รายงานวิจัยฉบับสมบูรณ์

โครงการ แบบจำลองทางจักรวาลวิทยาจากสนามทรีฟอร์มที่ทำให้ทั่วไป

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและ
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Abstract

Project Code: TRG5780046

Project Title: Cosmological Models due to Generalized Three-Form Field

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One of important observational evidences in cosmology indicatesthat the universe is expanding with acceleration. In order to describe these evidences, one can introduce an extra mysterious matter into the theory of gravitation namely "dark energy" or modify General Relativity at cosmological scale known as modified gravity theory. The aim of this research project is to construct the dark energy model due to a generalized three-form field. We found that it is possible to use the propose model to explain the late-time expansion of the universe.Moreover, the three-form can provide nonrelativistic matter content in the universe. For the fluid description of the three-form,the fluid can provide nonadiabatic pressure perturbations and corresponds to a system with nonconservation of the particle flux. Dynamical analysis for this model and the model with dark matter coupling are very interesting to investigate and we leave these points for further work. Along with dark energy model, modified gravity model, especially massive gravity, is also one of the aims for this research project. By considering dRGT massive gravity coupling to the k-essence scalar field,we found that the universe has the standard evolution and the graviton mass can play the role of both nonrelativistic matter and cosmological constant with helping of k-essence scalar field leading to unification of dark matter and dark energy. Since the stability and observational constraint for this model has not been investigated yet,it is worthwhile to examine and we leave it for further work.

Keywords :Dark energy, Three-form field, Massive gravity

บทคัดย่อ

รหัสโครงการ : TRG5780046

ชื่อโครงการ: แบบจำลองทางจักรวาลวิทยาจากสนามทรีฟอร์มที่ทำให้ทั่วไป

ชื่อนักวิจัย : พิทยุทธ วงศ์จันทร์ สังกัด มหาวิทยาลัยนเรศวร

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หลักฐานสำคัญอย่างหนึ่งจากผลการสังเกตุการณ์ทางจักรวาลวิทยาบ่งบอกว่าเอกภพกำลังขยายตัวด้วยความเร่งเราสามารถอธิบายเหตุการณ์นี้ได้ด้วยการใส่สสารพิเศษเข้ามาในทฤษฎีความโน้มถ่วงซึ่งโดยทั่วไปแล้วเราเรียกสสารพิเศษนี้ว่าพลังงานมืด หรือ อีกแนวทางหนึ่งคือการปรับเปลี่ยนทฤษฎีสัมพัทภาพทั่วไปซึ่งโดยทั่วไปแล้วเราเรียกทฤษฎีเหล่านี้ว่า ทฤษฎีโมดิฟรายด์กราวิตี ดังนั้น จุดประสงค์หลักของโครงการวิจัยนี้คือการสร้างแบบจำลองพลังงานมืดจากสนามทรีฟอร์มที่ทำให้ทั่วไป จากการศึกษาแบบจำลองดังกล่าวนั้น เราพบว่า มีความเป็นไปได้ที่จะใช้แบบจำลองนี้อธิบายการขยายตัวด้วยความเร่งของเอกภพได้ นอกจากนี้สนามทรีฟอร์มยังสามารถให้ค่าที่เป็นสสารแบบไม่สัมพัทธภาพได้ด้วย สำหรับการบรรยายในเชิงของไหลจากสนามทรีฟอร์มนั้น เราพบว่า สนามทรีฟอร์มสามารถให้การกระเพื่อมแบบไม่อเดียแบติกได้และยังพบอีกว่า ของไหลนี้สอดคล้องกับระบบที่ฟลักซ์ของอนุภาคของของไหลไม่อนุรักษ์ ประเด็นที่น่าสนใจต่อเนื่องจากโครงงานวิจัยในส่วนนี้คือการวิเคราะเชิงพลศาสตร์ในแบบจำลอง รวมไปถึงการศึกษาแบบจำลองที่มีการคู่ควบกับสสารมืดซึ่งผู้วิจัยจะศึกษาในลำดับต่อไป ในทำนองเดียวกันกับการศึกษาแบบจำลองพลังงานมืด การศึกษาแบบจำลองโมดิฟรายด์กราวิตีซึ่งเน้นที่ทฤษฎีแมสสีพกราวิตีนั้น ก็เป็นส่วนหนึ่งของโครงการวิจัยชิ้นนี้ จากการพิจารณาทฤษฎีแมสสีพกราวิตีแบบดีอาร์จีทีที่คู่ควบกับสนามสเกลาเคเอสเซนเราพบว่า เอกภพจะมีวิวัฒนาการในแบบมาตรฐานได้ นอกจากนี้เรายังพบว่ามวลของกราวิตอนที่ได้รับการเกื้อหนุนจากสนามสเกลาเคเอสเซนนี้ยังสามารถมีบทบาทเป็นได้ทั้งสสารแบบไม่สัมพัทธภาพและค่าคงที่จักรวาล ทั้งนี้จะนำไปสู่การการรวมเป็นหนึ่งเดียวของทั้งสสารมืดและพลังงานมืดได้ เนื่องด้วยสเถียรภาพและการหาข้อจำกัดจากผลสังเกตุการณ์สำหรับแบบจำลองนี้ยังไม่ได้ทำการศึกษา ดังนั้นการศึกษาในประเด็นดังกล่าวนี้ จึงเป็นเหตุที่คุ้มค่าที่จะค้นคว้าวิจัย ซึ่งผู้วิจัยได้ละไว้สำหรับการศึกษาในลำดับถัดไป

คำหลัก : พลังงงานมืด, สนามทรีฟอร์ม, แมสสีพกราวิตี

Executive Summary

งานวิจัยนี้ได้ถูกแบ่งออกเป็นสองส่วนหลักๆคือ แบบจำลองทางจักรวาลวิทยาจากสนามทรีฟอร์มที่ทำให้ทั่วไป และ แบบจำลองทางจักรวาลวิทยาสำหรับทฤษฎีแมสสีพกราวิตีที่คู่ควบกับสนามสเกลาร์เคเอสเซนซ์ โดยในส่วนแรกนั้นอยู่ในขั้นตอนรอการตอบรับจากวารสารวิจัย ทั้งนี้ เอกสารต้นฉบับนั้น สามารถดูรายละเอียดเพิ่มเติมได้ในภาคผนวกที่หนึ่ง ในส่วนที่สองนั้นบทความวิจัยได้ตีพิมพ์ที่วารสารระดับนานาๆชาติแล้วซึ่งบทความนี้ได้แนบมาในภาคผนวกที่สอง บทสรุปอย่างย่อของทั้งสองส่วนดังกล่าวได้เรียบเรียงพอสังเขบดังนี้

การทำวิจัยในส่วนแรกนั้นเริ่มต้นด้วยการหาแอคชัน (action) ที่มีความทั่วไปของสนามทรีฟอร์มโดยปรับเปลี่ยนจากสนามทรีฟอร์มที่มีพจน์จลน์เป็นแบบคาโนนิคอลและพจน์ศักย์ไปเป็นฟังก์ชันใดๆของพจน์จลน์และพจน์ศักย์ จากแอคชันนี้ เราสามารถหาสมการการเคลื่อนที่ และ เทนเซอร์พลังงาน-โมเมนตัม (energy-momentum tensor) ได้ซึ่งจากการวิเคราะห์ทั้ง สมการการเคลื่อนที่และเทนเซอร์พลังงาน-โมเมนตัมนี้ผู้วิจัยพบว่า ลักษณะของของไหลที่ได้นั้นสามารถแบ่งออกเป็นสองส่วนหลักๆคือ ของไหลที่มีพารามิเตอร์ของสมการสถานะ (equation of state parameter) คงที่และของไหลที่พารามิเตอร์ของสมการสถานะไม่คงที่ โดยในกรณีของพารามิเตอร์ของสมการสถานะคงที่นั้นเราสามารถให้พารามิเตอร์ของสมการสถานะ $w = 0$ได้โดยที่แอคชันยังสามารถหาค่าได้ซึ่งต่างจากในกรณีของสนามสเกลาที่แอคชันจะมีค่าเป็นอนันต์ถ้า$w = 0$ สำหรับในกรณีที่พารามิเตอร์ของสมการสถานะไม่คงที่นั้น จากการเปรียบเทียบความเร็วเสียงแบบอเดียแบติคกับความเร็วเสียงของการแพร่กระจายของการกระเพื่อม ผู้วิจัยพบว่า มีความเป็นไปได้ที่สนามทรีฟอร์มจะให้คุณลักษณะของการกระเพื่อมแบบไม่อเดียแบติคได้ ซึ่ง การกระเพื่อมแบบนี้ไม่สามารถเกิดขึ้นได้ในกรณีของของไหลแบบไม่สัมพัทธภาพจากสนามสเกลาร์เคเอสเซนซ์นอกจากนี้ผู้วิจัยได้ดำเนินการวิจัยต่อด้วยการหาความสอดคล้องทางอุณหพลศาสตร์ของของไหลและพบว่า ระบบของไหลที่ว่านี้สอดคล้องกับระบบที่จำนวนอนุภาคของระบบไม่อนุรักษ์

สำหรับงานวิจัยในส่วนที่สองนั้น ผู้วิจัยได้ศึกษาแบบจำลองทางจักรวาลวิทยาโดยที่ทฤษฎีความโน้มถ่วงสอดคล้องกับทฤษฎีที่กราวิตอนมีมวล หรือเรียกว่า ทฤษฎีแมสสีพกราวิตี (Massive Gravity Theory)จากการศึกษาผู้วิจัยพบว่า ทฤษฎีนี้ยังมีความบกพร่องอยู่หลักๆสองประการคือ1. การกำหนดเมทริคตัวช่วย (fiducial metric) ยังไม่สอดคล้องกับเมทริคกายภาพ (physical metric)2. คือ มวลของกราวิตรอนยังไม่มีความทั่วไปเพียงพอ จากข้อบกพร่องสองประการนี้ ผู้วิจัยได้ปรับปรุงทฤษฎีนี้โดยการกำหนดให้เมทริคตัวช่วยมีรูปแบบเหมือนกันกับเมทริคกายภาพนั่นคืออยู่ในรูปแบบของแอฟแอลอาร์ดับเบิลยู (FLRW metric) นอกจากนี้ผู้วิจัยยังกำหนดให้มวลของกราวิตอนขึ้นกับพจน์จนล์ของสนามสเกลาร์ จากการศึกษาดังกล่าว ผู้วิจัยพบว่า การกำหนดให้เมทริคตัวช่วยมีรูปแบบเหมือนกันกับเมทริคกายภาพนั้นทำให้มวลของกราวิตอนสามารถประพฤติตัวเป็นค่าคงที่จักรวาลเพื่อทำให้เอกภพขยายตัวด้วยความเร่งในช่วงเวลาปัจจุบันได้ นอกจากนี้การกำหนดมวลของกราวิตอนขึ้นกับพจน์จนล์นั้นยังสามารถช่วยให้เราตีความมวลของกราวิตอนเป็นสสารมืดในช่วงเวลาที่สสารมืดเด่นได้ด้วยทั้งนี้แบบจำลองนี้จะนำไปสู่การรวมเป็นหนึ่งเดียวของทั้งสสารมืดและพลังงานมืดได้จากมวลของกราวิตอน ยิ่งไปกว่านั้น จากการวิเคราะห์เชิงพลวัฒน์ในแบบจำลองนี้ ผู้วิจัยยังพบว่า ในช่วงของพารามิเตอร์ที่เหมาะสม เอกภพสามารถวิวัฒน์ไปสิ้นสุดที่อัตราส่วนของพลังงานมืดต่อสสารแบบไม่สัมพัทธภาพมีค่าโดยประมาณเป็น 7 ต่อ 3 ได้ ดังนั้นจึงพอสรุปได้ว่าแบบจำลองทางจักรวาลวิทยานี้สามารถแก้ปัญหาความบังเอิญทางจักรวาลวิทยาได้โดยที่ปัญหาเชิงทฤษฎีนี้กล่าวว่า ทำไมอัตราส่วนของพลังงานมืดต่อสสารแบบไม่สัมพัทธภาพจึงมีค่าพอๆกัน ณ ปัจจุบันทั้งๆที่การวิวัฒน์ของทั้งสองนั้นต่างกันอย่างมโหฬาร

เนื้อหางานวิจัย

1. บทนำ

ผลจากการสังเกตุการณ์ทางจักรวาลวิทยาบ่องบอกว่าเอกภพ ณ ปัจจุบันกำลังขยายตัวด้วยความเร่ง [1, 2] ทฤษฎีสัมพัทธภาพทั่วไปของไอน์สไตน์นั้นถือได้ว่าเป็นทฤษฎีความโน้มถ่วงที่คาดว่าจะอธิบายการวิวัฒน์ของเอกภพเช่นนี้ได้ อย่างไรก็ตาม เราพบว่า การพิจารณาทฤษฎีสัมพัทธภาพทั่วไปที่มีสสารพลังงานที่เรารู้จักในปัจจุบันนั้น ไม่สามารถทำให้เอกภพขยายตัวด้วยความเร่งได้ การที่จะทำให้เอกภพขยายตัวด้วยความเร่งบนพื้นฐานของทฤษฎีสัมพัทธภาพนี้สามารถทำได้ด้วยการใส่ สสารพลังงานพิเศษเข้ามาในทฤษฎี ซึ่งโดยทั่วไปแล้ว เรียกสสารพลังงานพิเศษนี้ว่า “พลังงานมืด” (dark energy) [3] แบบจำลองพลังงานมืดที่ว่านี้ได้ทำการศึกษาวิจัยการอย่างกว้างขวางไม่ว่าจะเป็นการตีความให้ สนามสเกลาร์ สนามเวกเตอร์ หรือสนามทรีฟอร์ม เป็นพลังงานมืด ส่วนหนึ่งของโครงการวิจัยนี้ได้มุ่งเน้นที่จะศึกษาพลังงานมืดจากสนามทรีฟอร์มที่ทำให้ทั่วไป สำหรับส่วนที่สองของโครงการวิจัยนี้ได้มุ่งเน้นที่จะอธิบายการขยายตัวด้วยความเร่งของเอกภพโดยการปรับเปลี่ยนทฤษฎีสัมพัทธภาพของไอน์สไตน์แทนการใส่สสารพลังงานพิเศษเข้ามาในทฤษฎี ซึ่งโดยทั่วไปเรียกแบบจำลองลักษณะนี้ว่า แบบจำลองโมดิฟรายกราวิตี (modified gravity) [4] ในทำนองเดียวกันกับแบบจำลองพลังงานมืด แบบจำลองโมดิฟรายกราวิตีนี้ก็ได้ศึกษาวิจัยกันอย่างแพร่หลาย อาทิเช่น แบบจำลองความโน้มถ่วงแบบ$f(R)$แบบจำลองสเกลาเทนเซอร์ แบบจำลองเวกเตอร์เทนเซอร์ แบบจำลองความโน้นถ่วงในมิติที่สูงกว่า หรือ แบบจำลองแมสสีพกราวิตี ซึ่งในส่วนนี้ผู้วิจัยได้สนใจศึกษาค้นคว้าในแบบจำลองแมสสีพกราวิตีแบบดีอาร์จีทีที่คู่ควบกับสนามสเกลาร์แบบเคเอสเซนซ์ (k-essence scalar field) ทั้งนี้ ผู้วิจัยได้แยกการพิจารณาโดยละเอียดของทั้งสองส่วนดังต่อไปนี้

2. พลังงานมืดจากสนามทรีฟอร์มที่ทำให้ทั่วไปและความหมายในเชิงของไหล

แรงจูงใจในการสร้างแบบจำลองทางจักรวาลวิทยาจากสนามทรีฟอร์มนั้นไม่ได้มาจากการที่ต้องการจะอธิบายผลจากการสังเกตุการณ์เท่านั้น แต่ยังมาจากเหตุผลเชิงทฤษฎีด้วย กล่าวคือ มีความเป็นไปได้ที่สนามทรีฟอร์มจะสามารถเกิดขึ้นได้จากทฤษฎีพื้นฐานอย่างเช่นทฤษฎีสตริงหรือทฤษฎีความโน้มถ่วงในมิติที่สูงกว่า ด้วยเหตุนี้ แบบจำลองทางจักรวาลวิทยาจากสนามทรีฟอร์มไม่ว่าจะเป็นแบบจำลองอินเฟลชันหรือแบบจำลองพลังงานมืดนั้นได้ทำการศึกษาวิจัยกันอย่างแพร่หลาย [5-11]แต่อย่างไรก็ตาม แบบจำลองเหล่านั้นยังศึกษาเฉพาะสนามทรีฟอร์มแบบคาโนนิคอล (canonical) ดังนั้นโครงงานวิจัยนี้จึงมุ่งเน้นที่จะศึกษาสนามทรีฟอร์มที่ทำให้ทั่วไปมากขึ้น

2.1 แบบจำลองที่นำเสนอและสมการพื้นหลัง

สำหรับการศึกษาแบบจำลองทางจักรวาลวิทยาจากสนามสเกลาร์ เราสามารถทำให้ทฤษฎีมีความทั่วไปมากขึ้นได้ โดยพิจารณาลากรางจ์ (Lagrangian) เป็นฟังก์ชันใดๆ ที่ขึ้นกับทั้งพจน์จลน์และพจน์ศักย์ แทนสนามสเกลาร์แบบคาโนนิคอล หรือที่เรียกโดยทั่วไปว่า สนามสเกลาร์แบบเคเอสเซนซ์ (k-essence) [12-14] โดยเทียบเคียงกับแบบจำลองพลังงานมืดจากสนามสเกลาร์แบบเคเอสเซนซ์นี้ เราสามารถพิจารณาแบบจำลองพลังงานมืดจากสนามทรีฟอร์มที่ทำให้ทั่วไปได้โดยการพิจารณา แอคชันที่มีลากรางจ์ เป็นฟังก์ชันใดๆ ที่ขึ้นกับทั้งพจน์จลน์และพจน์ศักย์ แทนสนามทรีฟอร์มแบบคาโนนิคอลกล่าวคือ

$$S=\int\sqrt{-g}d^4xP(K,y),$$

โดยที่ $K=-\frac{1}{48}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}$ คือพจน์จลน์แบบคาโนนิคอล $F_{\mu\nu\rho\sigma}=\nabla_{[\mu}A_{\nu\rho\sigma]}$และ $y=A_{\mu\nu\rho}A^{\mu\nu\rho}/12$ คือปริมาณสเกลาร์ของสนามทรีฟอร์ม $A_{\mu\nu\rho}$จากแอกชันนี้ เราสามารถหาสมการการเคลื่อนที่ และ เทนเซอร์พลังงาน-โมเมนตัม (energy-momentum tensor) ได้ดังนี้ตามลำดับ

$$\nabla_\mu\left(P_{,K}F^{\mu\nu\rho\sigma}\right)+P_{,y}A^{\nu\rho\sigma}=0,$$
$$T_{\mu\nu}=\frac{1}{6}P_{,K}F_{\mu\alpha\beta\gamma}F_{\nu}^{\alpha\beta\gamma}-\frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_{\nu}^{\rho\sigma}+Pg_{\mu\nu}$$
โดยที่ดัชนีล่างที่มีสัญลักษณ์ “,”นั้นหมายถึงการทำอนุพันธ์เทียบกับปริมาณที่เขียนตามสัญลักษณ์นั้น กล่าวคือ $P_{,X}=\frac{\partial P}{\partial X}$เมื่อตีความให้สนามทรีฟอร์มนี้เป็นของไหลในอุดมคติเราจะสามารถคำนวนหาความหนาแน่นพลังงานและความดันของของไหลนี้ได้โดยที่รายละเอียดการคำนวนสามารถดูได้จากภาคผนวก 1 จากผลที่ได้นี้เราสามารถเขียนความสัมพันธ์ของฟังก์ชัน P กับพารามิเตอร์ของสมการสถานะ w ได้ดังนี้
$$2yP_{,y}+(1+w)2KP_{,K}=(1+w)P$$
พารามิเตอร์ของสมการสถานะเป็นปริมาณที่มีความสำคัญอย่างยิ่งในการพิจารณาคุณสมบัติของของไหลซึ่งจะบ่งบอกถึงลักษณะการวิวัฒน์ของเอกภพได้กล่าวคือ ถ้า $w=0$ ของไหลจะมีพฤติกรรมเป็นแบบไม่สัมพัทธภาพ หรือเรียกอีกชื่อหนึ่งว่า ฝุ่น (dust) ถ้า $w=1/3$ ของไหลจะมีพฤติกรรมเป็นแบบสัมพัทธภาพ หรือเรียกอีกชื่อหนึ่งว่า รังสี (radiation) และถ้าต้องการให้เอกภพขยายตัวด้วยความเร่งพารามิเตอร์ของสมการสถานะต้องมีค่าเป็น $w<-1/3$ ดังนั้น เราสามารถวิเคราะห์คุณสมบัติของแบบจำลองนี้ได้จากการคำนวนหาความเป็นได้ของพารามิเตอร์ของสมการสถานะนั่นเอง ในที่นี้ เพื่อความสะดวกในการคำนวน เราสามารถจำกัดให้wคงที่ได้ซึ่งผลเฉลยของสมการเชิงอนุพันธ์ย่อยข้างบนสามารถเขียนได้เป็น
$$P=P_0K^\nu y^\mu,\qquad w=-1+\frac{2\mu}{1-2\nu},\qquad \nu\neq\frac{1}{2}$$
โดยที่P_0คือค่าคงที่การอินทิเกรต และ μ,ν คือค่าคงที่ใดๆ จากผลเฉลยนี้ จะเห็นได้ว่า เราสามารถเลือกค่า μ,νใดๆก็ได้เพื่อทำให้ $w=0$ โดยที่ P ยังคงหาค่าได้ และนี่คือข้อได้เปรียบของสนามทรีฟอร์มเมื่อเทียบกับสนามสเกลาร์ กล่าวคือ สำหรับสนามสเกลาร์แบบเคเอสเซนซ์นั้น เมื่อให้ $w=0$ แล้ว P จะหาค่าไม่ได้ โดยที่ $P=K_\phi^{\frac{1+w_\phi}{2w_\phi}}$ [15] (รายละเอียดการคำนวนสำหรับสนามสเกลาร์แบบเคเอสเซนส์นั้นสามารถดูได้จากภาคผนวก 1) นอกจากนี้จะเห็นได้ว่า ถ้า $\mu=0$ แล้ว $w=-1$นั่นคือเอกภพจะขยายตัวด้วยความเร่งได้ อย่างไรก็ตาม ในกรณีนี้แบบจำลองจะไม่ต่างกันกับแบบจำลองพลังงานมืดจากค่าคงที่จักรวาลซึ่งได้ศึกษาอย่างแพร่หลายมากนานแล้ว เราจึงไม่สนใจในกรณีนี้

สำหรับในกรณีที่ wไม่คงที่นั้น เราสามารถกำหนดให้มันขึ้นกับค่าของสนามทรีฟอร์มได้ $w=w(y)$และเพื่อความสะดวก เราอาจจะพิจารณาในกรณีที่สามารถหาผลเฉลยของ Pได้โดยง่ายโดยให้ $w=-1+\lambda y$โดยที่ λ คือค่าคงที่ใดๆ ซึ่งผลเฉลยสามารถเขียนได้ดังนี้
$$P=P_0K^\nu e^{\left(\frac{1-2\nu}{2}\right)\lambda y}$$
จากผลเฉลยนี้รวมทั้งรูปแบบของพารามิเตอร์ของสมการสถานะนี้ จะเห็นได้ว่า มีความเป็นได้ที่จะตีความให้สนามทรีฟอร์มเป็นพลังงานมืดเพื่อขับดันการขยายตัวด้วยความเร่งของเอกภพโดยที่ให้ค่า λ มีค่าน้อยและค่าของสนามทรีฟอร์ม y มีการวิวัฒน์จากมีค่ามากไปหาค่าน้อยๆ ยิ่งไปกว่านั้น จะเห็นได้ว่า เรามีโอกาสที่จะกำหนดให้ yวิวัฒน์ไปโดยที่ให้ wเปลี่ยนค่าจาก $w=0$ ไปเป็น $w=-1$ ซึ่งนี้จะนำไปสู่การรวมเป็นหนึ่งเดียวของทั้งพลังงานมืดและสสารมืด ทั้งการศึกษาความเป็นไปได้นี้สามารถทำได้โดยการวิเคราะห์เชิงพลศาสตร์ของระบบสมการซึ่งการคำนวนนี้ค่อนข้างยาวและซับซ้อน ผู้วิจัยจึงละไว้สำหรับงานวิจัยอื่นในลำดับต่อไป

2.2เพอเทอร์เบชันและเสียรภาพ

การวิเคราะห์สเถียรภาพของทฤษฎีทำได้โดยการทำเพอเทอร์เบชันอันดับสองที่แอคชัน การคำนวนนี้ประกอบด้วยสมการค่อนข้างยาว รายละเลียดของสมการที่คำนวนจึงไม่นำมาแสดง ณ ที่นี้แต่ผู้อ่านสามารถดูได้จากภาคผนวก 1 โดยสรุปแล้ว เราจะได้เงื่อนไขของการมีเสถียรภาพมาจากการพิจารณาพจน์จลน์ของสนามเพอเทอร์เบชันซึ่งถ้าพจน์นี้มีเครื่องหมายถูกต้องแสดงว่าทฤษฎีจะไม่มีความไม่เสถียรแบบโกสท์ (ghost) และ พจน์ที่ให้ความเร็วเสียงของการกระจายc_s (sound speed of propagation) ซึ่งโดยทั่วไปแล้วคำนวนได้จาก $c_s^2\geq 0$ จากการคำนวนเราได้ว่า สองเงื่อนไขนี้คือ

$$P_{,y} < 0, \qquad c_s^2 = 1 + \frac{2yP_{,yy}}{P_{,y}} - \frac{4KyP_{,Ky}^2}{P_{,y}\left(P_{,K} + 2KP_{,KK}\right)} \geq 0$$

จากสองเงื่อนไขนี้ จะเห็นได้ว่า ในกรณีของ $w = 0$ทฤษฎีจะมีสเถียรภาพ และ ยังสามารถตีความเป็นสสารแบบไม่สัมพัทธภาพถึงในระดับเพอเทอร์เบชัน อย่างไรก็ตาม สำหรับกรณี $w = -1 + \lambda y$ เราพบว่าทฤษฎีจะมีสเถียรภาพภายใต้เงื่อนไขดังต่อไปนี้ $P_0\lambda(2\nu - 1) > 0,\ c_s^2 = 1 + \lambda y \geq 0$ ซึ่งจะเห็นได้ว่า ในกรณีที่ λมีค่าน้อยๆและ yวิวัฒน์ไปโดยที่ w เปลี่ยนค่าจาก $w = 0$ ไปเป็น $w = -1$ นั้นทฤษฎียังคงมีสเถียรภาพได้ ซึ่งบอกเป็นนัยว่า แบบจำลองสำหรับพลังงานมืดจากสนามทรีฟอร์มที่ทำให้ทั่วไปนี้มีสเถียรภาพ

อีกหนึ่งปริมาณที่สำคัญเมื่อเปรียบเทียบผลกับในกรณีของสนามสเกลาร์คือความเป็นไปได้ที่จะมีการกระเพื่อมแบบไม่อะเดียแบติค (nonadiabatic perturbations) ซึ่งปริมาณนี้สามารถคำนวนได้จากผลต่างของความเร็วเสียงแบบอะเดียแบติค c_a (adiabatic sound speed) กับ ความเร็วเสียงของการกระจาย c_s โดยที่ถ้าความเร็วเสียงทั้งสองแบบนี้ไม่เท่ากัน การกระเพื่อมแบบไม่อะเดียแบติคจะสามารถเกิดขึ้นได้[16] จากการคำนวนความเร็วเสียงแบบอะเดียแบติค เราจะได้

$$c_a^2 = c_s^2 + \frac{4\sqrt{Ky}P_{,Ky}}{P_{,y}\dot{y} - 2\sqrt{Ky}P_{,Ky}}\left(1 + \frac{yP_{,yy}}{P_{,y}} - \frac{yP_{,y}^2 - 2KyP_{,Ky}^2}{P_{,y}\left(P_{,K} + 2KP_{,KK}\right)}\right)$$

