \leq [G : H]|L_m|$. Let $[G : H] = k$ for some $k \in \mathbb{N}$. Consider the set of all left cosets of H in G, $\{g_1H, g_2H, \ldots, g_kH\}$. Choose only one element from each left coset $g_1H, g_2H, \ldots, g_kH$, say $g_1h_1, g_2h_2, \ldots, g_kh_k$, respectively. Let $D_i = Cay(G \times \{l_i\}, p_1(A) \times \{l_i\})$ and $Y_i = \{g_1h_1, g_2h_2, \ldots, g_kh_k\} \times \{l_i\} \subseteq G \times \{l_i\}$. We prove that Y_i is a dominating set of D_i. Let $(g, l_i) \in (G \times \{l_i\}) \setminus Y_i$. Since $g \in G = \bigcup_{t=1}^{k} g_tH$, we get that $g \in g_jH$ for some $1 \leq j \leq k$. Then $g = g_jh$ for some $h \in H$. Thus $(g_jh_j, l_i) \in Y_i$ and $h_j^{-1}h \in H \subseteq p_1(A)$. So there exists $l_q \in p_2(A)$ such that $(h_j^{-1}h, l_q) \in A$ and we have $(g, l_i) = (g_jh, l_i) = ((g_jh_j)(h_j^{-1}h), l_i) = (g_jh_j, l_i)(h_j^{-1}h, l_q) \in Y_iA$. Hence Y_i is the dominating set of D_i and then $\gamma(D_i) \leq |Y_i| = k = [G : H]$. By Lemma 2.3, we can conclude that $\gamma(D) = \sum_{i=1}^{m} \gamma(D_i) = \gamma(D_i)|L_m| \leq [G : H]|L_m|$. Now, we will prove that $\gamma(D) \geq \frac{|G|}{|p_1(A)|}|L_m|$. By Lemma 2.3(1), we have

$$Cay(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong Cay(G \times \{l_j\}, p_1(A) \times \{l_j\}) \text{ for all } l_i, l_j \in L_m.$$

For each $1 \leq i \leq m$, we will consider the domination number of D_i and let X_i be the dominating set of D_i such that X_i is a $\gamma-$set. Since the identity of G lies in $p_1(A)$ and X_i is the dominating set of D_i, we get that $(X_i)(p_1(A) \times \{l_i\}) = G \times \{l_i\}$. Hence $|G| = |G \times \{l_i\}| = |(X_i)(p_1(A) \times \{l_i\})| \leq |X_i||p_1(A) \times \{l_i\}| = |X_i||p_1(A)|$. Thus $\gamma(D_i) = |X_i| \geq \frac{|G|}{|p_1(A)|}$. By Lemma 2.3(2), we obtain that $D = \dot{\bigcup\limits_{1 \leq i \leq m}} D_i$. Then we conclude that $\gamma(D) = \gamma(\dot{\bigcup\limits_{1 \leq i \leq m}} D_i) = \sum_{i=1}^{m} \gamma(D_i) = \gamma(D_i)|L_m| \geq \frac{|G|}{|p_1(A)|}|L_m|$. □

Corollary 3.14. *Let $S = G \times L_m$ be a left group and $A = K \times L_m$ a nonempty subset of S where K is any subgroup of G. Then $\gamma(D) = [G : K]|L_m|$.*

Proof. Since $A = K \times L_m$ where K is any subgroup of G, we obtain that the identity e of G lies in $K = p_1(A)$. Moreover, we get that K is the subgroup of G with a maximum cardinality that contained in $p_1(A)$. By Theorem 3.13, we obtain that $\frac{|G|}{|K|}|L_m| \leq \gamma(D) \leq [G : K]|L_m|$. Therefore, $\gamma(D) = [G : K]|L_m|$ since $[G : K] = \frac{|G|}{|K|}$. □

The following example gives the sharpness of bounds given in Theorem 3.13.

Example 3.15. *Let $\mathbb{Z}_6 \times L_2$ be a left group where $\mathbb{Z}_6$ is a group of integers modulo 6 under the addition and $L_2 = \{l_1, l_2\}$ is a left zero semigroup.*

(1). *Consider the Cayley digraph $Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.*

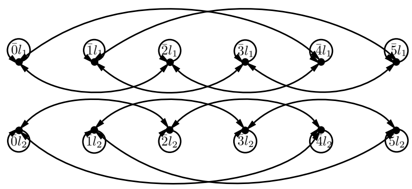

Figure 5: $Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.

