รายงานวิจัยฉบับสมบูรณ์

โครงการ ฟังก์ชันซึ่งรักษาระยะทางชนิดต่างๆ

โดย นายประพันธ์พงศ์ พงศ์ศรีเอี่ยม

มิถุนายน 2558

สัญญาเลขที่ TRG5680052

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ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากร

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัย

และมหาวิทยาลัยศิลปากร

(ความเห็นในรายงานนี้เป็นของผู้วิจัย สกว. และมหาวิทยาลัยศิลปากรไม่จำเป็นต้องเห็นด้วยเสมอไป)

กิตติกรรมประกาศ

กระผมขอขอบคุณสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยศิลปากร ที่ให้ทุนวิจัยแก่กระผมตามสัญญาเลขที่ TRG5680052 ขอบคุณ รศ.ดร.อิ่มจิตต์ เติมวุฒิพงษ์ ที่ให้คำแนะนำและช่วยหาข้อมูลการทำวิจัยซึ่งช่วยให้กระผมมีหัวข้อวิจัยเพิ่มขึ้น ขอบคุณ นางสาวธรรมธาดา เขมะรัชตกำธร ที่ช่วยพิมพ์เอกสารงานวิจัยให้อย่างรวดเร็วและเรียบร้อยดีซึ่งช่วยให้การทำงานวิจัยของกระผมเป็นไปได้โดยสะดวกและมีประสิทธิภาพ ขอบคุณ รศ.ดร.นวรัตน์ อนันต์ชื่น ที่ให้ห้องทำงานใหม่ซึ่งมีส่วนช่วยให้กระผมทำงานวิจัยในระยะยาวได้ด้วยความสะดวกสบาย

ประพันธ์พงศ์ พงศ์ศรีเอี่ยม

บทคัดย่อ

รหัสโครงการ: TGR5680052

ชื่อโครงการ: ฟังก์ชันซึ่งรักษาระยะทางชนิดต่าง ๆ

ชื่อนักวิจัย: นายประพันธ์พงศ์ พงศ์ศรีเอี่ยม

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยศิลปากร

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ระยะเวลาโครงการ: 2 ปี (3 มิถุนายน 2556 – 2 มิถุนายน 2558)

ระยะทาง d บนเซต X เป็นระยะทางอัลตรา ถ้า d สอดคล้องกับอสมการ $d(x,y) \le \max\{d(x,z), d(z,y)\}$ สำหรับทุก $x, y, z \in X$ ฟังก์ชัน $f:[0,\infty)\to[0,\infty)$ เป็นฟังก์ชันรักษาระยะทาง ถ้า $f\circ d$ เป็นระยะทางบน X สำหรับทุกปริภูมิอิงระยะทาง (X,d) ในโครงการนี้เราได้ศึกษาการเปลี่ยนแปลงของแนวคิดเรื่องฟังก์ชันรักษาระยะทางโดยการแทนระยะทางด้วยระยะทางอัลตราและเราได้ให้การประยุกต์บางประการของฟังก์ชันรักษาระยะทางในทฤษฎีจุดตรึง

คำหลัก: ระยะทาง, ระยะทางอัลตรา, ฟังก์ชันรักษาระยะทาง

Abstract

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Project Title: Functions which preserve various types of distances

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A metric d on a set X is said to be an ultrametric if d satisfies the inequality $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for all $x, y, z \in X$. A function $f:[0,\infty)\rightarrow [0,\infty)$ is said to be metric-preserving if $f \circ d$ is a metric on X for all metric spaces (X,d). In this project, we investigate a variation of the concept of metric-preserving functions where metrics are replaced by ultrametrics. We also show some applications of metric-preserving functions in fixed point theory.

Keywords: metric, ultrametric, metric-preserving function

1 Introduction

Under what conditions on a function $f : [0, \infty) \to [0, \infty)$ is it the case that for every metric space (X, d), $f \circ d$ is still a metric? It is well-known that for any metric d, $\frac{d}{1+d}$ and $\min\{1, d\}$ are bounded metrics topologically equivalent to d, while $\frac{d}{1+d^2}$ need not be a metric.

We call $f : [0, \infty) \to [0, \infty)$ *metric-preserving* if for all metric spaces (X, d), $f \circ d$ is a metric. Therefore the functions f and g given by $f(x) = \frac{x}{1+x}$ and $g(x) = \min\{1, x\}$ are metric-preserving but $h(x) = \frac{x}{1+x^2}$ is not. The concept of metric-preserving functions first appears in Wilson's article [35] and is thoroughly investigated by many authors, see for example, [1, 2, 6, 7, 10, 11, 12, 13, 14, 15, 18, 22, 23, 24, 28, 33, 34] and references therein.

However, other important types of distances such as ultrametrics, pseudometrics, pseudodistances [36, 37], w-distances, and τ-distances have not yet been developed in the connection with metric-preserving functions. These distances have many applications in mathematics, see for example, applications of w-distances and τ-distances in [17, 19, 20, 29, 30, 31, 32]. We will particularly concern with the ultrametrics which arise naturally in the study of p-adic numbers and non-archimedean analysis [3, 8], topology and dynamical system [4, 16, 21, 38], topological algebra [5], and theoretical computer science [27].

In connection with ultrametrics and metric-preserving functions, the problem arises to investigate the properties of the following functions and compare them with those of metric-preserving functions.

Definition 1. *Let* $f : [0, \infty) \to [0, \infty)$*. We say that*

(i) f *is ultrametric-preserving if for all ultrametric spaces* (X, d)*,* $f \circ d$ *is an ultrametric,*

(ii) f *is metric-ultrametric-preserving if for all metric spaces* (X, d)*,* $f \circ d$ *is an ultrametric, and*

(iii) f *is ultrametric-metric-preserving if for all ultrametric spaces* (X, d)*,* $f \circ d$ *is a metric.*

In this project, we obtain characterizations of the functions defined in Definition 1. We also obtain some applications of metric-preserving functions in fixed point theory. Some of our results are given in the next section.

2 Main Results

In this section, the results we obtain in this project are given. We refer the reader to our publications [25, 26] for more details.

Theorem 2. *Let $f : [0,\infty) \to [0,\infty)$. Then f is ultrametric-preserving if and only if f is amenable and increasing.*

Proof. Assume that f is ultrametric-preserving. It suffices to show that f is increasing. Let $a, b \in [0,\infty)$ and $a < b$. Let d_2 be the Euclidean metric on $\mathbb{R}^2$ and let $X = \{A, B, C\} \subseteq \mathbb{R}^2$ where $A = \left(-\frac{a}{2}, 0\right)$, $B = \left(\frac{a}{2}, 0\right)$, and $C = \left(0, \sqrt{\frac{4b^2-a^2}{4}}\right)$. Let $d = d_2 \mid_X$ be the restriction of d_2 on X. Then $d(A,B) = a$, $d(A,C) = d(B,C) = b$. Therefore (X,d) is an ultrametric space. Since f is ultrametric-preserving, $f \circ d$ is an ultrametric. Therefore

$$f(a) = f \circ d(A,B) \leq \max\{f \circ d(A,C), f \circ d(B,C)\} = f(b),$$

as required. Next assume that f is increasing and amenable. Let (X,d) be an ultrametric space, and let $x, y, z \in X$. Since f is amenable, $f \circ d(x,y) = 0$ if and only if $x = y$. Since d is an ultrametric, $d(x,z) \leq \max\{d(x,y), d(y,z)\}$. So $d(x,z) \leq d(x,y)$ or $d(x,z) \leq d(y,z)$. If $d(x,z) \leq d(x,y)$, then $f(d(x,z)) \leq f(d(x,y)) \leq \max\{f \circ d(x,y), f \circ d(y,z)\}$. If $d(x,z) \leq d(y,z)$, then $f(d(x,z)) \leq f(d(y,z)) \leq \max\{f \circ d(x,y), f \circ d(y,z)\}$. In any case $f \circ d(x,z) \leq \max\{f \circ d(x,y), f \circ d(y,z)\}$. Therefore $f \circ d$ is an ultrametric. This completes the proof. □

The proof of the following results are omitted because we already published it in mathematics journal [25, 26].

Corollary 3. *Let $f : [0,\infty) \to [0,\infty)$. Then the following statements hold:*

(i) If f is ultrametric-preserving and subadditive, then f is metric-preserving.

(ii) If f is metric-preserving and increasing on $[0,\infty)$, then f is ultrametric-preserving.

Theorem 4. *Let $f : [0,\infty) \to [0,\infty)$. If f is amenable and concave, then f is ultrametric-preserving.*

Theorem 5. *Let $f : [0,\infty) \to [0,\infty)$. Then f is metric-ultrametric-preserving if and only if f is amenable and f is a constant on $(0,\infty)$.*

Theorem 6. *Let $f : [0, \infty) \to [0, \infty)$ be amenable. Then the following statements are equivalent:*

(i) f *is ultrametric-metric-preserving,*

(ii) for each $(a, b, c) \in \Delta_\infty$, $(f(a), f(b), f(c)) \in \Delta$,

(iii) for each $0 \leq a \leq b$, $f(a) \leq 2f(b)$.

Theorem 7. *Let $f : [0, \infty) \to [0, \infty)$ be metric-preserving. The the following statements are equivalent:*

1) f *is continuous at* $[0, \infty)$,

2) f *is continuous at* 0,

3) For every $\varepsilon > 0$, *there exists and* $x > 0$ *such that* $f(x) < \varepsilon$,

4) f *is strongly metric-preserving,*

5) f *is uniformly continuous on* $[0, \infty)$,

6) f *is weakly continuous on* $[0, \infty)$,

7) f *is weakly continuous at* 0,

8) f *is quasi continuous on* $[0, \infty)$,

9) f *is quasi continuous at* 0,

10) f *is a.c.S on* $[0, \infty)$,

11) f *is a.c.S at* 0,

12) f *is a.c.H on* $[0, \infty)$,

13) f *is a.c.H at* 0.

Theorem 8. *Let f be ultrametric-metric-preserving. Then the following statements are equivalent:*

(i) f *is continuous at* 0,

(ii) f *is weakly continuous at* 0,

(iii) *for every* $\varepsilon > 0$*, there exists an* $x > 0$ *such that* $f(x) < \varepsilon$*,*

(iv) f *is quasi continuous at* 0*,*

(v) f *is a.c.S. at* 0*,*

(vi) f *is a.c.H. at* 0*.*

Theorem 9. *Let* (X, d) *be a metric space and let* $g : X \to X$*. Assume that there exists* $k \in (0, 1)$ *and a metric-preserving function* f *satisfying the following conditions:*

(a) *for each* $x \in X$*, there exists* $\varepsilon > 0$ *such that for every* $u \in X$

$$d(x, u) < \varepsilon \quad \Rightarrow \quad (f \circ d)(g(x), g(u)) \leq kd(x, u), \quad \textit{and}$$

(b) $f'(0) > k$*.*

Then g *is a local radial contraction.*

Theorem 10. *Suppose, in addition to the assumptions in Theorem 9,* X *is complete and rectifiably pathwise connected. Then* g *has a unique fixed point* x_0*, and* $\lim_{n\to\infty} g^n(x) = x_0$ *for each* $x \in X$*.*

Comments

There are many other types of distances that have not been investigated in connection with metric-preserving functions. So there are a lot of open problems that we can work on. We refer the reader to the encyclopedia of distances collected by Michel Marie Deza and Elena Deza [9] for a lot more information. For example, we may consider the relation between w-distances and metric-preserving functions in a similar way to what we did with ultrametrics [25]. In addition, differentiability of the functions given in [25] has not been studied. We believe that properties of these functions should be explored and this may lead to interesting and important mathematical research topics in the future.