นั่นแสดงให้เห็นว่า การกระเพื่อมแบบไม่อะเดียแบติคนั้นสามารถเกิดขึ้นได้ในกรณีของสนามทรีฟอร์ม ซึ่งต่างกันกับในกรณีของสนามสเกลาร์ซึ่งการกระเพื่อมแบบไม่อะเดียแบติคไม่สามารถเกิดขึ้นได้เนื่องจาก $c_a^2 = c_s^2$ และนี้คืออีกข้อได้เปรียบหนึ่งของสนามทรีฟอร์มเมื่อเทียบกับสนามสเกลาร์ ทั้งนี้โดยทั่วไปแล้ว ของไหลใดๆนั้นจะให้การกระเพื่อมแบบไม่อะเดียแบติคได้

2.3การบรรยายเชิงอุณหพลศาสตร์

สิ่งสำคัญประการหนึ่งสำหรับการศึกษาแบบจำลองทางจักรวาลวิทยาคือการวิเคราะห์คุณสมบัติของของไหลในแบบจำลองนั้น ทั้งนี้ ที่ระดับสเกลของจักรวาลวิทยานั้น องค์ประกอบในเอกภพจะสามารถตีความเป็นของไหลในอุดมคติเพื่อให้สอดคล้องกับหลักการทางจักรวาลวิทยา หรือ สอดคล้องกลับผลการสังเกตุการณ์ ดังนั้นในหัวข้อนี้จึงได้มุ่งเน้นเพื่อหาคุณสมบัติของของไหลในเชิงอุณหพลศาสตร์ในแบบจำลองที่สร้างจากสนามทรีฟอร์มที่ทำให้ทั่วไป การวิเคราะห์คุณสมบัติของของไหลนั้นสามารถทำได้โดยคำนวนหาปริมานทางอุณหพลศาสตร์ จากนั้นจึงเขียนสมการการเคลื่อนที่ของสนามให้อยู่ในรูปของปริมาณเหล่านี้ แล้วจึงเปรียบเทียบกับสมการทางอุณพลศาสตร์

สองปริมาณสำคัญทางอุณหพลศาสตร์ที่สามารถคำนวนได้จากเทนเซอร์พลังงาน-โมเมนตัมของสนามทรีฟอร์มที่ทำให้ทั่วไปกับเทนเซอร์พลังงานโมเมนตัมของของไหลในอุดมคติ คือ ความหนาแน่นพลังงาน ρ และความดัน p ซึ่งสามารถเขียนได้ดังนี้ $\rho = 2KP_{,K} - P,\ p = P - 2KP_{,K} - 2yP_{,y} = -\rho - 2yP_{,y}$ อีกสองปริมาณสำคัญที่สามารถคำนวนได้ คือ ความเร็วสี่มิติ (four-velocity) u^μและความหนาแน่นจำนวนอนุภาค (number density) n จากการคำนวนเราพบว่า

$$u^\mu = \frac{\epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}}{6\sqrt{2y}}, \qquad n = \sqrt{2y}P_{,y}$$

ทั้งนี้วิธีการคำนวนเพื่อให้ได้มาซึ่งสองปริมาณนี้ นั้นสามารถดูได้จากภาคผนวก 1 จากสองปริมาณนี้ เราสามารถหาเวกเตอร์ฟลักซ์อนุภาค (particle flux vector) j^μได้ดังนี้

$$j^\mu = \sqrt{-g}nu^\mu = \sqrt{-g}P_{,y}\frac{\epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}}{6}$$

จะเห็นได้ว่า จากสมการการเคลื่อนที่ของสนามทรีฟอร์มนั้น โดยทั่วไปแล้ว $\partial_\mu j^\mu \neq 0$นั่นคือ ฟลักซ์ของความหนาแน่นอนุภาคจะไม่อนุรักษ์นั่นเอง ซึ่งการไม่อนุรักษ์นี้จะนำไปสู่การการไม่อนุรักษ์ของความหนาแน่นเอ็นโทรปีของของไหลตามการไหลของมัน นอกจากนี้แล้ว การไม่อนุรักษ์นี้ยังนำไปสู่การไม่อนุรักษ์ของจำนวนอนุภาคของระบบในเชิงอุณหพลศาสตร์ด้วย ทั้งนี้ การไม่อนุรักษ์เหล่านี้ไม่ได้หมายความถึงว่าระบบของสนามทรีฟอร์มนั้นไม่อนุกรักษ์เทนเซอร์พลังงาน-โมเมนตัมแต่อย่างใด ทฤษฎีนี้ยังคงให้การอนุรักษ์ของเทน

เซอร์พลังงานโมเมนตัม $\nabla_\mu T^{\mu\nu} = 0$ การไม่อนุรักษ์เหล่านี้เกิดขึ้นมาจากการที่แอคชันไม่มีสมมาตรการเลือนของสนามเหมือนในกรณีของสเกลาร์เคเอสเซนซ์ ซึ่งนี้เป็นของแตกต่างอย่างหนึ่งระหว่าง สนามทรีฟอร์มที่ทำให้ทั่วไปกับสนามสเกลาร์เคเอสเซนซ์ ข้อได้เปรียบอย่างหนึ่งสำหรับสนามทรีฟอร์มนี้คือ เราสามารถใช้สนามทรีฟอร์มนี้เป็นตัวแทนของสสารแบบไม่สัมพัทธภาพในเอกภพเพื่อคำนวนและวิเคราะห์แบบจำลองที่พลังงานมืดและสสารมืดมีอัตรกิริยาต่อกัน ซึ่งหัวข้อนี้กำลังเป็นที่น่าสนใจในการศึกษาเพราะการมีอันตรกิริยาต่อกันนี้จะสามารถนำไปสู่การแก้ปัญหาความบังเอิญเชิงจักรวาลวิทยาได้

3. แบบจำลองแมสสีพกราวิตีแบบดีอาร์จีทีที่คู่ควบกับสนามสเกลาร์แบบเคเอสเซนซ์

ดังที่ได้กล่าวไปก่อนหน้านี้ การศึกษาแบบจำลองที่พลังงานมืดที่มีอันตรกิริยากับสสารมืดนั้นได้ทำกันอย่างแพร่หลาย ทั้งนี้เป้าหมายหลักอย่างหนึ่งคือการหาความเป็นไปได้ที่จะแก้ปัญหาความบังเอิญเชิงจักรวาลวิทยาที่กล่าวว่า “ทำไมอัตราส่วนของพลังงานมืดต่อสสารแบบไม่สัมพัทธภาพจึงมีค่าพอๆกัน ณ ปัจจุบัน ทั้งๆที่การวิวัฒน์ของทั้งสองนั้นต่างกันอย่างมโหฬาร” การศึกษาแบบจำลองที่พลังงานมืดมีอันตรกิริยากับสสารมืดนั้นจำเป็นจะต้องพิจารณาสสารมืดที่ซึ่งเป็นสสารแบบไม่สัมพัทธภาพในระดับแอคชัน อย่างไรก็ตาม ลากรางจ์ของสสารมืดนั้นยังไม่เป็นที่รู้จักกันอย่างแพร่หลาย จากทฤษฎีสนาม ตัวแทนอย่างง่ายที่พอจะใช้แทนสสารมืดคือ สนามสเกลาร์แบบเคเอสเซนซ์ ถึงแม้ว่า สนามสเกลานี้ยังมีข้อบกพร่องอยู่บ้างก็ตาม เช่น ลากรางจ์จะหาค่าไม่ได้ แต่ในการศึกษานี้ก็ยังเป็นที่นิยมใช้กันอยู่ ในหัวข้อนี้ ผู้วิจัยได้นำเสนอแบบจำลองโมดิฟรายกราวิตีแบบดีอาร์จีทีที่คู่ควบกับสนามสเกลาร์แบบเคเอสเซนซ์ ซึ่งสนามสเกลาร์นี้อาจจะถูกตีความไปเป็นสสารแบบไม่สัมพัทธภาพต่อไป

3.1 แบบจำลองแมสสีพกราวิตีแบบมวลแปรผัน

ทฤษฎีแมสสีพกราวิตีเป็นทฤษฎีโมดิฟรายกราวิตีอย่างหนึ่งซึ่งให้พจน์มวลเพิ่มเข้าไปในทฤษฎีสัมพัทธภาพทั่วไปของไอน์สไตน์ การสร้างทฤษฎีแมสสีพกราวิตีนี้ได้นำเสนอมาตั้งแต่ปี ค.ศ 1939 โดย เฟียร์ซและเพาลี [17]ทฤษฎีเริ่มแรกนี้ประสบความสำเร็จโดยเพิ่มพจน์มวลในระดับเชิงเส้นเข้าไปในทฤษฎี แต่อย่างไรก็ตาม ผลการทำนายปรากฏการณ์ทางดาราศาสตร์ให้ผลไม่ถูกต้องทฤษฎีนี้จึงไม่ได้รับความสนใจ[18, 19] จนกระทั่งมีความพยายามที่จะเพิ่มมวลในระดับที่ไม่เชิงเส้นเข้าไปในทฤษฎี [20] ถึงแม้ว่า การเพิ่มมวลไม่เชิงเส้นนี้ (หรือเรียกทั่วไปว่า ทฤษฎีแมสสีพกราวิตีแบบไม่เชิงเส้น) จะทำให้ผลการทำนายตรงกับผลการสังเกตุการณ์แต่โดยทั่วไปแล้วทฤษฎีจะไม่มีสเถียรภาพ[21] ในปี ค.ศ 2010 เดอราห์ม กาบาแดดเซ และ ทอลเลย์ (de RhamGabadadze and Tolley) ได้เสนอทฤษฎีแมสสีพกราวิตีแบบไม่เชิงเส้นที่มีเสถียรภาพขึ้นซึ่งทฤษฎีนี้ถูกเรียกภายหลังตามบุคคลที่นำเสนอว่า แมสสีพกราวิตีแบบดีอาร์จีที (dRGT massive gravity) [22, 23] การศึกษาทฤษฎีแมสสีพกราวิตีในเชิงจักรวาลวิทยาได้ทำวิจัยกันอย่างแพร่หลาย ทั้งนี้ เพื่อศึกษาความเป็นไปได้ที่จะอธิบายการขยายตัวด้วยความเร่งของเอกภพ จากการศึกษาเหล่านี้ อาจจะพอสรุปได้ว่า ถึงแม้ว่าทฤษฎีนี้จะนำไปใช้อธิบายการขยายตัวด้วยความเร่งของเอกภพได้ก็ตาม แต่ ทฤษฎีนี้จะยังคงไม่สมบูรณ์ [24] กล่าวคือ จำนวนดีกรีของความอิสระ(number of degree of freedom) ที่ได้ยังไม่สอดคล้องในเชิงทฤษฎีกับความหมายในเชิงฟิสิกส์อนุภาคซึ่ง กราวิตอนแบบมีมวล (massive graviton) จะต้องมีจำนวนดีกรีของความอิสระเท่ากับ 5 หนึ่งในการแก้ปัญหานี้คือ การทำให้มวลของกราวิตอนนี้เปลี่ยนค่าได้ [25, 26] ซึ่งเรียกทฤษฎีนี้ว่า ทฤษฎีแมสสีพกราวิตีแบบมวลแปรผัน (mass-varying massive gravity) แอคชันของทฤษฎีนี้เขียนได้ดังนี้

$$S = \int \sqrt{-g}d^4x \left(\frac{M_p^2}{2}R(g) + V(\phi)U(g,f) + P(\phi) \right)$$

โดยที่ $P(\phi) = -\frac{1}{2}\left(\nabla_\mu \phi\right)^2 - W(\phi)$คือลากราจ์ของสนามสเกลาร์แบบคาโนนิคอล $V(\phi)$คือฟังก์ชันที่สื่อถึงมวลของกราวิตอน $R(g)$คือ สเกลาร์ริคชี่ ซึ่งเป็นส่วนของทฤษฎีสัมพัทธภาพทั่วไปของไอน์สไตน์ และ $U(g,f)$คือ พจน์ที่มวลไม่เชิงเส้นที่เพิ่มเข้ามาซึ่งเขียนได้ดังนี้

$$U(g,f) = U_2 + \alpha_3 U_3 + \alpha_4 U_4$$

$$U_2 = \frac{1}{2}([K]^2 - [K^2]), \qquad U_3 = \frac{1}{6}([K]^3 - 3[K][K^2] + [K^3])$$

$$U_4 = \frac{1}{24}([K]^4 - 6[K]^2[K^2] + 3[K^2]^2 + 8[K][K^3] - 6[K^4])$$

โดยที่ $K^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_\nu$ ซึ่ง $g_{\mu\nu}$คือเมทริคเชิงกายภาพ (physical metric) $f_{\mu\nu}$ คือเมทริคตัวช่วย (fiducial metric) รากที่สองของเทนเซอร์นิยามได้ดังนี้ $\left(\sqrt{g^{-1}f}\right)^\mu_\rho\left(\sqrt{g^{-1}f}\right)^\rho_\nu = g^{\mu\rho}f_{\rho\nu}$และ สัญลักษณ์ $[X]$ หมายถึง เทรซ (trace) ของเทนเซอร์ $X_{\mu\nu}$ข้อสังเกตุสำคัญอย่างหนึ่งในแอคชันนี้คือ ในพจน์มวลของกราวิตอนนั้นจะขึ้นกับเมทริคตัวช่วยด้วย ซึ่งเมทริคตัวช่วยนี้ประพฤติตัวเหมือนกับลากรางจ์มัลติพลายเออร์ (Lagrange multiplier) ช่วยกำจัดโกสท์ซึ่งทำให้ทฤษฎีไม่มีสเถียรภาพ นอกจากนี้แล้ว $f_{\mu\nu}$ ยังประกอบไปด้วย สตัคเคลเบิร์กสเกลาร์ (struckelberg scalar) ซึ่งช่วยให้ทฤษฎีกลับมามีสมมาตรภายใต้การแปลงพิกัดทั่วไป สิ่งสำคัญอีกประการหนึ่งสำหรับ $f_{\mu\nu}$คือเราสามารถเลือกรูปแบบของมันได้เพื่อให้ง่ายและเหมาะสมกับการพิจารณาเมทริคกายภาพ การศึกษาแบบจำลองพลังงานมืดจากทฤษฎีโดยที่เลือกให้ เมทริคตัวช่วยนี้มีรูปแบบเป็นมินโควสกี (Minkowski) พบว่า จำนวนดีกรีความอิสระนั้นถูกต้องและเอกภพสามารถขยายตัวด้วยความเร่งได้ อย่างไรก็ตาม เราพบว่า การที่เอกภพขยายตัวด้วยความเร่งนั้นไม่ได้เกิดจากมวลของกราวิตอน แต่เกิดจากการที่เราใส่สนามสเกลาร์เพิ่มเข้าไปในทฤษฎี ซึ่งนี่เป็นข้อเสียของทฤษฎีนี้ เนื่องจากแบบจำลองนี้ไม่ได้แตกต่างกับแบบจำลองพลังงานมืดของสนามสเกลาร์แบบคาโนนิคอลที่ศึกษามาก่อนหน้านี้แล้ว ในโครงงานวิจัยนี้จึงมุ่งเน้นที่จะแก้ปัญหานี้พร้อม ๆกับหาความเป็นไปได้ที่จะแก้ปัญหาความบังเอิญเชิงจักรวาลวิทยาด้วย ทั้งนี้ผู้วิจัยได้ปรับเปลี่ยนทฤษฎีนี้สองประเด็นหลัก ๆคือ 1. เลือกเมทริคตัวช่วยให้เหมาะสม 2. ทำให้ฟังก์ชันมวลและสนามสเกลาร์มีความทั่วไปมากขึ้นโดยการพิจารณา สนามสเกลาร์แบบเคเอสเซนซ์ ซึ่งจะถกในรายละเอียดประเด็นนี้ในหัวข้อต่อไป

3.2 แบบจำลองแมสสีพกราวิตีแบบดีอาร์จีทีที่คู่ควบกับสนามสเกลาร์เคเอสเซนซ์

ตามที่ได้กล่าวไปในหัวข้อก่อนหน้านี้ โครงการวิจัยนี้มุ่งเน้นที่จะปรับเปลี่ยนแบบจำลองแมสสีพกราวิตีแบบมวลแปรผันให้มีความทั่วไปมากขึ้น โดยประเด็นแรกที่จะทำการปรับเปลี่ยนคือ การเลือกเมทริคตัวช่วยให้มีความเหมาะสม เนื่องด้วยงานวิจัยก่อนหน้านี้เลือกใช้เมทริคตัวช่วยในรูปแบบ มินโควสกี แต่เมทริคกายภาพนั้นเป็นอยู่ในรูปแบบของ Friedmann-Lermate-Rebertson-Walker (FLRW) เนื่องจากเมทริคตัวช่วยนั้นทำหน้าที่เหมือนกันเมทริคอ้างอิงเมื่อเทียบกับ เมทริคกายภาพ ดังนั้น เมทริคตัวช่วยนี้ควรจะมีรูปแบบที่คล้าย หรือ เหมือนกับกันกับเมทริคกายภาพ ดังนั้นในงานวิจัยนี้เราจะเลือก $f_{\mu\nu}$ให้อยู่ในรูปแบบ FLRW แบบแบนราบได้ดังนี้

$$f_{\mu\nu} = diag\left(-n^2(t), \alpha^2(t), \alpha^2(t), \alpha^2(t)\right)$$

โดยที่ n คือฟังก์ชันใด ๆ และ α คือ สเกลแฟคเตอร์เสมือน สำหรับเมทริคนี้ ทั้งนี้ รูปแบบเมทริคนี้ได้มาโดยการเลือกเกจแบบยูนิทารี สำหรับเมทริคกายภาพนั้นรูปแบบของมันได้มาจากเงื่อนไขของหลักการทางจักรวาลวิทยา ดังนี้

$$g_{\mu\nu} = diag\left(-N^2(t), a^2(t), a^2(t), a^2(t)\right)$$

โดยที่ N คือฟังก์ชันใด ๆ โดยทั่วไปเรียกว่า ฟังก์ชันแลพ (Lape function) และ a คือ สเกลแฟคเตอร์

สำหรับประเด็นที่สองนั้น เราจะทำให้สนามสเกลาร์นี้มีความทั่วไปมากขึ้นโดยการเปลี่ยน $P(\phi)$ให้มีความทั่วไปมากขึ้น $P(\phi) \to P(X,\phi)$โดยที่ $X = -\frac{1}{2}\left(\nabla_\mu \phi\right)^2$ คือพจน์จลน์ หรืออีกนัยหนึ่งคือ เปลี่ยนจากสนามสเกลาร์แบบคาโนนิคอล ไปเป็นสนามสเกลาร์แบบเคเอสเซนซ์ และปรับเปลี่ยนมวลของกราวิตอนให้มีความทั่วไปมากขึ้น $V(\phi) \to V(X,\phi)$ ดังนั้นแอกชันสำหรับแบบจำลองนี้สามารถเขียนใหม่ได้เป็น

$$S=\int\sqrt{-g}d^4x\left(\frac{M_p^2}{2}R(g)+V(X,\phi)U(g,f)+P(X,\phi)\right)$$

จากแอคชันนี้ เราสามารถคำนวนหาสมการการเคลื่อนที่ได้จากทั้งการทำแวริเอชัน (variation) ของแอคชันเทียบกับทั้งเมทริคกายภาพและเมทริคตัวช่วย อย่างไรก็ตามสมการที่ได้นั้นยาวและซับซ้อนมาก จึงไม่ได้นำสมการทั้งหมดมาแสดง ณ ที่นี้ แต่สามารถดูได้จาก ภาคผนวก 2 ในที่จะแสดงเฉพาะสมการสำคัญที่ใช้วิเคราะห์พลวัฒน์ของระบบ โดยสมการที่น่าสนใจสมการหนึ่งคือ

$$\dot{\rho}_X+3HN\rho_X=\frac{\dot{X}}{2X}\rho_X$$

โดยที่ $H=\frac{\dot{a}}{a}$ คือ พารามิเตอร์ฮับเบิล (Hubble parameter) และ $\rho_X=2XP_{,X}+6XV_{,X}(F-G\eta)$ เมื่อ $\eta=\frac{n}{N}$

$$F=\left(2+\frac{4}{3}\alpha_3+\frac{1}{3}\alpha_4\right)-(3+3\alpha_3+\alpha_4)\bar{X}+(1+2\alpha_3+\alpha_4)\bar{X}^2-\frac{1}{3}(\alpha_3+\alpha_4)\bar{X}^3$$

$$G=\frac{1}{3}(3+3\alpha_3+\alpha_4)-(1+2\alpha_3+\alpha_4)\bar{X}+(\alpha_3+\alpha_4)\bar{X}^2-\frac{1}{3}\alpha_4\bar{X}^3,\qquad \bar{X}=\frac{\alpha}{a}$$

จากสมการข้างบนนั้น เราสามารถหาผลเฉลยของสมการได้ดังนี้

$$\rho_X=C\frac{\sqrt{2X}}{a^3}$$

โดยที่ Cคือค่าคงที่การอินทิเกรต จากผลเฉลยนี้ เราจะเห็นได้ว่า ρ_X นี้มีพฤติกรรมคล้ายกับสสารแบบไม่สัมพัทธภาพ ในกรณีที่ Xคงที่ จะได้ว่า ρ_Xนี้คือสสารแบบไม่สัมพัทธภาพ เนื่องจาก ρ_Xประกอบด้วยทั้งส่วนที่มาจาก สนามสเกลาร์ P และมวลของกราวิตอน Vดังนั้นเราสามารถพิจารณาเฉพาะส่วนที่ขึ้นกับมวลของกราวิตอนได้ ซึ่งในกรณีนี้บ่งบอกเราได้ว่า ผลบางส่วนของมวลของกราวิตอนสามารถวิวัฒน์เหมือนกับสสารแบบไม่สัมพัทธภาพได้ ซึ่งสามารถตีความไปเป็นสสารมืดได้

นอกจากนี้แล้ว เรายังสามารถพิจารณาผลทั้งหมดที่เกิดจากมวลของกราวิตอนได้โดยเขียนความหนาแน่นพลังงานยังผลและความดันยังผลได้ดังนี้

$$\rho_g=-3VF+6XV_{,X}(F-G\eta)$$
$$p_g=3VF+VF_{,\bar{X}}(\bar{X}-\eta)$$

จากการวิเคราะห์สมการอนุพันธ์ของ ρ_g ซึ่งรายละเอียดอยู่ในภาคผนวก 2 จะได้ว่า การวิวัฒน์ของความหนาแน่นพลังงานนี้ให้ผลเฉลยที่เอกภพสามารถขยายตัวด้วยความเร่งได้ ดังนั้น จากการวิเคราะห์เบื้องต้นนี้จะเห็นได้ว่า มวลของกราวิตอนนั้นสามารถให้คุณลักษณะของทั้งพลังงานมืด และสสารมืดได้ นี่จึงเป็นหนึ่งข้อดีสำคัญของแบบจำลองนี้ คือ เราสามารถอธิบาย สสาร พลังงาน ที่เราไม่รู้จักได้ทั้งสองอย่างได้ด้วยเพียงมวลของกราวิตอนเท่านั้น ในหัวข้อถัดไปพฤติกรรมนี้จะถูกวิเคราะห์โดยละเอียดโดยใช้การวิเคราะห์แบบพลวัฒน์ ซึ่งการวิเคราะห์แบบนี้จะทำให้เราสามารถเข้าใจถึงความเป็นไปได้ที่จะแก้ปัญหาความบังเอิญทางจักรวาลวิทยาด้วย

3.2ระบบสมการพลวัฒน์

การวิเคราะห์ระบบสมการพลวัฒน์หรือเรียกอีกอย่างหนึ่งว่า ระบบสมการออโตโนมัติ (autonomous equation system)จะทำให้เรามองเห็นภาพของการวิวัฒน์ของเอกภพได้อย่างชัดเจน หลักการของระบบสมการนี้คือการเปลี่ยนตัวแปรพลวัฒน์ใหม่โดยให้ระบบสมการนั้นเป็นระบบสมการอนุพันธ์อันดับหนึ่ง จากสมการเหล่านี้เราจะสามารถคำนวนจุดวิกฤตหรือจุดตรึง (fixed point) ของระบบได้ ซึ่งโดยทั่วไปแล้วระบบจะวิวัฒน์ผ่านจุดตรึงนี้ ถ้าจุดตรึงนั้นมีเสถียรภาพ ระบบจะวิวัฒน์เข้าหาจุดนั้นเสมอ ซึ่งหลักการณ์นี้จะนำเราไปสู่วิถีทางที่จะแก้ปัญหาความบังเอิญทางจักรวาลวิทยา กล่าวคือ ถ้าจุดตรึงนั้นเป็นจุดที่เสถียร และให้อัตราส่วนของพลังงานมืดต่อสสารแบบไม่สัมพัทธภาพมีค่าโดยประมาณเป็น 7 ต่อ 3 ดังนั้นระบบจะวิวัฒน์เข้าจุดนี้เสมอ นั่นคือ ไม่ว่าพลังงานมืด กับ สสารแบบไม่สัมพัทธภาพ จะวิวัฒน์ต่างกันแค่ไหน มันจะวิวัฒน์มาจุดนี้เสมอ ซึ่งจะสามารถตอบคำถามของปัญหาความบังเอิญได้ จากที่ได้กล่าวไปข้างต้นนั้น ลำดับแรก เราจะต้องเปลี่ยนตัวแปรพลวัฒน์ให้เหมาะสม ซึ่งสามารถเขียนได้ดังนี้

$$x=-\frac{FV}{M_p^2H^2},\qquad y=\frac{\rho_X}{3M_p^2H^2},\quad z=-\frac{P}{3M_p^2H^2},\qquad \Omega_r=\frac{\rho_r}{3M_p^2H^2}$$

โดยที่ρ_r คือ ความหนาแน่นพลังงานของรังสี เราสามาถนิยามพารามิเตอร์ใหม่เพื่อความสะดวกในการเขียนสมการดังนี้

$$\lambda=-\frac{2XV_{,X}}{V},\qquad r=\frac{G\eta}{F},\qquad s=-\frac{F_{,\bar{X}}(\bar{X}-\eta)}{3F},\qquad \gamma=-\frac{2XP_{,X}}{P},\qquad \Gamma=\frac{XV_{,XX}}{V_{,X}}$$

จากตัวแปรพลวัฒน์เหล่านี้ สมการการเคลื่อนที่ทั้งหมดสามารถเขียนให้อยู่ในรูปสมการออโตโนมัติได้ดังนี้

$$x'=3x\left(y+sx-\frac{s}{r}+\frac{4}{3}\Omega_r\right)$$

$$y'=3y\left(y+sx-1-\frac{s}{\lambda r}+\frac{4}{3}\Omega_r\right)$$

$$\Omega'_r=3\Omega_r\left(y+sx+\frac{4}{3}(\Omega_r-1)\right)$$

$$\lambda'=\frac{6s}{r}\left(\frac{\lambda}{2}-(1+\Gamma)\right)$$

โดยที่มีสมการการจำกัดอยู่สองสมการคือ $x+y+z+\Omega_r=1$และ $y=-\lambda x(1-r)-z$ข้อสังเกตุคือตัวแปรที่เป็นผลมาจากมวลของกราวิตอนคือ xและ y ตัวแปร z คือผลจากสนามสเกลาร์เคเอสเซนซ์ ลำดับต่อไปคือการคำนวนหาจุดตรึงและเสถียรภาพของจุดนั้น ซึ่งสามารถสรุปได้ในตารางดังต่อไปนี้

Name	x	y	z	w_{eff}	existence	stability
(a)	0	0	1	-1	$\gamma=0$	$0\le\frac{s}{r}\le 1$
(b)	$\frac{1}{r}$	0	$1-\frac{1}{r}$	$-1+\frac{s}{r}$	$\gamma=\lambda$	$\frac{\lambda}{1-\lambda}\le\frac{s}{r}\le 0$
(c)	0	$1+\frac{s}{\lambda r}$	$-\frac{s}{\lambda r}$	$\frac{s}{\lambda r}$	$\gamma=1+\frac{\lambda r}{s}$	$\frac{\lambda}{1-\lambda}<\frac{s}{r}<-1$
(d)	$\frac{1}{1+\lambda(r-1)}$	$\frac{\lambda(r-1)}{1+\lambda(r-1)}$	0	$\frac{1}{\lambda-1}$	$\lambda=\frac{s}{s-r}$	$0<\lambda<1$
(e)	$\frac{1+(\lambda-1)z_0}{1+\lambda(r-1)}$	$-\frac{\lambda(1-r(z_0+1))}{1+\lambda(r-1)}$	z_0	$\frac{1}{\lambda-1}$	$\lambda=\gamma=\frac{s}{s-r}$	$0<\lambda<1$

