รายงานวิจัยฉบับสมบูรณ์

โครงการ วิธีการใหม่ของการควบคุมเชิงทำนายแบบจำลองคงทนที่มีรากฐานเป็นท่อแบบไม่เชื่อมตรงสำหรับกระบวนการเกิดพอลิเมอร์ที่มีความไม่แน่นอน (TRG5880084)

โดย ผู้ช่วยศาสตราจารย์ ดร.พรชัย บำรุงศรี
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สัญญาเลขที่ TRG5880084

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มหาวิทยาลัยมหิดล

สารบัญ

บทคัดย่อ

รหัสโครงการ: TRG5880084

ชื่อโครงการ: วิธีการใหม่ของการควบคุมเชิงทำนายแบบจำลองคงทนที่มีรากฐานเป็นท่อแบบไม่เชื่อมตรงสำหรับกระบวนการเกิดพอลิเมอร์ที่มีความไม่แน่นอน

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งานวิจัยนี้ได้ทำการพัฒนาวิธีใหม่ของการควบคุมโดยจำกัดเส้นทางเดินทั้งหมดของระบบที่มีความไม่แน่นอนให้อยู่ในลำดับของท่อดังนั้นจึงสามารถรับประกันเสถียรภาพความคงทนและข้อจำกัดของระบบได้ในกรณีที่เกิดตัวแปรซึ่งมีความไม่แน่นอนและตัวแปรรบกวน การคำนวณหาค่าเหมาะสมที่สุดจะดำเนินการก่อนการควบคุมดังนั้นจึงสามารถนำวิธีการควบคุมที่พัฒนาขึ้นไปใช้ในการควบคุมระบบที่มีพลวัตรวดเร็วได้ วิธีการควบคุมที่พัฒนาขึ้นได้มีการนำไปประยุกต์ใช้กับปัญหาการควบคุมกระบวนการพอลิเมอไรเซชันของพอลิโพรพิลีนที่มีความไม่แน่นอนซึ่งปฏิกิริยาพอลิเมอไรเซชันที่เกิดขึ้นมีความรวดเร็วและคายความร้อนสูง ในกรณีที่ค่าคงที่ของจลนพลศาสตร์และค่าความร้อนของปฏิกิริยามีความไม่แน่นอนจะพบว่าวิธีการควบคุมที่พัฒนาขึ้นสามารถควบคุมตัวแปรควบคุมซึ่งคือมวลของพอลิเมอร์และอุณหภูมิภายในถังปฏิกรณ์ให้เข้าสู่ค่าเป้าหมายที่ต้องการได้ วิธีการที่พัฒนาขึ้นสามารถจัดการกับตัวแปรที่มีความไม่แน่นอนและตัวแปรรบกวนดังนั้นจึงสามารถควบคุมกระบวนการพอลิเมอไรเซชันที่มีความไม่แน่นอนได้อย่างมีประสิทธิภาพ ในขั้นตอนสุดท้ายได้มีการนำวิธีการควบคุมที่พัฒนาขึ้นไปประยุกต์ใช้กับการควบคุมเครื่องปฏิกรณ์ที่มีการจำลองการคายความร้อนบางส่วนซึ่งปฏิบัติการแบบต่อเนื่อง ในกรณีที่เกิดตัวแปรซึ่งมีความไม่แน่นอน เช่น ค่าคงที่ของอัตราการเกิดปฏิกิริยา ความร้อนของปฏิกิริยา อุณหภูมิของน้ำหล่อเย็น และอัตราการป้อนของเครื่องปฏิกรณ์ จะพบว่าวิธีการควบคุมที่พัฒนาขึ้นสามารถควบคุมอุณหภูมิของเครื่องปฏิกรณ์ให้เข้าสู่ค่าเป้าหมายได้ซึ่งเป็นการรับประกันเสถียรภาพความคงทนของระบบ

คำหลัก: การควบคุมเชิงทำนายแบบจำลองคงทนที่มีรากฐานเป็นท่อแบบไม่เชื่อมตรง กระบวนการพอลิเมอไรเซชันที่มีความไม่แน่นอน ตัวแปรที่มีความไม่แน่นอน เครื่องปฏิกรณ์ที่มีการจำลองการคายความร้อนบางส่วน

Abstract

Project Code: TRG5880084

Project Title: A novel off-line tube-based robust model predictive control algorithm for uncertain polymerization processes

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Project Period: July 2015 to June 2017 (2 years)

This research develops a novel off-line tube-based robust model predictive control algorithm. All trajectories of uncertain systems are restricted to lie in a sequence of tubes so robust stability and constraint satisfaction can be guaranteed in the presence of both uncertain parameters and disturbances. All of the optimization problems are solved off-line so the developed algorithm is applicable to fast dynamic systems. The developed control algorithm is applied to the control problem of uncertain polymerization process for polypropylene where the polymerization reactions taking place are fast and highly exothermic. In the case when the kinetic constant for propagation rate and the heat of reaction are uncertain, the results show that the developed control algorithm is able to regulate the controlled variables, which are the mass of polymer in the reactor and the reactor temperature, to the desired set points. The developed algorithm can handle both uncertain parameters and disturbances so the uncertain polymerization process can be efficiently controlled. Finally, the developed control algorithm is applied to a partially simulated exothermic (PARSEX) reactor operated in the continuous mode. In the presence of uncertain parameters such as the reaction rate constant, heat of reaction, cooling water temperature and reactor feed rate, the results show that the reactor temperature can be regulated to the set point so robust stability of the system is ensured.

Keywords: Off-line tube-based robust model predictive control; uncertain polymerization process; uncertain parameters; partially simulated exothermic reactor

1. Executive summary

The control of systems in the presence of uncertain parameters and disturbances is a challenging control problem because it is difficult to guarantee both robust stability and constraint satisfaction. In this research, a novel off-line tube-based robust model predictive control algorithm is developed. The trajectories of uncertain systems are restricted to lie in a sequence of tubes so robust stability and constraint satisfaction can be guaranteed in the presence of both uncertain parameters and disturbances. All of the optimization problems are solved off-line so the developed algorithm is applicable to fast dynamic systems.

In order to demonstrate the applications of the developed algorithm, it is applied to a difficult control problem of uncertain polymerization process for polypropylene where the exact values of reaction rate constants are unknown. The polymerization reactions taking place are fast and highly exothermic so the presence of uncertain parameters might lead to an unexpected thermal runaway. In the case when the kinetic constant for propagation rate and the heat of reaction are uncertain, the results show that the developed control algorithm is able to regulate the controlled variables, which are the mass of polymer in the reactor and the reactor temperature, to the desired set point by manipulating the mass flow rate of propylene monomer and the mass flow rate of cooling water, respectively. Additionally, in the presence of the disturbances acting on the system, the developed control algorithm is able to regulate the mass of polymer in the reactor and the reactor temperature to the neighborhood of the desired set point so robust stability of the system can be guaranteed. The developed algorithm can handle both uncertain parameters and disturbances so the uncertain polymerization process can be efficiently controlled.

Finally, the developed control algorithm is applied to a partially simulated exothermic (PARSEX) reactor operated in the continuous mode. In the presence of uncertain parameters such as the reaction rate constant, heat of reaction, cooling water temperature and reactor feed rate, the results show that the reactor temperature can be regulated to the set point so robust stability of the system is ensured.

2. Objectives of this research

(1) To develop a novel off-line tube-based robust MPC algorithm that can guarantee robust stability and constraint satisfaction of the control systems in the presence of both uncertain parameters and disturbances. Additionally, all of the optimization problems are solved off-line so the developed off-line tube-based robust MPC algorithm is applicable to fast dynamic processes.

(2) To apply the developed off-line tube-based robust MPC algorithm to a challenging control problem of uncertain polypropylene polymerization process where the reaction rate constants are uncertain and the polymerization reactions taking place are fast and highly exothermic.

(3) To apply the developed off-line tube-based robust MPC algorithm to a partially simulated exothermic pilot plant reactor where the reaction heat of various uncertain polymerization reactions can be generated. Therefore, the effects of uncertainties and disturbances in polymerization processes can be investigated and controlled under realistic situations.

3. Research Methodology

In this project, a novel off-line tube-based robust MPC is developed. The proposed tube-based robust MPC algorithm can handle both uncertain parameters and disturbances. Moreover, all of the optimization problems are solved off-line so the proposed tube-based robust MPC algorithm is applicable to fast exothermic processes. In order to illustrate its effectiveness, the proposed tube-based robust MPC algorithm will be applied to a simulation case study of polypropylene polymerization process with uncertain and highly exothermic polymerization reactions. Then, the proposed tube-based robust MPC algorithm will be applied to an experimental case study of a partially simulated exothermic (PARSEX) pilot plant reactor where the reaction heat of polymerization processes is generated and controlled by the proposed tube-based robust MPC algorithm. The procedures of this project are shown in Fig. 3.1.

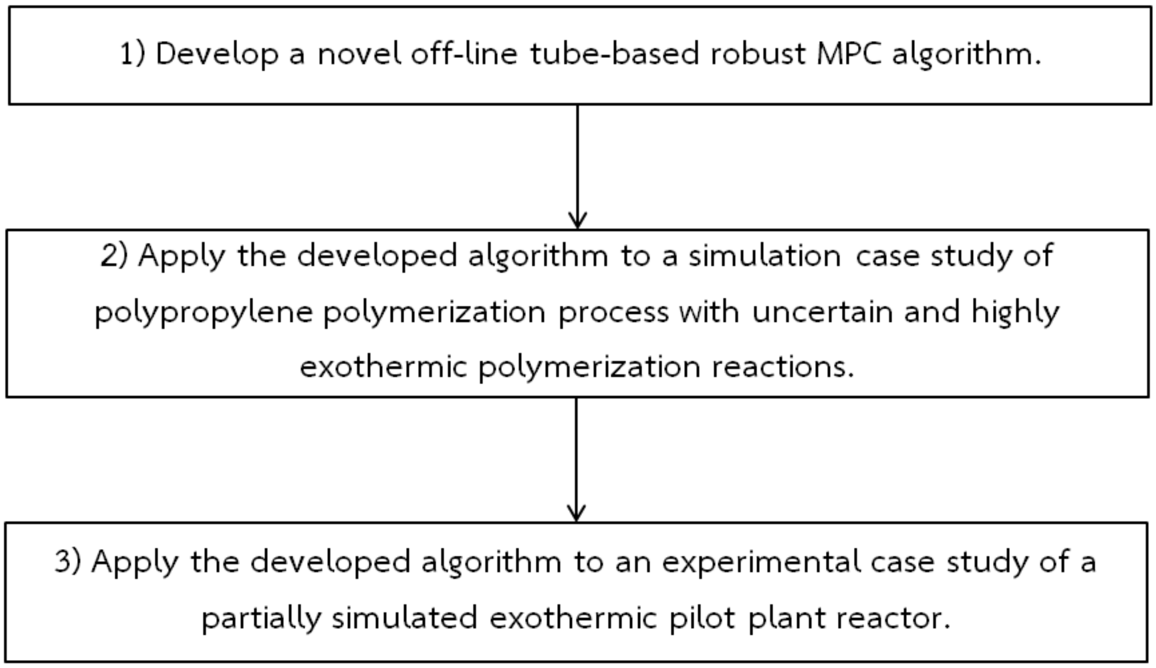


Fig. 3.1 The procedures of this project.

1) Develop a novel off-line tube-based robust MPC algorithm.

In the first experimental step, a novel off-line tube-based robust MPC algorithm is developed. No optimization problem needs to be solved on-line. Additionally, both model uncertainty and disturbance are included in the robust MPC formulation. Consider the following discrete-time system with model uncertainty and disturbance

$$x^{+} = A^{\lambda}x + B^{\lambda}u + w. \tag{3.1}$$

where $x \in \mathbb{R}^{n}$ is the state, $u \in \mathbb{R}^{m}$ is the control input, $w \in \mathbb{R}^{n}$ is the bounded disturbance and $x^{+} \in \mathbb{R}^{n}$ is the successor state. The system is subject to the state constraint $x \in \mathbb{X}$ and the control constraint $u \in \mathbb{U}$ where $\mathbb{X} \subset \mathbb{R}^{n}$ and $\mathbb{U} \subset \mathbb{R}^{m}$ are compact, convex and each set contains the origin as an interior point. The disturbance is bounded, i.e., $w \in \mathbb{W}$ where $\mathbb{W} \subset \mathbb{R}^{n}$ is compact, convex and contains the origin as an interior point. The objective is to robustly stabilize the system (3.1) while all of the constraints are satisfied. The presence of a persistent disturbance w means it is not possible to regulate the state x to the origin. The best that can be hoped for is to regulate the state to a neighborhood of the origin. The matrices A^{λ} and B^{λ} are not constant but vary with an uncertain parameter vector λ. An uncertain parameter vector λ can be measured at each sampling time but its future values are uncertain. We make the following assumption

Assumption 1. $[A^{\lambda}\ B^{\lambda}] \in \mathbf{Conv}\{[A_1\ B_1],\ldots,[A_L\ B_L]\}$ where $\mathbf{Conv}$ denotes the convex hull, $[A_j\ B_j]$ are vertices of the convex hull and L is the number of vertices of the convex hull. Any $[A^{\lambda}\ B^{\lambda}]$ can be written as $[A^{\lambda}\ B^{\lambda}] = \sum_{j=1}^{L} \lambda_j [A_j\ B_j]$ and the pair $[A_j\ B_j]$ is controllable.

Let the nominal system be defined by

$$x'^{+} = A^{\lambda}x' + B^{\lambda}u' \tag{3.2}$$

where $x' \in \mathbb{R}^{n}$ and $u' \in \mathbb{R}^{m}$ are the state and control input of the nominal system, respectively. The predicted state trajectory and control sequence when the initial state is x' are denoted by $\mathbf{x'} := \{x'_0, x'_1, \ldots, x'_N\}$ and $\mathbf{u'} := \{u'_0, u'_1, \ldots, u'_{N-1}\}$, respectively. Consider the following equation which is the difference between the systems (3.1) and (3.2)

$$x^{+} - x'^{+} = A^{\lambda}(x - x') + B^{\lambda}(u - u') + w. \tag{3.3}$$

In order to counteract the effect of disturbance, the control law $u = K(x - x') + u'$ is employed where K is the disturbance rejection gain. The system (3.3) is rewritten as

$$x^{+} - x'^{+} = (A^{\lambda} + B^{\lambda}K)(x - x') + w. \tag{3.4}$$

We will bound $x^{+} - x'^{+}$ by a robust positively invariant set Z. The definition of Z is as follows

Definition 1 The set $Z \subset \mathbb{R}^{n}$ is a robust positively invariant set of an uncertain system with disturbance $x^{+} = A^{\lambda}x + w$ if $A^{\lambda}Z \oplus \mathbb{W} \subseteq Z$ for $\forall x \in Z$, $\forall w \in \mathbb{W}$ and $\forall A^{\lambda} \in \mathrm{Conv}\{A_1, \ldots, A_L\}$ where $\oplus$ denotes the Minkowski set addition (Mayne et al., 2009).

For the system (3.4), it is clear that if K satisfies $(A_j + B_jK)^T P(A_j + B_jK) - P < 0, \forall j \in \{1, \ldots, L\}$ where P is a Lyapunov matrix, then $(A^{\lambda} + B^{\lambda}K)^T P(A^{\lambda} + B^{\lambda}K) - P < 0,\ \ \forall [A^{\lambda}\ B^{\lambda}] \in \mathrm{Conv}\{[A_j\ B_j], \forall j \in 1,2,\ldots,L\}$ and we can bound $x^{+} - x'^{+}$ by a robust positively invariant set Z satisfying $(A^{\lambda} + B^{\lambda}K)Z \oplus \mathbb{W} \subseteq Z$ for $\forall (x - x') \in Z$, $\forall w \in \mathbb{W}$ and $\forall [A^{\lambda}\ B^{\lambda}] \in \mathrm{Conv}\{[A_j\ B_j], \forall j \in 1,2,\ldots,L\}$. It is desirable that Z be as small as possible. The minimal Z of the uncertain system with disturbance (3.4) is $Z = \bigoplus_{i=0}^{\infty} (A^{\lambda} + B^{\lambda} K)^i \mathbb{W}$. Since $[A^{\lambda}\ B^{\lambda}] \in \mathrm{Conv}\{[A_j\ B_j], \forall j \in 1,2,\ldots,L\}$, the minimal Z can be calculated as

$$Z = \mathbb{W} \oplus \mathrm{Conv}\{(A_j + B_j K)\,\mathbb{W}, \forall j \in 1,2,\ldots,L\} \oplus \mathrm{Conv}\{(A_j + B_j K)(A_l + B_l K)\,\mathbb{W},$$
$$\forall j,l \in 1,2,\ldots,L\} \oplus \mathrm{Conv}\{(A_j + B_j K)(A_l + B_l K)(A_m + B_m K)\,\mathbb{W}, \forall j,l,m \in 1,2,\ldots,L\} \oplus \ldots \quad (3.5)$$

Defining $F_s := \bigoplus_{i=0}^{s-1} (A^{\lambda} + B^{\lambda} K)^i \mathbb{W}$, F_s can be properly scaled for some finite integer s to obtain the outer approximation of Z in (3.5) using the method in Raković et al. (2005). Since we can bound $x^+ - x'^+$ by Z, the following proposition can be established

Proposition 1. If $x \in x' \oplus Z$ and $u = K(x - x') + u'$, then $x^+ \in x'^+ \oplus Z$ for $\forall w \in \mathbb{W}$ and $\forall [A^{\lambda}\ B^{\lambda}] \in \mathrm{Conv}\{[A_j\ B_j], \forall j \in 1,2,\ldots,L\}$.

Proposition 1 states that the control law $u = K(x - x') + u'$ keeps the state x of an uncertain system with disturbance $x^+ = A^{\lambda} x + B^{\lambda} u + w$ close to the state x' of the nominal system $x'^+ = A^{\lambda} x' + B^{\lambda} u'$. It is clear that if we can regulate x' to the origin, then x must be regulated to a robust positively invariant set Z whose center is at the origin. An off-line robust MPC algorithm for the nominal system $x'^+ = A^{\lambda} x' + B^{\lambda} u'$ has been developed by Bumroongsri and Kheawhom (2012). The problem of regulating the state x' to the origin has been considered. In this approach, a sequence of stabilizing feedback gains F_i corresponding to a sequence of polyhedral invariant sets $P_i, i = \{1,\ldots,N_P\}$ where N_p is the number of polyhedral invariant sets, is precomputed off-line by solving the optimal control problems subject to LMI constraints (Boyd and Vandenberghe, 2004). At each sampling time, the state x' is measured and the smallest P_i containing x' is determined. Then, we set the real-time stabilizing feedback gain F equal to F_i and apply the control law $u' = Fx'$ to the process. The control law $u' = Fx'$ minimizes the following cost function

$$V_{\infty}(x'_0, \mathbf{u}') := \max_{[A^{\lambda}\ B^{\lambda}] \in \mathrm{Conv}\{[A_j\ B_j], \forall j \in 1,2,\ldots,L\}} \sum_{i=0}^{\infty} x_i'^T Q x'_i + (F x'_i)^T R (F x'_i) \quad (3.6)$$

where $x_i^{'}$ is the state of the nominal system at prediction time i and Q and R are the positive-definite weighting matrices. Additionally, the control law $u' = Fx'$ ensures that the Lyapunov function $V(x') := x'^T Px'$ is a strictly decreasing function satisfying

$$V(x'^{+}) - V(x') \le -x'^T Qx' - (Fx')^T R(Fx'),\ \forall [A^{\lambda}\ B^{\lambda}] \in \mathrm{Conv}\{[A_j\ B_j], \forall j \in 1,2,\ldots,L\} \quad (3.7)$$

where P is a Lyapunov matrix. At each sampling time, although the future values of an uncertain parameter vector λ in the prediction horizon (which is the infinite horizon in this case) are unknown, the satisfaction of (3.7) for the stabilizing feedback gain F ensures that robust stability of the nominal system $x'^{+} = (A^{\lambda} + B^{\lambda}F)x'$ is guaranteed. In order to guarantee satisfaction of the original state and control constraints, $x \in \mathbb{X}$ and $u \in \mathbb{U}$, we must employ tighter constraint sets for the nominal system, i.e., $x' \in \mathbb{X} \ominus Z$ and $Fx' \in \mathbb{U} \ominus KZ$ where $\ominus$ denotes the Minkowski set difference (Mayne et al., 2009). The control law $u = K(x - x') + u'$ is now rewritten as $u = K(x - x') + Fx'$. An important consequence is the following result

Proposition 2. If $x \in x' \oplus Z$, $x' \in \mathbb{X} \ominus Z$ and $Fx' \in \mathbb{U} \ominus KZ$, then the control law $u = K(x - x') + Fx'$ of an uncertain system with disturbance $x^{+} = A^{\lambda}x + B^{\lambda}u + w$ ensures satisfaction of the original constraints $x \in \mathbb{X}$, $u \in \mathbb{U}$ for $\forall w \in \mathbb{W}$ and $\forall [A^{\lambda}\ B^{\lambda}] \in \mathrm{Conv}\{[A_j\ B_j], \forall j \in 1,2,\ldots,L\}$.

Proposition 2 states that the control law $u = K(x - x') + Fx'$ ensures satisfaction of the original state and control constraints. In summary, the proposed off-line tube-based robust MPC for an uncertain system with disturbance $x^{+} = A^{\lambda}x + B^{\lambda}u + w$ can be formulated as follows

Off-line Step 1:
Calculate the disturbance rejection gain K satisfying $(A_j + B_jK)^T P(A_j + B_jK) - P < 0, \forall j \in \{1,\ldots,L\}$. Then, properly scale $F_s := \bigoplus_{i=0}^{s-1} (A^{\lambda} + B^{\lambda}K)^i\, \mathbb{W}$ for some finite integer s to obtain the outer approximation of Z in (3.5) using the method in Raković et al. (2005).

Off-line Step 2:

Calculate a sequence of stabilizing feedback gains F_i and the corresponding sequence of polyhedral invariant sets $P_i, i=\{1,\ldots,N_P\}$ using the method in Bumroongsri and Kheawhom (2012) with tighter constraint sets for the nominal system, i.e., $x' \in \mathbb{X} \ominus Z$ and $Fx' \in \mathbb{U} \ominus KZ$.

On-line:

At the first sampling time $(t=0)$, measure the state x and the uncertain parameter vector λ. Find the smallest polyhedral invariant set P_i containing the measured state x, set $F=F_i$ and apply the control law $u=Fx$ to the process. Then, calculate x'^{+} from $x'^{+}=(A^{\lambda}+B^{\lambda}F)x$ (Note that at the first sampling time, $x=x'$ so the control law $u=K(x-x')+Fx'$ is reduced to $u=Fx$).

At each sampling time $(t>0)$, measure the state x and the uncertain parameter vector λ. Find the smallest polyhedral invariant set P_i containing x' (which is calculated from the previous step), set $F=F_i$ and apply the control law $u=K(x-x')+Fx'$ to the process. Then, calculate x'^{+} from $x'^{+}=(A^{\lambda}+B^{\lambda}F)x'$.

We can now establish our main Theorem as follows

Theorem 1. The proposed tube-based MPC algorithm steers any initial state x of an uncertain system with disturbance $x^{+}=A^{\lambda}x+B^{\lambda}u+w$ in a sequence of polyhedral invariant sets $P_i, i=\{1,\ldots,N_P\}$ to a robust positively invariant set Z whose center is at the origin and thereafter maintains the state in Z for $\forall w \in \mathbb{W}$ and $\forall [A^{\lambda}\ B^{\lambda}] \in \text{Conv}\{[A_j\ B_j], \forall j \in 1,2,\ldots,L\}$.

Proof. Consider the following difference equation between $x^{+}=A^{\lambda}x+B^{\lambda}u+w$ and $x'^{+}=A^{\lambda}x'+B^{\lambda}u'$ where $u=K(x-x')+Fx'$ and $u'=Fx'$,

$$x^{+}-x'^{+}=(A^{\lambda}+B^{\lambda}K)(x-x')+w. \tag{3.8}$$

The disturbance rejection gain K satisfies $(A_j + B_j K)^T P(A_j + B_j K) - P < 0, \forall j \in \{1,\ldots,L\}$ so $x^+ - x'^+$ is bounded by a robust positively invariant set Z, i.e., $x^+ \in x'^+ \oplus Z$. Since the stabilizing feedback gain F ensures that the Lyapunov function is a strictly decreasing function satisfying (3.7), the state x'^+ must converge to the origin. Since $x^+ \in x'^+ \oplus Z$, x^+ must converge to a tube Z whose center is at the origin. Finally, the disturbance rejection controller $u = Kx$ keeps the state within a tube Z whose center is at the origin. □

Example: In this example, an implementation of the proposed off-line tube-based robust MPC algorithm is illustrated. Consider the following uncertain system with bounded disturbance

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w. \tag{3.9}$$

The state $x \in \mathbb{X}$ where $\mathbb{X} := \{x \in \mathbb{R}^2 | [0\ 1]x \le 2\}$, the control $u \in \mathbb{U}$ where $\mathbb{U} := \{u \in \mathbb{R} | |u| \le 1\}$, the disturbance $w \in \mathbb{W}$ where $\mathbb{W} := \{w \in \mathbb{R}^2 | [-0.1\ -0.1]^T \le w \le [0.1\ 0.1]^T\}$ and the uncertain parameter $\lambda \in \mathbb{L}$ where $\mathbb{L} := \{\lambda \in \mathbb{R} | 0.9 \le \lambda \le 1.1\}$. The weighting matrices in the cost function (3.6) are given as $Q = I$ and $R = 0.01$. The following nominal system

$$x'^+ = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} x' + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u' \tag{3.10}$$

is subject to tighter state and control constraints, i.e., $x' \in \mathbb{X} \ominus Z$ and $u' \in \mathbb{U} \ominus KZ$. The disturbance rejection gain $K = [-0.66\ -1.33]$ satisfies $(A_j + B_j K)^T P(A_j + B_j K) - P < 0, \forall j \in \{1,2\}$. The difference equation between (3.9) and (3.10) can be written as

$$x^+ - x'^+ = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} (x - x') + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} (u - u') + w. \tag{3.11}$$

The closed-loop system is simulated using the initial state $x = x' = [-5\ -2]^T$. The uncertain parameter λ and the disturbance w are varied as $\lambda = 1 + 0.1\sin(4k)$ and

$w = [0.1\sin(4k)\ 0.1\sin(4k)]^T$, respectively, where $k \in \{1,\ldots,19\}$ is the simulation horizon.

Figure 3.2 shows a robust positively invariant set Z precomputed off-line. The cross section of Z is shown in yellow. The blue line represents the trajectory of the difference equation (3.11). Starting from the origin, it is seen that the trajectory of the difference equation is restricted to lie within a tube Z.