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Output

1 ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1.1 P. Pongsriiam, I. Termwuttipong, Remarks on Ultrametrics and Metric-Preserving Functions, Abstract and Applied Analysis, Article ID 163258, 2014, 9 pages

1.2 P. Pongsriiam, I. Termwuttipong, On Metric-Preserving Functions and Fixed Point Theorems, Fixed Point Theory and Applications 2014, 2014:179, 14 pages

2 การนำผลงานวิจัยไปใช้ประโยชน์

- ใช้ในการพัฒนาการเรียนการสอน โดยนำเนื้อหาบางส่วนใส่ไว้ในเอกสารประกอบการสอน
- ใช้แนะแนวทางการทำวิจัยและนำส่วนที่เป็นปัญหาปลายเปิดให้นักศึกษาเลือกทำเป็นหัวข้อสัมมนาหรือวิทยานิพนธ์ในอนาคตได้

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Research Article

Remarks on Ultrametrics and Metric-Preserving Functions

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Functions whose composition with every metric is a metric are said to be metric-preserving. In this paper, we investigate a variation of the concept of metric-preserving functions where metrics are replaced by ultrametrics.

1. Introduction

Under what conditions on a function $f : [0,\infty) \to [0,\infty)$ is it the case that for every metric space (X, d), $f \circ d$ is still a metric? It is well known that for any metric d, $d/(1 + d)$ and $\min\{1, d\}$ are bounded metrics topologically equivalent to d, while $d/(1 + d^2)$ need not be a metric.

We call $f : [0,\infty) \to [0,\infty)$ *metric-preserving* if for all metric spaces (X, d), $f \circ d$ is a metric. Therefore, the functions f and g given by $f(x) = x/(1 + x)$ and $g(x) = \min\{1, x\}$ are metric-preserving but $h(x) = x/(1+x^2)$ is not. The concept of metric-preserving functions first appears in Wilson's article [1] and is thoroughly investigated by many authors; see for example, [2–18] and references therein.

However, other important types of distances such as ultrametrics, pseudometrics, pseudodistances [19, 20], w-distances, and τ-distances have not yet been developed in the connection with metric-preserving functions. These distances have many applications in mathematics; see, for example, applications of w-distances and τ-distances in [21–27]. We will particularly be concerned with the ultrametrics which arise naturally in the study of p-adic numbers and nonarchimedean analysis [28, 29], topology and dynamical system [30–33], topological algebra [34], and theoretical computer science [35].

In connection with ultrametrics and metric-preserving functions, the problem arises to investigate the properties of the following functions and compare them with those of metric-preserving functions.

Definition 1. Let $f : [0,\infty) \to [0,\infty)$. We say that

(i) f is ultrametric-preserving if for all ultrametric spaces (X, d), $f \circ d$ is an ultrametric;

(ii) f is metric-ultrametric-preserving if for all metric spaces (X, d), $f \circ d$ is an ultrametric;

(iii) f is ultrametric-metric-preserving if for all ultrametric spaces (X, d), $f \circ d$ is a metric.

For convenience, we also let $\mathscr{M}$ be the set of all metric-preserving functions, $\mathscr{U}$ the set of all ultrametric-preserving functions, $\mathscr{U}\mathscr{M}$ the set of all ultrametric-metric-preserving functions, and $\mathscr{M}\mathscr{U}$ the set of all metric-ultrametric-preserving functions.

We will give some basic definitions and useful results that will be used throughout this paper in the next section. We then give properties and characterizations of those functions in Sections 3, 4, and 5. We discuss and give some results on the continuity aspect of those functions in Section 6.

2. Preliminaries and Lemmas

In this section, we give some basic definitions and results for the convenience of the reader. First, we recall the definition of a metric space and an ultrametric space.

A *metric space* is a set X together with a function $d : X \times X \to [0,\infty)$ satisfying the following three conditions:

(M1) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$, (M2) for all $x, y \in X$, $d(x, y) = d(y, x)$, and

(M3) for all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

An *ultrametric space* is a metric space (X, d) satisfying the stronger inequality (called the ultrametric inequality):

(U3) for all $x, y, z \in X$, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

A metric space (X, d) is said to be *topologically discrete* if for every $x \in X$ there is an $\varepsilon > 0$ such that $B_d(x, \varepsilon) = \{x\}$, where $B_d(x, \varepsilon)$ denote the open ball center at x and radius ε. In addition, (X, d) is said to be *uniformly discrete* if there exists an $\varepsilon > 0$ such that $B_d(x, \varepsilon) = \{x\}$ for every $x \in X$.

Next we recall the definitions concerning certain behaviors of functions. Throughout, we let $f : [0, \infty) \to [0, \infty)$ and let $I \subseteq [0, \infty)$. Then f is said to be *increasing* on $I \subseteq [0, \infty)$ if $f(x) \leq f(y)$ for all $x, y \in I$ satisfying $x < y$, and f is said to be *strictly increasing* on $I \subseteq [0, \infty)$ if $f(x) < f(y)$ for all $x, y \in I$ satisfying $x < y$. The notion of *decreasing* or *strictly decreasing* functions is defined similarly.

The function f is said to be *amenable* if $f^{-1}(\{0\}) = \{0\}$, and f is said to be *tightly bounded* on $(0, \infty)$ if there is $v > 0$ such that $f(x) \in [v, 2v]$ for all $x > 0$. We say that f is *subadditive* if $f(a + b) \leq f(a) + f(b)$ for all $a, b \in [0, \infty)$, f is *convex* if $f((1 - t)x_1 + tx_2) \leq (1 - t)f(x_1) + tf(x_2)$ for all $x_1, x_2 \in [0, \infty)$ and $t \in [0, 1]$, and f is *concave* if $f((1-t)x_1 + tx_2) \geq (1 - t)f(x_1) + tf(x_2)$ for all $x_1, x_2 \in [0, \infty)$ and $t \in [0, 1]$. As mentioned earlier, we say that f is *metric-preserving* if for all metric spaces (X, d), $f \circ d$ is a metric. Furthermore, f is *strongly metric-preserving* if $f \circ d$ is a metric equivalent to d for every metric d.

Now we are ready to state the results which will be applied in the proof of our theorems.

Lemma 2. *Let $f : [0, \infty) \to [0, \infty)$. If f is amenable, subadditive, and increasing on $[0, \infty)$, then f is metric-preserving.*

Proof. The proof can be found, for example, in [4, 6]. □

Lemma 3. *If $f : [0, \infty) \to [0, \infty)$ is amenable and tightly bounded, then f is metric-preserving.*

Proof. The proof can be found, for example, in [3, 4]. □

The next lemma might be less well known, so we give a proof here for completeness.

Lemma 4. *If $f : [0, \infty) \to [0, \infty)$ is amenable and concave, then the function $x \mapsto f(x)/x$ is decreasing on $(0, \infty)$.*

Proof. Let $a, b \in (0, \infty)$ and $a < b$. Since f is concave, we obtain

$$\begin{aligned} f(a) &= f\left(\left(1 - \frac{a}{b}\right)(0) + \left(\frac{a}{b}\right)(b)\right) \\ &\geq \left(1 - \frac{a}{b}\right) f(0) + \frac{a}{b} f(b) \\ &= \frac{a}{b} f(b). \end{aligned} \tag{1}$$

Therefore, $f(a)/a \geq f(b)/b$, as desired. □

Lemma 5. *Let (X, d) be an ultrametric space. Then for every $x_1, x_2, \ldots, x_n \in X$,*

$$d(x_1, x_n) \leq \max\{d(x_1, x_2), d(x_2, x_3), \ldots, d(x_{n-1}, x_n)\}. \tag{2}$$

Proof. We have

$$\begin{aligned} d(x_1, x_n) &\leq \max\{d(x_1, x_2), d(x_2, x_n)\} \\ &\leq \max\{d(x_1, x_2), \max\{d(x_2, x_3), d(x_3, x_n)\}\} \\ &= \max\{d(x_1, x_2), d(x_2, x_3), d(x_3, x_n)\}. \end{aligned} \tag{3}$$

A repeated application of the ultrametric inequality as above gives the desired result. □

Next we give basic relations and properties of the functions in $\mathscr{M}$, $\mathscr{U}$, $\mathscr{MU}$, and $\mathscr{UM}$.

Proposition 6. *The following relations hold $\mathscr{MU} \overset{(S1)}{\subseteq} \mathscr{U} \cap \mathscr{M} \overset{(S2)}{\subseteq} \mathscr{U}, \mathscr{M} \overset{(S3)}{\subseteq} \mathscr{U} \cup \mathscr{M} \overset{(S4)}{\subseteq} \mathscr{UM}$.*

Proof. Since an ultrametric is a metric, $\mathscr{MU} \subseteq \mathscr{U}$ and $\mathscr{MU} \subseteq \mathscr{M}$. So $(S1)$ follows. Similarly, $\mathscr{U} \subseteq \mathscr{UM}$ and $\mathscr{M} \subseteq \mathscr{UM}$, so $(S4)$ holds. $(S2)$ and $(S3)$ are true in general. □

We will obtain characterization of the functions in $\mathscr{U}$, $\mathscr{MU}$, and $\mathscr{UM}$ in later section. Then we will show that the relation $\subseteq$ in Proposition 6 is in fact a proper subset. It is easy to see that if $f \in \mathscr{M}$, then f is amenable. We extend this to the case of any function $f \in \mathscr{M} \cup \mathscr{MU} \cup \mathscr{UM} \cup \mathscr{U}$.

Proposition 7. *If $f \in \mathscr{UM}$, then f is amenable.*

Proof. Assume that $f \in \mathscr{UM}$. To show that f is amenable, we let $x \in [0, \infty)$ be such that $f(x) = 0$. Let $X = \{A, B, C\} \subseteq \mathbb{R}^2$, where $A = (-x/2, 0)$, $B = (x/2, 0)$, and $C = (0, \sqrt{3}x/2)$. Let d_2 be the Euclidean metric on $\mathbb{R}^2$ and let $d = d_2|_X$ be the restriction of d_2 on X. Then $d(A, B) = d(A, C) = d(B, C) = x$. Therefore, (X, d) is an ultrametric space. So $f \circ d$ is a metric on X. Now $f(0) = f(d(A, A)) = (f \circ d)(A, A) = 0$, and $(f \circ d)(A, B) = f(d(A, B)) = f(x) = 0$, which implies $A = B$. That is, $(-x/2, 0) = (x/2, 0)$. Hence $x = 0$. This shows that f is amenable as desired. □

Corollary 8. *If $f : [0, \infty) \to [0, \infty)$ is in $\mathscr{M}$, $\mathscr{MU}$, $\mathscr{U}$, or $\mathscr{UM}$, then f is amenable.*

Proof. By Proposition 6, $\mathscr{M} \cup \mathscr{MU} \cup \mathscr{U} \cup \mathscr{UM} = \mathscr{UM}$. So the result follows from Proposition 7. □

3. Ultrametric-Preserving Functions

In this section, we obtain characterizations of ultrametric-preserving functions. Then we compare their properties with those of metric-preserving functions.

