รายงานวิจัยฉบับสมบูรณ์

Multivariate normal approximation on the unitary group

การประมาณเชิงปกติของหลายตัวแปรบนกรุปยูนิแทรี

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและ

มหาวิทยาลัยนเรศวร

Abstract

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Abstract : It has been proved that the rate of convergence of $\mathrm{Tr}(AU)$, where U is an $N \times N$ random unitary matrix and A is an $N \times N$ complex matrix, is bounded by $O(N^{-2+b})$, $0 \leq b < 1$. In this project, we focus on the rate of convergence of each matrix element of $\mathrm{Tr}(AU)$. Our essential tools are the method of moments, which requires the computation of moments and cumulants.

บทคัดย่อ : ได้มีการพิสูจน์แล้วว่าอัตราการลู่เข้าของ $\mathrm{Tr}(AU)$ เมื่อ U เป็นเมทริกซ์ยูนิแทรีสุ่มขนาด $N \times N$ และ A เป็นเมทริกซ์เชิงซ้อนขนาด $N \times N$ มีค่าไม่เกิน $O(N^{-2+b})$, $0 \leq b < 1$ ในโครงการนี้ เราได้ศึกษาอัตราการลู่เข้าของแต่ละสมาชิกของเมทริกซ์ $\mathrm{Tr}(AU)$ เครื่องมือจำเป็นที่เราใช้ก็คือกระบวนการของโมเมนต์ ซึ่งต้องใช้การคำนวณโมเมนต์และคิวมูแลนต์

keywords : rate of convergence, moment, cumulant

1. Abstract

It has been proved that the rate of convergence of $\mathrm{Tr}(AU)$, where U is an $N \times N$ random unitary matrix and A is an $N \times N$ complex matrix, is bounded by $O(N^{-2+b})$, $0 \leq b < 1$. In this project, we focus on the rate of convergence of each matrix element of $\mathrm{Tr}(AU)$. Our essential tools are the method of moments, which requires the computation of moments and cumulants.

2. Executive summary

2.1 Introduction to research

The problem of the value distributions of traces of random unitary matrices have been studied extensively by many authors. The convergence in distribution of $\mathrm{Tr}(U)$, where U is an $N \times N$ random unitary matrix distributed according to the Haar measure, to a standard normal complex random variable is proved by Diaconis and Shahshahani [7]. The convergence rate is very fast, either exponential or even superexponential.

Let A be an $N \times N$ complex matrix. The first few terms in the cumulant expansion of the real part of $\mathrm{Tr}(AU)$ (denoted by $Re\mathrm{Tr}(AU)$ are computed by Samuel [17] and Bars [1]. This shows the convergence in distribution to a normal random variable when $N \to \infty$. The rate of convergence of $Re\mathrm{Tr}(AU)$ is investigated be Meckes [13, 14] by using Stein's method of exchangeable pairs. Suppose A is normalized so that $\mathrm{Tr}(AA^*) = N$, where A^* is the conjugate transpose of A. Meckes proved that the distance of $Re\mathrm{Tr}(AU)$ to a normal random variable with mean zero and variance $\frac{1}{2}$ in the total variation metric on probability measures is of order $O(N^{-1})$. Later, Keating el al [10] improved this rate by using the method of moments. They obtained the exact moments for $Re\mathrm{Tr}(AU)$ and that the rate is of order $O(N^{-2+b})$, with $0 \leq b < 1$ depending only on the asymptotic behavior of the singular values of A.

2.2 Literature review

Let U be an $N \times N$ unitary matrix distributed according to Haar measure on the unitary group $U(N)$. Diaconis and Shahshahani [7] proved that the moments of the trace of U^j converge in distribution to the moments of a standard normal complex random variable. This is called the proof of limiting normality by the method of moment. They also conjecture that both convergences are very fast, either exponential or even super exponential.

Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ be the probability density function of a standard normal random variable and let

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\, dy$$

be the normal distribution. Let $F_N^{(j)}(x)$ be the distribution function of $\sqrt{\frac{2}{j}}\, Re\mathrm{Tr}\big(U^j\big)$, i.e.

$$F_N^{(j)}(x) = \int_{-\infty}^{x} f_N^{(j)}(t)\ dt$$

where the integrand $f_N^{(j)}(t)$ denotes the probability density function of $\sqrt{\frac{2}{j}}\, Re\mathrm{Tr}\big(U^j\big)$. Johansson [10] gave the rate of convergence to a standard normal random variable as following:

$$E(N) \coloneqq \sup_{x\in\mathbb{R}} \left|F_N^{(j)}(x) - \Phi(x)\right| = O\big(N^{-\delta_1 N}\big)$$

$$\int_{-\infty}^{\infty} \left|f_N^{(j)}(x) - \phi(x)\right| dx = O\big(N^{-\delta_2 N}\big)$$

where δ_1, δ_2 are positive constants independent on j and N. We can see that the convergence of $\mathrm{Tr}\big(U^j\big)$ is very fast. Many authors have continually improved Diaconis and Shahshahani's result.