ทั้งนี้ในตารางไม่ได้รวมค่าของ Ω_rเข้ามาด้วย แต่อย่างไรก็ตาม การวิเคราะห์ยังสามารถทำได้โดยทุกจุดที่แสดงนี้เป็นจุดที่ $\Omega_r=0$และเราสามารถเพิ่มอีกจุดเข้ามาได้โดยที่เป็นจุด $\Omega_r=1$ และตัวแปรอื่นมีค่าเป็นศูนย์หมด ซึ่งจุดนี้เป็นจุดที่ไม่เสถียรและสอดคล้องกับเอกภพในยุคที่รังสีเด่น จากนี้ เราสามารถวิเคราะห์ความสำคัญของแต่ละจุดได้โดยพิจารณาของข้อมูลในตารางนี้ สำหรับจุด (a) เป็นจุดที่พลังงานมืดเด่นซึ่งเห็นได้จาก $w_{eff}=-1$ แต่อย่างไรก็ตาม จุดนี้เป็นจุดที่ xและ y เป็นศูนย์ ดังนั้น พลังงานมืดที่เด่นนี้เป็นผลเนื่องมาจากสนามสเกลาร์เคเอสเซนซ์ สำหรับจุด (b) สามารถตีความเป็นจุดที่พลังงานมืดเด่นได้ โดยให้ $r\sim 1$และ $s\sim -0$ และจุดนี้จะเสถียรก็ต่อเมื่อ $\lambda>1$ สำหรับจุด (c)นี้สามารถตีความเป็นจุดที่สสารแบบไม่สัมพัทธภาพเด่นได้โดยที่ $\frac{s}{\lambda r}\sim -0$ซึ่งจุดนี้ $y\sim 1$ นั่นคือสสารแบบไม่สัมพัทธภาพนี้เกิดจากมวลของกราวิตอนนั่นเอง เนื่องจากจุดนี้เป็นจุดที่สอดคล้องกับจุดในอดีต เราจึงจำเป็นจะต้องให้จุดนี้ เป็นจุดที่ไม่เสถียร ซึ่งเงื่อนไขที่จุดนี้จะไม่เสถียรคือ $0<\lambda<1$ สำหรับจุด (d) เป็นจุดที่ $z=0$ และเป็นจุดที่พลังงานมืดเด่นได้ภายใต้เงื่อนไข $\lambda\sim 0$จากจุด (c) และจุด (d)นี้ เราสามารถเซตให้ $z=0$ ได้ซึ่งหมายความว่า เราไม่จำเป็นจะต้องมีสนามสเกลาร์เคเอสเซนซ์ เราก็สามารถอธิบายการวิวัฒน์ของเอกภพได้ตามมาตรฐานจากผลการสังเกตุการณ์ได้ ซึ่งนี่สอดคล้องกับที่เราวิเคราะห์ไว้ก่อนหน้านี้ว่า มวลของกราวิตอนสามารถประพฤติตัวเป็นได้ทั้งพลังงานมืด และ สสารมืด อย่างไรก็ตาม สำหรับจุดนี้เราไม่สามารถแก้ปัญหาความบังเอิญได้เพราะที่จุดตรึง (d) เป็นจุดที่ $x+y=1$ เมื่อพิจารณาจุด (e) จะเห็นได้ว่าจุดนี้เป็นจุดพลังงานมืดเด่นที่มีโอกาสจะเป็นจุดที่แก้ปัญหาความบังเอิญได้ ภายใต้เงื่อนไข $\lambda\sim 0$ การวิเคราะห์นี้ยังสามารถยืนยันความถูกต้องได้โดยใช้การคำนวนเชิงตัวเลข ซึ่งรายละเอียดนั้นสามารถดูได้จากภาคผนวก 2

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ภาคผนวก 1

A Perfect Fluid in Lagrangian Formulation due to Generalized Three-Form Field

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A Perfect Fluid in Lagrangian Formulation due to Generalized Three-Form Field

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Abstract. A Lagrangian formulation of perfect fluid due to a non-canonical three-form field is investigated. The thermodynamic quantities such as energy density, pressure and the four-velocity are obtained and then analyzed by comparing with the k-essence scalar field. The non-relativistic matter due to the generalized three-form field with the equation of state parameter being zero is realized while it might not be possible for the k-essence scalar field. We also found that non-adiabatic pressure perturbations can be possibly generated. The fluid dynamics of the perfect fluid due to the three-form field corresponds to the system in which the number of particles is not conserved. We argue that it is interesting to use this three-form field to represent the dark matter for the interaction theory between dark matter and dark energy.

Keywords: Three-Form Field, Perfect Fluid, Lagrangian Formulation

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1 Introduction

A theory of cosmological perturbations is one of important issues in cosmology nowadays. It provides us to understand how astronomical structures at large scales are generated and evolve. Also, it can provide us the resulting signatures of the theoretical model to compare with observational data. The theory of cosmological perturbations for a perfect fluid has been developed and studied intensively at the level of equations of motion, for example, a study of the perturbed Einstein field equations together with the equation of conservation of energy momentum tensor [1, 2]. Beside the cosmological perturbations at the level of the equations of motion, a study of the cosmological perturbations at the Lagrangian level has been investigated. The advantage point of the study at Lagrangian level is that it is useful to find the perturbed dynamical field as well as derive closed evolution equations. This can be clearly seen by considering the cosmological perturbations in $f(R, G)$ gravity theories where there are two dynamical fields for scalar perturbations [3, 4]. For the study in Lagrangian approach, one can straightforwardly identify which fields are dynamical or auxiliary and then immediately obtain the closed evolution equations.

A Lagrangian formulation for a perfect fluid in general relativity has been constructed and developed for a long time [5–7]. The Lagrangian of the fluid is simply written as its pressure [6] or energy density [7]. The advantage point of this formulation is that it naturally provides a consistent way to construct a covariant theory for dark energy and dark matter coupling. The study of dark energy and dark matter coupling has been widely investigated in order to describe a way out from the cosmic coincidence problem [8–12]. Moreover, the observation also provide a hint for the existence of the coupling [13]. However, in order to recover the standard thermodynamics equations, the Lagrangian must involve at least five independent functions. Even though this formulation can provide a consistent way for studying the perfect fluid in cosmology and is well known as a standard approach for the perfect fluid at the Lagrangian level, there might be disadvantage for this approach since the theory involves too many functions.

A simple Lagrangian approach for the perfect fluid has been investigated by using a non-canonical scalar field [14], namely k-essence field [15–17]. It was found that the k-essence scalar field can provide a description of the perfect fluid with constant equation of state parameter. Moreover, it was found that the cosmological perturbations of this kind of the scalar field is equivalent to those in perfect fluid. However, it cannot be properly used to describe a non-relativistic matter with the equation of state parameter being zero since the Lagrangian is not finite. It was also found that the non-adiabatic pressure perturbations cannot be generated [18] as well as a vector mode of the perturbations cannot be produced [19].

Beside the cosmological models due to the scalar field, a three-form field can be successfully used to describe both inflationary models and dark energy models [20–30]. Even though there is a duality between scalar field and three-form field, the cosmological models are significantly differed in both background and perturbation levels. At the perturbation level, it is obvious to see that the three-form field can generate intrinsic vector perturbations while it is not possible for the scalar field. Therefore, it might be worthy to find an equivalence between the three-form field with a perfect fluid. In the present work, by mimicking the k-essence scalar field, we consider a generalized version of the three-form field and then find a possible Lagrangian form to describe the perfect fluid at the background level. We found that a simple power-law of the canonical kinetic term can provide the constant equation of state parameter like in the case of k-essence. The advantage point of the three-form field compare with the scalar field is that it can provide a consistent description of the non-relativistic matter field where its equation of state parameter satisfies $w = 0$. The stability issue is also investigated and found that the non-relativistic matter field due to the three-form field is free-from ghost and Laplacian instabilities.

By using the equations of motion of the generalized three-form field, the thermodynamic quantities are identified and found that the perfect fluid due to the three-from field corresponds to fluid in which the number of particles is not conserved. By analyzing the speed of propagation of scalar perturbations and the adiabatic sound speed, we found that the non-adiabatic perturbations can be possibly generated. We argue that it is interesting to use this three-form field to represent the dark matter for the interaction theory between dark matter and dark energy.

This paper is organized as follows. In section 2, we propose a general form of the three-form field and then find the equation of motion as well as the energy momentum tensor. By working in FLRW metric, the energy density and the pressure as well as the equation of state parameter are found. Some specific forms of the Lagrangian satisfying the equations of motion are obtained and found that it can represent the non-relativistic matter. We also investigate the stability issue by using the perturbed action at second order in section 3. We found conditions to avoid ghost and Laplacian instabilities. In section 4, we investigate the thermodynamic properties of the model. We begin this section with review of some important idea of the Lagrangian formulation for the standard and k-essence scalar field and then find the thermodynamic properties due to the three-form fluid. Finally, the results are summarized and discussed in section 5.

2 Equations of motion and energy momentum tensor

Cosmological models due to a three-form field have been investigated not only in inflationary models but also dark energy models [20–30]. Moreover, at the end of inflationary period, a viable model due to the three-form field for the reheating period have been investigated [31]. A consistent mechanism to generate large scale cosmological magnetic fields by using the three-form field have been studied [32]. Recently, a generalized inflationary model by considering two three-form fields was also investigated [29]. All investigations of cosmological models due to three-form are considered only in canonical form. Since the non-canonical form of scalar field have been intensively investigated, it is interesting to investigate the cosmological model with a non-canonical form of the three-form field. In this section, we will consider a non-canonical form of the kinetic term of a three-form field, $A_{\alpha\beta\gamma}$, as follows

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R + P(K,y)\right], \tag{2.1}$$

where the kinetic term and scalar quantity of the three-form field are expressed as

$$K = -\frac{1}{48}F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}, \tag{2.2}$$
$$y = \frac{1}{12}A_{\alpha\beta\gamma}A^{\alpha\beta\gamma}, \tag{2.3}$$
$$F_{\mu\nu\rho\sigma} = \nabla_\mu A_{\nu\rho\sigma} - \nabla_\sigma A_{\mu\nu\rho} + \nabla_\rho A_{\sigma\mu\nu} - \nabla_\nu A_{\rho\sigma\mu}\,. \tag{2.4}$$

By varying the action with respect to the three-form field, the equations of motion of the three-form field can be written as

$$E_{\alpha\beta\gamma} = \nabla_\mu\left(P_{,K}F^{\mu}{}_{\alpha\beta\gamma}\right) + P_{,y}A_{\alpha\beta\gamma} = 0, \tag{2.5}$$

where the notation with subscript $P_{,x}$ denotes $P_{,x} = \partial_x P$. Due to the totally anti-symmetric property of the tensor $F_{\mu\alpha\beta\gamma}$, one found that there exist constraint equations as follows

$$\nabla_\mu\left(P_{,y}A^{\mu\alpha\beta}\right) = 0. \tag{2.6}$$

These equations suggest us that the conserved quantity is expressed in terms of three-form field. Note that for the k-essence scalar field, the conserved quantity is expressed in term of one-form or vector quantity. We will discuss on this issue in detail in section 4 where we investigate the fluid dynamics. The energy momentum tensor can be obtained by varying the action of the three-form field with respect to the metric as

$$T_{\mu\nu} = \frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} - \frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + Pg_{\mu\nu}. \tag{2.7}$$

For consistency of the derived equations, one can check that the conservation of the energy momentum tensor can be obtained up to the equation of motion as follows

$$\nabla_\mu T^\mu_\nu = \frac{1}{6}F_{\nu\alpha\beta\gamma}E^{\alpha\beta\gamma} = 0. \tag{2.8}$$

In order to capture the thermodynamics quantities such as the energy density and pressure due to the three-form field like the investigation in scalar field, let us consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) manifold whose metric element can be written as

$$ds^2 = -dt^2 + \gamma_{ij}dx^idx^j = -dt^2 + a(t)^2\delta_{ij}dx^idx^j\,. \tag{2.9}$$

By using this form of the metric and the constraint equation in Eq. (2.6), the components of the three-form field, $A_{\alpha\beta\gamma}$, can be written as

$$A_{0ij} = 0\,, \qquad A_{ijk} = \epsilon_{ijk}X(t) = \sqrt{\gamma}\varepsilon_{ijk}X(t) = a^3\varepsilon_{ijk}X(t), \tag{2.10}$$

where ε_{ijk} is the three-dimensional Levi-Civita symbol with $\varepsilon_{123} = 1$. By using this form of the metric, the components of energy momentum tensor can be expressed as

$$T^0_0 = P - 2KP_{,K}, \tag{2.11}$$
$$T^i_j = (P - 2KP_{,K} - 2yP_{,y})\delta^i_j\,. \tag{2.12}$$

By comparing these components of the energy momentum tensor of the three-form to one from the perfect fluid, the energy density and pressure of the three-form can be expressed as

$$\rho = 2KP_{,K} - P, \tag{2.13}$$
$$p = P - 2KP_{,K} - 2yP_{,y} = -\rho - 2yP_{,y}. \tag{2.14}$$

Note that we have used $y = X^2/2$ and $K = (\dot{X} + 3HX)^2/2$ where $H = \dot{a}/a$ is the Hubble parameter. From the energy density and the pressure above, the equation of state parameter of the three-form can be written as

$$w = \frac{p}{\rho} = -1 - \frac{2yP_{,y}}{\rho}. \tag{2.15}$$

The equation of motion of the three-form field in Eq. (2.5) can be written in flat FLRW background as

$$(2KP_{,KK} + P_{,K})\dot{K} + 2KP_{,yK}\dot{y} - 2\sqrt{K\,y}P_{,y} = 0. \tag{2.16}$$

From this point, one can check validity of the derived equations by reducing the general form of the action to the canonical one as setting $P = K - V(y)$. As a result, we found that all equations can be reduced to the canonical one investigated in [20–30]. Substituting ρ from Eq. (2.13) into Eq. (2.15), one obtains

$$2yP_{,y} + (1 + w)2KP_{,K} = (1 + w)P. \tag{2.17}$$

In order to find the form of P, one has to solve this equation. It is useful to solve this equation by considering a simple assumption such as taking the equation of state parameter to be a constant, $w = \text{const}$. By using separation of variable method, the solution can be written as

$$P = P_0K^\nu y^\mu, \tag{2.18}$$

where P_0 is an integration constant and μ, ν are the exponent constants obeying the relation

$$\nu = \frac{1 + w - 2\mu}{2(1 + w)}, \text{or } \; w = -1 + \frac{2\mu}{1 - 2\nu}, \quad \nu \neq \frac{1}{2}. \tag{2.19}$$

This form of the solution is very useful since one can interpret the three-form field as a non-relativistic matter or dark matter by setting the equation of state parameter as $w = 0$ while it cannot be properly used for k-essence scalar field case. We will show explicitly why we cannot properly use k-essence scalar field for the non-relativistic matter in section 4. In order to study the covariant coupling form between dark matter and dark energy as suggested from the observation [13], one can use the three-form as the dark matter with the consistent covariant interaction forms. Moreover, it may be interpreted as dark radiation by setting $w = 1/3$. Note that, in the case of $\nu = 1/2$, it corresponds to the trivial solution since the energy density of the field vanishes. It is important to note that the late-time acceleration of the unverse can also be achieved by setting $w = -1$. Even though this may not be distinguished to the cosmological constant at the background level, the cosmological perturbations due to this model of the three-form can be significantly deviated from the model of the cosmological constant.

Since the form of the Lagrangian P is obtained by assuming a constant equation of state parameter, the dark energy model from this three-form field cannot be proposed to solve the coincidence problem. One may allow the equation of state to be varying in order to overcome this issue. One of interesting solutions is assuming that the equation of state parameter depends on the three-form field $w = w(y)$. In order to solve Eq. (2.17) to obtain a suitable form of P, one may choose the equation of state parameter such as $w = -1 + \lambda y$, where λ is a constant. As a result, the solution can be written as

$$P = P_0 K^\nu e^{\frac{(1-2\nu)}{2}\lambda y}. \tag{2.20}$$

Naively, it is not difficult to obtain the dynamical dark energy due to the generalized three-form. One can set λ be effectively small and find the condition to provide an evolution of y such that it evolves from a large value to a small value. However, since it is not in the canonical form, the theory may be suffered from instabilities. In this work, the stability issue will be investigated in the next section. The investigation of the dark energy model due to the generalized three-form is left in further work.

3 Stability

In order to capture the stability conditions of the generalized three-form field, we may consider the perturbations of the field. Since the field minimally couples to the gravity, one has to take into account the metric perturbations. However, for simplicity but useful study, we will investigate the stabilities of the model only in a high-momentum limit. This will capture only some stability conditions. Nevertheless, this includes most of the necessary conditions as found in the canonical three-form field [27]. We leave the full investigation in further work where the cosmological perturbations are taken into account. For this purpose, the metric is held fixed as the Minkowski metric and the three-form field can be written as

$$A_{ijk} = \varepsilon_{ijk}(X(t) + \alpha(t,\vec{x})), \tag{3.1}$$
$$A_{0ij} = \varepsilon_{ijk}(\partial_k\beta(t,\vec{x}) + \beta_k(t,\vec{x})), \tag{3.2}$$

where α and β are perturbed scalar fields and β_k is a transverse vector obeying the relation $\partial_k\beta^k = 0$. This vector field will be responsible for the intrinsic vector perturbation of the three-form field. For the linear perturbations, the scalar and vector modes are decoupled and then they can be separately investigated. For the scalar modes, by expanding the action up to second order in the field, the second order action can be written as

$$S^{(2)} = \int d^4x \Big(\frac{1}{2}\frac{\dot{Q}^2}{(P_{,K} + 2KP_{,KK})} - \frac{1}{2}P_{,y}(\partial\beta)^2 + \frac{1}{2}P_{,y}c_s^2\alpha^2\Big), \tag{3.3}$$
$$\dot{Q} = (P_{,K} + 2KP_{,KK})\dot{\alpha} + 2\sqrt{K\,y}P_{,K}P_{,y}\alpha - (P_{,K} + 2KP_{,KK})\partial^2\beta, \tag{3.4}$$
$$c_s^2 = 1 + \frac{2yP_{,yy}}{P_{,y}} - \frac{4KyP_{,Ky}^2}{P_{,y}\,(2KP_{,KK} + P_{,K})}. \tag{3.5}$$

One can see that the field β is non-dynamical so that one can eliminate it by using its equation of motion. By applying the Euler-Lagrange equation to the above action, the equation of motion for the field β can be written as

$$(P_{,K} + 2KP_{,KK})\dot{\alpha} + 2\sqrt{K\,y}P_{,Ky}\alpha - (P_{,K} + 2KP_{,KK})\partial^2\beta - P_{,y}\beta = 0, \qquad (3.6)$$

From this equation of motion, we can replace the quantity $\dot{Q}$ as $\dot{Q} = P_{,y}\beta$. Note that this equation can be obtained by using the component $(0, i, j)$ of the covariant equation in Eq. (2.5). In order to find the solution for β, it is convenient to work in Fourier space so that the above equation can be algebraically solved. As a result, by substituting the solution of β into the action in Eq. (3.3), the second order action for the scalar perturbations can be rewritten as

$$S^{(2)} = \int dtd^3k\Big(F_1\dot{\alpha}^2 + F_2\dot{\alpha}\alpha + F_3\alpha^2\Big), \qquad (3.7)$$

where

$$F_1 = -\frac{P_{,y}\left(2KP_{,KK} + P_{,K}\right)}{2\left(k^2\left(2KP_{,KK} + P_{,K}\right) - P_{,y}\right)}, \qquad (3.8)$$

$$F_2 = -\frac{2\sqrt{K\,y}P_{,K\,y}P_{,y}}{\left(k^2\left(2KP_{,KK} + P_{,K}\right) - P_{,y}\right)}, \qquad (3.9)$$

$$F_3 = \frac{\left(2yP_{,yy} + P_{y}\right)\left(2k^2KP_{,KK} + k^2P_{,K} - P_{,y}\right) - 4k^2KyP_{,K\,y}^2}{2\left(k^2\left(2KP_{,KK} + P_{,K}\right) - P_{,y}\right)}. \qquad (3.10)$$

As we have discussed above, we will consider the stability conditions at high-momentum limit. Therefore, by taking the limit $k^2 \to \infty$, the second order action becomes

$$S^{(2)} = \int dtd^3k\,k^{-2}(-P_{,y})\Big(\frac{1}{2}\dot{\alpha}^2 - \frac{1}{2}k^2c_s^2\alpha^2 - \frac{1}{2}m_A^2\alpha^2\Big). \qquad (3.11)$$

where

$$m_A^2 = \frac{d}{dt}\Big(\frac{2\sqrt{K\,y}P_{,Ky}}{(P_{,K} + 2KP_{,KK})}\Big) - \frac{4K\,y\,P_{,Ky}^2}{(P_{,K} + 2KP_{,KK})^2}. \qquad (3.12)$$

Therefore, the condition to avoid ghost instabilities can be written as

$$P_{,y} < 0. \qquad (3.13)$$

This condition can be reduced to the canonical case by taking $P = K - V(y)$, which provides the result as $V_{,y} > 0$ consistently with the result in [27]. In order to avoid the Laplacian instability, one requires $c_s^2 \geq 0$ leading to the condition

$$1 + \frac{2yP_{,yy}}{P_{,y}} - \frac{4KyP_{,Ky}^2}{P_{,y}\left(2KP_{,KK} + P_{,K}\right)} \geq 0. \qquad (3.14)$$

To obtain a clear picture of this condition, one may specify the form of P. For the form with constant equation of state parameter, $P = P_0K^{\nu}y^{\mu}$, the sound speed square can be expressed as $c_s^2 = w$. Therefore, the three-form field can be interpreted as the non-relativistic matter up to a perturbation level since $c_s^2 = 0$ and $w = 0$. Moreover, it is obvious that the non-relativistic matter represented by the generalized three-form field is free from ghost and Laplacian instabilities. Note that the dark energy model with $w < -1/3$ for this form of the Lagrangian is suffered from Laplacian instabilities since the sound speed square is negative.

For another simple form of the Lagrangian with $P = P_0 K^\nu e^{\frac{1-2\nu}{2}\lambda y}$, the sound speed square and the equation of state parameter read $c_s^2 = 1 + \lambda y$ and $w = -1 + \lambda y$. From these expressions, one can see that the phantom expansion of the universe will provide a superluminality. The no-ghost condition can be expressed as $P_0\lambda(2\nu - 1) > 0$. At this point, it is possible to obtain a viable model of dark energy due to the generalized three-form field.

Now we will consider the vector mode of the perturbations by following the same step as in the scalar one. As a result, the second order action for the vector perturbations can be written as

$$S^{(2)} = \int d^4x \Big(-\frac{1}{2} P_{,y} \beta_i \beta^i \Big). \tag{3.15}$$

From this action, one can see that the vector mode does not propagate. A condition to avoid the instabilities coincides with the condition obtained in scalar mode.

In order to find possibility to obtain non-adiabatic perturbations due to the three-form field, one may find a difference between the speed of propagation of scalar perturbations, c_s^2, and the adiabatic sound speed, c_a^2. If these two kinds of the sound speed are equal, there are no non-adiabatic perturbations while it provides the possibility to generate non-adiabatic perturbations if they are not equal [18]. The speed of propagation of scalar perturbations is found in Eq. (3.5). For the adiabatic sound speed, one can derived as follows

$$c_a^2 \equiv \frac{\dot{p}}{\dot{\rho}} = 1 + 2\frac{yP_{,yy}\dot{y} + P_{,Ky}(y\dot{K} + 2\sqrt{Ky})}{P_{,y}\dot{y} - 2\sqrt{Ky}P_{,Ky}}, \tag{3.16}$$

$$= c_s^2 + \frac{4\sqrt{Ky}P_{,Ky}}{P_{,y}\dot{y} - 2\sqrt{Ky}P_{,Ky}} \left(1 + \frac{yP_{,yy}}{P_{,y}} + \frac{yP_{,y}^2 - 2KyP_{,Ky}}{P_{,y}(P_{,K} + 2KP_{,KK})} \right). \tag{3.17}$$

From this equation, one can see that the sound speed of scalar perturbations and the adiabatic sound speed are not generally equal. Therefore, it is possible to generate non-adiabatic perturbations from the generalized three-form field. This is one of advantage points of the generalized three-form field compare with the k-essence scalar field. Note that both kinds of the sound speed will coincide when the Lagrangian does not depend on y, $P = P(K)$. For this case, the non-adiabatic perturbations cannot be generated.

4 Fluid dynamics due to three-from field

In order to compare the results with the standard description of the fluid dynamics for the perfect fluid, let us briefly review an important concept of the standard version for the fluid dynamics. Since the perfect fluid dynamics due to the non-canonical scalar filed or k-essence field has been intensively investigated and interpreted as non-relativistic mater field, for example, in the case of massive gravity theory [33, 34], we will also review some important results of the k-essence scalar field before we discuss further on the three-form field.

4.1 Standard version and k-essence field

There are many approaches of the standard version for the perfect fluid Lagrangian. We will use Brown formulation [7] since it is more useful and has been widely used for recent studies in dark energy and dark matter couplings [9–12]. The Lagrangian of the perfect fluid can be written in terms of the energy density with Lagrange multipliers as

$$S_m = \int d^4x \left(-\sqrt{-g}\,\rho + j^\mu(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A\alpha^A_{,\mu})\right), \tag{4.1}$$

where $\rho = \rho(n,s)$ is the energy density of the fluid, n is a particle number density, s is an entropy density per particle and j^μ are components of the particle number flux. The second term which is contracted with j^μ is the Lagrange multiplier term with the Lagrange multiplier fields φ, θ and β_A where α_A are the Lagrangian coordinates of the fluid with index A running as $1, 2, 3$. j^μ can be written in terms of the four-velocity u^μ of the fluid as

$$j^\mu = \sqrt{-g}\,n\,u^\mu. \tag{4.2}$$

The four-velocity satisfies the relation $u_\mu u^\mu = -1$ where $n = |j|/\sqrt{-g}$ and $|j| = \sqrt{-j^\mu g_{\mu\nu} j^\nu}$. The standard energy momentum tensor of the perfect fluid can be obtained by varying the action with respect to the metric $g_{\mu\nu}$ as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\,g_{\mu\nu}, \tag{4.3}$$

where p is the pressure of the fluid defined as

$$p \equiv n\frac{\partial\rho}{\partial n} - \rho. \tag{4.4}$$

By varying the action with respect to the Lagrange multiplier fields θ and φ, the first law of Thermodynamics and the conservation of the particle number can be obtained respectively [7] as

$$dp = n d\mu - T ds, \tag{4.5}$$
$$\partial_\nu j^\nu = 0. \tag{4.6}$$

where T is a temperature and μ is a chemical potential defined as

$$\mu \equiv \frac{\rho + p}{n}. \tag{4.7}$$

From these equations of motion together with the conservation of the energy momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, all main thermodynamics equations can be obtained. For example, conservation of the entropy density can be obtained by using a projection of the conservation equation of the energy momentum tensor along the fluid flow as follows

$$u_\nu \nabla_\mu T^{\mu\nu} = -\frac{\mu}{\sqrt{-g}}\partial_\nu j^\nu - u^\nu T\partial_\nu s = 0. \tag{4.8}$$

From these equations, in the viewpoint of field theory, all main thermodynamics equations can be obtained if one can identify the main thermodynamics quantities in terms of the field such as energy density, pressure, four-velocity and chemical potential which give the form of energy momentum tensor as found in Eq. (4.3). We will show this procedure for instruction in the case of scalar field.