Theorem 9. *Let $f : [0, \infty) \to [0, \infty)$. Then f is ultrametric-preserving if and only if f is amenable and increasing.*

Proof. Assume that f is ultrametric-preserving. By Corollary 8, it suffices to show that f is increasing. Let $a, b \in [0, \infty)$ and $a < b$. Let d_2 be the Euclidean metric on $\mathbb{R}^2$ and let $X = \{A, B, C\} \subseteq \mathbb{R}^2$, where $A = (-a/2, 0)$, $B = (a/2, 0)$, and $C = (0, \sqrt{(4b^2 - a^2)/4})$. Let $d = d_2|_X$ be the restriction of d_2 on X. Then $d(A, B) = a$, $d(A, C) = d(B, C) = b$. Therefore, (X, d) is an ultrametric space. Since f is ultrametric-preserving, $f \circ d$ is an ultrametric. Therefore,

$$\begin{aligned} f(a) &= f \circ d(A, B) \\ &\leq \max\{f \circ d(A, C), f \circ d(B, C)\} \\ &= f(b), \end{aligned} \tag{4}$$

as required. Next assume that f is increasing and amenable. Let (X, d) be an ultrametric space, and let $x, y, z \in X$. Since f is amenable, $f \circ d(x, y) = 0$ if and only if $x = y$. Since d is an ultrametric, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. So $d(x, z) \leq d(x, y)$ or $d(x, z) \leq d(y, z)$. If $d(x, z) \leq d(x, y)$, then $f(d(x, z)) \leq f(d(x, y)) \leq \max\{f \circ d(x, y), f \circ d(y, z)\}$. If $d(x, z) \leq d(y, z)$, then $f(d(x, z)) \leq f(d(y, z)) \leq \max\{f \circ d(x, y), f \circ d(y, z)\}$. In any case $f \circ d(x, z) \leq \max\{f \circ d(x, y), f \circ d(y, z)\}$. Therefore, $f \circ d$ is an ultrametric. This completes the proof. □

Corollary 10. *Let $f : [0, \infty) \to [0, \infty)$. Then the following statements hold:*

(i) *if f is ultrametric-preserving and subadditive, then f is metric-preserving;*

(ii) *if f is metric-preserving and increasing on $[0, \infty)$, then f is ultrametric-preserving.*

Proof. We obtain that (i) follows from Theorem 9 and Lemma 2, and (ii) follows from Corollary 8 and Theorem 9. □

The next example shows that $\mathcal{M} \nsubseteq \mathcal{U}$ and $\mathcal{U} \nsubseteq \mathcal{M}$.

Example 11. Let $f, g : [0, \infty) \to [0, \infty)$ be given by

$$f(x) = x^2, \qquad g(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x \in \mathbb{Q} - \{0\}; \\ 2, & \text{if } x \in \mathbb{Q}^c. \end{cases} \tag{5}$$

By Theorem 9, f is ultrametric-preserving and g is not ultrametric-preserving. If d is the usual metric on $\mathbb{R}$, we see that

$$f \circ d(1, 3) = f(2) = 4 > 2 = f \circ d(1, 2) + f \circ d(2, 3). \tag{6}$$

So $f \circ d$ is not a metric and therefore f is not metric-preserving. Since $g(x) \in [1, 2]$ for all $x > 0$, g is tightly bounded, and therefore, by Lemma 3, g is metric-preserving. In conclusion, $f \in \mathcal{U}$, $f \notin \mathcal{M}$, $g \in \mathcal{M}$, and $g \notin \mathcal{U}$. This shows that $\mathcal{U} \nsubseteq \mathcal{M}$ and $\mathcal{M} \nsubseteq \mathcal{U}$. This example also shows that the relations $(S2)$ and $(S3)$ in Proposition 6 are proper subsets.

Next we give some results concerning concavity of the functions in $\mathcal{U} \cup \mathcal{M}$.

Theorem 12. *Let $f : [0, \infty) \to [0, \infty)$. If f is amenable and concave, then f is ultrametric-preserving.*

Proof. Assume that f is amenable and concave. We will show that f is increasing. First observe that if $y > 0$, then $f(y) > f(0)$ because f is amenable. Next let $y > x > 0$ and suppose for a contradiction that $f(y) < f(x)$. Let $t = f(y)/f(x)$, $x_1 = (yf(x) - xf(y))/(f(x) - f(y))$, and $x_2 = x$. Then $t \in (0, 1)$, and $x_1, x_2 \in (0, \infty)$. Since f is concave, we obtain

$$\begin{aligned} f(y) &= f((1 - t) x_1 + t x_2) \\ &\geq (1 - t) f(x_1) + t f(x_2) \\ &= (1 - t) f(x_1) + f(y). \end{aligned} \tag{7}$$

This implies that $f(x_1) = 0$ which contradicts the fact that $x_1 > 0$ and f is amenable. Hence f is increasing on $[0, \infty)$. By Theorem 9, f is ultrametric-preserving. □

Corollary 13. *If $f : [0, \infty) \to [0, \infty)$ is amenable and concave, then f is both ultrametric-preserving and metric-preserving.*

Proof. The first part comes from Theorem 12. The other part has appeared in the literature but we will give an alternative proof here. We know that f is increasing by Theorems 12 and 9. So by Lemma 2, it suffices to show that f is subadditive. Let $a, b \in (0, \infty)$. By Lemma 4, we have $f(a + b)/(a + b) \leq \min\{f(a)/a, f(b)/b\}$. Therefore,

$$\begin{aligned} f(a + b) &= a\left(\frac{f(a + b)}{a + b}\right) + b\left(\frac{f(a + b)}{a + b}\right) \\ &\leq a\frac{f(a)}{a} + b\frac{f(b)}{b} = f(a) + f(b), \end{aligned} \tag{8}$$

as required. This completes the proof. □

The next example shows that there exists a function which is both metric-preserving and ultrametric-preserving but not concave.

Example 14. Let $f : [0, \infty) \to [0, \infty)$ be defined by

$$f(x) = \begin{cases} x, & x \in [0, 1]; \\ 1, & x \in [1, 10]; \\ x - 9, & x \in (10, 11); \\ 2, & x \geq 11. \end{cases} \tag{9}$$

It is easy to see that f is amenable and increasing. So, by Theorem 9, f is ultrametric-preserving. Next we will show that f is metric-preserving. By Lemma 2, it suffices to show that f is subadditive. Observe that $f(x) \leq x$ and $f(x) \leq 2$ for every $x \in [0, \infty)$. We consider $a, b \in [0, \infty)$ in several cases.

If $a, b \in [0, 1]$, then $f(a) + f(b) = a + b \geq f(a + b)$.

If $a, b \in [1, 10]$, then $f(a) + f(b) = 2 \geq f(a + b)$.

Similarly, if $a, b > 10$, then $f(a) + f(b) > 2 \geq f(a + b)$.

If $a \in [0,1]$, $b \in [1,10]$, then

$$f(a) + f(b) = a + 1 \geq \max\{1, a+b-9\} \geq f(a+b). \tag{10}$$

If $a \in [0,1]$, $b \in [10,11]$, then $f(a)+f(b) = a+b-9 \geq f(a+b)$.

If $a \in [0,1]$, $b \in [11,\infty)$, then $f(a) + f(b) = a + 2 \geq 2 = f(a+b)$.

If $a \in [1,10]$, $b \in [10,\infty)$, then $f(a) + f(b) = b - 8 \geq 2 = f(a+b)$.

The other cases can be obtained similarly. Therefore, f is subadditive. Hence, f is metric-preserving. But $f((9+11)/2) < (f(9) + f(11))/2$, so f is not concave. That is, $f \in \mathscr{U} \cap \mathscr{M}$ but f is not concave. In addition, f is not a constant on $(0,\infty)$. So this example also shows that $\mathscr{U} \cap \mathscr{M} \nsubseteq \mathscr{M}\mathscr{U}$ and the relation $(S1)$ in Proposition 6 is a proper subset.

4. Metric-Ultrametric-Preserving Functions

In this section, we characterize the functions in $\mathscr{M}\mathscr{U}$. We will see that this notion is so strong that it forces the functions to be a constant on $(0,\infty)$. More precisely, we obtain the following theorem.

Theorem 15. *Let $f : [0,\infty) \to [0,\infty)$. Then f is metric-ultrametric-preserving if and only if f is amenable and f is a constant on $(0,\infty)$.*

Proof. First assume that f is amenable and is a constant on $(0,\infty)$. That is there exists a constant $c > 0$ such that

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ c, & \text{if } x > 0. \end{cases} \tag{11}$$

To show that f is metric-ultrametric-preserving, let (X, d) be a metric space and let $x, y, z \in X$. If $x = y$ or $x = z$ or $y = z$, then it is easy to see that $f \circ d(x,y) \leq \max\{f \circ d(x,z), f \circ d(z,y)\}$. If x, y, z are all distinct, then $f \circ d(x,y) = c = f \circ d(x,z) = f \circ d(y,z)$ and therefore

$$f \circ d(x,y) \leq \max\{f \circ d(x,z), f \circ d(z,y)\}. \tag{12}$$

This shows that $f \circ d$ is an ultrametric. In the other direction, we assume that $f \in \mathscr{M}\mathscr{U}$. By Corollary 8, it is enough to show that f is a constant on $(0,\infty)$. Throughout the proof, we let d be the usual metric on $\mathbb{R}$ and d_2 the Euclidean metric on $\mathbb{R}^2$. We will apply Lemma 5 repeatedly. First we will show that

$$f(1) = f\left(\frac{1}{n}\right) = f\left(\frac{m}{n}\right) \quad \text{for every } m, n \in \mathbb{N}. \tag{13}$$

So we let $m, n \in \mathbb{N}$ be arbitrary. Since $f \in \mathscr{M}\mathscr{U}$, $f \circ d$ is an ultrametric on $\mathbb{R}$. By Lemma 5, we have

$$\begin{aligned} f(1) &= f \circ d(0,1) \\ &\leq \max\left\{f \circ d\left(0, \frac{1}{n}\right), f \circ d\left(\frac{1}{n}, \frac{2}{n}\right), \ldots, \right. \\ &\qquad \left. f \circ d\left(\frac{n-1}{n}, 1\right)\right\} \\ &= \max\left\{f\left(\frac{1}{n}\right), f\left(\frac{1}{n}\right), \ldots, f\left(\frac{1}{n}\right)\right\} = f\left(\frac{1}{n}\right). \end{aligned} \tag{14}$$

Next let $A = (-1/2n, 0)$, $B = (1/2n, 0)$, $C = (0, \sqrt{(4-(1/n)^2)/4})$ be points in $\mathbb{R}^2$. Since $f \in \mathscr{M}\mathscr{U}$, $f \circ d_2$ is an ultrametric on $\mathbb{R}^2$. Therefore,

$$\begin{aligned} f\left(\frac{1}{n}\right) &= f \circ d_2(A,B) \leq \max\{f \circ d_2(A,C), f \circ d_2(C,B)\} \\ &= \max\{f(1), f(1)\} = f(1). \end{aligned} \tag{15}$$

Therefore, $f(1) = f(1/n)$. By a similar method, we obtain

$$\begin{aligned} f\left(\frac{m}{n}\right) &= f \circ d\left(0, \frac{m}{n}\right) \\ &\leq \max\left\{f \circ d\left(\frac{k-1}{n}, \frac{k}{n}\right) \mid k \in \{1, 2, \ldots, m\}\right\} \\ &= f\left(\frac{1}{n}\right). \end{aligned} \tag{16}$$

In addition, we let $A = (-1/2n, 0)$, $B = (1/2n, 0)$, $C = (0, \sqrt{(4(m/n)^2 - (1/n)^2)/4})$ be points in $\mathbb{R}^2$ so that

$$\begin{aligned} f\left(\frac{1}{n}\right) &= f \circ d_2(A,B) \leq \max\{f \circ d_2(A,C), f \circ d_2(C,B)\} \\ &= f\left(\frac{m}{n}\right). \end{aligned} \tag{17}$$

Therefore, $f(m/n) = f(1/n)$. Hence $f(m/n) = f(1/n) = f(1)$ for every $m, n \in \mathbb{N}$, as asserted. We conclude that

$$f(q) = f(1) \quad \text{for every } x \in \mathbb{Q} \cap (0,\infty). \tag{18}$$

Next let $a \in \mathbb{Q}^c \cap (0,\infty)$. We will show that $f(a) = f(1)$. Let $q_1, q_2 \in \mathbb{Q} \cap (0,\infty)$ be such that $q_1 < a < q_2$. Let $A_1 = (-q_1/2, 0)$, $B_1 = (q_1/2, 0)$, $C_1 = (0, \sqrt{(4a^2 - q_1^2)/4})$, $A_2 = (-a/2, 0)$, $B_2 = (a/2, 0)$, $C_2 = (0, \sqrt{(4q_2^2 - a^2)/4})$ be points in $\mathbb{R}^2$. By (18) and the fact that $f \circ d_2$ is an ultrametric on $\mathbb{R}^2$, we obtain

$$\begin{aligned} f(1) &= f(q_1) = f \circ d_2(A_1, B_1) \\ &\leq \max\{f \circ d_2(A_1, C_1), f \circ d_2(C_1, B_1)\} \\ &= f(a) = f \circ d_2(A_2, B_2) \\ &\leq \max\{f \circ d_2(A_2, C_2), f \circ d_2(C_2, B_2)\} \\ &= f(q_2) = f(1). \end{aligned} \tag{19}$$

This shows that

$$f(a) = f(1) \quad \forall a \in \mathbb{Q}^c \cap (0, \infty). \tag{20}$$

From (18) and (20), we see that $f(x) = f(1)$ for all $x \in (0, \infty)$. This completes the proof. □

Let f be a metric-preserving function and let d be a metric. Then either $f \circ d$ is a metric equivalent to d or $f \circ d$ is a uniformly discrete metric [3, 6]. In addition, f is continuous on $[0, \infty)$ if and only if it is continuous at 0 [3, 4, 6]. But by Theorem 15, every metric-ultrametric-preserving function f is always discontinuous at 0 and $f \circ d$ is always a uniformly discrete metric for all metric d. We record this in the next corollary.