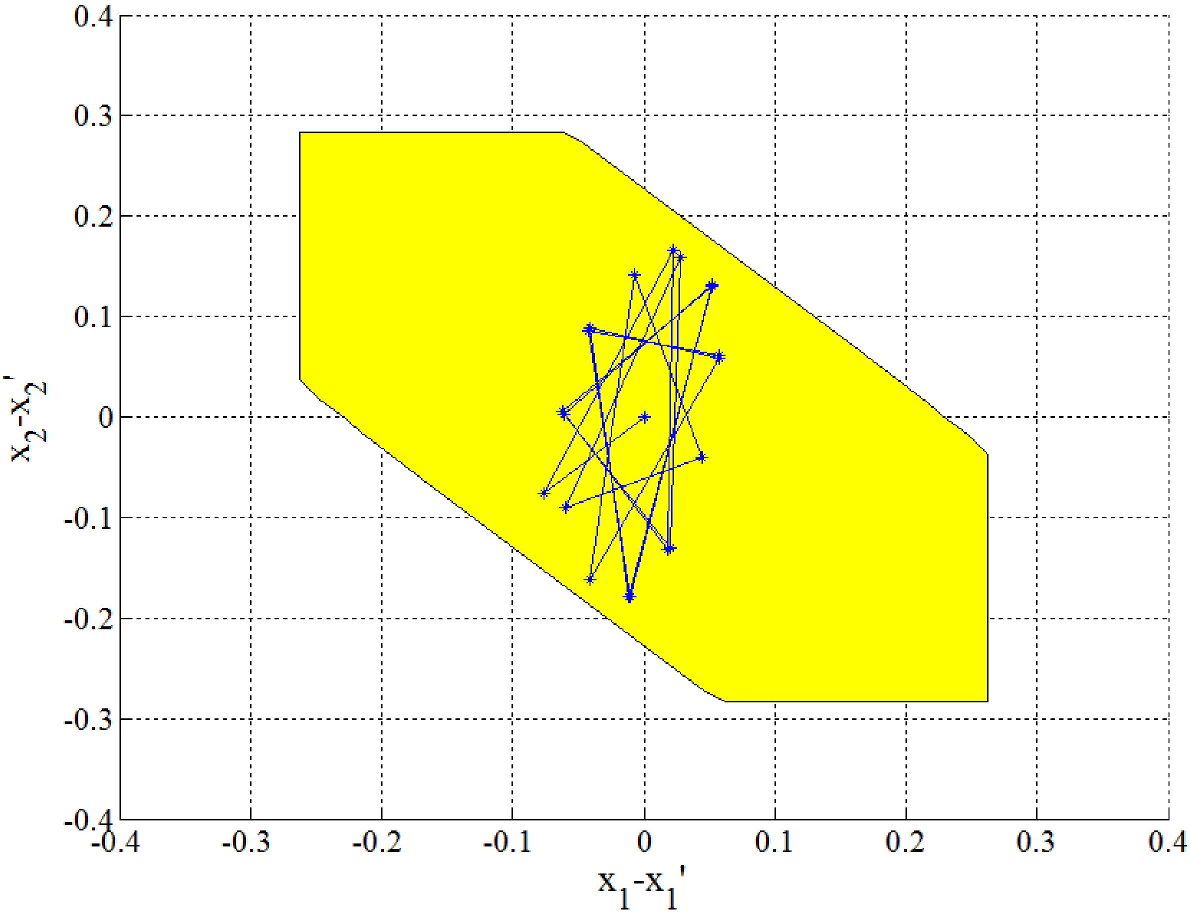


Fig. 3.2. A robust positively invariant set Z precomputed off-line.

Figure 3.3 shows a sequence of ten polyhedral invariant sets $P_i, i \in \{1,\ldots,10\}$ precomputed off-line. In this example, only ten polyhedral invariant sets are precomputed because P_i are almost constant for $i > 10$. The red line represents the trajectory of the nominal system (3.10). Starting from the initial point $x = x' = [-5\ -2]^T$, the state of the nominal system at each time step is restricted to lie within a sequence of ten polyhedral invariant sets $P_i, i \in \{1,\ldots,10\}$ precomputed off-line. Finally, the state of the nominal system converges to the origin.

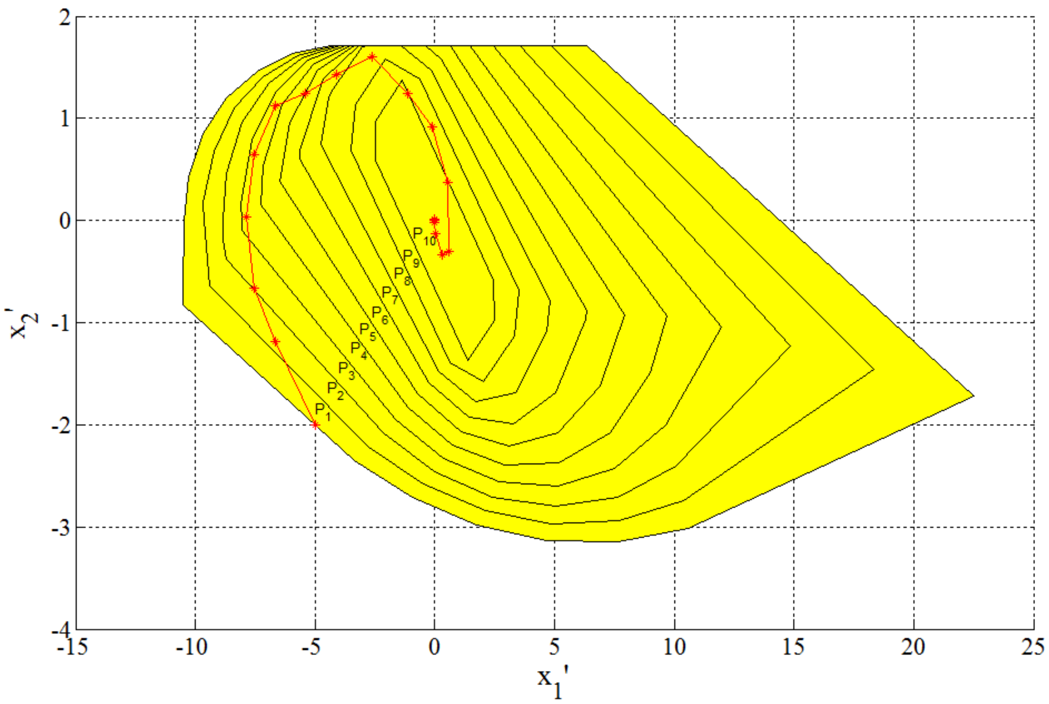


Fig. 3.3 A sequence of ten polyhedral invariant sets $P_i, i \in \{1,\ldots,10\}$ precomputed off-line.

The trajectory of the uncertain system with disturbance (3.9) is shown in Fig. 3.4. The region shown in green is the infeasible region of the state constraint $\mathbb{X} := \{x \in \mathbb{R}^2 \,|\, [0\ 1]x \leq 2\}$. The red line corresponds to the trajectory of the nominal system (3.10). The cross section of a tube Z precomputed off-line is shown in yellow. It is seen that the state of the uncertain system with disturbance at each time step is restricted to lie within a tube Z whose center is the state of the nominal system that converges to the origin. Finally, the state of the uncertain system with disturbance is kept within a tube Z whose center is at the origin.

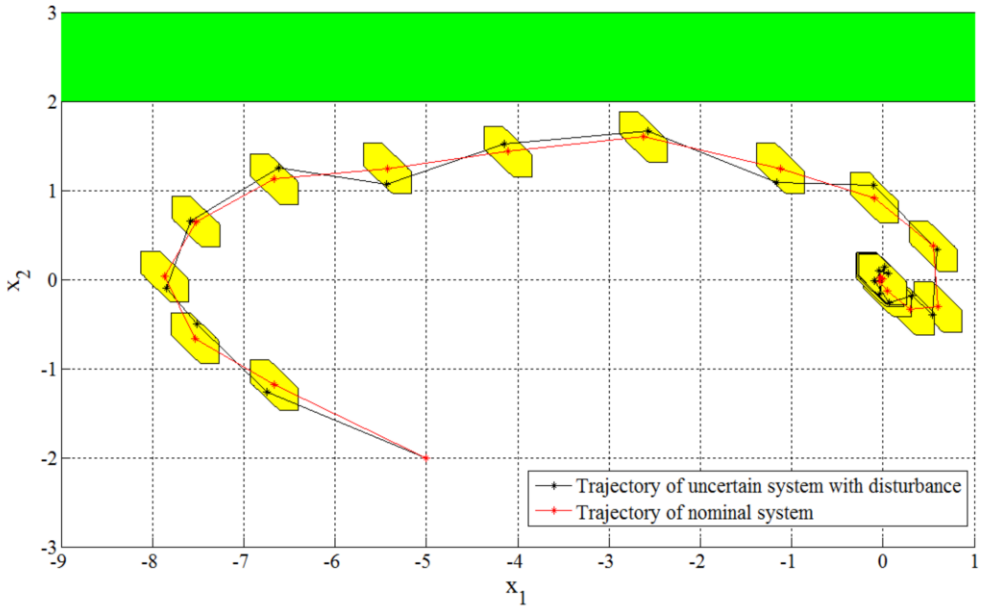


Fig. 3.4. The trajectory of the uncertain system with disturbance.

Figure 3.5 shows the control input as a function of sampling time. The region shown in yellow is $\mathbb{U} \ominus KZ$. The red line corresponds to the control input of the nominal system (3.10). The black line corresponds to the control input of the uncertain system with disturbance (3.9). It can be observed that the control input of the nominal system is restricted to lie within the region $\mathbb{U} \ominus KZ$ so that the control input of the uncertain system with disturbance satisfies the control constraint $\mathbb{U} := \{u \in \mathbb{R} \,|\, |u| \leq 1\}$.

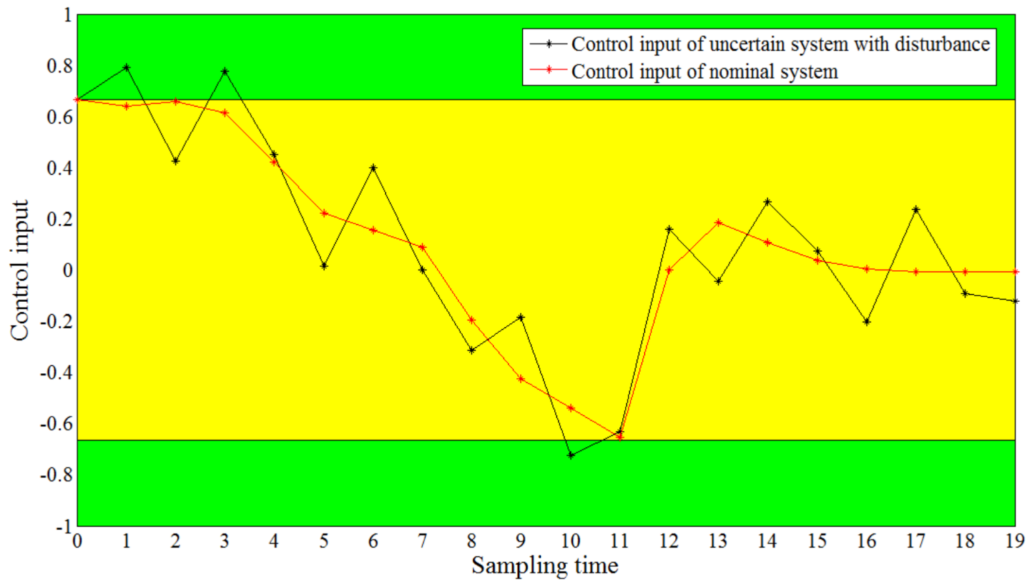


Fig. 3.5. The control input as a function of sampling time.

2) Apply the developed algorithm to a simulation case study of polypropylene polymerization process with uncertain and highly exothermic polymerization reactions.

In the second experimental step, the off-line tube-based robust MPC algorithm developed in the first step is applied to a simulation case study of polypropylene polymerization process. This process is characterized by its high value of reaction heat and uncertainty of reaction rate constants (Seki et al., 2001). Consider the bulk polymerization of polypropylene shown in Fig. 3.6. The process takes place in CSTR at 340 K and 30 atm using high-activity fourth generation Ziegler-Natta catalyst ($TiCl_4/MgCl_2$+p-ethylethoxy-benzoate+triethyl aluminum). The heat of polymerization reaction is removed by condensation of boiling propylene. In the presence of uncertain parameters and disturbances, the objective is to control the reactor temperature (T) and the cumulative mass average molecular weight of polymer (M_w^{ac}) by manipulating the mass flowrate of cooling water (m_w) and the mass flowrate of propylene monomer (m_e).

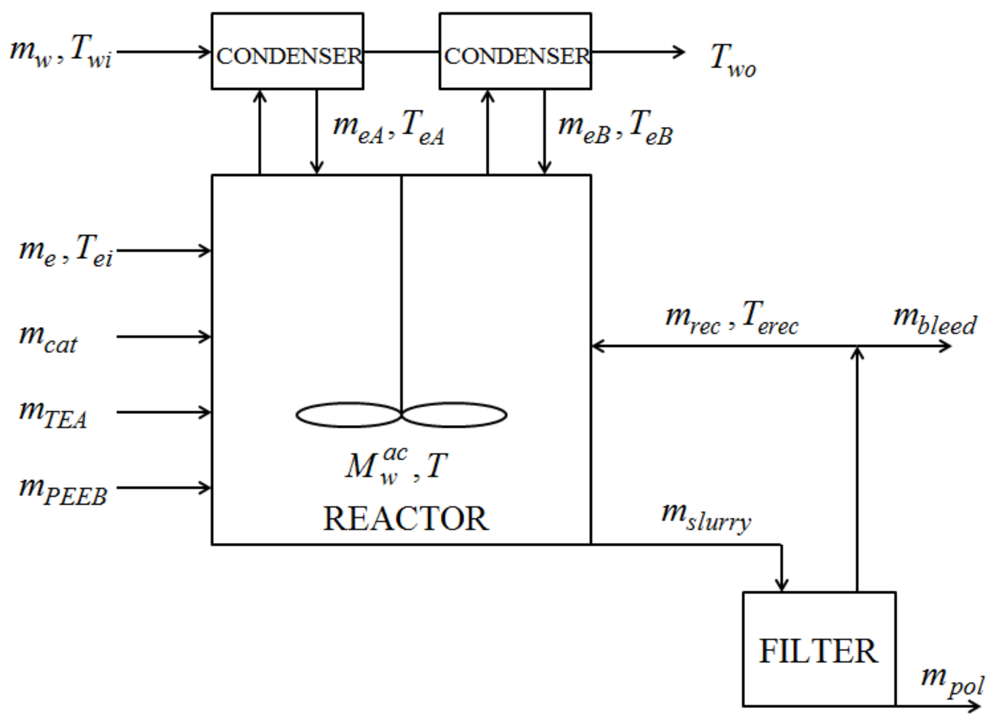


Fig.3.6. The bulk polymerization of polypropylene

The material balance of this process can be written as Eqs. (3.12) – (3.19) (Neto and Pinto, 2001; Prata et al., 2009)

$$\frac{dP_a}{dt} = m_e w_a - \left(\frac{P_a}{P_a + P_e}\right) m_{bleed} \tag{3.12}$$

$$\frac{dP_e}{dt} = m_e (1 - w_a) - R_{pol} - \left(\frac{P_e}{P_a + P_e}\right) m_{bleed} \tag{3.13}$$

$$\frac{dPol}{dt} = R_{pol} - m_{pol} \tag{3.14}$$

$$\frac{dCat}{dt} = m_{cat} - \left(\frac{Cat}{Pol}\right) m_{pol} - K_d Cat \tag{3.15}$$

$$\frac{dTEA}{dt} = m_{TEA} - \alpha\left(\frac{TEA}{P_e + P_a}\right) m_{bleed} - (1-\alpha)\left(\frac{TEA}{Pol}\right) m_{pol} \tag{3.16}$$

$$\frac{dPEEB}{dt} = m_{PEEB} - \left(\frac{PEEB}{Pol}\right) m_{pol} \tag{3.17}$$

$$\frac{dXS}{dt} = \frac{R_{pol}}{Pol}\left[XS^R + K_{XS}\left(\frac{TEA}{PEEB} - 1\right) - XS\right] \tag{3.18}$$

$$\frac{dN}{dt} = \frac{R_{pol}}{M_n^i} - \frac{m_{pol}}{M_n^{ac}} \tag{3.19}$$

where P_a is the mass of propane in the reactor (kg), m_e is the mass flowrate of propylene monomer (kg/h), w_a is the propane concentration in the feed stream (wt/wt), P_e is the mass of propylene in the reactor (kg), m_{bleed} is the mass flowrate of liquid bleed (kg/h), R_{pol} is the rate of polymerization (kg/h), Pol is the mass of polymer in the reactor (kg), m_{pol} is the mass flowrate of output polymer (kg/h), Cat is the mass of catalyst in the reactor (kg), m_{cat} is the mass flowrate of catalyst (kg/h), K_d is the catalyst deactivation constant (h^{-1}), TEA is the mass of triethyl aluminum in the reactor (kg), m_{TEA} is the mass flowrate of triethyl aluminum (kg/h), α is the recycle factor of triethyl aluminum (wt/wt), $PEEB$ is the mass of p-ethylethoxy-benzoate in the reactor (kg), m_{PEEB} is the mass flowrate of p-ethylethoxy-benzoate (kg/h), XS is the xylene extractable material (wt/wt), XS^R is the reference value of xylene extractable material (wt/wt), K_{XS} is the model parameter for XS correlation (dimensionless), N is the mol number of polymer (mol), M_n^i is the instantaneous number average molecular weight (kg/mol) and M_n^{ac} is the cumulative number average molecular weight (kg/mol). Some properties and unmeasured variables can be calculated from Eqs. (3.20) to (3.30).

$$m_{pol} = m_{slurry} w_{pol} \tag{3.20}$$

$$m_{slurry} = \frac{m_{rec} + m_{bleed}}{1 - w_{pol}} \tag{3.21}$$

$$w_{pol} = \frac{Pol}{M} \tag{3.22}$$

$$M = P_e + P_a + Pol \tag{3.23}$$

$$C_a = \frac{P_a}{P_a + P_e} \tag{3.24}$$

$$R_{pol} = \frac{(K_p)(Cat)(P_e)}{M} \tag{3.25}$$

$$MI = K_o (M_w^{ac})^{\lambda} \tag{3.26}$$

$$M_n^i = \frac{P_{MP_e}}{\gamma + C\dfrac{P_{H_2}}{(\dfrac{P_e}{P_a + P_e})}} \tag{3.27}$$

$$M_n^{ac} = \frac{Pol}{N} \tag{3.28}$$

$$M_w^{ac} = (PD)(M_n^{ac}) \tag{3.29}$$

$$V = (\frac{P_a}{\rho_a} + \frac{P_e}{\rho_e} + \frac{Pol}{\rho_{pol}}) \tag{3.30}$$

where m_{slurry} is the mass flowrate of slurry (kg/h), w_{pol} is the polymer concentration in the slurry (wt/wt), m_{rec} is the mass flowrate of the recycle (kg/h), M is the total mass in the reactor (kg), C_a is the propane concentration in the recycle (wt/wt), K_p is the kinetic constant for propagation rate (h^{-1}), MI is the melting index (kg/10 min), K_o is the model parameter for MI correlation ((kg/10 min)/(kg/mol)$^{\lambda}$), λ is the parameter for MI correlation (dimensionless), P_{MP_e} is the propylene molecular weight (kg/mol), γ is the parameter for M_n^i correlation (dimensionless), C is the kinetic constant for transfer rate to hydrogen (atm^{-1}), P_{H_2} is the partial pressure of hydrogen in the reactor (atm), M_w^{ac} is the cumulative mass average molecular weight (kg/mol), PD is the polydispersity (dimensionless), V is the reactor volume (m^3), ρ_a is the density of propane (kg/m^3), ρ_e is the density of propylene (kg/m^3) and ρ_{pol} is the density of polymer (kg/m^3). The energy balance of the polypropylene polymerization process shown in Fig. 3.6 can be written as Eqs. (3.31) to (3.33).

$$\frac{dT}{dt} = \frac{m_e C_{P_e}(T_{ei} - T) + m_{rec} C_{P_e}(T_{erec} - T) + (-\Delta H) R_{pol} - Q_e}{(P_a C_{P_a} + P_e C_{P_e} + Pol C_{P_{pol}})} + d_T \tag{3.31}$$

$$Q_e = m_{eA}\lambda_e + m_{eA} C_{P_e}(T - T_{eA}) + m_{eB}\lambda_e + m_{eB} C_{P_e}(T - T_{eB}) \tag{3.32}$$

$$T_{wo} = \frac{Q_e + m_w C_{P_w} T_{wi}}{m_w C_{P_w}} \tag{3.33}$$

where T is the reactor temperature (K), C_{P_e} is the heat capacity of propylene (kJ/kg.K), T_{ei} is the temperature of propylene monomer (K), T_{erec} is the temperature of the recycle (K), ΔH is the heat of reaction (kJ/kg), Q_e is the heat exchanged in the condenser (kJ/h), C_{P_a} is the heat capacity of propane (kJ/kg.K), $C_{P_{pol}}$ is the heat capacity of polymer (kJ/kg.K), m_{eA} is the mass flowrate of propylene reflux from the first condenser (kg/h), T_{eA} is the temperature of propylene reflux from the first condenser (K), m_{eB} is the mass flowrate of propylene reflux from the second condenser (kg/h), λ_e is the latent heat of vaporization of propylene (kJ/kg), T_{eB} is the temperature of propylene reflux from the second condenser (K), T_{wo} is the output temperature of cooling water (K), m_w is the mass flowrate of cooling water (kg/h), C_{P_w} is the heat capacity of cooling water (kJ/kg.K), T_{wi} is the input temperature of cooling water (K) and d_T is the disturbance from noises of temperature measurement.

Since the polymerization process is usually involved with complicated polymerization reactions, the values of some parameters such as reaction rate constants are not exactly known and they are considered to be uncertain (Prata et al., 2014). Table 3.1 shows the uncertain parameters present in the polypropylene polymerization process.

Table 3.1. The uncertain parameters present in the polypropylene polymerization process.

Parameters	Description	Lower Bound	Upper Bound	Unit
K_p	kinetic constant for propagation rate	$1.00\text{x}10^{4}$	$1.00\text{x}10^{5}$	h^{-1}
K_d	catalyst deactivation constant	4.00	8.00	h^{-1}
C	kinetic constant for transfer rate to hydrogen	$1.00\text{x}10^{-5}$	$1.00\text{x}10^{-3}$	atm^{-1}
ΔH	Heat of reaction	$4.94\text{x}10^{2}$	$6.94\text{x}10^{2}$	kJ/kg

The polymerization reaction is highly exothermic. In the presence of uncertain parameters such as reaction rate constants and disturbances such as measurement noises, an inefficient control of the polymerization process may lead to unexpected thermal runaway of the system. For this reason, the uncertain parameters and disturbances are explicitly taken into account in the proposed controller design as shown in the first experimental step.

3) Apply the developed algorithm to a partially simulated exothermic pilot plant reactor.

In this step, a partially simulated exothermic (PARSEX) pilot plant reactor operated in the continuous mode has been developed as shown in Fig. 3.7.

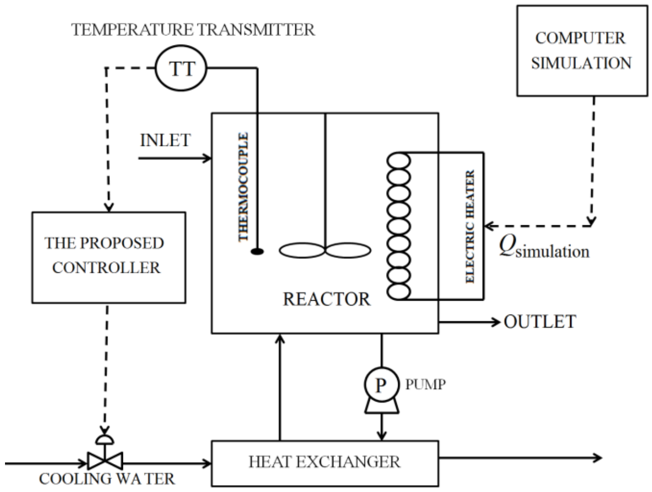


Figure 3.7. A partially simulated exothermic (PARSEX) pilot plant reactor.

The generation of heat ($Q_{\text{simulation}}$) of an exothermic polymerization reaction is calculated on-line according to the concerned kinetic model. Then, the heat is generated using an electric heater and the temperature of hot oil inside the reactor is controlled by adjusting the flow rate of cooling water. The conservation equations associated with the PARSEX reactor can be written as follows

$$\frac{dC}{dt} = -k_o C \exp(-\frac{E}{RT}) + \frac{F}{\rho V}(C_f - C) \tag{3.34}$$

$$\frac{dT}{dt} = \frac{Q_{simulation}}{V\rho C_p} + \frac{F}{\rho V}(T_f - T) + \frac{\dot{m}_{cw} C_{p,cw}}{V\rho C_p}(T_{cw} - T) \tag{3.35}$$

where the generation of heat is calculated according to $Q_{\text{simulation}} = -\Delta H k_o V C \exp(-\frac{E}{RT})$. The objective is to control the simulated reactor concentration C and the reactor temperature T by manipulating the simulated reactor feed concentration C_f and the flow rate of cooling water $\dot{m}_{cw}$, respectively. The reaction is simulated in the computer while the heat transfer is really occurred in the reactor. The parameter descriptions are presented as follows

Table 3.2. The parameter descriptions and nominal values

Parameters	Description	Values	Units
k_o	Pre-exponential factor	55,600	s^{-1}
E/R	Activation energy per universal gas constant	6,000	K
F	Reactor feed rate	0.125	kg/s
ρ	Oil density	910	kg/m^3
V	Reactor volume	0.15	m^3
ΔH	Heat of reaction	-150,000	kJ/kmol
C_p	Specific heat of oil	1.95	kJ/kg.K
$C_{p,cw}$	Specific heat of cooling water	4.18	kJ/kg.K
T_f	Reactor feed temperature	308	K
T_{cw}	Cooling water temperature	300	K
C_{ss}	Steady-state value of the simulated reactor concentration	1.03	$kmol/m^3$
$C_{f,ss}$	Steady-state value of the simulated reactor feed concentration	1.5	$kmol/m^3$
T_s	Steady-state value of the reactor temperature	320.60	K
$\dot{m}_{cw}$	Steady-state value of the flow rate of cooling water	0.08	kg/s

The following parameters are considered to be uncertain during the operation and their values are given as follows. The cooling water temperature cannot be further increased beyond +2% due to the limit in the heat transfer driving force to cool the reactor.

Table 3.3 The values of the uncertain parameters

Parameters	Description	Range from the nominal values	
		Minimum	Maximum
k_o	Pre-exponential factor	-20%	+20%
ΔH	Heat of reaction	-20%	+20%
T_{cw}	Cooling water temperature	-2%	+2%
F	Reactor feed rate	-20%	+20%

4. Results of this research

(1) Results from the development of a novel off-line tube-based robust MPC algorithm.

Case Study 1.1 In this case study, an implementation of the proposed off-line tube-based robust MPC algorithm is illustrated. Consider the following uncertain system with bounded disturbance

$$x^{+} = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w. \tag{4.1}$$

The state $x \in \mathbb{X}$ where $\mathbb{X} := \{x \in \mathbb{R}^2 \mid [0 \ 1]x \le 2\}$, the control $u \in \mathbb{U}$ where $\mathbb{U} := \{u \in \mathbb{R} \mid |u| \le 1\}$, the disturbance $w \in \mathbb{W}$ where $\mathbb{W} := \{w \in \mathbb{R}^2 \mid [-0.1 \ -0.1]^{\mathrm{T}} \le w \le [0.1 \ 0.1]^{\mathrm{T}}\}$ and the uncertain parameter $\lambda \in \mathbb{L}$ where $\mathbb{L} := \{\lambda \in \mathbb{R} \mid 0.9 \le \lambda \le 1.1\}$. The weighting matrices in the cost function are given as $Q = I$ and $R = 0.01$. The following nominal system

$$x'^{+} = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} x' + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u' \tag{4.2}$$

is subject to tighter state and control constraints, i.e., $x' \in \mathbb{X} \ominus Z$ and $u' \in \mathbb{U} \ominus KZ$. The disturbance rejection gain $K = [-0.66 \ -1.33]$ satisfies $(A_j + B_j K)^T P (A_j + B_j K) - P < 0, \forall j \in \{1,2\}$. The difference equation between (4.1) and (4.2) can be written as

$$x^{+} - x'^{+} = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} (x - x') + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} (u - u') + w. \tag{4.3}$$

The closed-loop system is simulated using the initial state $x = x' = [-5 \ -2]^{\mathrm{T}}$. The uncertain parameter λ and the disturbance w are varied as $\lambda = 1 + 0.1\sin(4k)$ and $w = [0.1\sin(4k) \ 0.1\sin(4k)]^T$, respectively, where $k \in \{1,\ldots,19\}$ is the simulation horizon.

