รายงานวิจัยฉบับสมบูรณ์

โครงการความสัมพันธ์กรีน อันดับบางส่วนและแรงก์ของกึ่งกรุปของการแปลงที่กำกัดเรนจ์โดยที่รักษาความสัมพันธ์สมมูล

โดย ผศ.ดร.กฤษฎา สังขนันท์

31 มีนาคม 2563

สัญญาเลขที่ TRG5880113

รายงานวิจัยฉบับสมบูรณ์

โครงการความสัมพันธ์กรีน อันดับบางส่วนและแรงก์ของกึ่งกรุป ของการแปลงที่กำกัดเรนจ์โดยที่รักษาความสัมพันธ์สมมูล

ผู้วิจัย ผศ.ดร.กฤษฎา สังขนันท์
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและ
มหาวิทยาลัยเชียงใหม่

บทคัดย่อ

ให้ $T(X,Y)$ เป็นกึ่งกรุปการแปลงที่กำกัดเรนจ์ กำหนดให้

$$T_E(X,Y)=\{\alpha\in T(X,Y):\forall (x,y)\in E,(x\alpha,y\alpha)\in E\}$$

เมื่อ E เป็นความสัมพันธ์สมมูลไม่ชัดบน X ในโครงการนี้เราจะให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับ $T_E(X,Y)$ เป็นกึ่งกรุปปรกติและหาลักษณะเฉพาะของความสัมพันธ์กรีนบน $T_E(X,Y)$ จากนั้นเราจะศึกษาอันดับบางส่วนธรรมชาติและพิจารณาว่าเมื่อใดที่สมาชิกสองตัวจะสัมพันธ์กันภายใต้อันดับดังกล่าว นอกจากนี้เราจะหาสมาชิกของ $T_E(X,Y)$ ที่มีสมบัติเข้ากันได้กับ $\leq$ และศึกษาสมาชิกค่ามากสุดและน้อยสุดด้วย ในท้ายที่สุดเราจะหาแรงก์ของ $T_E(X,Y)$ เมื่อ X เป็นเซตจำกัด

รหัสโครงการ : TRG5880113

ชื่อโครงการ : ความสัมพันธ์กรีน อันดับบางส่วนและแรงก์ของกึ่งกรุปของการแปลงที่กำกัดเรนจ์โดยที่รักษาความสัมพันธ์สมมูล

ชื่อนักวิจัย : ผศ.ดร.กฤษฎา สังขนันท์ ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยเชียงใหม่

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ระยะเวลาโครงการ : 2 ปี

คำสำคัญ : ความสัมพันธ์กรีน, อันดับบางส่วน, แรงก์, กึ่งกรุปการแปลง

Abstract

Let T (X,Y) be the full transformation semigroup with restricted range. Define

$$T_E(X,Y)=\{\alpha\in T(X,Y):\forall(x,y)\in E,(x\alpha,y\alpha)\in E\}$$

where E is a nontrivial equivalence on X. In this project, we give a necessary and sufficient condition for $T_E(X,Y)$ to be regular and characterize Green's relations on $T_E(X,Y)$. Then we study it with the so-called natural order and determine when two elements are related under this order. Moreover, we find elements of $T_E(X,Y)$ which are compatible with $\leq$. Also, the maximal and minimal elements are described. Finally, we find the ranks of $T_E(X,Y)$ when X is finite.

Project Code : TRG5880113

Project Title : Green's relations, partial orders and ranks of transformation semigroups with restricted range that preserve an equivalence

Investigator : Asst. Prof. Dr. Kritsada Sangkhanan, Department of Mathematics, Faculty of Science, Chiang Mai University

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Project Period : 2 years

Keywords : Green's ralations, Partial orders, Ranks, transformation semigroups

Executive Summary

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1. ความสำคัญและที่มาของปัญหา

Green's relations

Let S be a semigroup. The following definitions are due to J. A. Green. For any $a,b\in S$, he defined

$$aLb \text{ if and only if } S^1a=S^1b,$$

or equivalently; aLb if and only if $a=xb,b=ya$ for some $x,y\in S^1$.

Dually, he defined

$$aRb \text{ to mean } aS^1=bS^1,$$

or equivalently; aRb if and only if $a=bx,b=ay$ for some $x,y\in S^1$.

And he defined

$$aJb \text{ to mean } S^1aS^1=S^1bS^1,$$

or equivalently; aJb if and only if $a=xby,b=uav$ for some $x,y,u,v\in S^1$.

Finally, he defined

$$H=L\cap R \text{ and } D=L\circ R$$

and obtained that the composition of L and R is commutative. This follows that D is the join $L\vee R$, that is, D is the smallest equivalence relation containing $L\cup R$. Moreover, $H\subseteq L\subseteq D\subseteq J$ and $H\subseteq R\subseteq D\subseteq J$. But, in commutative semigroups, we have $H=L=R=D=J$. The relations L, R, H, D and J are called Green's relations on S. For each $a\in S$, we denote L-class, R-class, H-class, D-class and J-class containing a by L_a,R_a,H_a,D_a and J_a, respectively. These relations are very important in the semigroup theory because these are useful for understanding the nature of divisibility in a semigroup.

Natural partial order

The natural partial order on a semigroup has been developed in a number of steps. In the terminology of A. H. Clifford and G. B. Preston [1], a band B is a semigroup in which every element is an idempotent. On such a semigroup there is a natural (partial) order relation defined by the rule

$$e\leq f \text{ if and only if } e=ef=fe.$$

If the order relation ≤ is compatible with the multiplication in B, in the sense that $e \leq f$ and $g \leq h$ together imply that $eg \leq fh$, we shall say that B is a naturally ordered band. In 1966, J. M. Howie [3] described the structure of naturally ordered bands.

In the year 1952, V. Vagner [9] defined the natural order on an inverse semigroup S by

$$a \leq b \text{ if and only if } a=eb \text{ for some } e \in E(S)$$

where $E(S)$ is the set of all idempotents of S.

About 30 years later, R. E. Hartwig [2] and K. S. S. Nambooripad [5] independently discovered the generalization of the above orders. They defined it on a regular semigroup S by

$$a \leq b \text{ if and only if } a=eb=bf \text{ for some } e,f \in E(S). \quad (1)$$

In 1986, the natural order on a regular semigroup was further extended to any semigroup S by H. Mitsch [4]. He defined

$$a \leq b \text{ if and only if } a=xb=by \text{ and } xa=a \text{ for some } x,y \in S^1.$$

Let X be a set. A binary relation on the set X is a subset of $X \times X$. The set of all binary relations on X is denoted by $B(X)$. Let $\alpha,\beta \in B(X)$. We define the operation by the composition,

$$\alpha\beta=\{(x,y) \in X \times X : (x,z) \in \alpha \text{ and } (z,y) \in \beta \text{ for some } z \in X\}.$$

Then $B(X)$ under composition is a semigroup. In 2006, C. Namnak and P. Preechasilp [6] studied two natural partial orders, ≤ and ⊆, on $B(X)$ and characterized when two elements of $B(X)$ are related under these orders. The maximality, minimality, left compatibility and right compatibility of elements were considered with respect to each order.

Transformation Semigroups

In this part, we introduce the transformation semigroups and show some examples of these semigroups.

A partial transformation semigroup is the collection of functions from a subset of X into X with composition denoted by $P(X)$. In addition, the semigroup $T(X)$ and $I(X)$ are defined by:

$$T(X)=\{\alpha \in P(X): \text{dom } \alpha = X\},$$

$$I(X)=\{\alpha \in P(X): \alpha \text{ is injective}\}.$$

$T(X)$ and $I(X)$ are called the full transformation semigroup and the symmetric inverse semigroup, respectively. It is well-known that $P(X)$ and $T(X)$ are regular and $I(X)$ is an inverse semigroup.

To generalize the transformation semigroups, we introduce the transformation semigroups with restriced range which are the generalization of transformation semigroups. Let Y be a nonempty subset of X. We consider the subsemigroups of $P(X)$,$T(X)$ and $I(X)$ defined by:

$$PT(X,Y) = \{\alpha \in P(X) : X\alpha \subseteq Y\},$$

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\},$$

$$I(X,Y) = \{\alpha \in I(X) : X\alpha \subseteq Y\},$$

where $X\alpha$ denotes the range of α. In fact, if $Y=X$, then

$$PT(X,Y)=P(X),\ T(X,Y)=T(X) \text{ and } I(X,Y)=I(X).$$

Finally, we consider the linear transformation semigroup.

Let V be any vector space, $P(V)$ the set of all linear transformations $\alpha:S\longrightarrow T$ where S and T are subspaces of V. Then we have $P(V)$ under composition is a semigroup and it is called the partial linear transformation semigroup. Moreover, the semigroups $T(V)$ and $I(V)$ are defined by:

$$T(V)=\{\alpha \in P(V) : \text{dom}\,\alpha = V\},$$

$$I(V)=\{\alpha \in P(V) : \alpha \text{ is injective}\}.$$

It is well-known that $P(V)$ and $T(V)$ are regular and $I(V)$ is an inverse semigroup.

Similarly to $PT(X,Y)$, $T(X,Y)$ and $I(X,Y)$, we can defined the linear transformation semigroups with restricted range as follows.

Let W be a subspace of a vector space V. Define:

$$PT(V,W) = \{\alpha \in P(V) : V\alpha \subseteq W\},$$

$$T(V,W) = \{\alpha \in T(V) : V\alpha \subseteq W\},$$

$$I(V,W) = \{\alpha \in I(V) : V\alpha \subseteq W\}.$$

Let $T(X)$ be the full transformation semigroup on a set X and E be a nontrivial equivalence on X. Write

$$T_E(X) = \{f \in T(X) : \forall (x,y) \in E, (f(x), f(y)) \in E\},$$

then $T_E(X)$ is a subsemigroup of $T(X)$. In 2005, H. Pei [7] discussed regularity of elements and Green's relations for $T_E(X)$. Then, in 2007, L. Sun, H. Pei and Z. Cheng [8] endowed $T_E(X)$ with the so-called natural order and determined when two elements of $T_E(X)$

are related under this order, then found out elements of $T_E(X)$ which are compatible with ≤ on $T_E(X)$. Also, the maximal and minimal elements and the covering elements were described.

Rank of Semigroups

In this part, we introduce the finite transformation semigroups and the rank of any semigroups.

The rank of a semigroup S is the smallest number of elements required to generate S defined by

$$rank(S)=\min\{|A|:A\subseteq S,\langle A\rangle=S\}.$$

If S is generated by its idempotents $E(S)$, then it is possible to define the idempotent rank of S by

$$idrank(S)=\min\{|A|:A\subseteq E(S),\langle A\rangle=S\}.$$

To study the transformation semigroups on a finite set with restricted range, define

$$PT_{n,r}=PT(\{1,2,...,n\},\{1,2,...r\}),$$
$$T_{n,r}=T(\{1,2,...,n\},\{1,2,...r\}),$$
$$I_{n,r}=I(\{1,2,...,n\},\{1,2,...r\}).$$

It is well-known that the ranks of $P_n=PT_{n,n}$, $I_n=I_{n,n}$ and $T_n=T_{n,n}$ are equal to 4, 3 and 3, respectively.

In this project, we aim to generalize the results of H. Pei [7] and L. Sun, H. Pei and Z. Cheng [8] by define a transformation semigroup with restricted range that preserve an equivalence as follows. Let $T(X,Y)$ be the full transformation semigroup with restricted range and E a nontrivial equivalence on X. Define

$$T_E(X,Y)=\{f\in T(X,Y):\forall(x,y)\in E,(f(x),f(y))\in E\}.$$

We can see that if $X=Y$, then $T_E(X,Y)=T_E(X)$ which is concluded that $T_E(X)$ is a special case of $T_E(X,Y)$. In this research, we give a necessary and sufficient condition for $T_E(X,Y)$ to be regular and characterize Green's relations on $T_E(X,Y)$. Then we study it with the so-called natural order ≤ and then determine when two elements are related under this order. Moreover, we find elements of this semigroup which are compatible with ≤. Also, the maximal and minimal elements are described. Finally, we find the ranks of $T_E(X,Y)$ when X is fintie.

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2. วัตถุประสงค์

2.1 To obtain a necessary and sufficient condition for $T_E(X,Y)$ to be regular and the characterization of Green's relations on $T_E(X,Y)$.

2.2 To obtain the characterization of the natural partial order on $T_E(X,Y)$.

2.3 To obtain the rank of $T_E(X,Y)$ when X is finite.

3. ระเบียบวิธีวิจัย

3.1 Study the new research articles and books in the topic of the regularity and Green's relations on transformation semigroups with restricted range and the ranks of transformation semigroups with restricted range.

3.2 Send the work to the mentor for advise and revising.

3.3 Write an article and submit to the international journal.

4. แผนการดำเนินงานวิจัยตลอดโครงการในแต่ละช่วง 6 เดือน

Research Plan	Period month
1. Study the new research articles and books in the topic of the regularity and Green's relations on transformation semigroups with restricted range. 2. Obtain a necessary and sufficient condition for $T_E(X,Y)$ to be regular and the characterization of Green's relations on $T_E(X,Y)$. 3. Check, edit, revise the work and contact the mentor for an advise	**1-6**
4. Write an article and submit to the international journal. 5. Study the new research articles and books in the natural partial order on transformation semigroups with restricted range.	**7-12**
6. Obtain the characterization of the natural partial order on $T_E(X,Y)$. 7. Check, edit, revise the work and contact the mentor for an advise 8. Study the new research articles and books in the the ranks of transformation semigroups with restricted range.	**13-18**
9. Obtain the rank of $T_E(X,Y)$ when X is finite. 10. Check, edit, revise the work and contact the mentor for an advise 11. Write an article and submit to the international journal.	**19-24**

5. **ผลงาน/หัวข้อเรื่องที่ตีพิมพ์ในวารสารวิชาการระดับนานาชาติ**

ชื่อเรื่องที่ตีพิมพ์: Regularity and Green's relations on semigroups of transformations with restricted range that preserve an equivalence

ชื่อวารสารที่ตีพิมพ์: Semigroup Forum

6. **งบประมาณ**

	ปีที่ 1	ปีที่ 2	รวม
1. หมวดค่าตอบแทน - ค่าตอบแทนหัวหน้าโครงการ (เดือนละ 13,000 บาท)	156,000	156,000	312,000
2. หมวดค่าจ้าง - ค่าจ้างในการจัดเตรียมข้อมูลเพื่อใช้คำนวณทางกึ่งกรุป จำกัดโดยใช้โปรแกรม GAP	13,000	13,000	26,000
3. หมวดค่าวัสดุ - ค่าวัสดุสำนักงาน (อุปกรณ์เครื่องเขียนและกระดาษ) - ค่าวัสดุคอมพิวเตอร์ (ตลับผงหมึกสำหรับเครื่องพิมพ์)	5,000 5,000	5,000 5,000	10,000 10,000
4. หมวดครุภัณฑ์ - ค่าเครื่องคอมพิวเตอร์ (จำนวน 1 เครื่อง)* - ค่าเครื่องพิมพ์ (จำนวน 1 เครื่อง)**	50,000 10,000	- -	50,000 10,000
5. หมวดค่าใช้สอย - ค่าเดินทางเพื่อทำวิจัยในประเทศ - ค่าเดินทางเพื่อนำเสนอผลงานในประเทศ - ค่าสืบค้นผลงาน - ค่าอินเตอร์เน็ต - ค่าสำเนาเอกสาร - ค่าจัดทำรายงาน - ค่าไปรษณีย์ โทรศัพท์	10,000 15,000 10,000 6,000 10,000 5,000 5,000	40,000 35,000 20,000 6,000 10,000 5,000 5,000	50,000 50,000 30,000 12,000 20,000 10,000 10,000
รวมงบประมาณโครงการ	300,000	300,000	600,000

หมายเหตุ: เครื่องหมาย - หมายถึง ให้ใส่รายละเอียดรายการใช้จ่ายในแต่ละหมวด

* ใช้ในการคำนวณทางกึ่งกรุปจำกัด (finite semigroup) โดยใช้โปรแกรม GAP

** ใช้พิมพ์ผลลัพธ์ที่ได้จากการคำนวณ

1 Introduction

The full transformation semigroup is the collection of all functions from a set X into X with the composition which is denoted by $T(X)$. In 2008, J. Sanwong and W. Sommanee [7] studied the subsemigroup $T(X,Y)$ of $T(X)$ which is defined by

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$$

where Y is a fixed subset of X. In [7], they discussed the regularity of elements and then determined the Green's relations on $T(X,Y)$. Moreover, they obtained a class of maximal inverse subsemigroups of $T(X,Y)$. Furthermore, a natural partial order on $T(X,Y)$ was studied in some detail in [6, 10].

Let E be an equivalence on X. Write

$$T_E(X) = \{\alpha \in T(X) : \forall (x,y) \in E, (x\alpha, y\alpha) \in E\},$$

then $T_E(X)$ is a subsemigroup of $T(X)$. We see that $T_E(X)$ is $S(X)$, the semigroup of all continuous self-maps of the topological space X for which all E-classes form a basis. In 2005, H. Pei [5] studied regularity of elements and Green's relations for $T_E(X)$.

Now, we deal with the natural partial order or Mitsch's order [4] on any semigroup S defined by for $a,b \in S$,

$$a \leq b \text{ if and only if } a = xb = by,\ xa = a \text{ for some } x,y \in S^1$$

where the notation S^1 denotes a monoid obtained from S by adjoining an identity 1 if necessary ($S^1 = S$ for a monoid S). In [6] and [10], the authors characterized the natural partial order on the semigroup $T(X,Y)$. In addition, they studied compatibility of its elements and then found the maximal and minimal elements. In 2008, L. Sun, H. Pei and Z. Cheng [9] endowed $T_E(X)$ with Mitsch's natural order and investigated when two elements of $T_E(X)$ are related under this order, then found elements which are compatible. Finally, they described the maximal and minimal elements. Recently, in 2019, L. Sun [8] gave necessary and sufficient conditions for elements of $T_E(X)$ to be left or right compatible. In addition, H. Pei [2] studied the rank of $T_E(X)$. He showed that the rank of $T_E(X)$ is no more than 6.