Corollary 16. *Let $f : [0, \infty) \to [0, \infty)$ be metric-ultrametric-preserving. Then*

(i) *$f \circ d$ is a uniformly discrete metric for every metric d,*

(ii) *f is discontinuous at 0 and is continuous on $(0, \infty)$.*

Proof. By Theorem 15, there exists $c > 0$ such that

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ c, & \text{if } x > 0. \end{cases} \tag{21}$$

So (ii) follows immediately. If (X, d) is a metric space, then

$$f \circ d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ c, & \text{if } x \neq y. \end{cases} \tag{22}$$

So if we let $\varepsilon = c/2$, then $B_{f \circ d}(x, \varepsilon) = \{x\}$ for every $x \in X$. This proves (i). □

5. Ultrametric-Metric-Preserving Functions

In this section, we give a characterization of the functions in $\mathcal{UM}$ in terms of special type of triangle triplets. Recall that a triple (a, b, c) of nonnegative real numbers is called *triangle triplet* if $a \le b + c$, $b \le c + a$, and $c \le a + b$. We denote by Δ the set of all triangle triplets. We introduce a special type of triangle triplets that will be used to characterize ultrametric-metric-preserving functions in the next definition.

Definition 17. A triple (a, b, c) of nonnegative real numbers will be called ultra-triangle triplet if $a \le \max\{b, c\}$, $b \le \max\{c, a\}$, and $c \le \max\{a, b\}$. We denote by Δ_∞ the set of all ultra-triangle triplets.

Since we will compare the functions f in $\mathcal{UM}$ with those in $\mathcal{M}$, we first state a characterization of metric-preserving functions in terms of triangle triplets.

Theorem 18. *Let $f : [0, \infty) \to [0, \infty)$ be amenable. Then the following statements are equivalent:*

(i) *f is metric-preserving,*

(ii) *for each $(a, b, c) \in \Delta$, $(f(a), f(b), f(c)) \in \Delta$,*

(iii) *for each $(a, b, c) \in \Delta$, $f(a) \le f(b) + f(c)$.*

Proof. The proof can be found, for example, in [3, 4, 6]. □

Similar to Theorem 18, we obtain a characterization of the functions in $\mathcal{UM}$ in terms of ultra-triangle triplets as follows.

Theorem 19. *Let $f : [0, \infty) \to [0, \infty)$ be amenable. Then the following statements are equivalent:*

(i) *f is ultrametric-metric-preserving,*

(ii) *for each $(a, b, c) \in \Delta_\infty$, $(f(a), f(b), f(c)) \in \Delta$,*

(iii) *for each $0 \le a \le b$, $f(a) \le 2f(b)$.*

To prove Theorem 19, the following lemmas are useful.

Lemma 20. *If (X, d) is an ultrametric space and $x, y, z \in X$, then the triple $(d(x, y), d(x, z), d(z, y))$ is an ultra-triangle triplet. Conversely, if (a, b, c) is an ultra-triangle triplet, then there exist an ultrametric space (X, d) and $x, y, z \in X$ such that $(a, b, c) = (d(x, y), d(x, z), d(z, y))$.*

Lemma 21. *If $(a, b, c) \in \Delta_\infty$, then*

$$(i)\ a \le b = c \quad \text{or} \quad (ii)\ b \le c = a \quad \text{or} \quad (iii)\ c \le a = b. \tag{23}$$

We will prove Lemmas 21 and 20, and then Theorem 19, respectively.

Proof of Lemma 21. Let $(a, b, c) \in \Delta_\infty$. Suppose that a, b, c are all distinct. Without loss of generality, we can assume that $a < b < c$. Then $c > \max\{a, b\}$ which contradicts the fact that $(a, b, c) \in \Delta_\infty$. So a, b, c are not all distinct. If $a = b$, then $c \le \max\{a, b\} = a$ and (iii) holds. Similarly, if $a = c$, then (ii) holds and if $b = c$, then (i) holds. □

Proof of Lemma 20. The first part follows immediately from the ultrametric inequality of d. For the converse, we let $(a, b, c) \in \Delta_\infty$. By Lemma 21, we can assume that $a \le b = c$ (the other cases can be proved similarly). Let $X = \{A, B, C\} \subseteq \mathbb{R}^2$, where $A = (-a/2, 0)$, $B = (a/2, 0)$, and $C = (0, \sqrt{(4b^2 - a^2)/4})$. Let d_2 be the Euclidean metric on $\mathbb{R}^2$ and $d = d_2|_X$. Then (X, d) is an ultrametric space and $(a, b, c) = (d_2(A, B), d_2(A, C), d_2(C, B))$. □

Proof of Theorem 19. (i) → (ii) Let $f \in \mathcal{UM}$ and let $(a, b, c) \in \Delta_\infty$. Then by Lemma 20, there exist an ultrametric space (X, d) and $x, y, z \in X$ such that

$$(a, b, c) = (d(x, y), d(x, z), d(z, y)). \tag{24}$$

Since $f \in \mathcal{UM}$, $(X, f \circ d)$ is a metric space. It follows from the triangle inequality of $f \circ d$ that $(f \circ d(x, y), f \circ d(x, z), f \circ d(z, y))$ is a triangle triplet. That is, $(f(a), f(b), f(c)) \in \Delta$.

(ii) → (iii) Assume that (ii) holds. Let $0 \le a \le b$. Then, $(a, b, b) \in \Delta_\infty$. So $(f(a), f(b), f(b)) \in \Delta$ by (ii). Therefore, $f(a) \le f(b) + f(b) = 2f(b)$, as required.

(iii) → (i) Assume that (iii) holds. Let (X, d) be an ultrametric space. Since f is amenable, $f \circ d(x, y) = 0$ if and only if $x = y$. So it remains to show that the triangle inequality holds for $f \circ d$. Let $x, y, z \in X$. Then by Lemma 20, $(d(x, y), d(x, z), d(z, y)) \in \Delta_\infty$. Then by Lemma 21, we can assume that $d(x, y) \leq d(x,z) = d(z, y)$ (the other cases can be proved similarly). Then by (iii), we obtain

$$\begin{aligned} f \circ d(x, y) &= f(d(x, y)) \leq 2f(d(x,z)) \\ &= f(d(x,z)) + f(d(z, y)) \\ &= f \circ d(x,z) + f \circ d(z, y), \quad \text{as required.} \end{aligned} \tag{25}$$

Hence, the proof is complete. □

Next we give an example to show that the relation (*S*4) in Proposition 6 is a proper subset.

Example 22. Let $f : [0, \infty) \to [0, \infty)$ be given by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1; \\ \frac{1}{2}, & \text{if } x > 1. \end{cases} \tag{26}$$

Let d be the usual metric on $\mathbb{R}$. Then

$$\begin{aligned} f \circ d(1, 2) &= f(1) = 1 > \frac{1}{3} + \frac{1}{2} \\ &= f \circ d\left(1, \frac{2}{3}\right) + f \circ d\left(\frac{2}{3}, 2\right). \end{aligned} \tag{27}$$

So $f \circ d$ is not a metric and therefore $f \notin \mathscr{M}$. Since f is not increasing, $f \notin \mathscr{U}$. Next we will show that $f \in \mathscr{U}\mathscr{M}$, by applying Theorem 19. Let $0 \leq a \leq b$. If $b \geq 1/2$, then $f(b) \geq 1/2$ and therefore $2f(b) \geq 1 \geq f(x)$ for all $x \in [0, \infty)$. In particular, $2f(b) \geq f(a)$. If $b < 1/2$, then $a < 1/2$ and thus $f(a) = a \leq b = f(b) \leq 2f(b)$. In any case, we have $f(a) \leq 2f(b)$. Hence $f \in \mathscr{U}\mathscr{M}$ but $f \notin \mathscr{M}$ and $f \notin \mathscr{U}$. This example shows that $\mathscr{U}\mathscr{M} \nsubseteq \mathscr{U} \cup \mathscr{M}$ and the relation (*S*4) in Proposition 6 is in fact a proper subset.

Remark 23. (1) From Examples 11, 14, and 22, we now see that the relations (*S*1), (*S*2), (*S*3), and (*S*4) in Proposition 6 are in fact proper subsets.

(2) If we replace 1/2 in the definition of f in Example 22 by a constant c (that is, $f(x) = x$ if $x \leq 1$ and $f(x) = c$ if $x > 1$), then $f \in \mathscr{U}\mathscr{M}$ if and only if $c \geq 1/2$.

6. Continuity

In this section, we investigate the continuity aspect of the functions in $\mathscr{M}$, $\mathscr{U}$, $\mathscr{U}\mathscr{M}$, and $\mathscr{M}\mathscr{U}$. By Corollary 16, the continuity of metric-ultrametric-preserving functions is trivial: they are always discontinuous at 0 and continuous elsewhere. The continuity of metric-preserving functions has also been investigated by many authors [1–4, 6, 8, 18], but we can still extend it further in the next theorem.

Before we state the theorem, let us recall some definitions concerning generalized continuities. Let $f : [0, \infty) \to [0, \infty)$. Then f is said to be *weakly continuous* at $a \neq 0$ if and only if there are sequences (x_n) and (y_n) such that (x_n) is strictly increasing and converges to a, (y_n) is strictly decreasing and converges to a, and $f(x_n)$ and $f(y_n)$ converge to $f(a)$. If $a = 0$, then f is said to be *weakly continuous* at a if and only if there exists a strictly decreasing sequence (y_n) converging to a such that $f(y_n)$ converges to $f(a)$. We refer the reader to [36] for weak continuity of functions defined on a more general domain.

Unlike weak continuity, quasi continuity and almost continuity seem to be first given in a more general domain than a subset of $\mathbb{R}$. So we let X and Y be topological spaces and let $g : X \to Y$. Then g is said to be *quasi continuous* at $a \in X$ if for all open sets U of X and V of Y such that $a \in U$ and $f(a) \in V$, there is a nonempty open sets G of X such that $G \subseteq U$ and $f(G) \subseteq V$. The function g is said to be *almost continuous at x in the sense of Singal* (briefly a.c.S. at x) if for each open set V of Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq \operatorname{Int}(\overline{V})$ and g is said to be *almost continuous at x in the sense of Husain* (briefly a.c.H. at x) if for each open set V of Y containing $f(x)$, $\overline{f^{-1}(V)}$ is a neighborhood of x. The function g is said to be *quasi continuous* on $A \subseteq X$ (or a.c.S. on A, or a.c.H. on A) if it is quasi continuous at every $a \in A$ (a.c.S. at a for every $a \in A$, a.c.H. at a for every $a \in A$).

Remark 24. (1) The concepts of a.c.S. functions and a.c.H. functions are not equivalent as shown by Long and Carnahan [37].

(2) There are several other types of continuities in the literature. Some of them have the same name but different definition, see [38] for instance, a different definition of weak continuity. We refer the reader to [39–43] and the other references for additional details and information.