For the k-essence scalar field, we will follow [14] in which action of the k-essence field can be written as

$$S_\phi = \int d^4x\sqrt{-g}P(K_\phi), \tag{4.9}$$

where $K_\phi = -\nabla_\mu\phi\nabla^\mu\phi/2$ is the canonical kinetic term of the scalar field. The corresponding equations of motion of the scalar field can be expressed as

$$\nabla_\mu\Big(P'\nabla^\mu\phi\Big) = 0, \tag{4.10}$$

where prime denotes the derivative with respect to K_ϕ. The energy momentum tensor of the scalar field can be written as

$$T_{\mu\nu} = P'\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}P. \tag{4.11}$$

By comparing this energy momentum tensor with that in the perfect fluid in Eq. (4.3), the energy density, pressure and the four-velocity can be identified as

$$\rho_\phi = 2K_\phi P' - P, \tag{4.12}$$

$$p_\phi = P, \tag{4.13}$$

$$u^\mu = \frac{\nabla^\mu\phi}{\sqrt{2K_\phi}} \tag{4.14}$$

Therefore, the particle number density can be obtained in order to satisfy the conservation of the particle flux as $n_\phi = \sqrt{2K_\phi}P'$ while the chemical potential reads $\mu_\phi = \sqrt{2K_\phi}$. Therefore, one can check that the equation of motion in Eq. (4.10) satisfies the equation of the conservation of the particle flux as follows

$$\sqrt{-g}\nabla_\mu\Big(P'\nabla^\mu\phi\Big) = \partial_\mu\Big(\sqrt{-g}P'\nabla^\mu\phi\Big) = \partial_\mu\Big(\sqrt{-g}n_\phi u^\mu\Big) = \partial_\mu j^\mu_\phi = 0. \tag{4.15}$$

As a result, all fluid dynamics equations can be derived by using the results in the standard version. Note that the first law of thermodynamics is adopted for the scalar field while in the case of the standard version, it is obtained from the equation of motion. It is important to note that the conservation of the particle flux does not hold if we generalize the Lagrangian of the scalar field as $P = P(K_\phi, \phi)$ since the equations of motion in Eq. (4.10) becomes $\nabla_\mu\Big(P'\nabla^\mu\phi\Big) = -\partial P/\partial\phi$. This is not so surprisingly since the simple scalar field, such as quintessence field, is also equivalent to the system in which the particle flux is not conserved. This can be explicitly seen by taking $P = K_\phi - V(\phi)$.

By taking the equation of state parameter to be constant, the form of the Lagrangian obeys a relation

$$P(1 + w_\phi) = 2w_\phi K_\phi P'. \tag{4.16}$$

From this equation, one can find the exact form of the Lagrangian as

$$P = P_0 K_\phi^{\frac{1+w_\phi}{2w_\phi}}, \quad \text{where} \quad w_\phi \neq 0. \tag{4.17}$$

It is obviously that one cannot properly use this form of the scalar field to describe the non-relativistic matter since its equation of state parameter is zero, $w = 0$. This is one of drawbacks for the k-essence scalar field. As we have shown before, this does not happen in the case of generalized three-form field.

4.2 Generalized three-form field

As we have mentioned, one can find the equivalence between the energy momentum tensor of the three-form and the standard perfect fluid and then identify the fluid quantities such as ρ, p and the four-velocity u^μ in terms of the three-form field. By using these identifications, one can find the consequent thermodynamics equations of the three-form field as done in the scalar field case. The energy density and the pressure have been identified in Eq. (2.13) and Eq. (2.14) respectively. Now, we will identify the four-velocity of the three-form field by comparing the energy momentum tensor of the perfect fluid in Eq. (4.3) and the energy momentum tensor of the three-form in Eq. (2.7). As a result, the relation of the four-velocity and the three-form field can be written as

$$(\rho + p)u_\mu u_\nu = \frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} - \frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + (2KP_{,K} + 2yP_{,y})g_{\mu\nu}. \tag{4.18}$$

Since $F_{\mu\nu\rho\sigma}$ is a totally symmetric rank-4 tensor in 4-dimensional spacetime, it can be written in terms of a covariant tensor $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}$ where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in four-dimensional spacetime. By using the components of the three-form field in Eq. (2.10), the field strength tensor can be written as

$$F_{\mu\nu\rho\sigma} = (\dot{X} + 3HX)\epsilon_{\mu\nu\rho\sigma} = \sqrt{2K}\epsilon_{\mu\nu\rho\sigma}. \tag{4.19}$$

By using this equation, the first term in the right hand side of Eq. (4.18) can be rewritten as

$$\frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} = -2KP_{,K}g_{\mu\nu}. \tag{4.20}$$

Substituting this equation into Eq. (4.18), one obtains

$$\begin{aligned} (\rho + p)u_\mu u_\nu &= -\frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + 2yP_{,y}g_{\mu\nu}, \\ u_\mu u_\nu &= \frac{1}{4y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} - g_{\mu\nu}. \end{aligned} \tag{4.21}$$

One can check that the relation $u_\mu u^\mu = -1$ valid from this relation. Since the tensor $u_\mu u_\nu$ is constructed from two three-form fields, it plays the role of symmetric rank-2 tensor $S_{\mu\nu}$ instead of outer product of two four-velocity. Therefore, it is not trivial to find the form of the four-velocity of the three-form field. However, one may expect that the four-velocity may relate to the three-form field by the relation of the vector and the three-form in four dimensionality as $u^\mu \propto \epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}$. As a result, the four-velocity of the fluid can be written in terms of the three-form field as

$$u^\mu = \frac{\epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}}{3!\sqrt{2y}}, \tag{4.22}$$

where the three-form field can be written in terms of the four-velocity as

$$A^{\alpha\beta\gamma} = \sqrt{2y}\epsilon^{\mu\alpha\beta\gamma}u_\mu. \tag{4.23}$$

It is not trivial to find the conserved current density corresponding to three-form field. Actually, there are no conserved quantities obtained from invariance of the action under the shift of the field like the scalar field. However, one may find the conserved quantity from the constraint equation in Eq. (2.6) as follows

$$j^{\alpha\beta\gamma} = n^{\mu\alpha\beta\gamma}u_\mu = \sqrt{2y}P_{,y}\epsilon^{\mu\alpha\beta\gamma}u_\mu = P_{,y}A^{\alpha\beta\gamma}. \tag{4.24}$$

From this relation, the conserved quantity is now three-form field instead of vector field and the number density now is four-form field instead of scalar field. This equivalence comes from Hodge duality in four-dimensional spacetime. One may obtained the effective particle number density as

$$n = \sqrt{\frac{n_{\mu\alpha\beta\gamma}n^{\mu\alpha\beta\gamma}}{4!}} = \sqrt{2y}P_{,y}. \tag{4.25}$$

Therefore, the usual particle flux for the three-form field can be written as

$$j^\mu = \sqrt{-g}\,nu^\mu = \sqrt{-g}P_{,y}\frac{\epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}}{3!}. \tag{4.26}$$

This quantity does not trivially vanish due to the equation of motion in Eq. (2.16). Since $\partial_\mu j^\mu \neq 0$ together with Eq. (4.8), it is inferred that the entropy along the fluid flow is not conserved. The non-conservation of the particle flux for the three-form is due to the fact that the action is not invariant under shift of the field. In the scalar field case, the action is invariant under $\phi \to \phi + \xi$ where ξ is a constant. For general case of the scalar field with $P_\phi = P_\phi(K_\phi, \phi)$, this symmetry is also broken and then its dynamics will corresponds to the non-conservation of the particle flux like in the three-form case. For the three-form, if we restrict our attention to the case where $P = P(K)$ which is invariant under shift of the field, the particle number density, $n \propto \rho + p \propto P_{,y}$, will always vanish. Also, the equation of state parameter is always equal to -1 which cannot be responsible for the non-relativistic matter.

We also observe that condition of non-conservation of the entropy density along the fluid flow coincides with the condition of generation of non-adiabatic perturbations even though these conditions come from different approach. The conservation of the entropy density is derived from background equation while non-adiabatic perturbations are properties of the fluid at perturbation level. This argument also hold in both scalar field and three-form field cases. Therefore, this may shed light on the interplay between conserved quantities under shift of the filed and non-adiabatic perturbations.

Since the thermodynamics description for the generalized three-form field corresponds to the system in which the particle number is not conserved, it implies that the field may interact with other fields and then cause the non-conservation. It is important to note that the conservation of the energy momentum tensor of the three-form still valid, $\nabla_\mu T^\mu_\nu = 0$. The non-conservation quantities mentioned above are the thermodynamically effective quantities. Therefore, the interaction of the three-from field to the other fields is implied only in the description of the thermodynamical sense. As we have mentioned, the useful point of this three-form field is that it can represent the non-relativistic matter field with $w = 0$. Therefore, one may interpret it as dark matter. Since the particle number density is not conserved, it is worthwhile to investigate an interaction of this field to the dark energy. This may be useful approach for studies of dark energy and dark matter coupling since one can find the covariant interaction terms at the Lagrangian level and then the resulting closed evolution equations are obtained. This issue is of interest and we leave this detailed investigations for further work.

5 Summary

A Lagrangian formulation of perfect fluid is a powerful tool to study dynamics of the universe, especially interacting approach between dark energy and dark matter. A general description in this formulation invokes many functions and then it is not easy to handle. A k-essence scalar field can be used to describe the dynamics of the perfect fluid in cosmology. At the background level, even though the k-essence scalar field can be use to describe the perfect fluid with constant equation of state parameter, it cannot properly used for the non-relativistic matter with $w_\phi = 0$. At the perturbation level, the k-essence scalar field cannot provide non-adiabatic perturbations as well as intrinsic vector perturbations.

In the present paper, we propose an alternative way to alleviate these problems by using a generalized three-form field. The investigation is begun with proposing a general form of the action of the three-form field with a function depending on both the kinetic term and the field, $P = P(K, y)$, similarly to the k-essence scalar field. Equations of motion and energy momentum tensor of the three-form field in covariant form have been calculated. By working in FLRW background, the energy density and the pressure as well as the equation of state parameter are found. For the constant equation of state parameter, an exact form of the Lagrangian reads $P = P_0 K^\nu y^\mu$ where $w = -1 + \frac{2\mu}{1-2\nu}$ and $\nu \neq 1/2$. Therefore, one can set $w = 0$ by choosing proper values of the parameters μ and ν and then use the generalized three-form field to represent the non-relativistic matter. For non-constant equation of state parameter, we also point out that it is possible to construct an alternative model of dark energy. The stability analysis of the model is also performed. We found the conditions to avoid ghost and Laplacian instabilities. For the fluid with $w = 0$, it is free from ghost and Laplacian instabilities. For some specific model of dark energy, we argue that, to avoid the superluminality, the equation of state parameter must be greater than -1. In other words, the viable model of dark energy from the generalized three-form field cannot provide the phantom phase of the universe. Note that the no-ghost condition we found in this paper can be trusted only in the high momentum limit. We leave the full investigation for further work where we investigate the cosmological perturbations and observational constraint.

Thermodynamics properties due to the generalized three-form field are also investigated. It is found that this model corresponds to a system with non-conservation of the particle flux. This leads to a non-conservation of the entropy density along the fluid flow. This is not so surprisingly since many models of dark energy, for example quintessence model, also correspond to the non-conservation of the particle flux. We also found some links between non-conservation of the entropy density along the fluid flow which is a thermodynamically effective quantity at the background level and the generation of non-adiabatic perturbations which is a property of the model at perturbation level. This may shed light on the interplay between conserved quantities under shift of the filed and non-adiabatic perturbations. We can argue that this is an useful approach for a study of dark energy and dark matter coupling since one can find the covariant interaction terms at the Lagrangian level and then the resulting closed evolution equations are obtained. This issue is of interest and we leave this detailed investigations for further work.

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Regular Article - Theoretical Physics

Mass-varying massive gravity with k-essence

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Abstract For a large class of mass-varying massive-gravity models, the graviton mass cannot provide the late-time cosmic expansion of the universe due to its vanishing at late time. In this work, we propose a new class of mass-varying massive gravity models, in which the graviton mass varies according to a kinetic term of a k-essence field. By using a more general form of the fiducial metric, we found a solution such that a non-vanishing graviton mass can drive the accelerated expansion of the universe at late time. We also perform dynamical analyses of such a model and find that without introducing the k-essence Lagrangian, the graviton mass can be responsible for both dark contents of the universe, namely dark energy, which drives the accelerated expansion of the universe, and non-relativistic matter, which plays the role of dark matter. Moreover, by including the k-essence Lagrangian, we find that it is possible to alleviate the so-called cosmic coincidence problem.

1 Introduction

Massive gravity has its own series of developments as a modified gravity beyond general relativity. Back in 1939, Fierz and Pauli investigated a first model of massive gravity [1]. The model was a linearized general relativity, where the fluctuation of geometry propagates a spin-2 graviton, plus linear interactions, which, in particle physics language, corresponds to giving a non-zero mass to the graviton; hence the name "massive gravity". This model was supposed to coincide with general relativity in the massless limit but it faced a theoretical crisis when discontinuities in such a limit were found by van Dam et al. [2,3]. In particular, the discontinuities were found as different predictions between Fierz–Pauli massive gravity and general relativity. The problem remained unsolved for several years, until Vainshtein proposed a way out by introducing higher-order interactions into the Fierz–Pauli massive gravity [4]. In other words, he claimed that within a particular scale, coined the Vainshtein radius, any predictions from the linear theory cannot be trusted unless nonlinear contributions are taken into account. However, adding such nonlinearities, claimed by Boulware and Deser, not only fixes the discontinuity problem but also introduces a theoretical inconsistency, namely a Boulware–Deser ghost [5]. This ghost is an extra degree of freedom, apart from 5 degrees of freedom originally existing in the linear massive gravity, whose kinetic term has the wrong sign. The ghost problem had been a blockage for the massive-gravity theory until recently, in 2010, de Rham, Gabadadze, and Tolley found suitable nonlinear interactions which do not excite the Boulware–Deser mode; this is dubbed dRGT massive gravity [6,7]. Thus, massive gravity became again an active field of study.

Although it was just a generalization back then, massive gravity has its modern motivations. Introducing a non-zero mass to a graviton shrinks the scale at which the gravity works. In other words, the graviton mass weakens the gravitation at a large scale. As a result, it allows a cosmic acceleration and hence may be able to describe the mysterious dark energy in its language. This motivates cosmologists to study its cosmological implications. Moreover, since de Rham, Gabadadze, and Tolley found a healthy nonlinear massive gravity model, the theory had again opened a door to various researches on massive gravity; not only its cosmology but also the study of astrophysical objects in the theory, like black holes [8–13]. For cosmological models of massive gravity, it has been found that the solutions in the models with Minkowski fiducial metric do not admit the flat and closed FLRW solutions for the physical metric [14,15]. In order to obtain all kinds of FLRW solutions, one may consider a general form of the fiducial metric [16–20].

It has been found, however, that there are some inconsistencies when cosmology is taken into account. For example, some degrees of freedom cease to exist when the

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Friedmann–Lemaître–Robertson–Walker (FLRW) ansatz is assumed [19]. This leads to numerous studies beyond the dRGT massive gravity [21–39]. One of those is to generalize a constant graviton mass to be varied by other scalar field, dubbed mass-varying massive gravity [24–27]. The theory is proven to be free from a Boulware–Deser ghost. However, cosmological implications of such a model indicates a universe with subdominant contributions from massive gravity. In particular, the graviton mass is governed by the inverse of a scale factor of the universe which will vanish at late time. Consequently, such a model cannot give a proper explanation of the cosmic expansion caused by the massive graviton.

In this work, we propose an alternative way to construct a mass-varying massive gravity. The graviton mass is not only determined by a scalar field, but also by the kinetic term of the scalar field. Moreover, the scalar field is governed by a k-essence Lagrangian [40–42]. Under the FLRW ansatz, we found a solution whose the graviton mass do not necessarily vanish at late time. Moreover, by assuming both the k-essence and the graviton mass to behave as perfect fluids, we found that the graviton mass can give rise to a "dust-like" matter while combined with other contributions it is possible to have an equation of state parameter close to -1, as suggested by recent observations [43]. Such matter may be responsible for a dark matter, another mysterious content known to exist in addition to the ordinary matter. Since the graviton mass can give rise to both of the dark contents, it is tempting to consider as regards its evolution whether there exists an epoch in which the two contents in the dark sector are comparable, the so-called cosmic coincidence problem.

Our paper is organized as follows. In Sect. 2, the proposed model is addressed along with its equations of motion in the FLRW background. We also discuss some crucial properties of the model in this section where we have shown the existence of the dust-like matter expected to be responsible for the dark matter. With the help of appropriate assumptions, we show in Sect. 3 the solution to this model which corresponds to the dark energy and the non-vanishing characteristic of the graviton mass existing in this model. After sketching some perspectives, we begin the dynamical system analysis in Sect. 4 to find all possible fixed points and their stabilities, and the extended analyses are covered in Sect. 5. We conclude our work in the last section by the discussion of key ideas of our work and of whether or not the coincidence problem is alleviated.

2 The model and the background equations

We consider a mass-varying dRGT massive-gravity action where the graviton mass is varied by the k-essence field. Usually, one may consider the graviton mass as a function which varies as the scalar field propagates [24–27]. However, in this work, we will consider the graviton mass not only as a function of the scalar field ϕ but also its kinetic term $X \equiv -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. The action of such a model can be expressed as

$$S = \int d^4x\sqrt{-g}\bigg[\frac{M_p^2}{2}R[g] + V(X,\phi)(\mathcal{L}_2[g,f] + \alpha_3\mathcal{L}_3[g,f] + \alpha_4\mathcal{L}_4[g,f]) + P(X,\phi)\bigg], \quad (1)$$

where R is a Ricci scalar corresponding to a physical metric $g_{\mu\nu}$, $V(X,\phi)$ is a square of the graviton mass which depends on the scalar field and its kinetic term, $\mathcal{L}_i$ represents the interactions of the ith order of the massive graviton, and $P(X,\phi)$ is the Lagrangian of the k-essence field. In particular, those interactions of the massive graviton are constructed from two kinds of metrics and can be expressed as follows:

$$\mathcal{L}_2[g,f] = \frac{1}{2}\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right), \quad (2)$$

$$\mathcal{L}_3[g,f] = \frac{1}{3!}\left([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\right), \quad (3)$$

$$\mathcal{L}_4[g,f] = \frac{1}{4!}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]), \quad (4)$$

where the tensor $\mathcal{K}_{\mu\nu}$ is constructed from the physical metric $g_{\mu\nu}$ and an another metric $f_{\mu\nu}$ as

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_\nu, \quad (5)$$

where the square roots of those tensors are defined so that $\sqrt{g^{-1}f}^\mu{}_\rho\sqrt{g^{-1}f}^\rho{}_\nu = \left(g^{-1}f\right)^\mu{}_\nu$. In massive gravity, apart from the physical metric, there exists another kind of the metric tensor, $f_{\mu\nu}$, usually named "fiducial metric", which is an object introduced to the theory so that one can construct non-trivial interactions from the metric tensors as in Eqs. (2), (3), and (4). Those complicated combinations in the interactions, with arbitrary values of the parameters α_3, α_4, are to ensure the absence of the Boulware–Deser (ghostly) degree of freedom [6,7]. Moreover, thanks to the Stuckelberg tricks, the general covariance, or the gauge symmetry, can be well integrated into the massive gravity via

$$f_{\mu\nu} = \partial_\mu\varphi^\rho\partial_\nu\varphi^\sigma\tilde{f}_{\rho\sigma}, \quad (6)$$

provided that each of the fields φ^μ transforms as a scalar under any coordinate transformation. As for the $\tilde{f}_{ab}$, one can choose it to be any kind of metric which shares the symmetries of the physical metric. For example, one can have a four-dimensional Minkowski metric being the fiducial metric for a cosmological solution [15], or even a higher-dimensional kind of metric whose reduced four-dimensional metric is isotropic and homogeneous and is considered as the fiducial metric in the cosmological solution [20].

In this work, we consider the cosmological implications of the proposed model, where the isotropic and homogeneous universe is assumed whose spacetime is represented quite well by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric as follows:

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\Omega_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j, \tag{7}$$

where $N(t)$ is a lapse function, $a(t)$ represents a scale factor, which determines the scale of the spatial distance, and

$$\Omega_{ij}(\varphi) = \delta_{ij} + \frac{k\delta_{ia}\delta_{jb}\varphi^a\varphi^b}{1-k\delta_{lm}\varphi^l\varphi^m}, \tag{8}$$

is the spatial maximally symmetric metric whose spatial curvature is characterized by $k \in \{-1, 0, +1\}$ corresponding to the open, flat, and closed geometry, respectively. As claimed, the FLRW ansatz is also used as the fiducial metric,

$$\tilde{f}_{\mu\nu}\mathrm{d}\varphi^\mu\mathrm{d}\varphi^\nu = -n(\varphi^0)^2\left(\mathrm{d}\varphi^0\right)^2 + \alpha(\varphi^0)^2\Omega_{ij}(\varphi)\mathrm{d}\varphi^i\mathrm{d}\varphi^j, \tag{9}$$

where n and α are a lapse function and a scale factor in the fiducial sector. Plugging those in Eq. (1), the mini-superspace action of the model reads

$$S = \int \mathrm{d}^4x\sqrt{\frac{1}{1-kr^2}}\left[M_p^2\left(-3\frac{a\dot{a}^2}{N} + 3kNa\right) + 3Na^3V\left(F - G\frac{n}{N}\right) + Na^3P\right], \tag{10}$$

where

$$F \equiv \left(2 + \frac{4}{3}\alpha_3 + \frac{1}{3}\alpha_4\right) - (3 + 3\alpha_3 + \alpha_4)\bar{X} + (1 + 2\alpha_3 + \alpha_4)\bar{X}^2 - (\alpha_3 + \alpha_4)\frac{\bar{X}^3}{3}, \tag{11}$$

$$G \equiv \frac{1}{3}(3 + 3\alpha_3 + \alpha_4) - (1 + 2\alpha_3 + \alpha_4)\bar{X} + (\alpha_3 + \alpha_4)\bar{X}^2 - \alpha_4\frac{\bar{X}^3}{3}, \tag{12}$$

and we have defined

$$\bar{X} \equiv \frac{\alpha}{a}, \quad \eta \equiv \frac{n}{N}. \tag{13}$$

To determine the dynamics of the system, one can vary the action in Eq. (10) with respect to dynamical variables which are N, a, ϕ, and the Stuckelberg fields φ^μ. The corresponding equations of motion, assuming the unitary gauge $\varphi^\mu = x^\mu$ for simplicity, read

$$M_p^2\left(3H^2 + 3\frac{k}{a^2}\right) = -3VF + 6XV_{,X}(F - G\eta) + \left(2XP_{,X} - P\right), \tag{14}$$

$$M_p^2\left(\frac{2\dot{H}}{N} + 3H^2 + \frac{k}{a^2}\right) = -3VF + VF_{,\bar{X}}\left(\bar{X} - \eta\right) - P, \tag{15}$$

$$\frac{\dot{V}}{V} = NH\left(1 - h\bar{X}\right)\frac{F_{,\bar{X}}}{G}, \tag{16}$$

$$Na^3\left(3V_{,\phi}(F - G\eta) + P_{,\phi}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(a^3\sqrt{2X}\right)\left(3V_{,X}(F - G\eta) + P_{,X}\right)\right], \tag{17}$$

$$3HN(-2XP_{,X} - 6XV_{,X}(F - G\eta) + VF_{,\bar{X}}\left(\bar{X} - \eta\right)) = \frac{\mathrm{d}}{\mathrm{d}t}\left(-3VF + \left(2XP_{,X} + 6XV_{,X}(F - G\eta)\right) - P\right), \tag{18}$$

where the last equation is obtained from the conservation on the energy-momentum tensor; $\nabla_\mu T^\mu_\nu = 0$ and we have defined

$$h \equiv \frac{H_\alpha}{H}, \quad H_\alpha \equiv \frac{\dot{\alpha}}{\alpha n}. \tag{19}$$

From the above equations, one can see that Eq. (14) is a Friedmann equation with extra matter contents coming from the graviton mass. As a partner to the Friedmann equation, the so-called acceleration equation corresponds to Eq. (15). Since we have the Bianchi identity relating the equations of motion, these five equations of motion are not entirely independent. Note that this set of equations recovers the original self-accelerating cosmology when the square of a graviton mass V is constant by which the usual condition $F_{,\bar{X}}(1 - h\eta)$ is obtained readily from Eq. (16) [15]. However, as V is no longer constant, the equations of motion look even more complex than those in general relativity. To simplify the following calculations, we choose P such that the k-essence field behaves as a perfect fluid. The appropriate form of P, which satisfies such a behavior, is

$$P(X, \phi) = P_0X^{\frac{1+w}{2w}} = P_0X^{\gamma/2}, \tag{20}$$

where $\gamma \equiv 2XP_{,X}/P \equiv \frac{1+w}{2w}$, P_0 is a constant, and w is an equation of state parameter corresponding to the perfect fluid represented by the k-essence field [44]. Moreover, we let the graviton mass function mimic the perfect-fluid form as

$$V = V_0X^{\lambda/2}, \tag{21}$$

whose λ characterizes the power of the kinetic term as γ does for P, i.e. $\lambda \equiv 2XV_{,X}/V$ and V_0 is a constant. Note that under these assumptions, both P and V vary according to the kinetic term of ϕ but not the ϕ itself. Usually, in the quintessence model the continuity equation for the scalar field is obtained from the equation of motion of ϕ [45,46]. Taking that into account, we consider the equation of motion of ϕ in Eq. (17); then under the perfect-fluid assumptions for P and V in Eqs. (20) and (21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{a^3}{\sqrt{2X}}\right)\left(6XV_{,X}\left(F-G\eta\right)+2XP_{,X}\right)\right)=0. \tag{22}$$

After simple manipulations, the above equation gives the continuity equation for the k-essence field as

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_X+3HN\rho_X=\frac{\dot{X}}{2X}\rho_X, \tag{23}$$

where we have defined

$$\rho_X\equiv\left(2XP_{,X}+6XV_{,X}\left(F-G\eta\right)\right). \tag{24}$$

Equation (23) determines the dynamics of the matter of energy density ρ_X which resides in the Friedmann equation in Eq. (14). Interestingly, this looks exactly like a continuity equation of a "dust-like" matter with the interaction with the other matter sector determined by the flow rate of the form $\frac{\dot{X}}{2X}\rho_X$. One can also integrate Eq. (22) to find an expression for ρ_X in terms of the scale factor as

$$\rho_X=\frac{\sqrt{2X}C}{a^3}, \tag{25}$$

where C is an integration constant. In the case of a constant X, this ensures one of the properties that this matter shares with the dust; the energy density is inversely proportional to a^3 as the dust is. According to such characteristics, it is reasonable to interpret ρ_X as a dark matter. By doing so, this kind of dark matter possesses some interesting features. First of all, ρ_X is a dust-like matter which can arise naturally from the massive-gravity sector indicating that dark matter may be just an artifact of the varying graviton mass caused by the kinetic term of the k-essence field. Moreover, this claim is still valid even in the case of $P=0$. Since a graviton mass can represent dark energy in a generic class of the dRGT massive gravity, this suggests a unification of the dark sector, namely dark energy and dark matter, by such a varying graviton mass. Second, by having this kind of matter in the theory, we may expect this model of mass-varying massive gravity to solve the cosmic coincidence problem, where the universe is known to be composed mainly of comparable amounts of dark energy and dark matter. Thanks to the unification suggested above, it may be possible to provide an explanation on the coincidence problem by the existence of the graviton mass alone, while the cosmic acceleration also counts.

Since the equations of motion are coupled in a very cumbersome way, to get a picture of the whole of this system we need to perform a dynamical analysis, which is the main subject in the very last section. However, we can still get some rough descriptions, as a guideline to the dynamical analysis, by introducing some simple assumptions to the system, which is done in the next section.