We have $X = \{(\bar{0}, l_1), (\bar{0}, l_2), (\bar{1}, l_1), (\bar{1}, l_2)\}$ is a $\gamma-$set of $Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$. Thus $\gamma(Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})) = |X| = 4 = 2(2) = [\mathbb{Z}_6 : H]|L_2|$ where $H = \{\bar{0}, \bar{2}, \bar{4}\}$ is the subgroup with a maximum cardinality of $\mathbb{Z}_6$ that contained in $p_1(\{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.
Similarly, if $A = \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1), \ldots, (\overline{2k-2}, l_1)\}$ is a nonempty subset of $\mathbb{Z}_{2k} \times L_2$ where $k \in \mathbb{N}$, then $\{(\bar{0}, l_1), (\bar{0}, l_2), (\bar{1}, l_1), (\bar{1}, l_2)\}$ is a $\gamma-$set of $Cay(\mathbb{Z}_{2k} \times L_2, A)$. Hence $\gamma(Cay(\mathbb{Z}_{2k} \times L_2, A)) = 4 = [\mathbb{Z}_{2k} : H]|L_2|$ where $H = \{\bar{0}, \bar{2}, \bar{4}, \ldots, \overline{2k-2}\}$ is the subgroup with a maximum cardinality of $\mathbb{Z}_{2k}$ that contained in $p_1(A)$.

(2). *Consider the Cayley digraph* $Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{3}, l_2)\})$.

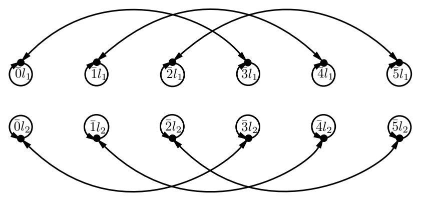

Figure 6: $Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{3}, l_2)\})$.

We have $Y = \{(\bar{0}, l_1), (\bar{1}, l_1), (\bar{2}, l_1), (\bar{0}, l_2), (\bar{1}, l_2), (\bar{2}, l_2)\}$ *is a* $\gamma-$*set of* $Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{3}, l_2)\})$ *and* $\gamma(Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{3}, l_2)\})) = |Y| = 6 = \frac{6}{2} \times 2 = \frac{|\mathbb{Z}_6|}{|p_1(\{(\bar{0}, l_1), (\bar{3}, l_2)\})|} \times |L_2|$.
Similarly, if $A = \{(\bar{0}, l_1), (\bar{k}, l_2)\}$ *is a nonempty subset of* $\mathbb{Z}_{2k} \times L_2$ *where* $k \in \mathbb{N}$*, then* $\{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{k-1}\} \times \{l_1, l_2\}$ *is a* $\gamma-$*set of* $Cay(\mathbb{Z}_{2k} \times L_2, A)$*. Hence* $\gamma(Cay(\mathbb{Z}_{2k} \times L_2, A)) = 2k = \frac{2k}{2} \times 2 = \frac{|\mathbb{Z}_{2k}|}{|p_1(A)|} \times |L_2|$.

Now, we show other results of the domination number of Cayley digraphs of $\mathbb{Z}_n$, the group of integers modulo n, with a connection set $\{\bar{1}, \bar{t}\} \subseteq \mathbb{Z}_n$ in order to apply to the domination number of Cayley digraphs of left groups $\mathbb{Z}_n \times L$ where L is a left zero semigroup. Recall that $N(x)$ is the set of all neighbours of x and let $N[x] = N(x) \cup \{x\}$.

In general, it is easy to verify that $\lceil \frac{n}{3} \rceil \leq \gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{t}\})) \leq \lceil \frac{n}{2} \rceil$.

Proposition 3.16. *Let* $n \geq 2$ *be a positive integer.*
Then $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{2}\})) = \lceil \frac{n}{3} \rceil$.

