



รายงานวิจัยฉบับสมบูรณ์

เครื่องหาปริพันธ์สำหรับฟังก์ชันกำหนดค่าเวกเตอร์และฟังก์ชัน
สาคิสสัณฐานของฟังก์ชันกำหนดค่าซีสตาร์

**Integrators for vector-valued regulated functions and
homomorphisms of C^* -valued regulated functions**

โดย จิตตารีย์ วุฒิจิริจิตติกาล

มิถุนายน 2560

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ผู้วิจัย
จิตรารีย์ วุฒิจริจิติกาล

สังกัด
คณะวิทยาศาสตร์ มหาวิทยาลัยอุบลราชธานี

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและมหาวิทยาลัยอุบลราชธานี

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Titarii Wootijirattikal

Abstract

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Investigator : Assistant Professor Titarii Wootijirattikal

Department of Mathematics Statistics and Computer, Faculty of Science,
Ubon Ratchathani University, Warinchamrap, Ubon Ratchathani 34190

E-mail Address : titarii.w@ubu.ac.th

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A function from a closed interval $[a, b]$, to a Banach space X , is said to be *regulated* if all one-side limits exist. A function α from $[a, b]$ to the space of all bounded linear transformations from X to a Banach space Y is an *integrator* for the regulated functions if for each regulated function f , Riemann-Stieltjes sums of f with respect to α converge to a vector in Y .

In the first year of the project, we use the Uniform boundedness principle to give a complete description of the class of all operator-valued integrators for vector-valued regulated functions. We prove that a subclass of such integrators is a Banach space under suitable norm. We give compactness criteria for integrator induced operators. We also show that each bounded linear map from a subspace of vector-valued regulated functions is represented by an operator-valued integrator.

In the second year, we give a complete description of the class of all integrators that induced operators that are homomorphisms, when X and Y are Banach algebras. The main result of L. Fernandes and R. Arbach [4] showed a special subclass of our characterization.

Keywords : Regulated functions, semivariation, Banach space, C^* -algebra

บทคัดย่อ

รหัสโครงการ : TRG5880051

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ชื่อนักวิจัย : ผู้ช่วยศาสตราจารย์ จิตารีย์ วุฒิจริฎติกาล
ภาควิชาคณิตศาสตร์ สถิติ และคอมพิวเตอร์ คณะวิทยาศาสตร์
มหาวิทยาลัยอุบลราชธานี อ.วารินชำราบ จ.อุบลราชธานี 34190

E-mail Address : titarii.w@ubu.ac.th

Project Period : 2 ปี

เราจะเรียกฟังก์ชันที่ส่งจากช่วงปิด $[a,b]$ ไปยังปริภูมิบานาค X ว่า ฟังก์ชันเรกิวเลท ถ้าทุกลิมิตด้านเดียวหาค่าได้ และเรียกฟังก์ชัน α ที่ส่งจาก $[a,b]$ ไปยังปริภูมิของการแปลงเชิงเส้นแบบมีขอบเขตทั้งหมด จากปริภูมิบานาค X ไปยัง ปริภูมิบานาค Y ว่า เครื่องหาปริพันธ์สำหรับฟังก์ชันเรกิวเลท ถ้าแต่ละฟังก์ชันเรกิวเลท f , ผลบวก Riemann-Stieltjes ของ f ที่สัมพันธ์กับ α ลู่เข้าสู่เวกเตอร์ใน Y

ในปีแรกของโครงการ เราใช้หลักการมีขอบเขตเอกรูปในการพรรณนาแบบบริบูรณ์ของคลาสของเครื่องหาปริพันธ์ค่าตัวดำเนินการทั้งหมดสำหรับฟังก์ชันเรกิวเลทค่าเวกเตอร์ เราพิสูจน์ว่า คลาสย่อยของเครื่องหาปริพันธ์เป็นปริภูมิบานาคภายใต้บรรทัดที่เหมาะสม เราให้เกณฑ์ความกระชับสำหรับเครื่องหาปริพันธ์ที่เหนี่ยวนำตัวดำเนินการ รวมถึงเราแสดงว่าแต่ละการส่งเชิงเส้นแบบมีขอบเขตจากปริภูมิย่อยของฟังก์ชันเรกิวเลทค่าเวกเตอร์ สามารถแทนได้ด้วยเครื่องหาปริพันธ์ค่าตัวดำเนินการ

ในปีที่สองของโครงการ ให้ X และ Y เป็นพีชคณิตบานาค เราพรรณนาแบบบริบูรณ์ของคลาสของเครื่องหาปริพันธ์ทั้งหมดที่เหนี่ยวนำตัวดำเนินการที่เป็นฟังก์ชันสาคิสต์ฐาน เราจะเห็นว่าผลลัพธ์หลักของ L. Fernandes and R. Arbach [4] เป็นเพียงคลาย่อยเฉพาะของผลลัพธ์หลักของเรา

Keywords : ฟังก์ชันเรกิวเลท , กิ่งการแปรผัน, ปริภูมิบานาค, พีชคณิตชีสตา

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Appendix No. 1

Operator Integrators for Regulated Vector Functions and Compactness

Appendix No. 2

Multiplicative Integrators of Algebra-Valued Regulated Functions

I Introduction

Many real-world problems are mathematically formulated as integral equations, such as

$$f(x) = \nu(x) + \lambda \int_{\Omega} K(x, t) \nu(t) dt.$$

Here it involves the identity as integrator function (which corresponds to the Lebesgue measure) and the unknown function to be found. Usually to solve an integral equation, we have to determine the admissible class of integrator functions and the domain of the integral operator. In many practical situations the class of integrator functions need to be as large as possible. And the functions in the domain of the integral operators also need to be widen from the continuous ones, as the real-world problems rarely arise continuously.

For real or complex-valued functions, a natural class of integrator functions is the class of functions of bounded variation. And the natural class of functions in the domains of such integration operators induced by functions of bounded variation is the space of regulated functions.

From statistical applications, vector-valued functions emerge naturally in many practical situations. This leads many researchers to consider vector-valued functions and operator-valued integrators. The notion of bounded variation for scalar functions has a most natural and obvious extension to operator-valued functions, namely replacing the absolute value, used in the calculation of variation, by operator norm. But in many practical situations, this is too restrictive. A generalized notion of bounded semivariation, an analogue of the calculation of the operator norm, emerges. These two notions (bounded variation and bounded semivariation) are equivalent in the scalar case. The space of vector-valued regulated functions has been used as the domain of integral operators induced by operator-valued functions of bounded semivariation. A lot of work has been done along this line. Our question is whether we can extend the class of integrator functions beyond the class of operator-valued functions of bounded semivariation. Answer to this question does not seem to be known even the very special case of scalar-valued functions. After answer that question, we will investigate homomorphisms of C^* -algebras of regulated functions taking values in a C^* -algebras.

A function f on a closed interval $[a, b]$, taking values in a Banach space X , is said to be *regulated* if all one-side limits exist at every point of the interval [3]. The space of regulated functions is much wider than the space of continuous functions. Given a function α on $[a, b]$ taking values in the space $B(X, Y)$ of bounded linear operators from X to a Banach space Y , an interesting and natural question is whether there is a vector in Y to which the internal (sampling points in interior of subintervals) Riemann-Stieltjes sums of f with respect to α converge. Those functions α with this property are called *integrators*. We give a complete characterization of all such operator-valued integrators. We show that a subclass of such integrators form a Banach space under suitable norm. We also derive criteria for the compactness of integrator induced operators. Finally, when the Banach spaces X and Y are replaced by Banach algebras, our question is when is the operator induced by the operator-valued integrator is multiplicative. The main result of L. Fernandes and R. Arbach [4] give a class of such examples. We will give a complete description of all such integrators.

II Objectives

The objectives for this research are as follows.

1. To determine the class of operator-valued integrators for the space of regulated functions taking values in a Banach space.
2. Use the results obtained to investigate homomorphisms of C^* -algebras of regulated functions taking values in a C^* -algebra.

III Methodologies

1. Study books and papers on integration of vector-valued functions with respect to operator-valued integrators, particularly [3, 6, 2] for the first phase and [7, 6, 4] for the second phase.
2. Construct examples of regulated functions and operator-valued functions that can help us gain deeper insight into these the concepts and questions.
3. Use known results and methods used therein and new examples from (2) to guess possible answers to devise more precise conjectures.
4. Use our modifications of techniques learned from studying the books and papers to come up with new techniques to prove the conjectures.
5. If it is unsuccessful, construct more examples, re-study key ideas in books and papers carefully, and re-examine the whole process to modify the conjectures.
6. Repeat these steps until valid proofs are obtained to turn the conjectures into theorems. Our techniques and conjectures will be modified repeatedly as we go along.
7. Write up results and follow up with revisions and revising repeatedly until they are ready for publication.
8. Communicate with mentor and co-mentor.

IV Results

Fix a Banach space X , and a closed interval $[a, b]$. A function $f : [a, b] \rightarrow X$ is said to be *regulated* if

$$\begin{aligned} f(c^+) &:= \lim_{t \rightarrow c^+} f(t), \quad f(c^-) := \lim_{t \rightarrow c^-} f(t) \quad \text{exist for all } c \in (a, b), \quad \text{and} \\ f(a^+) &:= \lim_{t \rightarrow a^+} f(t), \quad f(b^-) := \lim_{t \rightarrow b^-} f(t) \quad \text{exist.} \end{aligned}$$

Denote by $\text{Reg}([a, b], X) = \text{Reg}(X)$ the space of regulated functions on $[a, b]$ (taking values in X).

A *partition* P of the interval $[a, b]$ is given by a finite number of *division points*, t_j , in $[a, b]$:

$$P(a = t_0 < t_1 < \dots < t_{n(P)} = b), \quad n(P) \in \mathbb{N}.$$

The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b] =: \mathcal{P}$.

A function $g : [a, b] \rightarrow X$ is a *step function* if there are a partition

$$P(a = t_0 < t_1 < \dots < t_{n(P)} = b)$$

and vectors $x_j \in X$, $1 \leq j \leq n(P)$ such that $g(t) = x_j$ for all $t \in (t_{j-1}, t_j)$, $1 \leq j \leq n(P)$.

Since $n(P)$ is determined by g , in this case we denote $n(P)$ by $n(g)$.

Theorem 1. [J.Dieudonne] : A function $f : [a, b] \rightarrow X$ is regulated iff there exists a sequence $\{h_n\}$ of step functions such that $h_n \rightarrow f$ uniformly on $[a, b]$. Furthermore, if $f \in \text{Reg}(X)$, then

$$\|f\| := \sup_{t \in [a, b]} \|f(t)\| < \infty,$$

and $(\text{Reg}(X), \|\cdot\|)$ is a Banach space.

4.1 Operator Integrators for Regulated Vector Functions and Compactness

A function f on a closed interval $[a, b]$, taking values in a Banach space X , is said to be *regulated* if all one-side limits exist at every point of the interval [3]. The space of regulated functions is much wider than the space of continuous functions. Given a function α on $[a, b]$ taking values in the space $B(X, Y)$ of bounded linear operators from X to a Banach space Y , an interesting and natural question is whether there is a vector in Y to which the internal (sampling points in interior of subintervals) Riemann-Stieltjes sums of f with respect to α converge. Those functions α with this property are called *integrators*. We give a complete characterization of all such operator-valued integrators. We show that a subclass of such integrators form a Banach space under suitable norm. We also derive criteria for the compactness of integrator induced operators.

4.1.1 Integrators as bounded linear transformations

Let X, Y be Banach spaces. Denote $B(X, Y)$ by the space of bounded linear transformations from X to Y . An operator-valued function $\alpha : [a, b] \rightarrow B(X, Y)$ is said to be an *integrator* for the regulated functions if for each $f \in \text{Reg}(X)$ there is $y \in Y$ satisfies the following condition:

For every $\varepsilon > 0$ there is a partition $P_\varepsilon (a = s_0 < s_1 < \dots < s_{n(P_\varepsilon)} = b)$ such that for all partitions $P (a = t_0 < t_1 < \dots < t_{n(P)} = b)$ refining P_ε that is

$$\{s_k : 1 \leq k \leq n(P_\varepsilon)\} \subseteq \{t_j : 1 \leq j \leq n(P)\}$$

$$\text{and for all } t_j^* \in (t_{j-1}, t_j), 1 \leq j \leq n(P), \left\| y - \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})] (f(t_j^*)) \right\| < \varepsilon.$$

If α is an integrator for regulated functions, the vector $y \in Y$ to which integrates f is unique and is called to be *interior integral* (or *Dushnik integral*) of f and is denoted by

$$\int_a^b [d\alpha(t)] (f(t)) = y.$$

We use the Uniform Boundedness Principle to show that each integrator induces a bounded linear map from $\text{Reg}(X)$ to Y .

Theorem 2. Let $\alpha : [a, b] \rightarrow B(X, Y)$ be an integrator for $\text{Reg}(X)$. Then the map

$$\hat{\alpha} : f \mapsto \int_a^b [d\alpha(t)](f(t)), \quad f \in \text{Reg}(X)$$

is a bounded linear transformation from $\text{Reg}(X)$ to Y .