Let A be an $N \times N$ complex matrix normalized so that $\mathrm{Tr}(AA^*) = N$, where A^* is the conjugate transpose of A. Meckes [13, 14] proved that rate of convergence of $\mathrm{Tr}(AU)$ to a standard normal real random variable in the total is bounded by $\frac{c_N}{N}$, where c_N is asymptotic to $2\sqrt{2}$. Moreover, Chatterjee and Meckes [4] showed that rate of convergence of the multivariate version $\mathrm{Tr}(A_1U, A_2U, \dots, A_kU)$, where $A_1, A_2, \dots, A_k$ are $N \times N$ complex matrices, to a standard normal real random variable in the total is bounded by $\frac{c_N k}{N}$, where c_N is asymptotic to $2\sqrt{2}$. In 2011, Keating et al [11] using the method of moments to prove the convergence to central limit theorem of $\mathrm{Tr}(AU)$, and they also derived the rate of convergence $O(N^{-2+b})$ with $0 \leq b < 1$ from the Berry-Esseen inequality for the eigenvalues of random unitary matrices. In general, Berry-Esseen bounds are used to prove central limit theorems for sum of dependent or weakly dependent random variables. It is strikingly that such a bound exists for sums of eigenvalues of unitary matrices, which are strongly correlated.

2.3 Objectives

In this research, we aim to compute moments and cumulants of each matrix element of $\mathrm{Tr}(AU)$, where A is an $N \times N$ complex matrix and U is an $N \times N$ unitary matrix.

3. Results and discussion

Let A be an $N \times N$ complex matrix and U be an $N \times N$ random unitary matrix distributed according to the Haar measure on the unitary group $U(N)$. Let

$$\sigma^2 = \frac{(AA^*)_{\alpha\alpha}}{2N},$$

where A^* is the conjugate transpose of A and $(A)_{\alpha\beta}$ denote the entry in row α and column β of the matrix A. We define

$$X_N \coloneqq \frac{1}{\sigma} Re(AU)_{\alpha\beta} \quad \text{and} \quad Y_N \coloneqq \frac{1}{\sigma} Im(AU)_{\alpha\beta}.$$

Then

$$\frac{1}{\sigma}(AU)_{\alpha\beta} = X_N + i\, Y_N.$$

The invariance of the Haar measure on $U(N)$ under group action implies that the distribution of X_N and Y_N are the same. Therefore, we shall restrict our attention to only X_N.

The characteristic function of X_N is computed by

$$\begin{aligned}\psi_N(t) &= \mathbb{E}e^{itX_N} \\ &= \int_{U(N)} \exp\left(it\frac{1}{\sigma} Re\,(AU)_{\alpha\beta}\right) d\mu_H(U) \\ &= \int_{U(N)} \exp\left(\frac{it}{2\sigma}\left((AU)_{\alpha\beta} + \overline{(AU)_{\alpha\beta}}\right)\right) d\mu_H(U)\end{aligned}$$

$$= \int_{U(N)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{it}{2\sigma}\right)^n \left((AU)_{\alpha\beta} + \overline{(AU)_{\alpha\beta}}\right)^n \, d\mu_H(U)$$

$$= \int_{U(N)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{it}{2\sigma}\right)^n \sum_{m=0}^{n} \binom{n}{m} (AU)_{\alpha\beta}^{n-m} \left(\overline{(AU)_{\alpha\beta}}\right)^m \, d\mu_H(U).$$

Since $|\psi_N(t)| < 1$, we obtain that

$$\psi_N(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{it}{2\sigma}\right)^n \sum_{m=0}^{n} \binom{n}{m} \int_{U(N)} (AU)_{\alpha\beta}^{n-m} \left(\overline{(AU)_{\alpha\beta}}\right)^m \, d\mu_H(U).$$