Figure 4.1 shows a robust positively invariant set Z precomputed off-line. The cross section of Z is shown in yellow. The blue line represents the trajectory of the difference equation (4.3). Starting from the origin, it is seen that the trajectory of the difference equation is restricted to lie within a tube Z .

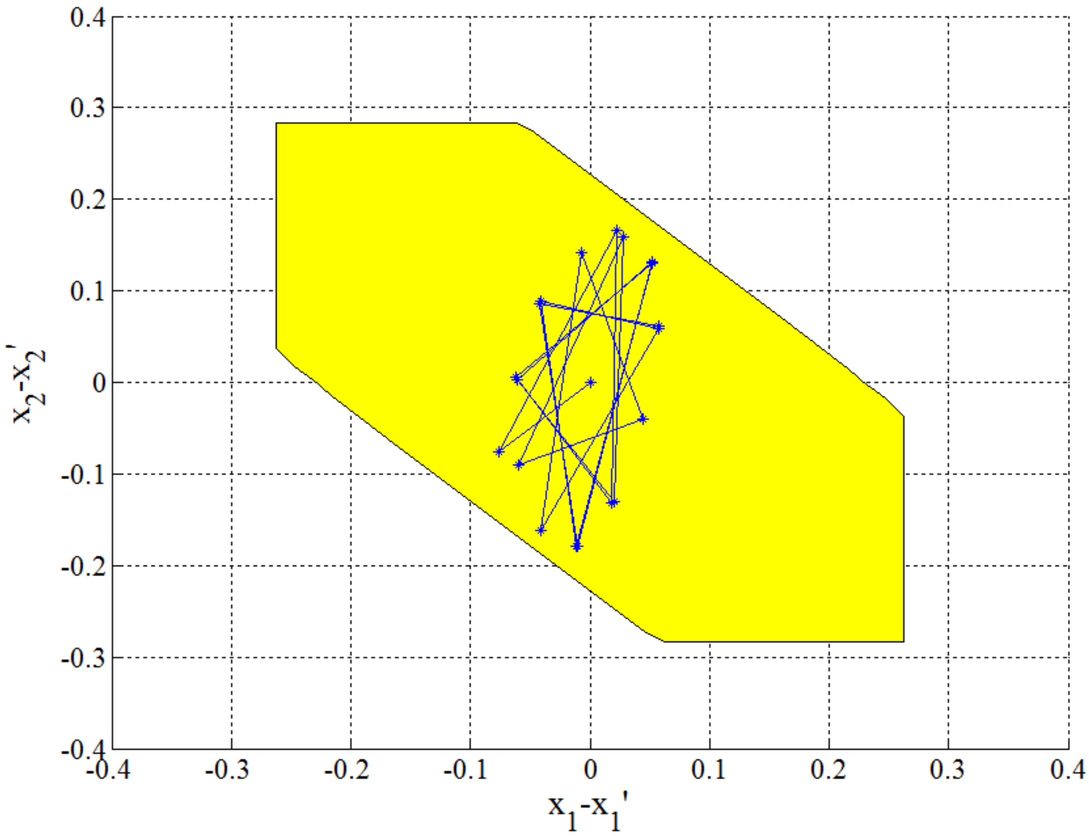


Fig. 4.1. A robust positively invariant set Z precomputed off-line.

Figure 4.2 show a sequence of ten polyhedral invariant sets $P_i, i \in \{1,...,10\}$ precomputed off-line. In this example, only ten polyhedral invariant sets are precomputed because P_i are almost constant for $i > 10$. The red line represents the trajectory of the nominal system (4.2). Starting from the initial point $x = x' = [\text{-}5\ \text{-}2]^{\mathrm{T}}$, the state of the nominal system at each time step is restricted to lie within a sequence of ten polyhedral invariant sets $P_i, i \in \{1,...,10\}$ precomputed off-line. Finally, the state of the nominal system converges to the origin.

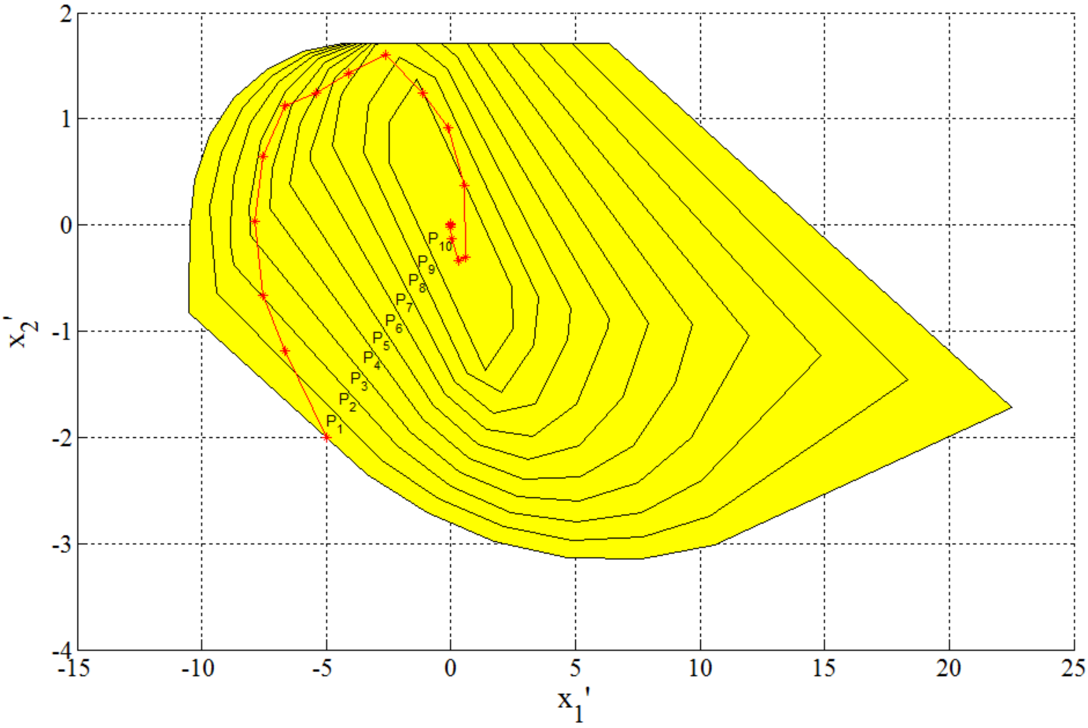


Fig. 4.2. A sequence of ten polyhedral invariant sets $P_i, i \in \{1,\ldots,10\}$ precomputed off-line.

The trajectory of the uncertain system with disturbance (4.1) is shown in Fig. 4.3. The region shown in green is the infeasible region of the state constraint $\mathbb{X} := \{x \in \mathbb{R}^2 \mid [0\ 1]x \le 2\}$. The red line corresponds to the trajectory of the nominal system (4.2). The cross section of a tube Z precomputed off-line is shown in yellow. It is seen that the state of the uncertain system with disturbance at each time step is restricted to lie within a tube Z whose center is the state of the nominal system that converges to the origin. Finally, the state of the uncertain system with disturbance is kept within a tube Z whose center is at the origin.

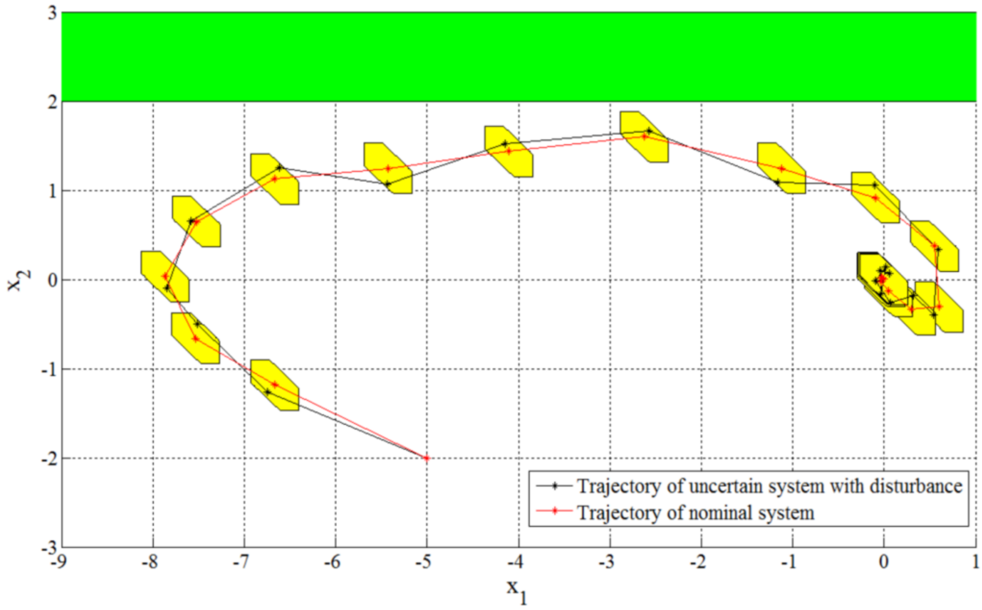


Fig. 4.3. The trajectory of the uncertain system with disturbance.

Figure 4.4 shows the control input as a function of sampling time. The region shown in yellow is $\mathbb{U} \ominus KZ$. The red line corresponds to the control input of the nominal system (4.2). The black line corresponds to the control input of the uncertain system with disturbance (4.1). It can be observed that the control input of the nominal system is restricted to lie within the region $\mathbb{U} \ominus KZ$ so that the control input of the uncertain system with disturbance satisfies the control constraint $\mathbb{U} := \{u \in \mathbb{R} \mid |u| \leq 1\}$.

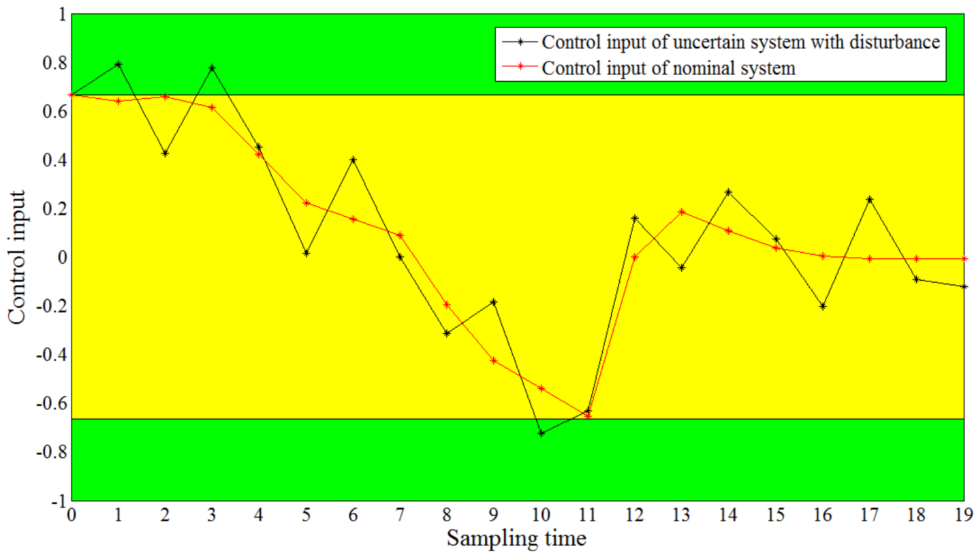


Fig. 4.4. The control input as a function of sampling time.

Case study 1.2 In this case study, the proposed algorithm is applied to a non-isothermal continuous stirred tank reactor (CSTR) in which an irreversible exothermic reaction $\mathrm{A} \rightarrow \mathrm{B}$ takes place. The dimensionless modeling equations of this CSTR can be written as (Nagrath et al., 2002; Silva and Kwong, 1999)

$$\frac{dx_1}{d\tau} = q(x_{1f} - x_1) - \phi x_1 \exp\left(\frac{x_2}{1+\frac{x_2}{\gamma}}\right) + w_1 \quad (4.4)$$

$$\frac{dx_2}{d\tau} = q(x_{2f} - x_2) - \delta(x_2 - x_3) + \beta\phi x_1 \exp\left(\frac{x_2}{1+\frac{x_2}{\gamma}}\right) + w_2 \quad (4.5)$$

$$\frac{dx_3}{d\tau} = \delta_1[q_c(x_{3f} - x_3) + \delta\delta_2(x_2 - x_3)] + w_3 \quad (4.6)$$

where x_1 is the dimensionless concentration of reactant A, x_2 is the dimensionless reactor temperature and x_3 is the dimensionless cooling jacket temperature. The manipulated variable is the dimensionless coolant flow rate q_c. The disturbances acting on the system are w_1, w_2 and w_3. By linearizing and discretizing (4.4) to (4.6) with a sampling period ΔT, the following discrete-time state space model is obtained

$$\begin{bmatrix} \bar{x}_1(k+1) \\ \bar{x}_2(k+1) \\ \bar{x}_3(k+1) \end{bmatrix} = \begin{bmatrix} 1+\Delta T[-q-\phi\kappa(x_{2S})] & -\Delta T[\frac{\phi x_{1S}\kappa(x_{2S})}{(1+\frac{x_{2S}}{\gamma})^2}] & 0 \\ \Delta T[\beta\phi\kappa(x_{2S})] & 1+\Delta T[-q-\delta+\frac{\beta\phi\kappa(x_{2S})x_{1S}}{(1+\frac{x_{2S}}{\gamma})^2}] & \Delta T\delta \\ 0 & \Delta T\delta\delta_1\delta_2 & 1-\Delta T[\delta_1 q_{cS}+\delta\delta_1\delta_2] \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \\ \bar{x}_3(k) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ \Delta T\delta_1[x_{3f} - x_{3S}] \end{bmatrix} \bar{q}_c(k) + \Delta T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{w}_1(k) \\ \bar{w}_2(k) \\ \bar{w}_3(k) \end{bmatrix} \quad (4.7)$$

where $\bar{x}_1(k) = x_1(k) - x_{1S}$, $\bar{x}_2(k) = x_2(k) - x_{2S}$, $\bar{x}_3(k) = x_3(k) - x_{3S}$, $\bar{q}_c(k) = q_c(k) - q_{cS}$, $\bar{w}_1(k) = w_1(k) - w_{1S}$, $\bar{w}_2(k) = w_2(k) - w_{2S}$, $\bar{w}_3(k) = w_3(k) - w_{3S}$ and

$\kappa(x_{2S}) = \exp(\frac{x_{2S}}{1+\frac{x_{2S}}{\gamma}})$. The model parameter values are shown in Table 4.1. The Damkohler number ϕ is considered to be uncertain and its value is randomly time-varying between $\phi_{\min} = 0.0648$ and $\phi_{\max} = 0.0792$. The disturbances $\overline{w}_1(k)$, $\overline{w}_2(k)$ and $\overline{w}_3(k)$ are randomly time-varying between -0.01 to 0.01. The constraints are $|\overline{x}_1(k)| \le 0.5$ and $|\overline{q}_c(k)| \le 1.0$. The weighting matrices in the cost function are $Q = I$ and $R = 0.1$. The objective is to regulate the state from $(\overline{x}_1(0), \overline{x}_2(0), \overline{x}_3(0)) = (0, 5, 0)$ to the neighborhood of the origin by manipulating $\overline{q}_c(k)$.

Table 4.1 The model parameter values.

Parameter	Value	Parameter	Value
q	1.0	δ	0.3
x_{1f}	1.0	β	8.0
ϕ	0.0648-0.0792	δ_1	10
γ	20	x_{3f}	-1.0
x_{2f}	0.0	δ_2	1.0
x_{1S}	0.8933	w_{1S}	0.0
x_{2S}	0.5193	w_{2S}	0.0
x_{3S}	-0.5950	w_{3S}	0.0
q_{cS}	1.65		

Figure 4.5 shows a sequence of four polyhedral invariant sets $P_i, i \in \{1,\ldots,4\}$ precomputed off-line. The polyhedral invariant sets are shown in yellow.

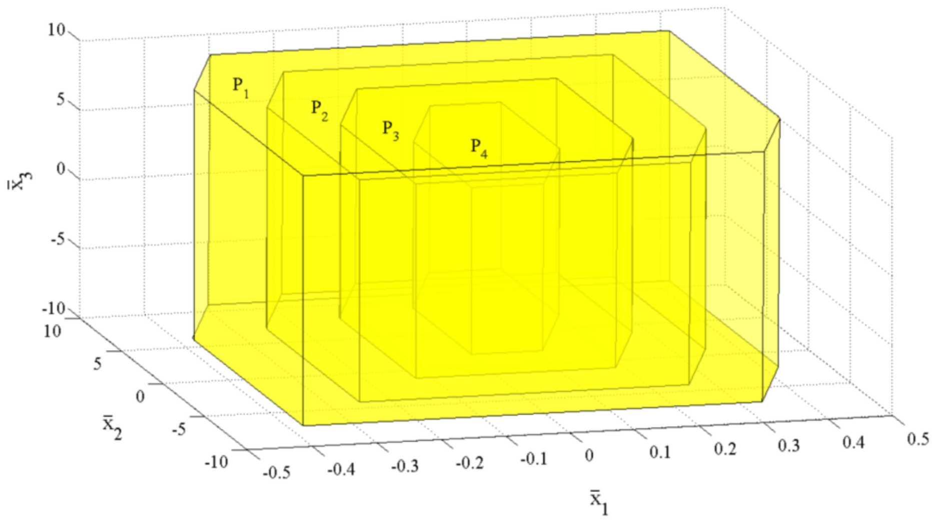


Fig.4.5. A sequence of four polyhedral invariant sets $P_i, i \in \{1,\ldots,4\}$ precomputed off-line.

Figure 4.6 shows the trajectory of the uncertain CSTR. The black line is the trajectory of the uncertain CSTR with disturbances (CSTR containing both time-varying parameter and disturbances). The red line is the trajectory of the uncertain CSTR with no disturbances (CSTR containing only time-varying parameter). It can be observed that the trajectory of the uncertain CSTR with disturbances lies in a sequence of tubes shown in yellow. Finally, the state of the uncertain CSTR with disturbances is steered to a tube whose center is at the origin.

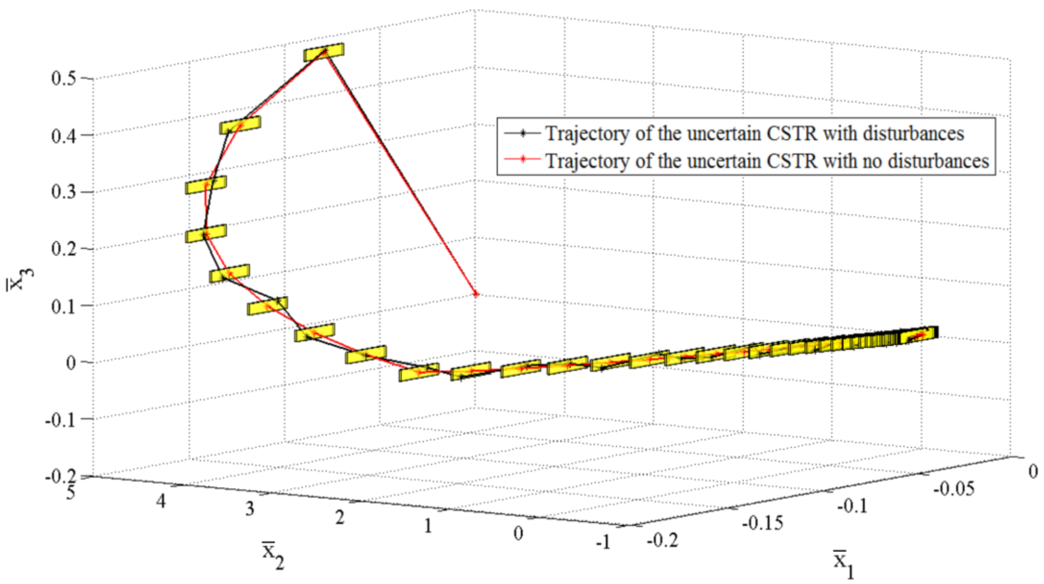
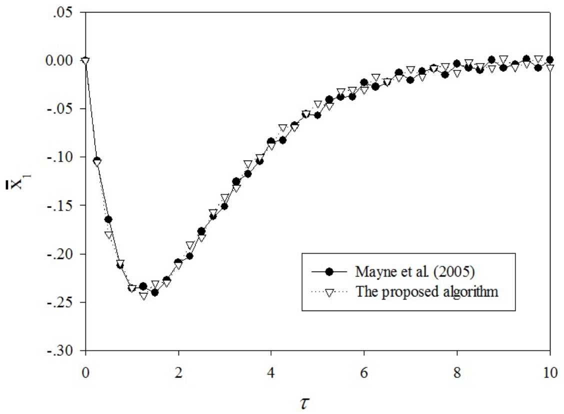
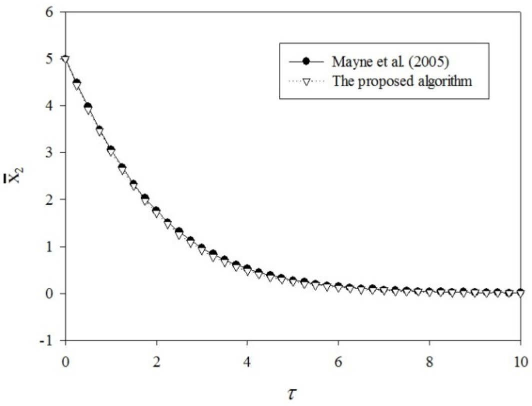


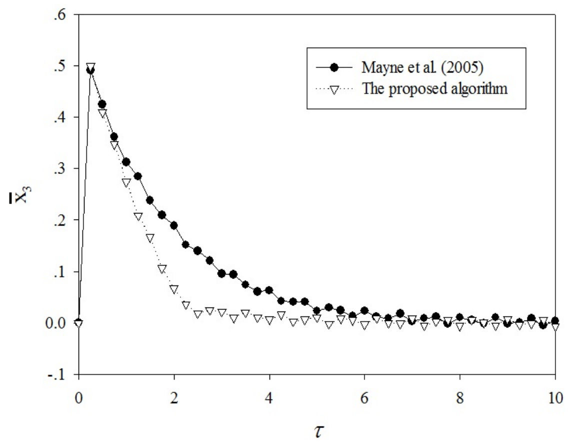
Fig. 4.6. The trajectory of the uncertain CSTR (The cross section of tube is shown in yellow).

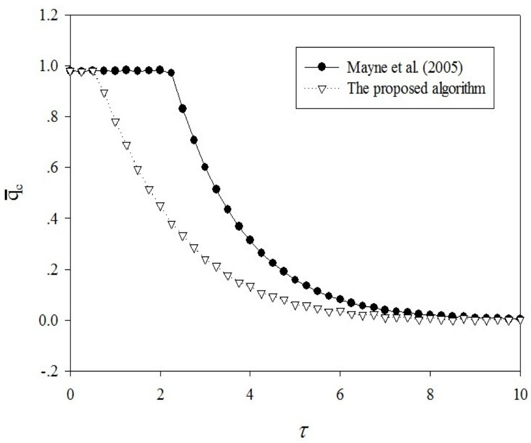
The proposed algorithm will be compared with tube-based robust MPC algorithm of Mayne et al. (2005) in which the on-line optimization problem must be solved at each sampling time. In Mayne et al. (2005), only disturbances are included in the controller design so there is a mismatch between the model and the process when the time-varying parameter is present. From Figure 4.7, it is seen that the proposed algorithm is able to steer the state of the uncertain CSTR with disturbances to the neighborhood of the origin faster than the algorithm of Mayne et al. (2005).





(a) dimensionless concentration of reactant A (b) dimensionless reactor temperature





(c) dimensionless cooling jacket temperature (d) dimensionless coolant flow rate

Fig.4.7. The control performance (a) dimensionless concentration of reactant A (b) dimensionless reactor temperature (c) dimensionless cooling jacket temperature and (d) dimensionless coolant flow rate.

The on-line computational time is shown in Table 4.2. It is seen that the proposed algorithm requires significantly less on-line computational time. The computations are performed using Intel Core 2 Duo (2.53 GHz), 2 GB RAM.

Table 4.2 The on-line computational time.

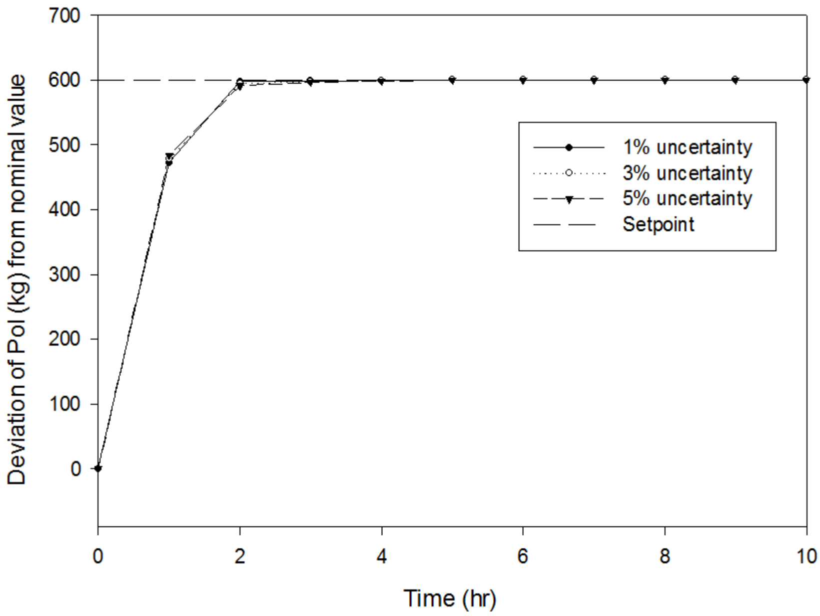
Algorithm	On-line computational time for each step (seconds)
Mayne et al. (2005)	0.067
The proposed algorithm	0.015

(2) Results from the application of the developed algorithm to a simulation case study of polypropylene polymerization process with uncertain and highly exothermic polymerization reactions.

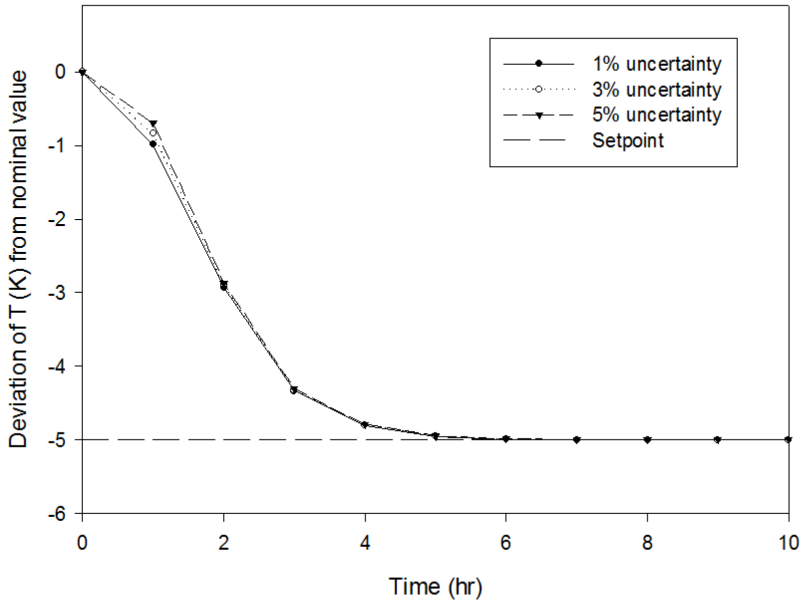
In this section, the developed off-line tube-based robust MPC algorithm is applied to a simulation case study of polypropylene polymerization process containing uncertain parameters and disturbances. The objective is to control the reactor temperature (T) and the cumulative mass average molecular weight of polymer (M_w^{ac}) by manipulating the mass flowrate of cooling water (m_w) and the mass flowrate of propylene monomer (m_e). The cumulative mass average molecular weight of polymer (M_w^{ac}) can be controlled by monitoring the mass of polymer in the reactor (Pol). The setpoints (with respect to the nominal values) for the mass of polymer in the reactor (Pol) and the reactor temperature (T) are 600 kg and -5 K, respectively. The results are presented in the form of deviation variables for the ease of understanding.