4.1.2 Equivalent formulation of integrators

We will give an equivalent formulation for integrators.

Lemma 3. Let $g : [a, b] \rightarrow X$ be a step function and let $\alpha : [a, b] \rightarrow B(X, Y)$ be an integrator for $\text{Reg}(X)$. Then there exist a partition $P_0 (a = u_0 < u_1 < \dots < u_{n(P_0)} = b)$ and $x_k \in [X]_1$, $1 \leq k \leq n(P)$, such that

$$\int_a^b [d\alpha(t)](g(t)) = \sum_{k=1}^{n(P_0)} [\alpha(u_k) - \alpha(u_{k-1})]x_k = \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](g(t_j^*))$$

for all partitions $P (a = t_0 < t_1 < \dots < t_{n(P)} = b)$ satisfying

$\{u_k : 1 \leq k \leq n(P_0)\} \subseteq \{t_j : 1 \leq j \leq n(P)\}$, and for all choices of $t_j^* \in (t_{j-1}, t_j)$, $1 \leq j \leq n(P)$.

Theorem 4. Let $\alpha : [a, b] \rightarrow B(X, Y)$ be an integrator for $\text{Reg}(X)$. Then for all partition $P (a = t_0 < t_1 < \dots < t_{n(P)} = b)$ and for all choices of $x_j \in [X]_1$, $1 \leq j \leq n(P)$,

$$\left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})]x_j \right\| \leq \|\hat{\alpha}\|.$$

Theorem 4 leads us to the following Definitions.

A function $\alpha : [a, b] \rightarrow B(X, Y)$ is said to be of *bounded semivariation* if there exists an $M > 0$ such that for all partitions $P (a = t_0 < t_1 < \dots < t_{n(P)} = b) \in \mathcal{P}[a, b]$, and for

all $x_k \in [X]_1$ (the closed unit ball of X), $1 \leq k \leq n(P)$, $\left\| \sum_{k=1}^{n(P)} [\alpha(t_k) - \alpha(t_{k-1})]x_k \right\| \leq M$.

The *semivariation* of a function $\alpha : [a, b] \rightarrow B(X, Y)$ of bounded semivariation is given by

$$V_s(\alpha) := \sup \left\{ \left\| \sum_{k=1}^{n(P)} [\alpha(t_k) - \alpha(t_{k-1})] x_k \right\| : P(a = t_0 < t_1 < \dots < t_{n(P)} = b) \in \mathcal{P}; \right. \\ \left. x_k \in [X]_1, 1 \leq k \leq n(P) \right\}.$$

From Theorem 4. , we get that each integrator β is of bounded semivariation with $V_s(\beta) \leq \|\hat{\beta}\|$. The backward of Theorem 4 need the following Lemmas.

Lemma 5. Let $\alpha : [a, b] \rightarrow B(X, Y)$ be of bounded semivariation and $t \in [a, b]$. Then

$$\|\alpha(t) - \alpha(a)\| \leq V_s(\alpha).$$

Lemma 6. Let $g, h : [a, b] \rightarrow X$ be step functions given by

$$a = u_0 < u_1 < \dots < u_{n(g)} = b, g(t) = x_j \in X, t \in (u_{j-1}, u_j), 1 \leq j \leq n(g), \text{ and} \\ a = v_0 < v_1 < \dots < v_{n(h)} = b, h(t) = w_k \in X, t \in (v_{k-1}, v_k), 1 \leq k \leq n(h).$$

Let $\alpha : [a, b] \rightarrow B(X, Y)$ be of bounded semivariation; and let

$$y_g = \sum_{j=1}^{n(g)} [\alpha(u_j) - \alpha(u_{j-1})] x_j \text{ and } y_h = \sum_{k=1}^{n(h)} [\alpha(v_k) - \alpha(v_{k-1})] w_k.$$

Then $\|y_g - y_h\| \leq \|g - h\| V_s(\alpha)$.

Theorem 7. Let $\alpha : [a, b] \rightarrow B(X, Y)$ be a function of bounded semivariation. Then α is an integrator for $\text{Reg}(X)$.

Combining Theorem 7 and 4, we get the following Corollary.

Corollary 8.

- (1) A function $\alpha : [a, b] \rightarrow B(X, Y)$ is an integrator for $\text{Reg}(X)$ if and only if α is of bounded semivariation.
- (2) For each integrator α for $\text{Reg}(X)$, $V_s(\alpha) \leq \|\hat{\alpha}\|$.

The following Theorem shows that the class of integrators is a Banach space under the suitable norm.

Theorem 9. The space $\mathcal{B}_{sv}^0([a, b], B(X, Y))$ of all function $\alpha : [a, b] \rightarrow B(X, Y)$ of bounded semivariation that satisfies $\alpha(a) = 0$, is a Banach space with pointwise addition and scalar multiplication and norm

$$\|\alpha\|_{sv} = \sup \left\{ \left\| \sum_{j=1}^n [\alpha(t_j) - \alpha(t_{j-1})] x_j \right\| : n \in \mathbb{N}, a = t_0 < t_1 < \dots < t_n = b, \right. \\ \left. x_j \in X, \|x_j\| \leq 1, 1 \leq j \leq n \right\}.$$

4.1.3 Integrators as compact operators

For a given $\alpha : [a, b] \rightarrow B(X, Y)$ define the set $[\alpha]_1$ as follows.

$$[\alpha]_1 := \left\{ \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})] x_j : P(a = t_0 < t_1 < \dots < t_{n(P)} = b) \in \mathcal{P}, x_j \in [X]_1, 1 \leq j \leq n(P) \right\}.$$

Note that α is an integrator iff $[\alpha]_1$ is bounded in Y .

Theorem 10. Let $\alpha : [a, b] \rightarrow B(X, Y)$ be an integrator for $\text{Reg}(X)$ that vanishes at a (i.e., $\alpha(a) = 0$). Then the following conditions on α from $\text{Reg}(X)$ to Y are equivalent.

1. $\hat{\alpha}$ is compact.
2. There is a compact operator K from a Banach space Z to Y such that $[\alpha]_1 \subseteq K([Z]_1)$.
3. There is a one-to-one compact operator K from a Banach space Z to Y such that $K^{-1}(\alpha(t))$ is bounded for all t and $K^{-1} \circ \alpha$ is an integrator.
4. The set $[\alpha]_1$ is a totally bounded subset of Y .

4.1.4 Operators as integrators

Let $\text{Reg}_r(X)$ be the set of all right continuous functions in $\text{Reg}(X)$ that are also left continuous at b ; i.e., $f \in \text{Reg}_l(X)$ iff $f(t) = f(t^+)$ for all $t \in [a, b)$ and $f(b) = 0$.

The following Theorem shows the Banach space $B(\text{Reg}_r(X), Y)$ and $\mathcal{B}_{sv}^0([a, b], B(X, Y))$ are isometrically isomorphic.

Theorem 11. Let $T \in B(\text{Reg}_r(X), Y)$. Then there exists a unique

$$\alpha_T \in \mathcal{B}_{sv}^0([a, b], B(X, Y)) \text{ such that } T(f) = \int_a^b [d\alpha(t)](f(t)) \quad \forall f \in \text{Reg}_r(X).$$

Moreover, $\hat{\alpha}_T = T$ and $\|T\| = \|\hat{\alpha}_T\| = V_s(\alpha_T)$.

4.2 Multiplicative Integrators of Algebra-valued Regulated Functions

A function f on a closed interval $[a, b]$, taking values in a Banach space X , is said to be *regulated* if all one-side limits exist. The class of regulated functions is much wider than the class of continuous functions. A function α on $[a, b]$ taking values in the space $B(X, Y)$ of bounded linear operators from X to a Banach space Y is an integrator if, for each regulated function f , there is a vector in Y to which Riemann-Stieltjes sums of f with sampling points from the interiors of subintervals, with respect to α converge. An operator valued function α is an *integrator* if and only if α is of bounded semivariation. When the Banach spaces X and Y are replaced by Banach algebras $\mathcal{A}$ and $\mathcal{B}$, a natural question is when is the operator induced by the operator-valued integrator is multiplicative. In 2012, L. Fernandes and R. Arbach proved that each function in the form

$$\alpha_c^T = \chi_{(c,b]} T = \begin{cases} 0 & \text{for } t \in [a, c] \\ T & \text{for } t \in (c, b] \end{cases}$$

with T a Banach algebra homomorphism from $\mathcal{A}$ to $\mathcal{B}$, induces a Banach algebra homomorphism from $\text{Reg}(\mathcal{A})$ to $\mathcal{B}$. Their result give a class of such examples. Our work will give a complete description of all such integrators.

4.2.1 Integrators as multiplicative maps

Let $\mathcal{A}$, $\mathcal{B}$ be Banach algebras with $1_{\mathcal{A}}$, $1_{\mathcal{B}}$ respectively (over $\mathbb{C}$). With the same process in the previous result, we can show that $\text{Reg}(\mathcal{A})$ is also a Banach algebra with identity 1_R . the constant function $1_{\text{Reg}(\mathcal{A})}(t) = 1_{\mathcal{A}}$, for all $t \in [a, b]$. Our question is : for which $\alpha \in \mathcal{B}_{sv}([a, b], B(\mathcal{A}, \mathcal{B}))$, is $\hat{\alpha}$ a Banach algebra homomorphism? [Inparticular: when is $\hat{\alpha}(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, and $\hat{\alpha}(fg) = \hat{\alpha}(f)\hat{\alpha}(g)$, $\forall f, g \in \text{Reg}(\mathcal{A})$?]

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with T a Banach algebra homomorphism from $\mathcal{A}$ to $\mathcal{B}$, induces a Banach algebra homomorphism from $\text{Reg}(\mathcal{A})$ to $\mathcal{B}$.

The collection of all such integrators have the following full descriptions.

Theorem 12. Let $\mathcal{A}, \mathcal{B}$ and α be as above. Suppose $\hat{\alpha} \neq 0$. Then $\hat{\alpha}$ is a homomorphism from $\text{Reg}(\mathcal{A})$ to $\mathcal{B}$ if and only if there exist $c \in [a, b]$, $S, T \in B(\mathcal{A}, \mathcal{B})$ such that T is a homomorphism and

$$\alpha(t) = \begin{cases} S & \text{for } a \leq t \leq c \\ S + T & \text{for } c < t \leq b \end{cases}$$

or

$$\alpha(t) = \begin{cases} S & \text{for } a \leq t < c \\ S + T & \text{for } c \leq t \leq b \end{cases}$$

(That is : there are $c \in [a, b]$, a homomorphism T and a bounded operator S from $\mathcal{A}$ to $\mathcal{B}$ such that $\alpha = S + \chi_{(c, b]}T$ or $\alpha = S + \chi_{[c, b]}T$.)

Now, suppose $\mathcal{A}, \mathcal{B}$ are C^* -algebras with identities $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$. Then $\text{Reg}(\mathcal{A})$ is a C^* -algebra with identity $1_{\text{Reg}(\mathcal{A})}(t) = 1_{\mathcal{A}}$, for all $t \in [a, b]$. The involution on $\text{Reg}(\mathcal{A})$ is defined pointwise: for each $f \in \text{Reg}(\mathcal{A})$, and each $t \in [a, b]$, $f^*(t) = (f(t))^*$.

Theorem 13. Let $\alpha : [a, b] \rightarrow B(\mathcal{A}, \mathcal{B})$ be an integrator for $\text{Reg}(\mathcal{A})$. Then $\hat{\alpha}$ is a $*$ -homomorphism from $\text{Reg}(\mathcal{A})$ to $\mathcal{B}$ if and only if there exist $c \in [a, b]$, $S, T \in B(\mathcal{A}, \mathcal{B})$ such that T is a $*$ -homomorphism and

$$\alpha(t) = \begin{cases} S & \text{for } a \leq t \leq c \\ S + T & \text{for } c < t \leq b \end{cases}$$

or

$$\alpha(t) = \begin{cases} S & \text{for } a \leq t < c \\ S + T & \text{for } c \leq t \leq b \end{cases}$$

(That is : there are $c \in [a, b]$, a $*$ -homomorphism T and a bounded operator S from $\mathcal{A}$ to $\mathcal{B}$ such that $\alpha = S + \chi_{(c, b]}T$ or $\alpha = S + \chi_{[c, b]}T$.)

V Conclusions and Suggestions

As stated in the objective no.1, we would like to determine the class of operator-valued integrators for the space of regulated functions taking values in a Banach space. We found that, we cannot extend the class of integrator functions apart from the class of operator-valued regulated functions of bounded semivariation. In the objective no.2, we plan to use the results obtained to investigate homomorphisms of C^* -algebras of regulated functions taking values in a C^* -algebra. After studied and worked on the homomorphism inducing integrators, we found that the main result of L. Fernandes and R. Arbach [4], is very special case. We have worked a complete description of all such integrators.

Since the class of Riesz operators is much wider than the class compact operators, but shares many nice properties of the compacts, it will be our follow up project along this line to investigate whether there are analogous criteria for integrator induced operators to be Riesz operators.