Since Haar measure is left and right invariant, the integral in this sum is zero unless $n = 2m$. Therefore,

$$\psi_N(t) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(\frac{it}{2\sigma}\right)^{2m} \binom{2m}{m} \int_{U(N)} (AU)_{\alpha\beta}^{m} \left(\overline{(AU)_{\alpha\beta}}\right)^m \, d\mu_H(U)$$

$$= \sum_{m=0}^{\infty} \frac{(it)^{2m}}{(2m)!} \left[\left(\frac{(2m-1)!!}{(2\sigma^2)^m m!}\right) \int_{U(N)} (AU)_{\alpha\beta}^{m} \left(\overline{(AU)_{\alpha\beta}}\right)^m \, d\mu_H(U)\right].$$

We obtain that all odd moments are zero and even moments are given by

$$\mu_{2m} = \frac{(2m-1)!!}{(2\sigma^2)^m m!} \, I_N^m(A), \tag{1}$$

where

$$I_N^m(A) = \int_{U(N)} (AU)_{\alpha\beta}^{m} \left(\overline{(AU)_{\alpha\beta}}\right)^m \, d\mu_H(U).$$

Lemma 1. *Let $m \le N$ and $\lambda = (1^{r_1} \ldots m^{r_m})$ denote a partition of m. We have*

$$I_N^m(A) = m! \sum_{\lambda \vdash m} g_\lambda M_\lambda(N) \, (AA^*)_{\alpha\alpha}^m. \tag{2}$$

Proof. Samuel [8] studied averages of the form

$$\int_{U(N)} U_{i_1 j_1} \ldots U_{i_m j_m} \overline{U_{k_1 l_1}} \ldots \overline{U_{k_m l_m}} \, d\mu_H(U)$$

$$= \sum_{\sigma,\tau \in S_m} M_{\sigma,\tau}(N) \delta_{i_1 k_{\sigma(1)}} \ldots \delta_{i_m k_{\sigma(m)}} \delta_{j_1 l_{\tau(1)}} \ldots \delta_{j_m l_{\tau(m)}}, \tag{3}$$

where S_m is the set of all permutations of the numbers $1, 2, \ldots, m$. The right-hand side of equation (3) can be rewritten as

$$\sum_{\rho,\tau \in S_m} M_\rho(N) \delta_{i_1 k_{\sigma\tau(1)}} \ldots \delta_{i_m k_{\sigma\tau(m)}} \delta_{j_1 l_{\tau(1)}} \ldots \delta_{j_m l_{\tau(m)}}$$

where we have shifted the index in the sum by setting $\rho = \sigma\tau^{-1}$ and used the fact that $M_{\sigma,\tau}(N)$ depends only on $\sigma\tau^{-1}$. Setting $j_m = \beta$ for all m and $l_m = \beta$ for all m, we get that

$$\int_{U(N)} U_{i_1 \beta} \ldots U_{i_m \beta} \overline{U_{k_1 \beta}} \ldots \overline{U_{k_m \beta}} \, d\mu_H(U)$$

$$= \sum_{\rho,\tau \in S_m} M_\rho(N) \delta_{i_1 k_{\rho\tau(1)}} \ldots \delta_{i_m k_{\rho\tau(m)}} \underbrace{\delta_{\beta\beta} \ldots \delta_{\beta\beta}}_{m \text{ terms}}. \tag{4}$$

By multiplying equation (4) by $A_{\alpha i_1}, \ldots, A_{\alpha i_m}$ and $\overline{A_{\alpha k_1}}, \ldots, \overline{A_{\alpha k_m}}$ and summing over indices $i_1, \ldots, i_m, k_1, \ldots, k_m$, we obtain that

$$I_N^m(A) = \sum_{\rho,\tau \in S_m} M_\rho(N) \sum_{\tau \in S_m} \sum_{a,b,c,\ldots,y,z} A_{\alpha a} \overline{A_{ba}} A_{bc} \overline{A_{dc}} \ldots A_{yz} \overline{A_{z\beta}}.$$

We can see that

$$\sum_{a,b,c,\dots,y,z} A_{\alpha a}\overline{A_{ba}}A_{bc}\overline{A_{dc}}\dots A_{yz}\overline{A_{z\beta}} = (AA^*)^m_{\alpha\alpha}.$$

The sum over τ is the number of elements in S_m that is equal to $m!$. Since λ depends on cycle structure of ρ and the number of the same cycle structure is g_λ. Hence we derive formula (1) as required. ■

Let us denote the moments of $\mathcal{N}(0,1)$, the normal random variable with mean 0 and variance 1, by μ^G_{2m}, i.e.