Case study 2.1 The control performance when the kinetic constant for propagation rate (K_p) and the heat of reaction (ΔH) are uncertain.

In this case study, the developed control algorithm is applied to a simulation case study of polypropylene polymerization process in which the kinetic constant for propagation rate (K_p) and the heat of reaction (ΔH) are uncertain. Figure 4.8 shows the simulation results when the values of the kinetic constant for propagation rate (K_p) and the heat of reaction (ΔH) are both increased by 1%, 3% and 5% from the nominal values. It is seen that the controlled variables, which are the mass of polymer in the reactor (Pol) and the reactor temperature (T), can be regulated to the setpoints $Pol = 600$ kg and $T = -5$ K despite increased values of uncertainties. This is due to the fact that the developed control algorithm is able to ensure robust stability in the presence of uncertain parameters.



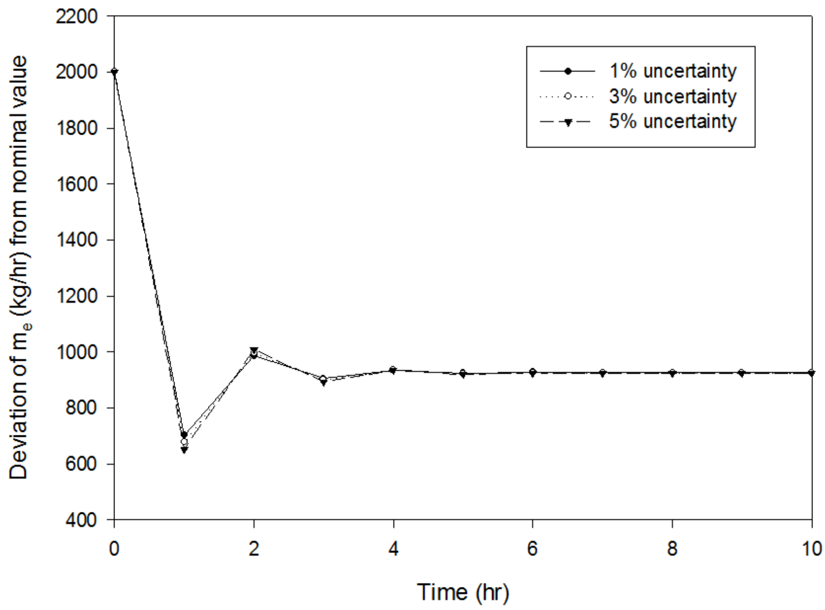
a) The mass of polymer in the reactor (Pol)



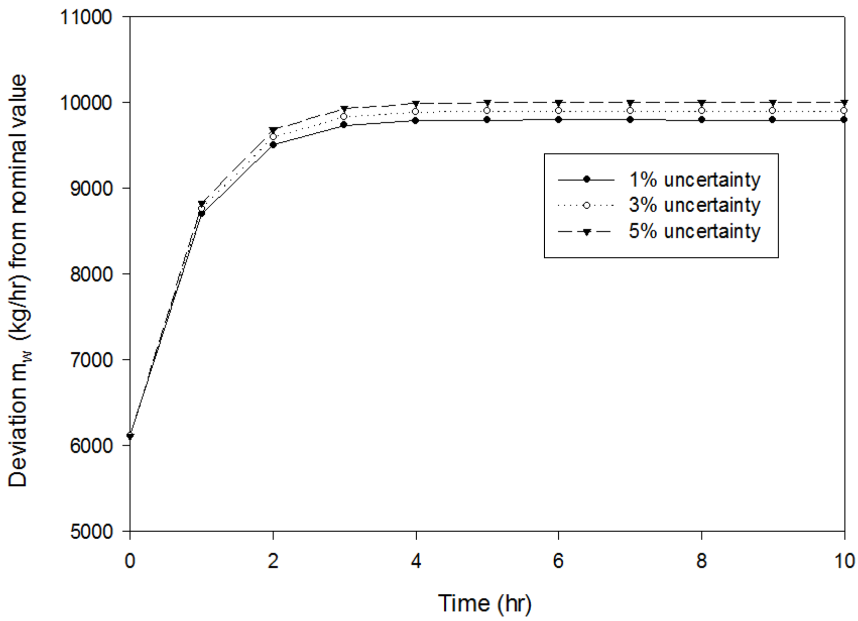
b) The reactor temperature (T)

Fig. 4.8 The controlled variables: a) the mass of polymer in the reactor (Pol) and b) the reactor temperature (T) when the values of the kinetic constant for propagation rate (K_p) and the heat of reaction (ΔH) are uncertain.

Figure 4.9 shows the manipulated variables, which are the mass flowrate of propylene monomer (m_e) and the mass flowrate of cooling water (m_w), when the values of the kinetic constant for propagation rate (K_p) and the heat of reaction (ΔH) are both increased by 1%, 3% and 5% from the nominal values. It can be observed from Fig. 4.9(a) that the values of the mass flowrate of propylene monomer (m_e) are high at the beginning of the process in order to steer the mass of polymer in the reactor (Pol) to the setpoint. Then, the values of the mass flowrate of propylene monomer (m_e) reach a steady state after 6 hr. Figure 4.9(b) shows the mass flowrate of cooling water (m_w) as a function of time. It is seen that values of the mass flowrate of cooling water (m_w) reach a steady state after 6 hr as the reactor temperature (T) is regulated to the setpoint.



a) The mass flowrate of propylene monomer (m_e)



b) The mass flowrate of cooling water (m_w)

Fig. 4.9. The manipulated variables: a) the mass flowrate of propylene monomer (m_e) and b) the mass flowrate of cooling water (m_w) when the values of the kinetic constant for propagation rate (K_p) and the heat of reaction (ΔH) are uncertain.

The cumulative mass average molecular weight of polymer (M_w^{ac}) in the reactor is shown in Fig. 4.10. It is seen that the cumulative mass average molecular weight of polymer (M_w^{ac}) can be regulated to the desired value of 1.57x10^5 g/mol for all cases.

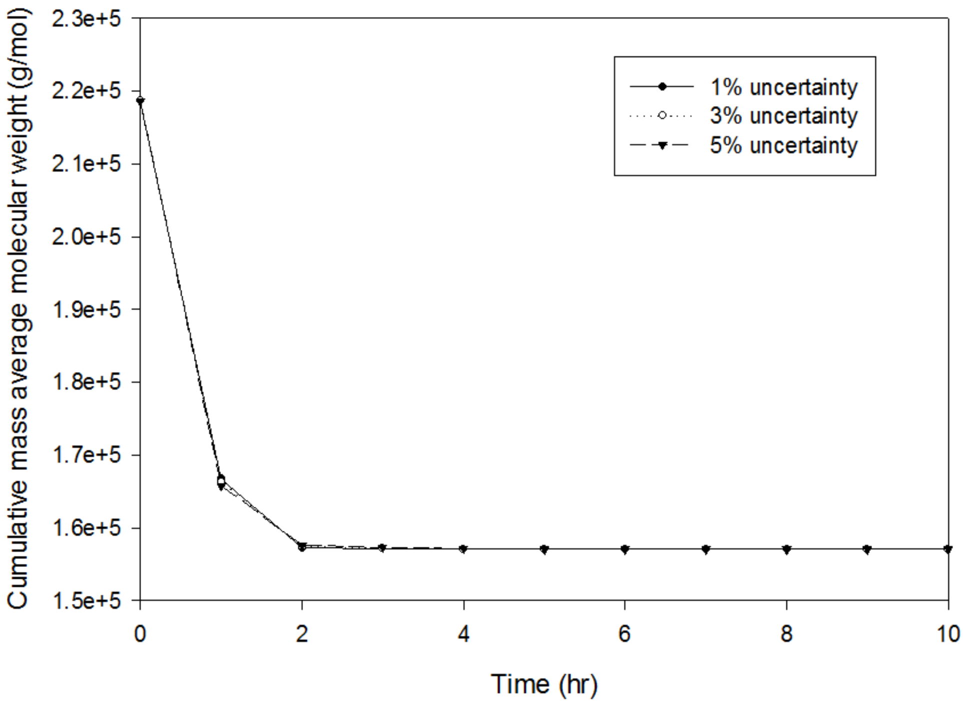
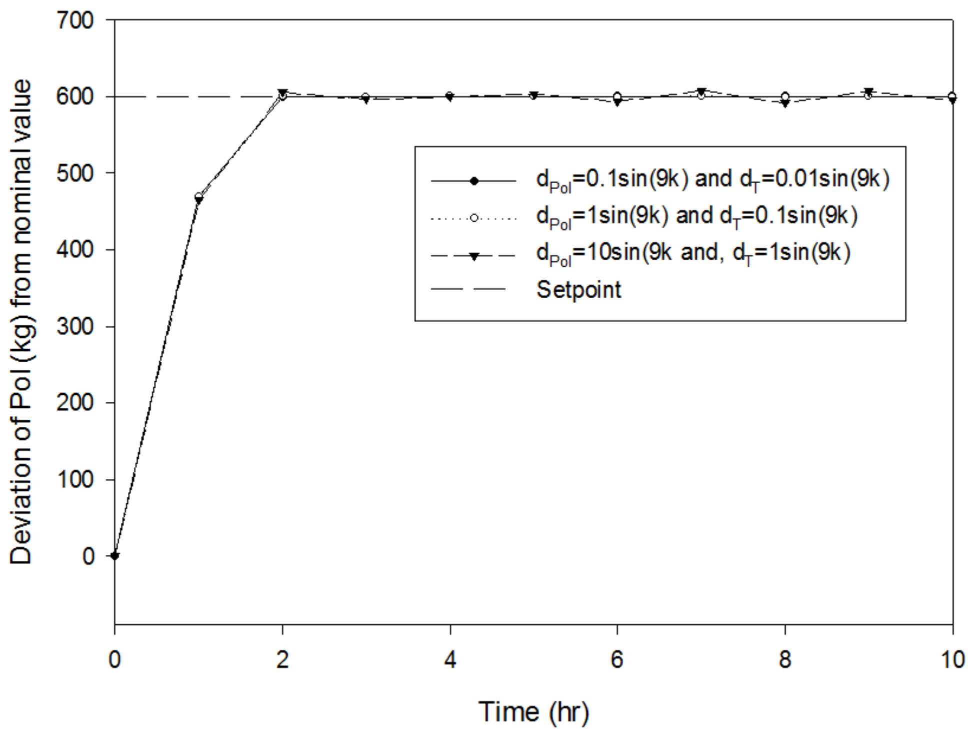


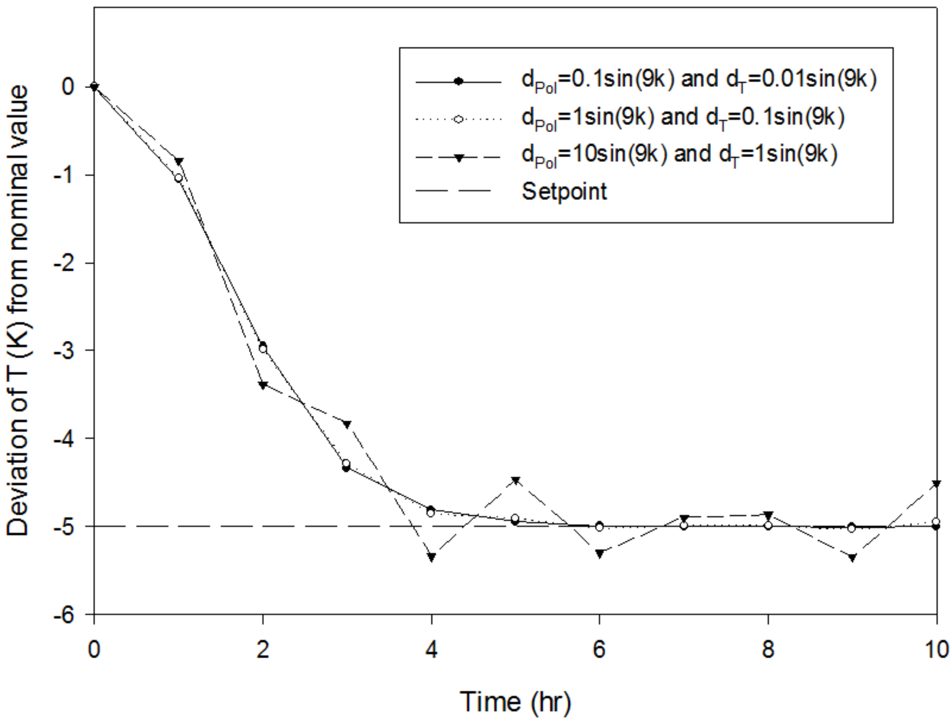
Fig. 4.10. The cumulative mass average molecular weight of polymer (M_w^{ac}) when the values of the kinetic constant for propagation rate (K_p) and the heat of reaction (ΔH) are uncertain.

Case study 2.2 The control performance in the presence of the disturbance acting on the mass of polymer in the reactor (d_{Pol}) and the disturbance acting on the reactor temperature (d_T).

In this case study, the developed control algorithm is applied to a simulation case study of polypropylene polymerization process in the presence of the disturbance acting on the mass of polymer in the reactor (d_{Pol}) and the disturbance acting on the reactor temperature (d_T). Figure 4.11 shows the simulation results when the disturbance acting on the mass of polymer in the reactor (d_{Pol}) and the disturbance acting on the reactor temperature (d_T) are varied as $d_{Pol} = 0.1\sin(9k), 1\sin(9k), 10\sin(9k)$ and $d_T = 0.01\sin(9k), 0.1\sin(9k), 1\sin(9k)$ where k is the simulation time step (hr). It is seen that the controlled variables, which are the mass of polymer in the reactor (Pol) and the reactor temperature (T), are regulated to the neighborhood of the setpoints $Pol = 600$ kg and $T = -5$ K. It should be noted that in the presence of the additive disturbances, the controlled variables can only be regulated to the neighborhood of the setpoints due to the nature of the disturbances.



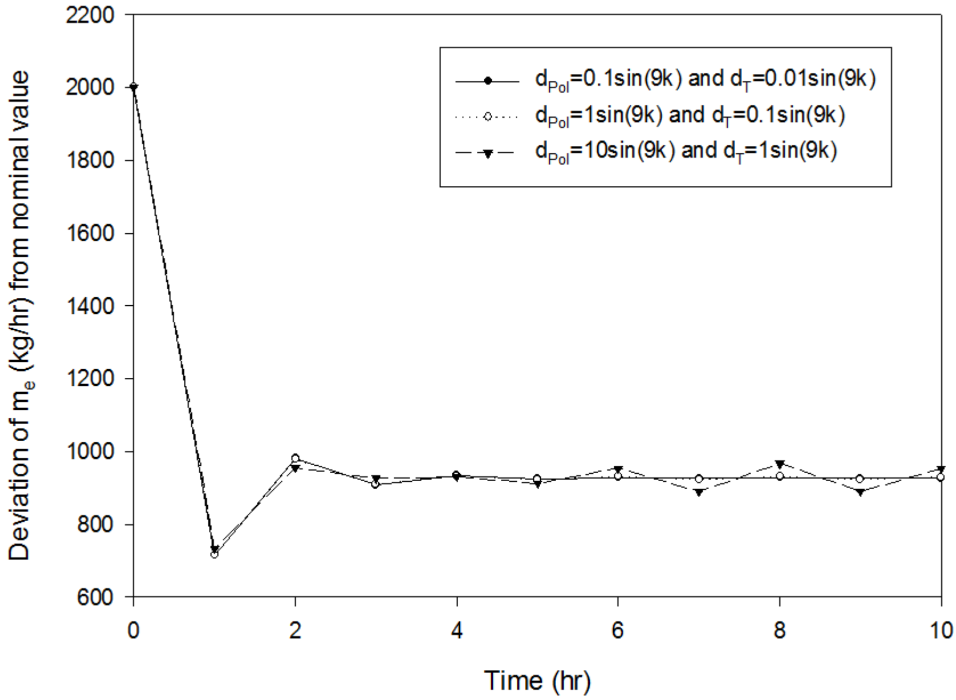
a) The mass of polymer in the reactor (Pol)



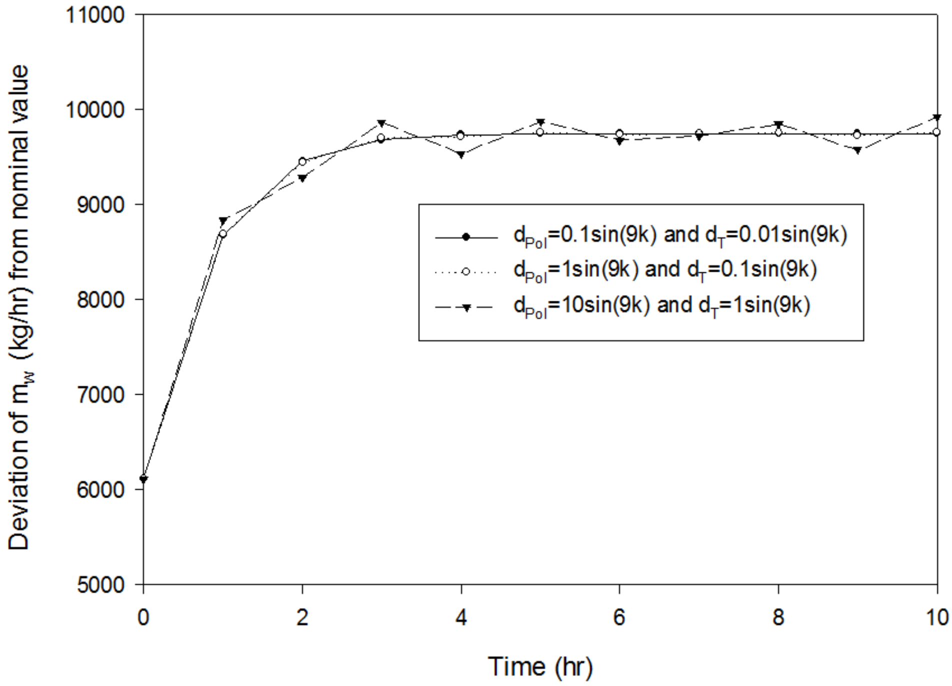
b) The reactor temperature (T)

Fig. 4.11. The controlled variables: a) the mass of polymer in the reactor (Pol) and b) the reactor temperature (T) in the presence of the disturbance acting on the mass of polymer in the reactor (d_{Pol}) and the disturbance acting on the reactor temperature (d_T).

The manipulated variables, which are the mass flowrate of propylene monomer (m_e) and the mass flowrate of cooling water (m_w), are shown in Fig. 4.12. In the presence of the additive disturbances, it can be observed that the mass flowrate of propylene monomer (m_e) and the mass flowrate of cooling water (m_w) are both fluctuating around the steady-state values as the controlled variables are regulated to the neighborhood of the setpoints due to the effects of the disturbances.



a) The mass flowrate of propylene monomer (m_e)



b) The mass flowrate of cooling water (m_w)

Fig. 4.12. The manipulated variables: a) the mass flowrate of propylene monomer (m_e) and b) the mass flowrate of cooling water (m_w) in the presence of the disturbance acting on the mass of polymer in the reactor (d_{Pol}) and the disturbance acting on the reactor temperature (d_T).

The cumulative mass average molecular weight of polymer (M_w^{ac}) in the reactor is shown in Fig. 4.13. It is seen that the cumulative mass average molecular weight of polymer (M_w^{ac}) can be regulated to the neighborhood of the desired value of 1.57x10^5 g/mol despite the time-varying disturbances acting on the system.

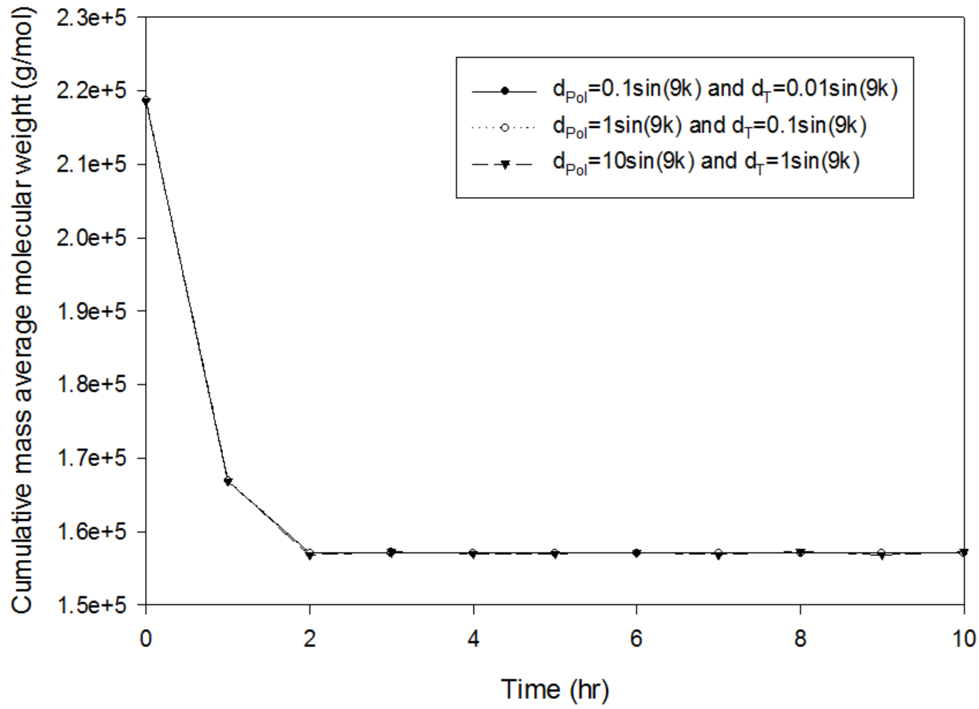


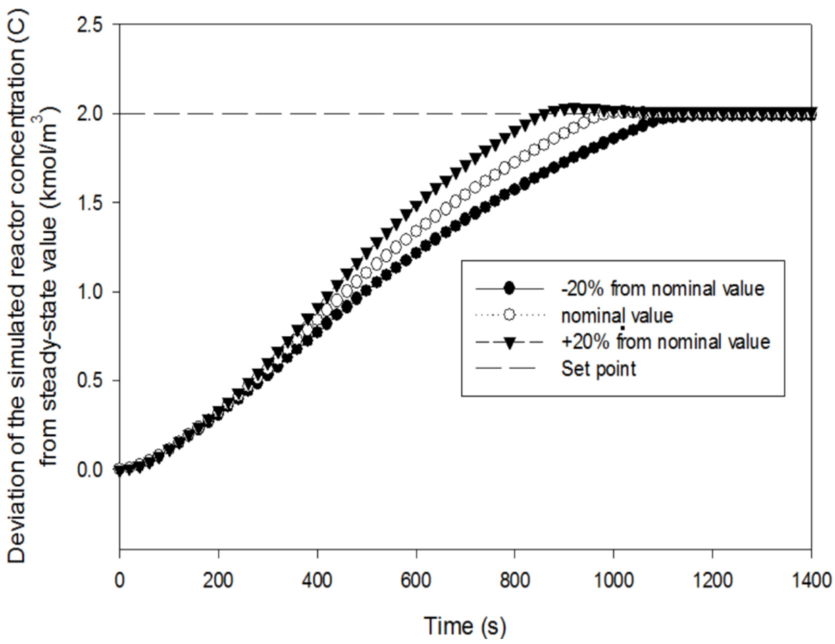
Fig. 4.13. The cumulative mass average molecular weight of polymer (M_w^{ac}) in the presence of the disturbance acting on the mass of polymer in the reactor (d_{Pol}) and the disturbance acting on the reactor temperature (d_T).

(3) Results from the application of the developed algorithm to a partially simulated exothermic pilot plant reactor.

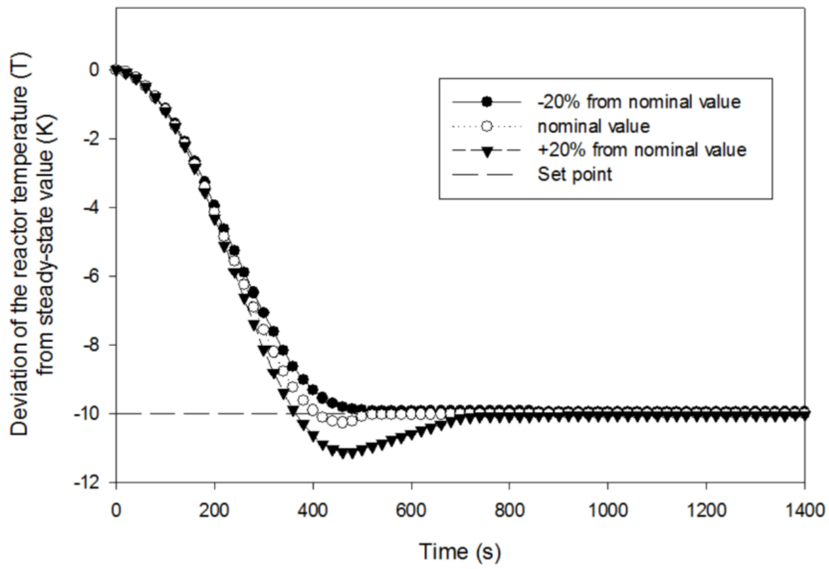
In this section, the developed control algorithm is applied to a partially simulated exothermic pilot plant (PARSEX) reactor. The objective is to control the simulated reactor concentration C and the reactor temperature T by manipulating the simulated reactor feed concentration C_f and the flow rate of cooling water $\dot{m}_{cw}$, respectively. The reaction is simulated in the computer while the heat transfer is really occurred in the reactor. Two case studies are conducted as follows

Case study 3.1 The control performance when the pre-exponential factor (k_o) and the heat of reaction (ΔH) are uncertain.

In this case study, the developed control algorithm is applied to the control of the simulated reactor concentration C and the reactor temperature T when the pre-exponential factor k_o and the heat of reaction ΔH are uncertain. The values of the pre-exponential factor k_o and the heat of reaction ΔH are in the range between -20% to +20% from the nominal values. Figure 4.14 shows the simulated reactor concentration C and the reactor temperature T when the values of pre-exponential factor are uncertain in the range between -20% to +20% from the nominal values. The developed controller can drive the simulated reactor concentration and the reactor temperature to the set point despite the presence of uncertainty. It can be observed that high value of the pre-exponential factor (+20%) leads to the overshoot for both state variables.



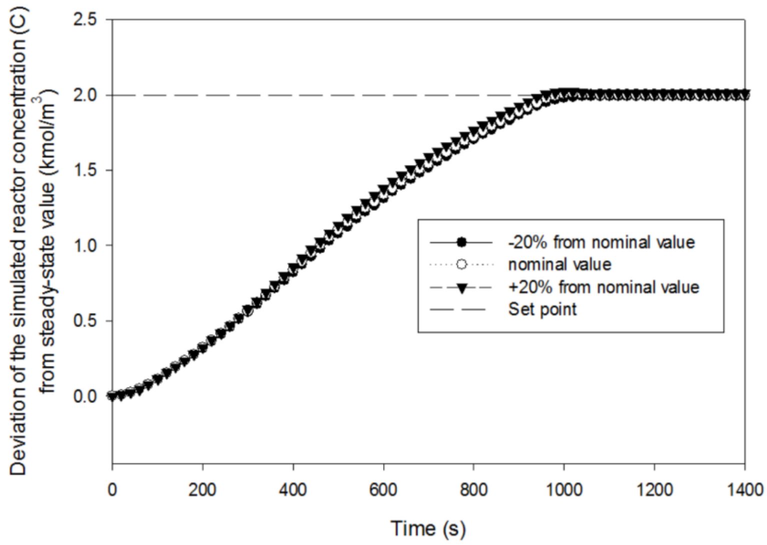
a) The simulated reactor concentration



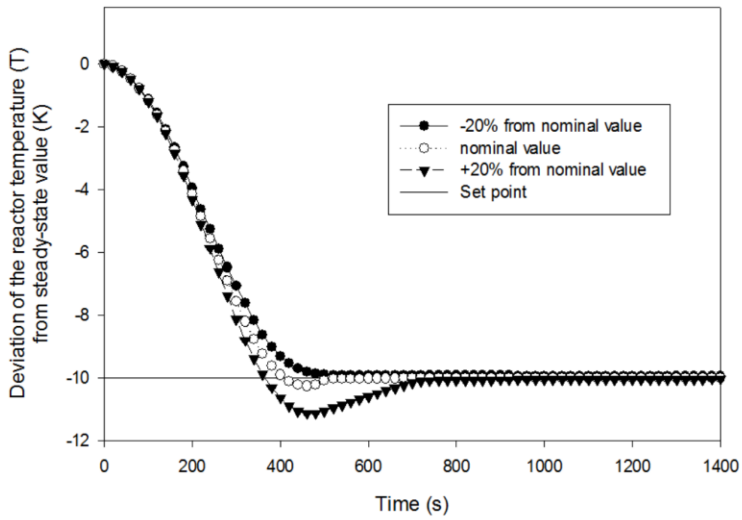
b) The reactor temperature

Figure 4.14 The simulated reactor concentration C and the reactor temperature T when the values of pre-exponential factor k_o are uncertain in the range between -20% to +20% from the nominal values.