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VII Output

ผลงานที่คาดว่าจะตีพิมพ์ในอนาคต

1. Wootijirattikal, T., Ong, S.-C., Lenbury, Y., Operator Integrators for Regulated Vector Functions and Compactness. (Submitted) (See Appendix #1)
2. Wootijirattikal, T., Ong, S.-C., Multiplicative Integrators of Algebra-Valued Regulated Functions. (Appendix #2)

การนำผลงานวิจัยไปใช้ประโยชน์

มีเครือข่ายความร่วมมือการทำวิจัยร่วมกับนักวิจัยอาวุโส นักวิจัยต่างสถาบัน นักวิจัยต่างชาติ

การเสนอผลงานในที่ประชุม

นำเสนอผลงานวิจัยแบบโปสเตอร์ ชื่อเรื่อง “Operator-valued Integrators for Vector-valued Regulated Functions” ในงานประชุมประจำปี ในโครงการ การประชุม “นักวิจัยรุ่นใหม่...พบ...เมธีวิจัยอาวุโส สกว.” ครั้งที่ 16 ระหว่างวันที่ 11 – 13 มกราคม 2560 ณ โรงแรมเดอะรีเจนท์ ชะอำ บีช รีสอร์ท จังหวัดเพชรบุรี

VIII Appendix

OPERATOR INTEGRATORS FOR REGULATED VECTOR FUNCTIONS AND COMPACTNESS

TITARII WOOTIJIRATTIKAL^{*}, SING-CHEONG ONG AND YONGWIMON LENBURY

Submitted to Operators and Matrices

1. Introduction

A function f on a closed interval $[a, b]$, taking values in a Banach space X , is said to be *regulated* if all one-sided limits exist at every point of the interval [3, §7.6, p.139]. The space of regulated functions, which has been extensively studied [10, 5, 9, 8, 7, 2, 6, 1, 4], is much wider than the space of continuous functions. Given a function α on $[a, b]$ taking values in the space $\mathfrak{B}(X, Y)$ of bounded linear operators from X to a Banach space Y , an interesting and natural question is whether there is a vector in Y to which the internal (sampling points in interior of subintervals) Riemann-Stieltjes sums (with respect to α) of f with respect to α converge. Those functions α with this property are called *integrators*. We give a complete characterization of all such operator-valued integrators. We show that a subclass of such integrators form a Banach space under a suitable norm. We also derive criteria for the compactness of integrator induced operators. All throughout the paper, we use only very elementary methods from basic functional analysis.

The paper is organized as follows. We introduce equivalent formulations for regulated functions in section 2. In section 3, we introduce the notion of integrators and use the uniform boundedness principle to prove that each integrator induces a bounded linear operator. In section 4 we show that the space of integrators vanishing on the left endpoint is a Banach space. Section 5 is devoted to the compactness of integrator induced operators. In section 6, we prove that each operator from a certain subspace of the space of regulated functions is an integrator induced operator.

2. Notation and Preliminaries

Fix real numbers $a < b$ and a Banach space X . A function $f : [a, b] \rightarrow X$ is said to be *regulated* (see [3, §7.6, p.139] and [6, p. 16]) if one-sided limits $f(c^+) := \lim_{t \rightarrow c^+} f(t)$ exist for all $c \in [a, b)$, and $f(c^-) := \lim_{t \rightarrow c^-} f(t)$ exist for all $c \in (a, b]$. Denote

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by $\text{Reg}([a, b], X) = \text{Reg}(X)$ the space of regulated functions on $[a, b]$ (taking values in X).

A *partition* P of the interval $[a, b]$ is given by a finite number of *division points* in $[a, b]$:

$$P(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b), \quad n(P) \in \mathbb{N}.$$

The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$ or simply $\mathcal{P}$, whenever no confusion can arise.

A function $g : [a, b] \rightarrow X$ is a *step function* if there are a partition

$$P(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b)$$

and vectors $x_j \in X$, $1 \leq j \leq n(P)$ such that $g(t) = x_j$ for all $t \in (t_{j-1}, t_j)$, $1 \leq j \leq n(P)$; i.e., g takes on constant values on each open subinterval in the partition. Since $n(P)$ is determined by g , we also denote $n(P)$ by $n(g)$. Regulated functions have the following useful characterization.

THEOREM 1. [3, Th 7.6.1, p.139] *A function $f : [a, b] \rightarrow X$ is regulated iff there exists a sequence $\{h_n\}$ of step functions, from $[a, b]$ to X , such that*

$$\lim_{n \rightarrow \infty} \left[\sup_{t \in [a, b]} \|h_n(t) - f(t)\| \right] = 0 \quad (\text{i. e., } h_n \rightarrow f \text{ uniformly on } [a, b].)$$

Furthermore, if $f \in \text{Reg}(X)$, then

$$\|f\| := \sup_{t \in [a, b]} \|f(t)\| < \infty,$$

and $(\text{Reg}(X), \|\cdot\|)$ is a Banach space.

3. Integrators as bounded linear transformations

Let X, Y be Banach spaces. Denote by $\mathfrak{B}(X, Y)$ the space of bounded linear transformations from X to Y . An operator-valued function $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ is called an *integrator* for the regulated functions if for each $f \in \text{Reg}(X)$, there is a $y \in Y$ that satisfies the following condition:

(†) for every $\varepsilon > 0$ there is a partition

$$P_\varepsilon(a = s_0 < s_1 < s_2 < \cdots < s_{n(P_\varepsilon)} = b)$$

such that for every partition

$$P(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b) \text{ that refines } P_\varepsilon :$$

$$\{s_k : 1 \leq k \leq n(P_\varepsilon)\} \subseteq \{t_j : 1 \leq j \leq n(P)\} \quad \text{and}$$

$$\text{for all selections of } t_j^* \in (t_{j-1}, t_j), \quad 1 \leq j \leq n(P),$$

$$\left\| y - \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) \right\| < \varepsilon.$$

Given an integrator α and an $f \in \text{Reg}(X)$, a routine verification reveals that the vector $y \in Y$ associated with f by α is unique. The vector y is called the *interior integral* (or *Dushnik integral* [6, p. 7]) of f with respect to α and is denoted by

$$\int_a^b [d\alpha(t)](f(t)) = y.$$

We show that each integrator induces a bounded linear map from $\text{Reg}(X)$ to Y .

THEOREM 2. *Let $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ be an integrator for $\text{Reg}(X)$. Then the map*

$$\hat{\alpha} : f \mapsto \int_a^b [d\alpha(t)](f(t)), \quad f \in \text{Reg}(X)$$

is a bounded linear transformation from $\text{Reg}(X)$ to Y .

Proof.

We omit the routine verification of linearity of $\hat{\alpha}$. First we show that α is a bounded function. Since $\alpha_0 = \alpha - \alpha(a)$ and α give the same integral for each function in $\text{Reg}(X)$. We assume, without loss of generality, that $\alpha(a) = 0$. Suppose α is not bounded. Then, inductively, there is a sequence $\{t_k\}_{k \in \mathbb{N}}$ in $[a, b]$ such that

$$\|\alpha(t_1)\| > 4, \quad \|\alpha(t_k)\| > 2^{2^k} \|\alpha(t_{k-1})\| \quad \text{for all } k \geq 2.$$

By compactness, we may assume without loss of generality, that the sequence $\{t_k\}$ is monotonically increasing to a limit $t_\infty \in [a, b]$. (The decreasing case is handled similarly.) For each $k \geq 1$ there is an $x_k \in [X]_1$ such that

$$\|[\alpha(t_1)]x_1\| > 4, \quad \|[\alpha(t_k)]x_k\| > \max \left\{ 2^{2^k} \|\alpha(t_{k-1})\|, \frac{3}{4} \|\alpha(t_k)\| \right\} \quad \forall k \geq 2.$$

Let $t_0 = a$. For each $k \in \mathbb{N}$, define f_k by putting $f_k(t) = 2^{-j} x_j$ for $t_{j-1} < t < t_j$, $1 \leq j \leq k$, and $f_k(t) = 0$ for all other $t \in [a, b]$. Then for $k < l$ in $\mathbb{N}$, we have

$$\|f_l - f_k\| = \max \left\{ 2^{-j} \|x_j\| : k+1 \leq j \leq l \right\} \leq 2^{-k-1} \rightarrow 0 \quad \text{as } k, l \rightarrow \infty.$$

Thus there is a function $f \in \text{Reg}(X)$ such that $\|f_k - f\| \rightarrow 0$. Note that $f(t) = f_k(t)$ for all $t \in [a, t_k]$, since $f_k|_{[a, t_k]} = f_l|_{[a, t_k]}$ for all $l \geq k$.

Let $y \in Y$. (We show that for every partition P_0 there are a refinement P of P_0 and interior sampling points such that the associated Riemann-Stieltjes sum of f has norm more than $\|y\| + 1$. Therefore y cannot be the interior integral of f , for any $y \in Y$, and hence the interior integral of f does not exist.) Let $P_0 (a = s_0 < s_1 < s_2 < \cdots < s_{n(P_0)} = b)$

be arbitrarily given. Choose $k \in \mathbb{N}$ such that $2^k > \|y\| + \|\alpha(b)\| + 1$ and $t_k > s_{n(P_0)-1}$ (since $t_l \nearrow b$). Let $P(a = u_0 < u_1 < u_2 < \dots < u_{n(P)} = b)$ be such that

$$\{s_j : 1 \leq j \leq n(P_0)\} \cup \{t_l : 1 \leq l \leq k\} = \{u_i : 1 \leq i \leq n(P)\}.$$

Then $u_{n(P)-1} = t_k$. Arbitrarily choose $u_i^* \in (u_{i-1}, u_i)$ for $1 \leq i < n(P) - 1$; and $u_{n(P)}^* \in (t_k, t_{k+1})$.

$$\begin{aligned} & \left\| y - \sum_{i=1}^{n(P)} [\alpha(u_i) - \alpha(u_{i-1})](f(u_i^*)) \right\| \\ & \geq \left\| \sum_{j=1}^k \left[\sum_{(u_{i-1}, u_i) \subseteq (t_{j-1}, t_j)} [\alpha(u_i) - \alpha(u_{i-1})](f(u_i^*)) \right] \right. \\ & \quad \left. + [\alpha(b) - \alpha(t_k)](f(u_{n(P)}^*)) \right\| - \|y\| \\ & = \left\| \sum_{j=1}^k \left[\sum_{(u_{i-1}, u_i) \subseteq (t_{j-1}, t_j)} [\alpha(u_i) - \alpha(u_{i-1})](2^{-j}x_j) \right] \right. \\ & \quad \left. + [\alpha(b) - \alpha(t_k)](2^{-k-1}x_{k+1}) \right\| - \|y\| \\ & = \left\| \sum_{j=1}^k \left[\sum_{(u_{i-1}, u_i) \subseteq (t_{j-1}, t_j)} [\alpha(u_i) - \alpha(u_{i-1})] \right] (2^{-j}x_j) \right. \\ & \quad \left. + [\alpha(b) - \alpha(t_k)](2^{-k-1}x_{k+1}) \right\| - \|y\| \\ & = \left\| \sum_{j=1}^k [\alpha(t_j) - \alpha(t_{j-1})](2^{-j}x_j) + [\alpha(b) - \alpha(t_k)](2^{-k-1}x_{k+1}) \right\| - \|y\| \\ & \geq \left\| [\alpha(t_k)](2^{-k}x_k) \right\| - \left\| [\alpha(t_{k-1})](2^{-k}x_k) \right\| \\ & \quad - \sum_{j=1}^{k-1} \left[\left\| [\alpha(t_j)](2^{-j}x_j) \right\| + \left\| [\alpha(t_{j-1})](2^{-j}x_j) \right\| \right] \\ & \quad - \left\| [\alpha(b)](2^{-k-1}x_{k+1}) \right\| - \left\| [\alpha(t_k)](2^{-k-1}x_{k+1}) \right\| - \|y\| \\ & > 2^{-k} \left[\frac{3}{4} \|\alpha(t_k)\| \right] - 2^{-k-1} \|\alpha(b)\| - 2^{-k-1} \|\alpha(t_k)\| \\ & \quad - \sum_{j=1}^{k-1} 2^{-j} (\|\alpha(t_j)\| + \|\alpha(t_{j-1})\|) - \|y\| \end{aligned}$$

$$\begin{aligned}
&> 2^{-k-2} \|\alpha(t_k)\| - \|\alpha(b)\| - 2^{-k} \|\alpha(t_{k-1})\| - \sum_{j=1}^{k-1} 2^{-j} (2 \|\alpha(t_{k-1})\|) - \|y\| \\
&> 2^{2^k - k - 2} \|\alpha(t_{k-1})\| - 2^k \|\alpha(b)\| - 2^{-k} \|\alpha(t_{k-1})\| - 2 \|\alpha(t_{k-1})\| - \|y\| \\
&\geq 2^{2^k - k - 3} \|\alpha(t_{k-1})\| - 2^{-k} \|\alpha(b)\| - \|y\| > 1.
\end{aligned}$$

This shows that every $y \in Y$ cannot be the interior integral of f with respect to α , contradicting our assumption on α . Therefore α is bounded.