$$\mu^G_{2m} \coloneqq (2m-1)!!.$$

Theorem 2. *We have the following bounds*

$$\mu_{2m} = \mu^G_{2m}\left(1 + O(m!\,N^{-1})\right).$$

Proof. From equations (1) and (2), we have

$$\mu_{2m} = \frac{(2m-1)!!}{(2\sigma^2)^m}\sum_{\lambda\vdash m} g_\lambda M_\lambda (AA^*)^m_{\alpha\alpha}.$$

Since $(AA^*)^m_{\alpha\alpha} = (2\sigma^2)^m N^m$, we get

$$\mu_{2m} = (2m-1)!!\sum_{\lambda\vdash m} g_\lambda M_\lambda N^m. \tag{5}$$

Denote by $\lambda_e = (1^m)$ the cycle-type of the identity in S_m. The sum in equation (5) can be split as follows:

$$\mu_{2m} = (2m-1)!!\,M_{\lambda_e}N^m + (2m-1)!!\sum_{\substack{\lambda\vdash m\\ \lambda\neq\lambda_e}} g_\lambda M_\lambda N^m.$$

From Brouwer and Beenakker [2], the large-N expansion of M_λ is

$$M_{j_1,\dots,j_k} = \prod_{i=1}^k M_{j_i} + O(N^{k-2m-2}),$$
$$M_j = \frac{1}{j}N^{1-2j}(-1)^{j-1}\binom{2j-2}{j-1} + O\left(N^{-1-2j}\right).$$

Then

$$M_{1^{r_1},\dots,m^{r_m}} = \prod_{j=1}^m \left[M_j\right]^{r_j} = \prod_{j=1}^m\left[O\left(N^{1-2j}\right)\right]^{r_j} = O\left(N^{l(\lambda)-2m}\right).$$

Therefore,

$$\mu_{2m} = (2m-1)!!\left[\frac{1}{N^m} + O(N^{-m-2})\right]N^m$$
$$+ (2m-1)!!\sum_{\substack{\lambda\vdash m\\ \lambda\neq\lambda_e}} g_\lambda\left[\frac{1}{N^m} + O(N^{-m-2})\right]N^m.$$

The sum in the second term can be estimated by the greatest value of $l(\lambda)$ which is $m-1$, and we know that $g_\lambda \le m!$. So we have

$$\mu_{2m} = (2m-1)!!\left[\frac{1}{N^m} + O(N^{-m-2})\right]N^m + (2m-1)!!\,m!\,[O(N^{-m-1})]N^m$$

$$= \mu_{2m}^{G}\big(1 + m!\, O(N^{-1})\big).$$

as required. ■

Remark 3. *If we take $N \to \infty$, we will get $\mu_{2m} = \mu_{2m}^{G}$. This demonstrates that X_N converges in distribution to a standard normal.*

Remark 4. *When $A = I$, we get*

$$I_N^m(I) = m! \sum_{\lambda \vdash m} g_\lambda M_\lambda.$$

Then

$$\begin{aligned}
\mu_{2m} &= \frac{(2m-1)!!}{(2\sigma^2)^m} \sum_{\lambda \vdash m} g_\lambda M_\lambda \\
&= \frac{(2m-1)!!}{(2\sigma^2)^m} \left[M_{\lambda_e} + \sum_{\substack{\lambda \vdash m \\ \lambda \neq \lambda_e}} g_\lambda M_\lambda \right] \\
&= \frac{(2m-1)!!}{(2\sigma^2)^m} \left[\frac{1}{N^m} + O(N^{-m-2}) + m!\, O(N^{-m-1}) \right] \\
&= \frac{(2m-1)!!}{(2\sigma^2)^m} O(N^{-m}).
\end{aligned}$$

4. Conclusion

In summary, we obtain that all odd moments of $\frac{1}{\sigma}\, Re(AU)_{\alpha\beta}$ are zero and even moments are given by

$$\mu_{2m} = \frac{(2m-1)!!}{(2\sigma^2)^m m!}\, I_N^m(A),$$

where

$$I_N^m(A) = m! \sum_{\lambda \vdash m} g_\lambda M_\lambda(N)\, (AA^*)_{\alpha\alpha}^m,$$

$m \leq N$ *and* $\lambda = (1^{r_1} \ldots m^{r_m})$ is *a partition of* m. Moreover, *we derive the bounds*

$$\mu_{2m} = \mu_{2m}^{G}\big(1 + O(m!\, N^{-1})\big).$$

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