3 Dark energy solution for the self-accelerating universe

It is widely known that our universe is expanding with an acceleration for which dark energy is responsible. There is recent observational evidence indicating that the observed effective equation of state parameter of the dark energy is close to -1 [43]. In this section, we shall adopt this characteristic by treating all the contributions from the graviton mass to have such a property. We define

$$\rho_g\equiv-3VF+6XV_{,X}\left(F-G\eta\right), \tag{26}$$

$$p_g\equiv3VF-VF_{,\bar{X}}\left(\bar{X}-\eta\right). \tag{27}$$

From the above definition, the corresponding equation of state parameter is defined as

$$w_g\equiv\frac{p_g}{\rho_g}. \tag{28}$$

By treating ρ_g as an energy density of dark energy, we set $w_g=-1$ and then we have the following condition:

$$6XV_{,X}\left(F-G\eta\right)=VF_{,\bar{X}}\left(\bar{X}-\eta\right). \tag{29}$$

To simplify the calculation, we use the perfect-fluid form of V in Eq. (21). Consequently, Eq. (29) becomes

$$3\lambda\left(F-G\eta\right)=F_{,\bar{X}}\left(\bar{X}-\eta\right), \tag{30}$$

$$\lambda=\frac{F_{,\bar{X}}\left(\bar{X}-\eta\right)}{3\left(F-G\eta\right)}. \tag{31}$$

Equation (31) is a requirement for the exponent λ to have a solution with the equation of state equal to -1. To get a picture of this characteristic, let us assume

$$\bar{X}=\text{constant}, \tag{32}$$

$$\eta=\text{constant}, \tag{33}$$

$$\text{then}\quad h=\frac{1}{\eta}. \tag{34}$$

Under these assumptions, the exponent λ in Eq. (31) is just a constant. To investigate this further, we consider Eq. (16) under the previous assumptions,

$$\frac{\dot{V}}{V} = NH\left(1 - h\bar{X}\right)\frac{F_{,\bar{X}}}{G},$$
$$\frac{\lambda\dot{X}}{2X} = NH\left(1 - \frac{\bar{X}}{\eta}\right)\frac{F_{,\bar{X}}}{G},$$
$$= -\left(\bar{X} - \eta\right)\frac{F_{,\bar{X}}}{G\eta}\frac{\dot{a}}{a}. \quad (35)$$

From the condition of λ in Eq. (31),

$$\frac{\dot{X}}{X} = -\frac{6\,(F - G\eta)}{G\eta}\frac{\dot{a}}{a}. \quad (36)$$

Since $\bar{X}$, η, and hence F and G, are constant, this equation can be integrated easily,

$$\int\frac{\mathrm{d}X}{X} = -\frac{6\,(F - G\eta)}{G\eta}\int\frac{\mathrm{d}a}{a},$$
$$X = C_0 a^{-\frac{6(F-G\eta)}{G\eta}} \quad (37)$$

where C_0 is an integration constant. Now we have

$$V = V_0 X^{-\frac{\left(1-\frac{\bar{X}}{\eta}\right)\eta F_{,\bar{X}}}{6(F-G\eta)}} = V_0 C_0 a^{\left(1-\frac{\bar{X}}{\eta}\right)\frac{F_{,\bar{X}}}{G}}. \quad (38)$$

Furthermore, Eq. (37) possibly determines a relation between the scale factor and the rate of change of the scalar field, since

$$X = \frac{\dot{\phi}^2}{2N^2} = C_0 a^{-\frac{6(F-G\eta)}{G\eta}}. \quad (39)$$

The expression of V in Eq. (38) shows the evolution of the (square of the) graviton mass as a evolves. In the previous model of mass-varying massive gravity [24–27], in which the Minkowski fiducial metric is used, the varying graviton mass shrinks as the scale factor grows. In this model, however, the exponent in Eq. (38) determines whether the graviton mass will shrink or not as the scale factor grows, or whether it will remain constant in the case that the exponent vanishes. Note that this crucial difference is caused by the different form of the fiducial metric, which is the FLRW metric in this case, to be compared with the Minkowski one in the previous models. This result indicates the sensitivity of the fiducial metric existing in the generic dRGT massive gravity where different fiducial metrics set different stages for the system and provide different solutions [16–20].

One more crucial point of this analysis is that the contributions from the graviton mass can have the same equation of state parameter as dark energy, while one of those contributions possesses the characteristic of dust, namely the term $6XV_{,X}\,(F - G\eta)$. From Eq. (23), such a term belongs to the dark matter ρ_X. This may be a way out for the cosmic coincidence problem, since we may infer that varying graviton mass is responsible for a dark matter via the term like $6XV_{,X}\,(F - G\eta)$, as we have claimed in the previous section, while it can still drive the accelerating expansion. To verify this idea, and to seek a finer description of this model, we will perform a dynamical analysis, which can be found in the next section.

4 Dynamical system

In this section, we will consider the dynamics of the universe to be governed by this new class of mass-varying massive gravity models using the method of the autonomous system. Due to the complexity of the graviton mass, we will begin this section with a simple analysis by considering the flat FLRW where $k = 0$ and assuming that $\bar{X}$, η are constant over time, thus $h = 1/\eta$. From this assumption, the evolution of X is simply determined by Eq. (16) such that

$$X' = \frac{\dot{X}}{HNX} = \frac{2}{\lambda}\frac{F_{,\bar{X}}}{G}\left(1 - h\bar{X}\right) = -\frac{6s}{\lambda r}, \quad (40)$$
$$\lambda \equiv \frac{2XV_{,X}}{V}, \quad (41)$$

where the prime denotes the derivative with respect to $\ln a$. The parameters r and s are constant and defined as

$$r \equiv \frac{G\eta}{F}, \quad s \equiv \frac{F_{,\bar{X}}(\bar{X} - \eta)}{3F}. \quad (42)$$

In order to obtain a suitable autonomous system, let us define dimensionless variables as follows:

$$x = -\frac{FV}{M_p^2 H^2}, \quad (43)$$
$$z = -\frac{P}{3M_p^2 H^2}, \quad (44)$$
$$y = \frac{2XP_{,X} + 6XV_{,X}F(1 - r)}{3M_p^2 H^2} = \frac{\rho_X}{3M_p^2 H^2}, \quad (45)$$
$$\gamma \equiv \frac{2XP_{,X}}{P}. \quad (46)$$

By using these variables, the equations of motion can be written in the form of autonomous equations as

$$x' = 3x\left(y + sx - \frac{s}{r}\right), \quad (47)$$
$$y' = 3y\left(y + sx - 1 - \frac{s}{\lambda r}\right), \quad (48)$$
$$\lambda' = \frac{6s}{r}\left(\frac{\lambda}{2} - (1 + \Gamma)\right), \quad (49)$$

Table 1 Summary of the properties of the fixed points

Name	x	y	z	w_{eff}	Existence	Stability
(a)	0	0	1	-1	$\gamma = 0$	$0 \leq \frac{s}{r} \leq 1$
(b)	$\frac{1}{r}$	0	$1 - \frac{1}{r}$	$-1 + \frac{s}{r}$	$\gamma = \lambda$	$\frac{\lambda}{1-\lambda} \leq \frac{s}{r} < 0$
(c)	0	$1 + \frac{s}{\lambda r}$	$-\frac{s}{\lambda r}$	$\frac{s}{\lambda r}$	$\gamma = 1 + \frac{\lambda r}{s}$	$\frac{\lambda}{1-\lambda} < \frac{s}{r} < -1$
(d)	$\frac{1}{1+\lambda(r-1)}$	$\frac{\lambda(r-1)}{1+\lambda(r-1)}$	0	$\frac{1}{\lambda-1}$	$\lambda = \frac{s}{s-r}$	$0 < \lambda < 1$
(e)	$\frac{1+(\lambda-1)z_0}{1+\lambda(r-1)}$	$-\frac{\lambda(1-r(z_0+1))}{1+\lambda(r-1)}$	z_0	$\frac{1}{\lambda-1}$	$\lambda = \gamma = \frac{s}{s-r}$	$0 < \lambda < 1$

$$1 = x + y + z, \tag{50}$$

$$y = -\lambda x(1 - r) - z\gamma, \tag{51}$$

where $\Gamma \equiv XV_{,XX}/V_{,X}$. Since we have five variables with two constraints, it is sufficient to consider only three equations. Note that the constraint in Eq. (50) is derived from Eq. (14), while the constraint in Eq. (51) is obtained from the definition of y in Eq. (45). The equation of λ in Eq. (49) is not directly dependent on the other variables. Therefore, in principle, we can solve it separately. For simplicity, we can consider λ as a parameter and then consider only the autonomous equations with two variables, x and y. We will extend our analysis to a more general case with λ being the variable in the next section. The effective equation of state parameter can be written in terms of the dimensionless variables as

$$w_{\text{eff}} = \frac{P + 3VF - VF_{,\bar{X}}(\bar{X} - \eta)}{3M_p^2 H^2} = -z - x + xs$$
$$= -1 + y + xs. \tag{52}$$

From these autonomous equations, the corresponding fixed points can be found by evaluating $x' = 0$ and $y' = 0$ in Eqs. (47) and (48), respectively. The properties of all the fixed points are summarized in Table 1, while the analyses are separately discussed for each of the fixed points below.

4.1 Fixed point (a)

From Eqs. (47) and (48), it is obvious that the system has a fixed point $(x, y) = (0, 0)$. By using the constraint equations, one obtains $z = 1$ and $\gamma = 0$. This means that the function P is constant and then this point corresponds to general relativity with a cosmological constant where the universe is dominated by the cosmological constant. To ensure such a claim, one can compute the corresponding effective equation of state parameter, which yields $w_{\text{eff}} = -1$. This is exactly the equation of state parameter of the cosmological constant which drives the accelerating de Sitter expansion.

The stability of the fixed point can be found by analyzing the eigenvalues of the linearly perturbed autonomous equations. By performing the linear perturbations, the eigenvalues can be written as $(\mu_1, \mu_2) = (-3s/r, -3 - 3s/r)$. The stability requires both of the eigenvalues to be negative, or otherwise the fixed point is said to be unstable or to be a saddle fixed point. In this case, the signs of those eigenvalues are determined by the value of the term $\frac{s}{r} = \left(\bar{X} - \eta\right) \frac{F_{,\bar{X}}}{G\eta}$, which means $0 \leq \frac{s}{r} \leq 1$ for the stable fixed point. Note that in the case of vanishing eigenvalues, like $s = 0$, one has to consider the perturbations up to second order or use a numerical investigation in order to determine the stability. In this analysis, we ensure the stability in this case by the numerical method and we have found that it is stable.

Even though this fixed point can provide a period of late-time expansion, it is not much of interest due to the disappearance of the graviton mass. This resulting property is one of the drawbacks in the previous model of mass-varying massive gravity [24–27].

4.2 Fixed point (b)

One of possible fixed points may be in the form $(x, y) = (x_0, 0)$ by which the universe is governed mainly by massive gravity alone. From Eq. (47), one can find x_0 as follows:

$$x_0 = \frac{1}{r}. \tag{53}$$

According to Eq. (45), there are two possible solutions for this kind of fixed point. One is $r = 1$ in which $x_0 = 1$, $z_0 = 0$, and another one is $\lambda = \gamma$ in which $x_0 = \frac{1}{r}$, $z_0 = 1 - \frac{1}{r}$. The effective equation of state parameter can be written as

$$w_{\text{eff}} = -1 + \frac{F_{,\bar{X}}(\bar{X} - \eta)}{3G\eta} = -1 + \frac{s}{r}. \tag{54}$$

Interestingly, $w_{\text{eff}} = -1$ as $F_{,\bar{X}} = 0$ or $(\bar{X} - \eta) = 0$. This characteristic is a usual cosmological solution of the original massive gravity. In particular, this condition indicates that the graviton mass ceases to vary, according to Eq. (16). Moreover, since in this case $z = 1 - \frac{1}{r}$, the pressure of the k-essence field is non-zero for $r > 1$, which means the k-essence field is supposed to be a form of matter with non-zero pressure (not dust).

In order to find the stability condition for this fixed point, one can find the eigenvalues of the linearly perturbed autonomous equations, which can be written as

$$(\mu_1, \mu_2) = \left(3\frac{s}{r}, -3 + 3\frac{(\lambda - 1)s}{\lambda r}\right). \tag{55}$$

Again, both of the eigenvalues contain the term s/r, and then the fixed point will be stable if $\frac{\lambda}{1-\lambda} \leq \frac{s}{r} < 0$. Note that, for this fixed point, it is possible to provide $w_{\rm eff} < -1$ to satisfy the observation, which indicates that the mean value of the equation of state parameter is slightly less than -1 [43].

4.3 Fixed point (c)

One can obtain a fixed point such that $(x, y) = (0, y_0)$. From Eq. (47), one can find y_0 as follows:

$$y_0 = 1 + \frac{s}{\lambda r}. \tag{56}$$

By using the constraint equation in Eq. (50), one obtains $z_0 = -\frac{s}{\lambda r}$. From the constraint equation in Eq. (51), we have

$$\gamma = -\frac{y}{z} = -1 + \frac{1}{z} = 1 + \frac{1}{w_m}, \tag{57}$$

where w_m is the equation of state parameter of the fluid contributed from $P(X) = P_0 X^{(1+w_m)/2w_m}$. The effective equation of state parameter can be written as

$$w_{\rm eff} = -z = \frac{s}{\lambda r}. \tag{58}$$

Again, there exist two significant branches of the solution such that this fixed point is a matter-dominated point. If $z = 0$, this corresponds to $w_{\rm eff} = 0$, which leads to the universe being in a matter-dominated period.

The eigenvalues of the autonomous system can be written as

$$(\mu_1, \mu_2) = \left(3 + 3\frac{s}{\lambda r}, 3 - 3\frac{s(\lambda - 1)}{\lambda r}\right). \tag{59}$$

If one requires this point to represent the matter-dominated epoch, one must set the parameters so that this point is unstable. This means the universe should evolve through this point to end up in other stable points since we know the matter-dominated epoch should exist in the universe's timeline but not nowadays. One can see that, for small negative value of s/r, the universe can evolve in the standard history at which fixed point (c) corresponds to a matter-dominated period with $w_{\rm eff} \sim 0$, and fixed point (b) corresponds to the late-time expansion of the universe due to the contribution from the graviton mass. However, it is not possible to alleviate the coincidence problem, since the contribution of non-relativistic matter vanishes at late time.

4.4 Fixed point (d)

According to Eqs. (47) and (48), one may consider the fixed point corresponding to the non-zero x and y. This point can be obtained by evaluating both (non-zero) x and y from Eqs. (50), (51), and (47), while a constraint on the parameters by which the non-zero (x, y) exist can be obtained from Eqs. (47) and (48). After simple manipulation, we have

$$x = \frac{1}{1 + \lambda (r - 1)}, \qquad y = \frac{\lambda (r - 1)}{1 + \lambda (r - 1)}, \qquad \text{and } z = 0, \tag{60}$$

where γ is arbitrary and λ is fixed to be $\lambda = \frac{s}{s-r}$. The effective equation of state parameter can be written as

$$w_{\rm eff} = \frac{1}{\lambda - 1}. \tag{61}$$

To determine the stability of this point, we find the eigenvalues of the system of equations. Interestingly, this point renders the two autonomous equations degenerate. This can be seen by computing the linear perturbed equations for both x and y evaluated at this fixed point. The eigenvalues of this autonomous system are expressed as

$$(\mu_1, \mu_2) = \left(0, \frac{3\lambda}{\lambda - 1}\right). \tag{62}$$

The vanishing eigenvalue here is nothing but an artifact of the degeneracy due to this fixed point. In particular, it is possible to redefine the variables such that the problem is reduced into a one-dimensional system. With such a redefinition, the stability of this fixed point is due to the non-zero eigenvalue in Eq. (62), which can be negative when $0 < \lambda < 1$. If this condition is taken into account, requiring the fixed point (c) to represent the matter-dominated era will restrict the combination $\frac{s}{r}$ to vanish.

This fixed point seems to provide a possible way to alleviate the coincidence problem due to the non-zero y. However, it cannot be used since, at the late-time expansion, $w_{\rm eff}$ must approach -1 and then lead to the fact that $(x, y) \to (1, 0)$. Nevertheless, it still provides an interesting result. For the case of $s = 0$ and $0 < \lambda \ll 1$, this fixed point is stable, while the fixed point (b) is unstable and then we can use this fixed point as the one for the late-time expansion of the universe. For this condition the fixed point (c) is still used for the matter-dominated period with $z = 0$. Therefore, this means that it is possible to obtain $z = 0$ for the whole history of the universe. This leads to the fact that, without providing an extra non-relativistic matter field such as dark matter, the contribution from the graviton mass can play the role of both dark matter and dark energy. This is one of the crucial properties of this model, since it can unify the two main unknown contents of the universe, dark matter and dark energy, by using only a graviton mass.

4.5 Fixed point (e)

Similarly to the derivation in fixed point (d), one can solve an algebraic equation by imposing $\gamma = \lambda$ and requiring non-zero x and y. As the result, the fixed point can be expressed as

$$x = \frac{1+(\lambda-1)z_0}{1+\lambda(r-1)}, \quad y = -\frac{\lambda(1-r(z_0+1))}{1+\lambda(r-1)}, \quad z = z_0, \tag{63}$$

where $\gamma = \lambda = \frac{s}{s-r}$ and z_0 is arbitrary. The effective equation of state parameter is the same as the one in the fixed point (d), which can be written as

$$w_{\text{eff}} = \frac{1}{\lambda - 1}. \tag{64}$$

Moreover, the eigenvalues for the stability analysis are still the same asfor the fixed point (d) and then the stability condition for this fixed point can be expressed as $0 < \lambda < 1$. Even though this fixed point shares most properties with fixed point (d), it cannot provide the unification of the two dark components, since z must have a non-zero value.

From the above analyses, we experienced the incompatibility between matter domination and the present dark energy domination. One may see that for a large λ, the fixed point (c) can represent the matter-dominated epoch, while the small value of λ is needed in the fixed point (d) or (e) to solve the coincidence problem. It is natural to generalize the theory further by allowing λ to change appropriately in time. This idea will be adopted and carefully analyzed in the next section.

5 Extended analyses

As we have mentioned, even though the model can be used to unify the dark contents of the universe, it still cannot be used to solve the coincidence problem. According to our analysis, this is due to the fact that λ is set to be a constant. In this section, we will show the possibility to solve the coincidence problem when λ is set as a dynamical variable. For completeness, we will add radiation into our consideration and then use numerical method to show that the radiation does not affect the unification in the dark sector. Note that the equation of motion for the radiation is obtained by using the conservation of its energy-momentum tensor or the continuity equation. By including the radiation and taking λ as a dynamical variable, the autonomous equations can be written as

$$x' = 3x\left(y + sx - \frac{s}{r} + \frac{4}{3}\Omega_r\right), \tag{65}$$

$$y' = 3y\left(y + sx - 1 - \frac{s}{\lambda r} + \frac{4}{3}\Omega_r\right), \tag{66}$$

$$\Omega_r' = 3\Omega_r\left(y + sx + \frac{4}{3}(\Omega_r - 1)\right), \tag{67}$$

$$\lambda' = \frac{6s}{r}\left(\frac{\lambda}{2} - (1+\Gamma)\right), \tag{68}$$

$$1 = x + y + z + \Omega_r, \tag{69}$$

$$y = -\lambda x(1-r) - z\gamma, \tag{70}$$

$$\Omega_r \equiv \frac{\rho_r}{3M_p^2 H^2}, \tag{71}$$

where ρ_r is the energy density of the radiation. The effective equation of state parameter can be written as

$$w_{\text{eff}} = -1 + y + xs + \frac{4}{3}\Omega_r. \tag{72}$$

From Eq. (67), we can see that all fixed points we found in the previous section still exist with $\Omega_r = 0$. Also, there exists the unstable fixed point such that $\Omega_r = 1$, while x and z (hence y) vanish. From Eq. (68), one can see that λ does not couple to the others and the fixed point takes place at $\lambda = 2(\Gamma + 1)$. For simplicity, one can set Γ as a constant. In order to confirm the claim in the previous section that there exists a standard evolution without introducing a k-essence Lagrangian or in the case of $z = 0$, we use numerical methods to evaluate the equations above by setting $s = 0$. The evolutions of x, y, and Ω_r are illustrated in the left panel of Fig. 1, and the evolution of the effective equation of state parameter is shown in the right panel of Fig. 1. We can see that there exists non-relativistic matter, inferred as dark matter represented by the variable y, while the variable x represents the dark energy that drives the late-time expansion of the universe. Both x and y are contributed from the graviton mass.

Now, let us consider the possibility to solve the coincidence problem. Let us use the fixed point (e) to be one corresponding to the late-time expansion of the universe. For this fixed point, the parameters s, r, and Γ are obtained by giving the initial conditions for the dynamical variables. In order to obtain the dynamics of all variables, we have to put the initial conditions slightly away from the fixed point. It is sufficient to put λ slightly above the fixed point, since we need λ to grow as time goes backward to ensure that it will have a high enough value for the matter-dominated period. In order to obtain $w_{\text{eff}} \sim -1$ at the present time, we have to set the value of the variable λ at the fixed point as $\lambda_f \to 0$. As a result, $\frac{s}{r} = \frac{\lambda_f}{\lambda_f - 1} \to 0$. In order to obtain a proper matter-dominated period, one has to put the initial value of λ far away from the fixed point. This situation makes the fixed point (b) stable and then the system evolves to the point (b)

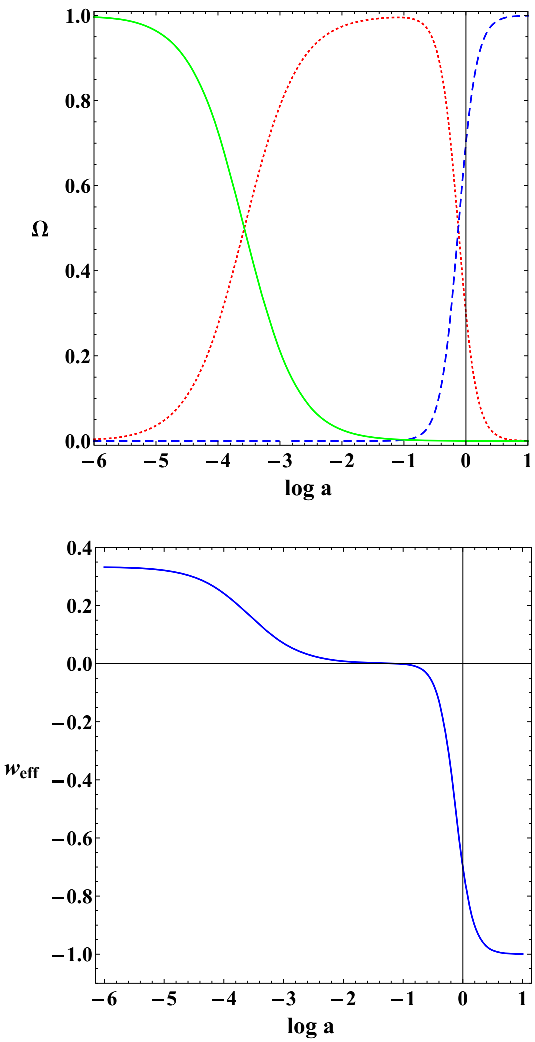

Fig. 1 The *left panel* shows the evolution of x, y, and Ω_r. The *dotted-red line* represents the evolution of x, the *dashed-blue line* represents the evolution of x, and the *solid-green line* represents the evolution of Ω_r. For the *right panel*, the evolution of $w_{\rm eff}$ is represented

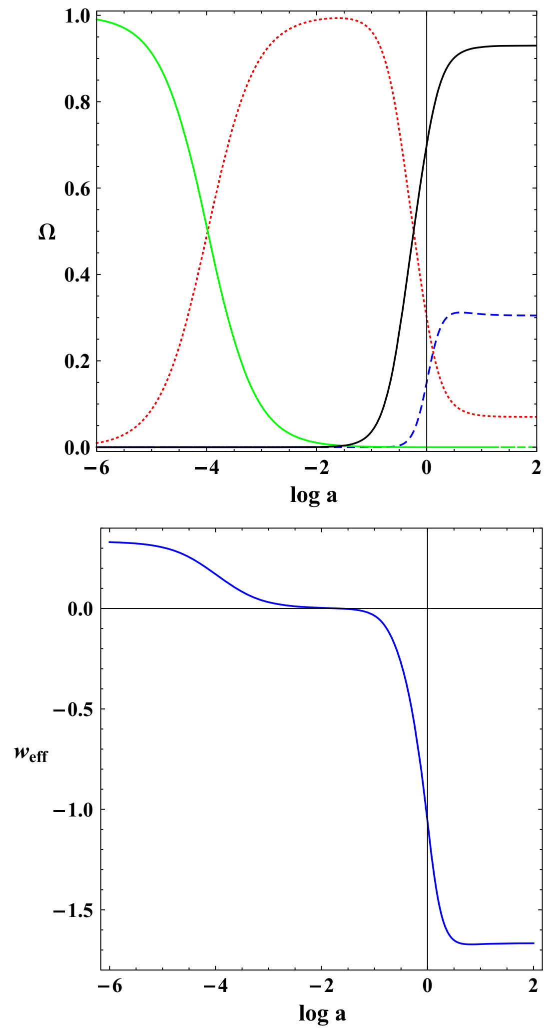

Fig. 2 The *left panel* shows the evolution of x, y, $x + z$, and Ω_r. The *dotted-red line* represents the evolution of x, the *dashed-blue line* represent the evolution of x, the *solid-black line* represents the evolution of $x + z$ and the *solid-green line* represents the evolution of Ω_r. For the *right panel*, the evolution of $w_{\rm eff}$ is represented. We set the parameters such that $\lambda_f = 0.4$ and $\lambda_0 = 1.0$ where λ_f is the value at the fixed point and λ_0 is one at the present time

eventually. Therefore, in order to have the fixed point (e) at late time, one has to set $w_{\rm eff}$ below -1 at the fixed point, so that the point (e) becomes a stable point. According to this setting, we show the evolution of the dynamical variables reaching the fixed point (e) to alleviate the coincidence problem in Fig. 2. Note that we set $\lambda_f = 0.4$, leading to $w_{\rm eff} = -1.67$ and $\lambda_0 = 1.0$.

In order to overcome the incompatibility among the fixed points, one may extend the analysis by allowing s, Γ or r to be dynamical variables. This will make the dynamical system more complicated. We found another possibility to overcome this incompatibility by imposing the constraint $\lambda = \gamma$ for the entire evolution. As a result, we have only three independent equations for six variables and three constraints. The dynamical variable λ can be written in terms of other variables as

$$\lambda = \frac{y}{rx + y + \Omega_r - 1}. \tag{73}$$

As a result, by setting the initial condition at the radiation dominated period, the evolution of the dynamical variables and the effective equation of state are shown in Fig. 3. From this figure, one can see that the evolution of the universe reaches the fixed point (e) at late time while the matter and radiation period are also properly presented. For the plot in this figure, we set $\lambda_f = 0.02$, and then the consequent results are $\Gamma = -0.99$ and $w_{\rm eff} \sim -1.02$. Note that the behavior of the resulting plot in Fig. 3 is sensitive to the initial value of x at the radiation dominated period where we set it choosing $x_i \sim 10^{-16}$.

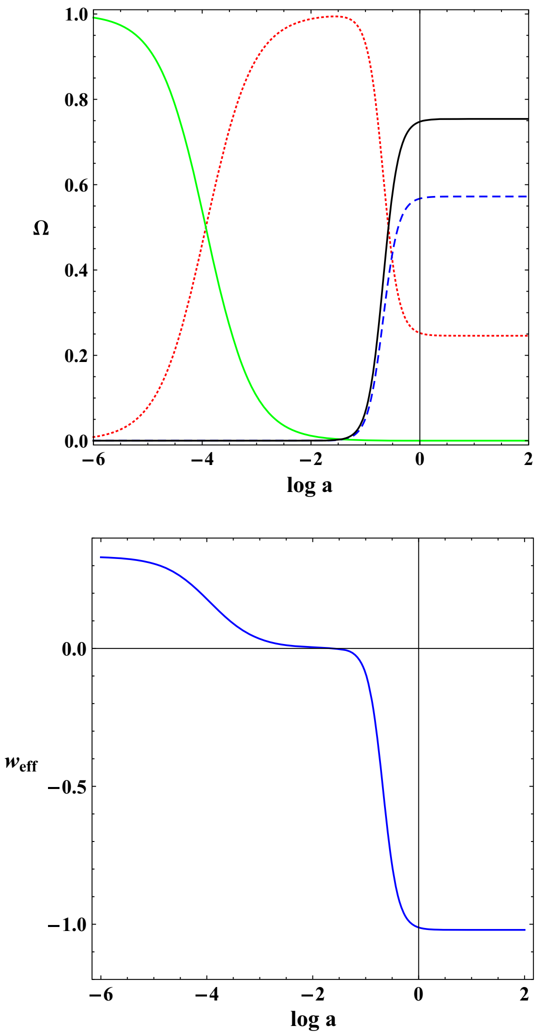

Fig. 3 The *left panel* shows the evolution of x, y, $x + z$, and Ω_r. The *dotted-red line* represents the evolution of x, the *dashed-blue line* represent the evolution of x, the *solid-black line* represents the evolution of $x + z$, and the *solid-green line* represents the evolution of Ω_r. For the *right panel*, the evolution of $w_{\rm eff}$ is presented

6 Conclusion

We have constructed a new class of mass-varying massive gravity models, in which not only the k-essence field but also its kinetic term determines the variation of the graviton mass. We have shown in Sect. 2 that there is a possibility for the graviton mass to live at late time compared with the previous model whose the graviton mass only depends on the scalar field and shrinks as the universe grows [24–27]. After simple manipulations and under particular assumptions, we found that a "dust-like" matter which behaves like a non-relativistic dust can naturally result from the graviton mass and it is a possible candidate for dark matter. This can be seen more clearly in the case $P = 0$ in which the dark matter comes solely from the varying graviton mass. Having such matter in the system, this model of massive gravity can describe the cosmic accelerating expansion with the equation of state parameter close to −1, while the universe is not entirely dominated by the dark energy part contributed also by the graviton mass. This property signals a possibility of having the universe composed of comparable amounts between dark energy and dark matter, known as the cosmic coincidence problem. To obtain a finer description on this, the usual method of the dynamical analysis is performed by taking the dark matter candidate into account and the results are carefully investigated as regards the issue of the coincidence problem. For the first simple case, the exponent of the kinetic term in the graviton mass λ is kept constant. We found the fixed points which correspond to various epochs in the history of the universe such as the matter-dominated period and massive-gravity-dominated periods. However, to have those fixed points with the appropriate stabilities in the evolution of the universe, the results suggest a system with λ as additional variable. The more general case, where λ is allowed to vary, is investigated where the radiation is included. While the result covers all the fixed points in the constant λ case, this allows the evolution in which there exists a matter-dominated period as well as a late-time expansion epoch. There are several crucial points in this investigation. First, we obtain the universe in which the graviton mass serves as both dark energy and dark matter, while it can still drive the cosmic acceleration. Second, to solve the coincidence problem, we obtain a universe with the effective equation of state parameter significantly below −1 unless both λ and γ are set equal with one another for the entire evolution of the universe. Since the analyses are under particular assumptions, this model still has room for study in more complicated ways. For example, one can exclude the assumptions proposed in this work for a more complex system or one can consider this model in a different aspect, like its astrophysical implications. Not only as regards the applications, but also studying the theoretical consistency, whether there exists a ghost instability or not, is a worthy challenge which we leave for future work. Apart from the constraints mentioned, one may think of constraining the model with various observations. This idea is also interesting, since the observations may judge the fate of this model by tightening it with constraints.