Now, we aim to generalize the results of $T_E(X)$ and $T(X,Y)$ by defining a transformation semigroup with restricted range that preserve an equivalence as follows. Let $T(X,Y)$ be the full transformation semigroup with restricted range and E an equivalence on X. Define

$$T_E(X,Y) = \{\alpha \in T(X,Y) : \forall (x,y) \in E, (x\alpha, y\alpha) \in E\} = T_E(X) \cap T(X,Y).$$

Then $T_E(X,Y)$ is a subsemigroup of $T(X)$. It is clear that if $X = Y$, then $T_E(X,Y) = T_E(X)$, which means that $T_E(X)$ is a special case of $T_E(X,Y)$. Furthermore, if E is the universal relation, $E = X \times X$, then $T_E(X,Y)$ becomes $T(X,Y)$. Moreover, it is not difficult to check that $T_E(X,Y)$ is the semigroup of all continuous self-maps of the topological space X for which all E-classes form a basis carrying X into a subspace Y and is referred to as a semigroup of continuous functions (see [3] for details).

In this project, we give a necessary and sufficient condition for $T_E(X,Y)$ to be regular and characterize Green's relations on $T_E(X,Y)$. Moreover, we study $T_E(X,Y)$ with Mitsch's natural partial order $\leq$ and then determine when two elements are related under this order. Also, we find elements of this semigroup which are compatible and then describe maximal and minimal elements. Finally, we study the rank of a subsemigroup of $T_E(X,Y)$ in a special case.

Let X/E denote the quotient set of X and let Y be a subset of X. The restriction of the equivalence E on Y, denoted by E_Y, is defined by

$$E_Y = \{(x,y) : x,y \in Y, (x,y) \in E\} = E \cap (Y \times Y).$$

For each $\alpha \in T_E(X,Y)$, let

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}$$

be the partition of X induced by α. As in [5], for each $A \subseteq X$, we write

$$\pi_A(\alpha) = \{M \in \pi(\alpha) : M \cap A \neq \emptyset\}.$$

We also define

$$\tilde{\pi}_A(\alpha) = \{M \in \pi(\alpha) : M \cap A \cap Y \neq \emptyset\}.$$

It is clear that $\tilde{\pi}_A(\alpha)$ is an appropriate extension of $\pi_A(\alpha)$ in the sense that if $Y = X$, then $\tilde{\pi}_A(\alpha) = \pi_A(\alpha)$. Obviously, $\tilde{\pi}_A(\alpha) \subseteq \pi_A(\alpha)$. For each $\alpha \in T_E(X,Y)$, define a function $\alpha_* : \pi(\alpha) \to X\alpha$ by

$$P\alpha_* = x\alpha \text{ for each } P \in \pi(\alpha) \text{ and each } x \in P.$$

We obtain the following lemma which will prove useful.

Lemma 1.1. *Let $\alpha \in T_E(X,Y)$. Then for each $B \in X/E$, there exists some $B' \in X/E$ such that $B\alpha \subseteq B' \cap Y \subseteq B'$. Consequently, for each $A \in X/E$, $A\alpha^{-1}$ is either the empty set or a union of some E-classes.*

For each $\alpha \in T_E(X,Y)$, let

$$E(\alpha) = \{A\alpha^{-1} : A \in X/E \text{ and } A\alpha^{-1} \neq \emptyset\}.$$

We can see that $E(\alpha)$ is also a partition of X and x,y are contained in the same $U \in E(\alpha)$ if and only if $(x\alpha, y\alpha) \in E$. Moreover, we define

$$E_Y(\alpha) = \{U \cap Y : U \in E(\alpha) \text{ and } U \cap Y \neq \emptyset\}.$$

Obviously, if $X = Y$, then $E_Y(\alpha) = E(\alpha)$.

Let E be an equivalence relation on a set X and U,V subsets of X. Let $\alpha : U \to V$. If $(u,u') \in E$ implies $(u\alpha, u'\alpha) \in E$ for each $u,u' \in U$, then α is said to be *E-preserving*. In addition, for each $u,u' \in U$, if $(u,u') \in E$ if and only if $(u\alpha, u'\alpha) \in E$, then α is called *E^*-preserving*. We remark that if α is an E^*-preserving bijection, then so is α^{-1}.

2 Regularity

In this section, we characterize regular elements in $T_E(X,Y)$ and then give a necessary and sufficient condition for $T_E(X,Y)$ to be regular.

Theorem 2.1. *Let $\alpha \in T_E(X,Y)$. Then α is regular if and only if for all $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha \subseteq (B \cap Y)\alpha$.*

Proof. Fix $y_0 \in Y$. For each $A \in X/E$, define a function β as follows. If $A \cap X\alpha = \emptyset$, define $x\beta = y_0$ for all $x \in A$. If $A \cap X\alpha \neq \emptyset$, by assumption, there is a class $B \in X/E$ such that $A \cap X\alpha \subseteq (B \cap Y)\alpha$. Fix $b_0 \in B \cap Y$. For each $x \in A \cap X\alpha$, there is an element $b_x \in B \cap Y$ such that $x = b_x\alpha$. Define

$$x\beta = \begin{cases} b_x & \text{if } x \in A \cap X\alpha \\ b_0 & \text{if } x \in A \setminus X\alpha. \end{cases}$$

It is not hard to see that β preserves E and $X\beta \subseteq Y$. Hence, $\beta \in T_E(X,Y)$. We claim that $\alpha = \alpha\beta\alpha$. Let $x \in X$. Then $x\alpha \in A \cap X\alpha$ for some E-class A and, by assumption, $A \cap X\alpha \subseteq (B \cap Y)\alpha$. By construction, $x\alpha = b_{x\alpha}\alpha$, where $b_{x\alpha} \in B \cap Y$. Note that

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = b_{x\alpha}\alpha = x\alpha.$$

Therefore, $\alpha = \alpha\beta\alpha$ since x is arbitrary, as claimed.

Conversely, suppose that α is regular. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_E(X,Y)$. Let $A \in X/E$. Then $A\beta \subseteq B \cap Y$ for some $B \in X/E$. We claim that $A \cap X\alpha \subseteq (B \cap Y)\alpha$. Let $y \in A \cap X\alpha$. Then $y \in A$ and $y = x\alpha$ for some $x \in X$. We obtain that $y\beta \in A\beta \subseteq B \cap Y$. Hence $x\alpha\beta \in B \cap Y$ which implies that $y = x\alpha = x\alpha\beta\alpha \in (B \cap Y)\alpha$ and the proof completes. □

By the above theorem, if $X = Y$, then α is regular if and only if for all $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha \subseteq B\alpha$. Hence Theorem 2.1 is a generalization of Corollary 2.3 in [5].

In [7], the authors defined a subset F of $T(X,Y)$ by

$$F = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$$

and proved that F is the set of all regular elements in $T(X,Y)$. Moreover, they also proved that F is the largest regular subsemigroup of $T(X,Y)$.

Now, we define a subset F_E of $T_E(X,Y)$ by $\alpha \in F_E$ if $\alpha \in T_E(X,Y)$ and for each $A \in X/E$, there exists $B \in X/E$ such that $A\alpha \subseteq (B \cap Y)\alpha$. It is easy to see that $F = F_E$ if $E = X \times X$ and $F_E = T_E(X)$ if $X = Y$. In general, F_E is a proper subset of $F \cap T_E(X,Y)$. To see this, consider the following example.

Let $X = \{1,2,3,4,5\}$ and $Y = \{1,2,4\}$. Define $X/E = \{A,B\}$ by $A = \{1,2,3\}$ and $B = \{4,5\}$. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

Clearly, $\alpha \in F \cap T_E(X,Y)$. We see that $A\alpha = \{1,2\}, (A \cap Y)\alpha = \{1\}$ and $(B \cap Y)\alpha = \{2\}$. Thus $\alpha \notin F_E$.

The subset F_E plays an essential role in the characterization of Green's relations, as shown in Section 3.

Lemma 2.2. *F_E is a right ideal of $T_E(X,Y)$. Consequently, it is a subsemigroup of $T_E(X,Y)$.*

Proof. Let $\alpha \in F_E$ and $\beta \in T_E(X,Y)$. Then for each $A \in X/E$, there is $B \in X/E$ such that

$$A\alpha\beta = (A\alpha)\beta \subseteq ((B \cap Y)\alpha)\beta = (B \cap Y)\alpha\beta.$$

Thus $\alpha\beta \in F_E$. □

Remark 2.3. *F_E contains the set of all regular elements in $T_E(X,Y)$.*

Proof. Let $\alpha \in T_E(X,Y)$ be a regular element and $A \in X/E$. Then $A\alpha \subseteq B \cap X\alpha$ for some $B \in X/E$ and so there exists $C \in X/E$ such that $A\alpha \subseteq B \cap X\alpha \subseteq (C \cap Y)\alpha$ since α is regular. Hence $\alpha \in F_E$. □

In general, the set F_E is not a regular subsemigroup of $T_E(X,Y)$. For example, let E be an equivalence on $X = \{1,2,3,4\}$ where $X/E = \{\{1,2\},\{3\},\{4\}\}$ and $Y = \{1,2,3\}$. Define $\alpha \in T_E(X,Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

It is easy to check that $\alpha \in F_E$ but not regular. Consequently, the set of all regular elements in $T_E(X,Y)$ is a proper subset of F_E.

Next, we give a necessary and sufficient condition for $T_E(X,Y)$ to be regular. Note that if $|Y| = 1$, then $T_E(X,Y)$ contains exactly one element and so $T_E(X,Y)$ is regular. Here, $\Delta(Y)$ stands for the diagonal relation on Y, that is, $\Delta(Y) = \{(y,y) \colon y \in Y\}$.

Theorem 2.4. *Let $Y \subseteq X$ such that $|Y| > 1$. Then $T_E(X,Y)$ is regular if and only if the following statements hold.*

(1) For each $G \in X/E$, $G \cap Y$ is non-empty.

(2) Either $E_Y = \Delta(Y)$ or $E = X \times X$ and $X = Y$.

Proof. ($\Rightarrow$) We prove by contrapositive. Suppose that there exists a class G with $G \cap Y$ is empty. Since $|Y| > 1$, there are $a,b \in Y$ such that $a \neq b$. Define a function $\alpha : X \to Y$ by $x\alpha = a$ if $x \in G$ and $x\alpha = b$ if $x \notin G$. We can see that $\alpha \in T_E(X,Y)$. Let A be the class containing a. We obtain $a \in A \cap X\alpha$. By the definition of α, for each $B \in X/E$ such that $B \cap Y \neq \emptyset$, we have $(B \cap Y)\alpha = \{b\}$ since $(B \cap Y) \cap G = \emptyset$. Hence $a \notin (B \cap Y)\alpha$ which implies that $A \cap X\alpha \nsubseteq (B \cap Y)\alpha$. Therefore, α is not regular by Theorem 2.1.

Assume that $E_Y \neq \Delta(Y)$ and $E \neq X \times X$. Then there is a class $A \neq X$ such that $|A \cap Y| > 1$. Let $a,b \in A \cap Y$ such that $a \neq b$. Define a function $\alpha : X \to Y$ by $x\alpha = a$, $\forall x \in A \cap Y$ and $x\alpha = b$, $\forall x \notin A \cap Y$. We can see that $\alpha \in T_E(X,Y)$. For each $B \in X/E$, $(B \cap Y)\alpha = \{a\}$ if $B = A$ and $(B \cap Y)\alpha = \{b\}$ if $B \neq A$. We obtain $A \cap X\alpha = \{a,b\} \nsubseteq (B \cap Y)\alpha$ for all class B. Hence α is not regular by Theorem 2.1.

Suppose that $E_Y \neq \Delta(Y)$ and $X \neq Y$. Then there exists $y \in X \setminus Y$. Since $E_Y \neq \Delta(Y)$, there is a class A such that $|A \cap Y| > 1$. Let $a,b \in A \cap Y$ such that $a \neq b$. Define a function $\alpha : X \to Y$ by $x\alpha = a$ if $x = y$ and $x\alpha = b$ if $x \neq y$. It is obvious that $\alpha \in T_E(X,Y)$. We see that for each $B \in X/E$, $(B \cap Y)\alpha = \{b\}$. Therefore, $A \cap X\alpha = \{a,b\} \nsubseteq \{b\} = (B \cap Y)\alpha$ which implies that α is not regular.

($\Leftarrow$) We can see that if $E = X \times X$ and $X = Y$, then $T_E(X,Y) = T(X)$ is regular. Now, we suppose that $E_Y = \Delta(Y)$. Let $\alpha \in T_E(X,Y)$ and $A \in X/E$. Then $A \cap X\alpha \subseteq A \cap Y = \{a\}$ for some a by (1). If $A \cap X\alpha \neq \emptyset$, then $A \cap X\alpha = \{a\}$ which implies that $a = x\alpha$ for some $x \in X$. Let B be a class containing x. Then $B \cap Y$ is non-empty from which it follows that $B \cap Y = \{b\}$ for some b. Further, since $(b,x) \in E$, we get $(b\alpha, a) = (b\alpha, x\alpha) \in E$ which follows that $b\alpha \in A \cap X\alpha = \{a\}$. Thus $A \cap X\alpha = \{a\} = \{b\alpha\} = (B \cap Y)\alpha$. Therefore, α is regular. □

By Theorem 2.5 of [5], we obtain some properties of regular elements in $T_E(X,Y)$ as follows.

Theorem 2.5. *Let α and β be regular elements in $T_E(X,Y)$. Then the following statements hold.*

(1) If $\pi(\alpha) = \pi(\beta)$, then $E(\alpha) = E(\beta)$.

(2) If $X\alpha = X\beta$, then for each $A \in X/E$, there are $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

3 Green's Relations

In this section, we characterize Green's relations on $T_E(X,Y)$. We start this section by recalling the definition of Green's relations.

Let S be a semigroup. The following definitions are due to J. A. Green. For any $a, b \in S$, Define

$$(a,b) \in \mathscr{L} \text{ if and only if } S^1a = S^1b,$$

or equivalently; $(a,b) \in \mathscr{L}$ if and only if $a = xb, b = ya$ for some $x, y \in S^1$.

Dually,

$$(a,b) \in \mathscr{R} \text{ to mean } aS^1 = bS^1,$$

or equivalently; $(a,b) \in \mathscr{R}$ if and only if $a = bx, b = ay$ for some $x, y \in S^1$ and then define

$$(a,b) \in \mathscr{J} \text{ to mean } S^1aS^1 = S^1bS^1,$$

or equivalently; $(a,b) \in \mathscr{J}$ if and only if $a = xby, b = uav$ for some $x, y, u, v \in S^1$.

Finally,

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R} \text{ and } \mathscr{D} = \mathscr{L} \circ \mathscr{R}.$$

Note that the above relations are equivalence relations. The relation $\mathscr{D}$ is the join $\mathscr{L} \vee \mathscr{R}$, that is, $\mathscr{D}$ is the smallest equivalence relation containing $\mathscr{L} \cup \mathscr{R}$. It is well-known that $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. Moreover, $\mathscr{H} \subseteq \mathscr{L} \subseteq \mathscr{D} \subseteq \mathscr{J}$ and $\mathscr{H} \subseteq \mathscr{R} \subseteq \mathscr{D} \subseteq \mathscr{J}$. But, in commutative semigroups, we have $\mathscr{H} = \mathscr{L} = \mathscr{R} = \mathscr{D} = \mathscr{J}$. The relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ and $\mathscr{J}$ are called *Green's relations* on S. For each $a \in S$, we denote $\mathscr{L}$-class, $\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class and $\mathscr{J}$-class containing a by L_a, R_a, H_a, D_a and J_a, respectively.

In general, if $X \neq Y$, then the semigroup $T_E(X,Y)$ does not contain the identity element. Hence $T_E(X,Y)^1 \neq T_E(X,Y)$.

Now, we prove the following theorem which extends Theorem 3.1 of [5].

Theorem 3.1. *Let $\alpha, \beta \in T_E(X,Y)$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{R}$.

(2) $\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$.

(3) There exists a bijective E^*-preserving $\phi : X\alpha \to X\beta$ such that $\beta = \alpha\phi$.

Proof. (1)⇒(2). Suppose that $(\alpha, \beta) \in \mathscr{R}$. Then there are $\gamma, \mu \in T_E(X,Y)^1$ such that $\alpha = \beta\gamma$ and $\beta = \alpha\mu$. If $\alpha = \beta$, then $\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$. If $\alpha \neq \beta$, then both γ and μ belong to $T_E(X,Y)$ which implies that $\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$ by Theorem 3.1 of [5].

(2)⇒(3). The proof also follows from Theorem 3.1 of [5].