Proof. We will consider the case $n \equiv 1 \ (mod\ 3)$.
It is easy to see that $\{\bar{1}, \bar{4}, \bar{7}, \dots, \overline{n-3}, \bar{n}\}$ is a dominating set of $Cay(\mathbb{Z}_n, \{\bar{1}, \bar{2}\})$. Hence $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{2}\})) \leq |\{\bar{1}, \bar{4}, \bar{7}, \dots, \overline{n-3}, \bar{n}\}| = \frac{n+2}{3} = \lceil \frac{n}{3} \rceil$.
Suppose that there exists a dominating set X such that $|X| < \frac{n+2}{3}$, i.e., $|X| \leq \frac{n-1}{3}$. Since $|N[x]| \leq 3$ for all $x \in X$, we obtain that $|\bigcup_{x \in X} N[x]| \leq 3|X| \leq n - 1 < n$ which is a contradiction. Therefore, $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{2}\})) = \lceil \frac{n}{3} \rceil$ and we can prove other cases similarly. □

Lemma 3.17. *Let* $n \geq 3$ *be a positive integer and* X *a dominating set of* $Cay(\mathbb{Z}_n, \{\bar{1}, \bar{3}\})$*. For each* $x \in X, |N[x] \cap N[v]| \geq 1$ *for some* $v \in X \setminus \{x\}$.

Proof. Let X be a dominating set of $Cay(\mathbb{Z}_n, \{\bar{1}, \bar{3}\})$ and $x \in X$.
Then $N[x] = \{x, x+1, x+3\}$. Since $x+2 \notin N[x]$ and $x+2$ has to be dominated, we can conclude that $x+2 \in X$ or $x+2 \in N[y]$ for some $y \in X$.

If $x+2 \in X$, then $N[x+2] = \{x+2, x+3, x+5\}$, i.e., $x+3 \in N[x] \cap N[x+2]$ which implies that $|N[x] \cap N[x+2]| \geq 1$.

If $x+2 \in N[y]$, then $y = x+1$ or $y = x-1$.

If $y = x+1$, then $x+1 \in X$. Thus $x+1 \in N[x] \cap N[x+1]$ which leads to $|N[x] \cap N[x+1]| \geq 1$.

If $y = x-1$, then $x-1 \in X$. Thus $x \in N[x] \cap N[x-1]$ which implies that $|N[x] \cap N[x-1]| \geq 1$. □

Proposition 3.18. *Let $n \geq 3$ be a positive integer.*
Then $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{3}\})) = \begin{cases} 2\lceil \frac{n}{5} \rceil - 1 & \text{if } n \equiv 1, 2 \ (mod\ 5), \\ 2\lceil \frac{n}{5} \rceil & \text{if } n \equiv 0, 3, 4 \ (mod\ 5). \end{cases}$

Proof. We will consider the case $n \equiv 1$ (mod 5). In this case, we can conclude that $T = \{\bar{1}, \bar{2}, \bar{6}, \bar{7}, \overline{11}, \overline{12}, \ldots, \overline{n-5}, \overline{n-4}, \bar{n}\}$ is a dominating set which implies that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{3}\})) \leq |T| = \frac{2n+3}{5} = 2\lceil \frac{n}{5} \rceil - 1$. Next, suppose that there exists a dominating set X such that $|X| \leq 2\lceil \frac{n}{5} \rceil - 2 = \frac{2(n-1)}{5}$. For each $x \in X$, we have by Lemma 3.17 that $N[x] \cap N[y] \geq 1$ for some $y \in X \setminus \{x\}$. Since $|N[x]| \leq 3$, we have $|\bigcup\limits_{x \in X} N[x]| \leq 3|X| - \lceil \frac{|X|}{2} \rceil \leq \frac{5|X|}{2} \leq n - 1 < n$, that is, $\bigcup\limits_{x \in X} N[x] \subsetneq \mathbb{Z}_n$. Hence X does not dominate $\mathbb{Z}_n$ which is a contradiction. Therefore, $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{3}\})) = |T| = 2\lceil \frac{n}{5} \rceil - 1$.
Similarly, we can prove the case $n \equiv 2$ (mod 5).