Next we show that each fixed partition induces a bounded linear transformation from $\text{Reg}(X)$ to Y . Let $P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b)$ and let

$$\mathbf{t}_P^* = \left\{ t_j^* \in (t_{j-1}, t_j), 1 \leq j \leq n(P) \right\}.$$

A straightforward application of triangle inequality shows that the map $T_{P, \mathbf{t}_P^*}$ defined by

$$T_{P, \mathbf{t}_P^*}(f) = \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) \quad \forall f \in \text{Reg}(X)$$

is a bounded linear transformation from $\text{Reg}(X)$ to Y (norm $\leq 2(n(P)) \|\alpha\|_\infty$).

Let $f \in \text{Reg}(X)$. We show that there is an $M_f > 0$ such that

$$\begin{aligned}
\left\| T_{P, \mathbf{t}_P^*}(f) \right\| &\leq M_f \quad \forall P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b) \in \mathcal{P}[a, b], \\
&\quad \forall t_j^* \in (t_{j-1}, t_j) : 1 \leq j \leq n(P).
\end{aligned}$$

Since α is an integrator, $y = \int_a^b [d\alpha(t)](f(t)) \in Y$ exists. With $\varepsilon = 1$ in the definition of interior integral ($\dagger$), there is a partition

$$P_0(a = s_0 < s_1 < s_2 < \dots < s_{n(P_0)} = b)$$

such that

$$\begin{aligned}
&\text{if } P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b) \text{ satisfies} \\
&\quad \{s_j : 1 \leq j \leq n(P_0)\} \subseteq \{t_i : 1 \leq i \leq n(P)\} \quad \text{and} \\
&\quad t_j^* \in (t_{j-1}, t_j), 1 \leq j \leq n(P)
\end{aligned}$$

then

$$\left\| y - \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) \right\| < 1$$

Let

$P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b)$ be an arbitrary partition, and let

$$\mathbf{t}_P^* = \left\{ t_j^* \in (t_{j-1}, t_j), 1 \leq j \leq n(P) \right\}.$$

Let Q be the minimum common refinement of P and P_0 , i. e.,

$$\begin{aligned} Q(a = u_0 < u_1 < \dots < u_{n(Q)} = b), \\ \{s_i : 1 \leq i \leq n(P_0)\} \cup \{t_j : 1 \leq j \leq n(P)\} = \{u_l : 1 \leq l \leq n(Q)\}. \end{aligned}$$

Choose $u_l^* \in (u_{l-1}, u_l)$ in such a way that $u_l^* = t_j^*$ if $t_j^* \in (u_{l-1}, u_l)$ for some j , and arbitrary $u_l^* \in (u_{l-1}, u_l)$ if $t_j^* \notin (u_{l-1}, u_l)$ for all $1 \leq j \leq n(P)$.

Since each u_l is either an s_i or a t_j , after cancellation of like terms, all remaining terms in the difference

$$\sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) - \sum_{l=1}^{n(Q)} [\alpha(u_l) - \alpha(u_{l-1})](f(u_l^*))$$

involve at least one s_i of the form $[\alpha(s_i) - \alpha(t_j)](f(t_j^*) - f(u_l^*))$ or its two other variants. Therefore there can be at most $2(n(P_0))$ terms of this form (since each s_i can appear in at most two such terms), and hence

$$\begin{aligned} \left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) - \sum_{l=1}^{n(Q)} [\alpha(u_l) - \alpha(u_{l-1})](f(u_l^*)) \right\| \\ \leq 2(n(P_0))(2\|\alpha\|_\infty)(2\|f\|). \end{aligned}$$

Since Q is a refinement of P_0 , we have

$$\begin{aligned} \left\| T_{P, \mathbf{t}_P^*}(f) \right\| &= \left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) \right\| \\ &\leq \left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) - \sum_{l=1}^{n(Q)} [\alpha(u_l) - \alpha(u_{l-1})](f(u_l^*)) \right\| \\ &\quad + \left\| \sum_{l=1}^{n(Q)} [\alpha(u_l) - \alpha(u_{l-1})](f(u_l^*)) - y \right\| + \|y\| \\ &< 8(n(P_0))\|\alpha\|_\infty\|f\| + 1 + \|y\|, \end{aligned}$$

which is the desired M_f , independent of the partition P and the choice of sampling points $\mathbf{t}_P^* = \{t_j^*\}$. Since the foregoing argument holds for each fixed f , by the uniform boundedness principle, there exists an $M > 0$ such that

$$\left\| T_{P, \mathbf{t}_P^*} \right\| \leq M \quad \text{for all partitions } P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b)$$

and for all choices of $\mathbf{t}_p^* = \{t_j^* \in (t_{j-1}, t_j) : 1 \leq j \leq n(P)\}$.

It follows that, for each $f \in \text{Reg}(X)$ and each $\varepsilon > 0$, there is a partition P and choice of sampling points $\mathbf{t}_p^*$ such that

$$\begin{aligned} \|\widehat{\alpha}(f)\| &= \left\| \int_a^b [d\alpha(t)](f(t)) \right\| \leq \left\| \int_a^b [d\alpha(t)](f(t)) - T_{P, \mathbf{t}_p^*}(f) \right\| + \left\| T_{P, \mathbf{t}_p^*}(f) \right\| \\ &< \varepsilon + \left\| T_{P, \mathbf{t}_p^*} \right\| \|f\| \leq \varepsilon + M \|f\| \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the map

$$\widehat{\alpha} : f \mapsto \int_a^b [d\alpha(t)](f(t)) \quad \forall f \in \text{Reg}(X)$$

is a bounded linear map from $\text{Reg}(X)$ to Y with norm $\|\widehat{\alpha}\| \leq M$.

4. Equivalent formulation of integrators

In this section we give an equivalent formulation for integrators.

LEMMA 3. *Let $g : [a, b] \rightarrow X$ be a step function and let $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ be an integrator for $\text{Reg}(X)$. Then there exist*

a partition $P_0(a = u_0 < u_1 < \dots < u_{n(P_0)} = b)$ and $x_k \in [X]_1$, $1 \leq k \leq n(P)$,

such that

$$\int_a^b [d\alpha(t)](g(t)) = \sum_{k=1}^{n(P_0)} [\alpha(u_k) - \alpha(u_{k-1})]x_k = \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](g(t_j^*))$$

for all partitions $P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b)$ satisfying $\{u_k : 1 \leq k \leq n(P_0)\} \subseteq \{t_j : 1 \leq j \leq n(P)\}$, and for all choices of $t_j^ \in (t_{j-1}, t_j)$, $1 \leq j \leq n(P)$,*

Proof. By the definition of step functions, there exist

$$P_0(a = u_0 < u_1 < u_2 < \dots < u_{n(P_0)} = b), \quad \text{and} \quad x_j \in X, \quad 1 \leq j \leq n(P_0)$$

such that

$$g(t) = x_j, \quad \forall t \in (u_{j-1}, u_j), \quad \forall 1 \leq j \leq n(P_0).$$

We show that

$$y = \sum_{j=1}^{n(P_0)} [\alpha(u_j) - \alpha(u_{j-1})]x_j = \int_a^b [d\alpha(t)](g(t))$$

and the partition P_0 satisfies the stated conditions.

To that end, let $P(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b)$ satisfy

$$\{u_j : 1 \leq j \leq n(P_0)\} \subseteq \{t_k : 1 \leq k \leq n(P)\}.$$

For each $k = 1, 2, \dots, n(P)$, arbitrarily choose $t_k^* \in (t_{k-1}, t_k)$. Observe that, for each $1 \leq j \leq n(P_0)$, since

$$\bigcup_{(t_{k-1}, t_k) \subseteq (u_{j-1}, u_j)} (t_{k-1}, t_k] = (u_{j-1}, u_j],$$

by cancellations of intermediate terms,

$$\sum_{(t_{k-1}, t_k) \subseteq (u_{j-1}, u_j)} [\alpha(t_k) - \alpha(t_{k-1})] = \alpha(u_{j-1}) - \alpha(u_j).$$

Therefore, since $t_k^* \in (u_{j-1}, u_j)$ and $g(t_k^*) = x_j$ whenever $(t_{k-1}, t_k) \subseteq (u_{j-1}, u_j)$,

$$\begin{aligned} & \sum_{k=1}^{n(P)} [\alpha(t_k) - \alpha(t_{k-1})] (g(t_k^*)) \\ &= \sum_{j=1}^{n(P_0)} \left[\sum_{(t_{k-1}, t_k) \subseteq (u_{j-1}, u_j)} [\alpha(t_k) - \alpha(t_{k-1})] (g(t_k^*)) \right] \\ &= \sum_{j=1}^{n(P_0)} \left[\sum_{(t_{k-1}, t_k) \subseteq (u_{j-1}, u_j)} [\alpha(t_k) - \alpha(t_{k-1})] x_j \right] \\ &= \sum_{j=1}^{n(P_0)} \left[\left(\sum_{(t_{k-1}, t_k) \subseteq (u_{j-1}, u_j)} [\alpha(t_k) - \alpha(t_{k-1})] \right) x_j \right] \\ &= \sum_{j=1}^{n(P_0)} [\alpha(u_j) - \alpha(u_{j-1})] x_j = y. \end{aligned}$$

Furthermore this also shows that the vector y is also the interior integral of g with respect to α .

THEOREM 4. *Let $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ be an integrator for $\text{Reg}(X)$. Then for all partition $P(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b)$ and for all choices of $x_j \in [X]_1$, $1 \leq j \leq n(P)$,*

$$\left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})] x_j \right\| \leq \|\widehat{\alpha}\|.$$

Proof. First note that by Theorem 2 the map, $\widehat{\alpha} : f \mapsto \int_a^b [d\alpha(t)](f(t))$, is a bounded linear transformation from $\text{Reg}(X)$ to Y . To see that $\|\widehat{\alpha}\|$ has the asserted property, let a partition $P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b)$ and $x_j \in [X]_1$, $1 \leq j \leq n(P)$, be given. Define

$$g(t) = x_j \text{ for } t \in (t_{j-1}, t_j), \ 1 \leq j \leq n(P), \text{ and } g(t) = 0 \text{ for all other } t\text{'s}.$$

Then g is a step function and hence is in $\text{Reg}(X)$ with $\|g\| \leq 1$. Thus, by Lemma 3,

$$\left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})]x_j \right\| = \left\| \int_a^b [d\alpha(t)](g(t)) \right\| = \|\widehat{\alpha}(g)\| \leq \|\widehat{\alpha}\| \|g\| \leq \|\widehat{\alpha}\|.$$

This result leads us to the following definition.

A function $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ is said to be of *bounded semivariation* if

there exists $M > 0$ such that

$$\begin{aligned} &\text{for all partitions } P(a = t_0 < t_1 < \dots < t_{n(P)} = b) \in \mathcal{P}[a, b], \quad \text{and} \\ &\text{for all } x_k \in [X]_1 \text{ (the closed unit ball of } X), \quad 1 \leq k \leq n(P), \end{aligned}$$

$$\left\| \sum_{k=1}^{n(P)} [\alpha(t_k) - \alpha(t_{k-1})]x_k \right\| \leq M.$$

The *semivariation* of a function $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ of bounded semivariation is given by

$$\begin{aligned} V_s(\alpha) := \sup \left\{ \left\| \sum_{k=1}^{n(P)} [\alpha(t_k) - \alpha(t_{k-1})]x_k \right\| : \right. \\ \left. P(a = t_0 < t_1 < \dots < t_{n(P)} = b) \in \mathcal{P}; x_k \in [X]_1, \ 1 \leq k \leq n(P) \right\}. \end{aligned}$$

It follows from Theorem 4 that each integrator β is of bounded semivariation with $V_s(\beta) \leq \|\widehat{\beta}\|$. We will need the following lemmas to the converse.