(3)⇒(1). The proof is an appropriate modification of the proof of (3)⇒(1) in Theorem 3.1 of [5]. In fact, suppose that there exists a bijective E^*-preserving $\phi : X\alpha \to X\beta$ such that $\beta = \alpha\phi$. For each $A \in X/E$, let $A' = A \cap X\alpha$. If $A' = \emptyset$, define $x\mu = y_0$ for each $x \in A$ and $y_0 \in Y$ is fixed. If $A' \neq \emptyset$, then $A'\phi = (A \cap X\alpha)\phi \subseteq B \cap X\beta$ for some class B since ϕ is E^*-preserving. In the case $A' \neq \emptyset$, we fix $b_0 \in B \cap X\beta$ and define μ by

$$x\mu = \begin{cases} x\phi & \text{if } x \in A'; \\ b_0 & \text{if } x \in A \setminus A'. \end{cases}$$

It is easy to verify that $\mu \in T_E(X,Y)$ and $\beta = \alpha\mu$. Similarly, we can show that $\alpha = \beta\gamma$ for some $\gamma \in T_E(X,Y)$. □

Lemma 3.2. *Let $\alpha, \beta \in T_E(X,Y)$. If $\pi(\alpha) = \pi(\beta)$, then either both α and β are in F_E, or neither is in F_E.*

Proof. Assume that $\pi(\alpha) = \pi(\beta)$ and let $\alpha \in F_E$. It suffices to show $\beta \in F_E$. Let $A \in X/E$. Then $A\alpha \subseteq (B \cap Y)\alpha$ for some class B. We claim that $A\beta \subseteq (B \cap Y)\beta$. Indeed, let $a \in A$. Then there is $x \in X$ such that $(a\beta)\beta^{-1} = (x\alpha)\alpha^{-1}$ since $(a\beta)\beta^{-1} \in \pi(\beta) = \pi(\alpha)$. Obviously, $a \in (a\beta)\beta^{-1} = (x\alpha)\alpha^{-1}$ which implies that $x\alpha = a\alpha \in A\alpha \subseteq (B \cap Y)\alpha$. Thus $x\alpha = b\alpha$ for some $b \in B \cap Y$. Hence $b \in (x\alpha)\alpha^{-1} = (a\beta)\beta^{-1}$ and so $a\beta = b\beta \in (B \cap Y)\beta$. Therefore, $\beta \in F_E$. □

By Theorem 3.1 and Lemma 3.2, we have the following corollary.

Corollary 3.3. *For $\alpha \in T_E(X,Y)$, the following statements hold.*

(1) If $\alpha \in F_E$, then $R_\alpha = \{\beta \in F_E : \pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)\}$.

(2) If $\alpha \in T_E(X,Y) \setminus F_E$, then

$$R_\alpha = \{\beta \in T_E(X,Y) \setminus F_E : \pi(\alpha) = \pi(\beta) \text{ and } E(\alpha) = E(\beta)\}.$$

Now, we have already characterized Green's $\mathscr{R}$-relation of $T_E(X,Y)$. To study the remaining Green's relations, we introduce some definitions for using throughout this paper. Actually, we extend the notions of E-admissibility and E^*-admissibility presented in [5].

Let $\alpha, \beta \in T_E(X,Y)$ and let ϕ be a mapping from $\pi(\alpha)$ into $\pi(\beta)$. We say that ϕ is $\tilde{E}$-*admissible* if and only if for each $A \in X/E$, there exists $B \in X/E$ such that

$$\pi_A(\alpha)\phi \subseteq \tilde{\pi}_B(\beta).$$

Equivalently, $\phi : \pi(\alpha) \to \pi(\beta)$ is $\tilde{E}$-admissible if and only if for each $A \in X/E$, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $B \cap P\phi \cap Y \neq \emptyset$.

If ϕ is a bijection such that ϕ and ϕ^{-1} are $\tilde{E}$-admissible, then ϕ is called $\tilde{E}^*$-*admissible*.

We remark that if $X = Y$, then the notions of E-admissibility (resp. $\tilde{E}$-admissibility) and $\tilde{E}$-admissibility (resp. $\tilde{E^*}$-admissibility) are the same.

Now, we determine Green's $\mathscr{L}$-relation on $T_E(X,Y)$. The proof of the following lemma is straightforward and so it is omitted.

Lemma 3.4. *Let $\alpha, \beta \in T_E(X,Y)$. If for each $A \in X/E$, there are $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$, then $X\alpha = X\beta$.*

Theorem 3.5. *Let $\alpha, \beta \in F_E$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{L}$ in $T_E(X,Y)$.

(2) For each $A \in X/E$, there are $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(3) There is a bijective $\tilde{E}^*$-admissible $\phi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \phi\beta_*$.

Proof. (1)⇒(2). Suppose that $(\alpha,\beta) \in \mathscr{L}$. Then $\alpha = \gamma\beta$ and $\beta = \mu\alpha$ for some $\gamma, \mu \in T_E(X,Y)^1$. If $\alpha = \beta$, then (2) holds. If $\alpha \neq \beta$, then $\gamma, \mu \in T_E(X,Y)$. The item (2) follows by [5, Theorem 3.2].

(2)⇒(3). Suppose (2) holds. Note that $X\alpha = X\beta$ by Lemma 3.4. Then for each $P \in \pi(\alpha)$, we have $P\alpha_* \in X\alpha = X\beta$. We can see that $(P\alpha_*)\beta^{-1} \in \pi(\beta)$. Define $\phi : \pi(\alpha) \to \pi(\beta)$ by $P\phi = (P\alpha_*)\beta^{-1}$. It is clear that ϕ is well-defined and $\alpha_* = \phi\beta_*$. Now, we show that ϕ is injective. Let $P_1, P_2 \in \pi(\alpha)$ be such that $P_1\phi = P_2\phi$. Let $y \in P_1$. Then $P_1\alpha_* = y\alpha$ and so $P_1\phi = (P_1\alpha_*)\beta^{-1} = (y\alpha)\beta^{-1}$. Since $P_1\phi = P_2\phi$, we have $(y\alpha)\beta^{-1} = P_2\phi = (P_2\alpha_*)\beta^{-1}$ which implies that $y\alpha = P_2\alpha_*$. Thus $y \in P_2$ and then $P_1 \subseteq P_2$. Similarly, we can show that $P_2 \subseteq P_1$. To show that ϕ is surjective, let $Q \in \pi(\beta)$. Then $Q = x\beta^{-1}$ for some $x \in X\beta = X\alpha$. Choose $P = x\alpha^{-1} \in \pi(\alpha)$. We obtain $Q = x\beta^{-1} = (P\alpha_*)\beta^{-1} = P\phi$. Therefore, ϕ is a bijection.

Next, we show that ϕ is $\tilde{E}$-admissible. Let $A \in X/E$. Then there is $B \in X/E$ such that $A\alpha \subseteq B\beta$ by the assumption. Hence there is a class D such that $B\beta \subseteq (D \cap Y)\beta$ since $\beta \in F_E$. Thus $A\alpha \subseteq (D \cap Y)\beta$. Let $P \in \pi_A(\alpha)$. Then $P \in \pi(\alpha)$ and $P \cap A \neq \emptyset$. Choose $x \in P \cap A$. Then $P\alpha_* = x\alpha \subseteq A\alpha \subseteq (D \cap Y)\beta$ from which it follows that $x\alpha = y\beta$ for some $y \in D \cap Y$. We have $y \in D \cap (y\beta)\beta^{-1} \cap Y$ and then

$$y \in D \cap (y\beta)\beta^{-1} \cap Y = D \cap (x\alpha)\beta^{-1} \cap Y = D \cap (P\alpha_*)\beta^{-1} \cap Y = D \cap P\phi \cap Y.$$

Thus ϕ is $\tilde{E}$-admissible since $D \cap P\phi \cap Y$ is non-empty. Finally, we prove that ϕ^{-1} is $\tilde{E}$-admissible. Let $P \in \pi(\alpha)$ and $Q \in \pi(\beta)$ be such that $Q = P\phi$. We obtain $Q = (P\alpha_*)\beta^{-1}$ which implies that $Q\beta_* = P\alpha*$. Hence

$$Q\phi^{-1} = P = (P\alpha_*)\alpha^{-1} = (Q\beta_*)\alpha^{-1}.$$

By the same argument as ϕ, we obtain ϕ^{-1} is also $\tilde{E}$-admissible. Therefore, ϕ is $\tilde{E}^*$-admissible.

(3)⇒(1). Assume that (3) holds. For each $A \in X/E$, there is $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $B \cap P\phi \cap Y \neq \emptyset$. For each $x \in A$, let $P_x = (x\alpha)\alpha^{-1}$. We can see that $P_x \in \pi_A(\alpha)$. Then $B \cap P_x\phi \cap Y \neq \emptyset$. Choose $d_x \in B \cap P_x\phi \cap Y$ and define $x\gamma = d_x$. First, we show that $\gamma \in T_E(X,Y)$. Let $(a,b) \in E$. Then $a, b \in A$ for some $A \in X/E$. By the definition of γ, we obtain $a\gamma, b\gamma \in B$ for some $B \in X/E$. Thus $(a\gamma, b\gamma) \in E$. Next, we prove that $\alpha = \gamma\beta$. Let $x \in A$ for some $A \in X/E$. Then $x\gamma\beta = d_x\beta$ where $d_x \in B \cap P_x\phi \cap Y$ for some $B \in X/E$. Moreover, since $d_x \in P_x\phi$, we get $d_x\beta = (P_x\phi)\beta_*$. Hence

$$x\gamma\beta = d_x\beta = (P_x\phi)\beta_* = P_x\alpha_* = ((x\alpha)\alpha^{-1})\alpha_* = x\alpha.$$

Similarly, we can show that $\beta = \mu\alpha$ for some$\mu \in T_E(X,Y)$. Therefore, $(\alpha,\beta) \in \mathscr{L}$. □

By the above theorem, if $X = Y$, then we obtain Theorem 3.2 of [5].

Theorem 3.6. *For $\alpha \in T_E(X,Y)$, the following statements hold.*

(1) If $\alpha \in T_E(X,Y) \setminus F_E$, then $L_\alpha = \{\alpha\}$.

(2) If $\alpha \in F_E$, then

$$L_\alpha = \{\beta \in F_E : (\forall A \in X/E)(\exists B, C \in X/E)\ A\alpha \subseteq B\beta \text{ and } A\beta \subseteq C\alpha\}.$$

Proof. Let $\alpha \in T_E(X,Y)$ and let $\beta \in L_\alpha$. Then $(\alpha,\beta) \in \mathscr{L}$ which implies that $\alpha = \gamma\beta$ and $\beta = \mu\alpha$ for some $\gamma,\mu \in T_E(X,Y)^1$. If $\gamma,\mu \in T_E(X,Y)$, then for each $A \in X/E$, there are $B,C \in X/E$ such that $A\alpha = A\gamma\beta = A\gamma\mu\alpha \subseteq (B\cap Y)\alpha$ and $A\beta = A\mu\alpha = A\mu\gamma\beta \subseteq (C\cap Y)\beta$. Thus $\alpha,\beta \in F_E$.

(1) If $\alpha \in T_E(X,Y)\setminus F_E$, then $\gamma = 1$ or $\mu = 1$ and hence $\alpha = \beta$.

(2) If $\alpha \in F_E$, then there are two cases to consider. The case $\alpha = \beta$ is clear. If $\alpha \neq \beta$, then $\gamma,\mu \in T_E(X,Y)$ and hence $\beta \in F_E$. In addition, for each $A \in X/E$, there are $B,C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$ by Theorem 3.5. The other containment is obvious. □

As a direct consequence of Corollary 3.3, Theorems 3.1, 3.5 and 3.6, we have the following theorems.

Theorem 3.7. *Let $\alpha,\beta \in F_E$. Then the following statements are equivalent.*

(1) $(\alpha,\beta) \in \mathscr{H}$ *in* $T_E(X,Y)$.

(2) $\pi(\alpha) = \pi(\beta)$, $E(\alpha) = E(\beta)$ *and for each* $A \in X/E$, *there are* $B,C \in X/E$ *such that* $A\alpha \subseteq B\beta$ *and* $A\beta \subseteq C\alpha$.

(3) There exist a bijective $\tilde{E}^*$*-admissible* $\phi : \pi(\alpha) \to \pi(\beta)$ *such that* $\alpha_* = \phi\beta_*$ *and a bijective* E^**-preserving* $\psi : X\alpha \to X\beta$ *such that* $\beta = \alpha\psi$.

Theorem 3.8. *For* $\alpha \in T_E(X,Y)$, *the following statements hold.*

(1) If $\alpha \in T_E(X,Y)\setminus F_E$, *then* $H_\alpha = \{\alpha\}$.

(2) If $\alpha \in F_E$, *then*

$$\begin{aligned} H_\alpha \;=\; & \{\beta \in F_E : \pi(\alpha) = \pi(\beta), E(\alpha) = E(\beta) \text{ and} \\ & (\forall A \in X/E)(\exists B,C \in X/E)\; A\alpha \subseteq B\beta \text{ and } A\beta \subseteq C\alpha\}. \end{aligned}$$

Now, we characterize Green's $\mathscr{D}$ relation.

Theorem 3.9. *Let $\alpha,\beta \in F_E$. Then the following statements are equivalent.*

(1) $(\alpha,\beta) \in \mathscr{D}$ *in* $T_E(X,Y)$.

(2) There is an E^**-preserving bijection* $\Phi : X\alpha \to X\beta$ *such that for each* $A \in X/E$, *there exist* $B,C \in X/E$ *with*

$$A\beta \subseteq (B\cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C\cap Y)\beta\Phi^{-1}.$$

(3) There exist a bijective $\tilde{E}^*$*-admissible* $\phi : \pi(\alpha) \to \pi(\beta)$ *and a bijective* E^**-preserving* $\psi : X\alpha \to X\beta$ *such that* $\alpha_*\psi = \phi\beta_*$.

Proof. (1)⇒(3). Suppose that $(\alpha,\beta) \in \mathscr{D}$ in $T_E(X,Y)$. Then there exists $\gamma \in T_E(X,Y)$ such that $(\alpha,\gamma) \in \mathscr{R}$ and $(\gamma,\beta) \in \mathscr{L}$. By Theorem 3.1 and 3.5, there exist a bijective E^*-preserving $\psi : X\alpha \to X\gamma$ such that $\gamma = \alpha\psi$ and a bijective $\tilde{E}^*$-admissible $\phi : \pi(\gamma) \to \pi(\beta)$ such that $\gamma_* = \phi\beta_*$. In addition, we obtain $\pi(\alpha) = \pi(\gamma)$ and $X\gamma = X\beta$. Hence $\phi : \pi(\alpha) \to \pi(\beta)$ and $\psi : X\alpha \to X\beta$. Now, we show that $\alpha_*\psi = \phi\beta_*$. We claim that $\gamma_* = \alpha_*\psi$. Indeed, let $x\gamma^{-1} \in \pi(\gamma)$. Then $(x\gamma^{-1})\gamma_* = x$. From $\pi(\alpha) = \pi(\gamma)$, there is $y \in X\alpha$ such that $x\gamma^{-1} = y\alpha^{-1}$ which implies that $y = (y\alpha^{-1})\alpha_* = (x\gamma^{-1})\alpha_*$. Since

$y \in X\alpha$, we get $y = z\alpha$ for some $z \in X$. Thus $z \in y\alpha^{-1} = x\gamma^{-1}$ from which it follows that $z\gamma = x = (x\gamma^{-1})\gamma_*$. We obtain

$$(x\gamma^{-1})\alpha_*\psi = y\psi = z\alpha\psi = z\gamma = (x\gamma^{-1})\gamma_*.$$

Therefore, $\alpha_*\psi = \gamma_* = \phi\beta_*$.

(3)⇒(2). Assume that there exist a bijective $\tilde{E}^*$-admissible $\phi : \pi(\alpha) \to \pi(\beta)$ and a bijective E^*-preserving $\psi : X\alpha \to X\beta$ such that $\alpha_*\psi = \phi\beta_*$. Define an E^*-preserving bijection $\Phi : X\alpha \to X\beta$ by $x\Phi = x\psi$ for all $x \in X\alpha$. Let $A \in X/E$. We have $\pi_A(\beta)\phi^{-1} \subseteq \tilde{\pi}_B(\alpha)$ for some $B \in X/E$ since $\phi^{-1} : \pi(\beta) \to \pi(\alpha)$ is $\tilde{E}^*$-admissible. We claim that $A\beta \subseteq (B \cap Y)\alpha\Phi$. Indeed, let $a \in A$. Then $(a\beta)\beta^{-1} \in \pi_A(\beta)$ which implies that $(a\beta)\beta^{-1}\phi^{-1} \in (\pi_A(\beta))\phi^{-1} \subseteq \tilde{\pi}_B(\alpha)$. Hence $(a\beta)\beta^{-1}\phi^{-1} = (b\alpha)\alpha^{-1}$ for some $b \in X$ and $(b\alpha)\alpha^{-1} \cap B \cap Y \neq \emptyset$. There exists $y \in (b\alpha)\alpha^{-1} \cap B \cap Y$ and then $b\alpha = y\alpha$. We see that

$$a\beta = (a\beta)\beta^{-1}\beta_* = (a\beta)\beta^{-1}\phi^{-1}\alpha_*\psi = (b\alpha)\alpha^{-1}\alpha_*\psi = b\alpha\psi$$

and so $a\beta = b\alpha\psi = y\alpha\psi \in (B \cap Y)\alpha\psi = (B \cap Y)\alpha\Phi$. Therefore, $A\beta \subseteq (B \cap Y)\alpha\Phi$. Similarly, we can show that $A\alpha \subseteq (C \cap Y)\beta\Phi^{-1}$ for some class C.

