รายงานวิจัยฉบับสมบูรณ์

วิธีการแบบแยกส่วนและแบบปกติสำหรับการแก้ปัญหาค่าเหมาะสม

The regularization and splitting methods for solving optimization problems

โดย ผู้ช่วยศาสตราจารย์ ดร.ประสิทธิ์ ช่อลำเจียก

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ศาสตราจารย์ ดร.สุเทพ สวนใต้ มหาวิทยาลัยเชียงใหม่ (นักวิจัยที่ปรึกษา)

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยพะเยา

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อีเมล์: prasitch2008@yahoo.com

ระยะเวลาโครงการ: 1 มิถุนายน 2557 ถึงวันที่ 31 พฤษภาคม 2559

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Abstract

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Investigator: Assistant Professor Dr.Prasit Cholamjiak, University of Phayao

E-mail Address: prasitch2008@yahoo.com

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Abstract: The purposes of this research are to introduce new kinds of iterative methods for solving the optimization problems as well as the related problems and to investigate the convergence theorems in Hilbert spaces and Banach spaces.

Keywords: Splitting method / Regularization method / Optimization problem / Convergence theorem / Algorithm

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CHAPTER I

INTRODUCTION

In optimization theory, one of the most important and interesting problems in the theory of maximal monotone operators is to find zeroes of maximal monotone operators. To be more precise, many problems that involve convexity can be expressed as the variational inclusion problem of maximal monotone operators. These problems include convex minimization, min-max problems, complementarity problems and variational inequalities as special cases. The regularization is one of the most important techniques in handling ill-posed problems and inverse problems. The Tikhonov regularization and proximal point methods are widely used to deal with one maximal monotone operator. The proximal point algorithm (PPA) initiated by Martinet in 1970 and subsequently studied by Rockafellar in 1976 is often referred. However, since the PPA does not necessarily converges strongly, many researchers have conducted worthwhile work on modifying the PPA so that the strong convergence is guaranteed, for examples, the relaxed proximal point algorithm (RPPA) and the contraction-proximal point algorithm (CPPA). The Tikhonov regularization is another method commonly used for solving this problem. In general, many practical nonlinear problems arising in applied areas such as inverse problems especially signal processing, image recovery, and machine learning can be formulated as finding the zeroes of the operator decomposed as the sum of two maximal monotone operators. The splitting methods play a central role in the analysis and the numerical solution of such problems. The Forward-Backward and Douglas-Rachford splitting algorithms are classical methods for computing those reliable solutions. Due to its applications, there have been several modifications and generalizations of these methods suggested and invented independently for solving the problem in many different contexts.

It is therefore the main objective in this research to study the modified forward-backward splitting methods and also to investigate strong convergence theorems for solving variational inclusion problems and to give some optimization problem including its numerical experiments. The main results can improve and extend the corresponding results in this area and, of course, can be applied to solve major problems existed in science, engineering, economics and other related branches. To be more precise, we apply our main results to the minimization optimization problem and the linear inverse problem.

CHAPTER II

LITERATURE REVIEW

Let H be a real Hilbert space and let $T : H \to 2^H$ be a maximal monotone operator. A fundamental problem of monotone operators is that of finding an $x \in D(A)$ such that

$$0 \in Tx. \tag{2.1}$$

where $D(T)$ denotes the domain of T. Denote by $J_r^T = (I + rT)^{-1},\ r > 0$ the resolvent of a maximal monotone operator T.

A classical method for solving this problem introduced by Martinet [33] is the well-known proximal point algorithm (PPA), which generates, for any initial guess $x_0 \in H$, an iterative sequence as

$$x_n \in x_{n+1} + r_n T x_{n+1},\ n \geq 1 \tag{2.2}$$

where $\{r_n\}$ is a positive real sequence. Note that (2.2) is equivalent to

$$x_{n+1} = J_{r_n}^T x_n,\ n \geq 1 \tag{2.3}$$

where $\{r_n\}$ is a positive real sequence. It was shown, in a real Hilbert space, that the sequence generated by (2.3) converges weakly to a zero of T. As pointed in Eckstein [15], the ideal form of the method is often impractical since, in many cases the exact iteration (2.3) may require a computation as difficult as solving the original problem (2.1). Rockafellar [39] has given a more practical method which is an inexact variant of the method: $x_0 \in H$

$$x_n + e_n \in x_{n+1} + r_n T x_{n+1},\ n \geq 1 \tag{2.4}$$

where $\{e_n\}$ is an error sequence. Note that the algorithm (2.4) can be rewritten as

$$x_{n+1} = J_{r_n}^T (x_n + e_n),\ n \geq 1 \tag{2.5}$$

This is an inexact proximal point algorithm. It was shown that, if $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then the sequence $\{x_n\}$ defined by (2.5) converges weakly to a zero of T.

Eckstein and Bertsekas [16] constructed the relaxed proximal point algorithm (RPPA):

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)J_{r_n}^T x_n + e_n, \; n \geq 1 \tag{2.6}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ and $\{e_n\}$ is an error sequence. The weak convergence of (2.6) is guaranteed provided that $\{\alpha_n\}$ and $\{e_n\}$ satisfy some mild conditions. It is noted that Guler [17] obtained an example to show that Rockafellar's proximal point algorithm does not converge strongly, in general. Since then, there are many modifications on the PPA.

Theorem 2.1.1. *Let K be a bounded closed convex subset of a Hilbert space H and $T : K \to K$ be a nonexpansive mapping. Let $u \in K$ be arbitrary. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}$, $\theta \in (0,1)$. Deine a sequence $\{x_n\}$ in K by $x_1 \in K$,*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \; n \geq 1. \tag{2.7}$$

Then, $\{x_n\}$ converges strongly to the element of $F(T) := \{x \in K \; : \; Tx = x\}$ nearest to u.

This method is called Halpern's iteration process. Employing the Halpern's iteration, to obtain the strong convergence, in 2004, Marino and Xu [32] proposed the contraction-proximal point algorithm (CPPA): $x_0, u \in H$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_{r_n}^T x_n + e_n, \; n \geq 1 \tag{2.8}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ and $\{e_n\}$ is an error sequence. Under suitable conditions, the CPPA (2.8) converges strongly to a zero of T.

Yao and Noor [55] extended the CPPA to the following form:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{r_n}^T x_n + e_n, \; n \geq 1 \tag{2.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are a real sequence in $(0,1)$ and $\{e_n\}$ is an error sequence. It was proved that the sequence $\{x_n\}$ generated by (2.9) strongly converges to a zero of T.

In recent years, many researchers attempt to construct new algorithms and study convergence of the sequence and also to generalize and improve the works on this direction (see, for instance, [4, 6, 7, 19, 21, 37, 45, 46]).

Another powerful and successful technique to obtain strong convergence is the Tikhonov regularization method which is generated a sequence$\{x_n\}$ by the following manner:

$$x_{n+1} = J_{r_n}^{T} u, \ n \geq 1, \tag{2.10}$$

where $u \in H$ and $r_n > 0$ such that $r_n \to \infty$. The strong convergence was investigated in Hilbert spaces.

In [26], Lehdili and Moudafi combined the technique of the proximal mapping and the Tikhonov regularization to introduce the prox-Tikhonov method which generates the sequence $\{x_n\}$ by the algorithm

$$x_{n+1} = J_{r_n}^{T_n} x_n, \ n \geq 1 \tag{2.11}$$

where $T_n = u_n I + T, \ u_n > 0$ is viewed as a Tikhonov regularization of T. Using the concept of variational distance, the strong convergence is obtained under some mild conditions.

Subsequently, algorithm (2.11) was extended by Xu [52] in the following: $x_0, u \in H$

$$x_{n+1} = J_{r_n}^{T}(\alpha_n u + (1-\alpha_n)x_n + e_n), \ n \geq 1 \tag{2.12}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ and $\{e_n\}$ is an error sequence. The strong convergence was proved in a Hilbert space. Some modifications and generalizations of the Tikhonov regularization can also be found in [44, 41, 46, 50].

The problems can be solved via the proximal point algorithm. But one of the major drawbacks of this algorithm is the need to evaluate the resolvent

$$J_{r_n}^{T} = (I + r_n T)^{-1}. \tag{2.13}$$

However, in some cases, the operator T can be decomposed into the sum of two maximal monotone operators A and B whose resolvents $J^A_{r_n}$ and $J^B_{r_n}$ are easier to evaluate than $J^T_{r_n}$. In this case, the strategy is to find a zero of T by using only $J^A_{r_n}$ and $J^B_{r_n}$ rather than $J^T_{r_n}$. Such a method is called an operator splitting method.

In 1955-1956, Peaceman and Rachford [35] and Douglas and Rachford [13] introduced the splitting methods for linear equations.

In 1969-1991, Kellogg [25] and Lions and Mercier [27] (see also [34, 48, 8]) extend this method to nonlinear equations in Hilbert spaces.

The central problem is to iteratively find a zero of the sum of two monotone operators A and B in a Hilbert space H, namely, a solution to inclusion problem: find $x \in H$ such that

$$0 \in (A + B)x. \tag{2.14}$$

where $A : H \to H$ is an operator and $B : H \to 2^H$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem.

A splitting method for (2.14) means an iterative method for which each iteration involves only with the individual operators A and B, but not the sum $A+B$. To solve (2.14), Peaceman and Rachford [35] introduced the Forward-Backward Splitting Method (FBSM) which generate a sequence $\{x_n\}$ by the recursion

$$x_{n+1} = (2J^A_{r_n} - I)(2J^B_{r_n} - I)x_n, \ n \geq 1 \tag{2.15}$$

where $J^A_{r_n}, \ J^B_{r_n}$ are resolvents of $A, \ B$. It was shown that $\{x_n\}$ defined by (2.15) converges to the zeroes of $A + B$. In 1956, Douglas and Rachford [13] introduced Douglas-Rachford Splitting Method (DRSM) as follows: $x_0 \in H$ and

$$x_{n+1} = J^A_{r_n}(2J^B_{r_n} - I)x_n + (I - J^B_{r_n})x_n, \ n \geq 1 \tag{2.16}$$

where $J^A_{r_n}, \ J^B_{r_n}$ are resolvents of $A, \ B$. It was shown that $\{x_n\}$ defined by (2.16) converges to the zeroes of $A + B$.

The nonlinear Peaceman-Rachford algorithm (2.15) fails, in general, to converge (even in the weak topology in the infinite-dimensional setting). This is due to the fact that the generating operator $(2J^A_{r_n} - I)(2J^B_{r_n} - I)$ for the algorithm (2.15) is merely nonexpansive. However, the mean averages of $\{u_n\}$ can be weakly convergent [34]. The nonlinear Douglas-Rachford algorithm (2.16) always converges in the weak topology to a point u and $u = J^B_{r_n} v$ is a solution to (2.14), since the generating operator $J^A_{r_n}(2J^A_{r_n} - I) + (I - J^B_{r_n})$ for this algorithm is firmly nonexpansive, namely, the operator is of the form $\frac{(I+T)}{2}$, where T is nonexpansive.

In 2012, Takahashi et al. [47] proved some strong convergence theorems of Halpern's type in a Hilbert spacec H, which is defined by the following manner: $x_1 \in H$ and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J^B_{r_n}(x_n - r_n A x_n)), \tag{2.17}$$

wher $u \in H$ is a fixed and A is an α-inverse strongly monotone mapping on H and B is an maximal monotone operator on H. They proved that if $\{r_n\} \subseteq (0, \infty)$, $\{\beta_n\} \subseteq (0, 1)$ and $\{\alpha_n\} \subseteq (0, 1)$ satisfy

1. $0 < a \leq r_n \leq 2\alpha$,
2. $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$,
3. $0 < c \leq \beta_n \leq d < 1$,
4. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then $\{x_n\}$ generated by (2.17) converges strongly to a solution of $A + B$.

Let X be a Banach space. It shold be noted that there is a few works concerning the split method established in a Banach space setting.

Recently, López et al. [28] introduced the following Halpern-type forward-backward method: $x_1 \in X$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J^B_{r_n}(x_n - r_n(Ax_n + a_n)) + b_n), \ n \geq 1, \tag{2.18}$$

where $u \in X$, A is an α-inverse strongly accretive mapping on X and B is an m-accretive operator on X, $\{r_n\} \subseteq (0, \infty)$, $\{\alpha_n\} \subseteq (0, 1]$ and $\{a_n\}, \{b_n\}$ are error sequences in X. It was proved that the sequence $\{x_n\}$ generated by (2.18) strongly converges to a zero point of the sum of A and B under some appropriate conditions. There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces); see [11, 42, 43, 47, 49, 56].

In our research, we aim to study the forward-backward splitting methods for solving (2.14) for nonlinear operators in a certain Banach space. Furthermore, we establish the strong convergence theorem under suitable conditions. Also, we discuss a results to the minimization optimization problem and related problems including the numerical experiments. Our results generalize and improve some known others appeared in the literature.

CHAPTER III

PRELIMINARIES

3.1 Preliminaries and lemmas

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel.

Definition 3.1.1. (Fixed point)

Let X be a nonempty set and $T : X \to X$ a self-mapping. We say that $x \in X$ is a fixed point of T if

$$T(x) = x \tag{3.19}$$

and denote by $Fix(T)$ the set of all fixed points of T.

Example 3.1.2. *1. If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $Fix(T) = \{-2\}$;*

2. If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $Fix(T) = \{0, 2\}$;

3. If $X = \mathbb{R}$ and $T(x) = x + 5$, then $Fix(T) = \emptyset$;

4. If $X = \mathbb{R}$ and $T(x) = x$, then $Fix(T) = \mathbb{R}$;.

Definition 3.1.3. (Metric space)

Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function. Then d is called a *metric* on X if the following properties hold:

1. $d(x, y) \geq 0$ for all $x, y \in X$;
2. $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
3. $d(x, y) = d(y, x)$ for all $x, y \in X$;
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric d at (x, y) is called distance between x and y, and the ordered pair (X, d) is called a *metric space.*

Example 3.1.4. *The real line $\mathbb{R}$ and define*

$$d(x, y) = |x - y| \; for \; all \; x, y \in \mathbb{R}. \tag{3.20}$$

Then $(\mathbb{R}, d)$ *is a metric space.*

Example 3.1.5. *The Euclidean plane* $\mathbb{R}^2$ *and define*

$$d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} \tag{3.21}$$

where $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2) \in \mathbb{R}^2$. *Then* $(\mathbb{R}^2, d)$ *is a metric space.*

Example 3.1.6. *The Euclidean space* $\mathbb{R}^n$ *and define*

$$d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2 + ... + (\xi_n - \eta_n)^2} \tag{3.22}$$

where $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n), y = (\eta_1, \eta_2, \eta_3, ..., \eta_n) \in \mathbb{R}^n$. *Then* $(\mathbb{R}^n, d)$ *is a metric space.*

Example 3.1.7. *Let* X *be the set of all bounded sequences of complex numbers; that is every element of* X *is a complex sequence*

$$x = (\xi_1, \xi_2, ...)$$

such that $|\xi_j| \leq c_x$ *for all* $j = 1, 2, ...$ *and* c_x *is a real number which may depend on* x, *but does not depend on* j *and define*

$$d(x,y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| \tag{3.23}$$

where $y = (\eta_j) \in X$ *and* $\mathbb{N} = 1, 2, ...$ *Then* (X, d) *is a metric space.*

Definition 3.1.8. (Closed set)

Let (X, d) be a metric space. A subset $U \subseteq X$ is open if for every $x \in X$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A set U is closed if its complement $X \setminus U$ is open.

Theorem 3.1.9. *Let* M *be a nonempty subset of a metric space* X. *Then* M *is closed if and only if there exists a sequence* $\{x_n\} \subseteq M$ *and* $x_n \to x$ *implies that* $x \in M$.

Definition 3.1.10. (Convergent sequence)

A sequence $\{x_n\}$ in a metric space X is said to be convergent to $x \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ if $n > N$ then $d(x, y) < \epsilon$. In this case, we write $x_n \to x$

Definition 3.1.11. (Cauchy sequence)

A sequence $\{x_n\}$ in a metric space X is said to be Cauchy if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ if $m, n > N$ then $d(x_m, x_n) < \epsilon$.

Definition 3.1.12. (Bounded sequence)

A sequence $\{x_n\}$ in X is bounded if there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Definition 3.1.13. (Lipschitzian mapping)

Let (X, d) be a metric space. Then a map $T : X \to X$ is called a lipschitzian mapping on X if there exists $L > 0$ such that

$$d(T(x), T(y)) \leq Ld(x, y) \ for \ all \ x, y \in X.$$

Definition 3.1.14. (Nonexpansive mapping)

Let (X, d) be a metric space. Then a map $T : X \to X$ is called a nonexpansive mapping on X if

$$d(T(x), T(y)) \leq d(x, y) \ for \ all \ x, y \in X.$$

Definition 3.1.15. (Contraction mapping)

Let (X, d) be a metric space. Then a map $T : X \to X$ is called a contraction mapping on X if there exists $q \in [0, 1)$ such that

$$d(T(x), T(y)) \leq qd(x, y) \ for \ all \ x, y \in X.$$

Theorem 3.1.16. ***(The Banach contraction principle)***

Let X be a complete metric space and let T be a contraction of X into itself. Then T has a unique fixed point.

Definition 3.1.17. (Vector space)

A vector space or linear space X over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is a set X together with an internal binary operation "+" called addition and a scalar multiplication carrying (α, x) in $\mathbb{K} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

1. $x + y = y + x$;
2. $(x + y) + z = x + (y + z)$;
3. there exists an element $0 \in X$ call the *zero vector* of X such that $x + 0 = x$ for all $x \in X$;
4. for every element $x \in X$, there exists an element $-x \in X$ called *the additive inverse* or *the negative* of x such that $x + (-x) = 0$;
5. $\alpha(x + y) = \alpha x + \alpha y$;
6. $(\alpha + \beta)x = \alpha x + \beta y$;
7. $(\alpha\beta)x = \alpha(\beta x)$;
8. $1 \cdot x = x$.

The elements of a vector space X are called vectors, and the elements of $\mathbb{K}$ are called scalars.

Example 3.1.18. *The Euclidean space $\mathbb{R}^n$ and define*

$$\begin{aligned} x + y &= (\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3, ..., \xi_n + \eta_n) \\ \alpha x &= (\alpha\xi_1, \alpha\xi_2, \alpha\xi_3, ..., \alpha\xi_n) \end{aligned}$$

where $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n), y = (\eta_1, \eta_2, \eta_3, ..., \eta_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then, space $\mathbb{R}^n$ is a real vector space.

Definition 3.1.19. (Convex set)

Let C be a subset of a linear space X. Then C is said to be convex if $(1 - \lambda)x + \lambda y \in C$ for all x, y and all scalar $\lambda \in [0, 1]$.

Example 3.1.20. *1. Every subspace of vector space is convex set.*

2. $\overline{B}(x; r) = \{x : \|x\| \leq r\}$ is convex set.

3. $[0, 1]^N = [1, 0] \times [1, 0] \times ... \times [1, 0]$ is convex set in $\mathbb{R}^N$.

Proposition 3.1.21. *Let C be a subset of a linear space X. Then C is convex if and only if $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \in C$ for any finite set $\{x_1, x_2, ..., x_n\} \subseteq C$ and scalars $\lambda_i \geq 0$ with $\lambda_1 + \lambda_2 + ... + \lambda_n = 1$.*

Definition 3.1.22. (Convex function)

Let X be a linear space and $f : X \to (-\infty, \infty]$ a function. Then f is said to be convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in X$ and $\lambda \in [0, 1]$.

Example 3.1.23. *1.* $F(x) = |x|^p$ *where* $p \geq 1$ *is convex function in* $\mathbb{R}$.

2. $F(x) = x^3 - x^2$ *is convex function in* $[\frac{1}{3}, \infty)$.

3. $F(x) = x \log x$ *where* $p \geq 1$ *is convex function in* $\mathbb{R}^+$.

Definition 3.1.24. (Normed space)

let X be a norm linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\| \cdot \| : X \to \mathbb{R}^+$ a function. Then $\| \cdot \|$ is said to be a norm if the following properties hold:

1. $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;
2. $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$;
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The ordered pair $(X, \| \cdot \|)$ is called a normed space.

Example 3.1.25. $\mathbb{R}^n$ *is a normed space with the following norms:*

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \; for\ all \;\; x = (x_1, x_2, .., x_n) \in \mathbb{R}^n; \\ \|x\|_p &= \Big(\sum_{i=1}^{n} |x_i|^p \Big)^{1/p} \; for\ all \;\; x = (x_1, x_2, .., x_n) \in \mathbb{R}^n \; and\ p \in (1, \infty); \\ \|x\|_\infty &= \max_{1 \leq i \leq n} |x_i| \; for\ all \;\; x = (x_1, x_2, .., x_n) \in \mathbb{R}^n. \end{aligned}$$

Remark 3.1.26. 1. $\mathbb{R}^n$ equipped with the norm defined by $\|x\|_p = \big(\sum_{i=1}^{n} |x_i|^p \big)^{1/p}$ is denoted by l_q^n for all $1 \leq p < \infty$.

2. $\mathbb{R}^n$ equipped with the norm defined by $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ is denoted by l_∞^n.

Example 3.1.27. *Let* $X = l_1$*, the linear space whose elements consist of all absolutely convergent sequences* $(x_1, x_2, ..., x_i, ...)$ *of scalars* ($\mathbb{R}$ *or* $\mathbb{C}$),

$$l_1 = \{x : x = (x_1, x_2, ..., x_i, ...) \; and \; \sum_{i=1}^{\infty} |x_i| < \infty\}.$$

Then l_1 *is a normed space with the norm defined by* $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$.

Example 3.1.28. *let $X = l_p$ $(1 < p < \infty)$, the linear space whose elements consist of all p-summable sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ or $\mathbb{C}$),*

$$l_p = \{x : x = (x_1, x_2, ..., x_i, ...) \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}.$$

Then l_p is a normed space with the norm defined by $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$.

Example 3.1.29. *let $X = l_\infty$, the linear space whose elements consist of all bounded sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ or $\mathbb{C}$),*

$$l_\infty = \{x : x = (x_1, x_2, ..., x_i, ...) \text{ and } \{x_i\}_{i=1}^{\infty} \text{ is bounded}\}.$$

Then l_∞ is a normed space with the norm defined by $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

Definition 3.1.30. (Completeness)

The space X is said to be complete if every Cauchy sequence in X converges.

Example 3.1.31. *The Euclidean space $\mathbb{R}^n$ is complete with*

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2 + ... + (\xi_n - \eta_n)^2} \tag{3.24}$$

where $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n), y = (\eta_1, \eta_2, \eta_3, ..., \eta_n) \in \mathbb{R}^n$.

Example 3.1.32. *The sequence space l_∞ is complete.*

Example 3.1.33. *The sequence space l_p is complete.*

Definition 3.1.34. (Banach space)

A normed space which is complete with respect to the metric induced by the norm is called a Banach space.

Example 3.1.35. *The Euclidean space $\mathbb{R}^n$ is a Banach space with the norm defined by*

$$\|x\| = \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2},$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Example 3.1.36. *The space l_p, $1 \leq p < \infty$ is a Banach space with the norm defined by*

$$\|x\|_p = \Big(\sum_{i=1}^{\infty} |x_i|^p\Big)^{1/p},$$

where $x = (x_1, x_2, ..., x_n, ...)$ and $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

Example 3.1.37. *The space l_∞ of all bounded sequence $x = (x_1, x_2, ..., x_n, ...)$ is a Banach space with the norm defined by*

$$\|x\| = \sup_i |x_i|.$$

Definition 3.1.38. (Inner product space)

An inner product space is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{K}$ of X; that is, with every pair of vectors x and y there is associated a scalar which is written

$$\langle x, y \rangle \tag{3.25}$$

and is called the inner product of x and y, such that for all vectors x, y, z and scalars a we have

(IP1) $\langle x, x \rangle \geq 0$;

(IP2) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$;

(IP3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$;

(IP4) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;

(IP5) $\langle x + y, z \rangle = \langle x, z \rangle + \langle z, y \rangle$.