Now, we will consider the case $n \equiv 3$ (mod 5). We can obtain that $T = \{\bar{1}, \bar{2}, \bar{6}, \bar{7}, \overline{11}, \overline{12}, \ldots, \overline{n-2}, \overline{n-1}\}$ is a dominating set. Then $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{3}\})) \leq |T| = \frac{2n+4}{5} = 2\lceil \frac{n}{5} \rceil$. Assume in the contrary that there exists a dominating set X such that $|X| \leq 2\lceil \frac{n}{5} \rceil - 1 = \frac{2n-1}{5}$. Again by Lemma 3.17, we have $|\bigcup\limits_{x \in X} N[x]| \leq \frac{5|X|}{2} \leq \frac{2n-1}{2} < \frac{2n}{2} = n$. Whence X does not dominate $\mathbb{Z}_n$ which contradicts to the property of the dominating set X. So we can conclude that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{3}\})) = |T| = 2\lceil \frac{n}{5} \rceil$. For the cases $n \equiv 0, 4$ (mod 5), we can prove them similarly. □

Proposition 3.19. *Let $n \geq 4$ be a positive integer.*
Then $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{4}\})) \leq \begin{cases} 3\lceil \frac{n}{7} \rceil & \text{if } n \equiv 0, 6 \ (mod\ 7), \\ 3\lceil \frac{n}{7} \rceil - 1 & \text{if } n \equiv 4, 5 \ (mod\ 7), \\ 3\lceil \frac{n}{7} \rceil - 2 & \text{if } n \equiv 1, 2, 3 \ (mod\ 7). \end{cases}$

Proof. Let $n \geq 4$ be a positive integer.

For $n \equiv 0$ (mod 7), we obtain that X_0 is a dominating set where
$X_0 = \{\bar{1}, \bar{8}, \overline{15}, \overline{22}, \ldots, \overline{n-6}\} \cup \{\bar{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-4}\} \cup \{\bar{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-1}\}$ which implies that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{4}\})) \leq |X_0| = 3\lceil \frac{n}{7} \rceil$.

For $n \equiv 1$ (mod 7), we obtain that X_1 is a dominating set where
$X_1 = \{\bar{1}, \bar{8}, \overline{15}, \overline{22}, \ldots, \bar{n}\} \cup \{\bar{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-5}\} \cup \{\bar{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-2}\}$ which implies that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{4}\})) \leq |X_1| = 3\lceil \frac{n}{7} \rceil - 2$.

For $n \equiv 2$ (mod 7), we obtain that X_2 is a dominating set where
$X_2 = \{\bar{1}, \bar{8}, \overline{15}, \overline{22}, \ldots, \overline{n-1}\} \cup \{\bar{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-6}\} \cup \{\bar{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-3}\}$ which implies that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{4}\})) \leq |X_2| = 3\lceil \frac{n}{7} \rceil - 2$.

For $n \equiv 3$ (mod 7), we obtain that X_3 is a dominating set where
$X_3 = \{\bar{1}, \bar{8}, \overline{15}, \overline{22}, \ldots, \overline{n-2}\} \cup \{\bar{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-7}\} \cup \{\bar{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-4}\}$ which implies that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{4}\})) \leq |X_3| = 3\lceil \frac{n}{7} \rceil - 2$.

For $n \equiv 4$ (mod 7), we obtain that X_4 is a dominating set where
$X_4 = \{\bar{1}, \bar{8}, \overline{15}, \overline{22}, \ldots, \overline{n-3}\} \cup \{\bar{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-1}\} \cup \{\bar{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-5}\}$ which implies that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{4}\})) \leq |X_4| = 3\lceil \frac{n}{7} \rceil - 1$.

For $n \equiv 5$ (mod 7), we obtain that X_5 is a dominating set where
$X_5 = \{\bar{1}, \bar{8}, \overline{15}, \overline{22}, \ldots, \overline{n-4}\} \cup \{\bar{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-2}\} \cup \{\bar{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-6}\}$ which implies that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{4}\})) \leq |X_5| = 3\lceil \frac{n}{7} \rceil - 1$.

For $n \equiv 6$ (mod 7), we obtain that X_6 is a dominating set where

$X_6 = \{\bar{1}, \bar{8}, \overline{15}, \overline{22}, \ldots, \overline{n-5}\} \cup \{\bar{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-3}\} \cup \{\bar{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \bar{n}\}$ which implies that $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{4}\})) \leq |X_6| = 3\lceil \frac{n}{7} \rceil$. □

Proposition 3.20. *Let $n \geq 5$ be a positive integer. Then $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{5}\})) \leq \lceil \frac{n}{3} \rceil + 1$.*

Proof. Let $n \geq 5$ be a positive integer.