Now we are ready to state our theorem. We will see that there is a similarity and dissimilarity between continuity of the functions in $\mathscr{M}$ and $\mathscr{U}\mathscr{M}$.

Theorem 25. *Let $f : [0, \infty) \to [0, \infty)$ be metric-preserving. The following statements are equivalent:*

(1) *f is continuous at $[0, \infty)$,*

(2) *f is continuous at 0,*

(3) *For every $\varepsilon > 0$, there exists and $x > 0$ such that $f(x) < \varepsilon$,*

(4) *f is strongly metric-preserving,*

(5) *f is uniformly continuous on $[0, \infty)$,*

(6) *f is weakly continuous on $[0, \infty)$,*

(7) *f is weakly continuous at 0,*

(8) *f is quasi continuous on $[0, \infty)$,*

(9) *f is quasi continuous at 0,*

(10) *f is a.c.S on $[0, \infty)$,*

(11) *f is a.c.S at 0,*

(12) *f is a.c.H on $[0, \infty)$,*

(13) *f is a.c.H at 0.*

Proof. The equivalence of (1), (2), (3), and (4) is proved in [4, 6]. With a bit more observation, we can prove that (1) to (11) are all equivalent. First we notice that

$$\left|f(a) - f(b)\right| \leq f(|a - b|) \quad \forall a, b \in [0, \infty). \tag{28}$$

To prove (28), we let $a, b \in [0, \infty)$. Then $(a, b, |a - b|)$ is a triangle triplet. So by Theorem 18, $(f(a), f(b), f(|a - b|))$ is a triangle triplet. Therefore,

$$f(a) \leq f(b) + f(|a - b|), \qquad f(b) \leq f(a) + f(|a - b|). \tag{29}$$

Thus, $|f(a) - f(b)| \leq f(|a - b|)$, as asserted. Now we will prove that (2), (5), (6), (7), and (3) are equivalent.

(2)→(5) Assume that f is continuous at 0. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$\text{if } a \in [0, \delta), \quad \text{then } f(a) < \varepsilon. \tag{30}$$

Now if $x, y \in [0, \infty)$ and $|x - y| < \delta$, then by (28) and (30), we obtain

$$\left|f(x) - f(y)\right| \leq f\left(\left|x - y\right|\right) < \varepsilon. \tag{31}$$

This shows that f is uniformly continuous on $[0, \infty)$.

It is easy to see that (5) implies (6) and (6) implies (7).

(7) → (3) We assume that (7) holds. Let (x_n) be the sequence in $(0, \infty)$ such that (x_n) is strictly decreasing and converges to 0, and $(f(x_n))$ converges to $f(0) = 0$. Therefore, if $\varepsilon > 0$ is given, there exists $N \in \mathbb{N}$ such that

$$f(x_N) = f(x_N) - f(0) < \varepsilon. \tag{32}$$

This proves (3). Since (3) and (2) are equivalent, we see that (2), (5), (6), (7), and (3) are equivalent, as asserted.

It is true in general that every continuous function is quasi continuous. So it is easy to see that (1) implies (8) and (8) implies (9). Next assume that (9) holds. To show (3), let $\varepsilon > 0$ be given. Let $V = U = [0, \varepsilon)$. Then V and U are open set in $[0, \infty)$ containing $f(0) = 0$ and 0, respectively. Since f is quasi continuous at 0, there exists a nonempty open set $G \subseteq U$ such that $f(G) \subseteq V$. Now we can choose $x \in G - \{0\}$ so that $x > 0$ and $f(x) < \varepsilon$. This gives (3). Since (1) and (3) are equivalent, we obtain that (1), (8), (9), and (3) are equivalent. Similarly, it is easy to see that (1) implies (10), (10) implies (11), (1) implies (12), and (12) implies (13). Since (1) and (3) are equivalent, it now suffices to show that each of (11) and (13) implies (3). First assume that (11) holds. Let $\varepsilon > 0$ and let $V = [0, \varepsilon)$. Then V is open in $[0, \infty)$ and contains $f(0)$. Since f is a.c.S. at 0, there exists an open set U containing 0 such that

$$f(U) \subseteq \operatorname{Int}\left(\overline{V}\right) = \operatorname{Int}[0, \varepsilon] = [0, \varepsilon). \tag{33}$$

Now we can choose $x \in U - \{0\}$ so that $x > 0$ and $f(x) < \varepsilon$. Similarly if (13) holds, then $\overline{f^{-1}(V)}$ is a neighborhood of 0, so $f^{-1}(V) \neq \{0\}$, and therefore we can choose $x \in f^{-1}(V) - \{0\}$ so that $f(x) < \varepsilon$ and $x > 0$. This completes the proof. □

The function f in Example 22 shows that in the case of ultrametric-metric-preserving functions, the global continuity on $[0, \infty)$ and the local continuity at 0 are not equivalent. In addition, the uniform continuity on $[0, \infty)$ and continuity on $[0, \infty)$ are not equivalent as can be seen from the function f in Example 11. However, we still have the following result for the continuity at 0.

Theorem 26. *Let f be ultrametric-metric-preserving. Then the following statements are equivalent:*

(i) *f is continuous at 0,*
(ii) *f is weakly continuous at 0,*
(iii) *for every $\varepsilon > 0$, there exists an $x > 0$ such that $f(x) < \varepsilon$,*
(iv) *f is quasi continuous at 0,*
(v) *f is a.c.S. at 0,*
(vi) *f is a.c.H. at 0.*

Proof. We have that (i) implies (ii) is true in general. By the same argument that (7) implies (3) in Theorem 25, we see that (ii) implies (iii). Next assume that (iii) holds. To show that f is continuous at 0, let $\varepsilon > 0$ be given. Then by (iii), there exists $x_0 > 0$ such that $f(x_0) < \varepsilon/2$. Let $\delta = x_0$ and let $x \in [0, \delta)$. Since $0 \leq x < \delta$ and $f \in \mathcal{UM}$, we obtain by Corollary 8 and Theorem 19 that

$$\left|f(x) - f(0)\right| = f(x) \leq 2f(\delta) = 2f\left(x_0\right) < \varepsilon. \tag{34}$$

This gives (i). Therefore, (i), (ii), and (iii) are equivalent. Since (i) implies (iv), (v), and (vi), it suffices to show that each of (iv), (v), and (vi) implies (iii). Since $f \in \mathcal{UM}$, it is amenable and we can use the same argument of the proof of Theorem 25 to show that (iv) implies (iii) (the same as (9) implies (3)), (v) implies (iii) (the same as (11) implies (3)), and (vi) implies (iii) (the same as (13) implies (3)). This completes the proof. □

Corollary 27. *Let $f \in \mathcal{UM}$. If f is discontinuous at 0, then there exists an $\varepsilon > 0$ such that $f(x) > \varepsilon$ for all $x > 0$.*

Proof. This follows from (i) and (iii) in Theorem 26. □

Example 28. Let $f, g : [0, \infty) \to [0, \infty)$ be given by

$$f(x) = \begin{cases} x, & x \leq 1; \\ 1, & x > 1,\ x \in \mathbb{Q}; \\ 2, & x > 1,\ x \notin \mathbb{Q}, \end{cases} \qquad g(x) = \begin{cases} x, & x < 1; \\ 2, & x \geq 1. \end{cases} \tag{35}$$

First we will show that $f \in \mathcal{UM}$ by applying Theorem 19. So we let $0 \leq a \leq b$. If $b > 1$, then $2f(b) \geq 2 \geq f(x)$ for every $x \in [0, \infty)$. In particular, $2f(b) \geq f(a)$. If $b \leq 1$, then $f(a) = a \leq b \leq 2b = 2f(b)$. So $f \in \mathcal{UM}$. It is easy to see that f is weakly continuous at 1 but is not continuous at 1. In fact f is weakly continuous at every $x \geq 0$ and is not continuous at any $x \geq 1$. This shows that we cannot replace continuity at 0 in Theorem 26 by continuity at any other point $x \neq 0$. Similarly, $g \in \mathscr{U}\mathscr{M}$ and is quasi continuous on $[0, \infty)$ but g is not continuous at 1.

Conflict of Interests

The authors declare that they have no competing interests.

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On metric-preserving functions and fixed point theorems

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Abstract
Kirk and Shahzad have recently given, in this journal, fixed point theorems concerning local radial contractions and metric transforms. In this article, we replace the metric transforms by metric-preserving functions. This in turn gives several extensions of the main results given by Kirk and Shahzad. Several examples are given. The fixed point sets of metric transforms and metric-preserving functions are also investigated.

Keywords: metric-preserving function; metric transform; local radial contraction; rectifiably pathwise connected space; uniform local multivalued contraction

1 Introduction

The concept of metric transforms is introduced by Blumenthal [1, 2] in 1936 while the concept of metric-preserving functions seems to be introduced by Wilson [3] in 1935 and is investigated in detail by many authors [4–20]. Recently, Petruşel *et al.* [14] have shown the role of equivalent metrics and metric-preserving functions in fixed point theory. In addition, Kirk and Shahzad [21] have given results concerning metric transforms and fixed point theorems. Their main results are as follows:

Theorem 1 (Kirk and Shahzad [21, Theorem 2.2]) *Let* (X,d) *be a metric space and* $g : X \to X$. *Suppose there exist a metric transform* ϕ *on* X *and a number* $k \in (0,1)$ *such that the following conditions hold*:

(a) *For each* $x \in X$ *there exists* $\varepsilon_x > 0$ *such that for every* $u \in X$

$$d(x,u) < \varepsilon \quad \Rightarrow \quad (\phi \circ d)\big(g(x), g(u)\big) \leq kd(x,u).$$

(b) *There exists* $c \in (0,1)$ *such that for all* $t > 0$ *sufficiently small*

$$kt \leq \phi(ct).$$

Then g *is a local radial contraction on* (X,d).

Theorem 2 (Kirk and Shahzad [21, Theorem 2.3]) *Suppose, in addition to the assumptions in Theorem* 1, X *is complete and rectifiably pathwise connected*. *Then* g *has a unique fixed point* x_0, *and* $\lim_{n\to\infty} g^n(x) = x_0$ *for each* $x \in X$.

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Our purpose is to show that the metric transform ϕ in Theorem 1 can be replaced by a metric-preserving function. This in turn gives extensions to the main results given by Kirk and Shahzad in [21, Theorem 2.2, Theorem 2.3, Theorem 2.8, Theorem 3.4, and Theorem 3.6]. Now let us recall some basic definitions that will be used throughout this article.

Definition 3 Let $f : [0,\infty) \to [0,\infty)$. Then

(i) f is said to be *a metric transform* if $f(0) = 0$, f is strictly increasing on $[0,\infty)$, and f is concave on $[0,\infty)$,
(ii) f is said to be *a metric-preserving function* if for all metric spaces (X,d), $f \circ d$ is a metric on X,
(iii) f is said to be *amenable* if $f^{-1}(\{0\}) = \{0\}$,
(iv) f is said to be *tightly bounded* if there exists $u > 0$ such that $f(x) \in [u, 2u]$ for all $x > 0$,
(v) f is said to be *subadditive* if $f(a+b) \le f(a) + f(b)$ for all $a, b \in [0,\infty)$.

Definition 4 Let (X,d) be a metric space and $g : X \to X$. Then g is said to be *a local radial contraction* if there exists $k \in (0,1)$ such that for each $x \in X$, there exists $\varepsilon > 0$ such that for every $u \in X$,

$$d(x,u) < \varepsilon \quad \Rightarrow \quad d\bigl(g(x), g(u)\bigr) \le kd(x,u).$$

Definition 5 Let (X,d) be a metric space and γ be a path in X, that is, a continuous map $\gamma : [a,b] \to X$. A *partition* Y of $[a,b]$ is a finite collection of points $Y = \{y_0, \ldots, y_N\}$ such that $a = y_0 \le y_1 \le y_2 \le \cdots \le y_N = b$. The supremum of the sums

$$\sum Y = \sum_{i=1}^{N} d\bigl(\gamma(y_{i-1}), \gamma(y_i)\bigr)$$

over all the partitions Y of $[a,b]$ is called the *length* of γ. A path is said to be *rectifiable* if its length is finite. A metric space is said to be *rectifiably pathwise connected* if each two points of X can be joined by a rectifiable path.