LEMMA 5. Let $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ be of bounded semivariation and $t \in [a, b]$. Then

$$\|\alpha(t) - \alpha(a)\| \leq V_s(\alpha).$$

Proof. Observe that

$$\begin{aligned} \|\alpha(t) - \alpha(a)\| &= \sup_{x \in [X]_1} \|[\alpha(t) - \alpha(a)]x\| \\ &= \sup_{x \in [X]_1} \|[\alpha(t) - \alpha(a)]x + [\alpha(b) - \alpha(t)]0\| \leq V_s(\alpha). \end{aligned}$$

LEMMA 6. Let $g, h : [a, b] \rightarrow X$ be step functions given by

$$a = u_0 < u_1 < \cdots < u_{n(g)} = b, \quad g(t) = x_j \in X, \quad t \in (u_{j-1}, u_j), \quad 1 \leq j \leq n(g),$$

and

$$a = v_0 < v_1 < \cdots < v_{n(h)} = b, \quad h(t) = w_k \in X, \quad t \in (v_{k-1}, v_k), \quad 1 \leq k \leq n(h).$$

Let $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ be of bounded semivariation; and let

$$y_g = \sum_{j=1}^{n(g)} [\alpha(u_j) - \alpha(u_{j-1})] x_j \quad \text{and} \quad y_h = \sum_{k=1}^{n(h)} [\alpha(v_k) - \alpha(v_{k-1})] w_k.$$

Then

$$\|y_g - y_h\| \leq \|g - h\| V_s(\alpha).$$

Proof. Relabel the elements of $\{u_j : 0 \leq j \leq n(g)\} \cup \{v_k : 0 \leq k \leq n(h)\}$ as $a = t_0 < t_1 < \cdots < t_N = b$ to eliminate duplicates. Then, for each $1 \leq j \leq n(g)$ and $1 \leq k \leq n(h)$,

$$\bigcup_{(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)} (t_{i-1}, t_i] = (u_{j-1}, u_j], \quad \text{and} \\ \bigcup_{(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)} (t_{i-1}, t_i] = (v_{k-1}, v_k].$$

Thus

$$\sum_{(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)} [\alpha(t_i) - \alpha(t_{i-1})] = \alpha(u_j) - \alpha(u_{j-1}) \quad \text{and} \\ \sum_{(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)} [\alpha(t_i) - \alpha(t_{i-1})] = \alpha(v_k) - \alpha(v_{k-1}).$$

Choose $t_i^* \in (t_{i-1}, t_i)$. Then $g(t_i^*) = x_j$ whenever $(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)$, and $h(t_i^*) = w_k$ whenever $(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)$. Since

$$\left\| \|g - h\|^{-1} (g(t_i^*) - h(t_i^*)) \right\| \leq 1, \quad 1 \leq i \leq N,$$

it follows that

$$\begin{aligned} \|y_g - y_h\| &= \left\| \sum_{j=1}^{n(g)} [\alpha(u_j) - \alpha(u_{j-1})] x_j - \sum_{k=1}^{n(h)} [\alpha(v_k) - \alpha(v_{k-1})] w_k \right\| \\ &= \left\| \sum_{j=1}^{n(g)} \left[\sum_{(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)} [\alpha(t_i) - \alpha(t_{i-1})] \right] x_j \right\| \end{aligned}$$

$$\begin{aligned}
& - \sum_{k=1}^{n(h)} \left[\sum_{(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)} [\alpha(t_i) - \alpha(t_{i-1})] \right] w_k \Big\| \\
&= \left\| \sum_{j=1}^{n(g)} \left[\sum_{(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)} [\alpha(t_i) - \alpha(t_{i-1})] \right] (g(t_i^*)) \right. \\
&\quad \left. - \sum_{k=1}^{n(h)} \left[\sum_{(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)} [\alpha(t_i) - \alpha(t_{i-1})] \right] (h(t_i^*)) \right\| \\
&= \left\| \sum_{j=1}^{n(g)} \left[\sum_{(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)} [\alpha(t_i) - \alpha(t_{i-1})] (g(t_i^*)) \right] \right. \\
&\quad \left. - \sum_{k=1}^{n(h)} \left[\sum_{(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)} [\alpha(t_i) - \alpha(t_{i-1})] (h(t_i^*)) \right] \right\| \\
&= \left\| \sum_{i=1}^N [\alpha(t_i) - \alpha(t_{i-1})] (g(t_i^*)) - \sum_{i=1}^N [\alpha(t_i) - \alpha(t_{i-1})] (h(t_i^*)) \right\| \\
&= \left\| \sum_{i=1}^N [\alpha(t_i) - \alpha(t_{i-1})] (g(t_i^*) - h(t_i^*)) \right\| \\
&= \left\| \sum_{i=1}^N [\alpha(t_i) - \alpha(t_{i-1})] \left[\|g - h\|^{-1} (g(t_i^*) - h(t_i^*)) \right] \right\| \|g - h\| \\
&\leq V_s(\alpha) \|g - h\|.
\end{aligned}$$

THEOREM 7. Let $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ be a function of bounded semivariation. Then α is an integrator for $\text{Reg}(X)$.

Proof. Let $\varepsilon > 0$. By Theorem 1 there exists a sequence $\{g_k\}_{k \in \mathbb{N}}$ of step functions $g_k : [a, b] \rightarrow X$ such that

$$\|f - g_k\| = \sup_{a \leq t \leq b} \|f(t) - g_k(t)\|_X \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

For each $k \in \mathbb{N}$, let $y_k \in Y$ be the vector associated with g_k by Lemma 6. The lemma also gives us

$$\|y_k - y_l\|_Y \leq V_s(\alpha) \|g_k - g_l\| \leq V_s(\alpha) [\|g_k - f\| + \|f - g_l\|] \rightarrow 0 \text{ as } k, l \rightarrow \infty.$$

Thus $\{y_k\}$ is a Cauchy sequence in Y . By the completeness of Y , there is a $y \in Y$ such that $\|y - y_k\|_Y \rightarrow 0$ as $k \rightarrow \infty$. We show that y has the property $(\dagger)$, i.e., is the integral of f . Let $\varepsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that

$$\|y - y_k\|_Y < \frac{\varepsilon}{2} \quad \text{and} \quad \|f - g_k\| < \min \left\{ 1, \frac{\varepsilon}{2[V_s(\alpha) + 1]} \right\} \quad \forall k \geq N.$$

Let $g = g_N$. Since g is a step function, there exist

$$a = u_0 < u_1 < u_2 < \cdots < u_{n(g)} = b, \quad x_j \in X, \quad 1 \leq j \leq n(g)$$

such that $g(t) = x_j$, for $t \in (u_{j-1}, u_j)$, $1 \leq j \leq n(g)$. Let P_0 be the partition defined by the division points u_j , i.e., $P_0(a = u_0 < u_1 < \cdots < u_{n(g)} = b)$. Let

$$P(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b)$$

be a partition that satisfies

$$\{u_j : 1 \leq j \leq n(g)\} \subseteq \{t_k : 1 \leq k \leq n(P)\}.$$

Let $t_j^* \in (t_{j-1}, t_j)$, $1 \leq j \leq n(g)$, be arbitrarily chosen. Then since $g = g_N$ is a step function, by Lemma 3

$$\sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})](g(t_i^*)) = \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})](g_N(t_i^*)) = y_N.$$

Since $\left\| \|g - f\|^{-1} (g(t_i^*) - f(t_i^*)) \right\| \leq 1$, $1 \leq i \leq n(P)$, we have

$$\begin{aligned} & \left\| y - \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})](f(t_i^*)) \right\| \\ & \leq \left\| y - \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})]g(t_i^*) \right\| + \left\| \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})](g(t_i^*) - f(t_i^*)) \right\| \\ & = \|y - y_N\| + \left\| \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})] \left[\|g - f\|^{-1} (g(t_i^*) - f(t_i^*)) \right] \right\| \|g - f\| \\ & < \frac{\varepsilon}{2} + V_s(\alpha) \|g - f\| < \frac{\varepsilon}{2} + \frac{\varepsilon V_s(\alpha)}{2[V_s(\alpha) + 1]} < \varepsilon. \end{aligned}$$

Combining Theorems 7 and 4, we have the following.

COROLLARY 8. 1. A function $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ is an integrator for $\text{Reg}(X)$ if, and only if, α is of bounded semivariation.

2. For each integrator α for $\text{Reg}(X)$, $V_s(\alpha) = \|\widehat{\alpha}\|$.

Proof. Part (1) follows directly from Theorems 7 and 4.

For part (2), we have already noted above that $V_s(\alpha) \leq \|\widehat{\alpha}\|$. For the opposite inequality, let $f \in \text{Reg}(X)$, and let $\varepsilon > 0$. Let P_0 and all s_j be as in ($\dagger$). Choose any $s_j^* \in (s_{j-1}, s_j)$, for $1 \leq j \leq n(P_0)$. Then, since $\left\| \|f\|^{-1} f(s_j^*) \right\| \leq 1$ for all $1 \leq j \leq n(P_0)$, we have

$$\|\widehat{\alpha}(f)\| \leq \left\| \int_a^b [d\alpha(t)](f(t)) - \left[\sum_{j=1}^{n(P_0)} [\alpha(s_j) - \alpha(s_{j-1})](f(s_j^*)) \right] \right\|$$

$$\begin{aligned}
& + \left\| \sum_{j=1}^{n(P_0)} [\alpha(s_j) - \alpha(s_{j-1})](f(s_j^*)) \right\| \\
& < \varepsilon + \left\| \sum_{j=1}^{n(P_0)} [\alpha(s_j) - \alpha(s_{j-1})](\|f\|^{-1} f(s_j^*)) \right\| \cdot \|f\| \leq \varepsilon + V_s(\alpha) \|f\|
\end{aligned}$$

Since ε is arbitrary, we have $\|\hat{\alpha}\| \leq V_s(\alpha)$, and hence equality follows.

Denote by $\mathcal{B}_{sv}([a, b], \mathfrak{B}(X, Y))$ (or simply $\mathcal{B}_{sv}$ whenever no ambiguity, i.e., only one interval $[a, b]$ and two fixed spaces X and Y in sight) the space of all functions of bounded semivariation from $[a, b]$ to $\mathfrak{B}(X, Y)$. It is not hard to see that V_s is not a norm, as the semivariation of any constant function is 0. However every $\alpha \in \mathcal{B}_{sv}$ and its normalized form $\alpha_0 = \alpha - \alpha(a)$ define the same integrator. The set of all such normalized functions is a Banach space.

THEOREM 9. *The space $\mathcal{B}_{sv}^0([a, b], \mathfrak{B}(X, Y))$ of all functions vanishing at the point a is a Banach space under the norm $\|\alpha\| = V_s(\alpha)$. Furthermore, the space $\mathcal{B}_{sv}^0([a, b], \mathfrak{B}(X, Y))$ is isometrically isomorphic to the close subspace*

$$\left\{ \hat{\alpha} : \alpha \in \mathcal{B}_{sv}^0 \right\} \quad \text{of } \mathfrak{B}(\text{Reg}(X), Y) \quad \text{via the map } \alpha \mapsto \hat{\alpha}.$$

Proof. We prove only the completeness and leave all other verifications to the readers. Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}_{sv}^0$ (i.e., $V_s(\alpha_n - \alpha_k) \rightarrow 0$ as $n, k \rightarrow \infty$). (We are assuming that all $\alpha_n(a) = 0$, $n \in \mathbb{N}$.) Then for each $t \in [a, b]$, by Lemma 5

$$\begin{aligned}
\|\alpha_n(t) - \alpha_k(t)\| &= \|(\alpha_n(t) - \alpha_k(t)) - (\alpha_n(a) - \alpha_k(a))\| \\
&< V_s(\alpha_n - \alpha_k) \rightarrow 0 \quad \text{as } n, k \rightarrow \infty
\end{aligned}$$

That is, for each $t \in [a, b]$, $\{\alpha_n(t)\}$ is a Cauchy sequence in the complete space $\mathfrak{B}(X, Y)$, and hence has a limit $\alpha(t) \in \mathfrak{B}(X, Y)$. Furthermore, since the right hand side of the inequality is independent of $t \in [a, b]$, the convergence of $\alpha_n(t) \rightarrow \alpha(t)$ is uniform on $[a, b]$.

We show that $\alpha \in \mathcal{B}_{sv}^0$ and $V_s(\alpha - \alpha_n) \rightarrow 0$. Since $\{\alpha_n\}$ is V_s -Cauchy, there is an $N \in \mathbb{N}$ such that

$$V_s(\alpha_n - \alpha_k) < 1 \quad \forall n, k \geq N.$$

Let $P(a = t_0 < t < \dots < t_{n(P)} = b) \in \mathcal{P}$, and $x_j \in [X]_1$, $1 \leq j \leq n(P)$. By the uniform convergence of $\|\alpha_n(t) - \alpha(t)\| \rightarrow 0$ for $t \in [a, b]$, there is an $N_1 > N$ such that

$$\|\alpha(t) - \alpha_k(t)\| < \frac{1}{n(P)} \quad \forall t \in [a, b], \forall k \geq N_1$$

Then, since $\|x_j\| \leq 1$, for all $1 \leq j \leq n(P)$,

$$\left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})]x_j \right\|$$

$$\begin{aligned}
&\leq \left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha_{N_1}(t_j)]x_j \right\| + \left\| \sum_{j=1}^{n(P)} [\alpha_{N_1}(t_j) - \alpha_N(t_j)]x_j \right\| \\
&\quad + \left\| \sum_{j=1}^{n(P)} [\alpha_N(t_j) - \alpha_N(t_{j-1})]x_j \right\| + \left\| \sum_{j=1}^{n(P)} [\alpha_N(t_{j-1}) - \alpha_{N_1}(t_{j-1})]x_j \right\| \\
&\quad + \left\| \sum_{j=1}^{n(P)} [\alpha_{N_1}(t_{j-1}) - \alpha(t_{j-1})]x_j \right\| \\
&\leq \sum_{j=1}^{n(P)} \left\| \alpha(t_j) - \alpha_{N_1}(t_j) \right\| + V_s(\alpha_{N_1} - \alpha_N) + V_s(\alpha_N) \\
&\quad + V_s(\alpha_N - \alpha_{N_1}) + \sum_{j=1}^{n(P)} \left\| \alpha_{N_1}(t_{j-1}) - \alpha(t_{j-1}) \right\| \\
&< V_s(\alpha_N) + 4
\end{aligned}$$

Since P and $x_j \in [X]_1$ are arbitrary, $\alpha \in \mathcal{B}_{sv}$ with $V_s(\alpha) \leq V_s(\alpha_N) + 4$.