(2)⇒(3). Assume that (2) holds. Define a function $\phi : \pi(\alpha) \to \pi(\beta)$ by $(x\alpha)\alpha^{-1}\phi = (x\alpha)\Phi\beta^{-1}$ for all $x \in X$. Then ϕ is well-defined. We first show that ϕ is injective. Let $(x\alpha)\alpha^{-1}, (y\alpha)\alpha^{-1}$ be such that $(x\alpha)\alpha^{-1}\phi = (y\alpha)\alpha^{-1}\phi$. Then $(x\alpha)\Phi\beta^{-1} = (y\alpha)\Phi\beta^{-1}$ and so $x\alpha = y\alpha$ since Φ is injective. Hence $(x\alpha)\alpha^{-1} = (y\alpha)\alpha^{-1}$. Next, we prove the surjectivity of ϕ. Let $(x\beta)\beta^{-1} \in \pi(\beta)$. Then $x\beta \in A\beta \subseteq (B \cap Y)\alpha\Phi$ for some classes A and B. Then there is $y \in B \cap Y$ such that $y\alpha\Phi = x\beta$. Thus $(y\alpha)\alpha^{-1}\phi = (y\alpha)\Phi\beta^{-1} = (x\beta)\beta^{-1}$. Therefore, ϕ is bijective.

Finally, we show that ϕ is $\tilde{E}^*$-admissible. Let $A \in X/E$. Then there exists $C \in X/E$ such that $A\alpha \subseteq (C \cap Y)\beta\Phi^{-1}$. Let $(x\alpha)\alpha^{-1} \in \pi_A(\alpha)$. Then $(x\alpha)\alpha^{-1} \cap A \neq \emptyset$ which implies that there is $a \in A$ such that $a\alpha = x\alpha$. Hence $x\alpha = a\alpha \in A\alpha \subseteq (C \cap Y)\beta\Phi^{-1}$ and so $x\alpha = c\beta\Phi^{-1}$ for some $c \in C \cap Y$. Thus $c \in x\alpha\Phi\beta^{-1} = (x\alpha)\alpha^{-1}\phi$ which implies that $c \in (x\alpha)\alpha^{-1}\phi \cap C \cap Y \neq \emptyset$ and so ϕ is $\tilde{E}$-admissible. On the other hand, let $A \in X/E$. Then there exists $B \in X/E$ such that $A\beta \subseteq (B \cap Y)\alpha\Phi$. Let $(x\beta)\beta^{-1} \in \pi_A(\beta)$. Then $(x\beta)\beta^{-1} \cap A \neq \emptyset$ which implies that there is $a \in A$ such that $a\beta = x\beta$. Hence $x\beta = a\beta \in A\beta \subseteq (B \cap Y)\alpha\Phi$ and so $x\beta = b\alpha\Phi$ for some $b \in B \cap Y$. Thus

$$b \in (b\alpha)\alpha^{-1} = (b\alpha)\alpha^{-1}\phi\phi^{-1} = (b\alpha)\Phi\beta^{-1}\phi^{-1} = (x\beta)\beta^{-1}\phi^{-1}$$

which implies that $b \in (x\beta)\beta^{-1}\phi^{-1} \cap B \cap Y \neq \emptyset$. Therefore, ϕ^{-1} is $\tilde{E}$-admissible. Moreover, we define a function $\psi : X\alpha \to X\beta$ by $x\psi = x\Phi$ for each $x \in X\alpha$. It remains to show that $\alpha_*\psi = \phi\beta_*$. Indeed, let $(x\alpha)\alpha^{-1} \in \pi(\alpha)$. Then

$$(x\alpha)\alpha^{-1}\alpha_*\psi = x\alpha\psi = x\alpha\Phi = (x\alpha)\Phi\beta^{-1}\beta_* = (x\alpha)\alpha^{-1}\phi\beta_*.$$

(3) ⇒(1). Assume that (3) holds. Define a function $\gamma : X \to Y$ by $x\gamma = x\alpha\psi$. Since ψ is E^*-preserving, we have $\gamma \in T_E(X,Y)$. We first show that $\pi(\gamma) = \pi(\alpha)$. Let $A = x\gamma^{-1} \in \pi(\gamma)$. Then $\{x\} = A\gamma = A\alpha\psi$ which implies that $A\alpha = x\psi^{-1}$ since ψ is a bijection. We obtain $A\alpha = \{y\}$ where $\{y\} = x\psi^{-1}$ and then $y\psi = x$. Hence $A \subseteq y\alpha^{-1}$. Let $z \in y\alpha^{-1}$. Then $z\alpha = y$ from which it follows that $z\gamma = z\alpha\psi = y\psi = x$. Thus $z \in x\gamma^{-1} = A$ implies $y\alpha^{-1} \subseteq A$. So $A = y\alpha^{-1} \in \pi(\alpha)$. We conclude that $\pi(\gamma) \subseteq \pi(\alpha)$. On the other hand, let $B = a\alpha^{-1} \in \pi(\alpha)$. Then $B\alpha = \{a\}$ which implies that $B\gamma = B\alpha\psi = \{a\}\psi = \{b\}$ for some $b \in Y$. Hence $B \subseteq b\gamma^{-1}$. Let $c \in b\gamma^{-1}$. Then $c\alpha\psi = c\gamma = b$ from which it follows that $\{c\alpha\} = b\psi^{-1} = \{a\}$ since ψ is a bijection.

Then $c\alpha = a$ implies $c \in a\alpha^{-1} = B$. So $b\gamma^{-1} \subseteq B$. We conclude that $B = b\gamma^{-1} \in \pi(\gamma)$. Therefore, $\pi(\alpha) \subseteq \pi(\gamma)$ and then $\pi(\gamma) = \pi(\alpha)$. Moreover, since $\phi : \pi(\gamma) \to \pi(\beta)$ is bijective $\tilde{E}^*$-admissible and $\gamma_* = \alpha_*\psi = \phi\beta_*$, we get $(\gamma,\beta) \in \mathscr{L}$ and $X\gamma = X\beta$ by Theorem 3.5. Since $\psi : X\alpha \to X\gamma$ is bijective E^*-preserving and $\gamma = \alpha\psi$, we obtain $(\alpha,\gamma) \in \mathscr{R}$ by Theorem 3.1. Therefore, $(\alpha,\beta) \in \mathscr{D}$. □

The above result extends Theorem 3.4 of [5]. In fact, we obtain an additional characterization of the $\mathscr{D}$-relation in the case when $X = Y$, as shown in the following corollary.

Corollary 3.10. *Let $\alpha,\beta \in T_E(X)$. Then the following statements are equivalent.*

(1) $(\alpha,\beta) \in \mathscr{D}$.

(2) There is an E^*-preserving bijection $\Phi : X\alpha \to X\beta$ such that for each $A \in X/E$, there exist $B,C \in X/E$ with

$$A\beta \subseteq B\alpha\Phi \text{ and } A\alpha \subseteq C\beta\Phi^{-1}.$$

We remark that for condition (2) of Theorem 3.9 to be true, it suffices to check that Φ is injective since the surjectivity of Φ follows from the fact that for each $A \in X/E$, there exists $B \in X/E$ with

$$A\beta \subseteq (B \cap Y)\alpha\Phi.$$

Theorem 3.11. *For $\alpha \in T_E(X,Y)$, the following statements hold.*

(1) If $\alpha \in T_E(X,Y) \setminus F_E$, then

$$D_\alpha = \{\beta \in T_E(X,Y) \setminus F_E : \pi(\alpha) = \pi(\beta) \text{ and } E(\alpha) = E(\beta)\} = R_\alpha.$$

(2) If $\alpha \in F_E$, then

$$D_\alpha = \{\beta \in F_E : \beta \text{ satisfies (2) of Theorem 3.9}\}.$$

Proof. (1) Let $\alpha \in T_E(X,Y) \setminus F_E$ and let $\beta \in D_\alpha$. Then there exists $\gamma \in T_E(X,Y)$ such that $(\alpha,\gamma) \in \mathscr{R}$ and $(\gamma,\beta) \in \mathscr{L}$. Hence $\gamma \in T_E(X,Y) \setminus F_E$ such that $\pi(\alpha) = \pi(\gamma)$ and $E(\alpha) = E(\gamma)$. Moreover, we obtain $\beta = \gamma$ since $L_\gamma = \{\gamma\}$. Thus $\beta \in T_E(X,Y) \setminus F_E$ such that $\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$. The other containment is clear since $\mathscr{R} \subseteq \mathscr{D}$.

(2) Let $\alpha \in F_E$ and let $\beta \in D_\alpha$. Then there exists $\gamma \in T_E(X,Y)$ such that $(\alpha,\gamma) \in \mathscr{R}$ and $(\gamma,\beta) \in \mathscr{L}$. By Corollary 3.3 and Theorem 3.6, we obtain both γ and β belong to F_E. It is easy to see that β satisfies (2) of Theorem 3.9. The other containment is clear. □

For each $x \in X$, denote the equivalence class containing x by $[x]$. Now, we characterize Green's $\mathscr{J}$-relation. First, we prove the following lemmas.

Lemma 3.12. *Let $\alpha,\beta \in T_E(X,Y)$. Then the following statements are equivalent.*

(1) There is an E-preserving surjection $\Phi : X\alpha \to X\beta$ such that for each $A \in X/E$ there exists $B \in X/E$ with $A\beta \subseteq (B \cap Y)\alpha\Phi$.

(2) $\beta = \gamma\alpha\mu$ for some $\gamma,\mu \in T_E(X,Y)$.

Proof. (1)⇒(2). Suppose that (1) holds. Note that for each $A \in X/E$, $(A \cap X\alpha)\Phi \subseteq B$ for some $B \in X/E$ since Φ is E-preserving. Furthermore, $(A \cap X\alpha)\Phi \subseteq X\beta \subseteq Y$ so that $(A \cap X\alpha)\Phi \subseteq B \cap Y$. Now, we define a function μ as follows. For each $A \in X/E$ such that $A \cap X\alpha \neq \emptyset$, $(A \cap X\alpha)\Phi \subseteq B \cap Y$ for some $B \in X/E$. Choose $b \in B \cap Y$ and define

$$x\mu = \begin{cases} b & \text{if } x \in A \setminus X\alpha; \\ x\Phi & \text{if } x \in A \cap X\alpha. \end{cases}$$

Let $x\mu = x\beta$ if $x \in \bigcup\{A \in X/E : A \cap X\alpha = \emptyset\}$. Clearly, $\mu \in T_E(X,Y)$. Next, we define a function γ as follows. Let $A \in X/E$. By the condition of Φ, there exists $B \in X/E$ such that $A\beta \subseteq (B \cap Y)\alpha\Phi$. Let $x \in A$. Then $x\beta = y\alpha\Phi$ for some $y \in B \cap Y$. Define $x\gamma = y$. It then follows that $\gamma \in T_E(X,Y)$ and $\beta = \gamma\alpha\mu$.

(2)⇒(1). Let $\beta = \gamma\alpha\mu$ for some $\gamma, \mu \in T_E(X,Y)$. Fix $y_0 \in X\beta$. If $[x] \cap X\gamma\alpha \neq \emptyset$, choose $x_0 \in [x] \cap X\gamma\alpha$ for each $x \in X\alpha$. Define a function $\Phi : X\alpha \to X\beta$ by

$$x\Phi = \begin{cases} x\mu & \text{if } x \in X\gamma\alpha; \\ x_0\mu & \text{if } x \notin X\gamma\alpha \text{ and } [x] \cap X\gamma\alpha \neq \emptyset; \\ y_0 & \text{if } [x] \cap X\gamma\alpha = \emptyset. \end{cases}$$

It is easy to verify that Φ is E-preserving. Let $A \in X/E$. There exists $B \in X/E$ such that $A\gamma \subseteq B \cap Y$. Therefore,

$$A\beta = A\gamma\alpha\mu = A\gamma\alpha\Phi \subseteq (B \cap Y)\alpha\Phi.$$

In addition, we obtain that Φ is surjective. □

As a direct consequence of Theorem 3.9, we obtain the following lemma.

Lemma 3.13. *Let $\alpha, \beta \in F_E$. If $(\alpha, \beta) \in \mathscr{D}$ in $T_E(X,Y)$, then there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with*

$$A\beta \subseteq (B \cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C \cap Y)\beta\Psi.$$

Theorem 3.14. *Let $\alpha, \beta \in T_E(X,Y)$. Then $(\alpha, \beta) \in \mathscr{J}$ if and only if either*

(1) $\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$; or

(2) there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with

$$A\beta \subseteq (B \cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C \cap Y)\beta\Psi.$$

Proof. Assume that $(\alpha, \beta) \in \mathscr{J}$. Then there exist $\gamma, \delta, \lambda, \mu \in T_E(X,Y)^1$ such that $\alpha = \gamma\beta\delta$ and $\beta = \lambda\alpha\mu$. If $\gamma = 1 = \lambda$, then $\alpha = \beta\delta$ and $\beta = \alpha\mu$ which implies $(\alpha, \beta) \in \mathscr{R} \subseteq \mathscr{D}$. If $\delta = 1 = \mu$, then $\alpha = \gamma\beta$ and $\beta = \lambda\alpha$ which implies $(\alpha, \beta) \in \mathscr{L} \subseteq \mathscr{D}$. Hence (1) or (2) holds by Theorem 3.11 and Lemma 3.13.

If $\{\gamma, \lambda\} \neq \{1\}$ and $\{\delta, \mu\} \neq \{1\}$, then we have $\alpha = \eta\beta\zeta$ and $\beta = \rho\alpha\sigma$ for some $\eta, \zeta, \rho, \sigma \in T_E(X,Y)$. For example, if $\gamma = 1 = \mu$ and $\lambda, \delta \in T_E(X,Y)$, then

$$\alpha = \beta\delta = \lambda\alpha\delta \text{ and } \beta = \lambda\alpha = \lambda\beta\delta.$$

We have (2) holds by Lemma 3.12. The converse is clear by Theorem 3.1 and Lemma 3.12. □

By using Lemma 3.13 with the same proof as given in the above theorem, we obtain the following result.

Corollary 3.15. *Let $\alpha,\beta \in F_E$. Then $(\alpha,\beta) \in \mathscr{J}$ in $T_E(X,Y)$ if and only if there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B,C \in X/E$ with*

$$A\beta \subseteq (B\cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C\cap Y)\beta\Psi.$$

The above result leads to the following corollary which extends the result in [5].

Corollary 3.16. *Let $\alpha,\beta \in T_E(X)$. Then $(\alpha,\beta) \in \mathscr{J}$ if and only if there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B,C \in X/E$ with*

$$A\beta \subseteq B\alpha\Phi \text{ and } A\alpha \subseteq C\beta\Psi.$$

Next, we aim to characterize the $\mathscr{J}$-relation on the finite case.

Lemma 3.17. *Let $\alpha,\beta \in T_E(X,Y)$ be such that $\pi(\alpha)$ is finite. If $\theta : \pi(\alpha) \to \pi(\beta)$ and $\varphi : \pi(\beta) \to \pi(\alpha)$ are bijective $\tilde{E}$-admissible, then θ is $\tilde{E}^*$-admissible.*

Proof. The proof of this lemma follows the same steps as in Lemma 3.7 of [5]. For the sake of completeness, we give the proof as follows. Note that $\theta\varphi$ is a bijection from $\pi(\alpha)$ onto $\pi(\alpha)$ which implies that $\theta\varphi$ is a permutation on $\pi(\alpha)$. Moreover, since $\pi(\alpha)$ is finite, we obtain $(\theta\varphi)^m$ is the identity permutation for some natural number m. We claim that $(\theta\varphi)^n$ is $\tilde{E}$-admissible for all natural number n. Indeed, let $A \in X/E$. There are $B,C \in X/E$ such that $\pi_A(\alpha)\theta \subseteq \tilde{\pi}_B(\beta)$ and $\pi_B(\beta)\varphi \subseteq \tilde{\pi}_C(\alpha)$ since $\theta : \pi(\alpha) \to \pi(\beta)$ and $\varphi : \pi(\beta) \to \pi(\alpha)$ are $\tilde{E}$-admissible. We have

$$\pi_A(\alpha)\theta\varphi \subseteq \tilde{\pi}_B(\beta)\varphi \subseteq \pi_B(\beta)\varphi \subseteq \tilde{\pi}_C(\alpha).$$

Thus $\theta\varphi$ is $\tilde{E}$-admissible. By induction, we conclude that $(\theta\varphi)^n$ is $\tilde{E}$-admissible for all natural number n.

Since $(\theta\varphi)^m$ is the identity permutation on $\pi(\alpha)$, we obtain $(\theta\varphi)^m = \theta\varphi(\theta\varphi)^{m-1}$ which implies that $\theta^{-1} = \varphi(\theta\varphi)^{m-1}$. Moreover, since φ and $(\theta\varphi)^{m-1}$ are $\tilde{E}$-admissible, by the same argument as above, we can show that θ^{-1} is $\tilde{E}$-admissible. Therefore, θ is $\tilde{E}^*$-admissible. □

Lemma 3.18. *Let $U,V \subseteq X$ be finite. If $\alpha : U \to V$ and $\beta : V \to U$ are surjective E-preserving, then α and β are bijective E^*-preserving.*

Proof. It is clear that α and β are bijective since U and V are finite sets. Let $U/E_U = \{A_1,A_2,\dots,A_m\}$ and $V/E_V = \{B_1,B_2,\dots,B_n\}$ be the sets of all distinct equivalence classes of U and V, respectively. For each class B_i, we see that $B_i\alpha^{-1}$ is non-empty since α is surjective. Note that $B_i\alpha^{-1}$ is a union of some classes in U/E_U. Let k_i be the number of classes which are contained in $B_i\alpha^{-1}$. It is not difficult to see that $k_i \geq 1$ for all i and $k_1+k_2+\cdots+k_n = m$. Hence $m \geq n$. Similarly, we can show that $n \geq m$ by using β and so $m = n$. Moreover, we have $k_1+k_2+\cdots+k_n = n$ which implies that $k_1 = k_2 = \cdots = k_n = 1$ and hence $U/E_U = \{B_1\alpha^{-1},B_2\alpha^{-1},\dots,B_n\alpha^{-1}\}$. Now, we show that α is E^*-preserving. Let $(x\alpha,y\alpha) \in E$. Then $x\alpha,y\alpha \in B$ for some class $B \in V/E_V$ from which it follows that $(x\alpha)\alpha^{-1},(y\alpha)\alpha^{-1} \subseteq B\alpha^{-1}$. Thus $x,y \in B\alpha^{-1} \in U/E_U$ and so $(x,y) \in E$. Similarly, we can show that β is also E^*-preserving. □

From the above lemma, we obtain the following corollary.