Example 3.1.39. *The function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \; for \; all \; x = (x_1, x_2, ..., x_n), \; y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n \tag{3.26}$$

is an inner product on $\mathbb{R}^n$. In this case $\mathbb{R}^n$ with this inner product is called real Euclidean n-space.

Example 3.1.40. *Let $\mathbb{C}^n$ be the set of n-tuples of complex numbers. Then the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} \; for \; all \; x = (x_1, x_2, ..., x_n), \; y = (y_1, y_2, ..., y_n) \in \mathbb{C}^n \tag{3.27}$$

is an inner product on $\mathbb{C}^n$*. In this case* $\mathbb{C}^n$ *with this inner product is called complex Euclidean n-space.*

Example 3.1.41. *Let* l_2 *be the set of all sequences of complex numbers* $(a_1, a_2, \ldots, a_i, \ldots)$ *with* $\sum_{i=1}^{\infty} |a_i|^2 < \infty$*. Then the function* $\langle \cdot, \cdot \rangle : l_2 \times l_2 \to \mathbb{C}$ *defined by*

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} \text{ for all } x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in l_2 \tag{3.28}$$

is an inner product on l_2*.*

Proposition 3.1.42. ***(The Cauchy-Schwarz inequality)***

Let X *be an inner product space. Then the following holds:*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \text{ for all } x, y \in X, \tag{3.29}$$

i.e.,

$$|\langle x, y \rangle| \leq \|x\| \|y\| \text{ for all } x, y \in X. \tag{3.30}$$

Definition 3.1.43. (Hilbert space)

An inner product space which is complete with respect to the induced norm is called a Hilbert space.

Example 3.1.44. *The Euclidean space* $\mathbb{R}^n$ *is a Hilbert space with inner product defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

where $x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

Example 3.1.45. *The space* l_2 *is a Hilbert space with inner product defined by*

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x_j \overline{y_j},$$

where $x, y \in l_2$*.*

Definition 3.1.46. (Proper function)

Let function $f : X \to (-\infty, \infty]$. Then f is said to be proper if there exists $x \in X$ with $f(x) < \infty$.

Definition 3.1.47. (Lower semicontinuous function)

Let X be a linear space and $f : X \to (-\infty, \infty]$ a proper function. Then f is said to be lower semicontinuous (l.s.c.) at $x_0 \in X$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x_0) = \sup_{V \in U_{x_0}} \inf_{x \in V} f(x), \tag{3.31}$$

where U_{x_0} is a base of neighborhoods of the point $x_0 \in X$. f is said to be lower semicontinuous on X if it is lower semicontinuous on each point of X, *i.e.*, for each $x \in X$,

$$x \to x_0 \Rightarrow f(x) \leq \liminf_{n \to \infty} f(x_n). \tag{3.32}$$

Example 3.1.48. *Let $(X, \|\cdot\|)$ be normed space. If $F(x) = \|x\|$ for all $x \in X$ then F is lower semicontinuous function.*

Definition 3.1.49. (Bounded linear operator)

Let X and Y be normed spaces and $T : X \to Y$ a linear operator. The operator T is said to be bounded if there is a real number c such that for all $x \in X$,

$$\|Tx\| \leq c\|x\|. \tag{3.33}$$

Definition 3.1.50. (Strict convexity)

A Banach space X is said to be strictly convex if

$$x, y \in S_X \textit{ with } x \neq y \Rightarrow \|(1-\lambda)x + \lambda y\| < 1 \textit{ for all } \lambda \in (0,1). \tag{3.34}$$

This says that the midpoint $(x+y)/2$ of two distinct points x and y in the unit sphere S_X of X does not lie on S_X. In other words, if $x, y \in S_X$ with $\|x\| = \|y\| = \|(x+y)/2\|$, then $x = y$.

Example 3.1.51. *Let $X = \mathbb{R}^n, n \geq 2$ with norm $\|x\|_2$ defined by*

$$\|x\|_2 = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}, \ x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n. \tag{3.35}$$

Then X is strictly convex.

The *modulus of convexity* of a Banach space X is the function $\delta_X(\epsilon)$: $(0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\Big\}.$$

Then X is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for any $\epsilon \in (0,2]$.

Example 3.1.52. *Every Hilbert space H is a uniformly convex space. In fact, the parallelogram law gives us*

$$\|x+y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x-y\|^2 \text{ for all } x, y \in H. \tag{3.36}$$

Suppose $x, y \in B_H$ with $x \neq y$ and $\|x-y\| \geq \epsilon$. Then

$$\|x+y\|^2 \leq 4 - \epsilon^2,$$

so it follows that

$$\|(x+y)/2\|^2 \leq 1 - \delta(\epsilon),$$

where $\delta(\epsilon) = 1 - \sqrt{1-\epsilon^2/4}$. Therefore, H is uniformly convex.

The *modulus of smoothness* of X is the function $\rho_X(t) : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_X(t) = \sup\Big\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : \|x\| = 1, \|y\| = 1\Big\}.$$

Then X is said to be *uniformly smooth* if $\rho'_X(0) = \lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$. For any $q \in (1,2]$, a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c_q > 0$ such that $\rho_X(t) > c_q t^q$ for any $t > 0$.

Example 3.1.53. *The l_p spaces $(1 < p \leq 2)$ are uniformly smooth. In fact,*

$$\lim_{t\to 0} \frac{\rho_{l_p(t)}}{t} = \lim_{t\to 0} \frac{(1+t^p)^{1/p} - 1}{t} = 0.$$

The *subdifferential* of a proper convex function $f : X \to (-\infty, +\infty]$ is the set-valued operator $\partial f : X \to 2^X$ defined as

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle + f(x) \leq f(y)\}.$$

If f is proper convex and lower semicontinuous, then the subdifferential $\partial f(x) \neq \emptyset$ for any $x \in int\mathcal{D}(f)$, the interior of the domain of f.

The *generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{j(x) \in X^* : \langle j_q(x), x\rangle = \|x\|^q,\ \|j_q(x)\| = \|x\|^{q-1}\}.$$

If $q = 2$, then the corresponding duality mapping is called the *normalized duality mapping* and denoted by J. We know that the following subdifferential inequality holds: for any $x, y \in X$,

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle,\ j_q(x + y) \in J_q(x + y). \tag{3.37}$$

In particular, it follows that, for all $x, y \in X$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,\ j(x + y) \in J(x + y). \tag{3.38}$$

Lemma 3.1.54. [[53], Corollary $1'$] *Let $1 < q \leq 2$ and X be a smooth Banach space. Then the following statements are equivalent:*

(i) X is q-uniformly smooth.

(ii) There is a constant $k_q > 0$ such that for all $x, y \in X$

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + k_q\|y\|^q. \tag{3.39}$$

The best constant k_q will be called the q-uniform smoothness coefficient of X.

Theorem 3.1.55. *Let E be a Banach space and let J be the duality mapping of E. Then:*

1. *For $x \in E$, $J(x)$ is nonempty, bounded, closed and convex;*
2. *$J(0)=\{0\}$;*
3. *for $x \in E$ and a real α, $J(\alpha x) = \alpha J(x)$;*
4. *for $x, y \in E$, $f \in J(x)$ and $g \in J(y)$, $\langle x - y, f - g\rangle \geq 0$;*
5. *for $x, y \in E$, $f \in J(y)$, $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, f\rangle$.*

Proposition 3.1.56. ([10]) *Let $1 < q < \infty$. Then we have the following:*

1. The Banach space X is smooth if and only if the duality mapping J_q is single valued.

2. The Banach space X is uniformly smooth if and only if the duality mapping J_q is single valued and norm-to-norm uniformly continuous on bounded sets of X.

A set-valued operator $A : X \to 2^X$ with the domain $\mathcal{D}(A)$ and the range $\mathcal{R}(A)$ is said to be *accretive* if, for all $t > 0$ and $x, y \in \mathcal{D}(A)$,

$$\|x - y\| \leq \|x - y + t(u - v)\| \tag{3.40}$$

for all $u \in Ax$ and $v \in Ay$.

Recall that A is accretive if and only if, for each $x, y \in \mathcal{D}(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0 \tag{3.41}$$

for all $u \in Ax$ and $v \in Ay$. An accretive operator A is said to be *m-accretive* if the range

$$\mathcal{R}(I + \lambda A) = X$$

for some $\lambda > 0$. It can be shown that an accretive operator A is m-accretive if and only if

$$\mathcal{R}(I + \lambda A) = X$$

for all $\lambda > 0$.

For any $\alpha > 0$ and $q \in (1, \infty)$, we say that an accretive operator A is *α-inverse strongly accretive* (shortly, *α-isa*) of order q if, for each $x, y \in \mathcal{D}(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq \alpha\|u - v\|^q \tag{3.42}$$

for all $u \in Ax$ and $v \in Ay$.

Let C be a nonempty closed and convex subset of a real Banach space X and K be a nonempty subset of C. A mapping $T : C \to K$ is called a *retraction* of C onto K if $Tx = x$ for all $x \in K$. We say that T is *sunny* if, for each $x \in C$ and $t \geq 0$,

$$T(tx + (1-t)Tx) = Tx, \tag{3.43}$$

whenever $tx+(1-t)Tx \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive.

Theorem 3.1.57. ([38]) *Let X be a uniformly smooth Banach space and $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define a mapping $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D.*

Lemma 3.1.58. ([29], Lemma 3.1) *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+$, $\{\alpha_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be the sequences such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n + c_n$$

for all $n \geq 1$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

1. *If $b_n \leq \alpha_n M$ where $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
2. *If $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 3.1.59. ([20]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n \tau_n$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n$$

for all $n \geq 1$, where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\tau_n\}$ and $\{\rho_n\}$ are real sequences such that

1. $\sum_{n=1}^{\infty} \gamma_n = \infty$;

2. $\lim_{n\to\infty} \rho_n = 0$;

3. $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ *for any subsequence* $\{n_k\} \subset \{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 3.1.60. [[31], p.63] *Let* $q > 1$. *Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}} \tag{3.44}$$

for arbitrary positive real numbers a *and* b.

Lemma 3.1.61. ([28], Lemma 3.1) *For any* $r > 0$, *if*

$$T_r := J_r^B(I - rA) = (I + rB)^{-1}(I - rAx),$$

then $Fix(T_r) = (A + B)^{-1}(0)$.

Lemma 3.1.62. ([28], Lemma 3.2) *For any* $s \in (0, r]$ *and* $x \in X$, *we have*

$$\|x - T_s x\| \leq 2\|x - T_r x\|.$$

Lemma 3.1.63. ([28], Lemma 3.3) *Let* X *be a uniformly convex and* q*-uniformly smooth Banach space for some* $q \in (1, 2]$. *Assume that* A *is a single-valued* α*-isa of order* q *in* X. *Then, for any* $s > 0$, *there exists a continuous, strictly increasing and convex function* $\phi_q : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi_q(0) = 0$ *such that, for all* $x, y \in \mathcal{B}_r$,

$$\begin{aligned} \|T_r x - T_r y\|^q &\leq \|x - y\|^q - r(\alpha q - r^{q-1}\kappa_q)\|Ax - Ay\|^q \\ &- \phi_q(\|(I - J_r)(I - rA)x - (I - J_r)(I - rA)y\|), \end{aligned} \tag{3.45}$$

where κ_q *is the* q*-uniform smoothness coefficient of* X.

Remark 3.1.64. For any $p \in [2, \infty)$, L^p is 2-uniformly smooth with $\kappa_2 = p - 1$ and, for any $p \in (1, 2]$, L^p is p-uniformly smooth with $\kappa_p = (1 + t_p^{p-1})(1 + t_p)^{1-p}$, where t_p is the unique solution to the equation

$$(p - 2)t^{p-1} + (p - 1)t^{p-2} - 1 = 0$$

for any $t \in (0, 1)$.

CHAPTER IV

MAIN RESULTS

4.1 The modified forward-backward splitting method for solving quasi inclusion problem

In this section, we first establish some crucial propositions and then prove our main theorem.

Proposition 4.1.1. *Let $q > 1$ and let X be a real smooth Banach space with the generalized duality mapping j_q. Let $m \in \mathbb{N}$ be fixed. Let $\{x_i\}_{i=1}^m \subset X$ and $t_i \geq 0$ for all $i = 1, 2, ..., m$ with $\sum_{i=1}^m t_i \leq 1$. Then we have*

$$\|\sum_{i=1}^m t_i x_i\|^q \leq \frac{\sum_{i=1}^m t_i\|x_i\|^q}{q-(q-1)(\sum_{i=1}^m t_i)}. \tag{4.46}$$

Proof. By definition of the generalized duality mapping j_q and Lemma 3.1.60, we can estimate the following:

$$\begin{aligned}
\|\sum_{i=1}^m t_i x_i\|^q &= \langle \sum_{i=1}^m t_i x_i, j_q(\sum_{i=1}^m t_i x_i)\rangle \\
&= \sum_{i=1}^m t_i \langle x_i, j_q(\sum_{i=1}^m t_i x_i)\rangle \\
&\leq \sum_{i=1}^m t_i\|x_i\|\|\sum_{i=1}^m t_i x_i\|^{q-1} \\
&\leq \sum_{i=1}^m t_i(\frac{1}{q}\|x_i\|^q + \frac{q-1}{q}\|\sum_{i=1}^m t_i x_i\|^q) \\
&= \frac{1}{q}\sum_{i=1}^m t_i\|x_i\|^q + \frac{q-1}{q}\|\sum_{i=1}^m t_i x_i\|^q(\sum_{i=1}^m t_i),
\end{aligned}$$

which implies that

$$(1-\frac{q-1}{q}\sum_{i=1}^m t_i)\|\sum_{i=1}^m t_i x_i\|^q \leq \frac{1}{q}\sum_{i=1}^m t_i\|x_i\|^q.$$

We see that $1-\frac{q-1}{q}\sum_{i=1}^m t_i$ is positive since $q > 1$ and $\sum_{i=1}^m t_i \leq 1$. It follows that

$$\|\sum_{i=1}^m t_i x_i\|^q \leq \frac{\sum_{i=1}^m t_i\|x_i\|^q}{q-(q-1)(\sum_{i=1}^m t_i)}.$$

□

Proposition 4.1.2. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X. Let $\{x_n\}$ be generated by $u, x_1 \in X$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J_{r_n}^B (x_n - r_n A x_n) + e_n, \ n \geq 1, \tag{4.47}$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, $0 < r_n \leq (\alpha q / k_q)^{1/(q-1)}$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. If $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$, then $\{x_n\}$ is bounded.

Proof. For each $n \in \mathbb{N}$, we put $T_n = J_{r_n}^B (I - r_n A)$ and let $\{y_n\}$ be defined by

$$y_{n+1} = \alpha_n u + \lambda_n y_n + \delta_n T_n y_n. \tag{4.48}$$

Firstly, we compute the following:

$$\begin{aligned}
\|x_{n+1} - y_{n+1}\| &= \|\lambda_n (x_n - y_n) + \delta_n (T_n x_n - T_n y_n) + e_n\| \\
&\leq \lambda_n \|x_n - y_n\| + \delta_n \|T_n x_n - T_n y_n\| + \|e_n\| \\
&\leq \lambda_n \|x_n - y_n\| + \delta_n \|x_n - y_n\| + \|e_n\| \\
&= (1 - \alpha_n) \|x_n - y_n\| + \|e_n\|.
\end{aligned}$$

By the assumptions and Lemma 3.1.58 (2), we conclude that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Let $z = Qu$, where Q is a sunny nonexpansive retraction of X onto Ω.

We next show that $\{y_n\}$ is bounded. Indeed

$$\begin{aligned}
\|y_{n+1} - z\| &= \|\alpha_n (u - z) + \lambda_n (y_n - z) + \delta_n (T_n y_n - z)\| \\
&\leq \alpha_n \|u - z\| + \lambda_n \|y_n - z\| + \delta_n \|T_n y_n - z\| \\
&\leq \alpha_n \|u - z\| + \lambda_n \|y_n - z\| + \delta_n \|y_n - z\| \\
&= \alpha_n \|u - z\| + (1 - \alpha_n) \|y_n - z\|.
\end{aligned}$$

This shows that $\{y_n\}$ is bounded by Lemma 3.1.58 (1) and hence $\{x_n\}$ is also bounded. $\square$

Theorem 4.1.3. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X. Let $\{x_n\}$ be generated by $u, x_1 \in X$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J_{r_n}^B (x_n - r_n A x_n) + e_n, \ n \geq 1, \tag{4.49}$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\alpha q / k_q)^{1/(q-1)}$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

Then $\{x_n\}$ strongly converges to $z = Qu$, where Q is a sunny nonexpansive retraction of X onto Ω.

Proof. Since, by Proposition 4.1.2, $\lim_{n\to\infty} \|x_n - y_n\| = 0$, it suffices to show that $\lim_{n\to\infty} y_n = z = Qu$. From (3.37), we have

$$\begin{aligned} \|y_{n+1} - z\|^q &= \|\alpha_n(u - z) + \lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\ &\leq \|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\ &\quad + q\alpha_n \big\langle u - z, j_q(y_{n+1} - z)\big\rangle. \end{aligned} \tag{4.50}$$

On the other hand, by Proposition 4.1.1 and Lemma 3.1.63, we obtain

$$\begin{aligned} &\|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\ \leq\ & \frac{1}{q - (q-1)(1-\alpha_n)} (\lambda_n \|y_n - z\|^q + \delta_n \|T_n y_n - z\|^q) \\ \leq\ & \frac{1}{q - (q-1)(1-\alpha_n)} \Big(\lambda_n \|y_n - z\|^q \\ & + \delta_n \big(\|y_n - z\|^q - r_n(\alpha q - r_n^{q-1} k_q) \|A y_n - A z\|^q \end{aligned}$$

$$
\begin{aligned}
& -\phi_q(\|y_n - r_nAy_n - T_ny_n + r_nAz\|)\Big)\Big) \\
= \; & \frac{1-\alpha_n}{q-(q-1)(1-\alpha_n)}\|y_n - z\|^q - \frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}\|Ay_n - Az\|^q \\
& -\frac{\delta_n}{q-(q-1)(1-\alpha_n)}\phi_q(\|y_n - r_nAy_n - T_ny_n + r_nAz\|). \qquad (4.51)
\end{aligned}
$$

Replacing (4.51) into (4.50), it follows that

$$
\begin{aligned}
\|y_{n+1} - z\|^q \;\leq\; & \Big(1 - \frac{\alpha_n q}{q-(q-1)(1-\alpha_n)}\Big)\|y_n - z\|^q \\
& -\frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}\|Ay_n - Az\|^q \\
& -\frac{\delta_n}{q-(q-1)(1-\alpha_n)}\phi_q(\|y_n - r_nAy_n - T_ny_n + r_nAz\|) \\
& +q\alpha_n\big\langle u - z, j_q(y_{n+1} - z)\big\rangle. \qquad (4.52)
\end{aligned}
$$

We can check that $\frac{\alpha_n q}{q-(q-1)(1-\alpha_n)}$ is in $(0,1)$ since $\{\alpha_n\} \subset (0,1)$ and $1 < q \leq 2$. Moreover, by condition (ii), $\frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}$ and $\frac{\delta_n}{q-(q-1)(1-\alpha_n)}$ are positive. From (4.52), we then have

$$
\|y_{n+1} - z\|^q \leq \Big(1 - \frac{\alpha_n q}{q-(q-1)(1-\alpha_n)}\Big)\|y_n - z\|^q + q\alpha_n\big\langle u - z, j_q(y_{n+1} - z)\big\rangle \qquad (4.53)
$$

and also

$$
\begin{aligned}
\|y_{n+1} - z\|^q \;\leq\; & \|y_n - z\|^q - \frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}\|Ay_n - Az\|^q \\
& -\frac{\delta_n}{q-(q-1)(1-\alpha_n)}\phi_q(\|y_n - r_nAy_n - T_ny_n + r_nAz\|) \\
& + q\alpha_n\big\langle u - z, j_q(y_{n+1} - z)\big\rangle. \qquad (4.54)
\end{aligned}
$$

For each $n \geq 1$, we set

$$
\begin{aligned}
s_n &= \|y_n - z\|^q, \;\; \gamma_n = \frac{\alpha_n q}{q-(q-1)(1-\alpha_n)}, \\
\tau_n &= \big(q-(q-1)(1-\alpha_n)\big)\big\langle u - z, j_q(y_{n+1} - z)\big\rangle, \\
\eta_n &= \frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}\|Ay_n - Az\|^q \\
& \quad + \frac{\delta_n}{q-(q-1)(1-\alpha_n)}\phi_q(\|y_n - r_nAy_n - T_ny_n + r_nAz\|), \\
\rho_n &= q\alpha_n\big\langle u - z, j_q(y_{n+1} - z)\big\rangle. \qquad (4.55)
\end{aligned}
$$

Then (4.53) and (4.54) are reduced to the following:

$$
s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\tau_n, \;\; n \geq 1
$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \ n \geq 1.$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, it follows that $\sum_{n=1}^{\infty} \gamma_n = \infty$. By the boundedness of $\{y_n\}$ and $\lim_{n\to\infty} \alpha_n = 0$, we see that $\lim_{n\to\infty} \rho_n = 0$. In order to complete the proof, using Lemma 3.1.59, it remains to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ for any subsequence $\{n_k\} \subset \{n\}$.

Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$. So, by our assumptions and the property of ϕ_q, we can deduce that

$$\lim_{k\to\infty} \|Ay_{n_k} - Az\| = \lim_{k\to\infty} \|y_{n_k} - r_{n_k} Ay_{n_k} - T_{n_k} y_{n_k} + r_{n_k} Az\| = 0.$$

This gives, by the triangle inequality, that

$$\lim_{k\to\infty} \|T_{n_k} y_{n_k} - y_{n_k}\| = 0. \tag{4.56}$$

Since $\liminf_{n\to\infty} r_n > 0$, there is $r > 0$ such that $r_n \geq r$ for all $n \geq 1$. In particular, $r_{n_k} \geq r$ for all $k \geq 1$. Lemma 3.1.62 yields that

$$\|T_r^{A,B} y_{n_k} - y_{n_k}\| \leq 2\|T_{n_k} y_{n_k} - y_{n_k}\|.$$

Then, by (4.56), we obtain

$$\limsup_{k\to\infty} \|T_r^{A,B} y_{n_k} - y_{n_k}\| \leq 2 \lim_{k\to\infty} \|T_{n_k} y_{n_k} - y_{n_k}\| = 0.$$

It follows that

$$\lim_{k\to\infty} \|T_r^{A,B} y_{n_k} - y_{n_k}\| = 0. \tag{4.57}$$

Let $z_t = tu + T_r^{A,B} z_t$, $t \in (0,1)$. Employing Theorem 3.1.57, we have $z_t \to Qu = z$ as $t \to 0$. So we obtain

$$\begin{aligned} \|z_t - y_{n_k}\|^q &= \|t(u - y_{n_k}) + (1-t)(T_r^{A,B} z_t - y_{n_k})\|^q \\ &\leq (1-t)^q \|T_r^{A,B} z_t - y_{n_k}\|^q + qt\big\langle u - y_{n_k}, j_q(z_t - y_{n_k})\big\rangle \\ &= (1-t)^q \|T_r^{A,B} z_t - y_{n_k}\|^q + qt\big\langle u - z_t, j_q(z_t - y_{n_k})\big\rangle \\ &\quad + qt\big\langle z_t - y_{n_k}, j_q(z_t - y_{n_k})\big\rangle \end{aligned}$$

$$
\begin{aligned}
&\le (1-t)^q(\|T_r^{A,B}z_t - T_r^{A,B}y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q \\
&\quad + qt\big\langle u - z_t, j_q(z_t - y_{n_k})\big\rangle + qt\|z_t - y_{n_k}\|^q \\
&\le (1-t)^q(\|z_t - y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q \\
&\quad + qt\big\langle u - z_t, j_q(z_t - y_{n_k})\big\rangle + qt\|z_t - y_{n_k}\|^q.
\end{aligned}
$$

This shows that

$$
\big\langle z_t - u, j_q(z_t - y_{n_k})\big\rangle \le \frac{(1-t)^q}{qt}(\|z_t - y_{n_k}\| + \|T_r^{A,B}y_{n_k} - y_{n_k}\|)^q + \frac{(qt-1)}{qt}\|z_t - y_{n_k}\|^q. \tag{4.58}
$$

From (4.58) and (4.57), we obtain

$$
\begin{aligned}
\limsup_{k\to\infty}\big\langle z_t - u, j_q(z_t - y_{n_k})\big\rangle &\le \frac{(1-t)^q}{qt}M^q + \frac{(qt-1)}{qt}M^q \\
&= \Big(\frac{(1-t)^q + qt - 1}{qt}\Big)M^q,
\end{aligned} \tag{4.59}
$$

where $M = \limsup_{k\to\infty}\|z_t - y_{n_k}\|, t \in (0,1)$. We see that $\frac{(1-t)^q+qt-1}{qt} \to 0$ as $t \to 0$. From Proposition 3.1.56 (2), we know that j_q is norm-to-norm uniformly continuous on bounded subsets of X. Since $z_t \to z$ as $t \to 0$, we have $\|j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\| \to 0$ as $t \to 0$. Observe that

$$
\begin{aligned}
&|\langle z_t - u, j_q(z_t - y_{n_k})\rangle - \langle z - u, j_q(z - y_{n_k})\rangle| \\
&\le |\langle z_t - z, j_q(z_t - y_{n_k})\rangle| + |\langle z - u, j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\rangle| \\
&\le \|z_t - z\|\|z_t - y_{n_k}\|^{q-1} + \|z - u\|\|j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\|.
\end{aligned}
$$

So, as $t \to 0$, we get

$$
\langle z_t - u, j_q(z_t - y_{n_k})\rangle \to \langle z - u, j_q(z - y_{n_k})\rangle.
$$

From (4.59), as $t \to 0$, it follows that

$$
\limsup_{k\to\infty}\big\langle z - u, j_q(z - y_{n_k})\big\rangle \le 0. \tag{4.60}
$$

On the other hand, by (4.48) and (4.56), we see that

$$
\|y_{n_k+1} - y_{n_k}\| \le \alpha_{n_k}\|u - y_{n_k}\| + \delta_{n_k}\|T_{n_k}y_{n_k} - y_{n_k}\| \to 0, \tag{4.61}
$$

as $k \to \infty$. Combining (4.60) and (4.61), we get that

$$\limsup_{k\to\infty} \left\langle z-u, j_q(z-y_{n_k+1})\right\rangle \leq 0.$$

It also follows that $\limsup_{k\to\infty} \tau_{n_k} \leq 0$. We conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 3.1.59. Hence $y_n \to z$ as $n \to \infty$. We thus complete the proof. □

By setting $\lambda_n = 0$ for all $n \geq 1$, we obtain the following result:

Corollary 4.1.4. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X. Let $\{x_n\}$ be generated by $u, x_1 \in X$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J^B_{r_n}(x_n - r_n A x_n) + e_n,\ n \geq 1, \tag{4.62}$$

where $J^B_{r_n} = (I + r_n B)^{-1}$, $\{r_n\} \subset (0,\infty)$ and $\{\alpha_n\}$ is a sequence in $[0,1]$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\alpha q/k_q)^{1/(q-1)}$;

(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

Then $\{x_n\}$ strongly converges to $z = Qu$, where Q is a sunny nonexpansive retraction of X onto Ω.

Remark 4.1.5. (1) Our results extend those of [3, 21, 32, 51, 54, 55] from Hilbert spaces to Banach spaces.

(2) We remove the conditions that $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ and $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$ in Theorem 3.3 of Yao-Noor [55] and the conditions that $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\lim_{n\to\infty} \left(\frac{1}{r_{n+1}} - \frac{1}{r_n}\right) = 0$ and $\sum_{n=1}^{\infty} \frac{|\alpha_{n+1}-\alpha_n|}{r_{n+1}} < \infty$ in Theorem 1 of Boikanyo-Moroşanu [3].

We now give an example in ℓ_3 space which is a uniformly convex and 2-uniformly smooth Banach space but not a Hilbert space.

Example 4.1.6. *Let $A : \ell_3 \to \ell_3$ be defined by $Ax = 2x + (1, 1, 1, 0, 0, 0, 0, ...)$ and let $B : \ell_3 \to \ell_3$ be defined by $Bx = 5x$ where $x = (x_1, x_2, x_3, ...) \in \ell_3$.*

We see that A is a 1/2-isa of order 2 and B is an m-accretive operator. Indeed, let $x, y \in \ell_3$, then

$$\begin{aligned}\langle Ax - Ay, j_2(x - y)\rangle &= \langle 2x - 2y, j_2(x - y)\rangle \\ &= 2\|x - y\|_{\ell_3}^2 \\ &= \frac{1}{2}\|Ax - Ay\|_{\ell_3}^2.\end{aligned}$$

We also have

$$\langle Bx - By, j_2(x - y)\rangle = 5\|x - y\|_{\ell_3}^2 \geq 0$$

and $R(I + rB) = \ell_3$ for all $r > 0$. By a direct calculation, we have for $s > 0$

$$\begin{aligned}J_s^B(x - sAx) &= (I + sB)^{-1}(x - sAx) \\ &= \frac{1 - 2s}{1 + 5s}x - \frac{s}{1 + 5s}(1, 1, 1, 0, 0, 0, 0, ...),\end{aligned}$$

where $x = (x_1, x_2, x_3, ...) \in \ell_3$. Since, in ℓ_3, $q = 2$, $k_2 = 2$ and $\alpha = 1/2$, we can choose $r_n = 0.1$ for all $n \in \mathbb{N}$. Let $\alpha_n = \frac{1}{1000n+1}$, $\lambda_n = \frac{1}{10n}$ and $\delta_n = (1 - \frac{1}{10n} - \frac{1}{1000n+1})$. Let $u = (-0.05, -0.08, -0.06, 0, 0, 0, 0, ...)$ and $e_n = (0, 0, 0, ...)$. Starting $x_1 = (1.2, 2.5, 3.4, 0, 0, 0, 0, ...)$ and computing iteratively algorithm (4.49) in Theorem 4.1.3, we obtain the following numerical results.

n	x_n	$\|x_{n+1} - x_n\|_{\ell_3}$
1	(1.2000000,2.5000000,3.4000000,0.0000000,0.0000000,...)	1.6937789E+00
10	(-0.1368743, -0.1311181, -0.1271209,0.0000000,0.0000000,...)	8.2573797E-03
20	(-0.1428340, -0.1428263, -0.1428159,0.0000000,0.0000000,...)	1.7112628E-05
30	(-0.1428499, -0.1428522, -0.1428507,0.0000000,0.0000000,...)	3.4600332E-07
40	(-0.1428519, -0.1428536, -0.1428524,0.0000000,0.0000000,...)	1.7301589E-07
50	(-0.1428530, -0.1428543, -0.1428534,0.0000000,0.0000000,...)	1.0843686E-07
60	(-0.1428537, -0.1428548, -0.1428541,0.0000000,0.0000000,...)	7.4310792E-08
70	(-0.1428542, -0.1428552, -0.1428545,0.0000000,0.0000000,...)	5.4092528E-08
80	(-0.1428546, -0.1428554, -0.1428549,0.0000000,0.0000000,...)	4.1132625E-08
90	(-0.1428549, -0.1428556, -0.1428551,0.0000000,0.0000000,...)	3.2329686E-08
100	(-0.1428551, -0.1428558, -0.1428553,0.0000000,0.0000000,...)	2.6078314E-08
⋮	⋮	⋮
200	(-0.1428561, -0.1428565, -0.1428562,0.0000000,0.0000000,...)	6.4012505E-09
250	(-0.1428563, -0.1428566, -0.1428564,0.0000000,0.0000000,...)	4.0821310E-09
300	(-0.1428565, -0.1428567, -0.1428565,0.0000000,0.0000000,...)	2.8280803E-09
350	(-0.1428566, -0.1428568, -0.1428566,0.0000000,0.0000000,...)	2.0742607E-09
400	(-0.1428566, -0.1428568, -0.1428567,0.0000000,0.0000000,...)	1.5860974E-09
450	(-0.1428567, -0.1428568, -0.1428567,0.0000000,0.0000000,...)	1.2519825E-09
500	(-0.1428567, -0.1428569, -0.1428568,0.0000000,0.0000000,...)	1.0133107E-09
550	(-0.1428568, -0.1428569, -0.1428568,0.0000000,0.0000000,...)	8.3691026E-10
600	(-0.1428568, -0.1428569, -0.1428568,0.0000000,0.0000000,...)	7.0286201E-10
650	(-0.1428568, -0.1428569, -0.1428569,0.0000000,0.0000000,...)	5.9861825E-10

Table 1 Numerical results of Example 4.1.6 for iteration process (4.49)

From Table 1, the solution is $(-0.142857, -0.142857, -0.142857, 0, 0, 0, 0, ...)$.

4.2 Applications and numerical examples

In this section, we discuss some concrete examples as well as the numerical results for supporting the main theorem.

4.2.1 Minimization Problem

In this subsection, we apply Theorem 4.1.3 to the convex minimization problem. Let H be a real Hilbert space. Let $F : H \to \mathbb{R}$ be a convex smooth function and $G : H \to \mathbb{R}$ be a convex, lower-semicontinuous and nonsmooth function. We consider the problem of finding $\hat{x} \in H$ such that

$$F(\hat{x}) + G(\hat{x}) \leq F(x) + G(x)$$

for all $x \in H$. This problem is equivalent, by Fermat's rule, to the problem of finding $\hat{x} \in H$ such that

$$0 \in \nabla F(\hat{x}) + \partial G(\hat{x}),$$

where ∇F is a gradient of F and ∂G is a subdifferential of G. In this point of view, we can set $A = \nabla F$ and $B = \partial G$ in Theorem 4.1.3. This is because if ∇F is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone [[1], Corollary 10]. Moreover, ∂G is maximal monotone [[40], Theorem A]. So we obtain the following result.

Theorem 4.2.1. *Let H be real Hilbert space. Let $F : H \to \mathbb{R}$ be a convex and differentiable function with $(1/L)$-Lipschitz continuous gradient ∇F and $G : H \to \mathbb{R}$ be a convex and lower semi-continuous function which $F+G$ attains a minimizer. Let $\{e_n\}$ be a sequence in H. Let $\{x_n\}$ be generated by $u, x_1 \in H$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J_{r_n}\big(x_n - r_n \nabla F(x_n)\big) + e_n, \ n \geq 1, \tag{4.63}$$

where $J_{r_n} = (I + r_n \partial G)^{-1}$, $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that *(i)* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2L$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

Then $\{x_n\}$ *strongly converges to a minimizer of* $F + G$.

Example 4.2.2. *Solve the following minimization problem:*

$$\min_{x\in\mathbb{R}^3} \|x\|_2^2 + (3, 5, -1)x + 9 + \|x\|_1, \tag{4.64}$$

where $x = (y_1, y_2, y_3) \in \mathbb{R}^3$.

For each $x \in \mathbb{R}^3$, we set $F(x) = \|x\|_2^2 + (3, 5, -1)x + 9$ and $G(x) = \|x\|_1$. Then $\nabla F(x) = 2x + (3, 5, -1)$. We can check that F is convex and differentiable on $\mathbb{R}^3$ with 2-Lipschitz continuous gradient ∇F. Moreover, G is convex and lower semi-continuous but not differentiable on $\mathbb{R}^3$. From [?] we know that, for $r > 0$,

$$(I + r\partial G)^{-1}(x) = (\max\{|y_1| - r, 0\}sign(y_1), ..., \max\{|y_3| - r, 0\}sign(y_3)).$$

We choose $\alpha_n = \frac{1}{100n+1}$, $\lambda_n = \frac{99n}{(n+1)(100n+1)}$, $\delta_n = \frac{n}{n+1}$ and $r_n = 0.2$. Let $e_n = (\frac{1}{n^2}, \frac{1}{n^2}, \frac{1}{n^2})$, $u = (2.553479, 5.187352, 1.903486)$ and $x_1 = (3.425859, 8.231258, 1.430561)$. Using algorithm (4.63) in Theorem 4.2.1, we obtain the following numerical results.

n	$x_n = (y_1^n, y_2^n, y_3^n)$	$F(x_n) + G(x_n)$	$\|x_{n+1} - x_n\|_2$
1	(3.425859,8.231258,1.430561)	153.6276069	1.4677578E+00
2	(3.332050, 6.954869, 2.149131)	128.1489310	2.7041086E+00
3	(2.156296, 4.541443, 1.824807)	73.4780908	2.4690302E+00
4	(1.021838, 2.392267, 1.388738)	36.1366456	1.8578516E+00
5	(0.270687, 0.736212, 1.008125)	16.1316179	9.8758376E-01
6	(-0.108319, -0.126974, 0.713871)	8.8129329	6.9838132E-01
7	(-0.381831, -0.732534, 0.498872)	6.2374757	4.8162425E-01
8	(-0.573595, -1.147294, 0.346679)	5.0291149	3.2683581E-01
9	(-0.705677, -1.426945, 0.240984)	4.4730915	2.1951923E-01
10	(-0.795670, -1.613558, 0.168420)	4.2194535	1.4654722E-01
⋮	⋮	⋮	⋮
654	(-0.999858, -1.999718, 0.000079)	4.0000001	5.1403155E-07
655	(-0.999858, -1.999718, 0.000079)	4.0000001	5.1241306E-07
656	(-0.999858, -1.999719, 0.000079)	4.0000001	5.1080220E-07
657	(-0.999858, -1.999719, 0.000079)	4.0000001	5.0919892E-07
658	(-0.999858, -1.999720, 0.000079)	4.0000001	5.0760317E-07
659	(-0.999859, -1.999720, 0.000078)	4.0000001	5.0601491E-07
660	(-0.999859, -1.999721, 0.000078)	4.0000001	5.0443408E-07
661	(-0.999859, -1.999721, 0.000078)	4.0000001	5.0286064E-07
662	(-0.999859, -1.999721, 0.000078)	4.0000001	5.0129456E-07
663	(-0.999860, -1.999722, 0.000078)	4.0000001	4.9973577E-07

Table 2 Numerical results of Example 4.2.2 for iteration process (4.63)

From Table 2, we see that $x_{663} = (-0.999860, -1.999722, 0.000078)$ is an approximation of the minimizer of $F + G$ with an error $4.9973577E - 07$ and its minimum value is approximately 4.0000001. In fact, the minimizer of $F + G$ is $(-1, -2, 0)$ and $(F + G)(-1, -2, 0) = 4$.

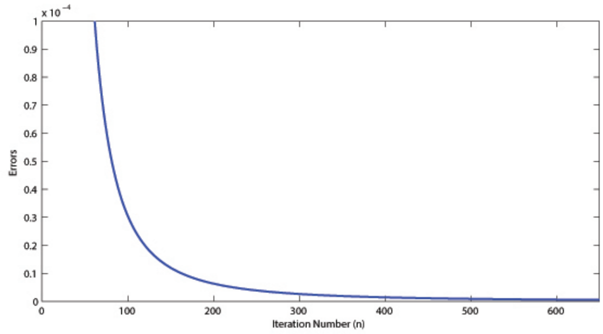

Figure 1 The error plotting of $\|x_{n+1} - x_n\|_2$ in Table 2

4.2.2 Linear Inverse Problem

In this subsection, we apply Theorem 4.1.3 to solve the unconstrained linear system

$$Cx = d \tag{4.65}$$

where C is a bounded linear operator on H and $d \in H$. For each $x \in H$, we define $F : H \to \mathbb{R}$ by

$$F(x) = \frac{1}{2}\|Cx - d\|^2. \tag{4.66}$$

From [?] we know that $\nabla F(x) = C^T(Cx - d)$ and ∇F is K-Lipschitz continuous with K the largest eigenvalue of $C^T C$. So we obtain the following result.

Theorem 4.2.3. *Let H be real Hilbert space. Let $C : H \to H$ be a bounded linear operator and $d \in H$ with K the largest eigenvalue of $C^T C$. Let $\{e_n\}$ be a sequence in H. Let $\{x_n\}$ be generated by $u, x_1 \in H$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n\big(x_n - r_n C^T(Cx_n - d)\big) + e_n, \ n \geq 1, \tag{4.67}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2/K$;

(iii) $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

If (4.65) is consistent, then $\{x_n\}$ *strongly converges to a solution of a linear system.*

Example 4.2.4. *Solve the following linear system:*

$$\begin{aligned} 2y_1 + y_2 - 3y_3 + 2y_4 &= 13 \\ y_1 - 2y_2 + 3y_3 + 5y_4 &= 9 \\ -3y_1 + 5y_2 + 4y_3 - 2y_4 &= -3 \\ 4y_1 + 2y_2 - y_3 - y_4 &= 6. \end{aligned} \tag{4.68}$$

Let $C = \begin{pmatrix} 2 & 1 & -3 & 2 \\ 1 & -2 & 3 & 5 \\ -3 & 5 & 4 & -2 \\ 4 & 2 & -1 & -1 \end{pmatrix}$, $x = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ and $d = \begin{pmatrix} 13 \\ 9 \\ -3 \\ 6 \end{pmatrix}$. Then

$$C^T C = \begin{pmatrix} 30 & -7 & -19 & 11 \\ -7 & 34 & 9 & -20 \\ -19 & 9 & 35 & 2 \\ 11 & -20 & 2 & 34 \end{pmatrix}.$$

The largest eigenvalue of C^TC is 65.5033. This allows us to choose the upper bound of $\{r_n\}$. We also note that since C^TC is symmetric, the largest eigenvalue K is less than mb, where m is the dimension of the matrix C^TC and b is its maximal element; see [[57], Theorem 1].

We choose $\alpha_n = \frac{1}{50n+1}$, $\lambda_n = \frac{49n}{(n+1)(50n+1)}$, $\delta_n = \frac{n}{n+1}$ and $r_n = 0.03$ for all $n \geq 1$. Let $e_n = (\frac{1}{n^3}, \frac{1}{n^3}, \frac{1}{n^3}, \frac{1}{n^3})^T$, $u = (3, 1, 1, 4)^T$ and $x_1 = (-1, 3, 2, 5)^T$. Using algorithm (4.67) in Theorem 4.2.3, we obtain the following numerical results.

n	$x_n = (y_1^n, y_2^n, y_3^n, y_4^n)^T$	$\|x_{n+1} - x_n\|_2$
1	(-1.000000,3.000000,2.000000,5.000000)	3.0154184E+00
2	(1.608431,3.435784,0.640392,5.500392)	1.7095286E+00
3	(1.626720,3.340500,-0.516476,4.245509)	8.4386757E-01
4	(1.339520,2.891206,-0.916370,3.727969)	5.0556755E-01
5	(1.099700,2.553623,-1.039057,3.465156)	2.9280323E-01
$\vdots$	$\vdots$	$\vdots$
238	(1.000364,1.999292,-0.999491,2.999184)	5.3648214E-06
239	(1.000363,1.999295,-0.999493,2.999188)	5.3189260E-06
240	(1.000361,1.999298,-0.999496,2.999191)	5.2735040E-06
241	(1.000359,1.999301,-0.999498,2.999195)	5.2287703E-06
242	(1.000358,1.999304,-0.999500,2.999198)	5.1844999E-06
243	(1.000356,1.999307,-0.999502,2.999202)	5.1408891E-06
244	(1.000355,1.999310,-0.999504,2.999205)	5.0977315E-06
245	(1.000353,1.999313,-0.999506,2.999208)	5.0552065E-06
246	(1.000352,1.999316,-0.999508,2.999212)	5.0131244E-06
247	(1.000350,1.999319,-0.999510,2.999215)	4.9716496E-06

Table 3 Numerical results of Example 4.2.4 for iteration process (4.67)

From Table 3 we see that the solution of a linear system (4.68) is $(1, 2, -1, 3)$.

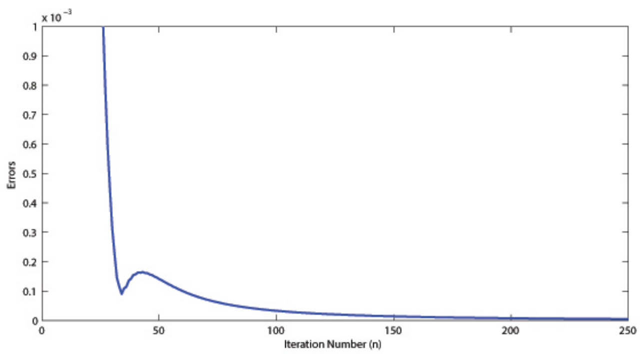

Figure 2 The error plotting of $\|x_{n+1} - x_n\|_2$ in Table 3

Remark 4.2.5. We remark that Theorem 4.1.3 can be further applied to the variational inequality problem, the split feasibility problem and the fixed point problem. See also [28, 47].

We next prove another strong convergence theorem which mainly extends and improves the results obtained by Takahashi et al. [47].

Theorem 4.2.6. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ be an m-accretive operator. Assume that $S = (A + B)^{-1}(0) \neq \emptyset$. We define a sequence $\{x_n\}$ by the iterative scheme: for any $x_1 \in X$,*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J^B_{r_n}(x_n - r_n Ax_n)) \tag{4.69}$$

for each $n \geq 1$, where $u \in X$, $J^B_{r_n} = (I + r_n B)^{-1}$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\{r_n\} \subset (0, +\infty)$. Assume that the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}$ converges strongly to a point $z = Q(u)$, where Q is the sunny nonexpansive retraction of X onto S.