For the convergence, let $\varepsilon > 0$. Since $\{\alpha_n\}$ is a V_s -Cauchy sequence, there is an N such that

$$V_s(\alpha_n - \alpha_k) < \frac{\varepsilon}{4} \quad \forall n, k \geq N.$$

Let $n \geq N$, $P(a = t_0 < t < \dots < t_{n(P)} = b) \in \mathcal{P}$, and $x_j \in [X]_1$, $j = 1, \dots, n(P)$ be arbitrary. Since $\alpha_n(t) \rightarrow \alpha(t)$ uniformly for $t \in [a, b]$, there exists an $N_1 > N$ such that

$$\left\| \alpha(t) - \alpha_k(t) \right\| < \frac{\varepsilon}{4[n(P)]} \quad \forall k \geq N_1, \forall t \in [a, b].$$

Since $n, N_1 \geq N$ and $x_j \in [X]_1$, $1 \leq j \leq n(P)$, we have

$$\begin{aligned}
&\left\| \sum_{j=1}^{n(P)} [(\alpha_n(t_j) - \alpha(t_j)) - (\alpha_n(t_{j-1}) - \alpha(t_{j-1}))]x_j \right\| \\
&\leq \left\| \sum_{j=1}^{n(P)} [(\alpha_n(t_j) - \alpha_{N_1}(t_j)) - (\alpha_n(t_{j-1}) - \alpha_{N_1}(t_{j-1}))]x_j \right\| \\
&\quad + \left\| \sum_{j=1}^{n(P)} [\alpha_{N_1}(t_j) - \alpha(t_j)]x_j \right\| + \left\| \sum_{j=1}^{n(P)} [\alpha_{N_1}(t_{j-1}) - \alpha(t_{j-1})]x_j \right\| \\
&\leq V_s(\alpha_n - \alpha_{N_1}) + \sum_{j=1}^{n(P)} \left\| \alpha_{N_1}(t_j) - \alpha(t_j) \right\| + \sum_{j=1}^{n(P)} \left\| \alpha_{N_1}(t_{j-1}) - \alpha(t_{j-1}) \right\| < \frac{3\varepsilon}{4}.
\end{aligned}$$

Since this holds for all partitions and all $x_j \in [X]_1$, we have $V_s(\alpha - \alpha_n) < \varepsilon$ for all $n \geq N$. The arbitrariness of ε implies that $V_s(\alpha - \alpha_n) \rightarrow 0$.

5. Integrators as compact operators

We first consider an example. Let $X = Y = \ell^2$ (the square summable Hilbert sequence space), and $[a, b] = [0, 1]$. For $n \in \mathbb{N}$, and $t \in (2^n, 2^{n+1}]$, define $\alpha(t)$ to be the rank one operator defined by the diagonal matrix, A_n , whose (n, n) -entry is $n^{-1/2}$ and all others 0, and $\alpha(0) = 0$. With e_j the j^{th} standard basis vector, and $N \in \mathbb{N}$, we have

$$\left\| \sum_{j=1}^N [\alpha(2^{N-j+1}) - \alpha(2^{N-j})] e_{N-j} \right\| = \left\| \sum_{j=1}^N A_{N-j} e_{N-j} \right\| = \left[\sum_{j=1}^N (N-j)^{-1} \right]^{1/2} \rightarrow \infty$$

as $N \rightarrow \infty$. Thus α is not an integrator, though each $\alpha(t)$ is of rank one and hence compact. However, if A_n is to have (n, n) -entry n^{-1} , $\hat{\alpha}$, then they are all mutually orthogonal rank one projections, $\hat{\alpha}([\text{Reg}(X)]_1)$ is contained in the image of the unit ball under the Hilbert-Schmidt diagonal operator with diagonal entries $\frac{1}{n}$. Hence $\hat{\alpha}$ is in fact compact.

Before stating criterion for compactness we introduce the following notation for convenience. For a given $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ define the set $[\alpha]_1$ as follows.

$$[\alpha]_1 := \left\{ \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})] x_j : P(a = t_0 < \dots < t_{n(P)} = b) \in \mathcal{P}, \right. \\ \left. x_j \in [X]_1, 1 \leq j \leq n(P) \right\}$$

Note that α is an integrator iff $[\alpha]_1$ is bounded (in Y).

THEOREM 10. *Let $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ be an integrator for $\text{Reg}(X)$ that vanishes at a (i.e., $\alpha(a) = 0$). Then the following conditions on α from $\text{Reg}(X)$ to Y are equivalent.*

1. $\hat{\alpha}$ is compact,
2. There is a compact operator K from a Banach space Z to Y such that $[\alpha]_1 \subseteq K([Z]_1)$.
3. There is a one-to-one compact operator K from a Banach space Z to Y such that $K^{-1}(\alpha(t))$ is bounded for all t and $K^{-1} \circ \alpha$ is an integrator.
4. The set $[\alpha]_1$ is a totally bounded subset of Y .

Proof. [(1) $\Rightarrow$ (2)] Suppose $\hat{\alpha}$ is compact. Each element of $[\alpha]_1$ determines a partition $P(a = t_0 < \dots < t_{n(P)} = b) \in \mathcal{P}[a, b]$ and a finite collection of vectors $x_j \in [X]_1, 1 \leq j \leq n(P)$. Then the function g defined by $g(t) = x_j$ for $t \in [t_{j-1}, t_j)$, and $g(b) = 0$ belongs in $[\text{Reg}(X)]_1$. Thus, by Lemma 3,

$$\sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})] x_j = \hat{\alpha}(g) \in \hat{\alpha}([\text{Reg}(X)]_1).$$

Thus $[\alpha]_1$ is contained in $\widehat{\alpha}([\text{Reg}(X)]_1) = K([Z]_1)$

[(2) $\Rightarrow$ (3)] Suppose $[\alpha]_1 \subseteq K([Z]_1)$ for some compact operator K from a Banach space Z to Y . Consider the compact operator $\tilde{K}$ induced on the quotient space $Z/\ker K$, we may assume that K is one-to-one. For $t \in [a, b]$ and $x \in [X]_1$, since the function

$$g_{t,x}(s) = \begin{cases} x & \text{for } s \in [a, t] \\ 0 & \text{for } s \in (t, b] \end{cases}$$

is in $[\text{Reg}(X)]_1$, we have

$$\begin{aligned} [\alpha(t)]x &= [\alpha(t) - \alpha(a)]x = \int_a^b [d\alpha(t)](g_{t,x}(t)) \\ &= \widehat{\alpha}(g_{t,x}) \in \widehat{\alpha}([\text{Reg}(X)]_1) \subseteq K([Z]_1). \end{aligned}$$

Thus $[\alpha(t)](X) \subseteq K(Z)$ for all $t \in [a, b]$. Hence $K^{-1}[\alpha(t)]$ is a closed operator from the Banach space X to Z . Thus each $K^{-1}[\alpha(t)]$ is a bounded operator from X to Z . Furthermore $\beta := K^{-1} \circ \alpha : [a, b] \rightarrow \mathfrak{B}(X, Z)$ satisfies

$$[\beta]_1 = [K^{-1} \circ \alpha]_1 = K^{-1}([\alpha]_1) \subseteq [Z]_1.$$

Therefore β is an integrator.

[(3) $\Rightarrow$ (4)] Suppose $\beta := K^{-1} \circ \alpha$ defines an integrator. Then $[\beta]_1 = K^{-1}([\alpha]_1)$ is a bounded subset of Z . Thus $[\alpha]_1 = K(K^{-1}([\alpha]_1)) = K([\beta]_1)$ is totally bounded.

[(4) $\Rightarrow$ (1)] Suppose $[\alpha]_1$ is totally bounded. We show that $\widehat{\alpha}([\text{Reg}(X)]_1)$ is contained in the closure of $[\alpha]_1$. Let $f \in [\text{Reg}(X)]_1$, and $\varepsilon > 0$. There is a step function g such that $\|f - g\| < \frac{\varepsilon}{\|\widehat{\alpha}\| + 1}$. Then there exist a partition

$$\begin{aligned} a = t_0 < \cdots < t_{n(g)} = b \text{ and } x_j \in [X]_1, 1 \leq j \leq n(g) \text{ such that} \\ g(t) = x_j \text{ for } t \in (t_{j-1}, t_j), 1 \leq j \leq n(g). \end{aligned}$$

By Lemma 3,

$$\widehat{\alpha}(g) = \int_a^b [d\alpha(t)](g(t)) = \sum_{j=1}^{n(g)} [\alpha(t_j) - \alpha(t_{j-1})]x_j \in [\alpha]_1.$$

Moreover,

$$\|\widehat{\alpha}(f) - \widehat{\alpha}(g)\| = \|\widehat{\alpha}(f - g)\| \leq \|\widehat{\alpha}\| \|f - g\| \leq \|\widehat{\alpha}\| \left[\frac{\varepsilon}{\|\widehat{\alpha}\| + 1} \right] < \varepsilon.$$

Therefore $\widehat{\alpha}(f) \in \overline{[\alpha]_1}$, that is $\widehat{\alpha}([\text{Reg}(X)]_1) \subseteq \overline{[\alpha]_1}$. Since $\overline{[\alpha]_1}$ is compact, and $\widehat{\alpha}([\text{Reg}(X)]_1) \subseteq \overline{[\alpha]_1}$, thus $\widehat{\alpha}$ is compact.

6. Operators as integrators

We prove next a Riesz representation type theorem. Let $\text{Reg}_r(X)$ be the set of all right continuous functions in $\text{Reg}(X)$ that are also left continuous at b ; i.e., $f \in \text{Reg}_l(X)$ iff $f(t) = f(t^+)$ for all $t \in [a, b)$ and $f(b) = 0$.

THEOREM 11. *Let $T \in \mathfrak{B}(\text{Reg}_r(X), Y)$. Then there exists a unique $\alpha_T \in \mathcal{B}_{sv}^0([a, b], \mathfrak{B}(X, Y))$ such that*

$$T(f) = \int_a^b [d\alpha(t)](f(t)) \quad \forall f \in \text{Reg}_r(X).$$

Moreover, $\hat{\alpha}_T = T$ and $\|T\| = \|\hat{\alpha}_T\| = V_s(\alpha_T)$.

This theorem can be restated as follows. The Banach spaces $\mathfrak{B}(\text{Reg}_r(X), Y)$ and $\mathcal{B}_{sv}^0([a, b], \mathfrak{B}(X, Y))$ are isometrically isomorphic.

Proof. Note that for each $x \in X$ and each $t \in [a, b]$, $\chi_{[a,t]}x$ is a step function (taking the value x on $[a, t)$ and 0 on $[t, b]$) belonging to $\text{Reg}_r(X)$. Define (with the convention that $\chi_{[a,a]} = 0$ is the zero function)

$$[\alpha_T(t)]x = T(\chi_{[a,t]}x) \quad \forall x \in X, t \in [a, b].$$

To show that $\alpha_T \in \mathcal{B}_{sv}$, let $P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b)$, and $x_j \in [X]_1$ be given. Note that the function g defined by

$$g = \sum_{j=1}^{n(P)} (\chi_{[a,t_j]} - \chi_{[a,t_{j-1}]})x_j \quad (*)$$

is a step function in $\text{Reg}_r(X)$, taking the value x_j on $[t_{j-1}, t_j)$, $1 \leq j \leq n(P)$, and

$$\|g\| \leq \max \{\|x_j\| : 1 \leq j \leq n(P)\} \leq 1.$$

Furthermore, by linearity of T ,

$$\begin{aligned} \left\| \sum_{j=1}^{n(P)} [\alpha_T(t_j) - \alpha_T(t_{j-1})]x_j \right\| &= \left\| \sum_{j=1}^{n(P)} T((\chi_{[a,t_j]} - \chi_{[a,t_{j-1}]})x_j) \right\| \\ &= \left\| T \left(\sum_{j=1}^{n(P)} (\chi_{[a,t_j]} - \chi_{[a,t_{j-1}]})x_j \right) \right\| = \|T(g)\| \leq \|T\|. \end{aligned}$$

That is $\alpha_T \in \mathcal{B}_{sv}$, and hence α_T an integrator for $\text{Reg}(X)$, and hence for $\text{Reg}_r(X)$, by Theorem 7.

Since each step function g in $\text{Reg}_r(X)$ can be written as a sum of the form (*) above, and the interior integral of a step function is independent of the values of the function at the division points for g , by Lemma 3,

$$\int_a^b [d\alpha(t)](g(t)) = \hat{\alpha}_T(g) = T(g).$$

From the density of the step functions, continuous from the right in $\text{Reg}_r(X)$, and the continuity of both T and $\hat{\alpha}_T$, we have $T = \hat{\alpha}_T$, and $\|T\| = \|\hat{\alpha}_T\|$.