Corollary 3.19. *If Z is a finite set and $\alpha : Z \to Z$ is bijective E-preserving, then α is E^*-preserving.*

Now, we have the following result which covers Theorem 3.8 of [5].

Theorem 3.20. *If $\alpha, \beta \in F_E$ such that both $X\alpha$ and $X\beta$ are finite, then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{D}$ *in* $T_E(X,Y)$.

(2) $(\alpha, \beta) \in \mathscr{J}$ *in* $T_E(X,Y)$.

(3) There are E^*-preserving bijections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with

$$A\beta \subseteq (B\cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C\cap Y)\beta\Psi.$$

Proof. The implication (1)⇒(2) is trivial.

(2)⇒(3). Assume that $(\alpha, \beta) \in \mathscr{J}$ in $T_E(X,Y)$. Then, by Corollary 3.15, there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with

$$A\beta \subseteq (B\cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C\cap Y)\beta\Psi.$$

By Lemma 3.18, we have Φ and Ψ are bijective E^*-preserving since $X\alpha$ and $X\beta$ are finite.

(3)⇒(1). Define $\theta : \pi(\alpha) \to \pi(\beta)$ by $(x\alpha)\alpha^{-1}\theta = (x\alpha)\Psi^{-1}\beta^{-1}$ for all $x \in X$. Then θ is a well-defined bijection such that $\theta\beta_* = \alpha_*\Psi^{-1}$. We only need to check that θ is $\tilde{E}^*$-admissible. Let $A \in X/E$. Then there is $C \in X/E$ such that $A\alpha \subseteq (C\cap Y)\beta\Psi$ by assumption. Let $P = (x\alpha)\alpha^{-1} \in \pi_A(\alpha)$. Then $P \cap A \neq \emptyset$ which implies that there exists $a \in P\cap A$. We obtain $x\alpha = a\alpha \in (P\cap A)\alpha \subseteq A\alpha \subseteq (C\cap Y)\beta\Psi$ from which it follows that $x\alpha = c\beta\Psi$ for some $c \in C\cap Y$. Hence

$$c \in (x\alpha)\Psi^{-1}\beta^{-1} \cap C \cap Y = (x\alpha)\alpha^{-1}\theta \cap C\cap Y = P\theta \cap C \cap Y \neq \emptyset.$$

Thus θ is bijective and $\tilde{E}$-admissible. On the other hand, define $\tau : \pi(\beta) \to \pi(\alpha)$ by $(x\beta)\beta^{-1}\tau = (x\beta)\Phi^{-1}\alpha^{-1}$ for all $x \in X$. Similarly, we can show that τ is also bijective and $\tilde{E}$-admissible. Hence θ is $\tilde{E}^*$-admissible by Lemma 3.17. Therefore, $(\alpha, \beta) \in \mathscr{D}$ by Theorem 3.9. □

Finally, we determine Green's relations on the regular elements in $T_E(X,Y)$.

Theorem 3.21. *Let α, β be regular elements in $T_E(X,Y)$. If there exists a bijective E^*-preserving $\psi : X\alpha \to X\beta$, then there is a bijective $\tilde{E}^*$-admissible $\phi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_*\psi = \phi\beta_*$.*

Proof. The proof of this theorem is a slight modification of the proof of Theorem 3.12 of [5]. Actually, we define $\phi : \pi(\alpha) \to \pi(\beta)$ by

$$P\phi = (P\alpha_*\psi)\beta_*^{-1} \text{ for all } P \in \pi(\alpha).$$

Since α_*, ψ and β_* are bijective, we obtain ϕ is also bijective. Moreover, we can see that $\alpha_*\psi = \phi\beta_*$. Now, we show that ϕ is $\tilde{E}^*$-admissible. Let $A \in X/E$ and $A\alpha = B'$.

Then $A\alpha = B' \subseteq B$ for some $B \in X/E$ by Lemma 1.1. We see that $B'\psi = C' \subseteq C$ for some $C \in X/E$ since ψ is E^*-preserving. Moreover, since β is regular, we obtain $C' \subseteq C \cap X\beta \subseteq (D\cap Y)\beta$ for some $D \in X/E$ by Theorem 2.1. We claim that $\pi_A(\alpha)\phi \subseteq \tilde{\pi}_D(\beta)$. Indeed, let $P \in \pi_A(\alpha)$. Then $P\alpha_* \in A\alpha = B'$. Hence $P\alpha_*\psi \in C' \subseteq (D\cap Y)\beta$ since $B'\psi = C'$ and $C' \subseteq (D\cap Y)\beta$. Hence $P\phi \cap D \cap Y = P\alpha_*\psi\beta^{-1} \cap D \cap Y \neq \emptyset$ which implies that $P\phi \in \tilde{\pi}_D(\beta)$. We conclude that ϕ is $\tilde{E}$-admissible. Similarly, we have ϕ^{-1} is also $\tilde{E}$-admissible since ϕ is bijective and $\phi^{-1} = \beta_*\psi^{-1}\alpha_*^{-1}$. Therefore, ϕ is $\tilde{E}^*$-admissible. □

Recall that the set of all regular elements in $T_E(X,Y)$ is contained in F_E. Then we get the following theorem.

Theorem 3.22. *Let α,β be regular elements in $T_E(X,Y)$. Then the following statements hold.*

(1) $(\alpha,\beta) \in \mathscr{R}$ if and only if $\pi(\alpha) = \pi(\beta)$.

(2) $(\alpha,\beta) \in \mathscr{L}$ if and only if $X\alpha = X\beta$.

(3) $(\alpha,\beta) \in \mathscr{H}$ if and only if $\pi(\alpha) = \pi(\beta)$ and $X\alpha = X\beta$.

(4) $(\alpha,\beta) \in \mathscr{D}$ if and only if there exists a bijective E^*-preserving $\psi : X\alpha \to X\beta$.

The proof of this theorem follows from Theorems 2.5 and 3.21 so it is omitted. Moreover, if $X = Y$, then Theorem 3.13 of [5] is true.

4 Partial Order

Now, we give some notations which will be used throughout this paper. For each $\alpha \in T_E(X,Y)$ and $A \subseteq X$, the restriction of α to A is denoted by $\alpha|_A$. We adopt the notation introduced in [1] namely, if $\alpha \in T_E(X,Y)$, then we write

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some unmentioned index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

Let U,V be subsets of X and $\alpha : U \to V$. If $(u,u') \in E$ implies $(u\alpha, u'\alpha) \in E$ for each $u,u' \in U$, then α is said to be *E-preserving*.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two collections of subsets of X. If for each $A \in \mathfrak{A}$, there exists $B \in \mathfrak{B}$ such that $A \subseteq B$, then $\mathfrak{A}$ is said to *refine* $\mathfrak{B}$.

Recall that the natural partial order on any semigroup S is defined by

$$a \leq b \text{ if and only if } a = xb = by,\ xa = a \text{ for some } x,y \in S^1,$$

or equivalently,

$$a \leq b \text{ if and only if } a = wb = bz, az = a \text{ for some } w,z \in S^1. \tag{1}$$

In this paper, we use (1) to define the partial order on the semigroup $T_E(X,Y)$, that is for each $\alpha,\beta \in T_E(X,Y)$

$$\alpha \leq \beta \text{ if and only if } \alpha = \gamma\beta = \beta\mu,\ \alpha = \alpha\mu \text{ for some } \gamma,\mu \in T_E(X,Y)^1.$$

We note that if $Y \subsetneq X$, then $T_E(X,Y)$ has no identity elements. Hence, in this case, $T_E(X,Y)^1 \neq T_E(X,Y)$. In addition, $\leq$ on $T_E(X,Y)$ does not coincide with the restriction of $\leq$ on $T_E(X)$. For example, let $X = \{1,2,3\}$, $Y = \{1,2\}$ and $X/E = \{\{1,2\},\{3\}\}$. Define

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}.$$

If we let

$$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \text{ and } \mu = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix},$$

then $\alpha = \gamma\beta = \beta\mu$, $\alpha = \alpha\mu$ which implies that $\alpha \leq \beta$ in $T_E(X)$ but we cannot find $\gamma' \in T_E(X,Y)^1$ such that $\alpha = \gamma'\beta$. Hence $\alpha \nleq \beta$ in $T_E(X,Y)$.

Now, we start this section with the characterization of $\leq$ on $T_E(X,Y)$ which extends Theorem 2.1 of [9].

Theorem 4.1. *Let $\alpha, \beta \in T_E(X,Y)$. Then $\alpha \leq \beta$ if and only if $\alpha = \beta$ or the following statements hold.*

(1) $E(\beta)$ refines $E(\alpha)$ and $\pi(\beta)$ refines $\pi(\alpha)$.

(2) For each $x \in X$, if $x\beta \in X\alpha$, then $x\alpha = x\beta$.

(3) For each $A \in X/E$, there is $B \in X/E$ such that $A\alpha \subseteq (B \cap Y)\beta$.

Proof. Assume that $\alpha \leq \beta$. Then there exist $\gamma, \mu \in T_E(X,Y)^1$ such that $\alpha = \gamma\beta = \beta\mu$ and $\alpha = \alpha\mu$. We can see that if $\gamma = 1$ or $\mu = 1$, then $\alpha = \beta$. Now, suppose that $\gamma, \mu \in T_E(X,Y)$. Then (1) and (2) follow by Theorem 2.1 of [9]. To see (3), let $A \in X/E$. Then there is $B \in X/E$ such that $A\gamma \subseteq B \cap Y$ which implies that $A\alpha = A\gamma\beta \subseteq (B \cap Y)\beta$, as required.

The proof of the converse is a slight modification of the proof of Theorem 2.1 of [9]. Actually, suppose that (1), (2) and (3) hold. We note that, by (3), $X\alpha \subseteq X\beta$. For each $A \in X/E$, define a function μ as follows.

If $A \cap X\beta = \emptyset$, fix $y_0 \in Y$ and define $z\mu = y_0$ for all $z \in A$.

If $A \cap X\beta \neq \emptyset$, we first define μ on $A \cap X\beta$. For each $z \in A \cap X\beta$, there is $x \in X$ such that $z = x\beta$, so define $z\mu = x\alpha$. To define μ on $A \setminus X\beta$, fix $z_0 \in A \cap X\beta$ and define $z\mu = z_0\mu$ for all $z \in A \setminus X\beta$.

To see that μ is well-defined on A. Suppose that there is $x' \in X$ such that $z = x'\beta = x\beta$. Then $x, x' \in z\beta^{-1}$. There exists $y \in X\alpha$ such that $z\beta^{-1} \subseteq y\alpha^{-1}$ since $\pi(\beta)$ refines $\pi(\alpha)$. Hence $x\alpha = x'\alpha = y$. Thus μ is well-defined on A and then it is also well-defined on X. Next, we show that μ is E-preserving. Let $x, y \in A$. If $A \cap X\beta = \emptyset$, then $(x\mu, y\mu) = (y_0, y_0) \in E$. For $A \cap X\beta \neq \emptyset$, there are three cases to consider.

If $x, y \in A \cap X\beta$, then $x = x'\beta$ and $y = y'\beta$ for some $x', y' \in X$ from which it follows that there exists $U \in E(\beta)$ such that $x', y' \in U$. Moreover, since $E(\beta)$ refines $E(\alpha)$, there is $V \in E(\alpha)$ such that $U \subseteq V$. Then $x', y' \in V = B\alpha^{-1}$ for some $B \in X/E$. Hence $x'\alpha, y'\alpha \in B$ implies $(x\mu, y\mu) = (x'\alpha, y'\alpha) \in E$.

If $x \in A \cap X\beta$ and $y \in A \setminus X\beta$, then $(x\mu, y\mu) = (x\mu, z_0\mu)$ where $z_0 \in A \cap X\beta$ is fixed. By the above case, we conclude that $(x\mu, y\mu) = (x\mu, z_0\mu) \in E$.

If $x, y \in A \setminus X\beta$, then $(x\mu, y\mu) = (z_0\mu, z_0\mu) \in E$.

Thus $\mu \in T_E(X,Y)$. It is not hard to see that $x\beta\mu = x\alpha$ for each $x \in X$ by the definition of μ. To show that $\alpha = \alpha\mu$, let $x \in X$. There is $y \in X$ such that $x\alpha = y\beta$ since $X\alpha \subseteq X\beta$. By (2), we have $y\alpha = y\beta$ which implies that

$$x\alpha\mu = y\beta\mu = y\alpha = y\beta = x\alpha.$$

Finally, we define a function $\gamma : X \to Y$ as follows. Let $A \in X/E$. Then there is $B \in X/E$ such that $A\alpha \subseteq (B \cap Y)\beta$ by (3). We obtain for each $z \in A$, there is $x \in B \cap Y$ such that $z\alpha = x\beta$ and define $z\gamma = x$. To show that γ is E-preserving, let $(x,y) \in E$. Then $x,y \in A$ for some E-class A. By the definition of γ, we obtain $x\gamma, y\gamma \in B$ for some $B \in X/E$ and so $(x\gamma, y\gamma) \in E$. To see that $\alpha = \gamma\beta$, let $z \in X$. Then $z\gamma = x$ for some x with $z\alpha = x\beta$. Thus $z\gamma\beta = x\beta = z\alpha$. Therefore, $\alpha \leq \beta$. □

Remark 4.2. *If $\alpha, \beta \in T_E(X,Y)$ and $\alpha \leq \beta$, then $X\alpha \subseteq X\beta$.*

By the same proof as given in Corollary 2.2 of [9], we have the following result immediately.

Corollary 4.3. *Let $\alpha, \beta \in T_E(X,Y)$ be such that $\alpha \leq \beta$. Then the following statements hold.*

(1) If $X\alpha = X\beta$, then $\alpha = \beta$.

(2) For each $P \in \pi(\alpha)$, there exists $P' \in \pi(\beta)$ such that $P' \subseteq P$ and $P\alpha = P'\beta$.

(3) If $\pi(\alpha) = \pi(\beta)$, then $\alpha = \beta$.

(4) For each $U = A\alpha^{-1} \in E(\alpha)$ such that $A \in X/E$, there is $V \in E(\beta)$ such that $V \subseteq U$ and $U\alpha = V\alpha \subseteq V\beta = A \cap X\beta$.

5 Compatibility

Recall that an element $\gamma \in T_E(X,Y)$ is said to be *left compatible* with $\leq$ if $\gamma\alpha \leq \gamma\beta$ for all $\alpha, \beta \in T_E(X,Y)$ such that $\alpha \leq \beta$. *Right compatibility* with $\leq$ is defined dually. In this part, we will find out elements of $T_E(X,Y)$ which are compatible with $\leq$ on $T_E(X,Y)$. We remark that if $|Y| = 1$, then $|T_E(X,Y)| = 1$ which implies that an element in $T_E(X,Y)$ is left and right compatible. From now on, we assume that $|Y| > 1$.

Lemma 5.1. *Let $\gamma \in T_E(X,Y)$. If Y/E_Y is finite and $Y\gamma = Y$, then γ is regular.*

Proof. Assume that Y/E_Y is finite and $Y\gamma = Y$. Write $Y/E_Y = \{A_1, A_2, \ldots, A_n\}$. We can see that for each $A_i \in Y/E_Y$, there exists unique $A_j \in Y/E_Y$ such that $A_i = A_j\gamma$. To show that γ is regular, let $A \in X/E$. If $A \cap Y = \emptyset$, then $A \cap X\gamma = \emptyset$, so we are done. If $A \cap Y \neq \emptyset$, then $A \cap Y \in Y/E_Y$, so $A \cap Y = (B \cap Y)\gamma$ for some $B \in X/E$. Thus $A \cap X\gamma \subseteq A \cap Y = (B \cap Y)\gamma$. □

For each $x \in X$, denote the equivalence E-class containing x by $[x]$.

Lemma 5.2. *If $\gamma \in T_E(X,Y)$ is left compatible with respect to $\leq$, then $Y\gamma = Y$.*

Proof. We prove by contrapositive. Assume that $Y\gamma \neq Y$. Then there is $y \in Y \setminus Y\gamma$.

Case 1. If $|[y] \cap Y| = 1$, there is $z \in Y$ such that $z \notin [y]$ since $|Y| > 1$. We define $\alpha, \beta \in T_E(X,Y)$ by $x\alpha = y$ for all $x \in X$ and

$$x\beta = \begin{cases} y & \text{if } x \in [y] \\ z & \text{otherwise.} \end{cases}$$

We have $E(\alpha) = \{X\} = \pi(\alpha)$ which implies that $E(\beta)$ refines $E(\alpha)$ and $\pi(\beta)$ refines $\pi(\alpha)$. Let $x\beta \in X\alpha$. Then $x\beta = y = x\alpha$. It is clear that $A\alpha = \{y\}$ for all $A \in X/E$ and hence $A\alpha \subseteq ([y] \cap Y)\beta$. Thus $\alpha \leq \beta$ by Theorem 4.1. We see that $A\gamma\alpha = \{y\} \nsubseteq$

$\{z\} = (B \cap Y)\gamma\beta$ for all classes A and B which implies that $\gamma\alpha \not\leq \gamma\beta$. Hence γ is not left compatible with $\leq$ on $T_E(X,Y)$.