Proof. Let $z = Q(u)$. Let $T_n = J^B_{r_n}(I - r_n A)$ and $z_n = \alpha_n u + (1 - \alpha_n)J^B_{r_n}(x_n - r_n Ax_n)$ for each $n \geq 1$. Then we obtain, by Lemma 3.1.61,

$$\begin{aligned} \|z_n - z\| &= \|\alpha_n u + (1 - \alpha_n)J^B_{r_n}(x_n - r_n Ax_n) - z\| \\ &= \|\alpha_n(u - z) + (1 - \alpha_n)(T_n x_n - z)\| \\ &\leq \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \end{aligned} \tag{4.70}$$

It follows from (4.70) that

$$\|x_{n+1} - z\| = \|\beta_n(x_n - z) + (1 - \beta_n)(z_n - z)\|$$

$$
\begin{aligned}
&\leq \beta_n\|x_n - z\| + (1-\beta_n)\|z_n - z\| \\
&\leq \beta_n\|x_n - z\| + (1-\beta_n)(\alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\|) \\
&= \beta_n\|x_n - z\| + (1-\beta_n)\alpha_n\|u - z\| + (1-\beta_n)(1-\alpha_n)\|x_n - z\| \\
&= (1-\alpha_n(1-\beta_n))\|x_n - z\| + (1-\beta_n)\alpha_n\|u - z\|.
\end{aligned}
$$

Hence we can apply Lemma 3.1.58 to claim that $\{x_n\}$ is bounded. Using the inequality (3.37) and Lemma 3.1.63, we derive that

$$
\begin{aligned}
\|z_n - z\|^q &= \|\alpha_n(u - z) + (1-\alpha_n)(J_{r_n}^B(x_n - r_nAx_n) - J_{r_n}^B(z - r_nAz))\|^q \\
&\leq (1-\alpha_n)^q\|J_{r_n}^B(x_n - r_nAx_n) - J_{r_n}^B(z - r_nAz)\|^q \\
&\quad + q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\
&= (1-\alpha_n)^q\|T_nx_n - T_nz\|^q + q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\
&\leq (1-\alpha_n)^q\Big[\|x_n - z\|^q - r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad - \phi_q(\|(I - J_{r_n}^B)(I - r_nA)x_n - (I - J_{r_n}^B)(I - r_nA)z\|)\Big] \\
&\quad + q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\
&= (1-\alpha_n)^q\|x_n - z\|^q - (1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad - (1-\alpha_n)^q\phi_q(\|(x_n - r_nAx_n - T_nx_n) - (z - r_nAz - T_nz)\|) \\
&\quad + q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\
&= (1-\alpha_n)^q\|x_n - z\|^q - (1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad - (1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&\quad + q\alpha_n\langle u - z, J_q(z_n - z)\rangle.
\end{aligned} \tag{4.71}
$$

It follows from (4.71) that

$$
\begin{aligned}
\|x_{n+1} - z\|^q &= \|\beta_n(x_n - z) + (1-\beta_n)(z_n - z)\|^q \\
&\leq \beta_n^q\|x_n - z\|^q + (1-\beta_n)^q\|z_n - z\|^q \\
&= \beta_n^q\|x_n - z\|^q + (1-\beta_n)^q\Big[(1-\alpha_n)^q\|x_n - z\|^q \\
&\quad - (1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad - (1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&\quad + q\alpha_n\langle u - z, J_q(z_n - z)\rangle\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \beta_n^q\|x_n - z\|^q + (1-\beta_n)^q(1-\alpha_n)^q\|x_n - z\|^q \\
&\quad -(1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad -(1-\beta_n)^q(1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&\quad +(1-\beta_n)^q q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\
&\leq (\beta_n + (1-\beta_n)(1-\alpha_n))\|x_n - z\|^q \\
&\quad -(1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad -(1-\beta_n)^q(1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&\quad +(1-\beta_n)^q q\alpha_n\langle u - z, J_q(z_n - z)\rangle \\
&= (1-(1-\beta_n)\alpha_n)\|x_n - z\|^q \\
&\quad -(1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad -(1-\beta_n)^q(1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&\quad +(1-\beta_n)^q q\alpha_n\langle u - z, J_q(z_n - z)\rangle. \qquad (4.72)
\end{aligned}
$$

We know that $(1-\beta_n)\alpha_n$ is in $(0,1)$ and $(1-\beta_n)^q(1-\alpha_n)^q$ are positive since $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$. Moreover, by the condition (c) and $1 < q \leq 2$, we can show that $(1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)$ is positive. Then, from (4.72), it follows that

$$
\begin{aligned}
\|x_{n+1} - z\|^q &\leq (1-(1-\beta_n)\alpha_n)\|x_n - z\|^q \\
&\quad +(1-\beta_n)^q q\alpha_n\langle u - z, J_q(z_n - z)\rangle \qquad (4.73)
\end{aligned}
$$

and also

$$
\begin{aligned}
\|x_{n+1} - z\|^q &\leq \|x_n - z\|^q - (1-\beta_n)^q(1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad -(1-\beta_n)^q(1-\alpha_n)^q\phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|) \\
&\quad +(1-\beta_n)^q q\alpha_n\langle u - z, J_q(z_n - z)\rangle. \qquad (4.74)
\end{aligned}
$$

For each $n \geq 1$, set

$$
\begin{aligned}
s_n &= \|x_n - z\|^q, \\
\gamma_n &= (1-\beta_n)\alpha_n,
\end{aligned}
$$

$$\begin{aligned}
\tau_n &= (1-\beta_n)^{q-1} q \langle u - z, J_q(z_n - z)\rangle, \\
\eta_n &= (1-\beta_n)^q (1-\alpha_n)^q r_n(\alpha q - r_n^{q-1}\kappa_q)\|Ax_n - Az\|^q \\
&\quad +(1-\beta_n)^q(1-\alpha_n)^q \phi_q(\|x_n - r_nAx_n - T_nx_n + r_nAz\|), \\
\rho_n &= (1-\beta_n)^q q\alpha_n \langle u - z, J_q(z_n - z)\rangle.
\end{aligned}$$

Then it follows from (4.73) and (4.74) that

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n\tau_n$$

and

$$s_{n+1} \le s_n - \eta_n + \rho_n$$

for each $n \ge 1$. Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, it follows that $\sum_{n=1}^{\infty}\gamma_n = \infty$. By the boundedness of $\{z_n\}$ and $\lim_{n\to\infty}\alpha_n = 0$, we see that $\lim_{n\to\infty}\rho_n = 0$.

In order to complete the proof, using Lemma 3.1.59, it remains to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \le 0$ for any subsequence $\{n_k\} \subset \{n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. So, by our assumptions and the property of ϕ_q, we can deduce that

$$\lim_{k\to\infty}\|Ax_{n_k} - Az\| = \lim_{k\to\infty}\|x_{n_k} - r_{n_k}Ax_{n_k} - T_{n_k}x_{n_k} + r_{n_k}Az\| = 0,$$

which gives, by the triangle inequality, that

$$\lim_{k\to\infty}\|T_{n_k}x_{n_k} - x_{n_k}\| = 0.$$

By the condition (c), there exists $\epsilon > 0$ such that $r_n \ge \epsilon$ for all $n > 0$. Then, by Lemma 3.1.62, we have

$$\|T_\epsilon x_{n_k} - x_{n_k}\| \le 2\|T_{n_k}x_{n_k} - x_{n_k}\|.$$

It follows that

$$\limsup_{k\to\infty}\|T_\epsilon x_{n_k} - x_{n_k}\| \le 2\limsup_{k\to\infty}\|T_{n_k}x_{n_k} - x_{n_k}\| = 0 \tag{4.75}$$

and so

$$\limsup_{k\to\infty}\|T_\epsilon x_{n_k} - x_{n_k}\| = 0. \tag{4.76}$$

Let $z_t = tu + (1-t)T_\epsilon z_t$ for any $t \in (0,1)$. Employing Theorem 3.1.57, we have $z_t \to Qu = z$ as $t \to 0$. So we obtain

$$\begin{aligned}
\|z_t - z_{n_k}\|^q &= \|t(u - z_{n_k}) + (1-t)(T_{r_{n_k}} z_t - z_{n_k})\|^q \\
&\leq (1-t)^q \|T_{r_{n_k}} z_t - z_{n_k}\|^q + qt\langle u - z_{n_k}, J_q(z_t - z_{n_k})\rangle \\
&= (1-t)^q \|T_{r_{n_k}} z_t - z_{n_k}\|^q + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle \\
&\quad + qt\langle z_t - z_{n_k}, J_q(z_t - z_{n_k})\rangle \\
&= (1-t)^q \|T_{r_{n_k}} z_t - T_{r_{n_k}} z_{n_k} + T_{r_{n_k}} z_{n_k} - z_{n_k}\|^q \\
&\quad + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q \\
&\leq (1-t)^q \Big[\|T_{r_{n_k}} z_t - T_{r_{n_k}} z_{n_k}\| + \|T_{r_{n_k}} z_{n_k} - z_{n_k}\|\Big]^q \\
&\quad + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q \\
&\leq (1-t)^q \Big[\|z_t - z_{n_k}\| + \|T_{r_{n_k}} z_{n_k} - z_{n_k}\|\Big]^q \\
&\quad + qt\langle u - z_t, J_q(z_t - z_{n_k})\rangle + qt\|z_t - z_{n_k}\|^q.
\end{aligned}$$

This shows that

$$\begin{aligned}
\langle z_t - u, J_q(z_t - z_{n_k})\rangle &\leq \frac{(1-t)^q}{qt}\Big[\|z_t - z_{n_k}\| + \|T_{r_{n_k}} z_{n_k} - z_{n_k}\|\Big]^q \\
&\quad + \frac{(qt-1)}{qt}\|z_t - z_{n_k}\|^q. \qquad (4.77)
\end{aligned}$$

So we have

$$\begin{aligned}
&\limsup_{k\to\infty}\langle z_t - u, J_q(z_t - z_{n_k})\rangle \\
\leq\;& \limsup_{k\to\infty}\frac{(1-t)^q}{qt}\Big[\|z_t - z_{n_k}\| + \|T_{r_{n_k}} z_{n_k} - z_{n_k}\|\Big]^q + \limsup_{k\to\infty}\frac{(qt-1)}{qt}\|z_t - z_{n_k}\|^q \\
=\;& \frac{(1-t)^q}{qt}M^q + \frac{(qt-1)}{qt}M^q \\
=\;& \Big(\frac{(1-t)^q + qt - 1}{qt}\Big)M^q, \qquad (4.78)
\end{aligned}$$

where $M = \limsup_{k\to\infty}\|z_t - z_{n_k}\|$, $t \in (0,1)$. We see that $\frac{(1-t)^q+qt-1}{qt} \to 0$ as $t \to 0$. From Proposition 3.1.56 (2), we know that J_q is norm-to-norm uniformly continuous on bounded subset of X. Since $z_t \to z$ as $t \to 0$, we have $\|J_q(z_t - x_{n_k}) - J_q(z - x_{n_k})\| \to 0$ as $t \to 0$. We see that

$$\Big|\langle z_t - u, J_q(z_t - z_{n_k})\rangle - \langle z - u, J_q(z - z_{n_k})\rangle\Big|$$

$$
\begin{aligned}
&= \left|\langle (z_t - z) + (z-u), J_q(z_t - z_{n_k})\rangle - \langle z-u, J_q(z - z_{n_k})\rangle\right| \\
&\le \left|\langle z_t - z, J_q(z_t - z_{n_k})\rangle\right| + \left|\langle z-u, J_q(z_t - z_{n_k})\rangle - \langle z-u, J_q(z - z_{n_k})\rangle\right| \\
&\le \|z_t - z\|\|z_t - x_{n_k}\|^{q-1} + \|z-u\|\|J_q(z_t - z_{n_k}) - J_q(z - z_{n_k})\|.
\end{aligned}
$$

So, as $t \to 0$, we get

$$
\langle z_t - u, J_q(z_t - z_{n_k})\rangle \to \langle z-u, J_q(z - z_{n_k})\rangle. \tag{4.79}
$$

From (4.78), as $t \to 0$, we see that

$$
\limsup_{k\to\infty} \langle z-u, J_q(z - z_{n_k})\rangle \le 0.
$$

This shows that $\limsup_{k\to\infty} \tau_{n_k} \le 0$. We conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 3.1.59 (iii). Hence $x_n \to z$ as $n \to \infty$. This completes the proof. □

We finally discuss some concrete examples as well as the numerical results for supporting the main theorem.

Theorem 4.2.7. *Let H be real Hilbert space. Let $F : H \to \mathbb{R}$ be a bounded linear operator with K-Lipschitz continuous gradient ∇F and $G : H \to \mathbb{R}$ be a convex and lower semi-continuous function which $F + G$ attains a minimizer. Let $J_{r_n}^{\partial G} = (I + r_n \partial G)^{-1}$ and $\{x_n\}$ be a sequence generated by $u, x_1 \in H$ and*

$$
x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u + (1-\alpha_n) J_{r_n}^{\partial G}(x_n - r_n \nabla F(x_n))) \tag{4.80}
$$

for each $n \ge 1$, where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{r_n\} \subset (0,+\infty)$. Assume that the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < \frac{2}{K}$.

Then the sequence $\{x_n\}$ converges strongly to a minimizer of $F + G$.

Example 4.2.8. Solve the following minimization:

$$\min_{x\in\mathbb{R}^4} \frac{1}{2}\|Cx-d\|_2^2 + \|x\|_1$$

where

$$C = \begin{bmatrix} 2 & 1 & 8 & 5 \\ 3 & -7 & -3 & -6 \\ -1 & 5 & -3 & 9 \\ 7 & -1 & -4 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad d = \begin{bmatrix} 26 \\ -6 \\ 7 \\ -6 \end{bmatrix}.$$

We set $F(x) = \frac{1}{2}\|Cx-d\|_2^2$ and $G(x) = \|x\|_1$. Then $\nabla F(x) = C^T(Cx-d)$ and $\nabla F(x)$ is K-Lipschitz continuous by [4]. From [18], for any $r > 0$,

$$J_r^{\partial G}(x) = \Big[\max\{|y_1 - r|, 0\}sign(y_1), ..., \max\{|y_4 - r|, 0\}sign(y_4)\Big].$$

We see that

$$C^TC = \begin{bmatrix} 63 & -31 & -18 & -3 \\ -31 & 76 & 18 & 90 \\ -18 & 18 & 98 & 23 \\ -3 & 90 & 23 & 146 \end{bmatrix}$$

and the largest eigenvalue of C^TC is 0.00915.

We choose $\alpha_n = \frac{1}{4000n+1}$, $\beta_n = \frac{1}{1500n}$, $r_n = 0.009, x_1 = (3,-5,1,3)^T$ and $u = (1,-1,-1,-2)^T$. Using algorithm (4.80) in Theorem 4.2.7, we obtain the following numerical results.

n	x_n	$F(x_n) + G(x_n)$	$\|x_{n+1} - x_n\|_2$
1	(3.000000, -5.000000, 1.000000, 3.000000)	1073.000000	4.806639E+00
50	(-0.926970, -2.533429, 2.102770, 3.138152)	24.821487	7.558257E-01
100	(-0.857996, -2.666656, 2.025993, 2.870673)	9.253030	1.423229E-01
150	(-0.845881, -2.693438, 2.011434, 2.821389)	8.701192	2.681196E-02
200	(-0.843740, -2.698758, 2.008675, 2.812280)	8.681616	5.052011E-03
250	(-0.843365, -2.699816, 2.008152, 2.810599)	8.680922	9.520138E-04
300	(-0.843304, -2.700034, 2.008053, 2.810294)	8.680898	1.794090E-04
⋮	⋮	⋮	⋮
700	(-0.843312, -2.700130, 2.008028, 2.810253)	8.680897	6.302689E-08
750	(-0.843314, -2.700132, 2.008028, 2.810254)	8.680897	5.458308E-08
800	(-0.843315, -2.700134, 2.008028, 2.810256)	8.680897	4.773015E-08
850	(-0.843315, -2.700136, 2.008028, 2.810257)	8.680897	4.209251E-08
900	(-0.843316, -2.700138, 2.008028, 2.810258)	8.680897	3.739877E-08
950	(-0.843317, -2.700139, 2.008028, 2.810259)	8.680897	3.344924E-08
1000	(-0.843318, -2.700140, 2.008028, 2.810259)	8.680897	3.009437E-08

Table 4

Form Table 4 we see that $x_{1000} = (-0.843318, -2.700140, 2.008028, 2.810259)$ is an approximation of the minimizer of $F + G$ with an error $3.009437E - 08$ and its minimum value is approximately 8.680897.

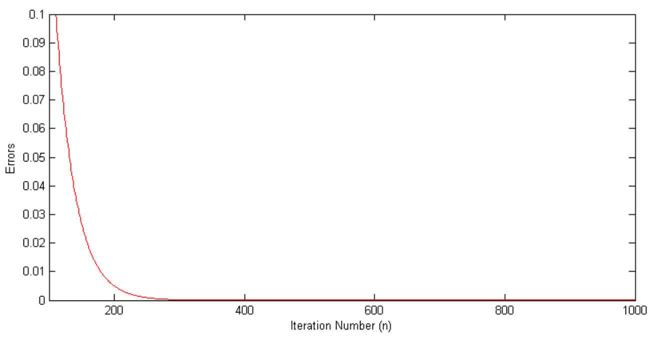

Figure 3

Example 4.2.9. Solve the following minimization:

$$\min_{x\in\mathbb{R}^3} \|Ax+c\|_2 + \frac{1}{2}x^T x + d^T x + 9 \tag{4.81}$$

where

$$A = \begin{bmatrix} -1 & 3 & 4 \\ 2 & -7 & 9 \\ -2 & -5 & -3 \end{bmatrix}, \quad x = (y_1, y_2, y_3)^T, \quad c = (11, 9, 6)^T, \quad d = (7, 6, 8)^T.$$

For each $x \in \mathbb{R}^3$, we set $F(x) = \frac{1}{2}x^T x + d^T x + 9$ and $G(x) = \|Ax + c\|_2$. Then $\nabla F(x) = x + (7, 6, 8)^T$. We can check that F is convex and differentiable on $\mathbb{R}^3$ with 1-Lipschitz continuous gradient ∇F and G is convex and lower semi-continuous. We choose $\alpha_n = \frac{1}{10n+1}$, $\beta_n = \frac{1}{5n}$, $r_n = 0.1, x_1 = (8, -2, 6)^T$ and $u = (-2, 3, 5)^T$. We have that, for $r > 0$,

$$(I + r\partial G)^{-1}(x) = \begin{cases} \left(\frac{1-r}{\|x\|_2}\right)x, & \text{if } \|x\|_2 \geq r, \\ 0, & \text{otherwise.} \end{cases}$$

Using algorithm (4.80) in Theorem 4.2.7, we obtain the following numerical results:

n	x_n	$F(x_n) + G(x_n)$	$\|x_{n+1} - x_n\|_2$
1	(8.000000, -2.000000, 6.000000)	161.316850	7.460748E+00
50	(-0.524837, -0.433635, -0.574738)	0.545773	3.947994E-04
100	(-0.520385, -0.438070, -0.582402)	0.497188	9.656413E-05
150	(-0.518942, -0.439522, -0.584907)	0.481252	4.261886E-05
200	(-0.518229, -0.440242, -0.586151)	0.473332	2.389088E-05
250	(-0.517803, -0.440673, -0.586894)	0.468594	1.525893E-05
300	(-0.517520, -0.440960, -0.587389)	0.465442	1.058212E-05
⋮	⋮	⋮	⋮
800	(-0.516640, -0.441852, -0.588928)	0.455624	1.481869E-06
850	(-0.516609, -0.441884, -0.588982)	0.455278	1.312465E-06
900	(-0.516582, -0.441911, -0.589030)	0.454971	1.170533E-06
950	(-0.516557, -0.441936, -0.589073)	0.454696	1.050439E-06
1000	(-0.516535, -0.441959, -0.589112)	0.454449	9.479211E-07

Table 5

Form Table 5, we see that $x_{1000} = (-0.516535, -0.441959, -0.589112)$ is an approximation of the minimizer of $F + G$ with an error $9.479211E-07$ and its minimum value is approximately 0.454449.

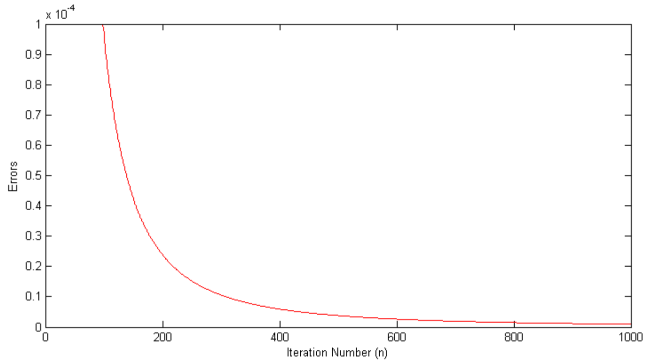

Figure 4

CHAPTER V

CONCLUSION

1. P. Cholamjiak, A Generalized Forward-backward Splitting Method for Solving Quasi Inclusion Problems in Banach Spaces, Numerical Algorithms, 71 (2016) 915-932

2. Y. Shehu and P. Cholamjiak, Another Look at the Split Common Fixed Point Problem for Demicontractive operators, RACSAM . 110 (2016) 201-218

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APPENDIX

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ORIGINAL PAPER

A generalized forward-backward splitting method for solving quasi inclusion problems in Banach spaces

Prasit Cholamjiak[1]

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Abstract We propose a new general type of splitting methods for accretive operators in Banach spaces. We then give the sufficient conditions to guarantee the strong convergence. In the last section, we apply our results to the minimization optimization problem and the linear inverse problem including the numerical examples.

Keywords Accretive operator · Maximal monotone operator · Banach space · Splitting method · Forward-Backward algorithm

Mathematics Subject Classification (2010) 47H09 · 47H10

1 Introduction

Let X be a real Banach space. We study the following inclusion problem: find $\hat{x} \in X$ such that

$$0 \in A\hat{x} + B\hat{x} \tag{1.1}$$

where $A : X \to X$ is an operator and $B : X \to 2^X$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this form.

✉ Prasit Cholamjiak
prasitch2008@yahoo.com

[1] School of Science, University of Phayao, Phayao, 56000, Thailand

A classical method for solving problem (1.1) is the forward-backward splitting method [10, 17, 24, 30] which is defined by the following manner: $x_1 \in X$ and

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n),\ n \geq 1, \tag{1.2}$$

where $r > 0$. We see that each step of iterates involves only with A as the forward step and B as the backward step, but not the sum of B. This method includes, in particular, the proximal point algorithm [5, 6, 13, 22, 27] and the gradient method [2, 12]. Lions-Mercier [17] introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n,\ n \geq 1 \tag{1.3}$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n,\ n \geq 1, \tag{1.4}$$

where $J_r^T = (I + rT)^{-1}$. The first one is often called Peaceman-Rachford algorithm [25] and the second one is called Douglas-Rachford algorithm [11]. We note that both algorithms can be weakly convergent in general [24].

Recently, López et al. [18] introduced the following Halpern-type forward-backward method: $x_1 \in X$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n), \tag{1.5}$$

where J_r^B is the resolvent of B, $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1]$ and $\{a_n\}$, $\{b_n\}$ are error sequences in X. It was proved that the sequence $\{x_n\}$ generated by (1.5) strongly converges to a zero point of the sum of A and B under some appropriate conditions. There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces); see [9, 29–31, 35].

In this work, we study a generalized forward-backward method for solving the inclusion problem (1.1) for an accretive and m-accretive operators in the framework of Banach spaces. We then prove its strong convergence under some mild conditions. Finally, we provide some numerical examples to support our main results.

2 Preliminaries and lemmas

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel.

The *modulus of convexity* of X is the function $\delta : (0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - l\left\|\frac{x+y}{2}\right\| :\ x, y \in X,\ \|x\| = \|y\| = 1,\ \|x - y\| \geq \varepsilon\right\}. \tag{2.1}$$

Then X is uniformly convex if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

The *modulus of smoothness* of X is the function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho(t) = \sup\left\{\frac{\|x + ty\| + \|x - ty\|}{2} - 1 :\ x, y \in X,\ \|x\| = \|y\| = 1\right\}. \tag{2.2}$$

Then X is uniformly smooth if $\lim_{t\to 0} \rho(t)/t = 0$. For $1 < q \leq 2$, a Banach space X is said to be q-uniformly smooth if there exists a constant $c_q > 0$ such that $\rho(t) \leq$

$c_q t^q$ for $t > 0$. We see that if X is q-uniformly smooth, then it is uniformly smooth. Let X^* be the dual space of X. Let $J_q (q > 1)$ denote the generalized duality mapping from X into 2^{X^*} given by $J_q(x) = \{j_q(x) \in X^* : \langle x, j_q(x)\rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\}$, $\forall x \in X$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. In particular, $J_2 := J$ is called the normalized duality mapping on X. It is also known (e.g., [[32], p.1128]) that

$$J_q(x) = \|x\|^{q-2} J(x), \; x \neq 0. \tag{2.3}$$

We next provide some properties of the duality mapping.

Proposition 2.1. (Cioranescu [8]) *Let* $1 < q < \infty$.

(i) The Banach space X is smooth if and only if the duality mapping J_q is single-valued.
(ii) The Banach space X is uniformly smooth if and only if the duality mapping J_q is single-valued and norm-to-norm uniformly continuous on bounded subsets of X.

Using the concept of sub-differentials, we know the following inequality:

Lemma 2.2. [[7], p.33] *Let $q > 1$ and X be a real normed space with the generalized duality mapping J_q. Then, for any $x, y \in X$, we have*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle \tag{2.4}$$

for all $j_q(x + y) \in J_q(x + y)$.

Lemma 2.3. [[32], Corollary $1'$] *Let $1 < q \leq 2$ and X be a smooth Banach space. Then the following statements are equivalent:*

(i) X is q-uniformly smooth.
(ii) There is a constant $k_q > 0$ such that for all $x, y \in X$

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + k_q\|y\|^q. \tag{2.5}$$

The best constant k_q will be called the q-uniform smoothness coefficient of X.