Figure 4.15 shows the simulated reactor concentration C and the reactor temperature T when the values of the heat of reaction ΔH are uncertain in the range between -20% to +20% from the nominal values. In the presence of uncertainty in the heat of reaction, the simulated reactor concentration and the reactor temperature can be regulated to the set point. However, high value of the heat of reaction (+20%) causes the overshoot of the reactor temperature.



a) The simulated reactor concentration

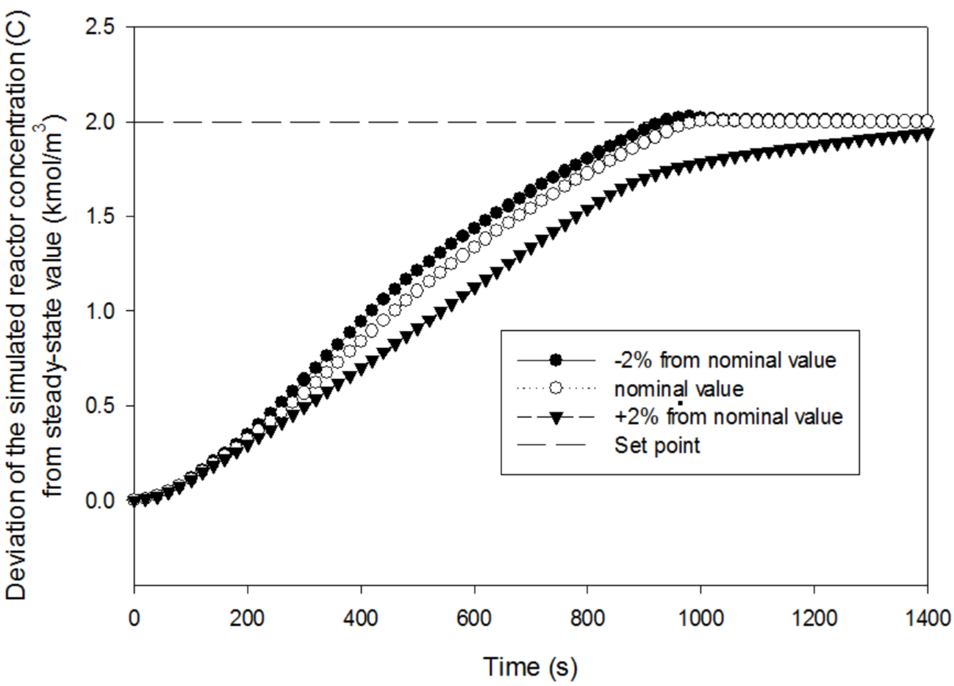


b) The reactor temperature

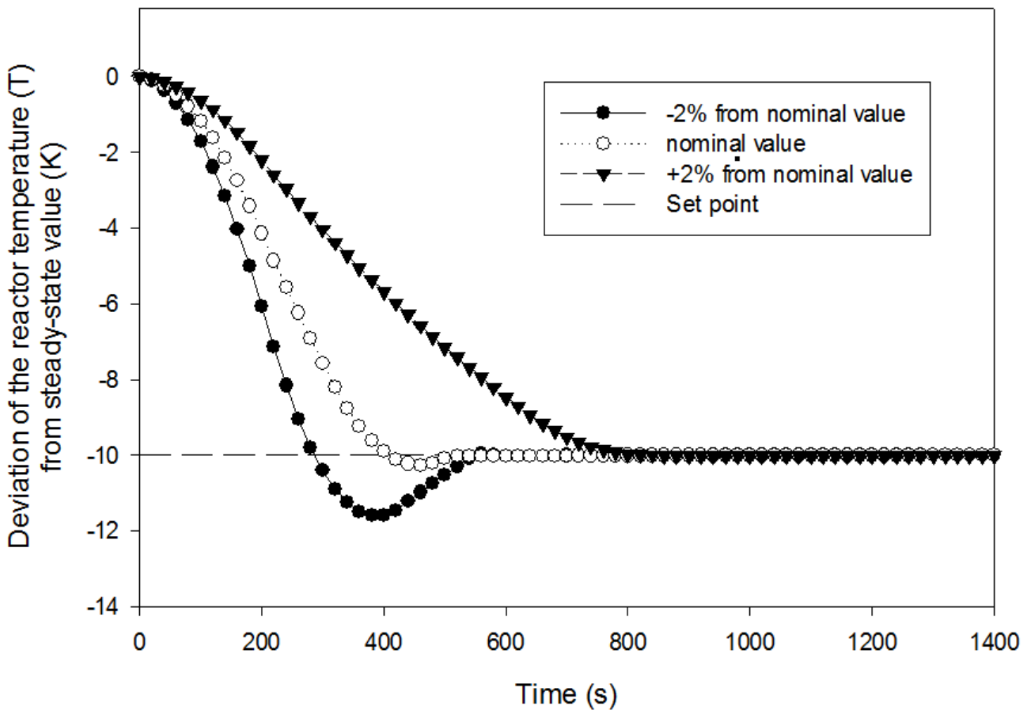
Figure 4.15 The simulated reactor concentration C and the reactor temperature T when the values of the heat of reaction ΔH are uncertain in the range between -20% to +20% from the nominal values.

Case study 3.2 The control performance when the cooling water temperature (T_{cw}) and the reactor feed rate (F) are uncertain.

In this case study, the developed control algorithm is applied to the control of the simulated reactor concentration C and the reactor temperature T when the cooling water temperature T_{cw} and the reactor feed rate F are uncertain. The values of the cooling water temperature T_{cw} are in the range between -2% to +2% from the nominal values (the cooling water temperature cannot be further increased beyond +2% due to the limit in the heat transfer driving force). The values of the reactor feed rate F are in the range between -20% to +20% from the nominal values. Figure 4.16 shows the simulated reactor concentration C and the reactor temperature T when the values of the cooling water temperature T_{cw} are uncertain in the range between -2% to +2% from the nominal values. High value of the cooling water temperature (+2%) has a small value of heat transfer driving force causing the sluggish response for the simulated reactor concentration and the reactor temperature.



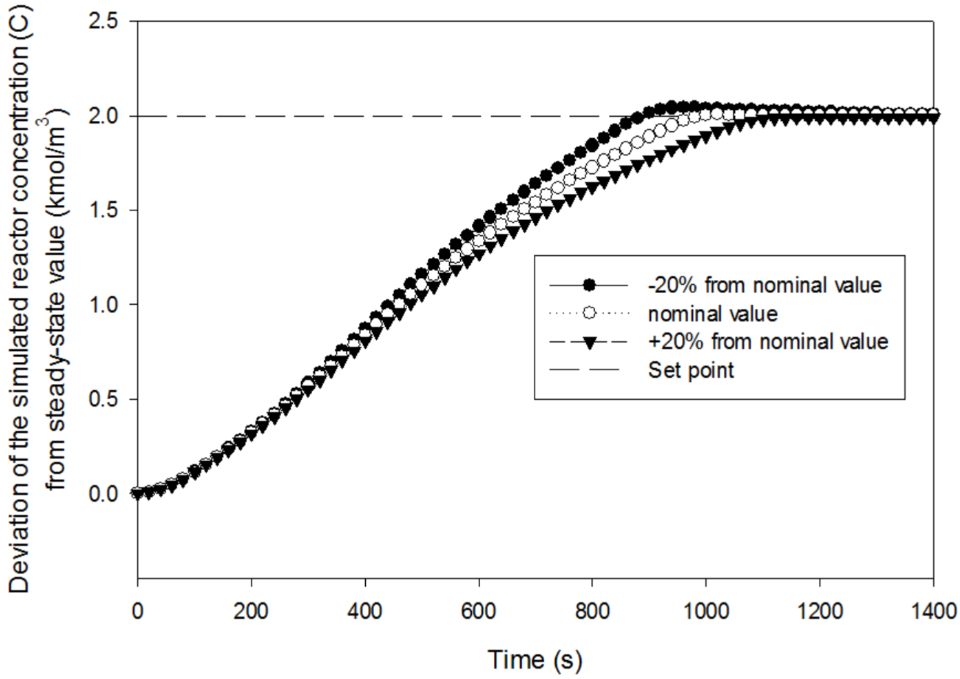
a) The simulated reactor concentration



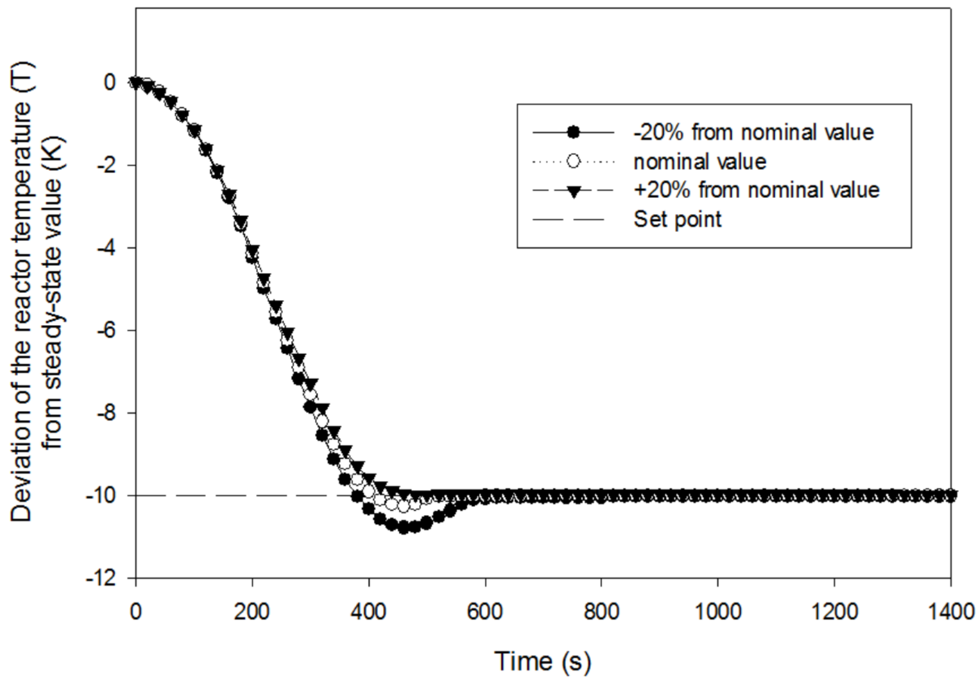
b) The reactor temperature

Figure 4.16 The simulated reactor concentration C and the reactor temperature T when the values of the cooling water temperature T_{cw} are uncertain in the range between -2% to +2% from the nominal values.

Figure 4.17 shows the simulated reactor concentration C and the reactor temperature T when the values of the reactor feed rate F are uncertain in the range between -20% to +20% from the nominal values. High value (+20%) of the reactor feed rate F leads to a small value of the residence time in the reactor. As the time that the substances spent in the reactor decrease, the slow responses of the simulated reactor concentration and the reactor temperature are obtained.



a) The simulated reactor concentration



b) The reactor temperature

Figure 4.17 The simulated reactor concentration C and the reactor temperature T when the values of the reactor feed rate F are uncertain in the range between -20% to +20% from the nominal values.

5. Conclusions

This research presents a novel off-line tube-based robust model predictive control algorithm. In the presence of both uncertain parameters and disturbances, robust stability and constraint satisfaction can be guaranteed due to the fact that the trajectories of uncertain systems are restricted to lie in a sequence of tubes. The on-line computational time is reduced because the optimization problems are solved off-line so the developed algorithm is applicable to fast dynamic systems.

The applications of the developed algorithm are demonstrated in the control problem of uncertain polymerization process for polypropylene in which the values of reaction rate constants are uncertain. The presence of uncertain parameters can lead to an unexpected thermal runaway because the polymerization reactions taking place are fast and highly exothermic. In the case of uncertain kinetic constant and uncertain heat of reaction, the developed control algorithm can regulate the controlled variables, which are the mass of polymer in the reactor and the reactor temperature, to the desired set point by manipulating the mass flow rate of propylene monomer and the mass flow rate of cooling water, respectively. In the presence of the disturbances acting on the system, the developed control algorithm is able to regulate the mass of polymer in the reactor and the reactor temperature to the neighborhood of the desired set point so robust stability of the system can be guaranteed. The developed algorithm can systematically handle both uncertain parameters and disturbances.

Finally, the developed control algorithm is applied to a partially simulated exothermic (PARSEX) reactor operated in the continuous mode. In the presence of uncertain parameters such as the reaction rate constant, heat of reaction, cooling water temperature and reactor feed rate, the developed control algorithm can regulate the reactor temperature to the set point despite uncertainty so robust stability of the system is ensured.

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6. Recommendations for future research

6.1 The state variables are assumed measurable in this research. In some applications, however, the state variables cannot be directly measured and only output variables can be measured. In next research, the output feedback controller should be developed to handle these situations.

6.2 The region of tube is constant in this research. The control performance can be further improved by using time-varying tubes.

7. Outputs of this research

7.1 ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

1) Bumroongsri, P., and Kheawhom, S. (2016). An off-line formulation of tube-based robust MPC using polyhedral invariant sets. Chem. Eng. Commun., 203, pp. 736-745. Impact Factor 1.104

2) Bumroongsri, P., and Kheawhom, S., Robust Model Predictive Control with time-varying tubes. (in press, DOI: 10.1007/s12555-016-0227-z) International Journal of Control, Automation and Systems. Impact Factor 0.954

7.2 การนำผลงานวิจัยไปใช้ประโยชน์

1) เชิงสาธารณะ โดยสถาบันการศึกษา
งานวิจัยที่เกิดขึ้นส่งเสริมให้เกิดการสร้างเครือข่ายความร่วมมือในการวิจัยระหว่างอาจารย์ต่างมหาวิทยาลัยที่ทำงานวิจัยในกลุ่มเดียวกัน โดยการดำเนินงานวิจัยภายใต้เครือข่ายพันธมิตรมหาวิทยาลัยเพื่อการวิจัย (Research University Network) ภายใต้กลุ่มคลัสเตอร์พลังงานซึ่งประกอบด้วย
1. รศ.ดร.สุรเทพ เขียวหอม
ภาควิชาวิศวกรรมเคมี คณะวิศวกรรมศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
2. ผศ.ดร.พรศิริ แก้วประดิษฐ์
ภาควิชาวิศวกรรมเคมี คณะวิศวกรรมศาสตร์ มหาวิทยาลัยสงขลานครินทร์
3. ผศ.ดร.พรชัย บำรุงศรี
ภาควิชาวิศวกรรมเคมี คณะวิศวกรรมศาสตร์ มหาวิทยาลัยมหิดล

2) เชิงวิชาการ โดยภาควิชาวิศวกรรมเคมี คณะวิศวกรรมศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
งานวิจัยนี้ก่อให้เกิดองค์ความรู้ใหม่ซึ่งนำไปใช้ในการพัฒนาการเรียนการสอนโดยการเป็นวิทยากรบรรยายพิเศษในหัวข้อ Current Development of MPC Technology สำหรับนักศึกษาปริญญาโทและเอก ภาควิชาวิศวกรรมเคมี คณะวิศวกรรมศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ผู้ประสานงาน
ผศ.ดร.อมรชัย อาภรณ์วิชานพ

ภาควิชาวิศวกรรมเคมี คณะวิศวกรรมศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

7.3 อื่นๆ

งานวิจัยนี้ก่อให้เกิดองค์ความรู้ใหม่ซึ่งได้รับเชิญเป็น Keynote speaker ณ. University of Tokyo ประเทศญี่ปุ่นในหัวข้อวิจัย

Bumroongsri, P., and Kheawhom, S. (2016). Tube-based robust model predictive control for systems with uncertain parameters and disturbances, The 7th International Symposium on Design, Operation and Control of Chemical Processes (PSE ASIA 2016), July 24-27, Tokyo, Japan.

Appendix

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Chemical Engineering Communications

ISSN: 0098-6445 (Print) 1563-5201 (Online) Journal homepage: http://www.tandfonline.com/loi/gcec20

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To cite this article: Pornchai Bumroongsri & Soorathep Kheawhom (2016) An Off-Line Formulation of Tube-Based Robust MPC Using Polyhedral Invariant Sets, Chemical Engineering Communications, 203:6, 736-745, DOI: 10.1080/00986445.2015.1089402

To link to this article: http://dx.doi.org/10.1080/00986445.2015.1089402

Accepted author version posted online: 17 Sep 2015.
Published online: 17 Sep 2015.

Chemical Engineering Communications, 203:736–745, 2016

ISSN: 0098-6445 print/1563-5201 online
DOI: 10.1080/00986445.2015.1089402

An Off-Line Formulation of Tube-Based Robust MPC Using Polyhedral Invariant Sets

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In this paper, an off-line formulation of tube-based robust model predictive control (MPC) using polyhedral invariant sets is proposed. A novel feature is the fact that no optimal control problem needs to be solved at each sampling time. Moreover, the proposed tube-based robust MPC algorithm can deal with the linear time-varying (LTV) system with bounded disturbance. The simulation results show that the state at each time step is restricted to lie within a tube whose center is the state of the nominal LTV system that converges to the origin. Finally, the state is kept within a tube whose center is at the origin, so robust stability is guaranteed. Satisfaction of the state and control constraints is guaranteed by employing tighter constraint sets for the nominal LTV system.

Keywords: Bounded disturbance; Linear time-varying system; Robust stability; Tube-based robust MPC

Introduction

Tube-based robust model predictive control (MPC) is an advanced control algorithm that can deal with model uncertainty. The basic idea of tube-based robust MPC is to maintain a state trajectory of an uncertain system inside a sequence of tubes (Rawlings and Mayne, 2009). Tube-based robust MPC is motivated by the fact that a real state trajectory differs from a state trajectory of a nominal system due to uncertainty (Mayne and Langson, 2001). Chisci et al. (2001) developed a tube-based robust model predictive controller for the linear time-invariant (LTI) system subject to bounded disturbance. The control law has the form $u = Kx + c$, where K is obtained by solving an unconstrained linear quadratic regulator (LQR) problem, x is the state, and c is the vanishing input, that is, $c_i = 0$ for $i \geq$ control horizon. The objective is to drive the state of an uncertain system to a terminal set while using c as little as possible. Constraint fulfillment is guaranteed by replacing the original constraints with more stringent ones. A larger control horizon implies better control performance at the price of a higher computational load, so a suitable trade-off is required. Langson et al. (2004) proposed tube-based robust MPC employing the time-varying control inputs instead of the LTI control law. A sequence of time-varying control inputs is obtained by solving an optimal control problem subject to the additional constraint sets in order to guarantee robust stability. Since the control inputs are time-varying, the proposed MPC algorithm can achieve better control performance than the conventional tube-based MPC algorithm using the LTI control law. The price to be paid is the computational complexity that increases with the prediction horizon.

Mayne et al. (2005) established robust exponential stability of the disturbance invariant set for the LTI system with bounded disturbance. The optimal control problem solved at each sampling time includes the initial state of the nominal model as a decision variable. The result is that the value function is zero in the disturbance invariant set so robust exponential stability of the disturbance invariant set can be established. The control law has the form $u = K(x - \bar{x}) + \bar{u}$, where $\bar{x}$ and $\bar{u}$ are the state and control inputs of the nominal system, respectively. Higher online computational time is required because the optimal control problem with increased decision variable has to be solved at each sampling time. In the case when the state of the LTI system with bounded disturbance is not exactly known, tube-based robust MPC can be implemented based on the observer state as proposed by Mayne et al. (2006). A simple Luenberger observer is employed to estimate the state. The state estimation and control errors at each time step are bounded by minimal robust positively invariant sets. Hence, the actual and observer states are restricted to lie within tubes whose center is the state of the nominal system. The control law has the form $u = K(\hat{x} - \bar{x}) + \bar{u}$, where $\hat{x}$ is the observer state. The controller is based on the observer state so the state $\bar{x}$ and control input $\bar{u}$ of the nominal system are subject to tighter constraint sets than the case when the state is exactly known. In Mayne et al. (2009), this idea is extended to the case when the initial state estimation error does not lie

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in the minimal robust positively invariant set but it lies in the time-varying set that converges to the minimal robust positively invariant set. In this case, higher online computational time is required because the time-varying set is computed online.

Tube-based robust MPC for tracking of LTI system with bounded disturbance was presented by Limon et al. (2010). The artificial steady variables are introduced as the decision variables in the optimal control problem. If the target is unreachable, the system will be steered to the neighborhood of the artificial target. The proposed MPC algorithm is suitable for the system whose target is significantly changed. However, the main drawback is that the proposed MPC algorithm requires high online computational time because some of the decision variables and constraints are introduced to the optimal control problem. Gonzalez et al. (2011) proposed tube-based robust MPC for tracking of linear time-varying (LTV) system subject to bounded disturbance. The proposed MPC algorithm requires an additional assumption that the time-varying parameter at each step within the prediction horizon is known a priori. Then, a reachable set at each time step is calculated instead of a disturbance invariant set in order to reduce the conservativeness. Although the conservativeness is reduced, the computational problem is more severe because both optimal control problem and reachable set are computed online.

In this paper, an off-line formulation of tube-based robust MPC using polyhedral invariant sets is proposed. The main contributions are that: (i) we propose tube-based robust MPC that solves all of the optimal control problems off-line, so no optimal control problem needs to be solved online; (ii) the proposed tube-based robust MPC algorithm can deal with LTV system subject to bounded disturbance. Unlike Gonzalez et al. (2011), the proposed algorithm does not require an additional assumption that the time-varying parameter at each step within the prediction horizon is known a priori. This article is organized as follows. The backgrounds of the conventional tube-based robust MPC are described in Backgrounds of Conventional Tube-Based Robust MPC section. In An Off-Line Formulation of Tube-Based Robust MPC Using Polyhedral Invariant Sets section, off-line tube-based robust MPC is proposed. In Illustrative Example section, the simulation results are presented. The conclusions are then drawn in Conclusions section.

Nomenclature

Given two subsets X and Y of $\mathbb{R}^n$, Minkowski set addition and set difference are defined, respectively, by $X \oplus Y := \{x + y | x \in X,\ y \in Y\}$ and $X \ominus Y := \{x | x \oplus Y \subseteq X\}$. The distance of a point $x \in \mathbb{R}^n$ from a set $Y \subseteq \mathbb{R}^n$ is denoted by $d(x, Y) := \inf\{|x - y| | y \in Y\}$ where $|\cdot|$ denotes the Euclidean norm. The distance of a point $x \in \mathbb{R}^n$ from a point $y \in \mathbb{R}^n$ is denoted by $d(x, y) := |x - y|$. For a matrix A, $A > 0$ means that A is a positive-definite matrix and $A < 0$ means that A is a negative-definite matrix. The spectral radius of a matrix A is denoted by $\rho(A)$. Conv$\{\cdot\}$ denotes the convex hull of the elements in $\{\cdot\}$.

Backgrounds of Conventional Tube-Based Robust MPC

In this section, some relevant backgrounds for the conventional tube-based robust MPC are presented. Consider the following discrete-time LTI system with disturbance

$$x^+ = Ax + Bu + w \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^n$ is the bounded disturbance, and $x^+ \in \mathbb{R}^n$ is the successor state. The system is subject to the state constraint $x \in \mathbb{X}$ and the control constraint $u \in \mathbb{U}$, where $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ are compact, convex, and each set contains the origin as an interior point. The disturbance is bounded, that is, $w \in \mathbb{W}$ where $\mathbb{W} \subset \mathbb{R}^n$ is compact, convex, and contains the origin as an interior point. The objective is to robustly stabilize the system in Equation (1). The presence of a persistent disturbance w means it is not possible to regulate the state x to the origin. The best that can be hoped for is to regulate the state to a neighborhood of the origin.

Let the nominal system be defined by

$$\bar{x}^+ = A\bar{x} + B\bar{u} \tag{2}$$

where $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in \mathbb{R}^m$ are the state and control inputs of the nominal system, respectively. The predicted nominal state trajectory and control sequence when the initial state is $\bar{x}_0$ are denoted by $\bar{\mathbf{x}} := \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_N\}$ and $\bar{\mathbf{u}} := \{\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{N-1}\}$, respectively, where N is the prediction horizon. Consider the following equation which is the difference between Equations (1) and (2)

$$x^+ - \bar{x}^+ = A(x - \bar{x}) + B(u - \bar{u}) + w \tag{3}$$

In order to counteract the effect of disturbance, the control law $u = K(x - \bar{x}) + \bar{u}$ is employed, where K is the disturbance rejection gain. The system in Equation (3) is rewritten as

$$x^+ - \bar{x}^+ = (A + BK)(x - \bar{x}) + w \tag{4}$$

We will bound $x^+ - \bar{x}^+$ by a robust positively invariant set Z. The definition of Z for the LTI system with disturbance is as follows:

Definition 1. The set $Z \subset \mathbb{R}^n$ is a robust positively invariant set of the LTI system with disturbance $x^+ = Ax + w$, if $AZ \oplus \mathbb{W} \subseteq Z$ for $\forall\ x \in Z$ and $\forall w \in \mathbb{W}$.

For the system in Equation (4), it is clear that if K is chosen such that $\rho(A + BK) < 1$, we can bound $x^+ - \bar{x}^+$ by a robust positively invariant set Z satisfying $(A + BK)Z \oplus \mathbb{W} \subseteq Z$ for $\forall (x - \bar{x}) \in Z$ and $\forall\ w \in \mathbb{W}$. It is desirable that Z be as small as possible. The minimal Z can be calculated as (Kolmanovsky and Gilbert, 1998):

$$Z = \bigoplus_{i=0}^{\infty} (A + BK)^i \mathbb{W} = \mathbb{W} \oplus (A + BK) \mathbb{W} \oplus (A + BK)^2 \mathbb{W} \oplus (A + BK)^3 \mathbb{W} \oplus \dots \tag{5}$$

If $(A+BK)$ is nilpotent with index s, that is, $(A+BK)^s=0$, then Z in Equation (5) can be finitely determined. In the case when $(A+BK)$ is not nilpotent, Z in Equation (5) can be approximated by using the method in Raković (2005) and Raković et al. (2005).

Since $x^+ - \bar{x}^+$ is bounded by Z, we can control the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$ in such a way that LTI system with disturbance $x^+ = Ax + Bu + w$ satisfies the original state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$, respectively. To achieve this, the tighter constraint sets for the nominal system are employed $\bar{x}_i \in \mathbb{X} \ominus Z$, $\bar{u}_i \in \mathbb{U} \ominus KZ$ for $i \in \{0, \ldots, N-1\}$. In order to ensure stability, an additional terminal constraint is employed $\bar{x}_N \in \overline{X}_f \subset \mathbb{X} \ominus Z$ where $\bar{X}_f$ is the terminal constraint set. The cost function for a trajectory of the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$ is

$$V_N(\bar{x}_0, \bar{u}) := \sum_{i=0}^{N-1} l(\bar{x}_i, \bar{u}_i) + V_f(\bar{x}_N) \tag{6}$$

where $l(\bar{x}_i, \bar{u}_i) := \frac{1}{2}[\bar{x}_i^T Q \bar{x}_i + \bar{u}_i^T R \bar{u}_i]$ is the stage cost; $V_f(\bar{x}_N) := \frac{1}{2}\bar{x}_N^T P \bar{x}_N$ is the terminal cost; Q, R, and P are the positive definite weighting matrices. The terminal constraint set and the terminal cost must satisfy the following usual assumptions (Mayne et al., 2000):

Assumption 1. $(A+BK)\bar{X}_f \subset \bar{X}_f, \bar{X}_f \subset \mathbb{X} \ominus Z, K\overline{X}_f \subset \mathbb{U} \ominus; KZ$.