We will give some auxiliary results in Section 2. Then we will give the results concerning metric-preserving functions, local radial contractions, and uniform local multivalued contractions in Section 3 and Section 4. Finally, we investigate the fixed point sets of metric transforms and metric-preserving functions in Section 5.

2 Lemmas

We need to use some properties of metric-preserving functions and some fixed point theorems. We give them in this section for the convenience of the reader. For more details of the metric-preserving functions, we refer the reader to [6, 8, 10].

Lemma 6 *Let* $f : [0,\infty) \to [0,\infty)$. *Then*

(i) *if* f *is metric preserving, then* f *is amenable,*
(ii) *if* f *is amenable and concave, then* f *is metric preserving.*

Proof The proof of (i) is easily obtained; see for example, in [5, Lemma 2.3]. The proof of (ii) is given in [5, Proposition 1.2] and [8, p.13]. See also [4, Proposition 2] and [6, p.311]. □

Lemma 7 *Let $f : [0,\infty) \to [0,\infty)$. If f is amenable, subadditive, and increasing, then f is metric preserving.*

Proof The proof can be found in [5, Proposition 1.1], [6, Proposition 2.3], and [8, p.9]. □

Lemma 8 *If $f : [0,\infty) \to [0,\infty)$ is amenable and tightly bounded, then f is metric preserving.*

Proof The proof is given in [5, Proposition 1.3], [6, Proposition 2.8], and [8, p.17]. □

Lemma 9 *If f is metric preserving and $0 \leq a \leq 2b$, then $f(a) \leq 2f(b)$.*

Proof The proof is given in [5, Lemma 2.5], and [8, p.16]. □

For a metric-preserving function f, let K_f denote the set

$$K_f = \{k > 0 \mid f(x) \leq kx \text{ for all } x \geq 0\}.$$

Recall also that we define $\inf \emptyset = +\infty$. Then we have the following result.

Lemma 10 *Let $f : [0,\infty) \to [0,\infty)$ be metric preserving. Then $f'(0) = \inf K_f$. In particular, $f'(0)$ always exists in $\mathbb{R} \cup \{+\infty\}$ and*

(i) *$f'(0) < +\infty$ if and only if $K_f \neq \emptyset$, and*
(ii) *$f'(0) = +\infty$ if and only if $K_f = \emptyset$.*

Proof The proof can be found in [4, Theorem 2], [6, Theorem 4.4], and [8, pp.37-39]. □

The next lemma is probably well known but we give a proof here for completeness.

Lemma 11 *If $f : [0,\infty) \to [0,\infty)$ is amenable and concave, then the function $x \mapsto \frac{f(x)}{x}$ is decreasing on $(0,\infty)$*

Proof Let $a, b \in (0,\infty)$ and $a < b$. Since f is concave, we obtain

$$f(a) = f\left(\left(1 - \frac{a}{b}\right)(0) + \left(\frac{a}{b}\right)(b)\right) \geq \left(1 - \frac{a}{b}\right)f(0) + \frac{a}{b}f(b) = \frac{a}{b}f(b).$$

Therefore $\frac{f(a)}{a} \geq \frac{f(b)}{b}$, as desired. □

Lemma 12 (Pokorný [16]) *Let $f : [0,\infty) \to [0,\infty)$. Assume that f is amenable and there is a periodic function g such that $f(x) = x + g(x)$ for all $x \geq 0$. Then f is metric preserving if and only if f is increasing and subadditive.*

Proof The proof can be found in [8, p.32] and [16, Theorem 1]. □

Lemma 13 (Hu and Kirk [22]) *Let* (X,d) *be a complete metric space for which each two points can be joined by a rectifiable path, and suppose* $g : X \to X$ *is a local radial contraction. Then* g *has a unique fixed point* x_0, *and* $\lim_{n\to\infty} g^n(x) = x_0$ *for each* $x \in X$.

As noted by Kirk and Shahzad [21], an assertion in the proof of Lemma 13 given in [22] was based on a false proposition of Holmes [23]. But Jungck [24] proved that the assertion itself is true. Hence the proof given in [22] with minor changes is true. Kirk and Shahzad [21] apply Tan's result [25] to extend some of their theorems. We will also apply Tan's result as well.

Lemma 14 (Tan [25]) *Let* X *be a topological space, let* $x_0 \in X$, *and let* $g : X \to X$ *be a mapping for which* $f := g^N$ *satisfies* $\lim_{n\to\infty} f^n(x) = x_0$ *for each* $x \in X$. *Then* $\lim_{n\to\infty} g^n(x) = x_0$ *for each* $x \in X$. (*Also if* x_0 *is the unique fixed point of* f, *it is also the unique fixed point of* g.)

We will use Nadler's result concerning set-valued mappings. So let us recall some more definitions. If $\varepsilon > 0$ is given, a metric space (X,d) is said to be *ε-chainable* if given $a, b \in X$ there exist $x_1, x_2, \ldots, x_n \in X$ such that $a = x_1$, $b = x_n$, and $d(x_i, x_{i+1}) < \varepsilon$ for all $i \in \{1, 2, \ldots, n-1\}$. The result of Nadler that we need is the following.

Lemma 15 (Nadler [26]) *Let* (X,d) *be a complete* ε*-chainable metric space. If* $T : X \to \mathcal{CB}(X)$ *is an* (ε,k)*-uniform local multivalued contraction, then* T *has a fixed point.*

3 Local radial contractions and metric-preserving functions

In this section, we will give a generalization of Theorem 1 where the metric transform ϕ is replaced by a metric-preserving function. In fact, we obtain a more general result as follows.

Theorem 16 *Let* (X,d) *be a metric space and let* $g : X \to X$. *Assume that there exist* $k \in (0,1)$ *and a metric-preserving function* f *satisfying the following conditions*:

(a) *for each* $x \in X$, *there exists* $\varepsilon > 0$ *such that for every* $u \in X$

$$d(x,u) < \varepsilon \quad \Rightarrow \quad (f \circ d)\big(g(x), g(u)\big) \le k d(x,u), \quad \textit{and}$$

(b) $f'(0) > k$.

Then g *is a local radial contraction.*

We know from Lemma 10 that $f'(0)$ always exists in $\mathbb{R} \cup \{+\infty\}$. So condition (b) in Theorem 16 makes sense. To prove this theorem, we will first show that g is continuous in the following lemma.

Lemma 17 *Suppose that the assumptions in Theorem* 16 *hold. Then the function* g *is continuous.*

As a consequence of Theorem 16, we can replace the metric transform ϕ in Theorem 1 by a metric-preserving function and obtain an extension of Theorem 1.

Theorem 18 *With the same assumptions in Theorem* 16 *except that condition* (b) *is replaced by* (b′): *there exists* $c \in (0,1)$ *such that* $f(ct) \geq kt$ *for all* $t > 0$ *sufficiently small. Then* g *is a local radial contraction.*

Remark 19 As noted by Kirk and Shahzad [21, Remark 2.5], [21, Proposition 2.6], metric transforms satisfying condition (b) in Theorem 1 are numerous. Proposition 20, Example 22, and Example 23 (to be given after the proof of Theorem 18) show that the class of metric-preserving functions satisfying condition (b) in Theorem 1 is larger than the class of metric transforms satisfying the same condition. Hence the class of such functions is even more numerous and Theorem 18 is indeed an extension of Theorem 1.

Now let us give the proof of Lemma 17, Theorem 16, and Theorem 18 as follows.

Proof of Lemma 17 Let $x \in X$ and let $\varepsilon > 0$. Since $k < f'(0) = \lim_{y\to 0^+} \frac{f(y)-f(0)}{y-0} = \lim_{y\to 0^+} \frac{f(y)}{y}$, there exists $\delta_1 > 0$ such that

$$0 < y \leq \delta_1 \quad \Rightarrow \quad \frac{f(y)}{y} > k. \tag{1}$$

By condition (a), there exists $\delta_2 > 0$ such that for every $u \in X$,

$$d(x,u) < \delta_2 \quad \Rightarrow \quad (f \circ d)\big(g(x), g(u)\big) \leq kd(x,u). \tag{2}$$

Let $\delta_3 = \min\{\delta_1, \delta_2, \varepsilon\}$. Then by (1), we obtain

$$\frac{f(\delta_3)}{\delta_3} > k. \tag{3}$$

Since f is metric preserving, we obtain by Lemma 9, and (3) that for every $b \in [0,\infty)$

$$b \geq \frac{\delta_3}{2} \quad \Rightarrow \quad f(b) \geq \frac{f(\delta_3)}{2} > \frac{k\delta_3}{2}. \tag{4}$$

Now let $\delta = \frac{\delta_3}{2}$ and $u \in X$ be such that $d(x,u) < \delta$. Then by (2), we obtain

$$f\big(d\big(g(x), g(u)\big)\big) \leq kd(x,u) < k\delta = \frac{k\delta_3}{2}.$$

Then by (4), $d(g(x), g(u)) < \frac{\delta_3}{2} \leq \frac{\varepsilon}{2} < \varepsilon$. This shows that g is continuous, as required. □

Proof of Theorem 16 Let $c = \frac{1}{2}(\frac{k}{f'(0)} + 1)$ where if $f'(0) = +\infty$, we define $\frac{k}{f'(0)}$ to be zero and $c = \frac{1}{2}(0+1) = \frac{1}{2}$. Then $0 \leq \frac{k}{f'(0)} < c < 1$. Consider

$$f'(0) = \lim_{y\to 0^+} \frac{f(y) - f(0)}{y - 0} = \lim_{y\to 0^+} \frac{f(y)}{y}.$$

Since $f'(0) > \frac{k}{c}$, there exists $\delta_1 > 0$ such that

$$0 < y < \delta_1 \quad \Rightarrow \quad \frac{f(y)}{y} > \frac{k}{c}. \tag{5}$$

To show that g is a local radial contraction with the contraction constant c, let $x \in X$. By Lemma 17, g is continuous at x. So there exists $\delta_2 > 0$ such that for every $u \in X$,

$$d(x,u) < \delta_2 \quad \Rightarrow \quad d\bigl(g(x),g(u)\bigr) < \delta_1. \tag{6}$$

By condition (a), there exists $\delta_3 > 0$ such that for every $u \in X$,

$$d(x,u) < \delta_3 \quad \Rightarrow \quad (f \circ d)\bigl(g(x),g(u)\bigr) \le kd(x,u). \tag{7}$$

Now let $\varepsilon = \min\{\delta_1,\delta_2,\delta_3\}$ and let $u \in X$ be such that $d(x,u) < \varepsilon$. We need to show that $d(g(x),g(u)) \le cd(x,u)$. If $d(g(x),g(u)) = 0$, then we are done. So assume that $d(g(x),g(u)) > 0$. Then $0 < d(x,u) < \varepsilon$ and we obtain by (7) that

$$\frac{(f \circ d)(g(x),g(u))}{d(x,u)} \le k. \tag{8}$$

The left hand side of (8) is

$$\begin{aligned}\frac{(f \circ d)(g(x),g(u))}{d(x,u)} &= \frac{f(d(g(x),g(u)))}{d(g(x),g(u))} \cdot \frac{d(g(x),g(u))}{d(x,u)} \\ &> \frac{k}{c}\frac{d(g(x),g(u))}{d(x,u)},\end{aligned} \tag{9}$$

where the above inequality is obtained from (6) and (5). From (8) and (9), we obtain

$$\frac{k}{c}\frac{d(g(x),g(u))}{d(x,u)} < k,$$

which implies the desired result. This completes the proof. □

Proof of Theorem 18 By Lemma 10, we know that $f'(0)$ exists in $\mathbb{R} \cup \{+\infty\}$ and by Theorem 16, it suffices to show that $f'(0) > k$. So we can assume further that $f'(0)$ exists in $\mathbb{R}$. Now $f'(0) = \lim_{y\to 0^+} \frac{f(y)-f(0)}{y-0} = \lim_{y\to 0^+} \frac{f(y)}{y}$. Since the limits involved in the following calculation exist, we obtain

$$\lim_{y\to 0^+} \frac{f(y)}{y} = \lim_{t\to 0^+} \frac{f(ct)}{ct} \ge \lim_{t\to 0^+} \frac{kt}{ct} = \frac{k}{c} > k.$$

Therefore $f'(0) > k$, as desired. □

As noted earlier, we will show that the class of metric-preserving functions and the class of metric-preserving functions satisfying condition (b) in Theorem 1 are, respectively, larger than the class of metric transforms and the class of metric transforms satisfying condition (b) in Theorem 1.