We define the domain and the range of an operator $A : X \to 2^X$ by $D(A) = \{x \in X : Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Az : z \in D(A)\}$, respectively. The inverse of A, denoted by A^{-1}, is defined by $x \in A^{-1}y$ if and only if $y \in Ax$. A set-valued operator A is said to be *accretive* if, for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\big\langle u - v, j(x - y)\big\rangle \geq 0, \; u \in Ax, \; v \in Ay. \tag{2.6}$$

An accretive operator A is said to be *m-accretive* if $R(I + rA) = X$ for all $r > 0$.

Given $\alpha > 0$ and $q \in (1, \infty)$, we say that an accretive operator A is α-inverse strongly accretive (α-isa) of order q if, for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq \alpha\|u - v\|^q, \; u \in Ax, \; v \in Ay. \tag{2.7}$$

Let C be a nonempty subset of a real Banach space X. Let $T : C \to C$ be a nonlinear mapping. We denote the fixed point set of T by $Fix(T)$, that is, $Fix(T) = \{x \in C : x = Tx\}$.

Let C be a nonempty, closed and convex subset of a real Banach space X and let D be a nonempty subset of C. A *retraction* from C to D is a mapping $Q : C \to D$ such that $Qx = x$ for all $x \in D$. A retraction Q from C to D is *nonexpansive* if $\|Qx - Qy\| \leq \|x - y\|$ for all $x, y \in C$. A retraction Q from C to D is *sunny* if , for each $x \in C$ and $t \geq 0$, we have

$$Q\big(tx + (1-t)Qx\big) = Qx, \tag{2.8}$$

whenever $tx + (1-t)Qx \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Reich [26] showed that if X is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Theorem 2.4. [[26], Corollary 1] *Let X be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to D$ by $Qu = s - lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D.*

In what follows, we shall use the following notation:

$$T_r^{A,B} = J_r^B(I - rA) = (I + rB)^{-1}(I - rA), \ r > 0. \tag{2.9}$$

Lemma 2.5. [[18], Lemma 3.1 and Lemma 3.2] *Let X be a Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator. Then we have*

(i) For $r > 0$, $Fix(T_r^{A,B}) = (A + B)^{-1}(0)$.
(ii) For $0 < s \leq r$ and $x \in X$, $\|x - T_s^{A,B}x\| \leq 2\|x - T_r^{A,B}x\|$.

Lemma 2.6. [[18], Lemma 3.3] *Let X be a uniformly convex and q-uniformly smooth Banach space for some $q \in (1, 2]$. Assume that A is a single-valued α-isa of order q in X. Then, given $r > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in B_r$,*

$$\begin{aligned}\|T_r^{A,B}x - T_r^{A,B}y\|^q \ \leq\ & \|x - y\|^q - r(\alpha q - r^{q-1}k_q)\|Ax - Ay\|^q \\ & -\phi_q(\|(I - J_r^B)(I - rA)x - (I - J_r^B)(I - rA)y\|),\end{aligned} \tag{2.10}$$

where k_q is the q-uniform smoothness coefficient of X.

Lemma 2.7. [[20], Lemma 3.1] *Let $\{a_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \ \ n \geq 1, \tag{2.11}$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$ *and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:*

(i) *If $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} b_n/\delta_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Employing the technique of Maingé [19], He-Yang [15] proved the following lemma.

Lemma 2.8. [[15], Lemma 8] *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\tau_n, \ n \geq 1$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \ n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$*, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\tau_n\}$, and $\{\rho_n\}$ are real sequences such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$*,*
(ii) $\lim_{n\to\infty} \rho_n = 0$*,*
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ *for any subsequence $\{n_k\} \subset \{n\}$. Then* $\lim_{n\to\infty} s_n = 0$.

Lemma 2.9. [[23], p.63] *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}} \tag{2.12}$$

for arbitrary positive real numbers a and b.

3 Main results

In this section, we first establish some crucial propositions and then prove our main theorem.

Proposition 3.1. *Let $q > 1$ and let X be a real smooth Banach space with the generalized duality mapping j_q. Let $m \in \mathbb{N}$ be fixed. Let $\{x_i\}_{i=1}^m \subset X$ and $t_i \geq 0$ for all $i = 1, 2, ..., m$ with $\sum_{i=1}^m t_i \leq 1$. Then we have*

$$\|\sum_{i=1}^m t_i x_i\|^q \leq \frac{\sum_{i=1}^m t_i\|x_i\|^q}{q-(q-1)(\sum_{i=1}^m t_i)}. \tag{3.1}$$

Proof By definition of the generalized duality mapping j_q and Lemma 2.9., we can estimate the following:

$$\|\sum_{i=1}^{m} t_i x_i\|^q = \big\langle \sum_{i=1}^{m} t_i x_i, j_q(\sum_{i=1}^{m} t_i x_i)\big\rangle \tag{3.2}$$

$$= \sum_{i=1}^{m} t_i \big\langle x_i, j_q(\sum_{i=1}^{m} t_i x_i)\big\rangle \tag{3.3}$$

$$\leq \sum_{i=1}^{m} t_i \|x_i\| \|\sum_{i=1}^{m} t_i x_i\|^{q-1} \tag{3.4}$$

$$\leq \sum_{i=1}^{m} t_i (\frac{1}{q}\|x_i\|^q + \frac{q-1}{q}\|\sum_{i=1}^{m} t_i x_i\|^q) \tag{3.5}$$

$$= \frac{1}{q}\sum_{i=1}^{m} t_i \|x_i\|^q + \frac{q-1}{q}\|\sum_{i=1}^{m} t_i x_i\|^q (\sum_{i=1}^{m} t_i), \tag{3.6}$$

which implies that

$$\big(1 - \frac{q-1}{q}\sum_{i=1}^{m} t_i\big)\|\sum_{i=1}^{m} t_i x_i\|^q \leq \frac{1}{q}\sum_{i=1}^{m} t_i \|x_i\|^q. \tag{3.7}$$

We see that $1 - \frac{q-1}{q}\sum_{i=1}^{m} t_i$ is positive since $q > 1$ and $\sum_{i=1}^{m} t_i \leq 1$. It follows that

$$\|\sum_{i=1}^{m} t_i x_i\|^q \leq \frac{\sum_{i=1}^{m} t_i \|x_i\|^q}{q - (q-1)(\sum_{i=1}^{m} t_i)}. \tag{3.8}$$

□

Proposition 3.2. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X. Let $\{x_n\}$ be generated by $u, x_1 \in X$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J_{r_n}^B (x_n - r_n A x_n) + e_n, \; n \geq 1, \tag{3.9}$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, $0 < r_n \leq (\alpha q/k_q)^{1/(q-1)}$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. If $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$, then $\{x_n\}$ is bounded.

Proof For each $n \in \mathbb{N}$, we put $T_n = J_{r_n}^B (I - r_n A)$ and let $\{y_n\}$ be defined by

$$y_{n+1} = \alpha_n u + \lambda_n y_n + \delta_n T_n y_n. \tag{3.10}$$

Firstly, using Lemma 2.6, we compute the following:

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &= \|\lambda_n(x_n - y_n) + \delta_n(T_n x_n - T_n y_n) + e_n\| && (3.11)\\ &\leq \lambda_n\|x_n - y_n\| + \delta_n\|T_n x_n - T_n y_n\| + \|e_n\| && (3.12)\\ &\leq \lambda_n\|x_n - y_n\| + \delta_n\|x_n - y_n\| + \|e_n\| && (3.13)\\ &= (1-\alpha_n)\|x_n - y_n\| + \|e_n\|. && (3.14) \end{aligned}$$

By the assumptions and Lemma 2.7 (ii), we conclude that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Let $z = Qu$, where Q is a sunny nonexpansive retraction of X onto Ω.

We next show that $\{y_n\}$ is bounded. Indeed

$$\begin{aligned} \|y_{n+1} - z\| &= \|\alpha_n(u-z) + \lambda_n(y_n - z) + \delta_n(T_n y_n - z)\| && (3.15)\\ &\leq \alpha_n\|u-z\| + \lambda_n\|y_n - z\| + \delta_n\|T_n y_n - z\| && (3.16)\\ &\leq \alpha_n\|u-z\| + \lambda_n\|y_n - z\| + \delta_n\|y_n - z\| && (3.17)\\ &= \alpha_n\|u-z\| + (1-\alpha_n)\|y_n - z\|. && (3.18) \end{aligned}$$

This shows that $\{y_n\}$ is bounded by Lemma 2.7 (i) and hence $\{x_n\}$ is also bounded. □

Theorem 3.3. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X. Let $\{x_n\}$ be generated by $u, x_1 \in X$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J^B_{r_n}(x_n - r_n A x_n) + e_n,\ n \geq 1, \qquad (3.19)$$

where $J^B_{r_n} = (I + r_n B)^{-1}$, $\{r_n\} \subset (0,\infty)$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\alpha q/k_q)^{1/(q-1)}$;
(iii) $\liminf_{n\to\infty}\delta_n > 0$;
(iv) $\sum_{n=1}^{\infty}\|e_n\| < \infty$ *or* $\lim_{n\to\infty}\|e_n\|/\alpha_n = 0$.

Then $\{x_n\}$ strongly converges to $z = Qu$, where Q is a sunny nonexpansive retraction of X onto Ω.

Proof Since, by Proposition 3.2, $\lim_{n\to\infty}\|x_n - y_n\| = 0$, it suffices to show that $\lim_{n\to\infty} y_n = z = Qu$. From Lemma 2.2, we have

$$\begin{aligned} \|y_{n+1} - z\|^q &= \|\alpha_n(u-z) + \lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q && (3.20)\\ &\leq \|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \\ &\quad + q\alpha_n\langle u - z, j_q(y_{n+1} - z)\rangle. && (3.21) \end{aligned}$$

On the other hand, by Proposition 3.1 and Lemma 2.6, we obtain

$$
\begin{aligned}
\|\lambda_n(y_n - z) + \delta_n(T_n y_n - z)\|^q \le\ & \frac{1}{q-(q-1)(1-\alpha_n)}(\lambda_n\|y_n - z\|^q \\
& + \delta_n\|T_n y_n - z\|^q) \qquad (3.22)\\
\le\ & \frac{1}{q-(q-1)(1-\alpha_n)}\Big(\lambda_n\|y_n - z\|^q \\
& +\delta_n\big(\|y_n - z\|^q - r_n(\alpha q - r_n^{q-1}k_q)\|Ay_n - Az\|^q \\
& -\phi_q(\|y_n - r_nAy_n - T_n y_n + r_nAz\|)\big)\Big) \qquad (3.23)\\
=\ & \frac{1-\alpha_n}{q-(q-1)(1-\alpha_n)}\|y_n - z\|^q \\
& -\frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}\|Ay_n - Az\|^q \qquad (3.24)\\
& -\frac{\delta_n}{q-(q-1)(1-\alpha_n)}\phi_q(\|y_n - r_nAy_n - T_n y_n + r_nAz\|).
\end{aligned}
$$

Replacing (3.24) into (3.21), it follows that

$$
\begin{aligned}
\|y_{n+1} - z\|^q \le\ & \left(1 - \frac{\alpha_n q}{q-(q-1)(1-\alpha_n)}\right)\|y_n - z\|^q \\
& -\frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}\|Ay_n - Az\|^q \\
& -\frac{\delta_n}{q-(q-1)(1-\alpha_n)}\phi_q(\|y_n - r_nAy_n - T_n y_n + r_nAz\|) \\
& +q\alpha_n\big\langle u - z, j_q(y_{n+1} - z)\big\rangle. \qquad (3.25)
\end{aligned}
$$

We can check that $\frac{\alpha_n q}{q-(q-1)(1-\alpha_n)}$ is in $(0, 1)$ since $\{\alpha_n\} \subset (0, 1)$ and $1 < q \le 2$. Moreover, by condition (ii), $\frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}$ and $\frac{\delta_n}{q-(q-1)(1-\alpha_n)}$ are positive. From (3.25), we then have

$$
\|y_{n+1} - z\|^q \le \left(1 - \frac{\alpha_n q}{q-(q-1)(1-\alpha_n)}\right)\|y_n - z\|^q + q\alpha_n\big\langle u - z, j_q(y_{n+1} - z)\big\rangle \qquad (3.26)
$$

and also

$$
\begin{aligned}
\|y_{n+1} - z\|^q \le\ & \|y_n - z\|^q - \frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}\|Ay_n - Az\|^q \\
& -\frac{\delta_n}{q-(q-1)(1-\alpha_n)}\phi_q(\|y_n - r_nAy_n - T_n y_n + r_nAz\|) \\
& + q\alpha_n\big\langle u - z, j_q(y_{n+1} - z)\big\rangle. \qquad (3.27)
\end{aligned}
$$

For each $n \geq 1$, we set

$$
\begin{aligned}
s_n &= \|y_n - z\|^q, \ \gamma_n = \frac{\alpha_n q}{q-(q-1)(1-\alpha_n)}, \\
\tau_n &= \big(q-(q-1)(1-\alpha_n)\big)\big\langle u-z, j_q(y_{n+1}-z)\big\rangle, \\
\eta_n &= \frac{\delta_n r_n(\alpha q - r_n^{q-1}k_q)}{q-(q-1)(1-\alpha_n)}\|Ay_n - Az\|^q \\
&\quad + \frac{\delta_n}{q-(q-1)(1-\alpha_n)}\phi_q(\|y_n - r_n Ay_n - T_n y_n + r_n Az\|), \\
\rho_n &= q\alpha_n\big\langle u-z, j_q(y_{n+1}-z)\big\rangle.
\end{aligned} \tag{3.28}
$$

Then (3.26) and (3.27) are reduced to the following:

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\tau_n, \ n \geq 1$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \ n \geq 1.$$

Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, it follows that $\sum_{n=1}^{\infty}\gamma_n = \infty$. By the boundedness of $\{y_n\}$ and $\lim_{n\to\infty}\alpha_n = 0$, we see that $\lim_{n\to\infty}\rho_n = 0$. In order to complete the proof, using Lemma 2.8, it remains to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \leq 0$ for any subsequence $\{n_k\} \subset \{n\}$.

Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. So, by our assumptions and the property of ϕ_q, we can deduce that

$$\lim_{k\to\infty}\|Ay_{n_k} - Az\| = \lim_{k\to\infty}\|y_{n_k} - r_{n_k}Ay_{n_k} - T_{n_k}y_{n_k} + r_{n_k}Az\| = 0. \tag{3.29}$$

This gives, by the triangle inequality, that

$$\lim_{k\to\infty}\|T_{n_k}y_{n_k} - y_{n_k}\| = 0. \tag{3.30}$$

Since $\liminf_{n\to\infty} r_n > 0$, there is $r > 0$ such that $r_n \geq r$ for all $n \geq 1$. In particular, $r_{n_k} \geq r$ for all $k \geq 1$. Lemma 2.5 (ii) yields that

$$\|T_r^{A,B}y_{n_k} - y_{n_k}\| \leq 2\|T_{n_k}y_{n_k} - y_{n_k}\|. \tag{3.31}$$

Then, by (3.30), we obtain

$$\limsup_{k\to\infty}\|T_r^{A,B}y_{n_k} - y_{n_k}\| \leq 2\lim_{k\to\infty}\|T_{n_k}y_{n_k} - y_{n_k}\| = 0. \tag{3.32}$$

It follows that

$$\lim_{k\to\infty}\|T_r^{A,B}y_{n_k} - y_{n_k}\| = 0. \tag{3.33}$$

Let $z_t = tu + T_r^{A,B} z_t$, $t \in (0, 1)$. Employing Theorem 2.4, we have $z_t \to Qu = z$ as $t \to 0$. So we obtain

$$\begin{aligned} \|z_t - y_{n_k}\|^q &= \|t(u - y_{n_k}) + (1-t)(T_r^{A,B} z_t - y_{n_k})\|^q & (3.34)\\ &\le (1-t)^q \|T_r^{A,B} z_t - y_{n_k}\|^q + qt\langle u - y_{n_k}, j_q(z_t - y_{n_k})\rangle & (3.35)\\ &= (1-t)^q \|T_r^{A,B} z_t - y_{n_k}\|^q + qt\langle u - z_t, j_q(z_t - y_{n_k})\rangle \\ &\quad + qt\langle z_t - y_{n_k}, j_q(z_t - y_{n_k})\rangle & (3.36)\\ &\le (1-t)^q (\|T_r^{A,B} z_t - T_r^{A,B} y_{n_k}\| + \|T_r^{A,B} y_{n_k} - y_{n_k}\|)^q \\ &\quad + qt\langle u - z_t, j_q(z_t - y_{n_k})\rangle + qt\|z_t - y_{n_k}\|^q & (3.37)\\ &\le (1-t)^q (\|z_t - y_{n_k}\| + \|T_r^{A,B} y_{n_k} - y_{n_k}\|)^q \\ &\quad + qt\langle u - z_t, j_q(z_t - y_{n_k})\rangle + qt\|z_t - y_{n_k}\|^q. & (3.38) \end{aligned}$$

This shows that

$$\langle z_t - u, j_q(z_t - y_{n_k})\rangle \le \frac{(1-t)^q}{qt}(\|z_t - y_{n_k}\| + \|T_r^{A,B} y_{n_k} - y_{n_k}\|)^q + \frac{(qt-1)}{qt}\|z_t - y_{n_k}\|^q. \tag{3.39}$$

From (3.39) and (3.33), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \langle z_t - u, j_q(z_t - y_{n_k})\rangle &\le \frac{(1-t)^q}{qt} M^q + \frac{(qt-1)}{qt} M^q & (3.40)\\ &= \left(\frac{(1-t)^q + qt - 1}{qt}\right) M^q, & (3.41) \end{aligned}$$

where $M = \limsup_{k\to\infty} \|z_t - y_{n_k}\|$, $t \in (0, 1)$. We see that $\frac{(1-t)^q + qt - 1}{qt} \to 0$ as $t \to 0$. From Proposition 2.1 (ii), we know that j_q is norm-to-norm uniformly continuous on bounded subsets of X. Since $z_t \to z$ as $t \to 0$, we have $\|j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\| \to 0$ as $t \to 0$. Observe that

$$\begin{aligned} &|\langle z_t - u, j_q(z_t - y_{n_k})\rangle - \langle z - u, j_q(z - y_{n_k})\rangle| \\ &\quad \le |\langle z_t - z, j_q(z_t - y_{n_k})\rangle| + |\langle z - u, j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\rangle| & (3.42)\\ &\quad \le \|z_t - z\|\|z_t - y_{n_k}\|^{q-1} + \|z - u\|\|j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\|. & (3.43) \end{aligned}$$

So, as $t \to 0$, we get

$$\langle z_t - u, j_q(z_t - y_{n_k})\rangle \to \langle z - u, j_q(z - y_{n_k})\rangle. \tag{3.44}$$

From (3.41), as $t \to 0$, it follows that

$$\limsup_{k\to\infty} \langle z - u, j_q(z - y_{n_k})\rangle \le 0. \tag{3.45}$$

On the other hand, by (3.10) and (3.30), we see that

$$\|y_{n_k+1} - y_{n_k}\| \le \alpha_{n_k}\|u - y_{n_k}\| + \delta_{n_k}\|T_{n_k} y_{n_k} - y_{n_k}\| \to 0, \tag{3.46}$$

as $k \to \infty$. Combining (3.45) and (3.46), we get that

$$\limsup_{k\to\infty} \langle z - u, j_q(z - y_{n_k+1})\rangle \le 0. \tag{3.47}$$

It also follows that $\limsup_{k\to\infty} \tau_{n_k} \leq 0$. We conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 2.8. Hence $y_n \to z$ as $n \to \infty$. We thus complete the proof. □

By setting $\lambda_n = 0$ for all $n \geq 1$, we obtain the following result:

Corollary 3.4. *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-isa of order q and $B : X \to 2^X$ an m-accretive operator such that $\Omega := (A+B)^{-1}(0) \neq \emptyset$. Let $\{e_n\}$ be a sequence in X. Let $\{x_n\}$ be generated by $u, x_1 \in X$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J^B_{r_n}(x_n - r_n A x_n) + e_n,\ n \geq 1, \tag{3.48}$$

where $J^B_{r_n} = (I + r_n B)^{-1}$, $\{r_n\} \subset (0,\infty)$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. *Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\alpha q/k_q)^{1/(q-1)}$;
(iii) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

Then $\{x_n\}$ strongly converges to $z = Qu$, where Q is a sunny nonexpansive retraction of X onto Ω.

Remark 3.5. (1) Our results extend those of [3, 16, 21, 31, 33, 34] from Hilbert spaces to Banach spaces.

(2) We remove the conditions that $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$ and $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$ in Theorem 3.3 of Yao-Noor [34] and the conditions that $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\lim_{n\to\infty} \left(\frac{1}{r_{n+1}} - \frac{1}{r_n}\right) = 0$ and $\sum_{n=1}^{\infty} \frac{|\alpha_{n+1}-\alpha_n|}{r_{n+1}} < \infty$ in Theorem 1 of Boikanyo-Moroşanu [3].

We now give an example in ℓ_3 space which is a uniformly convex and 2-uniformly smooth Banach space but not a Hilbert space.

Example 3.6. Let $A : \ell_3 \to \ell_3$ be defined by $Ax = 2x + (1, 1, 1, 0, 0, 0, 0, ...)$ and let $B : \ell_3 \to \ell_3$ be defined by $Bx = 5x$ where $x = (x_1, x_2, x_3, ...) \in \ell_3$.

We see that A is a 1/2-isa of order 2 and B is an m-accretive operator. Indeed, let $x, y \in \ell_3$, then

$$\begin{aligned}\langle Ax - Ay, j_2(x-y)\rangle &= \langle 2x - 2y, j_2(x-y)\rangle \\ &= 2\|x-y\|^2_{\ell_3} \\ &= \frac{1}{2}\|Ax - Ay\|^2_{\ell_3}.\end{aligned} \tag{3.49}$$

We also have

$$\langle Bx - By, j_2(x-y)\rangle = 5\|x-y\|^2_{\ell_3} \geq 0 \tag{3.50}$$

and $R(I + rB) = \ell_3$ for all $r > 0$. By a direct calculation, we have for $s > 0$

$$J_s^B(x - sAx) = (I + sB)^{-1}(x - sAx) = \frac{1-2s}{1+5s}x - \frac{s}{1+5s}(1, 1, 1, 0, 0, 0, 0, ...), \tag{3.51}$$

where $x = (x_1, x_2, x_3, ...) \in \ell_3$. Since, in ℓ_3, $q = 2$, $k_2 = 2$ and $\alpha = 1/2$, we can choose $r_n = 0.1$ for all $n \in \mathbb{N}$. Let $\alpha_n = \frac{1}{1000n+1}$, $\lambda_n = \frac{1}{10n}$ and $\delta_n = (1 - \frac{1}{10n} - \frac{1}{1000n+1})$. Let $u = (-0.05, -0.08, -0.06, 0, 0, 0, 0, ...)$ and $e_n = (0, 0, 0, ...)$. Starting $x_1 = (1.2, 2.5, 3.4, 0, 0, 0, 0, ...)$ and computing iteratively algorithm (3.19) in Theorem 3.3, we obtain the following numerical results.

From Table 1, the solution is $(-0.142857, -0.142857, -0.142857, 0, 0, 0, 0, ...)$.

4 Applications and numerical examples

In this section, we discuss some concrete examples as well as the numerical results for supporting the main theorem.