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T. Wootijirattikal
 Department of Mathematics, Statistics and Computer, Faculty of Science
 Ubon Ratchathani University
 Ubon Ratchathani 34190, Thailand
 Center of Excellence in Mathematics
 Bangkok 10400, Thailand
 e-mail: titarii.w@ubu.ac.th, ma_pintto@yahoo.com

S.-C. Ong
 Department of Mathematics
 Central Michigan University
 Mount Pleasant, MI 48859, USA
 e-mail: ong1s@cmich.edu, ong3pf@gmail.com

Y. Lenbury
 Department of Mathematics
 Mahidol University
 Bangkok 10400, Thailand
 Center of Excellence in Mathematics
 Bangkok 10400, Thailand
 e-mail: yongwimon.len@mahidol.ac.th, scylb@yahoo.com

Corresponding Author: T. Wootijirattikal

MULTIPLICATIVE INTEGRATORS OF ALGEBRA-VALUED REGULATED FUNCTIONS

TITARII WOOTIJIRATTIKAL* AND SING-CHEONG ONG

ABSTRACT. A function from a closed interval $[a, b]$ to a Banach space X is regulated if all one-sided limits exist. A function α from $[a, b]$ to the space of all bounded linear transformations from X to a Banach space Y is an integrator for the regulated functions if, for each regulated function f , the Riemann-Stieltjes sums of f , with sampling points from the interiors of subintervals, converge to a vector in Y . When X and Y are Banach algebra, we give a complete description of the class of all integrators that induce Banach algebra homomorphisms. The main result of L. Fernandes and R. Arbach [Ann. Funct. Anal. 3 (2012), no. 2, 21-31] exhibit a special subclass of our characterization.

1. INTRODUCTION

A function f on a closed interval $[a, b]$, taking values in a Banach space X , is said to be regulated if the all one-sided limits exist. The class of regulated functions is much wider than the class of continuous functions. A function α on $[a, b]$ taking values in the space $\mathfrak{B}(X, Y)$ of bounded linear operators from X to a Banach space Y is an *integrator* if, for each regulated function f , there is a vector in Y to which the Riemann-Stieltjes sums of f , with sampling points from the interiors of subintervals, with respect to α converge. An operator valued function α is an integrator if and only if α is of bounded semivariation [?]. When the Banach spaces X and Y are replaced by Banach algebras $\mathcal{A}$ and $\mathcal{B}$, a natural question to ask is when is the operator induced by the operator-valued integrator is multiplicative. The main result of L. Fernandes and R. Arbach [?] give a class of such examples. We shall give a complete description of all such integrators.

2. NOTATIONS AND PRELIMINARIES

Fix real numbers $a < b$ and a Banach space X . A function $f : [a, b] \rightarrow X$ is said to be *regulated* (see [?, §7.6, p.139] and [?, p. 16]) if

$$\begin{aligned} f(c^+) &:= \lim_{t \rightarrow c^+} f(t) \quad \text{and} \quad f(c^-) := \lim_{t \rightarrow c^-} f(t) \quad \text{exist for all } c \in (a, b), \quad \text{and} \\ f(a^+) &:= \lim_{t \rightarrow a^+} f(t) \quad \text{and} \quad f(b^-) := \lim_{t \rightarrow b^-} f(t) \quad \text{exist.} \end{aligned}$$

Denote by $\text{Reg}([a, b], X) = \text{Reg}(X)$ the space of regulated functions on $[a, b]$ (taking values in X).

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A partition P of the interval $[a, b]$ is given by a finite number of *division points*, t_j , in $[a, b]$:

$$P \left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b \right), \quad n(P) \in \mathbb{N}.$$

The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$ or simply $\mathcal{P}$, whenever no confusion arises.

A function $g : [a, b] \rightarrow X$ is a *step function* if there are a partition

$$P \left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b \right)$$

and vectors $x_j \in X$, $1 \leq j \leq n(P)$ such that $g(t) = x_j$ for all $t \in (t_{j-1}, t_j)$, $1 \leq j \leq n(P)$; i.e., g takes constant values on each open subinterval in the partition. Since $n(P)$ is determined by g , we also denote $n(P)$ by $n(g)$ in this case.

Theorem 1. [?, Th 7.6.1, p.139] *A function $f : [a, b] \rightarrow X$ is regulated iff there exists a sequence $\{h_n\}$ of step functions such that $h_n \rightarrow f$ uniformly on $[a, b]$. Furthermore, if $f \in \text{Reg}(X)$, then*

$$\|f\| := \sup_{t \in [a, b]} \|f(t)\| < \infty,$$

and $(\text{Reg}(X), \|\cdot\|)$ is a Banach space.

When X is a Banach algebra $\mathcal{A}$ with identity $\mathbb{1}_{\mathcal{A}}$, it is readily seen that $\text{Reg}(\mathcal{A})$ is also a Banach algebra with the pointwise product of regulated functions and multiplicative identity the function $\mathbb{1}_{\text{Reg}(\mathcal{A})}$ taking the value $\mathbb{1}_{\mathcal{A}}$ at every point in $[a, b]$ ([?, Th. 2.3]).

Let X, Y be Banach spaces. Denote by $\mathfrak{B}(X, Y)$ the space of bounded linear transformations from X to Y . An operator-valued function $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ is said to be an *integrator* for the regulated functions if for each $f \in \text{Reg}(X)$, there is a $y \in Y$ that satisfies the following condition:

$$\begin{aligned} (\ddagger) \text{ for every } \varepsilon > 0 \text{ there is a partition } P_0 \left(a = s_0 < s_1 < s_2 < \cdots < s_{n(P_0)} = b \right) \\ \text{such that for all partitions } P \left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b \right) \text{ refining } P_0 : \\ \{s_k : 1 \leq k \leq n(P_0)\} \subseteq \{t_j : 1 \leq j \leq n(P)\} \quad \text{and} \\ \text{for all } t_j^* \in (t_{j-1}, t_j), 1 \leq j \leq n(P), \\ \left\| y - \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) \right\| < \varepsilon. \end{aligned}$$

If α is an integrator for regulated functions, the vector y to which α “integrates” f is called the *interior integral* (or *Dushnik integral*) of f and is denoted by

$$\int_a^b [d\alpha(t)](f(t)) = y.$$

We gather some basic facts about integrators, that can be found in [?].

Theorem 2. *An operator-valued function $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ is an integrator if and only if α is of bounded semivariation in the following sense:*

there exists $M > 0$ such that

for all partitions $P(a = t_0 < t_1 < \dots < t_{n(P)} = b) \in \mathcal{P}[a, b]$, and
for all $x_k \in [X]_1$ (the closed unit ball of X), $1 \leq k \leq n(P)$,

$$\left\| \sum_{k=1}^{n(P)} [\alpha(t_k) - \alpha(t_{k-1})] x_k \right\| \leq M.$$

Denote by $\mathcal{B}_{sv}([a, b], \mathfrak{B}(X, Y))$ (or simply $\mathcal{B}_{sv}$ whenever no ambiguity, i.e., only one interval $[a, b]$ and two fixed spaces X and Y in sight) the space of all functions of bounded semivariation from $[a, b]$ to $\mathfrak{B}(X, Y)$.

The *semivariation* of a function $\alpha \in \mathcal{B}_{sv}([a, b], \mathfrak{B}(X, Y))$ is given by

$$V_s(\alpha) := \sup \left\{ \left\| \sum_{k=1}^{n(P)} [\alpha(t_k) - \alpha(t_{k-1})] x_k \right\| : P(a = t_0 < t_1 < \dots < t_{n(P)} = b) \in \mathcal{P}; \right. \\ \left. x_k \in [X]_1, 1 \leq k \leq n(P) \right\}.$$

Lemma 3. *Let $g : [a, b] \rightarrow X$ be a step function and let $\alpha : [a, b] \rightarrow \mathfrak{B}(X, Y)$ be of bounded semivariation. Then there exists a $y \in Y$ with the following property. There exists a partition $P_0(a = u_0 < u_1 < \dots < u_{n(P_0)})$ such that for all partitions*

$P(a = t_0 < t_1 < t_2 < \dots < t_{n(P)} = b)$ satisfying
 $\{u_k : 1 \leq k \leq n(P_0)\} \subseteq \{t_j : 1 \leq j \leq n(P)\}$, and all choices of
 $t_j^ \in (t_{j-1}, t_j)$, $1 \leq j \leq n(P)$,*

$$y = \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](g(t_j^*)).$$

3. INTEGRATORS AS MULTIPLICATIVE MAPS

Let $\mathcal{A}, \mathcal{B}$ be Banach algebras with identities $\mathbb{1}_{\mathcal{A}}, \mathbb{1}_{\mathcal{B}}$ respectively (over the complex field $\mathbb{C}$). A routine verification reveals that $\text{Reg}(\mathcal{A})$ is also a Banach algebra with identity $\mathbb{1}_R$ the constant function $\mathbb{1}_{\text{Reg}(\mathcal{A})}(t) = \mathbb{1}_{\mathcal{A}}$, for all $t \in [a, b]$ ([?, Th. 2.3]). A natural question is: for which integrator $\alpha : [a, b] \rightarrow \mathfrak{B}(\mathcal{A}, \mathcal{B})$, is $\hat{\alpha}$ a Banach algebra homomorphism? In particular: when is

$$\int_a^b [d\alpha(t)]((fg)(t)) = \hat{\alpha}(fg) = \hat{\alpha}(f)\hat{\alpha}(g) \\ = \left[\int_a^b [d\alpha(t)](f(t)) \right] \left[\int_a^b [d\alpha(t)](g(t)) \right], \quad \forall f, g \in \text{Reg}(\mathcal{A})?$$

Towards a characterization of all integrator-induced homomorphisms from $\text{Reg}(\mathcal{A})$ to $\mathcal{B}$, we will need the following simple fact.

Lemma 4. *If $S \neq T$ in $\mathfrak{B}(\mathcal{A}, \mathcal{B})$ are distinct Banach algebra homomorphisms, then $S - T$ is not multiplicative.*

Proof. Since $S - T \neq 0$, there is an $x \in \mathcal{A}$ such that $(S - T)x \neq 0$.

$$\begin{aligned} (S - T)(\mathbb{1}_{\mathcal{A}} \cdot x) &= (S - T)(x) \neq 0 & \text{whereas} \\ [(S - T)(\mathbb{1}_{\mathcal{A}})][(S - T)x] &= (\mathbb{1}_{\mathcal{B}} - \mathbb{1}_{\mathcal{B}})((S - T)x) = 0(Sx - Tx) = 0. \end{aligned} \quad \square$$

The main theorem in [?] states that if α has its value a fixed homomorphism on an interval $(c, b]$ and 0 elsewhere then it induces a homomorphism on $\text{Reg}(\mathcal{A})$. The collection of all such integrators have the following full descriptions.

Theorem 5. *Let $\mathcal{A}, \mathcal{B}$, and α be as above. Then $\hat{\alpha}$ is a homomorphism from $\text{Reg}(\mathcal{A})$ to $\mathcal{B}$ if, and only if there exist $c \in [a, b]$, $S, T \in \mathfrak{B}(\mathcal{A}, \mathcal{B})$ such that T is a homomorphism and*

$$\begin{aligned} \alpha(t) &= \begin{cases} S & \text{for } a \leq t \leq c \\ S + T & \text{for } c < t \leq b \end{cases} & \text{or} \\ \alpha(t) &= \begin{cases} S & \text{for } a \leq t < c \\ S + T & \text{for } c \leq t \leq b \end{cases} \end{aligned}$$

i.e., there are $a < c \in [a, b]$, a homomorphism T and a bounded operator S from $\mathcal{A}$ to $\mathcal{B}$ such that $\alpha = S + \chi_{(c, b]}T$ or $\alpha = S + \chi_{[c, b]}T$.

We denote the characteristic function of a set J (restricted to $[a, b]$) by χ_J .

Proof. [$\Rightarrow$] Since $\hat{\alpha}(\mathbb{1}_{\text{Reg}(\mathcal{A})}) = \mathbb{1}_{\mathcal{B}}$ which is nonzero by axioms of Banach algebra, $\hat{\alpha} \neq 0$. By assumption $\hat{\alpha}$ is also multiplicative. Let $S := \alpha(0)$. Then α and $\alpha - S$ define the same integrator from $\text{Reg}(\mathcal{A})$ to $\mathcal{B}$. Replacing α by $\alpha - \alpha(0)$, we may and shall assume that $\alpha(a) = 0$.

We then have

$$[\alpha(b)](\mathbb{1}_{\mathcal{A}}) = [\alpha(b) - \alpha(a)](\mathbb{1}_{\mathcal{A}}) = \int_a^b [d\alpha(t)](\mathbb{1}_{\text{Reg}(\mathcal{A})}(t)) = \hat{\alpha}(\mathbb{1}_{\text{Reg}(\mathcal{A})}) = \mathbb{1}_{\mathcal{B}}$$

Claim 1: $\alpha(s)$ is multiplicative for all $s \in (a, b]$.