Case 2. If $|[y] \cap Y| > 1$, there is $z \in [y] \cap Y$ such that $z \neq y$. We define $\alpha, \beta \in T_E(X,Y)$ by $x\alpha = y$ for all $x \in X$ and

$$x\beta = \begin{cases} y & \text{if } x = y \\ z & \text{otherwise.} \end{cases}$$

By the same argument as above, we obtain $\alpha \leq \beta$ but $\gamma\alpha \not\leq \gamma\beta$. Thus γ is not left compatible with $\leq$ on $T_E(X,Y)$. □

Now, we give a notation which will be used in this part. Let $A \in X/E$ and $\gamma \in T_E(X,Y)$. We define a subclass $\mathfrak{C}_\gamma(A)$ of $\mathscr{P}(A \cap X\gamma)$, the power set of $A \cap X\gamma$, by

$$\mathfrak{C}_\gamma(A) = \{C \subseteq A \cap X\gamma : C \nsubseteq (G \cap Y)\gamma \text{ for all } G \in X/E\}.$$

Clearly, $\emptyset \notin \mathfrak{C}_\gamma(A)$. Moreover, if $\mathfrak{C}_\gamma(A) \neq \emptyset$, then $A \cap X\gamma \in \mathfrak{C}_\gamma(A)$. We have known that γ is regular if and only if for each $A \in X/E$, there is $G \in X/E$ such that $A \cap X\gamma \subseteq (G \cap Y)\gamma$. Therefore, in this case, $\mathfrak{C}_\gamma(A)$ is empty for all class A. On the other hand, γ is not regular if and only if there is a class A such that $\mathfrak{C}_\gamma(A) \neq \emptyset$.

Theorem 5.3. *$\gamma \in T_E(X,Y)$ is left compatible with respect to $\leq$ if and only if the following statements hold.*

(1) $Y\gamma = Y$.

(2) For each class $A, B \in X/E$, $|B\gamma \setminus C| < |C \setminus B\gamma|$ for all $C \in \mathfrak{C}_\gamma(A)$.

Proof. ($\Rightarrow$) Suppose that γ is left compatible. Then $Y\gamma = Y$ by Lemma 5.2. Hence $X\gamma = Y = Y\gamma$. To show (2), we assume to the contrary that there are classes $A, B \in X/E$ and a set $C \in \mathfrak{C}_\gamma(A)$ such that $|B\gamma \setminus C| \geq |C \setminus B\gamma|$. Hence γ is not regular which implies that Y/E_Y is infinite by Lemma 5.1. We note that

$$\mathfrak{C}_\gamma(A) = \{C \subseteq A \cap Y : C \nsubseteq (G \cap Y)\gamma \text{ for all } G \in X/E\}.$$

Consider the following cases.

Case 1: $B\gamma \subseteq C$. Then $0 = |B\gamma \setminus C| \geq |C \setminus B\gamma|$ which implies that $C = B\gamma$. Write $C = \{c_i : i \in I\}$.

Subcase 1.1: $C = A \cap Y$. Choose $c \in C$ and $d \in D \cap Y$ for some class $D \neq A$ (since Y/E_Y is infinite). We define functions α and β by

$$\alpha = \begin{pmatrix} c_i & X \setminus C \\ c_i & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i & A \setminus C & X \setminus A \\ c_i & c & d \end{pmatrix}.$$

Subcase 1.2: $C \subsetneq A \cap Y$. Choose $a \in (A \cap Y) \setminus C$ and $d \in D \cap Y$ for some class $D \neq A$. We define functions α and β by

$$\alpha = \begin{pmatrix} c_i & X \setminus C \\ c_i & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i & A \setminus C & X \setminus A \\ c_i & a & d \end{pmatrix}.$$

Case 2: $C \subseteq B\gamma$. Then $|B\gamma \setminus C| \geq |C \setminus B\gamma| = 0$. If $|B\gamma \setminus C| = 0$, then it reduces to case 1. We assume that $C \subsetneq B\gamma$. Choose $b \in B\gamma \setminus C$ and $c \in C$. Write $C = \{c_i : i \in I\}$ and $B\gamma \setminus C = \{b_j : j \in J\}$. We define functions α and β by

$$\alpha = \begin{pmatrix} c_i & b_j & X \setminus B\gamma \\ c_i & c & b \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i & b_j & X \setminus B\gamma \\ c_i & b_j & b \end{pmatrix}.$$

Case 3: $B\gamma \nsubseteq C$ and $C \nsubseteq B\gamma$.

Subcase 3.1: $B\gamma \cap A = \emptyset$ and $C = A \cap Y$. Then $C \cap B\gamma = \emptyset$ which implies that $|B\gamma| = |B\gamma \setminus C| \geq |C \setminus B\gamma| = |C|$. Choose $c \in C$. Write $C = \{c_i : i \in I\}$ and $B\gamma = \{b_j : j \in J\}$. Since $|B\gamma| \geq |C|$, there is a surjection $\varphi : B\gamma \to C$. Hence we can find an element $b \in B\gamma$ such that $c = b\varphi$. Define functions α and β by

$$\alpha = \begin{pmatrix} c_i & b_j & X \setminus (C \cup B\gamma) \\ c_i & b_j\varphi & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i & b_j & A \setminus C & X \setminus (A \cup B\gamma) \\ c_i & b_j & c & b \end{pmatrix}.$$

Subcase 3.2: $B\gamma \cap A = \emptyset$ and $C \subsetneq A \cap Y$. Then $C \cap B\gamma = \emptyset$ which implies that $|B\gamma| = |B\gamma \setminus C| \geq |C \setminus B\gamma| = |C|$. Choose $a \in (A \cap Y) \setminus C$ and $c \in C$. Write $C = \{c_i : i \in I\}$ and $B\gamma = \{b_j : j \in J\}$. Since $|B\gamma| \geq |C|$, there is a surjection $\varphi : B\gamma \to C$. Hence we can find an element $b \in B\gamma$ such that $c = b\varphi$. Define functions α and β by

$$\alpha = \begin{pmatrix} c_i & b_j & X \setminus (C \cup B\gamma) \\ c_i & b_j\varphi & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i & b_j & A \setminus C & X \setminus (A \cup B\gamma) \\ c_i & b_j & a & b \end{pmatrix}.$$

Subcase 3.3: $B\gamma \subseteq A \cap Y$. We see that $C \cup B\gamma \subseteq A \cap Y$. Choose $c \in C \setminus B\gamma$. Write $C = \{c_i : i \in I\}$ and $B\gamma \setminus C = \{b_j : j \in J\}$. Since $|B\gamma \setminus C| \geq |C \setminus B\gamma|$, there is a surjection $\varphi : B\gamma \setminus C \to C \setminus B\gamma$. Hence we can find an element $b \in B\gamma \setminus C$ such that $c = b\varphi$. Define functions α and β by

$$\alpha = \begin{pmatrix} c_i & b_j & X \setminus (C \cup B\gamma) \\ c_i & b_j\varphi & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i & b_j & X \setminus (C \cup B\gamma) \\ c_i & b_j & b \end{pmatrix}.$$

For all cases, we see that $\alpha \leq \beta$. Then $C = B\gamma\alpha \subseteq (G \cap Y)\gamma\beta$ for some class G since γ is left compatible. We see that $C \subseteq (G \cap Y)\gamma$ which contradicts to $C \in \mathfrak{C}_\gamma(A)$. We conclude that for each class $A, B \in X/E$, $|B\gamma \setminus C| < |C \setminus B\gamma|$ for all $C \in \mathfrak{C}_\gamma(A)$.

($\Leftarrow$) Let $\alpha, \beta \in T_E(X,Y)$ be such that $\alpha \leq \beta$ with $\alpha \neq \beta$ and suppose that $\gamma \in T_E(X,Y)$ satisfies (1) and (2). We aim to show that $\gamma\alpha \leq \gamma\beta$ by using Theorem 4.1.

(1) To show that $E(\gamma\beta)$ refines $E(\gamma\alpha)$, let $U = A(\gamma\beta)^{-1} \in E(\gamma\beta)$ for some $A \in X/E$. We have $U\gamma\beta = A \cap X\gamma\beta \subseteq A$ and hence $U\gamma \subseteq A\beta^{-1} \in E(\beta)$. Moreover, since $\alpha \leq \beta$, we have $E(\beta)$ refines $E(\alpha)$ which implies that $A\beta^{-1} \subseteq B\alpha^{-1}$ for some $B \in X/E$ and so

$$U\gamma\alpha \subseteq (A\beta^{-1})\alpha \subseteq (B\alpha^{-1})\alpha \subseteq B.$$

Thus $U \subseteq B(\gamma\alpha)^{-1} \in E(\gamma\alpha)$. Therefore, $E(\gamma\beta)$ refines $E(\gamma\alpha)$. Similarly, we can show that $\pi(\gamma\beta)$ refines $\pi(\gamma\alpha)$.

(2) Let $x \in X$ and $x\gamma\beta \in X\gamma\alpha \subseteq X\alpha$. Since $\alpha \leq \beta$ and $(x\gamma)\beta \in X\alpha$, we have $x\gamma\beta = x\gamma\alpha$.

(3) Let $A \in X/E$. Then $A\gamma\alpha \subseteq (B \cap Y)\beta$ for some class B since $\alpha \leq \beta$. By the axiom of choice, we can find a subset C of $B \cap Y$ such that $\beta|_C : C \to A\gamma\alpha$ is a bijection. Hence $C\beta = A\gamma\alpha$. Note that $C \subseteq B \cap Y = B \cap Y\gamma \subseteq B \cap X\gamma$. We claim that $(C \setminus A\gamma)\beta \subseteq (A\gamma \setminus C)\alpha$. Indeed, let $x\beta \in (C \setminus A\gamma)\beta$ where $x \in C \setminus A\gamma$. Then $x\beta \in C\beta = A\gamma\alpha = (A\gamma \setminus C)\alpha \cup (A\gamma \cap C)\alpha$. If $x\beta \in (A\gamma \cap C)\alpha$, then $x\beta = y\alpha$ for some $y \in A\gamma \cap C$. Clearly, $x \neq y$. We see that $y\beta \in (A\gamma \cap C)\beta \subseteq C\beta = A\gamma\alpha \subseteq X\alpha$ which implies that $y\alpha = y\beta$ since $\alpha \leq \beta$. Hence $x\beta = y\beta$ which contradicts to the injectivity of $\beta|_C$. So $x\beta \in (A\gamma \setminus C)\alpha$. Thus

$$|A\gamma \setminus C| \geq |(A\gamma \setminus C)\alpha| \geq |(C \setminus A\gamma)\beta| = |C \setminus A\gamma|.$$

which implies that $C \notin \mathfrak{C}_\gamma(B)$ by assumption (2). Hence $C \subseteq (D \cap Y)\gamma$ for some class D and thus $A\gamma\alpha = C\beta \subseteq (D \cap Y)\gamma\beta$. $\square$

Consider the case $X = Y$. Let $A \in X/E$ and $\gamma \in T_E(X)$. We see that

$$\mathfrak{C}_\gamma(A) = \{C \subseteq A \cap X\gamma : C \nsubseteq G\gamma \text{ for all } G \in X/E\}.$$

By the above theorem, we obtain the following corollary which extends the result of [9].

Corollary 5.4. *$\gamma \in T_E(X)$ is left compatible with respect to $\leq$ if and only if the following statements hold.*

(1) γ is surjective.

(2) For each class $A, B \in X/E$, $|B\gamma \setminus C| < |C \setminus B\gamma|$ for all $C \in \mathfrak{C}_\gamma(A)$.

Now, we consider right compatibility.

Theorem 5.5. *Let $|Y/E_Y| = 2$, say $Y/E_Y = \{A, B\}$. Then $\gamma \in T_E(X,Y)$ is right compatible with $\leq$ if and only if $\gamma|_Y$ is injective or $|A\gamma| = 1 = |B\gamma|$.*

Proof. ($\Rightarrow$). We prove by contrapositive. Assume that $\gamma|_Y$ is not injective and $|A\gamma| \neq 1$. Then there are $b, c \in Y$ such that $b\gamma = y = c\gamma$ for some $y \in Y$. Consider the following cases.

Case 1: $b, c \in A$. Then there is $a \in A$ such that $a\gamma = x \neq y$ for some $x \in Y$ since $|A\gamma| > 1$. Define $\alpha, \beta \in T_E(X,Y)$ by

$$\alpha = \begin{pmatrix} [a] & X \setminus [a] \\ a & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} [a] \setminus \{b\} & b & X \setminus [a] \\ a & b & c \end{pmatrix}.$$

We see that $\alpha \leq \beta$ and $b\beta\gamma = y = z\alpha\gamma$ for some $z \in X \setminus [a]$. Hence $b\beta\gamma \in X\alpha\gamma$ but $b\alpha\gamma = x \neq y = b\beta\gamma$. Therefore $\alpha\gamma \nleq \beta\gamma$ which implies that γ is not right compatible with $\leq$ on $T_E(X,Y)$.

Case 2: $b \in B$ and $c \in A$. Then there is $a \in A$ such that $a\gamma = x \neq y$ for some $x \in Y$ since $|A\gamma| > 1$. Define $\alpha, \beta \in T_E(X,Y)$ by

$$\alpha = \begin{pmatrix} [a] & X \setminus [a] \\ a & b \end{pmatrix} \text{ and } \beta = \begin{pmatrix} [a] \setminus \{c\} & c & X \setminus [a] \\ a & c & b \end{pmatrix}.$$

We see that $\alpha \leq \beta$ and $c\beta\gamma = y = b\alpha\gamma$. Hence $c\beta\gamma \in X\alpha\gamma$ but $c\alpha\gamma = x \neq y = c\beta\gamma$. Therefore $\alpha\gamma \nleq \beta\gamma$ which implies that γ is not right compatible with $\leq$ on $T_E(X,Y)$.

Case 3: $b \in A$ and $c \in B$. The proof of this case is similar to Case 2.

Case 4: $b, c \in B$. Then there is $p, q \in A$ such that $p\gamma = m \neq n = q\gamma$ for some $m, n \in Y$ since $|A\gamma| > 1$. Define $\alpha, \beta \in T_E(X,Y)$ by

$$\alpha = \begin{pmatrix} X \setminus \{c, q\} & \{c, q\} \\ p & q \end{pmatrix} \text{ and } \beta = \begin{pmatrix} X \setminus ([b] \cup \{q\}) & q & [b] \setminus \{c\} & c \\ p & q & b & c \end{pmatrix}.$$

We see that $\alpha \leq \beta$. Moreover, we obtain $\{b, c\} \subseteq y(\beta\gamma)^{-1}$ but $b \in m(\alpha\gamma)^{-1}$ and $c \in n(\alpha\gamma)^{-1}$. Thus $\pi(\beta\gamma)$ does not refine $\pi(\alpha\gamma)$. Therefore, $\alpha\gamma \nleq \beta\gamma$ which implies that γ is not right compatible with $\leq$ on $T_E(X,Y)$.

($\Leftarrow$). Suppose that $\gamma|_Y$ is injective or $|A\gamma| = 1 = |B\gamma|$. Let $\alpha, \beta \in T_E(X,Y)$ be such that $\alpha \leq \beta$ with $\alpha \neq \beta$. Clearly, if $|Y\gamma| = 1$, then $\alpha\gamma = \beta\gamma$. Now, we assume that $|Y\gamma| \geq 2$. We aim to show that $\alpha\gamma \leq \beta\gamma$ by using Theorem 4.1.

(1) Let $U \in E(\beta\gamma)$. Then $U = C(\beta\gamma)^{-1}$ for some class C which implies that $U\beta\gamma \subseteq C$. Hence $U\beta \subseteq C\gamma^{-1} \cap Y \neq \emptyset$. We conclude that $C\gamma^{-1} \cap Y \in Y/E_Y$ or $C\gamma^{-1} \cap Y = Y$

since $|Y/E_Y| = 2$. If $C\gamma^{-1} \cap Y = Y$, then $C(\beta\gamma)^{-1} = X = C(\alpha\gamma)^{-1}$ from which it follows that $E(\alpha\gamma) = \{X\} = E(\beta\gamma)$. If $C\gamma^{-1} \cap Y \in Y/E_Y$, then there exists $D \in X/E$ such that $U\beta \subseteq C\gamma^{-1} \cap Y = D \cap Y$ and so $U \subseteq D\beta^{-1} \in E(\beta)$. Moreover, since $E(\beta)$ refines $E(\alpha)$, there is $V \in E(\alpha)$ such that $U \subseteq D\beta^{-1} \subseteq V$ which implies that $V = G\alpha^{-1}$ for some class G. Thus there is a class H such that $V\alpha\gamma \subseteq G\gamma \subseteq H$. We have $U \subseteq V \subseteq H(\alpha\gamma)^{-1} \in E(\alpha\gamma)$ and so $E(\beta\gamma)$ refines $E(\alpha\gamma)$.

Next, we prove that $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$. We consider two cases as follows.

Case 1: $\gamma|_Y$ is injective. Let $U = y(\beta\gamma)^{-1} \in \pi(\beta\gamma)$. Then $U\beta\gamma = \{y\}$ and so $U\beta \subseteq y\gamma^{-1}$. Then $U\beta = \{x\}$ for some $x \in Y$. Thus $U \subseteq x\beta^{-1} \in \pi(\beta)$ and then there is $z\alpha^{-1} \in \pi(\alpha)$ such that $U \subseteq x\beta^{-1} \subseteq z\alpha^{-1}$ since $\pi(\beta)$ refines $\pi(\alpha)$. We obtain $U\alpha = \{z\}$ which implies that $U\alpha\gamma = \{z\gamma\}$. Therefore, $U \subseteq (z\gamma)(\alpha\gamma)^{-1} \in \pi(\alpha\gamma)$ and so $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$.