Table 1 Numerical results of Example 3.6 for iteration process (3.19)

n	x_n	$\|x_{n+1} - x_n\|_{\ell_3}$
1	(1.2000000,2.5000000,3.4000000,0.0000000,0.0000000,0.0000000,...)	1.6937789E+00
10	(−0.1368743, −0.1311181, −0.1271209,0.0000000,0.0000000,0.0000000,...)	8.2573797E-03
20	(−0.1428340, −0.1428263, −0.1428159,0.0000000,0.0000000,0.0000000,...)	1.7112628E-05
30	(−0.1428499, −0.1428522, −0.1428507,0.0000000,0.0000000,0.0000000,...)	3.4600332E-07
40	(−0.1428519, −0.1428536, −0.1428524,0.0000000,0.0000000,0.0000000,...)	1.7301589E-07
50	(−0.1428530, −0.1428543, −0.1428534,0.0000000,0.0000000,0.0000000,...)	1.0843686E-07
60	(−0.1428537, −0.1428548, −0.1428541,0.0000000,0.0000000,0.0000000,...)	7.4310792E-08
70	(−0.1428542, −0.1428552, −0.1428545,0.0000000,0.0000000,0.0000000,...)	5.4092528E-08
80	(−0.1428546, −0.1428554, −0.1428549,0.0000000,0.0000000,0.0000000,...)	4.1132625E-08
90	(−0.1428549, −0.1428556, −0.1428551,0.0000000,0.0000000,0.0000000,...)	3.2329686E-08
100	(−0.1428551, −0.1428558, −0.1428553,0.0000000,0.0000000,0.0000000,...)	2.6078314E-08
⋮	⋮	⋮
200	(−0.1428561, −0.1428565, −0.1428562,0.0000000,0.0000000,0.0000000,...)	6.4012505E-09
250	(−0.1428563, −0.1428566, −0.1428564,0.0000000,0.0000000,0.0000000,...)	4.0821310E-09
300	(−0.1428565, −0.1428567, −0.1428565,0.0000000,0.0000000,0.0000000,...)	2.8280803E-09
350	(−0.1428566, −0.1428568, −0.1428566,0.0000000,0.0000000,0.0000000,...)	2.0742607E-09
400	(−0.1428566, −0.1428568, −0.1428567,0.0000000,0.0000000,0.0000000,...)	1.5860974E-09
450	(−0.1428567, −0.1428568, −0.1428567,0.0000000,0.0000000,0.0000000,...)	1.2519825E-09
500	(−0.1428567, −0.1428569, −0.1428568,0.0000000,0.0000000,0.0000000,...)	1.0133107E-09
550	(−0.1428568, −0.1428569, −0.1428568,0.0000000,0.0000000,0.0000000,...)	8.3691026E-10
600	(−0.1428568, −0.1428569, −0.1428568,0.0000000,0.0000000,0.0000000,...)	7.0286201E-10
650	(−0.1428568, −0.1428569, −0.1428569,0.0000000,0.0000000,0.0000000,...)	5.9861825E-10

4.1 Minimization problem

In this subsection, we apply Theorem 3.3 to the convex minimization problem. Let H be a real Hilbert space. Let $F : H \to \mathbb{R}$ be a convex smooth function and $G : H \to \mathbb{R}$ be a convex, lower-semicontinuous and nonsmooth function. We consider the problem of finding $\hat{x} \in H$ such that

$$F(\hat{x}) + G(\hat{x}) \leq F(x) + G(x) \tag{4.1}$$

for all $x \in H$. This problem (4.1) is equivalent, by Fermat's rule, to the problem of finding $\hat{x} \in H$ such that

$$0 \in \nabla F(\hat{x}) + \partial G(\hat{x}), \tag{4.2}$$

where ∇F is a gradient of F and ∂G is a subdifferential of G. In this point of view, we can set $A = \nabla F$ and $B = \partial G$ in Theorem 3.3. This is because if ∇F is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone [[1], Corollary 10]. Moreover, ∂G is maximal monotone [[28], Theorem A]. So we obtain the following result.

Theorem 4.1. *Let H be real Hilbert space. Let $F : H \to \mathbb{R}$ be a convex and differentiable function with $(1/L)$-Lipschitz continuous gradient ∇F and $G : H \to \mathbb{R}$ be a convex and lower semi-continuous function which $F + G$ attains a minimizer. Let $\{e_n\}$ be a sequence in H. Let $\{x_n\}$ be generated by $u, x_1 \in H$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J_{r_n}\big(x_n - r_n \nabla F(x_n)\big) + e_n, \; n \geq 1, \tag{4.3}$$

where $J_{r_n} = (I + r_n \partial G)^{-1}$, $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \lambda_n + \delta_n = 1$. Assume that

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2L$;
(iii) $\liminf_{n\to\infty} \delta_n > 0$;
(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

Then $\{x_n\}$ strongly converges to a minimizer of $F + G$.

Example 4.2. Solve the following minimization problem:

$$\min_{x\in\mathbb{R}^3} \|x\|_2^2 + (3, 5, -1)x + 9 + \|x\|_1, \tag{4.4}$$

where $x = (y_1, y_2, y_3) \in \mathbb{R}^3$.

For each $x \in \mathbb{R}^3$, we set $F(x) = \|x\|_2^2 + (3, 5, -1)x + 9$ and $G(x) = \|x\|_1$. Then $\nabla F(x) = 2x + (3, 5, -1)$. We can check that F is convex and differentiable on $\mathbb{R}^3$ with 2-Lipschitz continuous gradient ∇F. Moreover, G is convex and lower semi-continuous but not differentiable on $\mathbb{R}^3$. From [14] we know that, for $r > 0$,

$$(I + r\partial G)^{-1}(x) = (\max\{|y_1| - r, 0\} sign(y_1), \max\{|y_2| - r, 0\} sign(y_2), \max\{|y_3| - r, 0\} sign(y_3)).$$

We choose $\alpha_n = \frac{1}{100n+1}$, $\lambda_n = \frac{99n}{(n+1)(100n+1)}$, $\delta_n = \frac{n}{n+1}$ and $r_n = 0.2$. Let $e_n = (\frac{1}{n^2}, \frac{1}{n^2}, \frac{1}{n^2})$, $u = (2.553479, 5.187352, 1.903486)$ and $x_1 = (3.425859, 8.231258, 1.430561)$. Using algorithm (4.3) in Theorem 4.1, we obtain the following numerical results.

From Table 2, we see that $x_{663} = (-0.999860, -1.999722, 0.000078)$ is an approximation of the minimizer of $F + G$ with an error $4.9973577E - 07$ and its minimum value is approximately 4.0000001. In fact, the minimizer of $F + G$ is $(-1, -2, 0)$ and $(F + G)(-1, -2, 0) = 4$ (Figs. 1 and 2).

4.2 Linear inverse problem

In this subsection, we apply Theorem 3.3 to solve the unconstrained linear system

$$Cx = d \tag{4.5}$$

where C is a bounded linear operator on H and $d \in H$. For each $x \in H$, we define $F : H \to \mathbb{R}$ by

$$F(x) = \frac{1}{2}\|Cx - d\|^2. \tag{4.6}$$

Table 2 Numerical results of Example 4.2 for iteration process (4.3)

n	$x_n = (y_1^n, y_2^n, y_3^n)$	$F(x_n) + G(x_n)$	$\|x_{n+1} - x_n\|_2$
1	(3.425859,8.231258,1.430561)	153.6276069	1.4677578E+00
2	(3.332050, 6.954869, 2.149131)	128.1489310	2.7041086E+00
3	(2.156296, 4.541443, 1.824807)	73.4780908	2.4690302E+00
4	(1.021838, 2.392267, 1.388738)	36.1366456	1.8578516E+00
5	(0.270687, 0.736212, 1.008125)	16.1316179	9.8758376E−01
6	(−0.108319, −0.126974, 0.713871)	8.8129329	6.9838132E−01
7	(−0.381831, −0.732534, 0.498872)	6.2374757	4.8162425E−01
8	(−0.573595, −1.147294, 0.346679)	5.0291149	3.2683581E−01
9	(−0.705677, −1.426945, 0.240984)	4.4730915	2.1951923E−01
10	(−0.795670, −1.613558, 0.168420)	4.2194535	1.4654722E−01
⋮	⋮	⋮	⋮
654	(−0.999858, −1.999718, 0.000079)	4.0000001	5.1403155E−07
655	(−0.999858, −1.999718, 0.000079)	4.0000001	5.1241306E−07
656	(−0.999858, −1.999719, 0.000079)	4.0000001	5.1080220E−07
657	(−0.999858, −1.999719, 0.000079)	4.0000001	5.0919892E−07
658	(−0.999858, −1.999720, 0.000079)	4.0000001	5.0760317E−07
659	(−0.999859, −1.999720, 0.000078)	4.0000001	5.0601491E−07
660	(−0.999859, −1.999721, 0.000078)	4.0000001	5.0443408E−07
661	(−0.999859, −1.999721, 0.000078)	4.0000001	5.0286064E−07
662	(−0.999859, −1.999721, 0.000078)	4.0000001	5.0129456E−07
663	(−0.999860, −1.999722, 0.000078)	4.0000001	4.9973577E−07

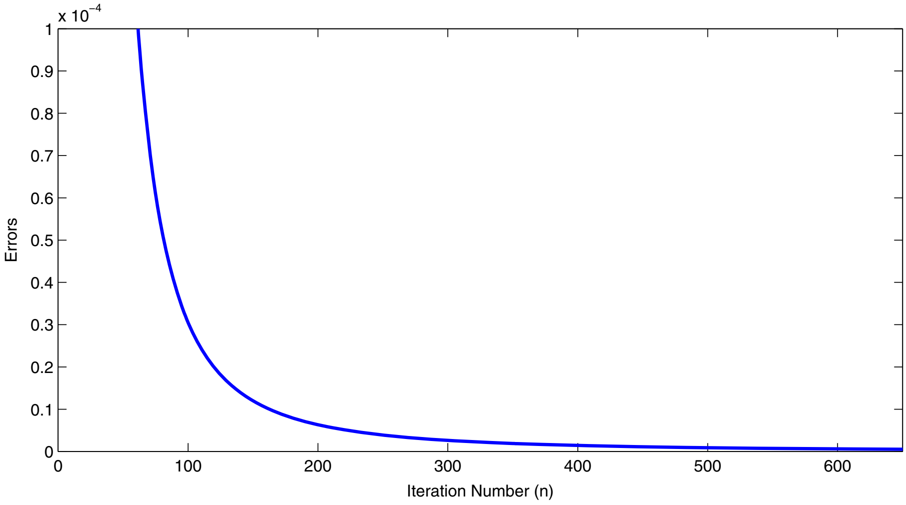

Fig. 1 The error plotting of $\|x_{n+1} - x_n\|_2$ in Table 2

From [4] we know that $\nabla F(x) = C^T(Cx - d)$ and ∇F is K-Lipschitz continuous with K the largest eigenvalue of $C^T C$. So we obtain the following result.

Theorem 4.3. *Let H be real Hilbert space. Let $C : H \to H$ be a bounded linear operator and $d \in H$ with K the largest eigenvalue of $C^T C$. Let $\{e_n\}$ be a sequence in H. Let $\{x_n\}$ be generated by $u, x_1 \in H$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n\big(x_n - r_n C^T(Cx_n - d)\big) + e_n, \ n \geq 1, \tag{4.7}$$

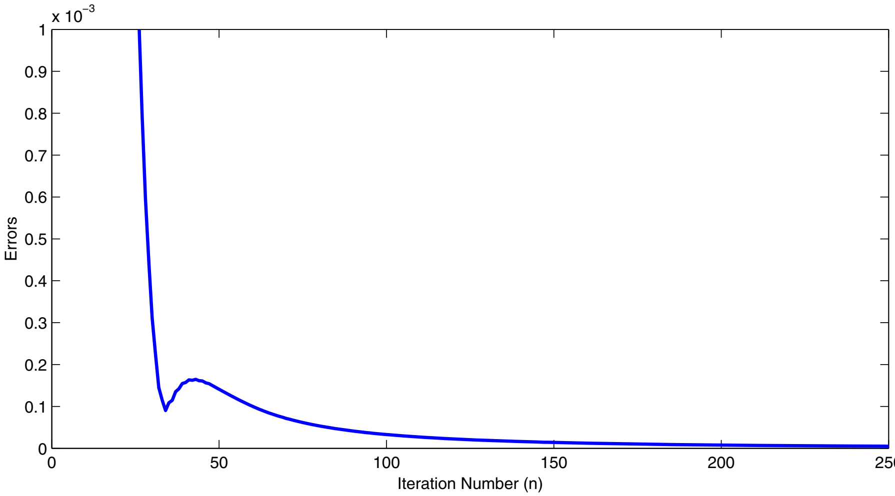

Fig. 2 The error plotting of $\|x_{n+1} - x_n\|_2$ in Table 3

where $\{r_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}$, $\{\lambda_n\}$, *and* $\{\delta_n\}$ *are sequences in* $[0, 1]$ *with* $\alpha_n + \lambda_n + \delta_n = 1$. *Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2/K$;
(iii) $\liminf_{n\to\infty} \delta_n > 0$;
(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \|e_n\|/\alpha_n = 0$.

If (4.5) is consistent, then $\{x_n\}$ *strongly converges to a solution of a linear system.*

Example 4.4. Solve the following linear system:

$$\begin{aligned} 2y_1 + y_2 - 3y_3 + 2y_4 &= 13 \\ y_1 - 2y_2 + 3y_3 + 5y_4 &= 9 \\ -3y_1 + 5y_2 + 4y_3 - 2y_4 &= -3 \\ 4y_1 + 2y_2 - y_3 - y_4 &= 6. \end{aligned} \tag{4.8}$$

Let $C = \begin{pmatrix} 2 & 1 & -3 & 2 \\ 1 & -2 & 3 & 5 \\ -3 & 5 & 4 & -2 \\ 4 & 2 & -1 & -1 \end{pmatrix}$, $x = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ and $d = \begin{pmatrix} 13 \\ 9 \\ -3 \\ 6 \end{pmatrix}$.

Table 3 Numerical results of Example 4.4 for iteration process (4.7)

n	$x_n = (y_1^n, y_2^n, y_3^n, y_4^n)^T$	$\|x_{n+1} - x_n\|_2$
1	(−1.000000,3.000000,2.000000,5.000000)	3.0154184E+00
2	(1.608431,3.435784,0.640392,5.500392)	1.7095286E+00
3	(1.626720,3.340500, −0.516476,4.245509)	8.4386757E-01
4	(1.339520,2.891206, −0.916370,3.727969)	5.0556755E-01
5	(1.099700,2.553623, −1.039057,3.465156)	2.9280323E-01
⋮	⋮	⋮
238	(1.000364,1.999292,−0.999491,2.999184)	5.3648214E-06
239	(1.000363,1.999295,−0.999493,2.999188)	5.3189260E-06
240	(1.000361,1.999298,−0.999496,2.999191)	5.2735040E-06
241	(1.000359,1.999301,−0.999498,2.999195)	5.2287703E-06
242	(1.000358,1.999304,−0.999500,2.999198)	5.1844999E-06
243	(1.000356,1.999307,−0.999502,2.999202)	5.1408891E-06
244	(1.000355,1.999310,−0.999504,2.999205)	5.0977315E-06
245	(1.000353,1.999313,−0.999506,2.999208)	5.0552065E-06
246	(1.000352,1.999316,−0.999508,2.999212)	5.0131244E-06
247	(1.000350,1.999319,−0.999510,2.999215)	4.9716496E-06

Then

$$C^T C = \begin{pmatrix} 30 & -7 & -19 & 11 \\ -7 & 34 & 9 & -20 \\ -19 & 9 & 35 & 2 \\ 11 & -20 & 2 & 34 \end{pmatrix}.$$

The largest eigenvalue of $C^T C$ is 65.5033. This allows us to choose the upper bound of $\{r_n\}$. We also note that since $C^T C$ is symmetric, the largest eigenvalue K is less than mb, where m is the dimension of the matrix $C^T C$ and b is its maximal element; see [[36], Theorem 1].

We choose $\alpha_n = \frac{1}{50n+1}$, $\lambda_n = \frac{49n}{(n+1)(50n+1)}$, $\delta_n = \frac{n}{n+1}$ and $r_n = 0.03$ for all $n \geq 1$. Let $e_n = (\frac{1}{n^3}, \frac{1}{n^3}, \frac{1}{n^3}, \frac{1}{n^3})^T$, $u = (3, 1, 1, 4)^T$ and $x_1 = (-1, 3, 2, 5)^T$. Using algorithm (4.7) in Theorem 4.3, we obtain the following numerical results.

From Table 3 we see that the solution of a linear system (4.8) is $(1, 2, -1, 3)$.

Remark 4.5. We remark that Theorem 3.3 can be further applied to the variational inequality problem, the split feasibility problem and the fixed point problem. See also [18, 29].

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ORIGINAL PAPER

Another look at the split common fixed point problem for demicontractive operators

Yekini Shehu[1] · Prasit Cholamjiak[2]

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Abstract In this article, we take a look at the recent results of Moudafi (Inverse Probl 26:587–600, 2010), Tang et al. (Math Model Anal 17:457–466, 2012) and Wang and Cui (Math Model Anal 18:537–542, 2013), which weak convergence results were obtained for the split common fixed point problem for demicontractive mappings. We introduce a new algorithm for solving the split common fixed point problem for demicontractive mappings and then prove strong convergence of the sequence in real Hilbert spaces. We also apply our results to the split common null point problem in real Hilbert spaces. Finally, we give numerical results to demonstrate its convergence.

Keywords Demicontractive mappings · Split common fixed point problems · Iterative scheme · Strong convergence · Hilbert spaces

Mathematics Subject Classification 47H06 · 47H09 · 47J05 · 47J25

1 Introduction

In this paper, we shall assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$. Let I denote the identity operator on H. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point

$$x \in C \quad \text{such that } Ax \in Q, \tag{1.1}$$

✉ Prasit Cholamjiak
prasitch2008@yahoo.com

Yekini Shehu
yekini.shehu@unn.edu.ng

1 Department of Mathematics, University of Nigeria, Nsukka, Nigeria

2 School of Science, University of Phayao, Phayao 56000, Thailand

where $A: H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was introduced by Censor and Elfving [9] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [4]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example [5,18,24,28,31,32,36] and the references therein).

Note that the SFP (1.1) can be formulated as a fixed point equation by using the following fact:

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*; \tag{1.2}$$

that is, x^* is a solution of the SFP (1.1) if and only if x^* is a solution of the fixed point Eq. (1.2) (see [27] for the details). This suggests that we can use fixed point algorithms (see [33,34,37]) to solve SFP. A well known algorithm used to solve the SFP (1.1) is Byrne's CQ algorithm [4] which is found to be a gradient projection method (GPM) in a convex minimization. Subsequently, Byrne [5] applied Krasnoselskii–Mann iteration to the CQ algorithm. Zhao and Yang [39] applied Krasnoselskii–Mann iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the Krasnoselskii–Mann algorithm for the SFP do not necessarily converge strongly in the infinite-dimensional Hilbert spaces.

We next provide some definitions which will be used in the sequel.

Let $T: H \to H$ be a mapping. A point $x \in H$ is said to be a *fixed point* of T provided that $Tx = x$. In this paper, we denote $F(T)$ by the fixed point set. The symbols $\to$ and $\rightharpoonup$ mean the strong convergence and the weak convergence, respectively.

Definition 1.1 The mapping $T: H \to H$ is said to be

(a) *nonexpansive* if

$$||Tx - Ty|| \le ||x - y||, \quad \forall x, y \in H.$$

(b) *quasi-nonexpansive* if

$$||Tx - Tp|| \le ||x - p||, \quad \forall x \in H, p \in F(T).$$

(c) *firmly nonexpansive* mapping if

$$||Tx - Ty||^2 \le ||x - y||^2 - ||(x - y) - (Tx - Ty)||^2, \quad \forall x, y \in H.$$

(d) *quasi-firmly nonexpansive* mapping if

$$||Tx - Tp||^2 \le ||x - p||^2 - ||x - Tx||^2, \quad \forall x \in H, p \in F(T).$$

(e) *strictly pseudocontractive* mapping if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \le ||x - y||^2 + k||(x - y) - (Tx - Ty)||^2, \quad \forall x, y \in H.$$

(f) *pseudocontractive* mapping if

$$||Tx - Ty||^2 \le ||x - y||^2 + ||(x - y) - (Tx - Ty)||^2, \quad \forall x, y \in H.$$

(g) *demicontractive (or k-demicontractive)* if there exists $k < 1$ such that

$$||Tx - Tp||^2 \le ||x - p||^2 + k||x - Tx||^2, \quad \forall x \in H, p \in F(T). \tag{1.3}$$

Remark 1.2 It is clear that, in a real Hilbert space H, (1.3) is equivalent to

$$\langle Tx - p, x - p\rangle ||x - p||^2 \ge \frac{1 - k_1}{2}||x - Tx||^2, \quad \forall x \in H, p \in F(T). \tag{1.4}$$

We denote by $\Im_N, \Im_{QN}, \Im_{FN}, \Im_{QF}, \Im_S, \Im_P, \Im_D$ (with $k \geq 0$) the classes of nonexpansive, quasi-nonexpansive, firmly-nonexpansive, quasi-firmly nonexpansive, strictly pseudocontractive, pseudocontractive and demicontractive mappings, respectively. It is easily seen that $\Im_{FN} \subsetneq \Im_N \subsetneq \Im_{QN} \subsetneq \Im_D$, $\Im_{FN} \subsetneq \Im_{QF} \subsetneq \Im_{QN} \subsetneq \Im_D$ and $\Im_{FN} \subsetneq \Im_N \subsetneq \Im_S \subsetneq \Im_D$ by the following examples.

The following example is the demicontractive mapping which is not pseudocontractive and also is not strictly pseudocontractive.

Example 1.3 [15] Let H be the real line and $C = [-1, 1]$. Define T on C by

$$Tx = \begin{cases} \frac{2}{3}x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{1.5}$$

The following example is the demicontractive mapping which is not quasi-nonexpansive and also is not pseudocontractive.

Example 1.4 [13] $f\colon [-2, 1] \to [-2, 1]$, $f(x) = -x^2 - x$.

Furthermore, $\Im_{FN}$ is well known to include the resolvent operator and the projection operator, while $\Im_{QF}$ contains the subgradient projection operator (see, e.g., [20] and the reference therein).

In this paper, we shall focus our attention on the following split common fixed point problem (SCFPP) for two operators:

$$\text{find } x \in C \quad \text{such that } Ax \in Q, \tag{1.6}$$

where $A\colon H_1 \to H_2$ is a bounded linear operator, $S\colon H_1 \to H_1$ and $T\colon H_2 \to H_2$ are two demicontractive operators with nonempty fixed point sets $F(S) = C$ and $F(T) = Q$. We denote the solution set of the SCFPP by

$$\Gamma := \{y \in C\colon Ay \in Q\} = C \cap A^{-1}(Q). \tag{1.7}$$

Recall that $F(S)$ and $F(T)$ are nonempty, closed and convex subsets of H_1 and H_2, respectively. If $\Gamma \neq \emptyset$, then Γ is a closed and convex subset of H_1. The SCFPP is a generalization of the SFP and the convex feasibility problem (CFP) (see [4,11]).

In order to solve (1.6), Censor and Segal [11] studied, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = S(x_n + \gamma A^t(T - I)Ax_n), \quad n \geq 1, \tag{1.8}$$

where $\gamma \in \left(0, \frac{2}{\lambda}\right)$, with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition). In 2011, Moudafi [21] introduced the following relaxed algorithm:

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Sy_n, \quad n \geq 1, \tag{1.9}$$

where $y_n = x_n + \gamma A^*(T - I)Ax_n$, $\beta \in (0, 1)$, $\alpha_n \in (0, 1)$, and $\gamma \in \left(0, \frac{1}{\lambda\beta}\right)$, with λ being the spectral radius of the operator A^*A. Moudafi proved weak convergence result of the algorithm (1.9) in Hilbert spaces where S and T are quasi-nonexpansive operators. We observe that strong convergence result can be obtained in the results of Moudafi [21] if a compactness type condition like demicompactness is imposed on the operator S. Furthermore, we can also obtain strong convergence result by suitably modifying the algorithm (1.9). Recently, Zhao and He [38] introduced the following viscosity approximation algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)((1 - w_n)x_n + w_n S(x_n + \gamma A^*(T - I)Ax_n)), \quad n \geq 1, \tag{1.10}$$

where $f: H_1 \to H_1$ is a contraction of modulus $\rho > 0$, $w_n \in \left(0, \frac{1}{2}\right)$, $\gamma \in \left(0, \frac{1}{\lambda}\right)$, with λ being the spectral radius of the operator A^*A. They proved strong convergence theorems concerning (1.6) for quasi-nonexpansive operators S and T in real Hilbert spaces. Inspired by the work of Zhao and He [38], Moudafi [23] quite recently revisited the viscosity-type approximation method. In fact, Moudafi gave a simple proof of the strong convergence of the iterative sequence $\{x_n\}$ defined by (1.10) based on attracting operator properties, then proposed its modification and finally proved its strong convergence (see Theorem 2.1 of [23]).