Proposition 20 *Every metric transform is metric preserving.*

Proof Let f be a metric transform. Since $f(0) = 0$ and f is strictly increasing, f is amenable. Since f is amenable and concave, we obtain by Lemma 6(ii) that f is metric preserving. □

Corollary 21 *Kirk and Shahzad's result* (*Theorem* 1) *holds.*

Proof This follows immediately from Proposition 20 and Theorem 18. □

Example 22 Let $f, g, h : [0,\infty) \to [0,\infty)$ be given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0 \text{ and } x \in \mathbb{Q}; \\ 2, & \text{if } x \in \mathbb{Q}^c, \end{cases} \qquad g(x) = \begin{cases} x, & \text{if } x \in [0,1]; \\ 1, & \text{if } x > 1, \end{cases}$$

$$h(x) = \begin{cases} x, & x \in [0,1]; \\ 1, & x \in [1,10]; \\ x-9, & x \in (10,11); \\ 2, & x \geq 11. \end{cases}$$

Since $f(x) \in [1,2]$ for all $x > 0$, f is tightly bounded. Therefore by Lemma 8, f is metric preserving. It is easy to see that f is not increasing (and is not concave either). So f is not a metric transform. It is easy to see that g is amenable and concave, so it is metric preserving, by Lemma 6(ii). In addition, if $c = k = \frac{1}{2} \in (0,1)$, then $g(ct) \geq kt$ for all $t \in [0,1]$. So g satisfies condition (b) in Theorem 1. But g is not a metric transform because it is not strictly increasing. For h, we proved in [27, Example 14] that h is metric preserving. Similar to g, the function h satisfies the condition (b) in Theorem 1. It is easy to see that h is neither strictly increasing nor concave. Therefore h is not a metric transform.

We can generate more functions similar to g given in Example 22 as follows.

Example 23 Let $a \geq 1$ and $b > 0$. Define $f_{a,b} : [0,\infty) \to [0,\infty)$ by

$$f_{a,b}(x) = \begin{cases} ax, & \text{if } x \in [0,b]; \\ ab, & \text{if } x > b. \end{cases}$$

Then $f_{a,b}$ is amenable and concave. So by Lemma 6(ii), $f_{a,b}$ is metric preserving. We also have $f'_{a,b}(0) = a \geq 1$. So it satisfies condition (b) in Theorem 16. However, $f_{a,b}$ is not a metric transform because it is not strictly increasing. In particular, if we let $X = [0,\infty)$, $k = \frac{1}{2}$, $f, g : X \to X$ given by $g(x) = \frac{1}{2}x$ and $f = f_{1,1}$, then f satisfies all the assumptions in Theorem 16.

Remark 24 Some natural questions concerning the relation of metric transforms, metric-preserving functions, and condition (b) can be answered by Example 22 and Example 23:

Q1: Is there a continuous metric-preserving function which is not a metric transform?
A1: Yes, g and h given in Example 22 and $f_{a,b}$ given in Example 23 are such functions.
Q2: Is there any nowhere continuous metric-preserving function which is not a metric transform?
A2: Yes, f given in Example 22 is such a function.
Q3: Is there a nowhere monotone metric-preserving function which is not a metric transform?
A3: Yes, f given in Example 22 is such a function.

Q4: Is there a metric-preserving function which is concave and satisfies condition (b) in Theorem 1 but it is not a metric transform?

A4: Yes, g given in Example 22 and $f_{a,b}$ given in Example 23 are such functions.

Now that we have obtained two extensions of Theorem 1, we give two generalizations of Theorem 2 as follows.

Theorem 25 *The following statements hold*:

(a) *Suppose, in addition to the assumptions in Theorem* 16, X *is complete and rectifiably pathwise connected. Then* g *has a unique fixed point* x_0, *and* $\lim_{n\to\infty} g^n(x) = x_0$ *for each* $x \in X$.

(b) *Suppose, in addition to the assumptions in Theorem* 18, X *is complete and rectifiably pathwise connected. Then* g *has a unique fixed point* x_0, *and* $\lim_{n\to\infty} g^n(x) = x_0$ *for each* $x \in X$.

Proof Part (a) follows immediately from Theorem 16 and Lemma 13. Part (b) follows immediately from Theorem 18 and Lemma 13. □

Finally, we remark that Kirk and Shahzad use Tan's result (Lemma 14) to extend Theorem 2 further [21, Theorem 2.3 and Theorem 2.8]. We similarly apply their argument to obtain the following.

Theorem 26 *Let* X *be a metric space which is complete and rectifiably pathwise connected, and suppose* $g : X \to X$ *is a mapping for which*

(a) g^N *satisfies the assumptions in Theorem* 16 *for some* $N \in \mathbb{N}$, *or*

(b) g^M *satisfies the assumptions in Theorem* 18 *for some* $M \in \mathbb{N}$.

Then g *has a unique fixed point* x_0, *and* $\lim_{n\to\infty} g^n(x) = x_0$ *for each* $x \in X$.

Proof This follows immediately from Theorem 16, Theorem 18, Lemma 13, and Lemma 14. □

Conclusion We have obtained extensions of the main results given by Kirk and Shahzad in [21, Theorem 2.2, Theorem 2.3, and Theorem 2.8]. We will obtain more results in the next section.

4 Set-valued contractions

Kirk and Shahzad [21] also give an analog of Theorem 1 and Theorem 2 for set-valued mappings. Our purpose in this section is to obtain an analog of Theorem 16 and Theorem 18 for set-valued mappings as well. First let us recall some definitions and results concerning set-valued mappings.

Let (X, d) be a metric space and let $\mathcal{CB}(X)$ be the family of nonempty, closed, and bounded subsets of X. The usual Hausdorff distance on $\mathcal{CB}(X)$ is defined as

$$H(A,B) = \max\bigl\{\rho(A,B), \rho(B,A)\bigr\},$$

where $A, B \in \mathcal{CB}(X)$, $\rho(A,B) = \sup_{x\in A} d(x,B)$, $\rho(B,A) = \sup_{x\in B} d(x,A)$.

Definition 27 Let $T : X \to \mathcal{CB}(X)$. Then

(i) T is called *a multivalued contraction mapping* if there exists a constant $k \in (0,1)$ such that $H(Tx, Ty) \le kd(x,y)$ for all $x, y \in X$.

(ii) For $\varepsilon > 0$ and $k \in (0,1)$, T is called *an* (ε, k)*-uniform local multivalued contraction* if for every $x, y \in X$

$$d(x,y) < \varepsilon \quad \Rightarrow \quad H(Tx, Ty) \le kd(x,y).$$

(iii) A point $x \in X$ is said to be *a fixed point of* T if $x \in Tx$.

Kirk and Shahzad's results on set-valued mappings which will be extended are as follows.

Theorem 28 (Kirk and Shahzad [21, Theorem 3.4]) *Let* (X,d) *be a metric space and* $T : X \to \mathcal{CB}(X)$. *Suppose there exist a metric transform* ϕ *and* $k \in (0,1)$ *such that the following conditions hold*:

(a) *For each* $x, y \in X$, $\phi(H(Tx, Ty)) \le kd(x,y)$.

(b) *There exists* $c \in (0,1)$ *such that for* $t > 0$ *sufficiently small*, $kt \le \phi(ct)$.

Then for $\varepsilon > 0$ *sufficiently small*, T *is an* (ε, c)*-uniform local multivalued contraction on* (X,d).

Theorem 29 (Kirk and Shahzad [21, Theorem 3.6]) *If, in addition to the assumptions of Theorem* 28, X *is complete and connected, then* T *has a fixed point*.

Our aim is to replace the metric transform ϕ in Theorem 28 by a metric-preserving function. We obtain the following theorem.

Theorem 30 *Let* (X,d) *be a metric space and* $T : X \to \mathcal{CB}(X)$. *Suppose there exist a metric-preserving function* f *and* $k \in (0,1)$ *such that the following conditions hold*:

(a) *For each* $x, y \in X$, $f(H(Tx, Ty)) \le kd(x,y)$.

(b) $f'(0) > k$.

Then for $\varepsilon > 0$ *sufficiently small*, T *is an* (ε, c)*-uniform local multivalued contraction on* (X,d).

Corollary 31 *With the same assumptions in Theorem* 30 *except that condition* (b) *is replaced by* (b′): *there exists* $c \in (0,1)$ *such that for* $t > 0$ *sufficiently small*, $kt \le f(ct)$. *Then for* $\varepsilon > 0$ *sufficiently small*, T *is an* (ε, c)*-uniform local multivalued contraction on* (X,d).

Theorem 32 *If, in addition to the assumptions of Theorem* 30 *or Corollary* 31, X *is complete and* ε*-chainable, then* T *has a fixed point. In particular, if* X *is complete and connected, then* T *has a fixed point*.

The proof of these results are similar to those in Section 3.

Proof of Theorem 30 We define $c = \frac{1}{2}(\frac{k}{f'(0)} + 1)$ as in the proof of Theorem 16. Then $0 \le \frac{k}{f'(0)} < c < 1$ and there exists $\delta_1 > 0$ such that for every $z \in [0, \infty)$

$$0 < z \le \delta_1 \quad \Rightarrow \quad \frac{f(z)}{z} > \frac{k}{c}. \tag{10}$$

To show that T is an (ε, c)-uniform local multivalued contraction for $\varepsilon > 0$ sufficiently small, we let $0 < \varepsilon < \frac{\delta_1}{2}$ and let $x, y \in X$ be such that $d(x,y) < \varepsilon$. By Lemma 9 and (10), we have for every $b \in [0, \infty)$

$$b \geq \frac{\delta_1}{2} \quad \Rightarrow \quad f(b) \geq \frac{f(\delta_1)}{2} > \frac{k\delta_1}{2c} > \frac{k\varepsilon}{c} > k\varepsilon. \tag{11}$$

By condition (a), we have $f(H(Tx, Ty)) \leq kd(x,y) < k\varepsilon$. Therefore we obtain by (11) that

$$H(Tx, Ty) < \frac{\delta_1}{2}. \tag{12}$$

If $d(x,y) = 0$ or $H(Tx, Ty) = 0$, then it is obvious that $H(Tx, Ty) \leq cd(x,y)$ and we are done. So assume that $H(Tx, Ty) > 0$ and $d(x,y) > 0$. Then

$$\frac{k}{c}\frac{H(Tx, Ty)}{d(x,y)} < \frac{f(H(Tx, Ty))}{H(Tx, Ty)} \cdot \frac{H(Tx, Ty)}{d(x,y)} = \frac{f(H(Tx, Ty))}{d(x,y)} \leq k,$$

where the first inequality is obtained by applying (12) and (10) and the last inequality is merely the condition (a). This implies $H(Tx, Ty) \leq cd(x,y)$, as desired. □

Proof of Corollary 31 We can imitate the proof of Theorem 18 to obtain $f'(0) > k$. So Corollary 31 follows immediately from Theorem 30. □

Proof of Theorem 32 This follows from Theorem 30, Corollary 31, and Lemma 15. The other part follows from the fact that a connected metric space is ε-chainable for every $\varepsilon > 0$. □

Conclusion We replace the metric transform ϕ by a metric-preserving function. Therefore we obtain theorems more general than those of Kirk and Shahzad [21, Theorem 2.2, Theorem 2.3, Theorem 2.8, Theorem 3.4, and Theorem 3.6].