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A Perfect Fluid in Lagrangian Formulation due to Generalized Three-Form Field

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Abstract. A Lagrangian formulation of perfect fluid due to a non-canonical three-form field is investigated. The thermodynamic quantities such as energy density, pressure and the four-velocity are obtained and then analyzed by comparing with the k-essence scalar field. The non-relativistic matter due to the generalized three-form field with the equation of state parameter being zero is realized while it might not be possible for the k-essence scalar field. We also found that non-adiabatic pressure perturbations can be possibly generated. The fluid dynamics of the perfect fluid due to the three-form field corresponds to the system in which the number of particles is not conserved. We argue that it is interesting to use this three-form field to represent the dark matter for the interaction theory between dark matter and dark energy.

Keywords: Three-Form Field, Perfect Fluid, Lagrangian Formulation

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1 Introduction

A theory of cosmological perturbations is one of important issues in cosmology nowadays. It provides us to understand how astronomical structures at large scales are generated and evolve. Also, it can provide us the resulting signatures of the theoretical model to compare with observational data. The theory of cosmological perturbations for a perfect fluid has been developed and studied intensively at the level of equations of motion, for example, a study of the perturbed Einstein field equations together with the equation of conservation of energy momentum tensor [1, 2]. Beside the cosmological perturbations at the level of the equations of motion, a study of the cosmological perturbations at the Lagrangian level has been investigated. The advantage point of the study at Lagrangian level is that it is useful to find the perturbed dynamical field as well as derive closed evolution equations. This can be clearly seen by considering the cosmological perturbations in $f(R, G)$ gravity theories where there are two dynamical fields for scalar perturbations [3, 4]. For the study in Lagrangian approach, one can straightforwardly identify which fields are dynamical or auxiliary and then immediately obtain the closed evolution equations.

A Lagrangian formulation for a perfect fluid in general relativity has been constructed and developed for a long time [5–7]. The Lagrangian of the fluid is simply written as its pressure [6] or energy density [7]. The advantage point of this formulation is that it naturally provides a consistent way to construct a covariant theory for dark energy and dark matter coupling. The study of dark energy and dark matter coupling has been widely investigated in order to describe a way out from the cosmic coincidence problem [8–12]. Moreover, the observation also provide a hint for the existence of the coupling [13]. However, in order to recover the standard thermodynamics equations, the Lagrangian must involve at least five independent functions. Even though this formulation can provide a consistent way for studying the perfect fluid in cosmology and is well known as a standard approach for the perfect fluid at the Lagrangian level, there might be disadvantage for this approach since the theory involves too many functions.

A simple Lagrangian approach for the perfect fluid has been investigated by using a non-canonical scalar field [14], namely k-essence field [15–17]. It was found that the k-essence scalar field can provide a description of the perfect fluid with constant equation of state parameter. Moreover, it was found that the cosmological perturbations of this kind of the scalar field is equivalent to those in perfect fluid. However, it cannot be properly used to describe a non-relativistic matter with the equation of state parameter being zero since the Lagrangian is not finite. It was also found that the non-adiabatic pressure perturbations cannot be generated [18] as well as a vector mode of the perturbations cannot be produced [19].

Beside the cosmological models due to the scalar field, a three-form field can be successfully used to describe both inflationary models and dark energy models [20–30]. Even though there is a duality between scalar field and three-form field, the cosmological models are significantly differed in both background and perturbation levels. At the perturbation level, it is obvious to see that the three-form field can generate intrinsic vector perturbations while it is not possible for the scalar field. Therefore, it might be worthy to find an equivalence between the three-form field with a perfect fluid. In the present work, by mimicking the k-essence scalar field, we consider a generalized version of the three-form field and then find a possible Lagrangian form to describe the perfect fluid at the background level. We found that a simple power-law of the canonical kinetic term can provide the constant equation of state parameter like in the case of k-essence. The advantage point of the three-form field compare with the scalar field is that it can provide a consistent description of the non-relativistic matter field where its equation of state parameter satisfies $w = 0$. The stability issue is also investigated and found that the non-relativistic matter field due to the three-form field is free-from ghost and Laplacian instabilities.

By using the equations of motion of the generalized three-form field, the thermodynamic quantities are identified and found that the perfect fluid due to the three-from field corresponds to fluid in which the number of particles is not conserved. By analyzing the speed of propagation of scalar perturbations and the adiabatic sound speed, we found that the non-adiabatic perturbations can be possibly generated. We argue that it is interesting to use this three-form field to represent the dark matter for the interaction theory between dark matter and dark energy.

This paper is organized as follows. In section 2, we propose a general form of the three-form field and then find the equation of motion as well as the energy momentum tensor. By working in FLRW metric, the energy density and the pressure as well as the equation of state parameter are found. Some specific forms of the Lagrangian satisfying the equations of motion are obtained and found that it can represent the non-relativistic matter. We also investigate the stability issue by using the perturbed action at second order in section 3. We found conditions to avoid ghost and Laplacian instabilities. In section 4, we investigate the thermodynamic properties of the model. We begin this section with review of some important idea of the Lagrangian formulation for the standard and k-essence scalar field and then find the thermodynamic properties due to the three-form fluid. Finally, the results are summarized and discussed in section 5.

2 Equations of motion and energy momentum tensor

Cosmological models due to a three-form field have been investigated not only in inflationary models but also dark energy models [20–30]. Moreover, at the end of inflationary period, a viable model due to the three-form field for the reheating period have been investigated [31]. A consistent mechanism to generate large scale cosmological magnetic fields by using the three-form field have been studied [32]. Recently, a generalized inflationary model by considering two three-form fields was also investigated [29]. All investigations of cosmological models due to three-form are considered only in canonical form. Since the non-canonical form of scalar field have been intensively investigated, it is interesting to investigate the cosmological model with a non-canonical form of the three-form field. In this section, we will consider a non-canonical form of the kinetic term of a three-form field, $A_{\alpha\beta\gamma}$, as follows

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R + P(K,y)\right], \tag{2.1}$$

where the kinetic term and scalar quantity of the three-form field are expressed as

$$K = -\frac{1}{48}F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}, \tag{2.2}$$

$$y = \frac{1}{12}A_{\alpha\beta\gamma}A^{\alpha\beta\gamma}, \tag{2.3}$$

$$F_{\mu\nu\rho\sigma} = \nabla_\mu A_{\nu\rho\sigma} - \nabla_\sigma A_{\mu\nu\rho} + \nabla_\rho A_{\sigma\mu\nu} - \nabla_\nu A_{\rho\sigma\mu}\,. \tag{2.4}$$

By varying the action with respect to the three-form field, the equations of motion of the three-form field can be written as

$$E_{\alpha\beta\gamma} = \nabla_\mu\left(P_{,K}F^{\mu}{}_{\alpha\beta\gamma}\right) + P_{,y}A_{\alpha\beta\gamma} = 0, \tag{2.5}$$

where the notation with subscript $P_{,x}$ denotes $P_{,x} = \partial_x P$. Due to the totally anti-symmetric property of the tensor $F_{\mu\alpha\beta\gamma}$, one found that there exist constraint equations as follows

$$\nabla_\mu\left(P_{,y}A^{\mu\alpha\beta}\right) = 0. \tag{2.6}$$

These equations suggest us that the conserved quantity is expressed in terms of three-form field. Note that for the k-essence scalar field, the conserved quantity is expressed in term of one-form or vector quantity. We will discuss on this issue in detail in section 4 where we investigate the fluid dynamics. The energy momentum tensor can be obtained by varying the action of the three-form field with respect to the metric as

$$T_{\mu\nu} = \frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} - \frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + Pg_{\mu\nu}. \tag{2.7}$$

For consistency of the derived equations, one can check that the conservation of the energy momentum tensor can be obtained up to the equation of motion as follows

$$\nabla_\mu T^\mu_\nu = \frac{1}{6}F_{\nu\alpha\beta\gamma}E^{\alpha\beta\gamma} = 0. \tag{2.8}$$

In order to capture the thermodynamics quantities such as the energy density and pressure due to the three-form field like the investigation in scalar field, let us consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) manifold whose metric element can be written as

$$ds^2 = -dt^2 + \gamma_{ij}dx^idx^j = -dt^2 + a(t)^2\delta_{ij}dx^idx^j\,. \tag{2.9}$$

By using this form of the metric and the constraint equation in Eq. (2.6), the components of the three-form field, $A_{\alpha\beta\gamma}$, can be written as

$$A_{0ij} = 0\,, \qquad A_{ijk} = \epsilon_{ijk}X(t) = \sqrt{\gamma}\varepsilon_{ijk}X(t) = a^3\varepsilon_{ijk}X(t), \tag{2.10}$$

where ε_{ijk} is the three-dimensional Levi-Civita symbol with $\varepsilon_{123} = 1$. By using this form of the metric, the components of energy momentum tensor can be expressed as

$$T^0_0 = P - 2KP_{,K}, \tag{2.11}$$
$$T^i_j = (P - 2KP_{,K} - 2yP_{,y})\delta^i_j \,. \tag{2.12}$$

By comparing these components of the energy momentum tensor of the three-form to one from the perfect fluid, the energy density and pressure of the three-form can be expressed as

$$\rho = 2KP_{,K} - P, \tag{2.13}$$
$$p = P - 2KP_{,K} - 2yP_{,y} = -\rho - 2yP_{,y}. \tag{2.14}$$

Note that we have used $y = X^2/2$ and $K = (\dot{X} + 3HX)^2/2$ where $H = \dot{a}/a$ is the Hubble parameter. From the energy density and the pressure above, the equation of state parameter of the three-form can be written as

$$w = \frac{p}{\rho} = -1 - \frac{2yP_{,y}}{\rho}. \tag{2.15}$$

The equation of motion of the three-form field in Eq. (2.5) can be written in flat FLRW background as

$$(2KP_{,KK} + P_{,K})\dot{K} + 2KP_{,yK}\dot{y} - 2\sqrt{K\,y}P_{,y} = 0. \tag{2.16}$$

From this point, one can check validity of the derived equations by reducing the general form of the action to the canonical one as setting $P = K - V(y)$. As a result, we found that all equations can be reduced to the canonical one investigated in [20–30]. Substituting ρ from Eq. (2.13) into Eq. (2.15), one obtains

$$2yP_{,y} + (1+w)2KP_{,K} = (1+w)P. \tag{2.17}$$

In order to find the form of P, one has to solve this equation. It is useful to solve this equation by considering a simple assumption such as taking the equation of state parameter to be a constant, $w = \text{const}$. By using separation of variable method, the solution can be written as

$$P = P_0 K^\nu y^\mu, \tag{2.18}$$

where P_0 is an integration constant and μ, ν are the exponent constants obeying the relation

$$\nu = \frac{1 + w - 2\mu}{2(1+w)}, \text{or } \; w = -1 + \frac{2\mu}{1-2\nu}, \quad \nu \neq \frac{1}{2}. \tag{2.19}$$

This form of the solution is very useful since one can interpret the three-form field as a non-relativistic matter or dark matter by setting the equation of state parameter as $w = 0$ while it cannot be properly used for k-essence scalar field case. We will show explicitly why we cannot properly use k-essence scalar field for the non-relativistic matter in section 4. In order to study the covariant coupling form between dark matter and dark energy as suggested from the observation [13], one can use the three-form as the dark matter with the consistent covariant interaction forms. Moreover, it may be interpreted as dark radiation by setting $w = 1/3$. Note that, in the case of $\nu = 1/2$, it corresponds to the trivial solution since the energy density of the field vanishes. It is important to note that the late-time acceleration of the unverse can also be achieved by setting $w = -1$. Even though this may not be distinguished to the cosmological constant at the background level, the cosmological perturbations due to this model of the three-form can be significantly deviated from the model of the cosmological constant.

Since the form of the Lagrangian P is obtained by assuming a constant equation of state parameter, the dark energy model from this three-form field cannot be proposed to solve the coincidence problem. One may allow the equation of state to be varying in order to overcome this issue. One of interesting solutions is assuming that the equation of state parameter depends on the three-form field $w = w(y)$. In order to solve Eq. (2.17) to obtain a suitable form of P, one may choose the equation of state parameter such as $w = -1 + \lambda y$, where λ is a constant. As a result, the solution can be written as

$$P = P_0 K^\nu e^{\frac{(1-2\nu)}{2}\lambda y}. \tag{2.20}$$

Naively, it is not difficult to obtain the dynamical dark energy due to the generalized three-form. One can set λ be effectively small and find the condition to provide an evolution of y such that it evolves from a large value to a small value. However, since it is not in the canonical form, the theory may be suffered from instabilities. In this work, the stability issue will be investigated in the next section. The investigation of the dark energy model due to the generalized three-form is left in further work.

3 Stability

In order to capture the stability conditions of the generalized three-form field, we may consider the perturbations of the field. Since the field minimally couples to the gravity, one has to take into account the metric perturbations. However, for simplicity but useful study, we will investigate the stabilities of the model only in a high-momentum limit. This will capture only some stability conditions. Nevertheless, this includes most of the necessary conditions as found in the canonical three-form field [27]. We leave the full investigation in further work where the cosmological perturbations are taken into account. For this purpose, the metric is held fixed as the Minkowski metric and the three-form field can be written as

$$A_{ijk} = \varepsilon_{ijk}(X(t) + \alpha(t,\vec{x})), \tag{3.1}$$
$$A_{0ij} = \varepsilon_{ijk}(\partial_k\beta(t,\vec{x}) + \beta_k(t,\vec{x})), \tag{3.2}$$

where α and β are perturbed scalar fields and β_k is a transverse vector obeying the relation $\partial_k\beta^k = 0$. This vector field will be responsible for the intrinsic vector perturbation of the three-form field. For the linear perturbations, the scalar and vector modes are decoupled and then they can be separately investigated. For the scalar modes, by expanding the action up to second order in the field, the second order action can be written as

$$S^{(2)} = \int d^4x \Big(\frac{1}{2}\frac{\dot{Q}^2}{(P_{,K} + 2KP_{,KK})} - \frac{1}{2}P_{,y}(\partial\beta)^2 + \frac{1}{2}P_{,y}c_s^2\alpha^2\Big), \tag{3.3}$$
$$\dot{Q} = (P_{,K} + 2KP_{,KK})\dot{\alpha} + 2\sqrt{K\,y}P_{,K}P_{,y}\alpha - (P_{,K} + 2KP_{,KK})\partial^2\beta, \tag{3.4}$$
$$c_s^2 = 1 + \frac{2yP_{,yy}}{P_{,y}} - \frac{4KyP_{,Ky}^2}{P_{,y}\,(2KP_{,KK} + P_{,K})}. \tag{3.5}$$

One can see that the field β is non-dynamical so that one can eliminate it by using its equation of motion. By applying the Euler-Lagrange equation to the above action, the equation of motion for the field β can be written as

$$(P_{,K} + 2KP_{,KK})\dot{\alpha} + 2\sqrt{K\,y}P_{,Ky}\alpha - (P_{,K} + 2KP_{,KK})\partial^2\beta - P_{,y}\beta = 0, \tag{3.6}$$

From this equation of motion, we can replace the quantity $\dot{Q}$ as $\dot{Q} = P_{,y}\beta$. Note that this equation can be obtained by using the component $(0, i, j)$ of the covariant equation in Eq. (2.5). In order to find the solution for β, it is convenient to work in Fourier space so that the above equation can be algebraically solved. As a result, by substituting the solution of β into the action in Eq. (3.3), the second order action for the scalar perturbations can be rewritten as

$$S^{(2)} = \int dtd^3k\Big(F_1\dot{\alpha}^2 + F_2\dot{\alpha}\alpha + F_3\alpha^2\Big), \tag{3.7}$$

where

$$F_1 = -\frac{P_{,y}\left(2KP_{,KK} + P_{,K}\right)}{2\left(k^2\left(2KP_{,KK} + P_{,K}\right) - P_{,y}\right)}, \tag{3.8}$$

$$F_2 = -\frac{2\sqrt{K\,y}P_{,K\,y}P_{,y}}{\left(k^2\left(2KP_{,KK} + P_{,K}\right) - P_{,y}\right)}, \tag{3.9}$$

$$F_3 = \frac{\left(2yP_{,yy} + P_y\right)\left(2k^2KP_{,KK} + k^2P_{,K} - P_{,y}\right) - 4k^2KyP_{,K\,y}^2}{2\left(k^2\left(2KP_{,KK} + P_{,K}\right) - P_{,y}\right)}. \tag{3.10}$$

As we have discussed above, we will consider the stability conditions at high-momentum limit. Therefore, by taking the limit $k^2 \to \infty$, the second order action becomes

$$S^{(2)} = \int dtd^3k\,k^{-2}(-P_{,y})\Big(\frac{1}{2}\dot{\alpha}^2 - \frac{1}{2}k^2c_s^2\alpha^2 - \frac{1}{2}m_A^2\alpha^2\Big). \tag{3.11}$$

where

$$m_A^2 = \frac{d}{dt}\Big(\frac{2\sqrt{K\,y}P_{,Ky}}{(P_{,K} + 2KP_{,KK})}\Big) - \frac{4K\,y\,P_{,Ky}^2}{(P_{,K} + 2KP_{,KK})^2}. \tag{3.12}$$

Therefore, the condition to avoid ghost instabilities can be written as

$$P_{,y} < 0. \tag{3.13}$$

This condition can be reduced to the canonical case by taking $P = K - V(y)$, which provides the result as $V_{,y} > 0$ consistently with the result in [27]. In order to avoid the Laplacian instability, one requires $c_s^2 \geq 0$ leading to the condition

$$1 + \frac{2yP_{,yy}}{P_{,y}} - \frac{4KyP_{,Ky}^2}{P_{,y}\left(2KP_{,KK} + P_{,K}\right)} \geq 0. \tag{3.14}$$

To obtain a clear picture of this condition, one may specify the form of P. For the form with constant equation of state parameter, $P = P_0K^\nu y^\mu$, the sound speed square can be expressed as $c_s^2 = w$. Therefore, the three-form field can be interpreted as the non-relativistic matter up to a perturbation level since $c_s^2 = 0$ and $w = 0$. Moreover, it is obvious that the non-relativistic matter represented by the generalized three-form field is free from ghost and Laplacian instabilities. Note that the dark energy model with $w < -1/3$ for this form of the Lagrangian is suffered from Laplacian instabilities since the sound speed square is negative.

For another simple form of the Lagrangian with $P = P_0 K^\nu e^{\frac{1-2\nu}{2}\lambda y}$, the sound speed square and the equation of state parameter read $c_s^2 = 1 + \lambda y$ and $w = -1 + \lambda y$. From these expressions, one can see that the phantom expansion of the universe will provide a superluminality. The no-ghost condition can be expressed as $P_0\lambda(2\nu - 1) > 0$. At this point, it is possible to obtain a viable model of dark energy due to the generalized three-form field.

Now we will consider the vector mode of the perturbations by following the same step as in the scalar one. As a result, the second order action for the vector perturbations can be written as

$$S^{(2)} = \int d^4x\Big(-\frac{1}{2}P_{,y}\beta_i\beta^i\Big). \tag{3.15}$$

From this action, one can see that the vector mode does not propagate. A condition to avoid the instabilities coincides with the condition obtained in scalar mode.

In order to find possibility to obtain non-adiabatic perturbations due to the three-form field, one may find a difference between the speed of propagation of scalar perturbations, c_s^2, and the adiabatic sound speed, c_a^2. If these two kinds of the sound speed are equal, there are no non-adiabatic perturbations while it provides the possibility to generate non-adiabatic perturbations if they are not equal [18]. The speed of propagation of scalar perturbations is found in Eq. (3.5). For the adiabatic sound speed, one can derived as follows

$$c_a^2 \equiv \frac{\dot{p}}{\dot{\rho}} = 1 + 2\frac{yP_{,yy}\dot{y} + P_{,Ky}(y\dot{K} + 2\sqrt{Ky})}{P_{,y}\dot{y} - 2\sqrt{Ky}P_{,Ky}}, \tag{3.16}$$

$$= c_s^2 + \frac{4\sqrt{Ky}P_{,Ky}}{P_{,y}\dot{y} - 2\sqrt{Ky}P_{,Ky}}\left(1 + \frac{yP_{,yy}}{P_{,y}} + \frac{yP_{,y}^2 - 2KyP_{,Ky}}{P_{,y}(P_{,K} + 2KP_{,KK})}\right). \tag{3.17}$$

From this equation, one can see that the sound speed of scalar perturbations and the adiabatic sound speed are not generally equal. Therefore, it is possible to generate non-adiabatic perturbations from the generalized three-form field. This is one of advantage points of the generalized three-form field compare with the k-essence scalar field. Note that both kinds of the sound speed will coincide when the Lagrangian does not depend on y, $P = P(K)$. For this case, the non-adiabatic perturbations cannot be generated.

4 Fluid dynamics due to three-from field

In order to compare the results with the standard description of the fluid dynamics for the perfect fluid, let us briefly review an important concept of the standard version for the fluid dynamics. Since the perfect fluid dynamics due to the non-canonical scalar filed or k-essence field has been intensively investigated and interpreted as non-relativistic mater field, for example, in the case of massive gravity theory [33, 34], we will also review some important results of the k-essence scalar field before we discuss further on the three-form field.

4.1 Standard version and k-essence field

There are many approaches of the standard version for the perfect fluid Lagrangian. We will use Brown formulation [7] since it is more useful and has been widely used for recent studies in dark energy and dark matter couplings [9–12]. The Lagrangian of the perfect fluid can be written in terms of the energy density with Lagrange multipliers as

$$S_m = \int d^4x \left(-\sqrt{-g}\,\rho + j^\mu(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A\alpha^A_{,\mu})\right), \tag{4.1}$$

where $\rho = \rho(n, s)$ is the energy density of the fluid, n is a particle number density, s is an entropy density per particle and j^μ are components of the particle number flux. The second term which is contracted with j^μ is the Lagrange multiplier term with the Lagrange multiplier fields φ, θ and β_A where α_A are the Lagrangian coordinates of the fluid with index A running as $1, 2, 3$. j^μ can be written in terms of the four-velocity u^μ of the fluid as

$$j^\mu = \sqrt{-g}\,n\,u^\mu. \tag{4.2}$$

The four-velocity satisfies the relation $u_\mu u^\mu = -1$ where $n = |j|/\sqrt{-g}$ and $|j| = \sqrt{-j^\mu g_{\mu\nu} j^\nu}$. The standard energy momentum tensor of the perfect fluid can be obtained by varying the action with respect to the metric $g_{\mu\nu}$ as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\,g_{\mu\nu}, \tag{4.3}$$

where p is the pressure of the fluid defined as

$$p \equiv n\frac{\partial\rho}{\partial n} - \rho. \tag{4.4}$$

By varying the action with respect to the Lagrange multiplier fields θ and φ, the first law of Thermodynamics and the conservation of the particle number can be obtained respectively [7] as

$$dp = n d\mu - T ds, \tag{4.5}$$
$$\partial_\nu j^\nu = 0. \tag{4.6}$$

where T is a temperature and μ is a chemical potential defined as

$$\mu \equiv \frac{\rho + p}{n}. \tag{4.7}$$

From these equations of motion together with the conservation of the energy momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, all main thermodynamics equations can be obtained. For example, conservation of the entropy density can be obtained by using a projection of the conservation equation of the energy momentum tensor along the fluid flow as follows

$$u_\nu \nabla_\mu T^{\mu\nu} = -\frac{\mu}{\sqrt{-g}}\partial_\nu j^\nu - u^\nu T\partial_\nu s = 0. \tag{4.8}$$

From these equations, in the viewpoint of field theory, all main thermodynamics equations can be obtained if one can identify the main thermodynamics quantities in terms of the field such as energy density, pressure, four-velocity and chemical potential which give the form of energy momentum tensor as found in Eq. (4.3). We will show this procedure for instruction in the case of scalar field.

For the k-essence scalar field, we will follow [14] in which action of the k-essence field can be written as

$$S_\phi = \int d^4x\sqrt{-g}P(K_\phi), \tag{4.9}$$

where $K_\phi = -\nabla_\mu\phi\nabla^\mu\phi/2$ is the canonical kinetic term of the scalar field. The corresponding equations of motion of the scalar field can be expressed as

$$\nabla_\mu\Big(P'\nabla^\mu\phi\Big) = 0, \tag{4.10}$$

where prime denotes the derivative with respect to K_ϕ. The energy momentum tensor of the scalar field can be written as

$$T_{\mu\nu} = P'\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}P. \tag{4.11}$$

By comparing this energy momentum tensor with that in the perfect fluid in Eq. (4.3), the energy density, pressure and the four-velocity can be identified as

$$\rho_\phi = 2K_\phi P' - P, \tag{4.12}$$
$$p_\phi = P, \tag{4.13}$$
$$u^\mu = \frac{\nabla^\mu\phi}{\sqrt{2K_\phi}} \tag{4.14}$$

Therefore, the particle number density can be obtained in order to satisfy the conservation of the particle flux as $n_\phi = \sqrt{2K_\phi}P'$ while the chemical potential reads $\mu_\phi = \sqrt{2K_\phi}$. Therefore, one can check that the equation of motion in Eq. (4.10) satisfies the equation of the conservation of the particle flux as follows

$$\sqrt{-g}\nabla_\mu\Big(P'\nabla^\mu\phi\Big) = \partial_\mu\Big(\sqrt{-g}P'\nabla^\mu\phi\Big) = \partial_\mu\Big(\sqrt{-g}n_\phi u^\mu\Big) = \partial_\mu j^\mu_\phi = 0. \tag{4.15}$$

As a result, all fluid dynamics equations can be derived by using the results in the standard version. Note that the first law of thermodynamics is adopted for the scalar field while in the case of the standard version, it is obtained from the equation of motion. It is important to note that the conservation of the particle flux does not hold if we generalize the Lagrangian of the scalar field as $P = P(K_\phi, \phi)$ since the equations of motion in Eq. (4.10) becomes $\nabla_\mu\Big(P'\nabla^\mu\phi\Big) = -\partial P/\partial\phi$. This is not so surprisingly since the simple scalar field, such as quintessence field, is also equivalent to the system in which the particle flux is not conserved. This can be explicitly seen by taking $P = K_\phi - V(\phi)$.

By taking the equation of state parameter to be constant, the form of the Lagrangian obeys a relation

$$P(1 + w_\phi) = 2w_\phi K_\phi P'. \tag{4.16}$$

From this equation, one can find the exact form of the Lagrangian as

$$P = P_0 K_\phi^{\frac{1+w_\phi}{2w_\phi}}, \quad \text{where} \quad w_\phi \neq 0. \tag{4.17}$$

It is obviously that one cannot properly use this form of the scalar field to describe the non-relativistic matter since its equation of state parameter is zero, $w = 0$. This is one of drawbacks for the k-essence scalar field. As we have shown before, this does not happen in the case of generalized three-form field.