Assumption 2. $V_f((A+BK)\bar{x}) + l(\bar{x}, K\bar{x}) \leq V_f(\bar{x}), \forall \bar{x} \in \bar{X}_f$.

In summary, at each sampling time the state x is measured and the following optimization problem is solved online:

$$\min_{\bar{x}_0, \bar{\mathbf{u}}} V_N(\bar{x}_0, \bar{\mathbf{u}}) \tag{7}$$

$$\text{such that } x \in \bar{x}_0 \oplus Z \tag{8}$$

$$\bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i, \; i \in \{0, \ldots, N-1\} \tag{9}$$

$$\bar{x}_i \in \mathbb{X} \ominus Z, \bar{u}_i \in \mathbb{U} \ominus KZ, i \in \{0, \ldots, N-1\} \tag{10}$$

$$\bar{x}_N \in \overline{X}_f \tag{11}$$

Then, the control law $u = K(x - \bar{x}) + \bar{u}$, $\bar{x} = \bar{x}_0$, $\bar{u} = \bar{u}_0$ is implemented to the process.

An Off-Line Formulation of Tube-Based Robust MPC Using Polyhedral Invariant Sets

It is seen that the conventional tube-based robust MPC in Backgrounds of Conventional Tube-Based Robust MPC section does not include a time-varying parameter in the problem formulation. Moreover, the optimal control problem must be solved at each sampling time. In this section, an off-line formulation of tube-based robust MPC is presented. No optimal control problem needs to be solved online. Additionally, the time-varying parameter is included in the problem formulation. Consider the following discrete-time LTV system with disturbance

$$x^+ = A^\lambda x + B^\lambda u + w \tag{12}$$

The descriptions for the state $x \in \mathbb{X}$, the control input $u \in \mathbb{U}$, and the disturbance $w \in \mathbb{W}$ are the same as in Backgrounds of Conventional Tube-Based Robust MPC section. The only difference is that, in this case, the matrices A^λ and B^λ are not constant but they vary with the time-varying parameter λ. The time-varying parameter λ can be measured at each sampling time but its future values are uncertain. We make the following assumption:

Assumption 3. $[A^\lambda B^\lambda] \in \text{Conv}\{[A_j B_j], \; \forall j \in 1, 2, \ldots, L\}$, where $[A_j B_j]$ are vertices of the convex hull and L is the number of vertices of the convex hull. The pair $[A_j \; B_j]$ is controllable.

Let the nominal LTV system be defined by

$$x'^+ = A^\lambda x' + B^\lambda u' \tag{13}$$

where $x' \in \mathbb{R}^n$ and $u' \in \mathbb{R}^m$ are the state and control inputs of the nominal LTV system, respectively. The predicted state trajectory and control sequence when the initial state is x'_0 are denoted by $\mathbf{x}' := \{x'_0, x'_1, \ldots, x'_N\}$ and $\mathbf{u}' := \{u'_0, u'_1, \ldots, u'_{N-1}\}$, respectively. Consider the following equation which is the difference between the systems in Equations (12) and (13):

$$x^+ - x'^+ = A^\lambda (x - x') + B^\lambda (u - u') + w \tag{14}$$

In order to counteract the effect of disturbance, the control law $u = K(x - x') + u'$ is employed where K is the disturbance rejection gain. The system in Equation (14) is rewritten as

$$x^+ - x'^+ = (A^\lambda + B^\lambda K)(x - x') + w \tag{15}$$

We will bound $x^+ - x'^+$ by a robust positively invariant set Z. The definition of Z for the LTV system with disturbance is as follows:

Definition 2. The set $Z \subset \mathbb{R}^n$ is a robust positively invariant set of the LTV system with disturbance $x^+ = A^\lambda x + w$, if $A^\lambda Z \oplus \mathbb{W} \subseteq Z$ for $\forall \; x \in Z$, $\forall w \in \mathbb{W}$, and $\forall A^\lambda \in \text{Conv}\{A_j, \forall j \in 1, 2, \ldots, L\}$.

For the system in Equation (15), it is clear that if K satisfies $(A_j + B_j K)^T P (A_j + B_j K) - P < 0, \forall j \in \{1, \ldots, L\}$ where P is a Lyapunov matrix, then $(A^\lambda + B^\lambda K)^T P (A^\lambda + B^\lambda K) - P < 0$, $\forall [A^\lambda B^\lambda] \in \text{Conv}\{[A_j B_j], \forall j \in 1, 2, \ldots, L\}$ and we can bound $x^+ - x'^+$ by a robust positively invariant set Z satisfying $(A^\lambda + B^\lambda K) Z \oplus \mathbb{W} \subseteq Z$ for $\forall (x - x') \in Z$, $\forall w \in \mathbb{W}$ and $\forall [A^\lambda B^\lambda] \in \text{Conv}\{[A_j B_j], \forall j \in 1, 2, \ldots, L\}$. It is desirable that Z be as small as possible. Unlike Equation (5), in the case of the LTI system with disturbance, the minimal Z of the LTV system with disturbance is $Z = \oplus_{i=0}^{\infty} (A^\lambda + B^\lambda K)^i \; \mathbb{W}$. Since $[A^\lambda B^\lambda] \in \text{Conv}\{[A_j B_j], \forall j \in 1, 2, \ldots, L\}$, the minimal

Z of the LTV system with disturbance can be calculated as

$$Z = \mathbb{W} \oplus \text{Conv}\{(A_j + B_jK)\mathbb{W}, \forall j \in 1, 2, \ldots, L\} \oplus \text{Conv}\{(A_j + B_jK)(A_l + B_lK), \mathbb{W} \forall j, l \in 1, 2, \ldots, L\} \oplus \{\text{Conv}(A_j + B_jK)(A_l + B_lK)(A_m + B_mK) \mathbb{W}, \forall j, l, m \in 1, 2, \ldots, L\} \oplus \ldots \quad (16)$$

Defining $F_s := \oplus_{i=0}^{s-1} (A^\lambda + B^\lambda K)^i \mathbb{W}$, F_s can be properly scaled for some finite integer s to obtain the outer approximation of Z in Equation (16). Since we can bound $x^+ - x'^+$ by Z, the following proposition can be established:

Proposition 1. If $x \in x' \oplus Z$ and $u = K(x - x') + u'$, then $x^+ \in x'^+ \oplus Z$ for $\forall w \in \mathbb{W}$ and $\forall [A^\lambda B^\lambda] \in \text{Conv}\{[A_jB_j], \forall j \in 1, 2, \ldots, L\}$.

Proposition 1 states that the control law $u = K(x - x') + u'$ keeps the state x of the LTV system with disturbance $x^+ = A^\lambda x + B^\lambda u + w$ close to the state x' of the nominal LTV system $x'^+ = A^\lambda x' + B^\lambda u'$. It is clear that if we can regulate x' to the origin, then x must be regulated to a robust positively invariant set Z whose center is at the origin. An off-line robust MPC algorithm for the nominal LTV system $x'^+ = A^\lambda x' + B^\lambda u'$ has been developed by Bumroongsri and Kheawhom (2012). The problem of regulating the state x' to the origin has been considered. In this approach, a sequence of stabilizing feedback gains F_i corresponding to a sequence of polyhedral invariant sets P_i, $i = \{1, \ldots, N_P\}$, where N_P is the number of polyhedral invariant sets, is precomputed off-line by solving the optimal control problems subject to LMI constraints (Boyd and Vandenberghe, 2004). At each sampling time, the state x' is measured and the smallest P_i containing x' is determined. Then, we set the real-time stabilizing feedback gain F equal to F_i and apply the control law $u' = Fx'$ to the process. The control law $u' = Fx'$ minimizes the following cost function:

$$V_\infty(x'_0, \mathbf{u}') := \max_{[A^\lambda B^\lambda] \in \text{Conv}\{[A_jB_j], \forall j \in 1,2,\ldots,L\}} \sum_{i=0}^{\infty} x_i'^T Qx'_i + (Fx'_i)^T R(Fx'_i) \quad (17)$$

where x'_i is the state of the nominal LTV system at prediction time i and Q and R are the positive-definite weighting matrices. Additionally, the control law $u' = Fx'$ ensures that the Lyapunov function $V(x') := x'^T Px'$ is a strictly decreasing function satisfying

$$V(x'^+) - V(x') \leq -x'^T Qx' - (Fx')^T R(Fx'), \forall [A^\lambda B^\lambda] \in \text{Conv}\{[A_jB_j], \forall j \in 1, 2, \ldots, L\} \quad (18)$$

where P is a Lyapunov matrix. At each sampling time, although the future values of the time-varying parameter λ in the prediction horizon (which is the infinite horizon in this case) are unknown, the satisfaction of Equation (18) for the stabilizing feedback gain F ensures that the time-varying set of all future states $R_{i+1} = (A^\lambda + B^\lambda F)R_i$, $R_o = \{x'_0\}$, converges to the origin $d(0, R_{i+1}) \to 0$, $\forall [A^\lambda B^\lambda] \in \text{Conv}\{[A_jB_j], \forall j \in 1, 2, \ldots, L\}$. Hence, robust stability of the nominal LTV system $x'^+ = (A^\lambda + B^\lambda F)x'$ is guaranteed. In order to guarantee satisfaction of the original state and control constraints, $x \in \mathbb{X}$ and $u \in \mathbb{U}$, we must employ tighter constraint sets for the nominal LTV system, that is, $x' \in \mathbb{X} \ominus Z$ and $Fx' \in \mathbb{U} \ominus KZ$. The control law $u = K(x - x') + u'$ is now rewritten as $u = K(x - x') + Fx'$. An important consequence is the following result:

Proposition 2 If $x \in x' \oplus Z$, $x' \in \mathbb{X} \ominus Z$, and $Fx' \in \mathbb{U} \ominus KZ$, then the control law $u = K(x - x') + Fx'$ of the LTV system with disturbance $x^+ = A^\lambda x + B^\lambda u + w$ ensures satisfaction of the original constraints $x \in \mathbb{X}$, $u \in \mathbb{U}$ for $\forall w \in \mathbb{W}$ and $\forall [A^\lambda B^\lambda] \in \text{Conv}\{[A_jB_j], \forall j \in 1, 2, \ldots, L\}$.

Proposition 2 states that the control law $u = K(x - x') + Fx'$ ensures satisfaction of the original state and control constraints. In summary, off-line tube-based robust MPC for LTV system with disturbance $x^+ = A^\lambda x + B^\lambda u + w$ can be formulated as follows:

Off-line:

Step 1: Calculate the disturbance rejection gain K satisfying $(A_j + B_jK)^T P(A_j + B_jK) - P < 0$, $\forall j \in \{1, \ldots, L\}$. Then, calculate a tube Z in Equation (16).

Step 2: Calculate a sequence of stabilizing feedback gains F_i and the corresponding sequence of polyhedral invariant sets P_i, $i = \{1, \ldots, N_P\}$ using the method in Bumroongsri and Kheawhom (2012) with tighter constraint sets for the nominal LTV system, that is, $x' \in \mathbb{X} \ominus Z$ and $Fx' \in \mathbb{U} \ominus KZ$.

Online:

At the first sampling time $(t = 0)$: Measure the state x and the time-varying parameter λ. Find the smallest polyhedral invariant set P_i containing the measured state x, set $F = F_i$ and apply the control law $u = Fx$ to the process. Then, calculate x'^+ from $x'^+ = (A^\lambda + B^\lambda F)x$. (Note that at the first sampling time, $x = x'$ so the control law $u = K(x - x') + Fx'$ is reduced to $u = Fx$.)

At each sampling time $(t > 0)$: Measure the state x and the time-varying parameter λ. Find the smallest polyhedral invariant set P_i containing x' (which is calculated from the previous step), set $F = F_i$, and apply the control law $u = K(x - x') + Fx'$ to the process. Then, calculate x'^+ from $x'^+ = (A^\lambda + B^\lambda F)x'$.

Theorem 1. The proposed tube-based MPC algorithm steers any initial state x of the system $x^+ = A^\lambda x + B^\lambda u + w$ in a sequence of polyhedral invariant sets P_i, $i = \{1, \ldots, N_P\}$ to a robust positively invariant set Z whose center is at the origin and thereafter maintains the state in Z for $\forall w \in \mathbb{W}$ and $\forall [A^\lambda B^\lambda] \in \text{Conv}\{[A_jB_j], \forall j \in 1, 2, \ldots, L\}$.

Proof. Consider the following difference equation between $x^+ = A^\lambda x + B^\lambda u + w$ and $x'^+ = A^\lambda x' + B^\lambda u'$, where $u = K(x - x') + Fx'$ and $u' = Fx'$,

$$x^+ - x'^+ = (A^\lambda + B^\lambda K)(x - x') + w \quad (19)$$

The disturbance rejection gain K satisfies $(A_j + B_j K)^T P(A_j + B_j K) - P < 0$, $\forall j \in \{1, \ldots, L\}$ so $x^+ - x'^+$ is bounded by a robust positively invariant set Z, that is, $x^+ \in x'^+ \oplus Z$. Since the stabilizing feedback gain F ensures that the Lyapunov function is a strictly decreasing function satisfying Equation (18), the state x'^+ must converge to the origin $d(x'^+, 0) \to 0$. Since $x^+ \in x'^+ \oplus Z$, x^+ must converge to a tube Z whose center is at the origin $d(x^+, Z) \to 0$. Finally, the disturbance rejection controller $u = Kx$ keeps the state within a tube Z whose center is at the origin. □

Corollary 1. The state of the LTV system with disturbance $x^+ = A^\lambda x + B^\lambda u + w$ at each time step is restricted to lie within a tube whose center is the state of the nominal LTV system $x'^+ = A^\lambda x' + B^\lambda u'$.

Remark 1. For any initial state x contained in the first polyhedral invariant set P_1 (which is largest in the sequence ofP_i, $i = \{1, \ldots, N_p\}$), there exists a control law that is able to steer the state to a tube Z whose center is at the origin by satisfying all state and control constraints $x \in \mathbb{X}$, $u \in \mathbb{U}$ for $\forall w \in \mathbb{W}$ and $\forall [A^\lambda B^\lambda] \in \text{Conv}\{[A_j\ B_j], \forall j \in 1, 2, \ldots, L\}$. Hence, the region of attraction for the proposed MPC algorithm is P_1.

Illustrative Example

Example 1. Consider the following LTV system with bounded disturbance

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w \qquad (20)$$

The state $x \in \mathbb{X}$, where $\mathbb{X}: = \{x \in \mathbb{R}^2 | [0\ 1] x \le 2\}$; the control $u \in \mathbb{U}$, where $\mathbb{U}: = \{u \in | |u| \le 1\}$; the disturbance $w \in \mathbb{W}$, where $\mathbb{W} := \{w \in \mathbb{R}^2 | [-0.1\ -0.1]^T \le w \le [0.1 0.1]^T\}$; and the time-varying parameter $\lambda \in \mathbb{L}$, where $\mathbb{L} := \{\lambda \in \mathbb{R} | 0.9 \le \lambda \le 1.1\}$. The weighting matrices in the cost function (Equation (17)) are given as $Q = I$ and $R = 0.01$. The following nominal LTV system:

$$x'^+ = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} x' + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u' \qquad (21)$$

is subject to tighter state and control constraints, that is, $x' \in \mathbb{X} \ominus Z$ and $u' \in \mathbb{U} \ominus KZ$. The disturbance rejection gain $K = [-0.66\ -1.33]$ satisfies $(A_j + B_j K)^T P(A_j + B_j K) - P < 0$, $\forall j \in \{1,2\}$. The difference equation between Equations (20) and (21) can be written as

$$x^+ - x'^+ = \begin{bmatrix} 1 & 1 \\ 0 & \lambda \end{bmatrix} (x - x') + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} (u - u') + w \qquad (22)$$

The closed-loop system is simulated using the initial state $x = x' = [-5\ -2]^T$. The time-varying parameter λ and the disturbance w are varied as $\lambda = 1 + 0.1\sin(4k)$ and $w = [0.1\sin(4k) 0.1\sin(4k)]^T$, respectively, where $k \in \{1, \ldots, 19\}$ is the simulation horizon.

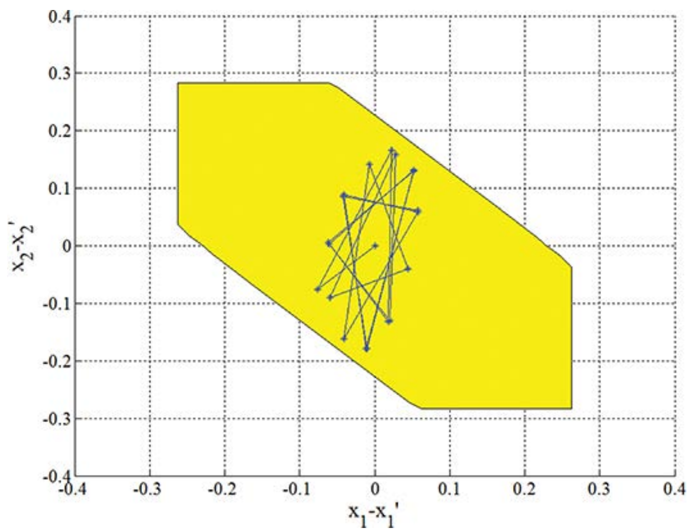

Fig. 1. The robust positively invariant set Z precomputed off-line. The set Z is shown in yellow.

Figure 1 shows a robust positively invariant set Z precomputed off-line. The set Z is shown in yellow. The blue line represents the trajectory of the difference Equation (22). Starting from the origin, it is seen that the trajectory of the difference equation is restricted to lie within the set Z.

Figure 2 shows a sequence of 10 polyhedral invariant setsP_i, $i \in \{1, \ldots, 10\}$ precomputed off-line. In this example, only 10 polyhedral invariant sets are precomputed because P_i are almost constant for $i > 10$. The red line represents the trajectory of the nominal LTV system (Equation (21)). Starting from the initial point $x = x' = [-5\ -2]^T$, the state of the nominal LTV system at each time step is restricted to lie within a sequence of 10 polyhedral invariant sets P_i, $i \in \{1, \ldots, 10\}$ precomputed off-line. Finally, the state of the nominal LTV system converges to the origin.

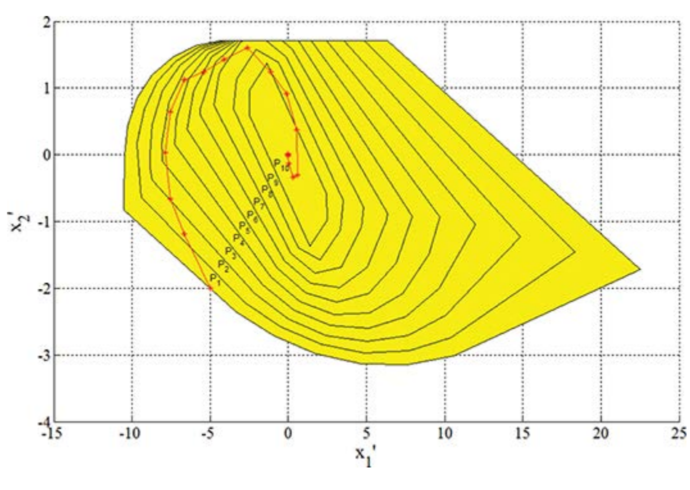

Fig. 2. A sequence of 10 polyhedral invariant sets P_i, $i \in \{1, \ldots, 10\}$ precomputed off-line. The polyhedral invariant sets are shown in yellow.

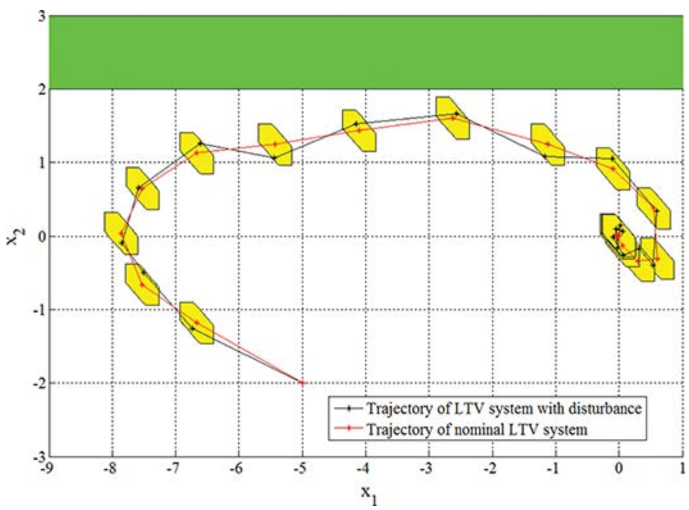

Fig. 3. The trajectory of the system when $\lambda = 1 + 0.1\sin(4k)$ and $w = [0.1\sin(4k)\ 0.1\sin(4k)]^T$. The infeasible region of state con-

The trajectory of the LTV system with disturbance (Equation (20)) is shown in Figure 3. The region shown in green is the infeasible region of the state constraint $\mathbb{X}:=\{x\in\mathbb{R}^2|[0\ 1]x\le 2\}$. The red line corresponds to the trajectory of the nominal LTV system (Equation (21)). The cross-section of a tube Z precomputed off-line is shown in yellow. It is seen that the state of the LTV system with disturbance at each time step is restricted to lie within a tube Z whose center is the state of the nominal LTV system that converges to the origin. Finally, the state of the LTV system with disturbance is kept within a tube Z whose center is at the origin.

Figure 4 shows the control input as a function of sampling time. The region shown in yellow is $\mathbb{U}\ominus KZ$. The red line corresponds to the control input of the nominal LTV system (Equation (21)). The black line corresponds to the control input of the LTV system with disturbance (Equation (20)). It can be observed that the control input of the

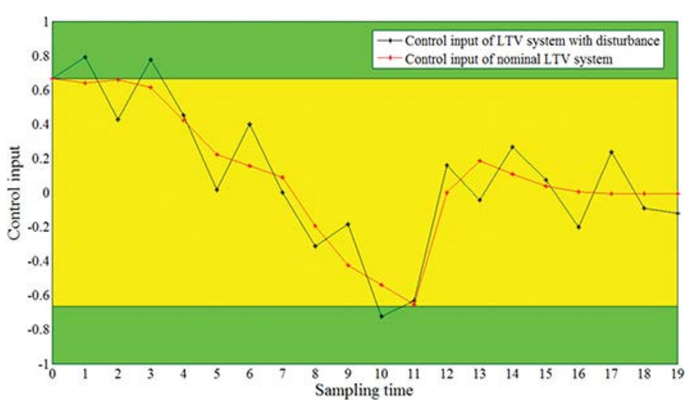

Fig. 4. The control input satisfying the input constraint $\mathbb{U}:=\{u\in\mathbb{R}\,||u|\le 1\}$. The tightened input constraint $\mathbb{U}\ominus KZ$ is shown in yellow. The original input constraint $\mathbb{U}$ is shown in green.

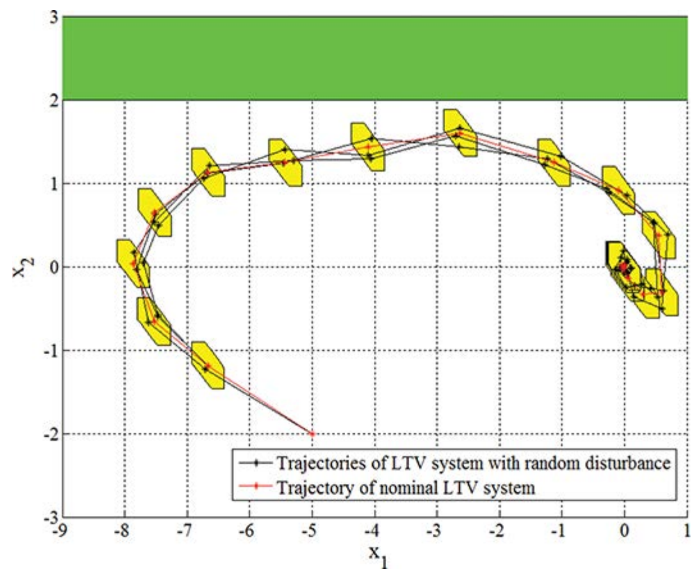

Fig. 5. The trajectories of the system when $\lambda = 1 + 0.1\sin(4k)$ and w are randomly time-varying. The infeasible region of state constraint is shown in green. The cross-section of tube is shown in yellow.

nominal LTV system is restricted to lie within the region $\mathbb{U}\ominus KZ$ so that the control input of the LTV system with disturbance satisfies the control constraint $\mathbb{U}:=\{u\in||u|\le 1\}$.

Figure 5 shows the trajectories of the LTV system with disturbance (Equation (20)) when the disturbance $w\in\mathbb{W}$ is randomly time-varying. At each time step, the states of the LTV system with random disturbance lie within a tube Z whose center is the state of the nominal LTV system that converges to the origin.

Example 2. In this example, the proposed algorithm is applied to a non-isothermal continuous stirred tank reactor (CSTR) in which an irreversible exothermic reaction $A\rightarrow B$ takes place. The dimensionless modeling equations of this CSTR can be written as (Silva and Kwong, 1999; Nagrath et al., 2002)

$$\frac{dx_1}{d\tau} = q(x_{1f} - x_1) - \varphi x_1 \exp\left(\frac{x_2}{1+\frac{x_2}{\gamma}}\right) + w_1 \tag{23}$$

$$\frac{dx_2}{d\tau} = q(x_{2f} - x_2) - \delta(x_2 - x_3) + \beta\varphi x_1 \exp\left(\frac{x_2}{1+\frac{x_2}{\gamma}}\right) + w_2 \tag{24}$$

$$\frac{dx_3}{d\tau} = \delta_1[q_c(x_{3f} - x_3) + \delta\delta_2(x_2 - x_3)] + w_3 \tag{25}$$

where x_1 is the dimensionless concentration of reactant A, x_2 is the dimensionless reactor temperature, and x_3 is the dimensionless cooling jacket temperature. The manipulated variable is the dimensionless coolant flow rate q_c. The disturbances acting on the system are w_1, w_2, and w_3. By linearizing and discretizing Equation (23) to Equation (25) with a sampling period ΔT, the following discrete-time state space model is obtained:

$$\begin{bmatrix} \bar{x}_1(k+1) \\ \bar{x}_2(k+1) \\ \bar{x}_3(k+1) \end{bmatrix} = \begin{bmatrix} 1+\Delta T[-q-\varphi\kappa(x_{2S})] & -\Delta T\left[\frac{\varphi x_{1S}\kappa(x_{2S})}{\left(1+\frac{x_{2S}}{\gamma}\right)^2}\right] & 0 \\ \Delta T[\beta\varphi\kappa(x_{2S})] & 1+\Delta T\left[-q-\delta+\frac{\beta\varphi\kappa(x_{2S})x_{1S}}{\left(1+\frac{x_{2S}}{\gamma}\right)^2}\right] & \Delta T\delta \\ 0 & \Delta T\delta\delta_1\delta_2 & 1-\Delta T[\delta_1 q_{cS}+\delta\delta_1\delta_2] \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \\ \bar{x}_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Delta T\delta_1[x_{3f}-x_{3S}] \end{bmatrix} \bar{q}_c(k) + \Delta T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{w}_1(k) \\ \bar{w}_2(k) \\ \bar{w}_3(k) \end{bmatrix} \quad (26)$$

where $\bar{x}_1(k) = x_1(k) - x_{1S}$, $\bar{x}_2(k) = x_2(k) - x_{2S}$, $\bar{x}_3(k) = x_3(k) - x_{3S}$, $\bar{q}_c(k) = q_c(k) - q_{cS}$, $\bar{w}_1(k) = w_1(k) - w_{1S}$, $\bar{w}_2(k) = w_2(k) - w_{2S}$, $\bar{w}_3(k) = w_3(k) - w_{3S}$, and $\kappa(x_{2S}) = \exp\left(x_{2S}/1+\frac{x_{2S}}{\gamma}\right)$. The model parameter values are shown in Table I. The Damkohler number ϕ is considered to be uncertain and its value is randomly time-varying between $\varphi_{\min} = 0.0648$ and $\varphi_{\max} = 0.0792$. The disturbances $\bar{w}_1(k)$, $\bar{w}_2(k)$, and $\bar{w}_3(k)$ are randomly time-varying between -0.01 and 0.01. The constraints are $|\bar{x}_1(k)| \leq 0.5$ and $|\bar{q}_c(k)| \leq 1.0$. The weighting matrices in the cost function in Equation (17) are $Q = I$ and $R = 0.1$. The objective is to regulate the state from $(\bar{x}_1(0), \bar{x}_2(0), \bar{x}_3(0)) = (0, 5, 0)$ to the neighborhood of the origin by manipulating $\bar{q}_c(k)$.