Case 2: $|A\gamma| = 1 = |B\gamma|$. Then $A\gamma \neq B\gamma$ since $|Y\gamma| \geq 2$. Note that $A = A' \cap Y$ and $B = B' \cap Y$ for some distinct classes A' and B' in X/E. We claim that $\pi(\alpha\gamma) = E(\alpha)$. Let $U = y(\alpha\gamma)^{-1} \in \pi(\alpha\gamma)$. Then $U\alpha\gamma = \{y\}$. We see that either $U\alpha \subseteq A$ or $U\alpha \subseteq B$. Assume that $U\alpha \subseteq A$. Then $U \subseteq A\alpha^{-1} \subseteq A'\alpha^{-1}$. We obtain $y \in U\alpha\gamma \subseteq A\gamma$ from which it follows that $A\gamma = \{y\}$ since $|A\gamma| = 1$. Let $x \in A'\alpha^{-1}$. Then $x\alpha \in A' \cap Y = A$ which implies that $x\alpha\gamma \in A\gamma = \{y\}$. Hence $x\alpha\gamma = y$ and so $x \in y(\alpha\gamma)^{-1} = U$. We conclude that $U = A'\alpha^{-1} \in E(\alpha)$. Similarly, if $U\alpha \subseteq B$, then $U = B'\alpha^{-1} \in E(\alpha)$. Thus $\pi(\alpha\gamma) \subseteq E(\alpha)$. On the other hand, let $V = C\alpha^{-1} \in E(\alpha)$. Then $V\alpha \subseteq C \cap Y$ which implies that either $V\alpha \subseteq C \cap Y = A$ or $V\alpha \subseteq C \cap Y = B$. Thus $V\alpha\gamma \subseteq A\gamma$ or $V\alpha\gamma \subseteq B\gamma$. If $V\alpha\gamma \subseteq A\gamma$, then $V\alpha\gamma = \{z\} = A\gamma$ for some $z \in Y$ since $|A\gamma| = 1$. Hence $V \subseteq z(\alpha\gamma)^{-1}$. Let $a \in z(\alpha\gamma)^{-1}$. Then $a\alpha\gamma = z$ and $a\alpha \in Y = A \cup B$. Clearly, $a\alpha \in A = C \cap Y$ since $A\gamma \neq B\gamma$. Hence $a \in C\alpha^{-1} = V$ and so $z(\alpha\gamma)^{-1} \subseteq V$. We conclude that $V = z(\alpha\gamma)^{-1} \in \pi(\alpha\gamma)$. Similarly, if $V\alpha\gamma \subseteq B\gamma$, then $V = w(\alpha\gamma)^{-1} \in \pi(\alpha\gamma)$ where $\{w\} = B\gamma$. Therefore, $\pi(\alpha\gamma) = E(\alpha)$. By the same proof, we obtain $\pi(\beta\gamma) = E(\beta)$. Thus $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$ since $E(\beta)$ refines $E(\alpha)$.

(2) Let $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y\alpha\gamma$ for some $y \in X$. If $\gamma|_Y$ is injective, then $x\beta = y\alpha \in X\alpha$ which implies that $x\beta = x\alpha$. Thus $x\beta\gamma = x\alpha\gamma$. If $|A\gamma| = 1 = |B\gamma|$, then $A\gamma \neq B\gamma$ since $|Y\gamma| \geq 2$. Thus $x\beta, y\alpha \in A$ or $x\beta, y\alpha \in B$. Now, we assume that $x\beta, y\alpha \in A$. Suppose to the contrary that $x\alpha \in B$. We see that $x\alpha = z\beta$ and $y\alpha = s\beta$ for some $z, s \in Y$ by Theorem 4.1 (3). Hence $z\beta, s\beta \in X\alpha$ implies $z\beta = z\alpha$ and $s\beta = s\alpha$. We obtain $\{s, x\} \subseteq A\beta^{-1}$ but $s \in A\alpha^{-1}$ and $x \in B\alpha^{-1}$ from which it follows that $E(\beta)$ does not refine $E(\alpha)$. It leads to a contradiction. Thus $x\alpha \in A$ which implies that $x\alpha\gamma = x\beta\gamma$ since $|A\gamma| = 1$. Similarly, if $x\beta, y\alpha \in B$, then we can show that $x\alpha\gamma = x\beta\gamma$.

(3) Let $A \in X/E$. By Theorem 4.1 (3), there is a class B such that $A\alpha \subseteq (B \cap Y)\beta$ and so $A\alpha\gamma \subseteq (B \cap Y)\beta\gamma$. □

Lemma 5.6. *Let $|Y/E_Y| = 1$. If $\gamma \in T_E(X,Y)$ is right compatible with $\leq$, then $|Y\gamma| = 1$ or $\gamma|_Y$ is injective.*

Proof. We prove by contrapositive. Assume that $|Y\gamma| > 1$ and $\gamma|_Y$ is not injective. Then there are $b, c \in Y$ with $b \neq c$ such that $b\gamma = y = c\gamma$ for some $y \in Y$. Moreover, there is $a \in Y$ such that $a\gamma = x \neq y$ since $|Y\gamma| > 1$. Clearly, a, b, c are distinct elements in Y and $[a] = [b] = [c] \in X/E$ since $|Y/E_Y| = 1$. Define $\alpha, \beta \in T_E(X,Y)$ by

$$\alpha = \begin{pmatrix} \{a,b\} & X \setminus \{a,b\} \\ a & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} a & b & X \setminus \{a,b\} \\ a & b & c \end{pmatrix}.$$

We see that $\alpha \leq \beta$ and $b\beta\gamma = y = c\alpha\gamma$. Hence $b\beta\gamma \in X\alpha\gamma$ but $b\alpha\gamma = x \neq y = b\beta\gamma$. Therefore $\alpha\gamma \not\leq \beta\gamma$ which implies that γ is not right compatible with $\leq$ on $T_E(X,Y)$. □

Lemma 5.7. *Let $|Y/E_Y| > 2$. If $\gamma \in T_E(X,Y)$ is right compatible with $\leq$, then $|Y\gamma| = 1$ or $\gamma|_Y$ is injective.*

Proof. We prove by contrapositive. Assume that $|Y\gamma| > 1$ and $\gamma|_Y$ is not injective. Then there exist $b,c \in Y$ with $b \neq c$ such that $b\gamma = c\gamma = y$ for some $y \in Y$. From $|Y\gamma| > 1$, there exists $a \in Y$ such that $a\gamma = x \neq y$ for some $x \in Y$. Clearly, $b \neq a \neq c$. We consider the following cases.

Case 1: $[a] \neq [b]$ in X/E. We see that $[a] \cup [b] \subsetneq X$ since $|X/E| \geq |Y/E_Y| > 2$. Define $\alpha, \beta \in T_E(X,Y)$ by

$$\alpha = \begin{pmatrix} [a]\cup[b] & X\setminus([a]\cup[b]) \\ a & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} [a] & [b] & X\setminus([a]\cup[b]) \\ a & b & c \end{pmatrix}.$$

We see that $\alpha \leq \beta$ and $b\beta\gamma = y = z\alpha\gamma$ for some $z \in X\setminus([a]\cup[b])$. Hence $b\beta\gamma \in X\alpha\gamma$ but $b\alpha\gamma = x \neq y = b\beta\gamma$. Therefore $\alpha\gamma \nleq \beta\gamma$ which implies that β is not right compatible with $\leq$ on $T_E(X,Y)$.

Case 2: $[a] = [b]$ in X/E. Define $\alpha, \beta \in T_E(X,Y)$ by

$$\alpha = \begin{pmatrix} [a] & X\setminus[a] \\ a & c \end{pmatrix} \text{ and } \beta = \begin{pmatrix} [a]\setminus\{b\} & b & X\setminus[a] \\ a & b & c \end{pmatrix}.$$

We see that $\alpha \leq \beta$ and $b\beta\gamma = y = z\alpha\gamma$ for some $z \in X\setminus[a]$. Hence $b\beta\gamma \in X\alpha\gamma$ but $b\alpha\gamma = x \neq y = b\beta\gamma$. Therefore $\alpha\gamma \nleq \beta\gamma$ which implies that γ is not right compatible with $\leq$ on $T_E(X,Y)$. □

Lemma 5.8. *If $\gamma \in T_E(X,Y)$ is right compatible, then $E_Y(\gamma) = Y/E_Y$ or $|E_Y(\gamma)| = 1$.*

Proof. We prove by contrapositve. Assume that $E_Y(\gamma) \neq Y/E_Y$ and $|E_Y(\gamma)| \geq 2$. Then there is $A \in X/E$ such that $A\gamma^{-1} \cap Y \in E_Y(\gamma)$ is a union of some E_Y-classes. Hence there are distinct classes $P,Q \in X/E$ such that $P\cap Y$ and $Q \cap Y$ are non-empty and contained in $A\gamma^{-1}\cap Y$. Moreover, there is a class B such that $B\gamma^{-1}\cap Y$ is non-empty and $A \neq B$ since $|E_Y(\gamma)| \geq 2$. Thus there is a class $R \subseteq B\gamma^{-1}$ such that $R \cap Y$ is non-empty. We note that $P \neq R \neq Q$. Choose $p \in P\cap Y$, $q \in Q\cap Y$ and $r \in R \cap Y$. Define functions $\alpha, \beta \in T_E(X,Y)$ by

$$\alpha = \begin{pmatrix} P & X\setminus P \\ p & r \end{pmatrix} \text{ and } \beta = \begin{pmatrix} P & Q & X\setminus(P\cup Q) \\ p & q & r \end{pmatrix}.$$

It is easy to verify that $\alpha \leq \beta$. We see that $P\beta\gamma = \{p\}\gamma \subseteq A$ and $Q\beta\gamma = \{q\}\gamma \subseteq A$ which implies that $P\cup Q \subseteq A(\beta\gamma)^{-1}$. Furthermore, $P\alpha\gamma = \{p\}\gamma \subseteq A$ and $Q\alpha\gamma = \{r\}\gamma \subseteq B$ from which it follows that $P \subseteq A(\alpha\gamma)^{-1}$ and $Q \subseteq B(\alpha\gamma)^{-1}$. Hence $E(\beta\gamma)$ does not refine $E(\alpha\gamma)$ and thus γ is not right compatible. □

Theorem 5.9. *Let $|Y/E_Y| \neq 2$. Then $\gamma \in T_E(X,Y)$ is right compatible with respect to $\leq$ if and only if the following statements hold.*

(1) $|Y\gamma| = 1$ *or* $\gamma|_Y$ *is injective.*

(2) $E_Y(\gamma) = Y/E_Y$ *or* $|E_Y(\gamma)| = 1$.

Proof. ($\Rightarrow$) Assume that γ is right compatible. Then (2) holds by Lemma 5.8. Moreover, if $|Y/E_Y| \neq 2$, then (1) holds by Lemmas 5.6 and 5.7.

($\Leftarrow$) Assume that (1) and (2) hold. Let $\alpha, \beta \in T_E(X,Y)$ be such that $\alpha \leq \beta$ with $\alpha \neq \beta$. It is clear that if $|Y\gamma| = 1$, then $\alpha\gamma = \beta\gamma$ for each $\alpha, \beta \in T_E(X,Y)$. Thus, in this case, γ is right compatible. Now, we suppose that $\gamma|_Y$ is injective. We aim to show that $\alpha\gamma \le \beta\gamma$ by using Theorem 4.1.

(1) Let $U \in E(\beta\gamma)$. Then $U = A(\beta\gamma)^{-1}$ for some class A which implies that $U\beta\gamma \subseteq A$. Hence $U\beta \subseteq A\gamma^{-1} \cap Y \in E_Y(\gamma)$. If $|E_Y(\gamma)| = 1$, then $Y \subseteq A\gamma^{-1}$. Thus $A(\beta\gamma)^{-1} = X = A(\alpha\gamma)^{-1}$ from which it follows that $E(\alpha\gamma) = \{X\} = E(\beta\gamma)$. If $E_Y(\gamma) = Y/E_Y$, then there exists $B \in X/E$ such that $U\beta \subseteq A\gamma^{-1} \cap Y = B \cap Y$ and so $U \subseteq B\beta^{-1} \in E(\beta)$. Moreover, since $E(\beta)$ refines $E(\alpha)$, there is $V \in E(\alpha)$ such that $U \subseteq V$ which implies that $V = C\alpha^{-1}$ for some class C. Thus there is a class D such that $V\alpha\gamma \subseteq C\gamma \subseteq D$. We have $U \subseteq V \subseteq D(\alpha\gamma)^{-1} \in E(\alpha\gamma)$ and so $E(\beta\gamma)$ refines $E(\alpha\gamma)$.

Let $U = y(\beta\gamma)^{-1} \in \pi(\beta\gamma)$. Then $U\beta\gamma = \{y\}$ and so $U\beta \subseteq y\gamma^{-1}$. By the injectivity of $\gamma|_Y$, we obtain $U\beta = \{x\}$ for some $x \in Y$. Thus $U \subseteq x\beta^{-1} \in \pi(\beta)$ and then there is $z\alpha^{-1} \in \pi(f)$ such that $U \subseteq x\beta^{-1} \subseteq z\alpha^{-1}$ since $\pi(\beta)$ refines $\pi(\alpha)$. We obtain $U\alpha = \{z\}$ which implies that $U\alpha\gamma = \{z\gamma\}$. Therefore, $U \subseteq (z\gamma)(\alpha\gamma)^{-1} \in \pi(\alpha\gamma)$ and so $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$.

(2) Let $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y\alpha\gamma$ for some $y \in X$. By the injectivity of $\gamma|_Y$, we have $x\beta = y\alpha \in X\alpha$ which implies that $x\beta = x\alpha$. Thus $x\beta\gamma = x\alpha\gamma$.

(3) Let $A \in X/E$. By Theorem 4.1 (3), there is a class B such that $A\alpha \subseteq (B \cap Y)\beta$ and so $A\alpha\gamma \subseteq (B \cap Y)\beta\gamma$. □

Finally, if $X = Y$, we have the following corollaries.

Corollary 5.10. *Let $|X/E| = 2$, say $X/E = \{A, B\}$. Then $\gamma \in T_E(X)$ is right compatible with $\le$ if and only if γ is injective or $|A\gamma| = 1 = |B\gamma|$.*

Corollary 5.11. *Let $|X/E| \neq 2$. Then $\gamma \in T_E(X)$ is right compatible with respect to $\le$ if and only if the following statements hold.*

(1) γ is a constant map or an injection.

(2) $E(\gamma) = X/E$ or $|E(\gamma)| = 1$.

6 Maximal and Minimal Elements

In this section, we study maximal and minimal elements in $T_E(X,Y)$ under the natural partial order $\le$. The subsemigroup F_E of $T_E(X,Y)$ plays an important role in the characterization of maximal elements.

Lemma 6.1. *Let $\alpha \in T_E(X,Y)$. If $\alpha \notin F_E$, then α is maximal.*

Proof. Let $\beta \in T_E(X,Y)$ be such that $\alpha \le \beta$. Assume that $\alpha \neq \beta$. Then for each $A \in X/E$, $A\alpha \subseteq (B \cap Y)\beta$ for some $B \in X/E$ by Theorem 4.1 (3). For each $a\alpha \in A\alpha \subseteq (B \cap Y)\beta$, there is $b \in B \cap Y$ such that $b\beta = a\alpha \in X\alpha$. Hence, by Theorem 4.1 (2), $a\alpha = b\beta = b\alpha \in (B \cap Y)\alpha$. We obtain $A\alpha \subseteq (B \cap Y)\alpha$ which is a contradiction. Therefore, $\alpha = \beta$. □

Now, we extend the notion of saturating presented in [9].

Definition 6.2. *Let $\alpha \in T_E(X,Y)$. A set $U \in E(\alpha)$ is said to be Y-saturated if $U\alpha = A \cap Y$ for some $A \in X/E$.*

We remark that if $X = Y$, then the notions of saturating and Y-saturating are the same.

Lemma 6.3. *Let $\alpha \in F_E$ be maximal and $U \in E(\alpha)$. Then U is Y-saturated or for each E-classes $A, C \subseteq U$, if $A\alpha \subseteq (C \cap Y)\alpha$, then $A = C$.*

Proof. We prove by contradiction. Assume that $U = G\alpha^{-1}$ is not Y-saturated and there are distinct E-classes $A, C \subseteq U$ such that $A\alpha \subseteq (C \cap Y)\alpha$. Then there is $b \in G \cap Y \setminus U\alpha$. We choose an element $a \in A$ and define a function $\beta : X \to Y$ by

$$x\beta = \begin{cases} b & \text{if } \ x = a; \\ x\alpha & \text{otherwise.} \end{cases}$$

It is clear that $\beta \in T_E(X,Y)$ and $\alpha \neq \beta$. We show that $\alpha \leq \beta$.

(1) Let $V \in E(\beta)$. Then $V = M\beta^{-1}$ for some E-class M. We assert that $V \subseteq M\alpha^{-1} \in E(\alpha)$. Let $x \in V$. Then $x\beta \in M$. If $x \neq a$, then $x\alpha = x\beta \in M$ which implies that $x \in M\alpha^{-1}$. If $x = a \in A$, then $x\alpha = a\alpha \in A\alpha \subseteq (C \cap Y)\alpha$ and so $x\alpha = c\alpha$ for some $c \in C \cap Y$. Hence $x\beta = a\beta = b \in G$. Then $x\beta \in M \cap G$. We obtain $M = G$ and $x\alpha = c\alpha \in C\alpha \subseteq U\alpha \subseteq G$ from which it follows that $x \in G\alpha^{-1} = M\alpha^{-1}$. Therefore, $E(\beta)$ refines $E(\alpha)$.