4.2 Generalized three-form field

As we have mentioned, one can find the equivalence between the energy momentum tensor of the three-form and the standard perfect fluid and then identify the fluid quantities such as ρ, p and the four-velocity u^μ in terms of the three-form field. By using these identifications, one can find the consequent thermodynamics equations of the three-form field as done in the scalar field case. The energy density and the pressure have been identified in Eq. (2.13) and Eq. (2.14) respectively. Now, we will identify the four-velocity of the three-form field by comparing the energy momentum tensor of the perfect fluid in Eq. (4.3) and the energy momentum tensor of the three-form in Eq. (2.7). As a result, the relation of the four-velocity and the three-form field can be written as

$$(\rho + p)u_\mu u_\nu = \frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} - \frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + (2KP_{,K} + 2yP_{,y})g_{\mu\nu}. \tag{4.18}$$

Since $F_{\mu\nu\rho\sigma}$ is a totally symmetric rank-4 tensor in 4-dimensional spacetime, it can be written in terms of a covariant tensor $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}$ where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in four-dimensional spacetime. By using the components of the three-form field in Eq. (2.10), the field strength tensor can be written as

$$F_{\mu\nu\rho\sigma} = (\dot{X} + 3HX)\epsilon_{\mu\nu\rho\sigma} = \sqrt{2K}\epsilon_{\mu\nu\rho\sigma}. \tag{4.19}$$

By using this equation, the first term in the right hand side of Eq. (4.18) can be rewritten as

$$\frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} = -2KP_{,K}g_{\mu\nu}. \tag{4.20}$$

Substituting this equation into Eq. (4.18), one obtains

$$\begin{aligned} (\rho + p)u_\mu u_\nu &= -\frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + 2yP_{,y}g_{\mu\nu}, \\ u_\mu u_\nu &= \frac{1}{4y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} - g_{\mu\nu}. \end{aligned} \tag{4.21}$$

One can check that the relation $u_\mu u^\mu = -1$ valid from this relation. Since the tensor $u_\mu u_\nu$ is constructed from two three-form fields, it plays the role of symmetric rank-2 tensor $S_{\mu\nu}$ instead of outer product of two four-velocity. Therefore, it is not trivial to find the form of the four-velocity of the three-form field. However, one may expect that the four-velocity may relate to the three-form field by the relation of the vector and the three-form in four dimensionality as $u^\mu \propto \epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}$. As a result, the four-velocity of the fluid can be written in terms of the three-form field as

$$u^\mu = \frac{\epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}}{3!\sqrt{2y}}, \tag{4.22}$$

where the three-form field can be written in terms of the four-velocity as

$$A^{\alpha\beta\gamma} = \sqrt{2y}\epsilon^{\mu\alpha\beta\gamma}u_\mu. \tag{4.23}$$

It is not trivial to find the conserved current density corresponding to three-form field. Actually, there are no conserved quantities obtained from invariance of the action under the shift of the field like the scalar field. However, one may find the conserved quantity from the constraint equation in Eq. (2.6) as follows

$$j^{\alpha\beta\gamma} = n^{\mu\alpha\beta\gamma}u_\mu = \sqrt{2y}P_{,y}\epsilon^{\mu\alpha\beta\gamma}u_\mu = P_{,y}A^{\alpha\beta\gamma}. \tag{4.24}$$

From this relation, the conserved quantity is now three-form field instead of vector field and the number density now is four-form field instead of scalar field. This equivalence comes from Hodge duality in four-dimensional spacetime. One may obtained the effective particle number density as

$$n = \sqrt{\frac{n_{\mu\alpha\beta\gamma}n^{\mu\alpha\beta\gamma}}{4!}} = \sqrt{2y}P_{,y}. \tag{4.25}$$

Therefore, the usual particle flux for the three-form field can be written as

$$j^\mu = \sqrt{-g}\, nu^\mu = \sqrt{-g}P_{,y}\frac{\epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}}{3!}. \tag{4.26}$$

This quantity does not trivially vanish due to the equation of motion in Eq. (2.16). Since $\partial_\mu j^\mu \neq 0$ together with Eq. (4.8), it is inferred that the entropy along the fluid flow is not conserved. The non-conservation of the particle flux for the three-form is due to the fact that the action is not invariant under shift of the field. In the scalar field case, the action is invariant under $\phi \to \phi + \xi$ where ξ is a constant. For general case of the scalar field with $P_\phi = P_\phi(K_\phi, \phi)$, this symmetry is also broken and then its dynamics will corresponds to the non-conservation of the particle flux like in the three-form case. For the three-form, if we restrict our attention to the case where $P = P(K)$ which is invariant under shift of the field, the particle number density, $n \propto \rho + p \propto P_{,y}$, will always vanish. Also, the equation of state parameter is always equal to -1 which cannot be responsible for the non-relativistic matter.

We also observe that condition of non-conservation of the entropy density along the fluid flow coincides with the condition of generation of non-adiabatic perturbations even though these conditions come from different approach. The conservation of the entropy density is derived from background equation while non-adiabatic perturbations are properties of the fluid at perturbation level. This argument also hold in both scalar field and three-form field cases. Therefore, this may shed light on the interplay between conserved quantities under shift of the filed and non-adiabatic perturbations.

Since the thermodynamics description for the generalized three-form field corresponds to the system in which the particle number is not conserved, it implies that the field may interact with other fields and then cause the non-conservation. It is important to note that the conservation of the energy momentum tensor of the three-form still valid, $\nabla_\mu T^\mu_\nu = 0$. The non-conservation quantities mentioned above are the thermodynamically effective quantities. Therefore, the interaction of the three-from field to the other fields is implied only in the description of the thermodynamical sense. As we have mentioned, the useful point of this three-form field is that it can represent the non-relativistic matter field with $w = 0$. Therefore, one may interpret it as dark matter. Since the particle number density is not conserved, it is worthwhile to investigate an interaction of this field to the dark energy. This may be useful approach for studies of dark energy and dark matter coupling since one can find the covariant interaction terms at the Lagrangian level and then the resulting closed evolution equations are obtained. This issue is of interest and we leave this detailed investigations for further work.

5 Summary

A Lagrangian formulation of perfect fluid is a powerful tool to study dynamics of the universe, especially interacting approach between dark energy and dark matter. A general description in this formulation invokes many functions and then it is not easy to handle. A k-essence scalar field can be used to describe the dynamics of the perfect fluid in cosmology. At the background level, even though the k-essence scalar field can be use to describe the perfect fluid with constant equation of state parameter, it cannot properly used for the non-relativistic matter with $w_\phi = 0$. At the perturbation level, the k-essence scalar field cannot provide non-adiabatic perturbations as well as intrinsic vector perturbations.

In the present paper, we propose an alternative way to alleviate these problems by using a generalized three-form field. The investigation is begun with proposing a general form of the action of the three-form field with a function depending on both the kinetic term and the field, $P = P(K, y)$, similarly to the k-essence scalar field. Equations of motion and energy momentum tensor of the three-form field in covariant form have been calculated. By working in FLRW background, the energy density and the pressure as well as the equation of state parameter are found. For the constant equation of state parameter, an exact form of the Lagrangian reads $P = P_0 K^\nu y^\mu$ where $w = -1 + \frac{2\mu}{1-2\nu}$ and $\nu \neq 1/2$. Therefore, one can set $w = 0$ by choosing proper values of the parameters μ and ν and then use the generalized three-form field to represent the non-relativistic matter. For non-constant equation of state parameter, we also point out that it is possible to construct an alternative model of dark energy. The stability analysis of the model is also performed. We found the conditions to avoid ghost and Laplacian instabilities. For the fluid with $w = 0$, it is free from ghost and Laplacian instabilities. For some specific model of dark energy, we argue that, to avoid the superluminality, the equation of state parameter must be greater than -1. In other words, the viable model of dark energy from the generalized three-form field cannot provide the phantom phase of the universe. Note that the no-ghost condition we found in this paper can be trusted only in the high momentum limit. We leave the full investigation for further work where we investigate the cosmological perturbations and observational constraint.

Thermodynamics properties due to the generalized three-form field are also investigated. It is found that this model corresponds to a system with non-conservation of the particle flux. This leads to a non-conservation of the entropy density along the fluid flow. This is not so surprisingly since many models of dark energy, for example quintessence model, also correspond to the non-conservation of the particle flux. We also found some links between non-conservation of the entropy density along the fluid flow which is a thermodynamically effective quantity at the background level and the generation of non-adiabatic perturbations which is a property of the model at perturbation level. This may shed light on the interplay between conserved quantities under shift of the filed and non-adiabatic perturbations. We can argue that this is an useful approach for a study of dark energy and dark matter coupling since one can find the covariant interaction terms at the Lagrangian level and then the resulting closed evolution equations are obtained. This issue is of interest and we leave this detailed investigations for further work.

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Regular Article - Theoretical Physics

Mass-varying massive gravity with k-essence

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Abstract For a large class of mass-varying massive-gravity models, the graviton mass cannot provide the late-time cosmic expansion of the universe due to its vanishing at late time. In this work, we propose a new class of mass-varying massive gravity models, in which the graviton mass varies according to a kinetic term of a k-essence field. By using a more general form of the fiducial metric, we found a solution such that a non-vanishing graviton mass can drive the accelerated expansion of the universe at late time. We also perform dynamical analyses of such a model and find that without introducing the k-essence Lagrangian, the graviton mass can be responsible for both dark contents of the universe, namely dark energy, which drives the accelerated expansion of the universe, and non-relativistic matter, which plays the role of dark matter. Moreover, by including the k-essence Lagrangian, we find that it is possible to alleviate the so-called cosmic coincidence problem.

1 Introduction

Massive gravity has its own series of developments as a modified gravity beyond general relativity. Back in 1939, Fierz and Pauli investigated a first model of massive gravity [1]. The model was a linearized general relativity, where the fluctuation of geometry propagates a spin-2 graviton, plus linear interactions, which, in particle physics language, corresponds to giving a non-zero mass to the graviton; hence the name "massive gravity". This model was supposed to coincide with general relativity in the massless limit but it faced a theoretical crisis when discontinuities in such a limit were found by van Dam et al. [2,3]. In particular, the discontinuities were found as different predictions between Fierz–Pauli massive gravity and general relativity. The problem remained unsolved for several years, until Vainshtein proposed a way out by introducing higher-order interactions into the Fierz–Pauli massive gravity [4]. In other words, he claimed that within a particular scale, coined the Vainshtein radius, any predictions from the linear theory cannot be trusted unless nonlinear contributions are taken into account. However, adding such nonlinearities, claimed by Boulware and Deser, not only fixes the discontinuity problem but also introduces a theoretical inconsistency, namely a Boulware–Deser ghost [5]. This ghost is an extra degree of freedom, apart from 5 degrees of freedom originally existing in the linear massive gravity, whose kinetic term has the wrong sign. The ghost problem had been a blockage for the massive-gravity theory until recently, in 2010, de Rham, Gabadadze, and Tolley found suitable nonlinear interactions which do not excite the Boulware–Deser mode; this is dubbed dRGT massive gravity [6,7]. Thus, massive gravity became again an active field of study.

Although it was just a generalization back then, massive gravity has its modern motivations. Introducing a non-zero mass to a graviton shrinks the scale at which the gravity works. In other words, the graviton mass weakens the gravitation at a large scale. As a result, it allows a cosmic acceleration and hence may be able to describe the mysterious dark energy in its language. This motivates cosmologists to study its cosmological implications. Moreover, since de Rham, Gabadadze, and Tolley found a healthy nonlinear massive gravity model, the theory had again opened a door to various researches on massive gravity; not only its cosmology but also the study of astrophysical objects in the theory, like black holes [8–13]. For cosmological models of massive gravity, it has been found that the solutions in the models with Minkowski fiducial metric do not admit the flat and closed FLRW solutions for the physical metric [14,15]. In order to obtain all kinds of FLRW solutions, one may consider a general form of the fiducial metric [16–20].

It has been found, however, that there are some inconsistencies when cosmology is taken into account. For example, some degrees of freedom cease to exist when the

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Friedmann–Lemaître–Robertson–Walker (FLRW) ansatz is assumed [19]. This leads to numerous studies beyond the dRGT massive gravity [21–39]. One of those is to generalize a constant graviton mass to be varied by other scalar field, dubbed mass-varying massive gravity [24–27]. The theory is proven to be free from a Boulware–Deser ghost. However, cosmological implications of such a model indicates a universe with subdominant contributions from massive gravity. In particular, the graviton mass is governed by the inverse of a scale factor of the universe which will vanish at late time. Consequently, such a model cannot give a proper explanation of the cosmic expansion caused by the massive graviton.

In this work, we propose an alternative way to construct a mass-varying massive gravity. The graviton mass is not only determined by a scalar field, but also by the kinetic term of the scalar field. Moreover, the scalar field is governed by a k-essence Lagrangian [40–42]. Under the FLRW ansatz, we found a solution whose the graviton mass do not necessarily vanish at late time. Moreover, by assuming both the k-essence and the graviton mass to behave as perfect fluids, we found that the graviton mass can give rise to a "dust-like" matter while combined with other contributions it is possible to have an equation of state parameter close to -1, as suggested by recent observations [43]. Such matter may be responsible for a dark matter, another mysterious content known to exist in addition to the ordinary matter. Since the graviton mass can give rise to both of the dark contents, it is tempting to consider as regards its evolution whether there exists an epoch in which the two contents in the dark sector are comparable, the so-called cosmic coincidence problem.

Our paper is organized as follows. In Sect. 2, the proposed model is addressed along with its equations of motion in the FLRW background. We also discuss some crucial properties of the model in this section where we have shown the existence of the dust-like matter expected to be responsible for the dark matter. With the help of appropriate assumptions, we show in Sect. 3 the solution to this model which corresponds to the dark energy and the non-vanishing characteristic of the graviton mass existing in this model. After sketching some perspectives, we begin the dynamical system analysis in Sect. 4 to find all possible fixed points and their stabilities, and the extended analyses are covered in Sect. 5. We conclude our work in the last section by the discussion of key ideas of our work and of whether or not the coincidence problem is alleviated.

2 The model and the background equations

We consider a mass-varying dRGT massive-gravity action where the graviton mass is varied by the k-essence field. Usually, one may consider the graviton mass as a function which varies as the scalar field propagates [24–27]. However, in this work, we will consider the graviton mass not only as a function of the scalar field ϕ but also its kinetic term $X \equiv -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. The action of such a model can be expressed as

$$S = \int d^4x\sqrt{-g}\bigg[\frac{M_p^2}{2}R[g] + V(X,\phi)(\mathcal{L}_2[g,f] + \alpha_3\mathcal{L}_3[g,f] + \alpha_4\mathcal{L}_4[g,f]) + P(X,\phi)\bigg], \quad (1)$$

where R is a Ricci scalar corresponding to a physical metric $g_{\mu\nu}$, $V(X,\phi)$ is a square of the graviton mass which depends on the scalar field and its kinetic term, $\mathcal{L}_i$ represents the interactions of the ith order of the massive graviton, and $P(X,\phi)$ is the Lagrangian of the k-essence field. In particular, those interactions of the massive graviton are constructed from two kinds of metrics and can be expressed as follows:

$$\mathcal{L}_2[g,f] = \frac{1}{2}\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right), \quad (2)$$

$$\mathcal{L}_3[g,f] = \frac{1}{3!}\left([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\right), \quad (3)$$

$$\mathcal{L}_4[g,f] = \frac{1}{4!}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]), \quad (4)$$

where the tensor $\mathcal{K}_{\mu\nu}$ is constructed from the physical metric $g_{\mu\nu}$ and an another metric $f_{\mu\nu}$ as

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_\nu, \quad (5)$$

where the square roots of those tensors are defined so that $\sqrt{g^{-1}f}^\mu{}_\rho\sqrt{g^{-1}f}^\rho{}_\nu = (g^{-1}f)^\mu{}_\nu$. In massive gravity, apart from the physical metric, there exists another kind of the metric tensor, $f_{\mu\nu}$, usually named "fiducial metric", which is an object introduced to the theory so that one can construct non-trivial interactions from the metric tensors as in Eqs. (2), (3), and (4). Those complicated combinations in the interactions, with arbitrary values of the parameters α_3, α_4, are to ensure the absence of the Boulware–Deser (ghostly) degree of freedom [6,7]. Moreover, thanks to the Stuckelberg tricks, the general covariance, or the gauge symmetry, can be well integrated into the massive gravity via

$$f_{\mu\nu} = \partial_\mu\varphi^\rho\partial_\nu\varphi^\sigma\tilde{f}_{\rho\sigma}, \quad (6)$$

provided that each of the fields φ^μ transforms as a scalar under any coordinate transformation. As for the $\tilde{f}_{ab}$, one can choose it to be any kind of metric which shares the symmetries of the physical metric. For example, one can have a four-dimensional Minkowski metric being the fiducial metric for a cosmological solution [15], or even a higher-dimensional kind of metric whose reduced four-dimensional metric is isotropic and homogeneous and is considered as the fiducial metric in the cosmological solution [20].

In this work, we consider the cosmological implications of the proposed model, where the isotropic and homogeneous universe is assumed whose spacetime is represented quite well by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric as follows:

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \Omega_{ij}(x) dx^i dx^j, \tag{7}$$

where $N(t)$ is a lapse function, $a(t)$ represents a scale factor, which determines the scale of the spatial distance, and

$$\Omega_{ij}(\varphi) = \delta_{ij} + \frac{k\delta_{ia}\delta_{jb}\varphi^a\varphi^b}{1-k\delta_{lm}\varphi^l\varphi^m}, \tag{8}$$

is the spatial maximally symmetric metric whose spatial curvature is characterized by $k \in \{-1, 0, +1\}$ corresponding to the open, flat, and closed geometry, respectively. As claimed, the FLRW ansatz is also used as the fiducial metric,

$$\tilde{f}_{\mu\nu} d\varphi^\mu d\varphi^\nu = -n(\varphi^0)^2 \left(d\varphi^0\right)^2 + \alpha(\varphi^0)^2 \Omega_{ij}(\varphi) d\varphi^i d\varphi^j, \tag{9}$$

where n and α are a lapse function and a scale factor in the fiducial sector. Plugging those in Eq. (1), the mini-superspace action of the model reads

$$S = \int d^4x \sqrt{\frac{1}{1-kr^2}} \left[M_p^2 \left(-3\frac{a\dot{a}^2}{N} + 3kNa \right) + 3Na^3 V \left(F - G\frac{n}{N} \right) + Na^3 P \right], \tag{10}$$

where

$$F \equiv \left(2 + \frac{4}{3}\alpha_3 + \frac{1}{3}\alpha_4 \right) - (3 + 3\alpha_3 + \alpha_4)\bar{X} + (1 + 2\alpha_3 + \alpha_4)\bar{X}^2 - (\alpha_3 + \alpha_4)\frac{\bar{X}^3}{3}, \tag{11}$$

$$G \equiv \frac{1}{3}(3 + 3\alpha_3 + \alpha_4) - (1 + 2\alpha_3 + \alpha_4)\bar{X} + (\alpha_3 + \alpha_4)\bar{X}^2 - \alpha_4\frac{\bar{X}^3}{3}, \tag{12}$$

and we have defined

$$\bar{X} \equiv \frac{\alpha}{a}, \quad \eta \equiv \frac{n}{N}. \tag{13}$$

To determine the dynamics of the system, one can vary the action in Eq. (10) with respect to dynamical variables which are N, a, ϕ, and the Stuckelberg fields φ^μ. The corresponding equations of motion, assuming the unitary gauge $\varphi^\mu = x^\mu$ for simplicity, read

$$M_p^2 \left(3H^2 + 3\frac{k}{a^2} \right) = -3VF + 6XV_{,X}(F - G\eta) + \left(2XP_{,X} - P\right), \tag{14}$$

$$M_p^2 \left(\frac{2\dot{H}}{N} + 3H^2 + \frac{k}{a^2} \right) = -3VF + VF_{,\bar{X}}\left(\bar{X} - \eta\right) - P, \tag{15}$$

$$\frac{\dot{V}}{V} = NH\left(1 - h\bar{X}\right)\frac{F_{,\bar{X}}}{G}, \tag{16}$$

$$Na^3\left(3V_{,\phi}(F - G\eta) + P_{,\phi}\right) = \frac{d}{dt}\left[\left(a^3\sqrt{2X}\right)\left(3V_{,X}(F - G\eta) + P_{,X}\right)\right], \tag{17}$$

$$3HN(-2XP_{,X} - 6XV_{,X}(F - G\eta) + VF_{,\bar{X}}(\bar{X} - \eta)) = \frac{d}{dt}\left(-3VF + \left(2XP_{,X} + 6XV_{,X}(F - G\eta)\right) - P\right), \tag{18}$$

where the last equation is obtained from the conservation on the energy-momentum tensor; $\nabla_\mu T^\mu_\nu = 0$ and we have defined

$$h \equiv \frac{H_\alpha}{H}, \quad H_\alpha \equiv \frac{\dot{\alpha}}{\alpha n}. \tag{19}$$

From the above equations, one can see that Eq. (14) is a Friedmann equation with extra matter contents coming from the graviton mass. As a partner to the Friedmann equation, the so-called acceleration equation corresponds to Eq. (15). Since we have the Bianchi identity relating the equations of motion, these five equations of motion are not entirely independent. Note that this set of equations recovers the original self-accelerating cosmology when the square of a graviton mass V is constant by which the usual condition $F_{,\bar{X}}(1 - h\eta)$ is obtained readily from Eq. (16) [15]. However, as V is no longer constant, the equations of motion look even more complex than those in general relativity. To simplify the following calculations, we choose P such that the k-essence field behaves as a perfect fluid. The appropriate form of P, which satisfies such a behavior, is

$$P(X, \phi) = P_0 X^{\frac{1+w}{2w}} = P_0 X^{\gamma/2}, \tag{20}$$

where $\gamma \equiv 2XP_{,X}/P \equiv \frac{1+w}{2w}$, P_0 is a constant, and w is an equation of state parameter corresponding to the perfect fluid represented by the k-essence field [44]. Moreover, we let the graviton mass function mimic the perfect-fluid form as

$$V = V_0 X^{\lambda/2}, \tag{21}$$

whose λ characterizes the power of the kinetic term as γ does for P, i.e. $\lambda \equiv 2XV_{,X}/V$ and V_0 is a constant. Note that under these assumptions, both P and V vary according to the kinetic term of ϕ but not the ϕ itself. Usually, in the quintessence model the continuity equation for the scalar field is obtained from the equation of motion of ϕ [45,46]. Taking that into account, we consider the equation of motion of ϕ in Eq. (17); then under the perfect-fluid assumptions for P and V in Eqs. (20) and (21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{a^3}{\sqrt{2X}}\right)\left(6XV_{,X}(F-G\eta)+2XP_{,X}\right)\right)=0. \quad (22)$$

After simple manipulations, the above equation gives the continuity equation for the k-essence field as

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_X+3HN\rho_X=\frac{\dot{X}}{2X}\rho_X, \quad (23)$$

where we have defined

$$\rho_X\equiv\left(2XP_{,X}+6XV_{,X}(F-G\eta)\right). \quad (24)$$

Equation (23) determines the dynamics of the matter of energy density ρ_X which resides in the Friedmann equation in Eq. (14). Interestingly, this looks exactly like a continuity equation of a "dust-like" matter with the interaction with the other matter sector determined by the flow rate of the form $\frac{\dot{X}}{2X}\rho_X$. One can also integrate Eq. (22) to find an expression for ρ_X in terms of the scale factor as

$$\rho_X=\frac{\sqrt{2X}C}{a^3}, \quad (25)$$

where C is an integration constant. In the case of a constant X, this ensures one of the properties that this matter shares with the dust; the energy density is inversely proportional to a^3 as the dust is. According to such characteristics, it is reasonable to interpret ρ_X as a dark matter. By doing so, this kind of dark matter possesses some interesting features. First of all, ρ_X is a dust-like matter which can arise naturally from the massive-gravity sector indicating that dark matter may be just an artifact of the varying graviton mass caused by the kinetic term of the k-essence field. Moreover, this claim is still valid even in the case of $P=0$. Since a graviton mass can represent dark energy in a generic class of the dRGT massive gravity, this suggests a unification of the dark sector, namely dark energy and dark matter, by such a varying graviton mass. Second, by having this kind of matter in the theory, we may expect this model of mass-varying massive gravity to solve the cosmic coincidence problem, where the universe is known to be composed mainly of comparable amounts of dark energy and dark matter. Thanks to the unification suggested above, it may be possible to provide an explanation on the coincidence problem by the existence of the graviton mass alone, while the cosmic acceleration also counts.

Since the equations of motion are coupled in a very cumbersome way, to get a picture of the whole of this system we need to perform a dynamical analysis, which is the main subject in the very last section. However, we can still get some rough descriptions, as a guideline to the dynamical analysis, by introducing some simple assumptions to the system, which is done in the next section.

3 Dark energy solution for the self-accelerating universe

It is widely known that our universe is expanding with an acceleration for which dark energy is responsible. There is recent observational evidence indicating that the observed effective equation of state parameter of the dark energy is close to -1 [43]. In this section, we shall adopt this characteristic by treating all the contributions from the graviton mass to have such a property. We define

$$\rho_g\equiv-3VF+6XV_{,X}(F-G\eta), \quad (26)$$
$$p_g\equiv3VF-VF_{,\bar{X}}\left(\bar{X}-\eta\right). \quad (27)$$

From the above definition, the corresponding equation of state parameter is defined as

$$w_g\equiv\frac{p_g}{\rho_g}. \quad (28)$$

By treating ρ_g as an energy density of dark energy, we set $w_g=-1$ and then we have the following condition:

$$6XV_{,X}(F-G\eta)=VF_{,\bar{X}}\left(\bar{X}-\eta\right). \quad (29)$$

To simplify the calculation, we use the perfect-fluid form of V in Eq. (21). Consequently, Eq. (29) becomes

$$3\lambda(F-G\eta)=F_{,\bar{X}}\left(\bar{X}-\eta\right), \quad (30)$$
$$\lambda=\frac{F_{,\bar{X}}\left(\bar{X}-\eta\right)}{3(F-G\eta)}. \quad (31)$$

Equation (31) is a requirement for the exponent λ to have a solution with the equation of state equal to -1. To get a picture of this characteristic, let us assume

$$\bar{X}=\text{constant}, \quad (32)$$
$$\eta=\text{constant}, \quad (33)$$
$$\text{then}\quad h=\frac{1}{\eta}. \quad (34)$$

Under these assumptions, the exponent λ in Eq. (31) is just a constant. To investigate this further, we consider Eq. (16) under the previous assumptions,

$$\begin{aligned}\frac{\dot{V}}{V} &= NH\left(1-h\bar{X}\right)\frac{F_{,\bar{X}}}{G},\\ \frac{\lambda\dot{X}}{2X} &= NH\left(1-\frac{\bar{X}}{\eta}\right)\frac{F_{,\bar{X}}}{G},\\ &= -\left(\bar{X}-\eta\right)\frac{F_{,\bar{X}}}{G\eta}\frac{\dot{a}}{a}.\end{aligned} \tag{35}$$

From the condition of λ in Eq. (31),

$$\frac{\dot{X}}{X} = -\frac{6\,(F-G\eta)}{G\eta}\frac{\dot{a}}{a}. \tag{36}$$

Since $\bar{X}$, η, and hence F and G, are constant, this equation can be integrated easily,

$$\begin{aligned}\int\frac{\mathrm{d}X}{X} &= -\frac{6\,(F-G\eta)}{G\eta}\int\frac{\mathrm{d}a}{a},\\ X &= C_0 a^{-\frac{6(F-G\eta)}{G\eta}}\end{aligned} \tag{37}$$

where C_0 is an integration constant. Now we have

$$V = V_0 X^{-\frac{\left(1-\frac{\bar{X}}{\eta}\right)\eta F_{,\bar{X}}}{6(F-G\eta)}} = V_0 C_0 a^{\left(1-\frac{\bar{X}}{\eta}\right)\frac{F_{,\bar{X}}}{G}}. \tag{38}$$

Furthermore, Eq. (37) possibly determines a relation between the scale factor and the rate of change of the scalar field, since

$$X = \frac{\dot{\phi}^2}{2N^2} = C_0 a^{-\frac{6(F-G\eta)}{G\eta}}. \tag{39}$$

The expression of V in Eq. (38) shows the evolution of the (square of the) graviton mass as a evolves. In the previous model of mass-varying massive gravity [24–27], in which the Minkowski fiducial metric is used, the varying graviton mass shrinks as the scale factor grows. In this model, however, the exponent in Eq. (38) determines whether the graviton mass will shrink or not as the scale factor grows, or whether it will remain constant in the case that the exponent vanishes. Note that this crucial difference is caused by the different form of the fiducial metric, which is the FLRW metric in this case, to be compared with the Minkowski one in the previous models. This result indicates the sensitivity of the fiducial metric existing in the generic dRGT massive gravity where different fiducial metrics set different stages for the system and provide different solutions [16–20].