In 2010, Moudafi [22] proposed an algorithm to solve the two-operator SCFPP (1.6) where S and T are demicontractive operators. The class of demicontractive operators is fundamental since many common types of operators arising in optimization belong to this class (see Remark 2 of Tang et al. [25]). Moudafi [22] proved that the sequence $\{x_n\}$ generated by Algorithm 1 *converges weakly* to the solution of (1.6).

Algorithm 1 [22] Let $x_0 \in H_1$ be arbitrary and let the sequence $\{x_n\}$ be defined by:

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S u_n, \quad n \geq 0, \tag{1.11}$$

where $u_n = x_n + \gamma A^*(T - I)Ax_n$, $\gamma \in \left(0, \frac{1-\mu}{\lambda}\right)$ with λ being the spectral radius of the operator A^*A and $\{\alpha_n\} \subset (0, 1)$.

Theorem 1.5 [22] *Given a bounded linear operator $A: H_1 \to H_2$, let $S: H_1 \to H_1$ and $T: H_2 \to H_2$ be demicontractive (with constants β, μ, respectively) with nonempty $F(S)$ and $F(T)$. Assume that $S - I$ and $T - I$ are demiclosed at 0. If the two sets of SCFPP* (1.6) *is nonempty, then any sequence $\{x_n\}$ generated by Algorithm* 1 *converges to a split common fixed point x^* of* (1.6)*, provided $\gamma \in \left(0, \frac{1-\mu}{\lambda}\right)$ and $\alpha_n \in (\delta, 1 - \beta - \delta)$ for a small enough $\delta > 0$.*

Recently, inspired and motivated by the result of Moudafi [22], Tang et al. [25] proposed a cyclic algorithm (Algorithm 2 below) to solve the SCFPP for demicontractive operators $\{S_i\}_{i=1}^p$ and $\{T_j\}_{j=1}^r$. Then they proved that the sequence generated by the proposed algorithm *converges weakly* to the solution of (SCFPP). Their work extends those of Moudafi [22], Censor and Segal [11] and others.

Algorithm 2 [25] Let $x_0 \in H_1$ be arbitrary and let the sequence $\{x_n\}$ be defined by:

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S_{i(n)} u_n, \quad n \geq 0, \tag{1.12}$$

where $u_n = x_n + \gamma A^*(T_{j(n)} - I)Ax_n$, $i(n) = n(mod\ p) + 1$ and $j(n) = n(mod\ r) + 1$, $\gamma \in \left(0, \frac{1-\mu}{\lambda}\right)$ with λ being the spectral radius of the operator A^*A and $\{\alpha_n\} \subset (0, 1)$.

Quite recently, Wang and Cui [26] presented a simple proof of the result of Tang et al. [25] and removed the continuity of the mapping. They obtained the weak convergence of the Algorithm 2 above.

We comment on the results of Moudafi [22], Tang et al. [25] and Wang and Cui [26] as follows:

(1) Theorem 1.5 gives a *weak convergence result* for two-operator SCFPP (1.6) when the operators S and T are demicontractive. In order to get strong convergence, one must impose a compactness type condition (demi-compactness) on the mapping S. But this compactness condition *appears strong*.

(2) Similarly, in order to obtain strong convergence result in those of Moudafi [22], Tang et al. [25] and Wang and Cui [26] without compactness type condition on the mappings S, a modification of (1.11) and (1.12) is necessary. This modification could be implicit iterative scheme or explicit iterative scheme. In the implicit iterative scheme, the computation of the next iteration x_{n+1} involves solving a nonlinear equation at every step of the iteration, a task which may pose the same difficulty level as the initial problem. Therefore, in order to get strong convergence result for two-operator SCFPP (1.6) when the operators S and T are demicontractive in infinitely dimensional Hilbert spaces without compactness type condition, a modification of (1.11) and (1.12) which is an explicit iterative scheme is necessary. This leads to this natural question.

Question Can we modify the iterative schemes (1.11) and (1.12) so that strong convergence is guaranteed without any further condition of compactness type on the operator?

Our aim in this work is to answer the above question. Thus, we propose a new algorithm to solve the two-operator SCFPP (1.6) when the operators S and T are demicontractive. Then we prove that the sequence generated by the proposed algorithm *converges strongly* to the solution of (1.6). Our work extends the results of Zhao and He [38], Moudafi [21,23], Censor and Segal [11] to the SCFPP when the operators S and T are demicontractive. Furthermore, our work improves the recent works of Moudafi [22], Tang et al. [25] and Wang and Cui [26].

2 Preliminaries

Definition 2.1 A mapping $T : H \to H$ is called *demiclosed at 0* if any sequence $\{x_n\}$ weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

Next, we state the following well-known lemmas which will be used in the sequel.

Lemma 2.2 *Let H be a real Hilbert space. Then the following results hold:*

$$(i)\quad ||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \quad \forall x, y \in H.$$
$$(ii)\quad ||x+y||^2 \le ||x||^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.3 (Xu [29]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \ge 0,$$

where

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \le 0$;
(iii) $\gamma_n \ge 0$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then, $a_n \to 0$ as $n \to \infty$.

3 Main results

In this section, we propose a new modification of (1.11) and then prove its strong convergence under some mild conditions.

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces, $A\colon H_1 \to H_2$ be a bounded linear operator and $A^*\colon H_2 \to H_1$ be a adjoint operator of A. Let $S\colon H_1 \to H_1$ be a k_1-demicontractive mapping such that $S - I$ is demi-closed at 0 and $C := F(S) \neq \emptyset$. Let $T\colon H_2 \to H_2$ be a k_2-demicontractive mapping such that $T - I$ is demi-closed at 0 and $Q := F(T) \neq \emptyset$. Suppose that SCFPP (1.6) has a nonempty solution set Ω. Let $\{\beta_n\}$ and $\{\lambda_n\}$ be two real sequences in $(0, 1)$ and $\gamma \in \left(0, \frac{1-k_2}{||A||^2}\right)$. Let $\{y_n\}$ and $\{x_n\}$ be generated by $x_1 \in H_1$ and*

$$\begin{cases} y_n = x_n + \gamma A^*(T - I)Ax_n \\ x_{n+1} = (1 - \beta_n)(\lambda_n y_n) + \beta_n S y_n, \quad n \geq 1. \end{cases} \tag{3.1}$$

Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \lambda_n = 1$*;*
(b) $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$*;*
(c) $\beta_n \in \left[\epsilon, \frac{\lambda_n(1-k_1)}{1-(1-k_1)(1-\lambda_n)}\right)$*, for some* $\epsilon > 0$ *and* $\limsup_{n\to\infty} \beta_n < 1 - k_1$.

Then $\{x_n\}$ converges strongly to an element x^* of Ω, where x^* is the minimum-norm solution of (1.6).

Proof Let $x^* \in \Omega$. From (3.1) and Lemma 2.2(i), we have

$$\begin{aligned} ||y_n - x^*||^2 &= ||x_n - x^* + \gamma A^*(T - I)Ax_n||^2 \\ &= ||x_n - x^*||^2 + 2\gamma\langle x_n - x^*, A^*(T - I)Ax_n\rangle + \gamma^2||A^*(T - I)Ax_n||^2. \end{aligned} \tag{3.2}$$

We see that

$$\begin{aligned} \gamma^2||A^*(T - I)Ax_n||^2 &= \gamma^2\langle A^*(T - I)Ax_n, A^*(T - I)Ax_n\rangle \\ &= \gamma^2\langle AA^*(T - I)Ax_n, (T - I)Ax_n\rangle \\ &\leq \gamma^2||A||^2||(T - I)Ax_n||^2. \end{aligned} \tag{3.3}$$

Since T is a demicontractive mapping and $Ax^* \in Q = F(T)$, we obtain

$$\begin{aligned} \langle x_n - x^*, A^*(T - I)Ax_n\rangle &= \langle A(x_n - x^*), (T - I)Ax_n\rangle \\ &= \langle A(x_n - x^*) + (T - I)Ax_n - (T - I)Ax_n, (T - I)Ax_n\rangle \\ &= \langle TAx_n - Ax^*, (T - I)Ax_n\rangle - ||(T - I)Ax_n||^2 \\ &= \frac{1}{2}\Big[||TAx_n - Ax^*||^2 + ||(T - I)Ax_n||^2 - ||Ax_n - Ax^*||^2\Big] \\ &\quad - ||(T - I)Ax_n||^2 \\ &\leq \frac{1}{2}\Big[||Ax_n - Ax^*||^2 + k_2||(T - I)Ax_n||^2\Big] \\ &\quad + \frac{1}{2}\Big[||(T - I)Ax_n||^2 - ||Ax_n - Ax^*||^2\Big] \\ &\quad - ||(T - I)Ax_n||^2 \\ &= \frac{k_2 - 1}{2}||(T - I)Ax_n||^2. \end{aligned} \tag{3.4}$$

Substituting (3.4) and (3.3) into (3.2), we have

$$||y_n - x^*||^2 \leq ||x_n - x^*||^2 - \gamma(1 - k_2 - \gamma||A||^2)||(T - I)Ax_n||^2. \tag{3.5}$$

From (3.1), we see that

$$\begin{aligned}||x_{n+1} - x^*|| &= ||(1 - \beta_n - (1 - \beta_n)(1 - \lambda_n))(y_n - x^*) \\ &\quad + \beta_n(Sy_n - x^*) - (1 - \beta_n)(1 - \lambda_n)x^*|| \\ &\leq ||(1 - \beta_n - (1 - \beta_n)(1 - \lambda_n))(y_n - x^*) \\ &\quad + \beta_n(Sy_n - x^*)|| + (1 - \beta_n)(1 - \lambda_n)||x^*||. \end{aligned} \tag{3.6}$$

Using condition (c), we also have

$$\begin{aligned}\beta_n &< \frac{\lambda_n(1-k_1)}{1-(1-k_1)(1-\lambda_n)} \Leftrightarrow \beta_n[1-(1-k_1)(1-\lambda_n)] < \lambda_n(1-k_1) \\ &\Leftrightarrow \beta_n - \beta_n(1-k_1)(1-\lambda_n) < \lambda_n(1-k_1) \\ &\Leftrightarrow \beta_n < \lambda_n(1-k_1) + \beta_n(1-k_1)(1-\lambda_n) \\ &\Leftrightarrow \beta_n < (1-(1-\lambda_n))(1-k_1) + \beta_n(1-k_1)(1-\lambda_n) \\ &\Leftrightarrow \beta_n < (1-k_1) - (1-\lambda_n)(1-k_1) + \beta_n(1-k_1)(1-\lambda_n) \\ &\Leftrightarrow \beta_n < (1-k_1) - (1-k_1)(1-\beta_n)(1-\lambda_n) \\ &\Leftrightarrow \beta_n < (1-k_1)(1-(1-\beta_n)(1-\lambda_n)) \\ &\Leftrightarrow \beta_n - (1-k_1)(1-(1-\beta_n)(1-\lambda_n)) < 0. \end{aligned} \tag{3.7}$$

Using Lemma 2.2(i), (1.3), (1.4) and (3.7), we obtain

$$\begin{aligned}&||(1-\beta_n-(1-\beta_n)(1-\lambda_n))(y_n-x^*) + \beta_n(Sy_n-x^*)||^2 \\ &\quad = (1-\beta_n-(1-\beta_n)(1-\lambda_n))^2||y_n-x^*||^2 \\ &\qquad + \beta_n^2||Sy_n-x^*||^2 + 2(1-\beta_n-(1-\beta_n)(1-\lambda_n))\beta_n\langle Sy_n-x^*, y_n-x^*\rangle \\ &\quad \leq (1-\beta_n-(1-\beta_n)(1-\lambda_n))^2||y_n-x^*||^2 + \beta_n^2[||y_n-x^*||^2 + k_1||y_n-Sy_n||^2] \\ &\qquad + 2(1-\beta_n-(1-\beta_n)(1-\lambda_n))\beta_n\left[||y_n-x^*||^2 - \frac{1-k_1}{2}||y_n-Sy_n||^2\right] \\ &\quad = (1-(1-\beta_n)(1-\lambda_n))^2||y_n-x^*||^2 \\ &\qquad + [k_1\beta_n^2 - (1-k_1)(1-\beta_n-(1-\beta_n)(1-\lambda_n))\beta_n]||y_n-Sy_n||^2 \\ &\quad = (1-(1-\beta_n)(1-\lambda_n))^2||y_n-x^*||^2 \\ &\qquad + \beta_n[\beta_n-(1-k_1)(1-(1-\beta_n)(1-\lambda_n))]||y_n-Sy_n||^2 \\ &\quad \leq (1-(1-\beta_n)(1-\lambda_n))^2||y_n-x^*||^2, \end{aligned} \tag{3.8}$$

which implies

$$\begin{aligned}&||(1-\beta_n-(1-\beta_n)(1-\lambda_n))(y_n-x^*) + \beta_n(Sy_n-x^*)|| \\ &\quad \leq (1-(1-\beta_n)(1-\lambda_n))||y_n-x^*||. \end{aligned} \tag{3.9}$$

From (3.5), (3.6) and (3.9), we get that

$$\begin{aligned}||x_{n+1}-x^*|| &\leq (1-(1-\beta_n)(1-\lambda_n))||y_n-x^*|| + (1-\beta_n)(1-\lambda_n)||x^*|| \\ &\leq (1-(1-\beta_n)(1-\lambda_n))||x_n-x^*|| + (1-\beta_n)(1-\lambda_n)||x^*|| \\ &\leq \max\{||x_n-x^*||, ||x^*||\}\end{aligned}$$

$$\begin{aligned}&\leq \vdots\\&\leq \max\{||x_1 - x^*||, ||x^*||\}.\end{aligned} \tag{3.10}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are bounded.
Now, for any $x \in H_1$, we have

$$\begin{aligned}&||Sx - x^*||^2 \leq ||x - x^*||^2 + k_1||x - Sx||^2\\&\quad \Rightarrow \langle Sx - x^*, Sx - x^*\rangle \leq \langle x - x^*, x - Sx\rangle + \langle x - x^*, Sx - x^*\rangle + k_1||x - Sx||^2\\&\quad \Rightarrow \langle Sx - x^*, Sx - x\rangle \leq \langle x - x^*, x - Sx\rangle + k_1||x - Sx||^2\\&\quad \Rightarrow \langle Sx - x, Sx - x\rangle + \langle x - x^*, Sx - x\rangle \leq \langle x - x^*, x - Sx\rangle + k_1||x - Sx||^2\\&\quad \Rightarrow (1 - k_1)||x - Sx||^2 \leq 2\langle x - x^*, x - Sx\rangle.\end{aligned} \tag{3.11}$$

Since $\beta_n < \frac{\lambda_n(1-k_1)}{1-(1-k_1)(1-\lambda_n)}$, it follows that $\beta_n < 1 - k_1$. Furthermore, by (3.5) and (3.11), we have

$$\begin{aligned}||y_{n+1} - x^*||^2 &\leq ||x_{n+1} - x^*||^2 = ||(1 - \beta_n)\lambda_n y_n + \beta_n S y_n - x^*||^2\\&= ||(1 - (1 - \beta_n)(1 - \lambda_n) - \beta_n)y_n + \beta_n S y_n - x^*||^2\\&= ||(y_n - x^*) - \beta_n(y_n - Sy_n) - (1 - \beta_n)(1 - \lambda_n)y_n||^2\\&\leq ||(y_n - x^*) - \beta_n(y_n - Sy_n)||^2 - 2(1 - \beta_n)(1 - \lambda_n)\langle y_n, x_{n+1} - x^*\rangle\\&= ||y_n - x^*||^2 - 2\beta_n\langle y_n - Sy_n, y_n - x^*\rangle + \beta_n^2||y_n - Sy_n||^2\\&\quad - 2(1 - \beta_n)(1 - \lambda_n)\langle y_n, x_{n+1} - x^*\rangle\\&\leq ||y_n - x^*||^2 - \beta_n(1 - k_1)||y_n - Sy_n||^2 + \beta_n^2||y_n - Sy_n||^2\\&\quad - 2(1 - \beta_n)(1 - \lambda_n)\langle y_n, x_{n+1} - x^*\rangle\\&= ||y_n - x^*||^2 - \beta_n[(1 - k_1) - \beta_n]||y_n - Sy_n||^2\\&\quad - 2(1 - \beta_n)(1 - \lambda_n)\langle y_n, x_{n+1} - x^*\rangle\\&\leq ||x_n - x^*||^2 - \beta_n[(1 - k_1) - \beta_n]||y_n - Sy_n||^2\\&\quad - 2(1 - \beta_n)(1 - \lambda_n)\langle y_n, x_{n+1} - x^*\rangle.\end{aligned} \tag{3.12}$$

Since $\{y_n\}$ and $\{x_n\}$ are bounded, there exists $M > 0$ such that $-2(1-\beta_n)\langle y_n, x_{n+1}-x^*\rangle \leq M$ for all $n \geq 0$. Hence, by (3.12), we have

$$||y_{n+1} - x^*||^2 - ||y_n - x^*||^2 + \beta_n[(1 - k_1) - \beta_n]||y_n - Sy_n||^2 \leq (1 - \lambda_n)M. \tag{3.13}$$

The rest of the proof will be divided into two cases.

Case 1 Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||y_n - x^*||\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{||y_n - x^*||\}_{n=1}^{\infty}$ converges and $||y_n - x^*||^2 - ||y_{n+1} - x^*||^2 \to 0$ as $n \to \infty$. From (3.13) and since $\lambda_n \to 1$, we have

$$||y_n - Sy_n|| \to 0 \tag{3.14}$$

as $n \to \infty$. From (3.5) and (3.10), we have

$$\begin{aligned}
&\gamma(1-k_2-\gamma||A||^2)||(T-I)Ax_n||^2\\
&\quad\le ||x_n-x^*||^2-||y_n-x^*||^2\\
&\quad\le \big(||y_{n-1}-x^*||+(1-\beta_{n-1})(1-\lambda_{n-1})||x^*||\big)^2-||y_n-x^*||^2\\
&\quad= ||y_{n-1}-x^*||^2-||y_n-x^*||^2+2(1-\beta_{n-1})(1-\lambda_{n-1})||x^*||||y_{n-1}-x^*||\\
&\qquad+((1-\beta_{n-1})(1-\lambda_{n-1}))^2||x^*||^2\\
&\quad\le ||y_{n-1}-x^*||^2-||y_n-x^*||^2+2(1-\lambda_{n-1})||x^*||||y_{n-1}-x^*||+(1-\lambda_{n-1})^2||x^*||^2.
\end{aligned}$$

Using condition (a), we get that

$$\gamma(1-k_2-\gamma||A||^2)||(T-I)Ax_n||^2 \to 0,$$

as $n \to \infty$. This shows that

$$||(T-I)Ax_n|| \to 0, \tag{3.15}$$

as $n \to \infty$. Also, we observe that

$$||y_n-x_n|| = \gamma||A^*(T-I)Ax_n|| \le \gamma||A^*||||(T-I)Ax_n|| \to 0,$$

as $n \to \infty$. Since $||y_n-x_n|| \to 0$ and $||y_n-Sy_n|| \to 0$ as $n \to \infty$, we have

$$||x_n-Sy_n|| \le ||y_n-x_n||+||y_n-Sy_n|| \to 0,$$

as $n \to \infty$. Moreover, we also have

$$\begin{aligned}
||\lambda_n y_n-Sy_n|| &= ||\lambda_n y_n-y_n+y_n-Sy_n||\\
&\le ||(\lambda_n-1)y_n||+||y_n-Sy_n||\\
&= (1-\lambda_n)||y_n||+||y_n-Sy_n|| \to 0,
\end{aligned}$$

as $n \to \infty$. So, from (3.1), we have

$$||x_{n+1}-Sy_n|| = (1-\beta_n)||\lambda_n y_n-Sy_n|| \to 0,$$

as $n \to \infty$. This implies that

$$||x_{n+1}-x_n|| \le ||x_{n+1}-Sy_n||+||x_n-Sy_n|| \to 0,$$

as $n \to \infty$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in H_1$. Using the fact that $x_{n_j} \rightharpoonup z \in H_1$ and $||y_n-x_n|| \to 0$, we conclude that $y_{n_j} \rightharpoonup z \in H_1$. By the demiclosedness principle of $S-I$ at zero and (3.14), we have that $z \in F(S)=C$. On the other hand, since A is a linear bounded operator and $x_{n_j} \rightharpoonup z \in H_1$, we have $Ax_{n_j} \rightharpoonup Az \in H_2$. Hence, by (3.15), we obtain

$$||TAx_{n_j}-Ax_{n_j}|| = ||TAx_{n_j}-Ax_{n_j}|| \to 0,$$

as $j \to \infty$. Since $T-I$ is demiclosed at zero, we get that $Az \in F(T)=Q$. Hence $z \in \Omega$.

Next, we prove that $\{x_n\}$ converges strongly to x^*. Setting $w_n=(1-\beta_n)y_n+\beta_n Sy_n$, $n \ge 1$, then from (3.1) we have that

$$x_{n+1} = w_n-(1-\beta_n)(1-\lambda_n)y_n.$$

It then follows that

$$\begin{aligned} x_{n+1} &= (1-(1-\beta_n)(1-\lambda_n))w_n - (1-\beta_n)(1-\lambda_n)(y_n - w_n) \\ &= (1-(1-\beta_n)(1-\lambda_n))w_n + (1-\beta_n)(1-\lambda_n)\beta_n(y_n - Sy_n). \end{aligned} \tag{3.16}$$

Also we have

$$\begin{aligned} ||w_n - x^*||^2 &= ||y_n - x^* - \beta_n(y_n - Sy_n)||^2 \\ &= ||y_n - x^*||^2 - 2\beta_n\langle y_n - Sy_n, y_n - x^*\rangle + \beta_n^2||y_n - Sy_n||^2 \\ &\le ||y_n - x^*||^2 - \beta_n[(1-k_1) - \beta_n]||y_n - Sy_n||^2 \\ &\le ||y_n - x^*||^2. \end{aligned} \tag{3.17}$$

Applying Lemma 2.2(ii) to (3.16), we have

$$\begin{aligned} &||y_{n+1} - x^*||^2 \le ||x_{n+1} - x^*||^2 \\ &\quad = ||(1-(1-\beta_n)(1-\lambda_n))(w_n - x^*) + (1-\beta_n)(1-\lambda_n)\beta_n(y_n - Sy_n) - (1-\beta_n)(1-\lambda_n)x^*||^2 \\ &\quad \le (1-(1-\beta_n)(1-\lambda_n))^2||w_n - x^*||^2 + 2(1-\beta_n)(1-\lambda_n)\langle\beta_n(y_n - Sy_n) - x^*, x_{n+1} - x^*\rangle \\ &\quad = (1-(1-\beta_n)(1-\lambda_n))^2||w_n - x^*||^2 + 2(1-\beta_n)(1-\lambda_n)\beta_n\langle y_n - Sy_n, x_{n+1} - x^*\rangle \\ &\qquad - 2(1-\beta_n)(1-\lambda_n)\langle x^*, x_{n+1} - x^*\rangle \\ &\quad \le (1-(1-\beta_n)(1-\lambda_n))^2||y_n - x^*||^2 + 2(1-\beta_n)(1-\lambda_n)\beta_n\langle y_n - Sy_n, x_{n+1} - x^*\rangle \\ &\qquad - 2(1-\beta_n)(1-\lambda_n)\langle x^*, x_{n+1} - x^*\rangle \\ &\quad \le (1-(1-\beta_n)(1-\lambda_n))||y_n - x^*||^2 + (1-\beta_n)(1-\lambda_n)[2\beta_n\langle y_n - Sy_n, x_{n+1} - x^*\rangle \\ &\quad - 2\langle x^*, x_{n+1} - x^*\rangle]. \end{aligned} \tag{3.18}$$

Clearly, $2\beta_n\langle y_n - Sy_n, x_{n+1} - x^*\rangle \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty}\left\{-2\langle x^*, x_{n+1} - x^*\rangle\right\} \le -2\limsup_{j\to\infty}\langle x^*, x_{n_j} - x^*\rangle = -2\langle x^*, z - x^*\rangle \le 0$ (here x^* is the minimum-norm solution of (1.6)). Now, using (3.18) and Lemma 2.3, we have $||y_n - x^*|| \to 0$. Thus $||x_n - x^*|| \to 0$ and $x_n \to x^*$ as $n \to \infty$.