Figure 6 shows a sequence of four polyhedral invariant sets P_i, $i \in \{1, \ldots, 4\}$ precomputed off-line. Figure 7 shows the trajectory of the uncertain CSTR. The black line is the trajectory of the uncertain CSTR with disturbances (CSTR containing both time-varying parameter and disturbances). The red line is the trajectory of the uncertain CSTR with no disturbances (CSTR containing only time-varying parameter). It can be observed that the trajectory of the uncertain CSTR with disturbances lies in a sequence of tubes

Table I. The model parameter values in Example 2

Parameter	Value	Parameter	Value
q	1.0	δ	0.3
x_{1f}	1.0	β	8.0
φ	0.0648–0.0792	δ_1	10
γ	20	x_{3f}	−1.0
x_{2f}	0.0	δ_2	1.0
x_{1S}	0.8933	w_{1S}	0.0
x_{2S}	0.5193	w_{2S}	0.0
x_{3S}	−0.5950	w_{3S}	0.0
q_{cS}	1.65		

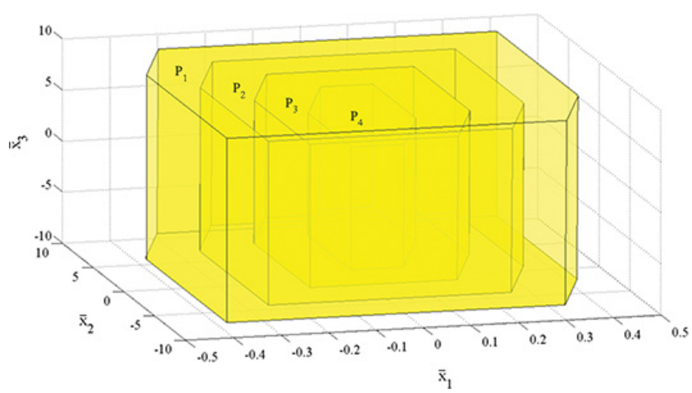

Fig. 6. A sequence of four polyhedral invariant sets P_i, $i \in \{1, \ldots, 4\}$ precomputed off-line. The polyhedral invariant sets are shown in yellow.

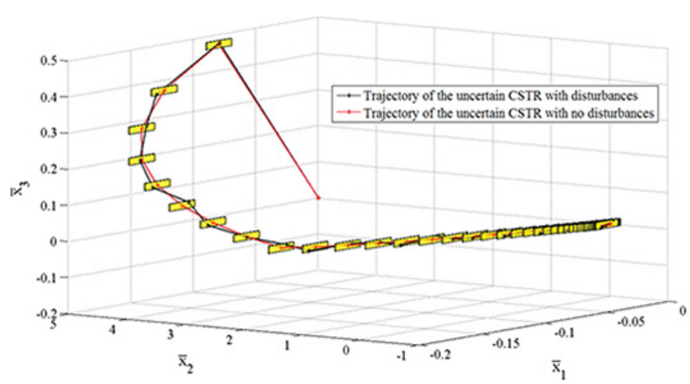

Figure 7. The trajectory of the uncertain CSTR. The cross-section of tube is shown in yellow.

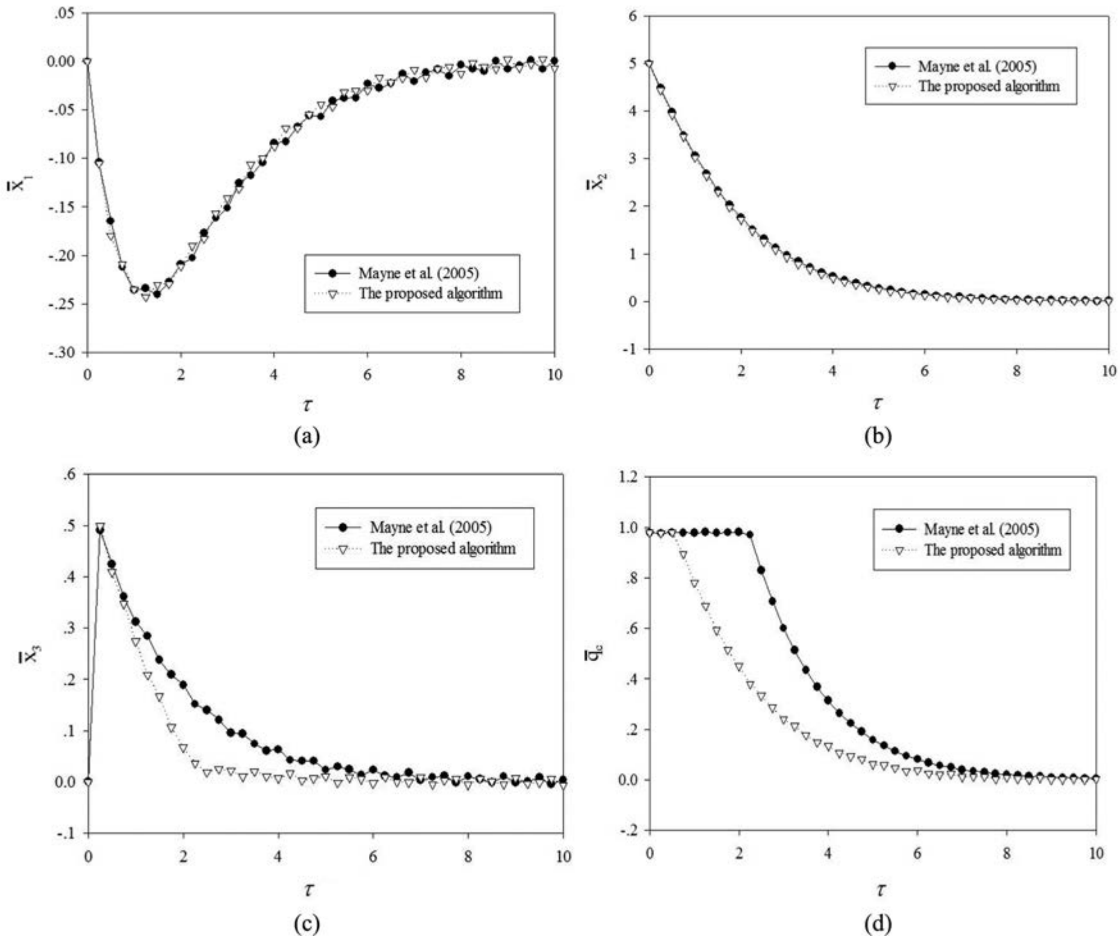

Fig. 8. The control performance (a) dimensionless concentration of reactant A; (b) dimensionless reactor temperature; (c) dimensionless cooling jacket temperature; and (d) dimensionless coolant flow rate.

shown in yellow. Finally, the state of the uncertain CSTR with disturbances is steered to a tube whose center is at the origin.

The proposed algorithm will be compared with tube-based robust MPC algorithm of Mayne et al. (2005) in which the online optimization problem must be solved at each sampling time. In Mayne et al. (2005), only disturbances are included in the controller design so there is a mismatch between the model and the process when the time-varying parameter is present. From Figure 8, it is seen that the proposed algorithm is able to steer the state of the uncertain CSTR with disturbances to the neighborhood of the origin faster than the algorithm of Mayne et al. (2005). Moreover, the proposed algorithm requires significantly less online computational time, as shown in Table II. The computations are performed using Intel Core 2 Duo (2.53 GHz), 2 GB RAM.

Table II. The online computational time

Algorithm	Online computational time for each step (s)
Mayne et al. (2005)	0.067
The proposed algorithm	0.015

Conclusions

In this paper, we present an offline tube-based robust MPC algorithm using polyhedral invariant sets. All of the optimal control problems are solved off-line so no optimal control problem needs to be solved online. The simulation results show that the state at each time step of the LTV system with disturbance is restricted to lie within a tube whose center is the state of the nominal LTV system that converges to the origin. Hence, the state of the LTV system with disturbance converges to a tube whose center is at the origin. Robust stability and satisfaction of the state and control constraints are guaranteed. In future work, the proposed algorithm can be extended to the nonlinear system with bounded disturbance.

Funding

This research project is supported by Mahidol University and Thailand Research Fund (TRF).

Nomenclature

x	state
u	input
c	vanishing input
$\bar{x}$	state of nominal system
$\hat{x}$	observer state
$\bar{u}$	input of nominal system
N	prediction horizon
K	disturbance rejection gain
Z	robust positively invariant set
w	disturbance
$\bar{X}_f$	terminal constraint set
Q	state weighting matrix
R	input weighting matrix
λ	time-varying parameter
x'	state of nominal LTV system
u'	input of nominal LTV system
P	Lyapunov matrix
P_i	polyhedral invariant set i
F_i	stabilizing feedback gain corresponding to P_i
F	real-time stabilizing feedback gain
$V(x')$	Lyapunov function of variable x'

Example 1

x_1	state 1 of LTV system with disturbance
x_2	state 2 of LTV system with disturbance
x'_1	state 1 of nominal LTV system
x'_2	state 2 of nominal LTV system
u	input of LTV system with disturbance
u'	input of nominal LTV system

Example 2

x_1	dimensionless concentration of reactant A
x_2	dimensionless reactor temperature
x_3	dimensionless cooling jacket temperature
q	dimensionless reactor feed-flow rate
q_c	dimensionless coolant flow rate
w_i	disturbance variable i
$\bar{x}_i(k)$	deviation form of state i at time k
x_{iS}	equilibrium point of state i
x_{1f}	dimensionless reactor feed concentration
x_{2f}	dimensionless reactor feed temperature
x_{3f}	dimensionless cooling jacket feed temperature
β	dimensionless heat of reaction
γ	dimensionless activation energy
δ	dimensionless heat transfer coefficient
δ_1	dimensionless volume ratio of reactor to cooling jacket
δ_2	dimensionless density $\times$ heat capacity ratio of reactor to cooling jacket
ϕ	Damkohler number
τ	dimensionless time

Mathematical Symbols

$X \oplus Y$	Minkowski set addition between X and Y
$X \ominus Y$	Minkowski set difference between X and Y
$d(x, Y)$	distance of a point x from a set Y
$d(x, y)$	distance of a point x from a point y
$\lvert\cdot\rvert$	Euclidean norm
$A > 0$	A is a positive-definite matrix
$A < 0$	A is a negative-definite matrix
$\rho(A)$	spectral radius of a matrix A
$\text{Conv}\{\cdot\}$	convex hull of the elements in $\{\cdot\}$

Abbreviations

MPC	model predictive control
LTI	linear time-invariant
LQR	linear quadratic regulator
LTV	linear time-varying
LMI	linear matrix inequality

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International Journal of Control, Automation and Systems 15(X) (2017) 1-6
http://dx.doi.org/10.1007/s12555-016-0227-z
ISSN:1598-6446 eISSN:2005-4092
http://www.springer.com/12555

Robust Model Predictive Control with Time-varying Tubes

Pornchai Bumroongsri* and Soorathep Kheawhom

Abstract: This paper focuses on the problem of robustly stabilizing uncertain discrete-time systems subject to bounded disturbances. The proposed tube-based model predictive controller ensures that all possible realizations of the state trajectory lie in the time-varying tubes so robust stability and satisfaction of the state and input constraints are guaranteed. The time-varying tubes are computed off-line so the on-line computational time is tractable. At each sampling time, the precomputed time-varying tubes are included in the optimal control problem as the constraints in the prediction horizon and only a quadratic programming problem is solved. In comparison to the algorithm that calculates the time-varying tubes on-line, the proposed algorithm can achieve the same level of control performance while the on-line computational time is greatly reduced.

Keywords: Bounded disturbances, model predictive controller, robust stability, time-varying tubes.

1. INTRODUCTION

Tube-based model predictive control (MPC) is an advanced control strategy that has been originally developed to deal with bounded disturbances. Its basic idea is to restrict all possible realizations of the state trajectory in a sequence of tubes so robust stability is ensured [1,2].

Mayne *et al*. [3] proposed a tube-based MPC algorithm that can ensure robust exponential stability of linear time-invariant (LTI) systems with bounded disturbances. An optimal control problem for the nominal systems (systems without disturbances) is solved at each sampling time by replacing the original constraint with more stringent one. A tube used as a constraint in each prediction horizon is a disturbance invariant set that includes all possible realizations of the disturbances for all future time steps. This leads to the reduction of size of the feasible region for the state and input constraints of the nominal system. Thus, the conservative result may be obtained.

Limon *et al*. [4] proposed a tube-based MPC algorithm for tracking of LTI systems with bounded disturbances. The artificial steady state variables are introduced in the optimal control problem as the decision variables. The proposed algorithm is suitable for the case when the target is significantly changed. If the target is unreachable, the state will be driven to the neighborhood of the artificial target instead. However, a tube used in the optimal control problem for each prediction horizon is also overestimated by including the effects of all possible realizations of the disturbances for all future time steps.

A tube-based MPC algorithm for linear time-varying (LTV) systems with bounded disturbances has been developed by Bumroongsri [5]. The proposed control law is a linear combination of two components. The first component steers the state of the nominal LTV systems to the origin. The second component keeps the state at each time step of the LTV systems with bounded disturbances within a tube whose center is the state of the nominal LTV systems. The disturbance invariant tube is precomputed off-line and the optimal control problem is solved on-line with tighter constraint sets. An off-line formulation of tube-based robust MPC has been developed by Bumroongsri and Kheawhom [6]. The proposed algorithm is also based on the disturbance invariant tube that takes into account the effects of all possible realizations of the disturbances for all future time steps.

An idea to reduce the conservativeness by computing on-line the time-varying tubes is introduced by Gonzalez *et al*. [7]. Instead of precomputing the disturbance invariant tube, the basic idea is to compute on-line the reachable set. In addition to the optimal control problem solved at each sampling time, the Minkowski sum algorithm has to be implemented on-line which leads to increased computational complexity. Mayne *et al*. [8] introduced an idea to use a time-varying tube for the case when the initial uncertainty set for the unknown state is large. The size of the tube is initially large and then converges to that of the disturbance invariant tube. After reaching the steady state, the size of the tube is still overestimated. This may lead to the conservativeness due to the fact that the effects of the

Manuscript received April 14, 2016; revised August 4, 2016; accepted August 31, 2016. Recommended by Associate Editor Yingmin Jia under the direction of Editor Myo Taeg Lim. This work was supported by Mahidol University and Thailand Research Fund (TRG5880084).

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disturbances are included for all future time steps.

In this paper, robust MPC with time-varying tubes is proposed. At each sampling time, the precomputed time-varying tubes are included in the optimal control problem as the constraints in the prediction horizon. The optimal control problem solved on-line is quadratic programming so the on-line computational time is tractable. The proposed algorithm can ensure robust stability and satisfaction of the state and input constraints for uncertain systems with bounded disturbances. The uncertain parameters and disturbances are not necessary to be measurable. This article is organized as follows: The problem statement is presented in Section 2. The proposed algorithm is presented in Section 3. An illustrative example is presented in Section 4. The conclusions are drawn in Section 5.

1.1. Nomenclature

Given two subsets X and Y of $\mathbb{R}^n$, Minkowski set addition and set difference are defined, respectively, by $X \oplus Y := \{x+y | x \in X, y \in Y\}$ and $X \ominus Y := \{x | x \oplus Y \subseteq X\}$.

2. PROBLEM STATEMENT

Consider the following uncertain discrete-time system with bounded disturbance

$$x^+ = A^\lambda x + B^\lambda u + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $w \in \mathbb{R}^n$ is the disturbance and $x^+ \in \mathbb{R}^n$ is the successor state. This system is subject to the state constraint $x \in \mathbb{X}$ and the input constraint $u \in \mathbb{U}$ where $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ are compact, convex and each set contains the origin as an interior point. The disturbance is bounded $w \in \mathbb{W}$ where $\mathbb{W} \subset \mathbb{R}^n$ is compact, convex and contains the origin as an interior point. The matrices A^λ and B^λ satisfy

$$[A^\lambda \; B^\lambda] \in \mathrm{Conv}\{[A_1 \; B_1], \ldots, [A_L \; B_L]\}, \tag{2}$$

where $\mathrm{Conv}\{\cdot\}$ denotes the convex hull of all elements in $\{\cdot\}$, $[A_j \; B_j]$ are the vertices of convex hull and L is the number of vertices of convex hull. Any $[A^\lambda \; B^\lambda]$ is the linear combination of the vertices such that

$$[A^\lambda \; B^\lambda] = \sum_{j=1}^{L} \lambda_j [A_j \; B_j], \tag{3}$$

where $[\lambda_1, \lambda_2, \ldots, \lambda_L]$ are the uncertain parameters satisfying $\sum_{j=1}^{L} \lambda_j = 1$. The uncertain parameters and the disturbances are not necessary to be measurable. The presence of bounded disturbance w means that it is only possible to regulate the state x to the neighborhood of the origin. The objective of this paper is to find the control law that is able to maintain all possible realizations of the state trajectory in the time-varying tubes despite the uncertain parameters and disturbances while satisfying all of the state and input constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$. The definition for the time-varying tube Z_i is given as follows:

Definition 1: The set $Z_i \subset \mathbb{R}^n$ is said to be a time-varying tube for the system $x^+ = Ax + w$ if $\forall x \in Z_i$ implies $x^+ \in Z_{i+1}$ for $\forall i \geq 0$ and $\forall w \in \mathbb{W}$.

Instead of computing a tube Z as an outer approximation of the disturbance invariant set as proposed by Raković *et al.* [9], a sequence of time-varying tubes is computed off-line in this paper. At each sampling time, the precomputed time-varying tubes are included in the optimal control problem as the constraints in the prediction horizon. By doing so, robust stability and satisfaction of the state and input constraints are guaranteed while the effect of the disturbance is not overestimated.

3. A SYNTHESIS APPROACH FOR ROBUST MPC WITH TIME-VARYING TUBES

This section presents a synthesis approach to robust MPC with time varying tubes. Consider the following nominal system (system without uncertain parameters $[\lambda_1, \lambda_2, \ldots, \lambda_L]$ and disturbance w)

$$\bar{x}^+ = A\bar{x} + B\bar{u}, \tag{4}$$

where $A = \frac{1}{L}\sum_{j=1}^{L} A_j$, $B = \frac{1}{L}\sum_{j=1}^{L} B_j$, $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in \mathbb{R}^m$ are the state and input of the nominal system, respectively. The predicted nominal state and input in the prediction horizon from the initial state $\bar{x}_0$ are denoted by $\overline{X} := \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N\}$ and $\overline{U} := \{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1}\}$, respectively, where N is the prediction horizon. Consider the following system which is the difference between (1) and (4)

$$x^+ - \bar{x}^+ = A(x - \bar{x}) + B(u - \bar{u}) + d, \tag{5}$$

where the disturbance d is given by $d := (A^\lambda - A)x + (B^\lambda - B)u + w$. Since $x \in \mathbb{X}$, $u \in \mathbb{U}$ and $w \in \mathbb{W}$, it is seen that $d \in \mathbb{D}$ where $\mathbb{D} := (A^\lambda - A)\mathbb{X} \oplus (B^\lambda - B)\mathbb{U} \oplus \mathbb{W}$. In order to deal with the disturbance d, the control law $u = \bar{u} + K(x - \bar{x})$ is employed where K is the disturbance rejection gain satisfying $(A^\lambda + B^\lambda K)^T P (A^\lambda + B^\lambda K) - P < 0$ for $\forall [A^\lambda \; B^\lambda] \in \mathrm{Conv}\{[A_1 \; B_1], \ldots, [A_L \; B_L]\}$. The Lyapunov matrix P and the disturbance rejection gain K can be found by solving the linear matrix inequality (LMI) optimization problem [10]. The difference equation (5) can be rewritten as

$$x^+ - \bar{x}^+ = (A + BK)(x - \bar{x}) + d. \tag{6}$$

The difference between $x^+ - \bar{x}^+$ is usually bounded within a disturbance invariant set $Z \subset \mathbb{R}^n$ [3–6]. The definition for the disturbance invariant set Z is given as follows:

Definition 2: The set $Z \subset \mathbb{R}^n$ is said to be a disturbance invariant set for the system (6) if $(A+BK)Z \oplus \mathbb{D} \subseteq Z$ for $\forall(x-\bar{x}) \in Z$ and $\forall d \in \mathbb{D}$.

According to Kolmanovsky and Gilbert [11], the minimal disturbance invariant tube Z is computed as

$$\begin{aligned} Z = &\bigoplus_{i=0}^{\infty}(A+BK)^i\mathbb{D} \\ =&\mathbb{D}\oplus(A+BK)\mathbb{D}\oplus(A+BK)^2\mathbb{D} \\ &\oplus(A+BK)^3\mathbb{D}\oplus\ldots. \end{aligned} \tag{7}$$

However, the use of the disturbance invariant tube Z in the robust MPC formulation may lead to the conservative results due to the fact that the disturbance invariant tube Z includes the effect of all possible realizations of the disturbances for all future time steps. In this paper, the time-varying tubes Z_i as defined in Definition 1 are computed off-line. At each sampling time, the precomputed time-varying tubes are included in the optimal control problem as the constraints in the prediction horizon. The time-varying tube at each time step of the system (6) can now be computed as follows

$$Z_{i+1} = (A+BK)Z_i \oplus \mathbb{D} \text{ for } \forall i \geq 0 \text{ and } \forall(x-\bar{x}) \in Z_i, \tag{8}$$

where $Z_0 = \{0\}$. It is seen that the time-varying tube Z_i converges to the disturbance invariant tube Z as $i \to \infty$. Thus, the use of the disturbance invariant tube Z in the robust MPC formulation is only a specific case of the synthesis approach of robust MPC presented in this paper. The idea of using the time-varying tubes is also developed by Gonzalez *et al.* [7]. However, its main drawback is that in addition to the optimal control problem solved at each sampling time, the Minkowski sum algorithm has to be implemented on-line which leads to increased computational complexity.

In order to ensure the satisfaction of the original state and input constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ for the proposed algorithm, tighter constraint sets are imposed on the prediction horizon of the nominal system, $\bar{x}_i \in \mathbb{X} \ominus Z_i$ and $\bar{u}_i \in \mathbb{U} \ominus KZ_i$. The cost function for the trajectory of the nominal system is

$$V_N(\bar{x}_0, \overline{U}) := \sum_{i=0}^{N-1} l(\bar{x}_i, \bar{u}_i) + V_f(\bar{x}_N), \tag{9}$$

where $l(\bar{x}_i, \bar{u}_i) := \frac{1}{2}[\bar{x}_i^T Q \bar{x}_i + \bar{u}_i^T R \bar{u}_i]$ is the stage cost, $V_f(\bar{x}_N) := \frac{1}{2}\bar{x}_N^T P_f \bar{x}_N$ is the terminal cost, Q, R and P_f are the positive definite weighting matrices. The proposed robust MPC algorithm with time-varying tubes can now be formulated as follows:

Off-line: Compute the disturbance rejection gain K satisfying $(A_j + B_j K)^T P (A_j + B_j K) - P < 0, \forall j \in \{1, \ldots, L\}$. Then, compute a sequence of time-varying tubes Z_i, $i = 1$, 2, ..., N according to (8) where N is the prediction horizon.

On-line: At each sampling time, the state x is measured and the following optimal control problem is solved

$$\min_{\bar{x}_0, \overline{U}} V_N(\bar{x}_0, \overline{U}) \tag{10}$$

$$\text{s.t. } \bar{x}_0 = x \tag{11}$$

$$\bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i, \;\; i \in \{0, \ldots, N-1\} \tag{12}$$

$$\bar{x}_i \in \mathbb{X} \ominus Z_i, \;\; \bar{u}_i \in \mathbb{U} \ominus KZ_i, \;\; i \in \{0, \ldots, N-1\} \tag{13}$$

$$\bar{x}_N \in \overline{X}_f \subset \mathbb{X} \ominus Z_N. \tag{14}$$

The additional terminal constraint (14) is employed to ensure robust stability where $\overline{X}_f$ is the terminal constraint set satisfying the following assumptions: (i) $(A+BK)\overline{X}_f \subset \overline{X}_f$, $\overline{X}_f \subset \mathbb{X} \ominus Z_N$, $K\overline{X}_f \subset \mathbb{U} \ominus KZ_N$, and (ii) $V_f((A+BK)\bar{x}) + l(\bar{x}, K\bar{x}) \leq V_f(\bar{x})$, $\forall \bar{x} \in \overline{X}_f$.

We can now establish our main result.

Theorem 1: The proposed robust MPC algorithm steers any initial state x of the uncertain system with bounded disturbance $x^+ = A^\lambda x + B^\lambda u + w$ in the time-varying tubes Z_i to the target set $\overline{X}_f \oplus Z_N$ while satisfying the original state and input constraints, $x \in \mathbb{X}$ and $u \in \mathbb{U}$.

Proof: Consider the optimal control problem (10) subject to the constraints (11) to (14), the constraint (14) ensures that the nominal state $\bar{x}$ must be driven to the terminal target set $\overline{X}_f$. Since all possible state trajectories lie in the time-varying tubes $x \in \bar{x} \oplus Z_i$, $i \in \{0, \ldots, N\}$, the state x must be driven to the target set $\overline{X}_f \oplus Z_N$. Lastly, the satisfaction of the tighten constraint sets for the state and input of the nominal system $\bar{x}_i \in \mathbb{X} \ominus Z_i$, $\bar{u}_i \in \mathbb{U} \ominus KZ_i$ ensures the satisfaction of the original state and input constraints.

4. AN ILLUSTRATIVE EXAMPLE

In this section, an implementation of the proposed algorithm is illustrated. Consider the following uncertain discrete-time system with bounded disturbance

$$\begin{aligned} x^+ =& A^\lambda x + B^\lambda u + w \\ =& \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w. \end{aligned} \tag{15}$$

The system matrices $A^\lambda = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B^\lambda = B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$. The state constraint is $\mathbb{X} := \{x \in \mathbb{R}^2 | [0 \;\; 1]x \leq 2\}$. The input constraint is $\mathbb{U} := \{u \in \mathbb{R} | \, |u| \leq 1\}$. The disturbance is bounded within $\mathbb{W} := \{w \in \mathbb{R}^2 | [-0.1 \;\; -0.1]^{\mathrm{T}} \leq w \leq [0.1 \;\; 0.1]^{\mathrm{T}}\}$. The nominal system can be written as

$$\bar{x}^+ = A\bar{x} + B\bar{u} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \bar{u}. \tag{16}$$

The difference between (15) and (16) can be written as

$$x^+ - \bar{x}^+ = A(x - \bar{x}) + B(u - \bar{u}) + d$$

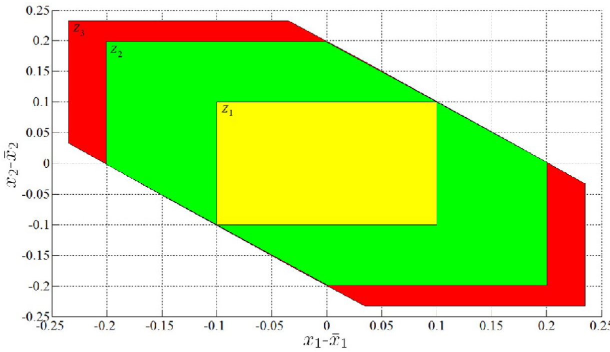


Fig. 1. The evolution of the size of the time-varying tube Z_i computed off-line.