5 Fixed point set of metric transforms and metric-preserving functions

Recall that for a function $f : X \to X$, we denote by $\operatorname{Fix} f$ the set of all fixed points of f. We begin this section with the following lemma.

Lemma 33 *Let* $f : [0, \infty) \to [0, \infty)$ *be a metric transform. If* $0 < a < b$, $f(a) = a$, *and* $f(b) = b$, *then* $[a, b] \subseteq \operatorname{Fix} f$.

Proof Since f is amenable and concave, the function $x \mapsto \frac{f(x)}{x}$ is decreasing on $(0, \infty)$ by Lemma 11. So if $a \leq x \leq b$, then $1 = \frac{f(a)}{a} \geq \frac{f(x)}{x} \geq \frac{f(b)}{b} = 1$, which implies $f(x) = x$. This shows that $[a, b] \subseteq \operatorname{Fix} f$. □

Lemma 34 *If* $f : [0, \infty) \to [0, \infty)$ *is a metric transform, then* $\operatorname{Fix} f$ *is a closed subset of* $[0, \infty)$.

Proof Let (a_n) be a sequence in $\operatorname{Fix} f$ and $a_n \to a$. If $a = 0$ or $a = a_n$ for some $n \in \mathbb{N}$, then $a \in \operatorname{Fix} f$ and we are done. So assume that $a > 0$ and $a \neq a_n$ for any $n \in \mathbb{N}$. Since $a > 0$ and $a_n \to a$, $a_n > 0$ for all large n. By passing to the subsequence, we can assume that $a_n > 0$

for every $n \in \mathbb{N}$. It is well known that every sequence of real numbers has a monotone subsequence (see *e.g.* [28, p.62]). By passing to the subsequence again, we can assume that (a_n) is monotone. Now suppose that (a_n) is increasing. Then by Lemma 33,

$$[a_1, a_n] \subseteq [a_1, a_2] \cup [a_2, a_3] \cup \cdots \cup [a_{n-1}, a_n] \subseteq \operatorname{Fix} f \quad \text{for every } n \in \mathbb{N}.$$

Since (a_n) is increasing and $a_n \to a$, if $a_1 \leq x < a$, then there exists $N \in \mathbb{N}$ such that $a_1 \leq x < a_N$, which implies that $x \in \operatorname{Fix} f$, by Lemma 33. This shows that $[a_1, a) \subseteq \operatorname{Fix} f$. Since f is increasing and $a_n < a$, $a_n = f(a_n) \leq f(a)$ for every $n \in \mathbb{N}$. Since $a_n \leq f(a)$ for every $n \in \mathbb{N}$ and $a_n \to a$, we have

$$a \leq f(a). \tag{13}$$

In addition, we obtain by Lemma 11 and the fact that $a \geq a_1$ that

$$\frac{f(a)}{a} \leq \frac{f(a_1)}{a_1} = 1. \tag{14}$$

From (13) and (14), we obtain $f(a) = a$, as required. The case where (a_n) is decreasing can be proved similarly. This completes the proof. □

Lemma 35 *Let* $f : [0, \infty) \to [0, \infty)$ *be a metric transform. Then* $\operatorname{Fix} f = [0, \infty)$ *if and only if* $\sup \operatorname{Fix} f = +\infty$.

Proof It is enough to show that $\sup \operatorname{Fix} f = +\infty$ implies $(0, \infty) \subseteq \operatorname{Fix} f$. So suppose that $\sup \operatorname{Fix} f = +\infty$ but there exists $x \in (0, \infty)$ such that $f(x) \neq x$. Since $\sup \operatorname{Fix} f = +\infty$, there exists $a > x$ such that $f(a) = a$. Similarly, there exists $b > a$ such that $f(b) = b$. Since f is amenable and concave, we obtain by Lemma 11

$$\frac{f(x)}{x} \geq \frac{f(a)}{a} = 1.$$

Since $f(x) \neq x$, $f(x) > x$. Since $x < a < b$, there exists $t \in (0, 1)$ such that $a = (1 - t)x + tb$. By the concavity of f, we obtain

$$a = f(a) = f\big((1 - t)x + tb\big) \geq (1 - t)f(x) + tf(b) > (1 - t)x + tb = a,$$

a contradiction. This completes the proof. □

Theorem 36 *If* $a > 0$, *then each set of the form* $\{0\}$, $\{0, a\}$, $[0, a]$, *and* $[0, \infty)$ *is a fixed point set of a metric transform. Conversely, if* f *is a metric transform, then* $\operatorname{Fix} f = \{0\}, \{0, a\}, [0, a]$, *or* $[0, \infty)$ *for some* $a \in (0, \infty)$.

Proof Define $f_1, f_2, f_3, f_4 : [0, \infty) \to [0, \infty)$ by

$$f_1(x) = \frac{x}{2}, \qquad f_2(x) = \sqrt{ax}, \qquad f_3(x) = x, \qquad f_4(x) = \begin{cases} x, & x \in [0, a]; \\ \frac{x+a}{2}, & x > a. \end{cases}$$

It is easy to verify that the functions f_1, f_2, f_3, f_4 are metric transforms and $\operatorname{Fix} f_1 = \{0\}$, $\operatorname{Fix} f_2 = \{0, a\}$, $\operatorname{Fix} f_3 = [0, \infty)$, and $\operatorname{Fix} f_4 = [0, a]$. This proves the first part.

Next let f be a metric transform such that $\operatorname{Fix} f \neq \{0\}$ and $\operatorname{Fix} f \neq [0, \infty)$. We let $a = \sup \operatorname{Fix} f$ and assert that $\operatorname{Fix} f = \{0, a\}$ or $[0, a]$. Note that since $\operatorname{Fix} f \neq \{0\}$, $a > 0$. It is obtained by Lemma 35 that $a < +\infty$. Now apply Lemma 34 to get $a \in \operatorname{Fix} f$. Therefore $\{0, a\} \subseteq \operatorname{Fix} f$. By the definition of a, we see that $x \notin \operatorname{Fix} f$ for every $x > a$. Now if $x \notin \operatorname{Fix} f$ for every $0 < x < a$, then $\operatorname{Fix} f = \{0, a\}$ and we are done. So assume that there exists $0 < x < a$ such that $x \in \operatorname{Fix} f$. We will show that $\operatorname{Fix} f = [0, a]$. Since $a = \sup \operatorname{Fix} f$, it is obvious that $\operatorname{Fix} f \subseteq [0, a]$. Suppose for a contradiction that there exists $0 < y < a$ such that $f(y) \neq y$. Since $0 < x < a$ and $x, a \in \operatorname{Fix} f$, we obtain by Lemma 33 that $y \notin [x, a]$. So $y < x$. By Lemma 11 we have

$$\frac{f(y)}{y} \geq \frac{f(x)}{x} = 1.$$

Since $f(y) \neq y$, $f(y) > y$. Since $y < x < a$, there exists $t \in (0, 1)$ such that $x = (1-t)y + ta$. By the concavity of f, we obtain

$$x = f(x) = f\big((1-t)y + ta\big) \geq (1-t)f(y) + tf(a) > (1-t)y + ta = x,$$

a contradiction. This completes the proof. □

Since every metric transform is metric preserving, we immediately obtain the result that each set of the form $\{0\}$, $\{0, a\}$, $[0, a]$, and $[0, \infty)$ is a fixed point set of a metric-preserving function. However, there is a metric-preserving function f where $\operatorname{Fix} f$ is not of this form. Let us show this more precisely.

Corollary 37 *If* $a > 0$, *then each set of the form* $\{0\}$, $\{0, a\}$, $[0, a]$, *and* $[0, \infty)$ *is a fixed point of a metric-preserving function.*

Proof This follows immediately from Theorem 36 and Proposition 20. □

Example 38 Let $f, g, h : [0, \infty) \to [0, \infty)$ be given by

$$f(x) = \lceil x \rceil, \qquad g(x) = \begin{cases} 0, & x = 0; \\ 1, & x \in \mathbb{Q} - \{0\}; \\ \sqrt{2}, & x \in \mathbb{Q}^c, \end{cases}$$

$$h(x) = \begin{cases} 0, & x = 0; \\ 1, & 0 < x < 1; \\ x, & x \in \mathbb{Q} \cap [1, 2]; \\ 2, & x \in (\mathbb{Q}^c \cap [1, 2]) \cup (2, \infty). \end{cases}$$

(Recall that $\lceil x \rceil$ is the smallest integer which is larger than or equal to x.) It is easy to verify that f is amenable, increasing, and subadditive. So by Lemma 7, f is metric preserving. Since g and h are amenable and tightly bounded, we obtain by Lemma 8 that g and h are metric-preserving. It is easy to see that $\operatorname{Fix} f = \mathbb{N} \cup \{0\}$, $\operatorname{Fix} g = \{0, 1, \sqrt{2}\}$, and $\operatorname{Fix} h = \{0\} \cup (\mathbb{Q} \cap [1, 2])$.

By generating a function similar to h we obtain a more general result as follows.

Proposition 39 *Let $A \subseteq [u, 2u]$ for some $u > 0$. Then $A \cup \{0\}$ is a fixed point set of a metric-preserving function.*

Proof We define $f : [0, \infty) \to [0, \infty)$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ x, & \text{if } x \in A; \\ u, & \text{if } x \notin A \wedge x \notin \{0, u\}, \end{cases}$$

and if $u \notin A$, then define $f(u) = 2u$. Then f is amenable and tightly bounded. Therefore, by Lemma 8, f is metric preserving. It is easy to see that $\operatorname{Fix} f = A \cup \{0\}$. This completes the proof. □

From Example 38 and Proposition 39, we see that the fixed point set of a metric-preserving function may not be of the form $\{0\}$, $\{0, a\}$, $[0, a]$, and $[0, \infty)$. Other natural questions and answers are the following:

Q1: Is there a metric-preserving function which does not satisfy the result in Lemma 33?

A1: Every function given in Example 38 is such a function.

Q2: Is there a metric-preserving function which does not satisfy the result in Lemma 34?

A2: The function h given in Example 38 and the function f given in Proposition 39 (with a suitable set A) are such functions.

Q3: Is there a metric-preserving function which does not satisfy the result in Lemma 35?

A3: The function f given in Example 38 is such a function.

We see that the fixed point sets of metric-preserving functions are quite difficult to be completely characterized. We leave this problem to the interested reader. Now we end this article by giving continuous metric-preserving functions which do not satisfy the results in Lemma 33 and Lemma 35.

Example 40 Let $f, g : [0, \infty) \to [0, \infty)$ be given by $f(x) = \lfloor x \rfloor + \sqrt{x - \lfloor x \rfloor}$ and $g(x) = x + |\sin x|$. (Recall that $\lfloor x \rfloor$ is the largest integer which is less than or equal to x.) We will use Lemma 12 to show that f and g are metric-preserving. First, the function $x \mapsto |\sin x|$ is periodic with period π.

$$\left|\sin(x + y)\right| = |\sin x \cos y + \cos x \sin y| \le |\sin x| + |\sin y|.$$

So the function $x \mapsto |\sin x|$ is also subadditive. From this, we easily see that g satisfies the condition in Lemma 12. So g is metric preserving. It is not difficult to verify that f is also satisfies the assumption in Lemma 12 and we will leave the details to the reader. It is also easy to see that $\operatorname{Fix} f = \mathbb{N} \cup \{0\}$ and $\operatorname{Fix} g = \{n\pi \mid n \in \mathbb{N} \cup \{0\}\}$. So f and g are continuous metric preserving functions of which fixed point sets do not satisfy the results in Lemma 33 and Lemma 35.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed significantly in writing this paper. All authors read and approved this final manuscript.

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