For $n \equiv 0$ $(mod\ 3)$, we obtain that $X_0 = \{\bar{1}, \bar{2}, \bar{4}, \bar{7}, \overline{10}, \overline{13}, \ldots, \overline{p-2}\}$ is a dominating set which leads to $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{5}\})) \leq |X_0| = \lceil \frac{n}{3} \rceil + 1$.

For $n \equiv 1$ $(mod\ 3)$, we obtain that $X_1 = \{\bar{1}, \bar{2}, \bar{4}, \bar{7}, \overline{10}, \overline{13}, \ldots, \bar{p}\}$ is a dominating set which leads to $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{5}\})) \leq |X_1| = \lceil \frac{n}{3} \rceil + 1$.

For $n \equiv 2$ $(mod\ 3)$, we obtain that $X_2 = \{\bar{1}, \bar{2}, \bar{4}, \bar{7}, \overline{10}, \overline{13}, \ldots, \overline{p-1}\}$ is a dominating set which leads to $\gamma(Cay(\mathbb{Z}_n, \{\bar{1}, \bar{5}\})) \leq |X_2| = \lceil \frac{n}{3} \rceil + 1$. □

Since a Cayley digraph of a left group can be considered as the disjoint union of Cayley digraphs of groups as shown in Lemma 2.3, we can directly obtain some results for the domination number of Cayley digraphs of left groups as follows.

Theorem 3.21. *Let $n \geq 2$ be a positive integer. If $p_1(A) = \{\bar{1}, \bar{2}\}$, then $\gamma(Cay(\mathbb{Z}_n \times L, A)) = |L|\lceil \frac{n}{3} \rceil$.*

Theorem 3.22. *Let $n \geq 3$ be a positive integer. If $p_1(A) = \{\bar{1}, \bar{3}\}$, then*
$$\gamma(Cay(\mathbb{Z}_n \times L, A)) = \begin{cases} |L|(2\lceil \frac{n}{5} \rceil - 1) & \text{if } n \equiv 1, 2\ (mod\ 5), \\ 2|L|\lceil \frac{n}{5} \rceil & \text{if } n \equiv 0, 3, 4\ (mod\ 5). \end{cases}$$

Theorem 3.23. *Let $n \geq 4$ be a positive integer. If $p_1(A) = \{\bar{1}, \bar{4}\}$, then*
$$\gamma(Cay(\mathbb{Z}_n \times L, A)) \leq \begin{cases} 3|L|\lceil \frac{n}{7} \rceil & \text{if } n \equiv 0, 6\ (mod\ 7), \\ |L|(3\lceil \frac{n}{7} \rceil - 1) & \text{if } n \equiv 4, 5\ (mod\ 7), \\ |L|(3\lceil \frac{n}{7} \rceil - 2) & \text{if } n \equiv 1, 2, 3\ (mod\ 7). \end{cases}$$

Theorem 3.24. *Let $n \geq 5$ be a positive integer. If $p_1(A) = \{\bar{1}, \bar{5}\}$, then $\gamma(Cay(\mathbb{Z}_n \times L, A)) \leq |L|(\lceil \frac{n}{3} \rceil + 1)$.*

Next, we give some results of the total domination number in Cayley digraphs of left groups with connection sets. We start with the lemma which gives the condition for the existence of the total dominating set in Cayley digraphs of left groups.

Lemma 3.25. *Let $S = G \times L_m$ be a left group and A a nonempty subset of S. Then the total dominating set of D exists if and only if $A \neq \emptyset$.*

Proof. Suppose that the total dominating set of D exists, say T. By the definition of T, we obtain that for each $(g, l) \in S$, (g, l) is dominated by (g_1, l_1) for some $(g_1, l_1) \in T$, i.e., $((g_1, l_1), (g, l)) \in E(D)$. Then $(g, l) = (g_1, l_1)(a_1, l_2)$ where $(a_1, l_2) \in A$ which implies that $A \neq \emptyset$.
Conversely, assume that the connection set $A \neq \emptyset$, i.e., there exists $(a, l) \in A$. Hence for each $(g_1, l_1) \in S$, we have $(g_1, l_1) = (g_1 a^{-1}, l_1)(a, l)$ where $(g_1 a^{-1}, l_1) \in S$. Thus (g_1, l_1) is dominated by $(g_1 a^{-1}, l_1)$ in S. If we take $T = S$, then we can conclude that T is a total dominating set of D, i.e., the total dominating set of D always exists when $A \neq \emptyset$. □

The following result gives us the total domination number of a Cayley digraph of a left group $G \times L_m$ with a connection set A in the term of a total domination number of a Cayley digraph of the subgroup $\langle p_1(A)\rangle$ of G.