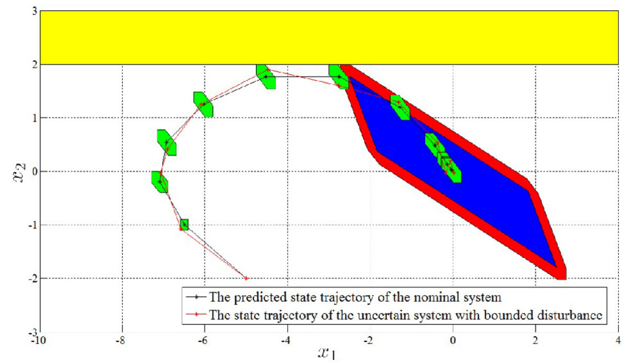


Fig. 2. The state trajectories of the system.

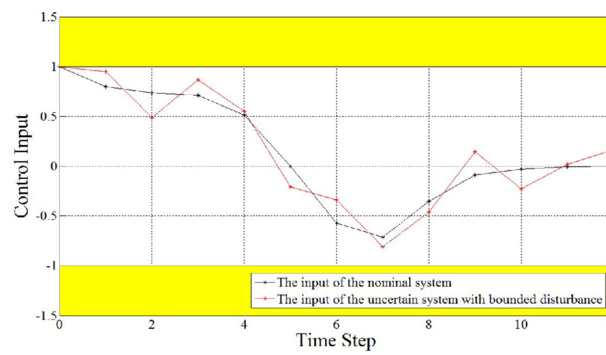


Fig. 3. The control inputs of the system.

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}(x-\bar{x}) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}(u-\bar{u}) + d, \tag{17}$$

where $d \in \mathbb{D}$, $\mathbb{D} := \mathbb{W}$. By using the control law $u = \bar{u} + K(x-\bar{x})$, the difference system (17) can be rewritten as

$$\begin{aligned} & x^+ - \bar{x}^+ \\ & = (A+BK)(x-\bar{x}) + d \\ & = (\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}[\ -0.66 \quad -1.33]\ )(x-\bar{x}) + d, \end{aligned} \tag{18}$$

where the $K = [\ -0.66 \quad -1.33]$ is the disturbance rejection gain. The closed-loop system is simulated using the initial state $x = \bar{x} = [-5\ \ -2]^{\mathrm{T}}$. The disturbance is varied as $d = w = [0.1\sin(4k)\ \ 0.1\sin(4k)]^T$ where k is the simulation time. The weighting matrices in the cost function are $Q = I$, $R = 0.01$ and $P_f = \begin{bmatrix} 2.0066 & 0.5099 \\ 0.5099 & 1.2682 \end{bmatrix}$. The prediction horizon is $N = 12$.

Fig. 1 shows the evolution of the size of the time-varying tube Z_i computed off-line where Z_1, Z_2 and Z_3 are shown in yellow, green and red, respectively. It is seen that the size of the time-varying tube Z_i increases as the time step i in the prediction horizon increases. However, the size of the time-varying tube is nearly constant beyond Z_3.

Fig. 2 shows the state trajectories of the system. The region shown in yellow is the infeasible region of the state constraint $\mathbb{X} := \{x \in \mathbb{R}^2 | [0\ \ 1]x \leq 2\}$. The terminal constraint set $\overline{X}_f$ for the nominal state $\bar{x}$ is shown in blue. The target set $\overline{X}_f \oplus Z_N$ for the state x is shown in red. The cross section of the precomputed time-varying tube Z_i is shown in green. The black line is the predicted state trajectory of the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$ with the prediction horizon $N = 12$. The red line is the state trajectory of the uncertain system with bounded disturbance. It is seen that the state trajectory of the uncertain system with bounded disturbance lies in the time-varying tube Z_i precomputed off-line. The center of the tube is the predicted state trajectory of the nominal system.

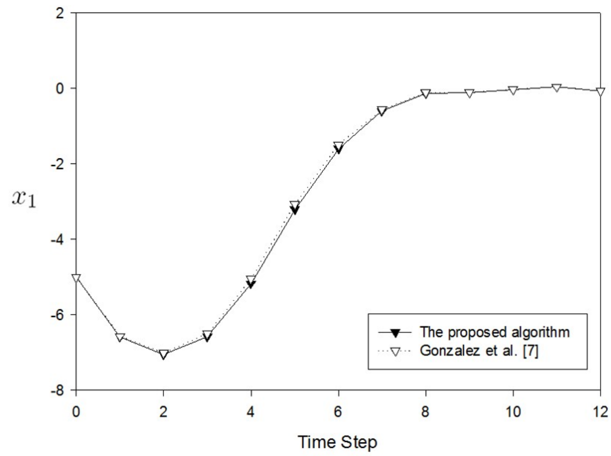
Fig. 3 shows the control inputs of the system. The infeasible region of the original input constraint $\mathbb{U} := \{u \in \mathbb{R} | |u| \leq 1\}$ is shown in yellow. The black line is the input $\bar{u}$ of the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$ with the prediction horizon $N = 12$. The red line is the input u of the uncertain system with bounded disturbance. It is seen that the original input constraint is satisfied while the input of the nominal system converges to the origin.

Table 1. The on-line computational time.

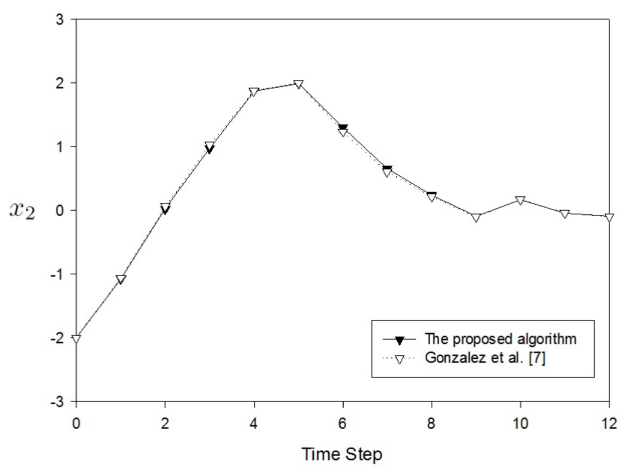
Algorithms	On-line CPU time per step
The proposed algorithm	0.043 s
Gonzalez *et al*. [7]	0.127 s

The proposed robust MPC algorithm with time-varying tubes is compared with the tube-based MPC algorithm proposed by Gonzalez *et al*. [7] where the Minkowski sum algorithm has to be implemented on-line to find the time-varying reachable sets. Fig. 4 shows the control performance for the uncertain system with bounded disturbance. It can be observed that nearly the same control performance can be obtained while the on-line computational time is greatly reduced as shown in Table 1.

The computations in Table 1 have been performed using Intel Core 2 Duo (2.53 GHz), 2 GB RAM.



(a) The state x_1.



(b) The state x_2.

Fig. 4. The control performance; (a) the state x_1 and (b) the state x_2.

In Gonzalez *et al*. [7], the disturbance rejection gain is computed on-line compensating the mismatch between the real and the nominal state at each step within the prediction horizon. The time-varying tube is computed at each sampling time in accordance with the disturbance rejection gain. In the computation of the time-varying tube, the Minkowski sum algorithm has to be implemented on-line which leads to increased computational complexity.

In our proposed algorithm, the disturbance rejection gain is computed off-line satisfying the Lyapunov stability constraint. A sequence of the time-varying tubes is computed off-line in accordance with the precomputed disturbance rejection gain. The reduction in the on-line computational burden is obtained because a sequence of the time-varying tubes is computed off-line so no Minkowski sum algorithm has to be implemented at each sampling time.

5. CONCLUSION

In this paper, robust MPC with time-varying tubes is presented. All possible realizations of the state trajectory of the uncertain systems with bounded disturbances are guaranteed to lie within the time-varying tubes. The time-varying tubes are computed off-line so no additional on-line computational time is required. Only the quadratic programming for the optimal control problem is solved at each sampling time. In comparison to the algorithm that calculates the time-varying tubes on-line, the proposed algorithm can achieve the same level of control performance while the on-line computational time is greatly reduced. The proposed algorithm can be further extended to the case when the exact state in unknown.

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Tube-Based Robust Model Predictive Control for Systems with Uncertain Parameters and Disturbances

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ABSTRACT: Chemical processes are multivariable processes that change one or more chemical compounds to the desired products. Chemical processes are involved with many complex chemical reactions. Additionally, they usually contain some uncertain parameters and disturbances. In order to efficiently control these uncertain chemical processes, a multivariable control algorithm that can handle both uncertain parameters and disturbances needs to be developed. However, the control of systems in the presence of uncertain parameters and disturbances is a challenging control problem because it is difficult to guarantee both robust stability and constraint satisfaction. In this paper, a novel tube-based robust model predictive control algorithm is developed. The trajectories of the systems are restricted to lie in a sequence of tubes so robust stability and constraint satisfaction can be guaranteed in the presence of both uncertain parameters and disturbances. All of the optimization problems are solved off-line so the developed algorithm is applicable to fast dynamic systems. In order to demonstrate the applications of the developed algorithm, it is applied to a control problem of the continuous stirred tank reactor with uncertain parameters and disturbances.

Keywords: Model Predictive Control; tubes; uncertain parameters; disturbances; robust stability; constraint satisfaction.

1 Introduction

Chemical processes are usually composed of a number of reactions. These reactions usually contain some uncertain parameters such as those in reaction rate constants and heat transfer coefficients. Moreover, they usually contain some disturbances such as measurement noises. Some chemical processes are fast and highly exothermic so inefficient handling of these uncertainties and disturbances may lead to unexpected thermal runaway of the system. For this reason, it is necessary to develop an efficient multivariable control algorithm that is able to ensure both robust stability and constraint satisfaction in the presence of uncertain parameters and disturbances. Additionally, it must be applicable to the control of fast dynamic reactions.

Model predictive control (MPC) is an advanced control algorithm for multivariable processes. MPC is widely used in many chemical processes because input and output constraints are considered in a systematic manner. At each sampling time, MPC solves a finite horizon optimal control problem based on an explicit model of the process. Although an optimal control sequence is determined, only the first control action is applied to the process. Since models are only approximations of real processes, it is extremely important for MPC to be robust to uncertain parameters and disturbances. Moreover, only small computational time should be required so that it can be applied to fast dynamic processes.

Off-line formulations of robust MPC have been developed to deal with uncertain parameters while ensuring the same level of control performance (Wan and Kothare, 2003; Ding *et al.*, 2007). The main idea was to solve an optimization problem off-line to find a sequence of controller gains and the associated ellipsoidal invariant sets. At each sampling time, the real-time state feedback gain was computed by linear interpolation between the pre-computed feedback gains. Only uncertain parameters were considered in the MPC formulation so robust stability was not guaranteed in the presence of disturbances. An off-line MPC algorithm for linear parameter-varying (LPV) systems was developed by Bumroongsri and Kheawhom (2012a). The real-time state feedback gain was calculated by linear interpolation between the pre-computed state feedback gains using the scheduling parameters. Bumroongsri and Kheawhom (2012b) proposed an off-line robust MPC algorithm based on polyhedral invariant sets instead of ellipsoidal invariant sets. A significantly larger stabilizable region was obtained. However, the effects of disturbances were neglected in the robust MPC formulation so robust stability could not be guaranteed in the presence of disturbances. In order to apply off-line robust MPC to uncertain chemical processes with disturbances, the effects of disturbances should be explicitly included in the MPC formulation.

In the context of tube-based robust MPC, the effects of disturbances are explicitly included in the MPC formulation. Tube-based robust MPC is motivated by the fact that a real state trajectory differs from a state trajectory of a nominal system due to the effects of disturbances. Chisci *et al*. (2001) developed an on-line tube-based robust model predictive controller for linear time-invariant (LTI) systems subject to bounded disturbances. The objective was to drive the state of LTI system with bounded disturbances to a terminal set. Constraint fulfillment was guaranteed by replacing the original constraints with more stringent ones. Langson *et al.* (2004) proposed an on-line tube-based robust MPC employing the time-varying control inputs instead of the LTI control law. A sequence of time-varying control inputs was obtained by solving an optimal control problem subject to additional constraint sets. The proposed MPC algorithm could achieve better control performance than the conventional tube-based MPC algorithm using LTI control law. The price to be paid was the computational complexity that increased with the prediction horizon.

Tube-based robust MPC for tracking of LTI system with bounded disturbances was presented by Limon *et al.* (2010). The artificial steady-state variables were introduced as the decision variables in the optimization problem. If the target was unreachable, the system would be steered to the neighborhood of the artificial steady-state point. High on-line computational time was required because some decision variables and constraints were introduced to the optimal control problem. Tube-based robust MPC for linear time-varying (LTV) systems with bounded disturbances was developed by Bumroongsri (2015). A novel feature was the fact that the developed algorithm could handle both time-varying parameters and disturbances. However, the optimization problem must be solved on-line at each sampling time. In order to apply tube-based robust MPC to fast dynamic processes, its off-line formulation needs to be developed.

In this paper, an off-line synthesis approach for tube-based robust MPC is presented. The trajectories of systems with both uncertain parameters and disturbances are restricted to lie in a sequence of tubes so robust stability and constraint satisfaction are guaranteed. Additionally, no optimization problem needs to be solved on-line so the developed tube-based robust MPC algorithm is applicable to fast dynamic processes. This paper is organized as follows. The problem statement is presented in Section 2. The proposed synthesis approach for tube-based robust MPC is presented in Section 3. A numerical example is presented in Section 4. The conclusions are drawn in Section 5.

2 Problem Statement

Consider the following discrete-time system with uncertain parameters and disturbances

$$x^+ = A^\lambda x + B^\lambda u + w \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $w \in \mathbb{R}^n$ is the disturbance and $x^+ \in \mathbb{R}^n$ is the successor state. The matrices A^λ and B^λ depend on an uncertain parameter vector λ that is $[A^\lambda \; B^\lambda] \in \mathrm{Conv}\{[A_j \; B_j], \forall j \in 1,2,\ldots,L\}$ where $\mathrm{Conv}\{\cdot\}$ is the convex hull of all elements in $\{\cdot\}$. The state constraint is $x \in \mathbb{X}$. The input constraint is $u \in \mathbb{U}$. The disturbances w lie in the set $\mathbb{W}$. Consider the following nominal system (the system without disturbances) defined by

$$x'^+ = A^\lambda x' + B^\lambda u' \tag{2}$$

where $x' \in \mathbb{R}^n$ and $u' \in \mathbb{R}^m$ are the state and input of the nominal system, respectively. The difference system between (1) and (2) is

$$x^+ - x'^+ = A^\lambda (x - x') + B^\lambda (u - u') + w. \tag{3}$$

In order to deal with both uncertain parameters and disturbances, the control law $u = K(x - x') + u'$ is employed where $u' = Fx'$ is the control law for the nominal system and K is the disturbance rejection gain. The difference system (3) can be written as

$$x^+ - x'^+ = (A^\lambda + B^\lambda K)(x - x') + w. \tag{4}$$

The trajectory of the difference system $x^+ - x'^+$ is bounded by a robust positively invariant set Z defined by

$$Z = \bigoplus_{i=0}^{\infty} \text{Conv}\{(A^{\lambda} + B^{\lambda} K)^{i}\, \mathbb{W}, \forall j \in 1,2,..., L\} \tag{5}$$

where the symbol $\oplus$ denotes the Minkowski set addition. The objective is to drive the nominal system (2) to the origin while maintaining the trajectory of the system with uncertain parameters and disturbances (1) in a sequence of tubes whose centers are the state of the nominal system (2).

3 A Synthesis Approach for Tube-Based Robust MPC

In this section, an off-line formulation for tube-based robust MPC is presented. If we can drive the state of the nominal system x' to the origin, the state of the uncertain system with uncertain parameters and disturbances x must be driven to a tube Z whose center is at the origin. An off-line formulation of robust MPC for nominal system was developed by Wan and Kothare (2003). A sequence of off-line feedback gains F_i corresponding to a sequence of invariant ellipsoids $\varepsilon_i, i = \{1,..., N_{\varepsilon}\}$ was pre-computed by solving the linear matrix inequality (LMI) optimization problems. The real-time feedback gain F was calculated at each sampling time by linear interpolation between the off-line feedback gains F_i. Therefore, the state of the nominal system x' can be driven to the origin using the nominal control law $u' = Fx'$.

In order to drive the state of the uncertain system with uncertain parameters and disturbances x to the origin, the control law $u = K(x - x') + Fx'$ is employed in this paper where K is the disturbance rejection gain. The task of the disturbance rejection gain K is to maintain the state of the uncertain system with uncertain parameters and disturbances x in a sequence of tubes whose centers are the state of the nominal system x'. When the nominal state x' is driven to the origin, the state of the uncertain system with uncertain parameters and disturbances x must be driven to the neighborhood of the origin. Therefore, robust stability can be ensured. In order to ensure constraint satisfaction $x \in \mathbb{X}$ and $u \in \mathbb{U}$, tighter constraint sets for the nominal system $x' \in \mathbb{X} \ominus Z$ and $F_i x' \in \mathbb{U} \ominus KZ$ must be imposed where the symbol $\ominus$ denotes the Minkowski set difference. In summary, the developed tube-based robust MPC algorithm can be written as follows:

Off-line step 1: Compute the disturbance rejection gain K satisfying the Lyapunov stability constraint $(A_j + B_j K)^T P (A_j + B_j K) - P < 0$.

Off-line step 2: Compute a sequence of off-line feedback gains F_i corresponding to a sequence of invariant ellipsoids $\varepsilon_i, i = \{1,..., N_{\varepsilon}\}$ with tighter constraint sets $x' \in \mathbb{X} \ominus Z$ and $F_i x' \in \mathbb{U} \ominus KZ$.

At the first sampling time $(t = 0)$: measure the state x and the uncertain parameter λ. Find the real-time feedback gain F (that is the off-line feedback gain F_i of the smallest invariant ellipsoid ε_i containing the measured state x). Apply the control law $u = Fx$ to the process and compute x'^{+} from $x'^{+} = (A^{\lambda} + B^{\lambda} F)x$ (At the first sampling time, $x = x'$ so the control law $u = K(x - x') + Fx'$ is equal to $u = Fx$).

At each sampling time $(t > 0)$: measure the state x and the uncertain parameter λ. Find the real-time feedback gain F (that is the off-line feedback gain F_i of the smallest invariant ellipsoid ε_i containing the nominal state x' computed from the previous step). Apply the control law $u = K(x - x') + Fx'$ to the process and compute x'^{+} from $x'^{+} = (A^{\lambda} + B^{\lambda} F)x'$.

Remark 1: It is assumed that the state x is measurable. For the case when the state x is unmeasurable, an observer should be included in the controller design (Mayne *et al.*, 2006).

4 Numerical Example

In this section, the developed algorithm is applied to a numerical case study of an uncertain CSTR where an irreversible exothermic reaction $\text{A} \rightarrow \text{B}$ is assumed to take place. A cooling jacket is used to adjust the reactor temperature. The material and energy balances are as follows:

$$\frac{dC_A}{dt} = \frac{Q}{V}(C_{Af} - C_A) - k_o \exp\left(\frac{-E_a}{RT}\right) C_A \tag{6}$$

$$\frac{dT}{dt} = \frac{Q}{V}(T_f - T) - \frac{UA}{V\rho C_p}(T - T_c) + \left(\frac{-\Delta H}{\rho C_p}\right) k_o \exp\left(\frac{-E_a}{RT}\right) C_A \tag{7}$$

$$\frac{dT_c}{dt} = \frac{Q_c}{V_c}(T_{cf} - T_c) + \frac{UA}{V_c \rho_c C_{pc}}(T - T_c)\,. \tag{8}$$

where C_A is the concentration of reactant A, Q is the feed stream flowrate, V is the reactor volume, C_{Af} is the concentration of reactant A in feed stream, k_o is the pre-exponential factor, E_a is the activation energy, R is the gas constant, T is the reactor temperature, T_f is the feed stream temperature, UA is the overall heat transfer coefficient, ΔH is the heat of reaction, C_p is the heat capacity, ρ is the density, T_c is the coolant temperature, T_{cf} is the coolant feed temperature, V_c is the cooling jacket volume, ρ_c is the coolant density, C_{pc} is the coolant heat capacity and Q_c is the coolant flowrate.

The material and energy balances can be written in the dimensionless form as follows (Nagrath *et al.*, 2002):

$$\frac{dx_1}{d\tau} = q(x_{1f} - x_1) - \phi x_1 \exp\left(\frac{x_2}{1 + \frac{x_2}{\gamma}}\right) \tag{9}$$

$$\frac{dx_2}{d\tau} = q(x_{2f} - x_2) - \delta(x_2 - x_3) + \beta\phi x_1 \exp\left(\frac{x_2}{1 + \frac{x_2}{\gamma}}\right) \tag{10}$$

$$\frac{dx_3}{d\tau} = \delta_1[q_c(x_{3f} - x_3) + \delta\delta_2(x_2 - x_3)] \tag{11}$$

where x_1 is the dimensionless concentration of reactant A, x_2 is the dimensionless reactor temperature and x_3 is the dimensionless coolant temperature. The manipulated variable is the dimensionless coolant flowrate q_c. The following discrete-time state-space model is obtained by linearizing and discretizing (9)-(11) with a sampling period of ΔT:

$$\begin{bmatrix} \bar{x}_1(k+1) \\ \bar{x}_2(k+1) \\ \bar{x}_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 + \Delta T[-q - \phi\kappa(x_{2S})] & -\Delta T[\frac{\phi x_{1S}\kappa(x_{2S})}{(1 + \frac{x_{2S}}{\gamma})^2}] & 0 \\ \Delta T[\beta\phi\kappa(x_{2S})] & 1 + \Delta T[-q - \delta + \frac{\beta\phi\kappa(x_{2S})x_{1S}}{(1 + \frac{x_{2S}}{\gamma})^2}] & \Delta T\delta \\ 0 & \Delta T\delta\delta_1\delta_2 & 1 - \Delta T[\delta_1 q_{cS} + \delta\delta_1\delta_2] \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \\ \bar{x}_3(k) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ \Delta T\delta_1[x_{3f} - x_{3S}] \end{bmatrix} \bar{q}_c(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \end{bmatrix} \tag{12}$$

where $\bar{x}_1(k) = x_1(k) - x_{1S}$, $\bar{x}_2(k) = x_2(k) - x_{2S}$, $\bar{x}_3(k) = x_3(k) - x_{3S}$, $\bar{q}_c(k) = q_c(k) - q_{cS}$ and $\kappa(x_{2S}) = \exp\left(\frac{x_{2S}}{1 + \frac{x_{2S}}{\gamma}}\right)$. The disturbances $w_1(k)$, $w_2(k)$ and $w_3(k)$ are added to the system to take into account the effects of unmeasured disturbances. Their values are time-varying between -0.01 and 0.01. The Damkohler number ϕ is uncertain and its value is time-varying between $\phi_{\min} = 0.0648$ and $\phi_{\max} = 0.0792$. The objective is to drive the dimensionless state from the initial point $(0, -5, 0)$ to the neighborhood of the origin by adjusting $\bar{q}_c(k)$. The state constraint is $\left|\bar{x}_1(k)\right| \leq 0.5$ and the input constraint is $\left|\bar{q}_c(k)\right| \leq 1.0$.The values of dimensionless parameters are shown in Table 1.

Table 1. The values of dimensionless parameters.

Parameters	Values	Parameters	Values	Parameters	Values	Parameters	Values
γ	20	q	1.0	q_{cS}	1.0	δ_1	10
β	8.0	δ	0.3	x_{1S}	0.8933	δ_2	1.0
x_{2S}	0.5193	x_{3S}	-0.5950	x_{3f}	-1.0		

A sequence of five invariant ellipsoids $\varepsilon_i, i = \{1,...,5\}$ computed off-line is shown in Fig.1. Each invariant ellipsoid ε_i has the corresponding off-line feedback gain F_i so the real-time feedback gain F can be updated based on the distance from the origin.

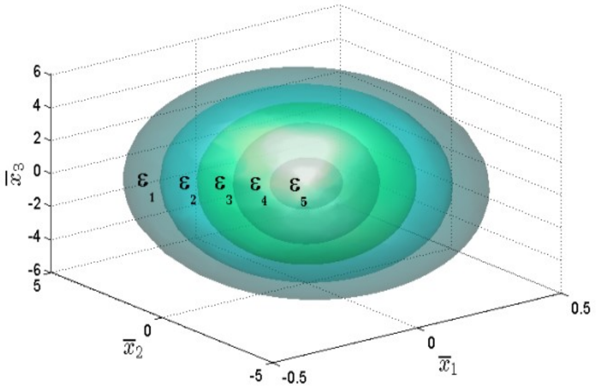

Figure 1. A sequence of five invariant ellipsoids.

Figure 2 shows the tracjectories of the system. The trajectory of the uncertain CSTR with disturbances lies in a sequence of tubes whose centers are the state of the uncertain CSTR with no disturbances (nominal system). Therefore, robust stability and constraint satisfaction are ensured.

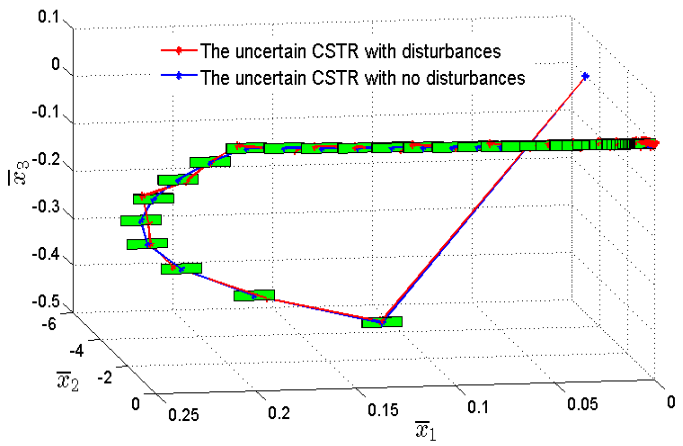

Figure 2. The tracjectories of the system.

Figure 3 shows the dimensionless state variables. It is seen that the system is robustly stabilized since all states are driven to the neighborhood of the origin.

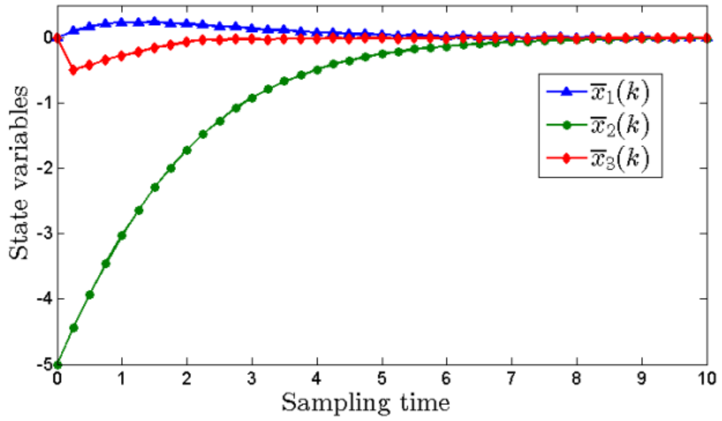

Figure 3. The dimensionless state variables of the system.

5 Conclusions

In this paper, a synthesis approach for robust MPC is presented. The trajectories of system with both uncertain parameters and disturbances are restricted to lie in a sequence of tubes so robust stability and constraint satisfaction are guaranteed. Additionally, no optimization problem needs to be solved on-line so the developed tube-based robust MPC algorithm is applicable to fast dynamic processes.

Acknowledgements

This research is supported by Thailand Research Fund and Mahidol University.

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