One more crucial point of this analysis is that the contributions from the graviton mass can have the same equation of state parameter as dark energy, while one of those contributions possesses the characteristic of dust, namely the term $6XV_{,X}\,(F-G\eta)$. From Eq. (23), such a term belongs to the dark matter ρ_X. This may be a way out for the cosmic coincidence problem, since we may infer that varying graviton mass is responsible for a dark matter via the term like $6XV_{,X}\,(F-G\eta)$, as we have claimed in the previous section, while it can still drive the accelerating expansion. To verify this idea, and to seek a finer description of this model, we will perform a dynamical analysis, which can be found in the next section.

4 Dynamical system

In this section, we will consider the dynamics of the universe to be governed by this new class of mass-varying massive gravity models using the method of the autonomous system. Due to the complexity of the graviton mass, we will begin this section with a simple analysis by considering the flat FLRW where $k = 0$ and assuming that $\bar{X}$, η are constant over time, thus $h = 1/\eta$. From this assumption, the evolution of X is simply determined by Eq. (16) such that

$$X' = \frac{\dot{X}}{HNX} = \frac{2}{\lambda}\frac{F_{,\bar{X}}}{G}\left(1-h\bar{X}\right) = -\frac{6s}{\lambda r}, \tag{40}$$

$$\lambda \equiv \frac{2XV_{,X}}{V}, \tag{41}$$

where the prime denotes the derivative with respect to $\ln a$. The parameters r and s are constant and defined as

$$r \equiv \frac{G\eta}{F}, \quad s \equiv \frac{F_{,\bar{X}}(\bar{X}-\eta)}{3F}. \tag{42}$$

In order to obtain a suitable autonomous system, let us define dimensionless variables as follows:

$$x = -\frac{FV}{M_p^2 H^2}, \tag{43}$$

$$z = -\frac{P}{3M_p^2 H^2}, \tag{44}$$

$$y = \frac{2XP_{,X} + 6XV_{,X}F(1-r)}{3M_p^2 H^2} = \frac{\rho_X}{3M_p^2 H^2}, \tag{45}$$

$$\gamma \equiv \frac{2XP_{,X}}{P}. \tag{46}$$

By using these variables, the equations of motion can be written in the form of autonomous equations as

$$x' = 3x\left(y + sx - \frac{s}{r}\right), \tag{47}$$

$$y' = 3y\left(y + sx - 1 - \frac{s}{\lambda r}\right), \tag{48}$$

$$\lambda' = \frac{6s}{r}\left(\frac{\lambda}{2} - (1+\Gamma)\right), \tag{49}$$

Table 1 Summary of the properties of the fixed points

Name	x	y	z	w_{eff}	Existence	Stability
(a)	0	0	1	-1	$\gamma = 0$	$0 \le \frac{s}{r} \le 1$
(b)	$\frac{1}{r}$	0	$1 - \frac{1}{r}$	$-1 + \frac{s}{r}$	$\gamma = \lambda$	$\frac{\lambda}{1-\lambda} \le \frac{s}{r} < 0$
(c)	0	$1 + \frac{s}{\lambda r}$	$-\frac{s}{\lambda r}$	$\frac{s}{\lambda r}$	$\gamma = 1 + \frac{\lambda r}{s}$	$\frac{\lambda}{1-\lambda} < \frac{s}{r} < -1$
(d)	$\frac{1}{1+\lambda(r-1)}$	$\frac{\lambda(r-1)}{1+\lambda(r-1)}$	0	$\frac{1}{\lambda-1}$	$\lambda = \frac{s}{s-r}$	$0 < \lambda < 1$
(e)	$\frac{1+(\lambda-1)z_0}{1+\lambda(r-1)}$	$-\frac{\lambda(1-r(z_0+1))}{1+\lambda(r-1)}$	z_0	$\frac{1}{\lambda-1}$	$\lambda = \gamma = \frac{s}{s-r}$	$0 < \lambda < 1$

$$1 = x + y + z, \tag{50}$$

$$y = -\lambda x(1-r) - z\gamma, \tag{51}$$

where $\Gamma \equiv XV_{,XX}/V_{,X}$. Since we have five variables with two constraints, it is sufficient to consider only three equations. Note that the constraint in Eq. (50) is derived from Eq. (14), while the constraint in Eq. (51) is obtained from the definition of y in Eq. (45). The equation of λ in Eq. (49) is not directly dependent on the other variables. Therefore, in principle, we can solve it separately. For simplicity, we can consider λ as a parameter and then consider only the autonomous equations with two variables, x and y. We will extend our analysis to a more general case with λ being the variable in the next section. The effective equation of state parameter can be written in terms of the dimensionless variables as

$$w_{\mathrm{eff}} = \frac{P + 3VF - VF_{,\bar{X}}(\bar{X}-\eta)}{3M_p^2H^2} = -z - x + xs$$
$$= -1 + y + xs. \tag{52}$$

From these autonomous equations, the corresponding fixed points can be found by evaluating $x' = 0$ and $y' = 0$ in Eqs. (47) and (48), respectively. The properties of all the fixed points are summarized in Table 1, while the analyses are separately discussed for each of the fixed points below.

4.1 Fixed point (a)

From Eqs. (47) and (48), it is obvious that the system has a fixed point $(x, y) = (0, 0)$. By using the constraint equations, one obtains $z = 1$ and $\gamma = 0$. This means that the function P is constant and then this point corresponds to general relativity with a cosmological constant where the universe is dominated by the cosmological constant. To ensure such a claim, one can compute the corresponding effective equation of state parameter, which yields $w_{\mathrm{eff}} = -1$. This is exactly the equation of state parameter of the cosmological constant which drives the accelerating de Sitter expansion.

The stability of the fixed point can be found by analyzing the eigenvalues of the linearly perturbed autonomous equations. By performing the linear perturbations, the eigenvalues can be written as $(\mu_1, \mu_2) = (-3s/r, -3 - 3s/r)$. The stability requires both of the eigenvalues to be negative, or otherwise the fixed point is said to be unstable or to be a saddle fixed point. In this case, the signs of those eigenvalues are determined by the value of the term $\frac{s}{r} = \left(\bar{X} - \eta\right) \frac{F_{,\bar{X}}}{G\eta}$, which means $0 \le \frac{s}{r} \le 1$ for the stable fixed point. Note that in the case of vanishing eigenvalues, like $s = 0$, one has to consider the perturbations up to second order or use a numerical investigation in order to determine the stability. In this analysis, we ensure the stability in this case by the numerical method and we have found that it is stable.

Even though this fixed point can provide a period of late-time expansion, it is not much of interest due to the disappearance of the graviton mass. This resulting property is one of the drawbacks in the previous model of mass-varying massive gravity [24–27].

4.2 Fixed point (b)

One of possible fixed points may be in the form $(x, y) = (x_0, 0)$ by which the universe is governed mainly by massive gravity alone. From Eq. (47), one can find x_0 as follows:

$$x_0 = \frac{1}{r}. \tag{53}$$

According to Eq. (45), there are two possible solutions for this kind of fixed point. One is $r = 1$ in which $x_0 = 1$, $z_0 = 0$, and another one is $\lambda = \gamma$ in which $x_0 = \frac{1}{r}$, $z_0 = 1 - \frac{1}{r}$. The effective equation of state parameter can be written as

$$w_{\mathrm{eff}} = -1 + \frac{F_{,\bar{X}}(\bar{X}-\eta)}{3G\eta} = -1 + \frac{s}{r}. \tag{54}$$

Interestingly, $w_{\mathrm{eff}} = -1$ as $F_{,\bar{X}} = 0$ or $(\bar{X} - \eta) = 0$. This characteristic is a usual cosmological solution of the original massive gravity. In particular, this condition indicates that the graviton mass ceases to vary, according to Eq. (16). Moreover, since in this case $z = 1 - \frac{1}{r}$, the pressure of the k-essence field is non-zero for $r > 1$, which means the k-essence field is supposed to be a form of matter with non-zero pressure (not dust).

In order to find the stability condition for this fixed point, one can find the eigenvalues of the linearly perturbed autonomous equations, which can be written as

$$(\mu_1, \mu_2) = \left(3\frac{s}{r}, -3 + 3\frac{(\lambda - 1)s}{\lambda r}\right). \quad (55)$$

Again, both of the eigenvalues contain the term s/r, and then the fixed point will be stable if $\frac{\lambda}{1-\lambda} \leq \frac{s}{r} < 0$. Note that, for this fixed point, it is possible to provide $w_{\rm eff} < -1$ to satisfy the observation, which indicates that the mean value of the equation of state parameter is slightly less than -1 [43].

4.3 Fixed point (c)

One can obtain a fixed point such that $(x, y) = (0, y_0)$. From Eq. (47), one can find y_0 as follows:

$$y_0 = 1 + \frac{s}{\lambda r}. \quad (56)$$

By using the constraint equation in Eq. (50), one obtains $z_0 = -\frac{s}{\lambda r}$. From the constraint equation in Eq. (51), we have

$$\gamma = -\frac{y}{z} = -1 + \frac{1}{z} = 1 + \frac{1}{w_m}, \quad (57)$$

where w_m is the equation of state parameter of the fluid contributed from $P(X) = P_0 X^{(1+w_m)/2w_m}$. The effective equation of state parameter can be written as

$$w_{\rm eff} = -z = \frac{s}{\lambda r}. \quad (58)$$

Again, there exist two significant branches of the solution such that this fixed point is a matter-dominated point. If $z = 0$, this corresponds to $w_{\rm eff} = 0$, which leads to the universe being in a matter-dominated period.

The eigenvalues of the autonomous system can be written as

$$(\mu_1, \mu_2) = \left(3 + 3\frac{s}{\lambda r}, 3 - 3\frac{s(\lambda - 1)}{\lambda r}\right). \quad (59)$$

If one requires this point to represent the matter-dominated epoch, one must set the parameters so that this point is unstable. This means the universe should evolve through this point to end up in other stable points since we know the matter-dominated epoch should exist in the universe's timeline but not nowadays. One can see that, for small negative value of s/r, the universe can evolve in the standard history at which fixed point (c) corresponds to a matter-dominated period with $w_{\rm eff} \sim 0$, and fixed point (b) corresponds to the late-time expansion of the universe due to the contribution from the graviton mass. However, it is not possible to alleviate the coincidence problem, since the contribution of non-relativistic matter vanishes at late time.

4.4 Fixed point (d)

According to Eqs. (47) and (48), one may consider the fixed point corresponding to the non-zero x and y. This point can be obtained by evaluating both (non-zero) x and y from Eqs. (50), (51), and (47), while a constraint on the parameters by which the non-zero (x, y) exist can be obtained from Eqs. (47) and (48). After simple manipulation, we have

$$x = \frac{1}{1 + \lambda(r - 1)}, \qquad y = \frac{\lambda(r - 1)}{1 + \lambda(r - 1)}, \qquad \text{and } z = 0, \quad (60)$$

where γ is arbitrary and λ is fixed to be $\lambda = \frac{s}{s-r}$. The effective equation of state parameter can be written as

$$w_{\rm eff} = \frac{1}{\lambda - 1}. \quad (61)$$

To determine the stability of this point, we find the eigenvalues of the system of equations. Interestingly, this point renders the two autonomous equations degenerate. This can be seen by computing the linear perturbed equations for both x and y evaluated at this fixed point. The eigenvalues of this autonomous system are expressed as

$$(\mu_1, \mu_2) = \left(0, \frac{3\lambda}{\lambda - 1}\right). \quad (62)$$

The vanishing eigenvalue here is nothing but an artifact of the degeneracy due to this fixed point. In particular, it is possible to redefine the variables such that the problem is reduced into a one-dimensional system. With such a redefinition, the stability of this fixed point is due to the non-zero eigenvalue in Eq. (62), which can be negative when $0 < \lambda < 1$. If this condition is taken into account, requiring the fixed point (c) to represent the matter-dominated era will restrict the combination $\frac{s}{r}$ to vanish.

This fixed point seems to provide a possible way to alleviate the coincidence problem due to the non-zero y. However, it cannot be used since, at the late-time expansion, $w_{\rm eff}$ must approach -1 and then lead to the fact that $(x, y) \to (1, 0)$. Nevertheless, it still provides an interesting result. For the case of $s = 0$ and $0 < \lambda \ll 1$, this fixed point is stable, while the fixed point (b) is unstable and then we can use this fixed point as the one for the late-time expansion of the universe. For this condition the fixed point (c) is still used for the matter-dominated period with $z = 0$. Therefore, this means that it is possible to obtain $z = 0$ for the whole history of the universe. This leads to the fact that, without providing an extra non-relativistic matter field such as dark matter, the contribution from the graviton mass can play the role of both dark matter and dark energy. This is one of the crucial properties of this model, since it can unify the two main unknown contents of the universe, dark matter and dark energy, by using only a graviton mass.

4.5 Fixed point (e)

Similarly to the derivation in fixed point (d), one can solve an algebraic equation by imposing $\gamma = \lambda$ and requiring non-zero x and y. As the result, the fixed point can be expressed as

$$x = \frac{1+(\lambda-1)z_0}{1+\lambda(r-1)}, \quad y = -\frac{\lambda(1-r(z_0+1))}{1+\lambda(r-1)}, \quad z = z_0, \tag{63}$$

where $\gamma = \lambda = \frac{s}{s-r}$ and z_0 is arbitrary. The effective equation of state parameter is the same as the one in the fixed point (d), which can be written as

$$w_{\text{eff}} = \frac{1}{\lambda - 1}. \tag{64}$$

Moreover, the eigenvalues for the stability analysis are still the same asfor the fixed point (d) and then the stability condition for this fixed point can be expressed as $0 < \lambda < 1$. Even though this fixed point shares most properties with fixed point (d), it cannot provide the unification of the two dark components, since z must have a non-zero value.

From the above analyses, we experienced the incompatibility between matter domination and the present dark energy domination. One may see that for a large λ, the fixed point (c) can represent the matter-dominated epoch, while the small value of λ is needed in the fixed point (d) or (e) to solve the coincidence problem. It is natural to generalize the theory further by allowing λ to change appropriately in time. This idea will be adopted and carefully analyzed in the next section.

5 Extended analyses

As we have mentioned, even though the model can be used to unify the dark contents of the universe, it still cannot be used to solve the coincidence problem. According to our analysis, this is due to the fact that λ is set to be a constant. In this section, we will show the possibility to solve the coincidence problem when λ is set as a dynamical variable. For completeness, we will add radiation into our consideration and then use numerical method to show that the radiation does not affect the unification in the dark sector. Note that the equation of motion for the radiation is obtained by using the conservation of its energy-momentum tensor or the continuity equation. By including the radiation and taking λ as a dynamical variable, the autonomous equations can be written as

$$x' = 3x\left(y + sx - \frac{s}{r} + \frac{4}{3}\Omega_r\right), \tag{65}$$

$$y' = 3y\left(y + sx - 1 - \frac{s}{\lambda r} + \frac{4}{3}\Omega_r\right), \tag{66}$$

$$\Omega_r' = 3\Omega_r\left(y + sx + \frac{4}{3}(\Omega_r - 1)\right), \tag{67}$$

$$\lambda' = \frac{6s}{r}\left(\frac{\lambda}{2} - (1+\Gamma)\right), \tag{68}$$

$$1 = x + y + z + \Omega_r, \tag{69}$$

$$y = -\lambda x(1-r) - z\gamma, \tag{70}$$

$$\Omega_r \equiv \frac{\rho_r}{3M_p^2 H^2}, \tag{71}$$

where ρ_r is the energy density of the radiation. The effective equation of state parameter can be written as

$$w_{\text{eff}} = -1 + y + xs + \frac{4}{3}\Omega_r. \tag{72}$$

From Eq. (67), we can see that all fixed points we found in the previous section still exist with $\Omega_r = 0$. Also, there exists the unstable fixed point such that $\Omega_r = 1$, while x and z (hence y) vanish. From Eq. (68), one can see that λ does not couple to the others and the fixed point takes place at $\lambda = 2(\Gamma + 1)$. For simplicity, one can set Γ as a constant. In order to confirm the claim in the previous section that there exists a standard evolution without introducing a k-essence Lagrangian or in the case of $z = 0$, we use numerical methods to evaluate the equations above by setting $s = 0$. The evolutions of x, y, and Ω_r are illustrated in the left panel of Fig. 1, and the evolution of the effective equation of state parameter is shown in the right panel of Fig. 1. We can see that there exists non-relativistic matter, inferred as dark matter represented by the variable y, while the variable x represents the dark energy that drives the late-time expansion of the universe. Both x and y are contributed from the graviton mass.

Now, let us consider the possibility to solve the coincidence problem. Let us use the fixed point (e) to be one corresponding to the late-time expansion of the universe. For this fixed point, the parameters s, r, and Γ are obtained by giving the initial conditions for the dynamical variables. In order to obtain the dynamics of all variables, we have to put the initial conditions slightly away from the fixed point. It is sufficient to put λ slightly above the fixed point, since we need λ to grow as time goes backward to ensure that it will have a high enough value for the matter-dominated period. In order to obtain $w_{\text{eff}} \sim -1$ at the present time, we have to set the value of the variable λ at the fixed point as $\lambda_f \to 0$. As a result, $\frac{s}{r} = \frac{\lambda_f}{\lambda_f - 1} \to 0$. In order to obtain a proper matter-dominated period, one has to put the initial value of λ far away from the fixed point. This situation makes the fixed point (b) stable and then the system evolves to the point (b)

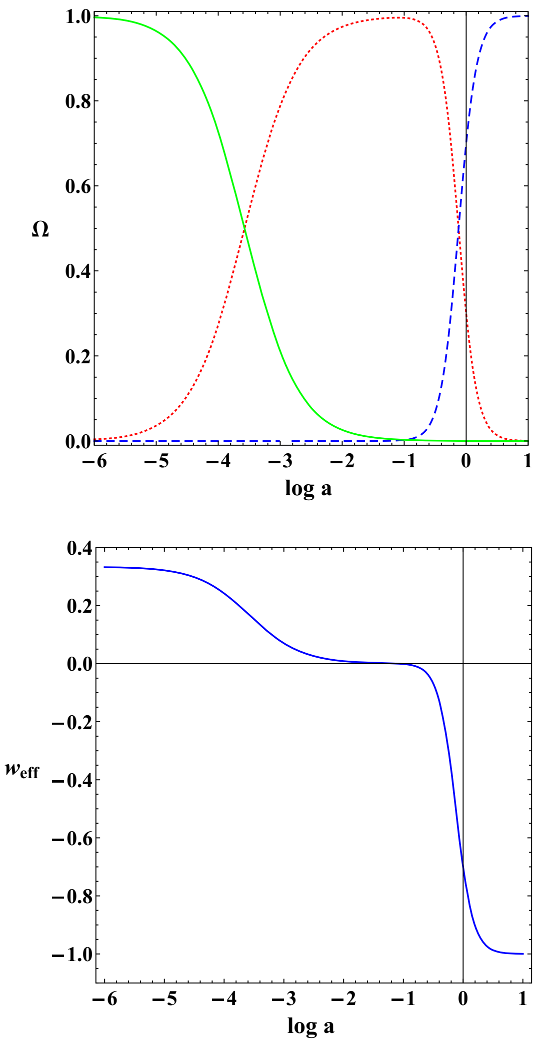

Fig. 1 The *left panel* shows the evolution of x, y, and Ω_r. The *dotted-red line* represents the evolution of x, the *dashed-blue line* represents the evolution of x, and the *solid-green line* represents the evolution of Ω_r. For the *right panel*, the evolution of $w_{\rm eff}$ is represented

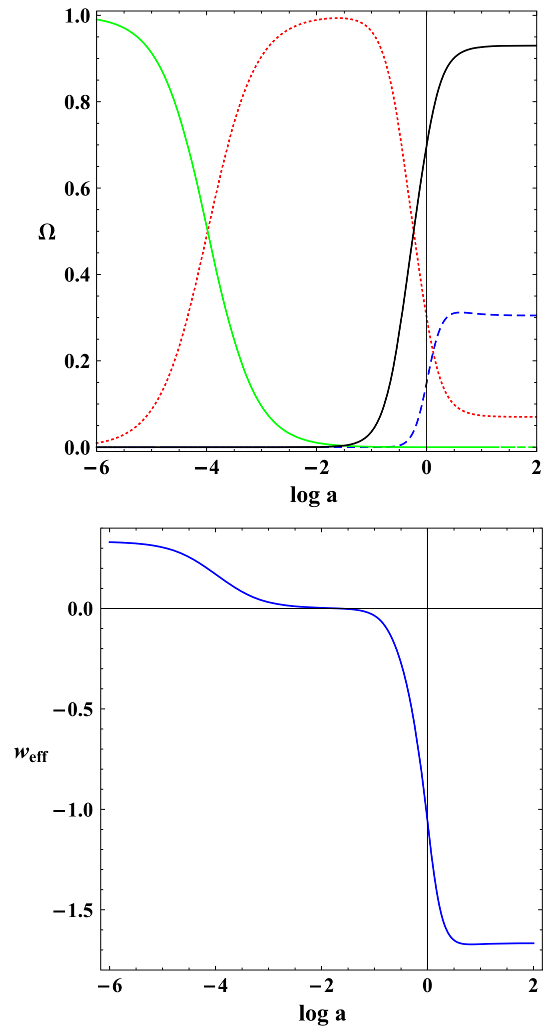

Fig. 2 The *left panel* shows the evolution of x, y, $x + z$, and Ω_r. The *dotted-red line* represents the evolution of x, the *dashed-blue line* represent the evolution of x, the *solid-black line* represents the evolution of $x + z$ and the *solid-green line* represents the evolution of Ω_r. For the *right panel*, the evolution of $w_{\rm eff}$ is represented. We set the parameters such that $\lambda_f = 0.4$ and $\lambda_0 = 1.0$ where λ_f is the value at the fixed point and λ_0 is one at the present time

eventually. Therefore, in order to have the fixed point (e) at late time, one has to set $w_{\rm eff}$ below -1 at the fixed point, so that the point (e) becomes a stable point. According to this setting, we show the evolution of the dynamical variables reaching the fixed point (e) to alleviate the coincidence problem in Fig. 2. Note that we set $\lambda_f = 0.4$, leading to $w_{\rm eff} = -1.67$ and $\lambda_0 = 1.0$.

In order to overcome the incompatibility among the fixed points, one may extend the analysis by allowing s, Γ or r to be dynamical variables. This will make the dynamical system more complicated. We found another possibility to overcome this incompatibility by imposing the constraint $\lambda = \gamma$ for the entire evolution. As a result, we have only three independent equations for six variables and three constraints. The dynamical variable λ can be written in terms of other variables as

$$\lambda = \frac{y}{rx + y + \Omega_r - 1}. \tag{73}$$

As a result, by setting the initial condition at the radiation dominated period, the evolution of the dynamical variables and the effective equation of state are shown in Fig. 3. From this figure, one can see that the evolution of the universe reaches the fixed point (e) at late time while the matter and radiation period are also properly presented. For the plot in this figure, we set $\lambda_f = 0.02$, and then the consequent results are $\Gamma = -0.99$ and $w_{\rm eff} \sim -1.02$. Note that the behavior of the resulting plot in Fig. 3 is sensitive to the initial value of x at the radiation dominated period where we set it choosing $x_i \sim 10^{-16}$.

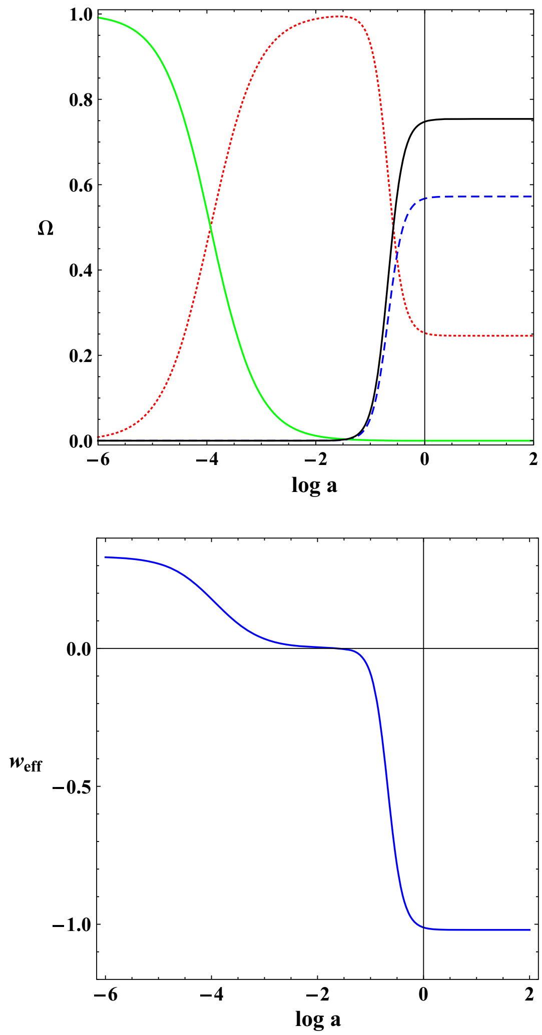

Fig. 3 The *left panel* shows the evolution of x, y, $x + z$, and Ω_r. The *dotted-red line* represents the evolution of x, the *dashed-blue line* represent the evolution of x, the *solid-black line* represents the evolution of $x + z$, and the *solid-green line* represents the evolution of Ω_r. For the *right panel*, the evolution of w_{eff} is presented

6 Conclusion

We have constructed a new class of mass-varying massive gravity models, in which not only the k-essence field but also its kinetic term determines the variation of the graviton mass. We have shown in Sect. 2 that there is a possibility for the graviton mass to live at late time compared with the previous model whose the graviton mass only depends on the scalar field and shrinks as the universe grows [24–27]. After simple manipulations and under particular assumptions, we found that a "dust-like" matter which behaves like a non-relativistic dust can naturally result from the graviton mass and it is a possible candidate for dark matter. This can be seen more clearly in the case $P = 0$ in which the dark matter comes solely from the varying graviton mass. Having such matter in the system, this model of massive gravity can describe the cosmic accelerating expansion with the equation of state parameter close to −1, while the universe is not entirely dominated by the dark energy part contributed also by the graviton mass. This property signals a possibility of having the universe composed of comparable amounts between dark energy and dark matter, known as the cosmic coincidence problem. To obtain a finer description on this, the usual method of the dynamical analysis is performed by taking the dark matter candidate into account and the results are carefully investigated as regards the issue of the coincidence problem. For the first simple case, the exponent of the kinetic term in the graviton mass λ is kept constant. We found the fixed points which correspond to various epochs in the history of the universe such as the matter-dominated period and massive-gravity-dominated periods. However, to have those fixed points with the appropriate stabilities in the evolution of the universe, the results suggest a system with λ as additional variable. The more general case, where λ is allowed to vary, is investigated where the radiation is included. While the result covers all the fixed points in the constant λ case, this allows the evolution in which there exists a matter-dominated period as well as a late-time expansion epoch. There are several crucial points in this investigation. First, we obtain the universe in which the graviton mass serves as both dark energy and dark matter, while it can still drive the cosmic acceleration. Second, to solve the coincidence problem, we obtain a universe with the effective equation of state parameter significantly below −1 unless both λ and γ are set equal with one another for the entire evolution of the universe. Since the analyses are under particular assumptions, this model still has room for study in more complicated ways. For example, one can exclude the assumptions proposed in this work for a more complex system or one can consider this model in a different aspect, like its astrophysical implications. Not only as regards the applications, but also studying the theoretical consistency, whether there exists a ghost instability or not, is a worthy challenge which we leave for future work. Apart from the constraints mentioned, one may think of constraining the model with various observations. This idea is also interesting, since the observations may judge the fate of this model by tightening it with constraints.

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