Theorem 3.26. *Let $S = G\times L_m$ be a left group, A a nonempty subset of S, and $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \dots, g_k\langle p_1(A)\rangle\}$. Then $\gamma_t((Cay(S, A)) = m\cdot k\cdot \gamma_t(Cay(\langle p_1(A)\rangle, p_1(A)))$.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.12. $\square$

Proposition 3.27. *Let $n \geq 3$ be an odd integer and $c = \frac{n-1}{2}$. Let $S = \mathbb{Z}_n \times L_m$ be a left group and A a nonempty subset of S such that $p_1(A) = \{c, n-c, c-1, n-(c-1), \dots, c-(k-1), n-(c-(k-1))\}$ where $1 \leq k \leq c$. Then $\gamma_t(D) = m\lceil \frac{n}{2k}\rceil$.*

Proof. The result of this proposition follows from Lemma 2.3 and Lemma 2.5, directly. $\square$

Proposition 3.28. *Let $n \geq 3$ be an even integer and $c = \lfloor\frac{n-1}{2}\rfloor$. Let $S = \mathbb{Z}_n \times L_m$ be a left group and A a nonempty subset of S such that $p_1(A) = \{\frac{n}{2}, c, n-c, c-1, n-(c-1), \dots, c-(k-1), n-(c-(k-1))\}$ where $1 \leq k \leq c$. Then $\gamma_t(D) = m\lceil\frac{n}{2k+1}\rceil$.*

Proof. This proposition follows from Lemma 2.3 and Lemma 2.6, directly. $\square$

Before we give the next lemmas, we will define some notations which are used in the proof. Let $I = [a, b]$ be an interval of consecutive integers x such that $a \leq x \leq b$. Furthermore, let (V, E) be a digraph and for each $u \in V$, let $N(u) = \{v \in V | (u, v) \in E\}$ be the set of all neighbours of a vertex u and $N(A) = \bigcup\limits_{a\in A} N(a)$ where A is a subset of V.

Lemma 3.29. *Let $n \geq 3$ be an odd integer. Let $m = \frac{n-1}{2}$ and k be a fixed number such that $1 \leq k \leq m$.*
If $A = \{m, m-1, m-2, \dots, m-(k-1)\} \subseteq \mathbb{Z}_n$, then $\gamma_t(Cay(\mathbb{Z}_n, A)) = \lceil\frac{n}{k}\rceil$.

Proof. Assume that $A = \{m, m-1, m-2, \dots, m-(k-1)\}$ and let $l = \lceil\frac{n}{k}\rceil$. Since every vertex in D has an out-degree k, from the definition of the total domination number, it follows that $\gamma_t(Cay(\mathbb{Z}_n, A)) \geq l$. Let $x = m+k+1$ and $X_t = \{x, x+k, x+2k, \dots, x+(l-1)k\}$. Note that $|X_t| = l$. Since $l = \lceil\frac{n}{k}\rceil$, we get that $n = (l-1)k+j$ for some $j \in \mathbb{N}$ with $1 \leq j \leq k$. Thus $V(Cay(\mathbb{Z}_n, A))$ can be partitioned into l intervals as follows:

$$I_1 = [1, k],\ I_2 = [k+1, 2k],\ I_3 = [2k+1, 3k], \dots, I_{l-1} = [(l-2)k+1, (l-1)k], \text{ and } I_l = [(l-1)k+1, n].$$

Note that $|I_i| = k$ for all i with $1 \leq i \leq l-1$ and $1 \leq |I_l| \leq k$. For any $0 \leq i \leq l-2$, we have $x+ik \in X_t$ and $I_{i+1} = [ik+1, (i+1)k]$. Since $(x+ik)+(m-(k-1)) \equiv ik+1\ (mod\ n)$ and A is a set of k consecutive integers with the least element $m-(k-1)$, we have $N(x+ik) = I_{i+1}$. Therefore,

$(x+(l-1)k)+m-(k-1) \equiv (l-1)k+1\ (mod\ n)$ and so $I_l \subseteq N(x+(l-1)k)$.