Let $x, y \in \mathcal{A}$. Let $f = \chi_{[a, s]}x$, $g = \chi_{[a, s]}y$. Then $f, g \in \text{Reg}(\mathcal{A})$, $fg = \chi_{[a, s]}xy \in \text{Reg}(\mathcal{A})$, and, by Lemma 3,

$$\begin{aligned} \hat{\alpha}(f) &= \int_a^b [d\alpha(t)](f(t)) = [\alpha(s) - \alpha(a)]x = [\alpha(s)]x, \\ \hat{\alpha}(g) &= \int_a^b [d\alpha(t)](g(t)) = [\alpha(s)]y, & \text{and} \\ \hat{\alpha}(fg) &= \int_a^b [d\alpha(t)]((fg)(t)) = [\alpha(s)](xy). \end{aligned}$$

Thus

$$[\alpha(s)](xy) = \widehat{\alpha}(fg) = (\widehat{\alpha}(f))(\widehat{\alpha}(g)) = ([\alpha(s)]x)([\alpha(s)]y).$$

That is $[\alpha(s)](xy) = ([\alpha(s)]x)([\alpha(s)]y)$. Since $x, y \in \mathcal{A}$ are arbitrary, $\widehat{\alpha}(s)$ is multiplicative. In particular, with $s = b$, we have $T := \alpha(b)$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Claim 2: for all $u < v$ in $[a, b]$, $\alpha(v) - \alpha(u)$ is multiplicative.

Let $x, y \in \mathcal{A}$. Set $f = \chi_{[u,v]}x$, $g = \chi_{[u,v]}y$. Then, $f, g, fg = \chi_{[u,v]}(xy) \in \text{Reg}(\mathcal{A})$, and

$$\widehat{\alpha}(f) = \int_a^b [d\alpha(t)](f(t)) = [\alpha(v) - \alpha(u)]x,$$

$$\widehat{\alpha}(g) = \int_a^b [d\alpha(t)](g(t)) = [\alpha(v) - \alpha(u)]x, \quad \text{and}$$

$$\widehat{\alpha}(fg) = \int_a^b [d\alpha(t)]((fg)(t)) = [\alpha(v) - \alpha(u)](xy).$$

Thus, since $\widehat{\alpha}$ is assumed to be multiplicative,

$$\begin{aligned} [\alpha(v) - \alpha(u)](xy) &= \widehat{\alpha}(fg) = [\widehat{\alpha}(f)][\widehat{\alpha}(g)] \\ &= [[\alpha(v) - \alpha(u)]x][[\alpha(v) - \alpha(u)]y]. \end{aligned}$$

Since $\widehat{\alpha}$ is assumed to be a nonzero homomorphism, $\alpha(t_0) \neq 0$ for some $t_0 \in [a, b]$. Let $c = \inf \{t \in [a, b] : \alpha(t) \neq 0\}$.

Case 1: $c = b$. As noted above, $T := \alpha(b)$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Since $\alpha(t) = 0$ for $t \in [a, b)$, by definition of c , we see that α has the second asserted form with $S = \alpha(a) = 0$ and $c = b$.

Case 2: $c < b$. We claim that α is constant on $(c, b]$. With $T = \alpha(b)$ we show that $\alpha = \chi_{(c,b]}T$ or $\alpha = \chi_{[c,b]}T$.

For $t \in (c, b)$, since $\alpha(b) - \alpha(t)$ is multiplicative by Claim 2, we have $\alpha(b) = \alpha(t)$ by Lemma 4. Thus $\alpha = \chi_{(c,b]}T$ or $\alpha = \chi_{[c,b]}T$. That is α takes one of the asserted forms.

[$\Leftarrow$] Conversely, suppose $\alpha = \chi_{[c,b]}T + S$ or $\alpha = \chi_{(c,b]}T + S$ for some Banach algebra homomorphism T from $\mathcal{A}$ to $\mathcal{B}$ and some bounded linear transformation $S \in \mathfrak{B}(\mathcal{A}, \mathcal{B})$. We show first that

$$\int_a^b [d\alpha(t)](f(t)) = \begin{cases} T(f(c^-)) & \text{if } \alpha = \chi_{[c,b]}T + S \\ T(f(c^+)) & \text{if } \alpha = \chi_{(c,b]}T + S \end{cases} \quad \forall f \in \text{Reg}(\mathcal{A}).$$

Let $f \in \text{Reg}(\mathcal{A})$ and let $\varepsilon > 0$. There exists a $\delta > 0$ such that

$$\begin{aligned} (\forall t \in [a, b]) \left[(c < t < c + \delta) \Rightarrow \left(\|f(c^+) - f(t)\| < \frac{\varepsilon}{\|T\| + 1} \right) \right] \quad \text{and} \\ (\forall t \in [a, b]) \left[(c - \delta < t < c) \Rightarrow \left(\|f(c^-) - f(t)\| < \frac{\varepsilon}{\|T\| + 1} \right) \right] \end{aligned}$$

Arbitrarily pick $c_1 \in (c - \delta, c)$ and $c_2 \in (c, c + \delta)$. Let

$$P(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b) \text{ be a partition of } [a, b] \text{ with}$$

$$\{c, c_1, c_2\} \subseteq \{t_1, \dots, t_{n(P)}\}.$$

To show that P is the P_ε in the definition of the integral, let

$$Q(a = s_0 < s_1 < s_2 < \cdots < s_{n(Q)} = b) \text{ a partition such that}$$

$$\{s_1, \dots, s_{n(Q)}\} \supseteq \{t_1, \dots, t_{n(P)}\} \quad \text{and} \quad s_j^* \in (s_{j-1}, s_j), \quad 1 \leq j \leq n(Q).$$

Then there is a $1 \leq j_0 < n(Q)$ with $t_{j_0} = c$. Thus $c_1 \leq t_{j_0-1}$ and $t_{j_0+1} \leq c_2$, and hence $[\alpha(s_j) - \alpha(s_{j-1})] = 0$ for all $j < j_0$, and all $j > j_0 + 2$. If $\alpha = \chi_{[c,b]}T + S$, we also have $\alpha(s_{j_0+1}) - \alpha(s_{j_0}) = 0$. Since $c - \delta < c_1 \leq t_{j_0-1} < s_{j_0}^* < s_{j_0} = c$, we have

$$\|f(c^-) - f(s_{j_0}^*)\| < \frac{\varepsilon}{\|T\|+1}, \text{ and hence}$$

$$\begin{aligned} & \left\| T(f(c^-)) - \sum_{j=1}^{n(Q)} [\alpha(s_j) - \alpha(s_{j-1})](f(s_j^*)) \right\| \\ &= \left\| T(f(c^-)) - [\alpha(s_{j_0}) - \alpha(s_{j_0-1})](f(s_{j_0}^*)) \right\| \\ &= \left\| T(f(c^-)) - T(f(s_{j_0}^*)) \right\| \leq \|T\| \|f(c^-) - f(s_{j_0}^*)\| < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have proved that $\int_a^b [d\alpha(t)](f(t)) = T(f(c^-))$, if $\alpha = \chi_{[c,b]}T + S$.

Likewise, if $\alpha = \chi_{(c,b]}T + S$, then $\alpha(s_{j_0}) - \alpha(s_{j_0-1}) = 0$. Since $c = s_{j_0} < s_{j_0+1}^* < s_{j_0+1} \leq c_2 < c + \delta$, we have $\|f(c^+) - f(s_{j_0+1}^*)\| < \frac{\varepsilon}{\|T\|+1}$, and hence

$$\begin{aligned} & \left\| T(f(c^+)) - \sum_{j=1}^{n(Q)} [\alpha(s_j) - \alpha(s_{j-1})](f(s_j^*)) \right\| \\ &= \left\| T(f(c^+)) - [\alpha(s_{j_0+1}) - \alpha(s_{j_0})](f(s_{j_0+1}^*)) \right\| \\ &= \left\| T(f(c^+)) - T(f(s_{j_0+1}^*)) \right\| \leq \|T\| \|f(c^+) - f(s_{j_0+1}^*)\| < \varepsilon. \end{aligned}$$

Again by the arbitrariness of ε , we have proved that $\int_a^b [d\alpha(t)](f(t)) = T(f(c^+))$, if $\alpha = \chi_{(c,b]}T + S$.

To see that $\widehat{\alpha}$ is multiplicative, let $f, g \in \text{Reg}(\mathcal{A})$. Then by what we just proved above,

$$\widehat{\alpha}(fg) = \int_a^b [d\alpha(t)]((fg)(t)) = \begin{cases} T((fg)(c^-)) & \text{for } \alpha = \chi_{[c,b]}T + S \\ T((fg)(c^+)) & \text{for } \alpha = \chi_{(c,b]}T + S \end{cases}$$

$$\widehat{\alpha}(f) = \int_a^b [d\alpha(t)](f(t)) = \begin{cases} T(f(c^-)) & \text{for } \alpha = \chi_{[c,b]}T + S \\ T(f(c^+)) & \text{for } \alpha = \chi_{(c,b]}T + S \end{cases}$$

$$\widehat{\alpha}(g) = \int_a^b [d\alpha(t)](g(t)) = \begin{cases} T(g(c^-)) & \text{for } \alpha = \chi_{[c,b]}T + S \\ T(g(c^+)) & \text{for } \alpha = \chi_{(c,b]}T + S \end{cases}$$

It follows from the continuity of the multiplication on $\mathcal{A}$ that $(fg)(c^-) = f(c^-)g(c^-)$, and $(fg)(c^+) = f(c^+)g(c^+)$. Since T is multiplicative, we have

$$\begin{aligned} T((fg)(c^+)) &= T([f(c^+)]g(c^+)) = [T(f(c^+))][T(g(c^+))] \\ T((fg)(c^-)) &= T([f(c^-)]g(c^-)) = [T(f(c^-))][T(g(c^-))] \end{aligned}$$

and hence

$$\widehat{\alpha}(fg) = [\widehat{\alpha}(f)][\widehat{\alpha}(g)].$$

To see that $\widehat{\alpha}(\mathbb{1}_{\text{Reg}(\mathcal{A})}) = \mathbb{1}_{\mathcal{B}}$, note that $\mathbb{1}_{\text{Reg}(\mathcal{A})}(t^\pm) = \mathbb{1}_{\mathcal{A}}$ for all $t \in (a, b)$.

$$\begin{aligned} \widehat{\alpha}(\mathbb{1}_{\text{Reg}(\mathcal{A})}) &= \int_a^b [d\alpha(t)](\mathbb{1}_{\text{Reg}(\mathcal{A})}(t)) = \begin{cases} T((\mathbb{1}_{\text{Reg}(\mathcal{A})})(c^-)) & \text{for } \alpha = \chi_{[c,b]}T + S \\ T((\mathbb{1}_{\text{Reg}(\mathcal{A})})(c^+)) & \text{for } \alpha = \chi_{(c,b]}T + S \end{cases} \\ &= T(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}. \end{aligned} \quad \square$$

Now suppose $\mathcal{A}$ and $\mathcal{B}$ are C^* -algebras with identities $\mathbb{1}_{\mathcal{A}}$ and $\mathbb{1}_{\mathcal{B}}$. Then $\text{Reg}(\mathcal{A})$ is a C^* -algebra with identity $\mathbb{1}_{\text{Reg}(\mathcal{A})}(t) = \mathbb{1}_{\mathcal{A}}$ for all $t \in [a, b]$. The involution on $\text{Reg}(\mathcal{A})$ is defined pointwise: for each $f \in \text{Reg}(\mathcal{A})$, and each $t \in [a, b]$, $f^*(t) = (f(t))^*$.

Then a routine verification reveals that $\text{Reg}(\mathcal{A})$ is a C^* -algebra. Since a $*$ -homomorphism from a C^* -algebra to another is just an algebra homomorphism that also preserves the adjoint, a direct adaptation of our proof yields the following.

Theorem 6. *Let $\alpha : [a, b] \rightarrow \mathfrak{B}(\mathcal{A}, \mathcal{B})$ be an integrator for $\text{Reg}(\mathcal{A})$. Then $\widehat{\alpha}$ is a $*$ -homomorphism from $\text{Reg}(\mathcal{A})$ to $\mathcal{B}$ if, and only if there exist $c \in [a, b]$, $S, T \in \mathfrak{B}(\mathcal{A}, \mathcal{B})$ such that T is a $*$ -homomorphism and*

$$\alpha(t) = \begin{cases} S & \text{for } a \leq t \leq c \\ S+T & \text{for } c < t \leq b \end{cases} \quad \text{or}$$

$$\alpha(t) = \begin{cases} S & \text{for } a \leq t < c \\ S+T & \text{for } c \leq t \leq b \end{cases}$$

i.e., there are a $c \in [a, b)$, a $$ -homomorphism T and a bounded operator S from $\mathcal{A}$ to $\mathcal{B}$ such that $\alpha = S + \chi_{(c, b]}T$ or $\alpha = S + \chi_{[c, b]}T$.*

REFERENCES

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER, FACULTY OF SCIENCE, UBON RATCHATHANI UNIVERSITY, UBON RATCHATHANI 34190, THAILAND

E-mail address: sctitawo@ubu.ac.th, ma_pintto@yahoo.com

DEPARTMENT OF MATHEMATICS, CENTRAL MICHIGAN UNIVERSITY, MOUNT PLEASANT, MI 48859, USA

E-mail address: ong1s@cmich.edu, ong3pf@gmail.com