Case 2 Assume that $\{||y_n - x^*||\}$ is not monotonically decreasing. Set $\Gamma_n = ||y_n - x^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \quad \Gamma_k \le \Gamma_{k+1}\}$$

for all $n \ge n_0$ (for some n_0 large enough). Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \ge 0, \quad \forall n \ge n_0.$$

From (3.13), it is easy to see that

$$||y_{\tau(n)} - Sy_{\tau(n)}||^2 \le \frac{(1-\lambda_{\tau(n)})M}{\beta_{\tau(n)}[(1-k_1) - \beta_{\tau(n)}]} \to 0,$$

as $n \to \infty$. Thus,

$$||y_{\tau(n)} - Sy_{\tau(n)}|| \to 0,$$

as $n \to \infty$. Furthermore, we can show that

$$\begin{aligned} &||(T-I)Ax_{\tau(n)}|| \to 0, \\ &||y_{\tau(n)} - x_{\tau(n)}|| = \gamma||A^*(T-I)Ax_{\tau(n)}|| \le \gamma||A^*||||(T-I)Ax_{\tau(n)}|| \to 0, \end{aligned}$$

and

$$||x_{\tau(n)+1} - x_{\tau(n)}|| \to 0,$$

as $n \to \infty$. Since $\{y_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{y_{\tau(n)}\}$, still denoted by $\{y_{\tau(n)}\}$, which converges weakly to $z \in H_1$. Observe that since $\lim_{n\to\infty} ||x_{\tau(n)} - y_{\tau(n)}|| = 0$, we also have $x_{\tau(n)} \rightharpoonup z$. Using the demiclosedness of $S - I$ at the origin and the fact that $||y_{\tau(n)} - Sy_{\tau(n)}|| \to 0$, we have that $z \in F(S) = C$. Similarly, we can show that $z \in F(T) = Q$. Hence $z \in \Omega$. We note that, for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq ||y_{\tau(n)+1} - x^*||^2 - ||y_{\tau(n)} - x^*||^2 \\ &\leq (1 - \beta_{\tau(n)})(1 - \lambda_{\tau(n)})[2\langle \beta_{\tau(n)}(y_{\tau(n)} - Sy_{\tau(n)}), x_{\tau(n)+1} - x^*\rangle \\ &\quad - 2\langle x^*, x_{\tau(n)+1} - x^*\rangle - ||y_{\tau(n)} - x^*||^2]. \end{aligned}$$

This implies that

$$||y_{\tau(n)} - x^*||^2 \leq 2\langle \beta_{\tau(n)}(y_{\tau(n)} - Sy_{\tau(n)}), x_{\tau(n)+1} - x^*\rangle - 2\langle x^*, x_{\tau(n)+1} - x^*\rangle. \quad (3.19)$$

Since $||y_{\tau(n)} - Sy_{\tau(n)}|| \to 0$ and $||x_{\tau(n)+1} - x_{\tau(n)}|| \to 0$, by (3.19), it follows that

$$\limsup_{n\to\infty} ||y_{\tau(n)} - x^*||^2 \leq -2\langle x^*, z - x^*\rangle \leq 0,$$

which also implies that

$$\lim_{n\to\infty} ||y_{\tau(n)} - x^*|| = 0.$$

Therefore

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain, for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

So $\lim_{n\to\infty} \Gamma_n = 0$ and $\{y_n\}$ converges strongly to x^*. Hence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Corollary 3.2 *Let H_1 and H_2 be two real Hilbert spaces, $A\colon H_1 \to H_2$ be a bounded linear operator and $A^*\colon H_2 \to H_1$ be the adjoint operator of A. Let $S\colon H_1 \to H_1$ be a quasi-nonexpansive mapping such that $S - I$ is demi-closed at 0 and $C := F(S) \neq \emptyset$. Let $T\colon H_2 \to H_2$ be a quasi-nonexpansive mapping such that $T - I$ is demi-closed at 0 and $Q := F(T) \neq \emptyset$. Suppose that SCFPP (1.6) has a nonempty solution set Ω. Let $\{\beta_n\}$ and $\{\lambda_n\}$ be two real sequences in $(0, 1)$ and $\gamma \in \left(0, \frac{1}{||A||^2}\right)$. Let $\{y_n\}$ and $\{x_n\}$ be generated by (3.1). Suppose the following conditions are satisfied:*

(a) $\lim_{n\to\infty} \lambda_n = 1$;
(b) $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$;
(c) $0 < \epsilon \leq \beta_n \leq b < 1$.

Then $\{x_n\}$ converges strongly to an element x^* of Ω, where x^* is the minimum-norm solution of (1.6).

Following the proof line in Theorems 3.1 and 2 of Tang et al. [25], we can easily prove the following theorem for multiple-set split feasibility problem (MSSFP) of demicontractive operators. The MSSFP is formulated as:

$$\text{find} \quad x \in \cap_{i=1}^{p} C_i \quad \text{such that } Ax \in \cap_{j=1}^{r} Q_j, \tag{3.20}$$

where $A\colon H_1 \to H_2$ is a bounded linear operator, C_i $(i = 1, 2, \dots, p)$ is a nonempty, closed and convex subset of a Hilbert space H_1 and Q_j $(j = 1, 2, \dots, r)$ is a nonempty, closed and convex subset of a Hilbert space H_2.

Theorem 3.3 *Let H_1 and H_2 be two real Hilbert spaces, $A\colon H_1 \to H_2$ be a bounded linear operator and $A^*\colon H_2 \to H_1$ be the adjoint operator of A. For each $i = 1, 2, \dots, p$ let $S_i\colon H_1 \to H_1$ be a ρ_i-demicontractive mapping such that $S_i - I$ is demi-closed at 0 and $C := \cap_{i=1}^{p} F(S_i) \neq \emptyset$. For each $j = 1, 2, \dots, r$ let $T_j : H_2 \to H_2$ be a μ_j-demicontractive mapping such that $T_j - I$ is demi-closed at 0 and $Q := \cap_{j=1}^{r} F(T_j) \neq \emptyset$. Let $k_1 := \max\{\rho_i : i = 1, 2, \dots, p\}$ and $k_2 := \max\{\mu_j : j = 1, 2, \dots, r\}$. Suppose that (MSSFP) (3.20) has a nonempty solution set Ω. Let $\{\beta_n\}$ and $\{\lambda_n\}$ be two real sequences in $(0, 1)$ and $\gamma \in \left(0, \frac{1-k_2}{||A||^2}\right)$. Let $\{y_n\}$ and $\{x_n\}$ be generated by $x_1 \in H_1$ and*

$$\begin{cases} y_n = x_n + \gamma A^*(T_{j(n)} - I)Ax_n \\ x_{n+1} = (1 - \beta_n)(\lambda_n y_n) + \beta_n S_{i(n)} y_n, \quad n \geq 1, \end{cases} \tag{3.21}$$

where $i(n) = n(\mathrm{mod}\ p) + 1$ and $j(n) = n(\mathrm{mod}\ r) + 1$. Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \lambda_n = 1$;
(b) $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$;
(c) $\beta_n \in \left[\epsilon, \frac{\lambda_n(1-k_1)}{1-(1-k_1)(1-\lambda_n)}\right)$, *for some* $\epsilon > 0$ *and* $\limsup_{n\to\infty} \beta_n < 1 - k_1$.

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element x^* of Ω, where x^* is the minimum-norm solution of (3.20).

4 An application

4.1 The split common null point problem

We now apply Theorem 3.1 to solve the split common null point problem (see, for example [6]) for set-valued mappings in Hilbert spaces. Let H_1 and H_2 be two real Hilbert spaces. Let $B_i\colon H_1 \to 2^{H_1}$ $(1 \leq i \leq p)$ and $F_j\colon H_2 \to 2^{H_2}$ $(1 \leq j \leq r)$ and let $A_j\colon H_1 \to H_2$ $(1 \leq j \leq r)$ be a bounded linear operator. This problem is formulated as follows: find a point $x^* \in H_1$ such that

$$0 \in \cap_{i=1}^{p} B_i(x^*) \tag{4.1}$$

and such that the point $y_j^* = A_j x^* \in H_2$ and solves

$$0 \in \cap_{j=1}^{r} F_j(y_j^*). \tag{4.2}$$

We denote by $SCNPP(p, r)$ the solution set of (4.1). Special case of $SCNPP(p, r)$ includes the split variational inequality problem (SVIP) in a real Hilbert space.

Let H_1 and H_2 be two real Hilbert spaces. Let $f\colon H_1 \to H_1$ and $g\colon H_2 \to H_2$. Let $A\colon H_1 \to H_2$ be a bounded linear operator. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. The SVIP (see, for example [10]) is formulated as follows: find a point $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C \tag{4.3}$$

and such that the point $y^* = Ax^* \in Q$ and solves

$$\langle g(y^*), y - y^* \rangle \geq 0 \quad \forall y \in Q. \tag{4.4}$$

We note that (4.3) is the classical variational Inequality problem (VIP) and also denote its solution set by $SOL(C, f)$. The SVIP is quite general and should enable split minimization between two spaces so that the image of a solution point of one minimization problem, under a given bounded linear operator, is a solution point of another minimization problem.

In this section, we prove strong convergence theorem for solving the split common null point problem (4.1)–(4.2) for the case when $p = r = 1$. That is, given two set-valued mappings $B_1 : H_1 \to 2^{H_1}$, and $F_1 : H_2 \to 2^{H_2}$ and a bounded linear operator $A : H_1 \to H_2$, we find a point $x^* \in H_1$ such that

$$0 \in B_1(x^*) \quad \text{and} \quad 0 \in F_1(A(x^*)). \tag{4.5}$$

We denote by Ω the solution set of (4.5).

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping M is said to be *maximal* if the graph $G(M)$ is not properly contained in the graph of any other monotone map, where $G(M) := \{(x, y) \in H \times H : y \in Mx\}$ for a multi-valued mapping M. It is also known that M is *maximal* if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$. The resolvent operator J_r associated with M and r is the mapping $J_r : H \to H$ defined by

$$J_r(u) = (I + rM)^{-1}(u), \quad u \in H, \quad r > 0. \tag{4.6}$$

It is known that the resolvent operator J_r is single-valued and nonexpansive (see, for example [3]) and that a solution of the problem: find $u \in H$ such that $0 \in M(u)$ is a fixed point of J_r, $\forall r > 0$ (see, for example [17]).

We now prove the following convergence theorem for the split common null point problem.

Theorem 4.1 *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given set-valued maximal monotone mappings $B_1 : H_1 \to 2^{H_1}$, and $F_1 : H_2 \to 2^{H_2}$, respectively. Assume that $\gamma \in \left(0, \frac{1}{||A||^2}\right)$. Assume that $\Omega \neq \emptyset$. Let $\{\beta_n\}$ and $\{\lambda_n\}$ be two real sequences in $(0, 1)$ satisfying:*

(a) $\lim_{n\to\infty} \lambda_n = 1$;
(b) $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$;
(b) $0 < \epsilon \leq \beta_n \leq b < 1$.

Let $r > 0$. Then $\{y_n\}$ and $\{x_n\}$ generated by $x_1 \in H_1$ and

$$\begin{cases} y_n = x_n + \gamma A^*(J_r^{F_1} - I)Ax_n \\ x_{n+1} = (1 - \beta_n)(\lambda_n y_n) + \beta_n J_r^{B_1} y_n, \quad n \geq 1 \end{cases} \tag{4.7}$$

converge strongly to a solution point x^* of Ω, where x^* is the minimum-norm solution of (4.5).

Proof Let $S = J_r^{B_1}$ and $T = J_r^{F_1}$. Then, we have S and T are nonexpansive and hence 0-demicontractive. We obtain the desired conclusion by following the proof line in Theorem 3.1. □

Remark 4.2 If $A = I$ in the problem (1.6), then (1.6) reduces to the CFP for demicontractive (quasi-nonexpansive) operators which the results in the papers [1,7,8,16,19,30] can be applied to solve. Furthermore, if $S = P_C$ and $T = P_Q$, then the problem (1.6) reduces to the SFP (1.1) which the results in [27] can applied to solve. Based on this remark, our results complement those of [1,7,8,16,19,27,30]. Moreover, our results can be used to solve the fixed point problem for demicontractive (quasi-nonexpansive) operators considered in [1,7,8,16,19,30] and also the SFP considered in [27].

Remark 4.3 In conclusion, we make the following comments which highlight our contributions in this paper.

(1) Theorem 4.1 complements Theorems 4.3 and 4.4 of [6]. In other words, Theorem 4.1 is another new strong convergence result for the split common null point problem in real Hilbert spaces. Furthermore, since the $SCNPP(p, r)$ generalizes the SVIP, then Theorem 4.1 includes all the applications to which SVIP applies (see Section 7 of [10]). In particular, it includes the SFP and the CFP.
(2) In this paper, we obtain strong convergence results for the split common fixed problems for demicontractive mappings without any extra conditions (such as demi-compactness or semi-compactness) on the operators or on the space (see, for example [12]).
(3) Our results extend the class of operators for the SCFPP considered in those of Moudafi [21,23], and Zhao and He [38] to a wider class of operators.
(4) On page 272 of [35], Yao and Cho made the following remark:"It is a very interesting topic of constructing some algorithms such that the strong convergence of proposed algorithms are guaranteed. For this purpose, in this article we present a modified Krasnoselskii–Mann method $x_{n+1} = \alpha_n(\lambda_n y_n) + (1 - \alpha_n)Ty_n$ for nonexpansive mappings in Hilbert spaces and show that the proposed method $x_{n+1} = \alpha_n(\lambda_n y_n) + (1 - \alpha_n)Ty_n$ has strong convergence. However, we note that in order to obtain the main result of Theorem 3, we have imposed some additional conditions $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ and $\sum_{n=1}^{\infty}(1 - \lambda_n)\alpha_n = \infty$. Hence this brings us a nature problem: could we weaken or drop these additional assumptions?" In our results here, the conditions $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ imposed in the results of Yao and Cho [35] are dispensed with even for a class of demicontractive mappings which are larger than the class of nonexpansive mappings considered in [35]. Thus, our results improve and extend those of Yao and Cho [35].
(5) In the results of Moudafi [22], Tang et al. [25], and Wang and Cui [26], weak convergence results were given concerning the SCFPP for demicontractive mappings while in this paper, we give *strong convergence* results for the SCFPP for demicontractive mappings.
(6) Since demicontractive operators include directed operators (an operator $T: H \to H$ is called directed if $\langle z - Tx, x - Tx\rangle \le 0, \forall z \in F(T), x \in H$), then all the results in this paper hold if S and T are directed operators. Please see, for example, Cui et al. [14] and Bauschke and Combettes [2] for more details.

Remark 4.4 The prototype for the iteration parameters are as follows:

$$\lambda_n = 1 - \frac{1}{\sqrt{n+1}}, \beta_n = \epsilon + \frac{1}{\sqrt{n+1}}[(1 - k_1) - \epsilon], \quad \forall n \ge 1.$$

It is easy to check that these choices satisfy all the conditions of Theorem 3.1.

5 Numerical example

In this section, we give a numerical example to demonstrate the convergence of our algorithm.

Let $H_1 = (\mathbb{R}^3, ||\cdot||_2) = H_2$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $S\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}$. It is clear that both T and S are 0-demicontractive mappings. Choose $\lambda_n = 1 - \frac{1}{\sqrt{n+1}}$ and $\beta_n = \frac{1}{2}\left(1 + \frac{1}{\sqrt{n+1}}\right)$ for all $n \geq 1$. The stopping criterion for our testing method is taken as: $||x_{n+1} - x_n||_2 < 10^{-6}$ where $x_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$. Let us assume that $A = \begin{pmatrix} 5 & -5 & -7 \\ -4 & 2 & -4 \\ -7 & -4 & 5 \end{pmatrix}$, then the iterative scheme (3.1) becomes

Table 1 Table for Case 1

n	a_n	b_n	c_n	$\|\|x_{n+1} - x_n\|\|_2$
2	0.33500	−0.23876	0.80418	4.5988115
4	0.04455	0.03399	0.06614	0.1472707
6	0.00758	0.01515	0.01864	0.0196816
8	0.00186	0.00490	0.00675	0.0060833
10	0.00059	0.00162	0.00242	0.0020533
⋮	⋮	⋮	⋮	⋮
26	0.00000	0.00000	0.00000	0.0000008

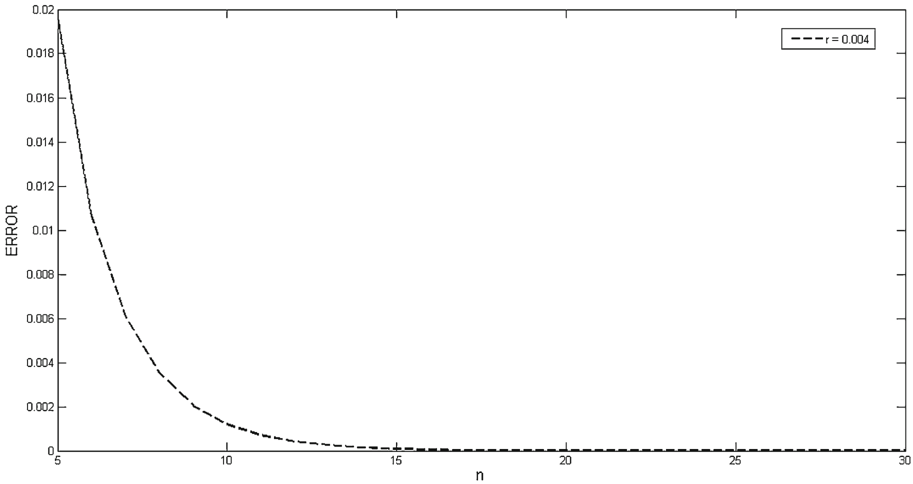

Fig. 1 Figure for Case 1

Table 2 Table for Case 2

n	a_n	b_n	c_n	$\|\|x_{n+1} - x_n\|\|_2$
2	0.27125	−0.32486	0.91888	4.4713265
4	0.03607	0.00803	0.07237	0.1752988
6	0.00615	0.00986	0.01610	0.0168615
8	0.00125	0.00357	0.00601	0.0046967
10	0.00031	0.00113	0.00229	0.0017221
⋮	⋮	⋮	⋮	⋮
25	0.00000	0.00000	0.00000	0.0000007

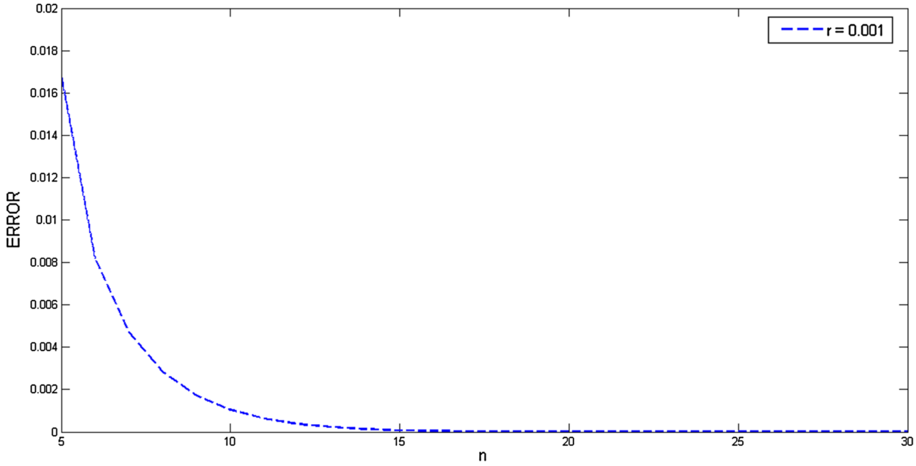

Fig. 2 Figure for Case 2

$$\begin{cases} y_n = x_n + \gamma A^T(T - I)Ax_n \\ x_{n+1} = \left(\frac{1}{2} - \frac{1}{\sqrt{n+1}}\right)\left(1 - \frac{1}{\sqrt{n+1}}\right) y_n + \frac{1}{2}\left(1 + \frac{1}{\sqrt{n+1}}\right) Sy_n, \quad n \geq 1. \end{cases} \tag{5.1}$$

In this example, we start with the initial point $x_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ and consider various choices of γ to see how our iterative scheme depends on the choice of γ. In the graphs below, we plot the number of iterations against $||x_{n+1} - x_n||_2$.

Case 1 Take $\gamma = 0.004$. Then using (5.1), we have the Table 1 and Fig. 1 below.

Case 2 Take $\gamma = 0.001$. Then using (5.1), we have the Table 2 and Fig. 2 below.

Case 3 Take $\gamma = 0.0001$. Then using (5.1), we have the Table 3 and Fig. 3 below.

Remark 5.1 We see that the smaller the choice of $\gamma > 0$ chosen, the less the number of iterations required.

Table 3 Table for Case 3

n	a_n	b_n	c_n	$\|\|x_{n+1} - x_n\|\|_2$
2	0.25213	−0.35068	0.95328	4.4334495
4	0.03178	−0.00232	0.07504	0.1859462
6	0.00532	0.00706	0.01449	0.0165695
8	0.00101	0.00279	0.00516	0.0039471
10	0.00021	0.00086	0.00198	0.0014463
⋮	⋮	⋮	⋮	⋮
24	0.00000	0.00000	0.00000	0.0000006

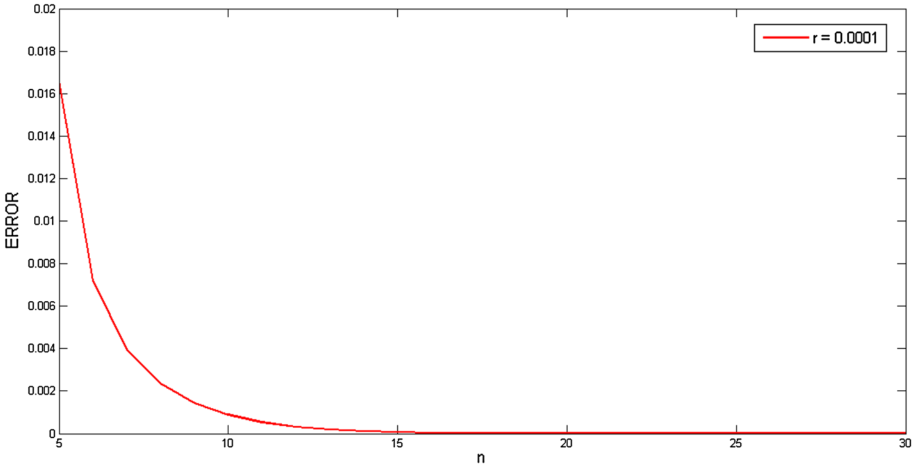

Fig. 3 Figure for Case 3

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