Consequently,

$$\begin{aligned} V(Cay(\mathbb{Z}_n, A)) &= I_1 \cup I_2 \cup \ldots \cup I_{l-1} \cup I_l \\ &\subseteq N(x) \cup N(x+k) \cup \ldots \cup N(x+(l-2)k) \cup N(x+(l-1)k) \\ &= \bigcup_{y \in X_t} N(y) \\ &= N(X_t). \end{aligned}$$

Thus X_t is a total dominating set of $Cay(\mathbb{Z}_n, A)$. Hence $\gamma_t(Cay(\mathbb{Z}_n, A)) \leq |X_t| = l$ and then $\gamma_t(Cay(\mathbb{Z}_n, A)) = l = \lceil \frac{n}{k} \rceil$. □

Lemma 3.30. *Let $n \geq 3$ be an even integer. Let $m = \lfloor \frac{n-1}{2} \rfloor$ and k be a fixed number such that $1 \leq k \leq m$.*
If $A = \{\frac{n}{2}, m, m-1, \ldots, m-(k-1)\} \subseteq \mathbb{Z}_n$, then $\gamma_t(Cay(\mathbb{Z}_n, A)) = \lceil \frac{n}{k+1} \rceil$.

Proof. Suppose that $A = \{\frac{n}{2}, m, m-1, \ldots, m-(k-1)\}$. Then $|A| = k+1$ and let $l = \lceil \frac{n}{k+1} \rceil$. Since every vertex of $Cay(\mathbb{Z}_n, A)$ has an out-degree $k+1$, we also have $\gamma_t(Cay(\mathbb{Z}_n, A)) \geq l$. Let $x = m+k+2$ and $X_t = \{x, x+(k+1), x+2(k+1), \ldots, x+(l-1)(k+1)\}$. By partitioning the set of all vertices in $Cay(\mathbb{Z}_n, A)$ into l intervals as follows :

$$I_1 = [1, k+1], I_2 = [(k+1)+1, 2(k+1)], \ldots, I_{l-1} = [(l-2)(k+1)+1, (l-1)(k+1)], \text{ and } I_l = [(l-1)(k+1)+1, n],$$

we can prove the remaining part of this lemma by applying the proof of the previous lemma, similarly. We also have $\gamma_t(Cay(\mathbb{Z}_n, A)) \leq |X_t| = l$. Thus $\gamma_t(Cay(\mathbb{Z}_n, A)) = l = \lceil \frac{n}{k+1} \rceil$. □

Now, we apply the above two lemmas to obtain the results for the total domination number of Cayley digraphs of a left group $\mathbb{Z}_n \times L_m$ with an according connection set.

Theorem 3.31. *Let $n \geq 3$ be an odd integer. Let $c = \frac{n-1}{2}$ and k be a fixed number such that $1 \leq k \leq c$. Let $S = \mathbb{Z}_n \times L_m$ be a left group and A a nonempty subset of S. If $p_1(A) = \{c, c-1, c-2, \ldots, c-(k-1)\}$, then $\gamma_t(D) = m\lceil \frac{n}{k} \rceil$.*

Proof. This theorem is a direct result from Lemma 2.3 and Lemma 3.29. □

Theorem 3.32. *Let $n \geq 3$ be an even integer. Let $c = \lfloor \frac{n-1}{2} \rfloor$ and k be a fixed number such that $1 \leq k \leq c$. Let $S = \mathbb{Z}_n \times L_m$ be a left group and A a nonempty subset of S. If $p_1(A) = \{\frac{n}{2}, c, c-1, \ldots, c-(k-1)\}$, then $\gamma_t(D) = m\lceil \frac{n}{k+1} \rceil$.*

Proof. This theorem follows from Lemma 2.3 and Lemma 3.30. □

4 Conclusion

In this paper, we give the backgrounds of the research and some preliminaries together with the notations in section 1 and section 2, respectively. In the third section, some results of the domination number and total domination number of Cayley digraphs of right groups and left groups with appropriate connection sets are obtained. In addition, we have the conditions for the existence of the total dominating sets of Cayley digraphs of left groups and right groups. Moreover, the sharpness of bounds for domination parameters in Cayley digraphs of left groups and right groups are also shown.

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