Let $(x\beta)\beta^{-1} \in \pi(\beta)$. Assume that $x \neq a$. Then $x\beta = x\alpha$. We claim that $(x\beta)\beta^{-1} \subseteq (x\alpha)\alpha^{-1} \in \pi(\alpha)$. Let $z \in (x\beta)\beta^{-1}$. Then $z\beta = x\beta$. If $z = a$, then $x\beta = z\beta = a\beta = b$ which is a contradiction since $x \neq a$. Hence $z \neq a$ and $z\alpha = z\beta = x\beta = x\alpha$. Thus $z \in (x\alpha)\alpha^{-1}$. Now, we suppose that $x = a$. We assert that $(x\beta)\beta^{-1} \subseteq (a\alpha)\alpha^{-1} \in \pi(\alpha)$. Indeed, let $y \in (x\beta)\beta^{-1}$. Then $y\beta = x\beta = a\beta = b$ which implies that $y = a$. Hence $y\alpha = a\alpha$ and so $y \in (a\alpha)\alpha^{-1}$. Thus $\pi(\beta)$ refines $\pi(\alpha)$.

(2) Let $x\beta \in X\alpha$. It is clear that $x \neq a$ and then $x\alpha = x\beta$.

(3) Let $D \in X/E$. Assume that $D \neq A$. Then $D\alpha \subseteq (H \cap Y)\alpha$ for some E-class H since $\alpha \in F_E$. Let $x\alpha \in D\alpha$ be such that $x \in D$. Then $x \neq a$ since $D \neq A$. Hence $x\alpha = y\alpha$ for some $y \in H \cap Y$. If $y = a$, then $x\alpha = y\alpha = a\alpha = b$ which is a contradiction since $x \neq a$. Thus $y \neq a$ and so $x\alpha = y\alpha = y\beta \in (H \cap Y)\beta$ which implies that $D\alpha \subseteq (H \cap Y)\beta$. Now, we suppose that $D = A$. Let $x\alpha \in D\alpha$ be such that $x \in D = A$. Then $x\alpha \in D\alpha = A\alpha \subseteq (C \cap Y)\alpha = (C \cap Y)\beta$ and hence $D\alpha \subseteq (C \cap Y)\beta$.

Therefore, α is not maximal which is a contradiction. □

Lemma 6.4. *Let $\alpha \in F_E$ be maximal and $U \in E(\alpha)$. Then U is Y-saturated or $\alpha|_A$ is injective for each E-class $A \subseteq U$.*

Proof. We prove by contradiction. Assume that U is not Y-saturated and there exists E-class $A \subseteq U$ such that $\alpha|_A$ is not injective. Moreover, $A\alpha \subseteq (C \cap Y)\alpha$ for some $C \in X/E$ since $\alpha \in F_E$. If $A \neq C$, then α is not maximal by Lemma 6.3. Now, we suppose that $A\alpha \subseteq (A \cap Y)\alpha$. Since U is not Y-saturated, we have $U\alpha \subsetneq B \cap Y$ for some $B \in X/E$ which implies that there is $c \in B \cap Y \setminus U\alpha$. In addition, since $\alpha|_A$ is not injective, there exist distinct elements $a \in A \cap Y$ and $b \in A$ such that $a\alpha = b\alpha$. Define $\beta : X \to Y$ by

$$x\beta = \begin{cases} c & \text{if } \ x = b; \\ x\alpha & \text{otherwise.} \end{cases}$$

We can easily see that $\beta \in T_E(X,Y)$ and $\alpha \neq \beta$. We prove that $\alpha \leq \beta$ by using Theorem 4.1.

(1) Let $V \in E(\beta)$. Then $V = C\beta^{-1}$ for some $C \in X/E$. We claim that $V = C\alpha^{-1} \in E(\alpha)$. Let $x \in V$. Then $x\beta \in C$. If $x \neq b$, then $x\alpha = x\beta \in C$ and so $x \in C\alpha^{-1}$. If $x = b$, then $x\beta = c \in B$ which implies that $x\beta \in B \cap C \neq \emptyset$. Hence $B = C$. Moreover, we obtain $x\alpha = b\alpha \in B = C$ and so $x \in C\alpha^{-1}$. Thus $V \subseteq C\alpha^{-1}$. Conversely, let $x \in C\alpha^{-1}$.

Then $x\alpha \in C$. If $x \neq b$, then $x\beta = x\alpha \in C$ and hence $x \in C\beta^{-1} = V$. If $x = b$, then $x\alpha = b\alpha \in B$ which implies that $B = C$. Moreover, we have $x\beta = c \in B = C$ and so $x \in C\beta^{-1} = V$. Thus $C\alpha^{-1} \subseteq V$ and then $V = C\alpha^{-1} \in E(\alpha)$. Therefore, $E(\beta)$ refines $E(\alpha)$.

Let $y\beta^{-1} \in \pi(\beta)$. Then there is $x \in y\beta^{-1}$. If $x \neq b$, then $x\alpha = x\beta = y$. We claim that $y\beta^{-1} \subseteq y\alpha^{-1}$. Let $p \in y\beta^{-1}$. Then $p\beta = y = x\alpha \in X\alpha$ from which it follows that $p \neq b$. Hence $p\alpha = p\beta = y$ and so $p \in y\alpha^{-1} \in \pi(\alpha)$. If $x = b$, then $c = b\beta = x\beta = y$. We claim that $y\beta^{-1} \subseteq (a\alpha)\alpha^{-1}$. Let $p \in y\beta^{-1} = c\beta^{-1}$. Then $p\beta = c$ which implies that $p = b$. Thus $p\alpha = b\alpha = a\alpha$ and so $p \in (a\alpha)\alpha^{-1} \in \pi(\alpha)$. Therefore, $\pi(\beta)$ refines $\pi(\alpha)$.

(2) Let $x\beta \in X\alpha$. It is easy to see that $x \neq b$ and then $x\alpha = x\beta$.

(3) Let $C \in X/E$. Then $C\alpha \subseteq (D \cap Y)\alpha$ for some E-class D since $\alpha \in F_E$. We claim that $(D \cap Y)\alpha \subseteq (D \cap Y)\beta$. Let $x\alpha \in (D \cap Y)\alpha$ be such that $x \in D \cap Y$. If $x \neq b$, then $x\alpha = x\beta \in (D \cap Y)\beta$. If $x = b$, then $x = b \in A$ and so $A = D$. Hence $x\alpha = b\alpha = a\alpha = a\beta \in (A \cap Y)\beta = (D \cap Y)\beta$. Thus, $C\alpha \subseteq (D \cap Y)\alpha \subseteq (D \cap Y)\beta$.

Therefore, α is not maximal which is a contradiction. ☐

Definition 6.5. *Let $\alpha \in T_E(X,Y)$. A set $U \in E(\alpha)$ is said to be Y-divisible if there exist $A,B,C \in X/E$ such that $A \subseteq U$, $C \subseteq U \setminus A$, $A\alpha \subseteq (C \cap Y)\alpha$, $B \cap X\alpha = \emptyset$ and $|B \cap Y| \geq |A\alpha|$.*

Lemma 6.6. *Let $\alpha \in F_E$. If α is maximal, then U is not Y-divisible for each $U \in E(\alpha)$.*

Proof. We prove by contrapositive. Assume that there is a Y-divisible set $U \in E(\alpha)$. Let A,B,C be as in Definition 6.5. Let $\phi : A\alpha \to B \cap Y$ be an arbitrary injection. Define $\beta : X \to Y$ by

$$x\beta = \begin{cases} x\alpha\phi & \text{if } \ x \in A; \\ x\alpha & \text{otherwise.} \end{cases}$$

It is clear that $\beta \in T_E(X,Y)$ and $\alpha \neq \beta$. Next, we prove that $\alpha \leq \beta$ by using Theorem 4.1.

(1) Let $V \in E(\beta)$. Then $V = D\beta^{-1}$ for some $D \in X/E$ which implies that $V\beta \subseteq D$. If $D = B$, then $V \subseteq A \subseteq U \in E(\alpha)$. If $D \neq B$, then $V \cap A = \emptyset$ from which it follows that $V\alpha = V\beta \subseteq D$. Hence $V \subseteq D\alpha^{-1} \in E(\alpha)$. Thus $E(\beta)$ refines $E(\alpha)$. Let $M \in \pi(\beta)$. Then $M = y\beta^{-1}$ for some $y = x\beta \in X\beta$. If $y \in B$, then $x \in A$ and $y = x\beta = x\alpha\phi$. Since ϕ is injective, we have $y\phi^{-1} = \{x\alpha\}$. Define $N = (y\phi^{-1})\alpha^{-1} = (x\alpha)\alpha^{-1} \in \pi(\alpha)$. Let $z \in M$. Then $z\beta = y$. Since $y \in B$, we have $z \in A$ which implies that $y = z\beta = z\alpha\phi$. Hence $z \in (y\phi^{-1})\alpha^{-1} = N$. Thus $M \subseteq N$. On the other hand, assume that $y \notin B$. Let $w \in M$. Then $y = w\beta$ and $w \notin A$. Hence $y = w\beta = w\alpha$ from which it follows that $w \in y\alpha^{-1} \in \pi(\alpha)$. We conclude that $\pi(\beta)$ refines $\pi(\alpha)$.

(2) Let $x \in X$ be such that $x\beta \in X\alpha$. It is obvious that $x \notin A$. Hence $x\alpha = x\beta$.

(3) Let $F \in X/E$. Then $F\alpha \subseteq (G \cap Y)\alpha$ for some $G \in X/E$ since $\alpha \in F_E$. If $G \neq A$, then $F\alpha \subseteq (G \cap Y)\alpha = (G \cap Y)\beta$. If $G = A$, then

$$F\alpha \subseteq (A \cap Y)\alpha \subseteq A\alpha \subseteq (C \cap Y)\alpha = (C \cap Y)\beta.$$

Therefore, α is not maximal. ☐

Theorem 6.7. *Let $\alpha \in F_E$. Then α is maximal if and only if for each $U \in E(\alpha)$, at least one of the following conditions holds.*

(1) Both $\alpha|_A$ is injective for each E-class $A \subseteq U$ and for each E-classes $A, C \subseteq U$, if $A\alpha \subseteq (C \cap Y)\alpha$, then $A = C$.

(2) U is not Y-divisible and U is Y-saturated.

Proof. Assume that α is maximal and let $U \in E(\alpha)$. If U is not Y-saturated, then (1) holds by Lemmas 6.3 and 6.4. Otherwise, U is not Y-divisible by Lemma 6.6.

Conversely, let $\beta \in T_E(X,Y)$ be such that $\alpha \leq \beta$ and $U \in E(\alpha)$. We aim to prove that $U\alpha = U\beta$. We consider the following two cases.

Case 1: Assume that (1) holds. Let $x \in U$ be such that $x \in A$ for some E-class $A \subseteq U$. Then $x\alpha \in A\alpha$. We claim that $A\alpha \subseteq (A \cap Y)\beta$. Let $a\alpha \in A\alpha$. Since $\alpha \leq \beta$, we have $A\alpha \subseteq (C \cap Y)\beta$ for some $C \in X/E$ which implies that $a\alpha = c\beta$ for some $c \in C \cap Y$. Hence $c\beta = c\alpha$ by Theorem 4.1 (2). Thus $a\alpha = c\alpha \in (C \cap Y)\alpha$ and so $A\alpha \subseteq (C \cap Y)\alpha$. By assumption (1), we get $A = C$ and hence $A\alpha \subseteq (A \cap Y)\beta$. So $x\alpha = x'\beta$ for some $x' \in A \cap Y$. Again by Theorem 4.1 (2), we have $x'\alpha = x'\beta$ and then $x\alpha = x'\alpha$. Thus $x = x'$ since $\alpha|_A$ is injective. Therefore, $x\alpha = x'\beta = x\beta$ and then $U\alpha = U\beta$.

Case 2: U is not Y-divisible and Y-saturated. Then $U = A\alpha^{-1}$ such that $U\alpha = A \cap Y$ for some $A \in X/E$. By Corollary 4.3 (4), there is $V \in E(\beta)$ such that $V \subseteq U$ and

$$A \cap Y = U\alpha = V\alpha \subseteq V\beta = A \cap X\beta \subseteq A \cap Y$$

which implies that $U\alpha = V\alpha = V\beta = A \cap Y$. We suppose to a contrary that $U \notin E(\beta)$. Since $E(\beta)$ refines $E(\alpha)$ and $U \notin E(\beta)$, there is $W \in E(\beta)$ such that $W \neq V$ and $W \subseteq U$. Let $B \in X/E$ be such that $B\beta^{-1} = W$. If $B \cap X\alpha \neq \emptyset$, then $B\alpha^{-1} \in E(\alpha)$ and there is $y \in B \cap X\alpha \subseteq B \cap X\beta = W\beta$. Hence $y = x\beta$ for some $x \in W$. We obtain $x\beta = y \in X\alpha$ implies $x\alpha = x\beta = y \in B$ by Theorem 4.1 (2). Thus $x \in B\alpha^{-1}$ from which it follows that $x \in W \cap B\alpha^{-1} \neq \emptyset$. We conclude that $W \subseteq B\alpha^{-1}$ since $E(\beta)$ refines $E(\alpha)$. Hence $W \subseteq U \cap B\alpha^{-1} \neq \emptyset$ and so $U = B\alpha^{-1}$ since $E(\alpha)$ is a partition of X. We obtain $A\alpha^{-1} = U = B\alpha^{-1}$ which implies that $A = B$. Thus $V = A\beta^{-1} = B\beta^{-1} = W$ which is a contradiction. Hence $B \cap X\alpha = \emptyset$. Let $D \in X/E$ be such that $D \subseteq W$. Since $W\beta \subseteq B \cap Y$ and $\pi(\beta)$ refines $\pi(\alpha)$, we have

$$|B \cap Y| \geq |W\beta| \geq |D\beta| \geq |D\alpha|.$$

Since $\alpha \leq \beta$, there is $C \in X/E$ such that $D\alpha \subseteq (C \cap Y)\beta$. Let $x \in D$. Then $x\alpha = y\beta$ for some $y \in C \cap Y$. By Theorem 4.1 (2), we obtain $y\alpha = y\beta$. Hence $x\alpha = y\alpha \in (C \cap Y)\alpha$ which implies that $D\alpha \subseteq (C \cap Y)\alpha$. We note that $D\alpha \subseteq W\alpha \subseteq U\alpha \subseteq A$ and so $\emptyset \neq (C \cap Y)\alpha \cap A \subseteq C\alpha \cap A$. Hence $C\alpha \subseteq A$ from which it follows that $C \subseteq A\alpha^{-1} = U$. If $C = D$, then

$$D\alpha \subseteq (D \cap Y)\beta \subseteq D\beta \subseteq W\beta \subseteq B$$

which is a contradiction since $B \cap X\alpha = \emptyset$. Thus $C \subseteq U \setminus D$. We conclude that U is Y-divisible which is also a contradiction. Therefore, $U \in E(\beta)$ from which it follows that $U = V$ and so $U\alpha = U\beta$.

By Corollary 4.3 (1), $\alpha = \beta$ and so α is maximal. □

Finally, we characterize minimal elements in $T_E(X,Y)$.

Theorem 6.8. *Let $\alpha \in T_E(X,Y)$. Then α is minimal if and only if α is a constant map.*

Proof. Assume that α is a constant map. Then $X\alpha = \{y\}$ for some $y \in Y$. Let $\beta \in T_E(X,Y)$ be such that $\beta \leq \alpha$. Then $X\beta \subseteq X\alpha = \{y\}$ which implies that $x\beta = y$ for all $x \in X$. Hence $\beta = \alpha$.

Conversely, suppose that α is not a constant map. Choose $y \in Y\alpha$. Define $\beta \in T_E(X,Y)$ by $x\beta = y$ for all $x \in X$. We can easily see that $\beta \lneq \alpha$. □

7 The Rank of $T_E(X,Y)$

In this section, we use the notation introduced in [2]. Assume that the set X is $\{1,2,\dots,mn\}$ where $m \geq 2$ and $n \geq 3$. The equivalence E is defined by

$$E = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \cdots \cup (A_m \times A_m)$$

where $A_i = \{(i-1)n+1, (i-1)n+2, \dots, in\}$, $i = 1,2,\dots,m$.

By $\tau = (12)$, we denote the permutation of X which maps 1 into 2, 2 into 1 and maps x into x for $y \neq 1,2$. Moreover, by $\xi = (1,2,\dots,n)$, we denote the permutation of X which maps x into $x+1$ for $1 \leq x \leq n-1$, n into 1 and maps x into x for $x \notin A_1$. Let τ_* be the permutation of Y defined by

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ n+1 & n+2 & \cdots & 2n & 1 & 2 & \cdots & n \end{pmatrix}$$

where the image of each $x > 2n$ is x. Let $\tau' = (n+1\ n+2)$ be the permutation of X which maps $n+1$ into $n+2$, $n+2$ into $n+1$ and maps x into x for $x \neq n+1, n+2$. Let $\xi' = (n+1, n+2, \dots, 2n)$ be the permutation of X which maps x into $x+1$ $(n+1 \leq x < 2n)$, $2n$ into $n+1$ and maps x into x for $x \notin A_2$.

Let $\pi \in T(X)$ be defined by

$$1\pi = 2, \quad x\pi = x \ (x \neq 1)$$

and denote $\pi = [12]$. In addition, define $\xi_* \in T_E(X)$ by

$$x\xi_* = \begin{cases} x+n & \text{if } \ x \notin A_m; \\ x-(m-1)n & \text{otherwise} \end{cases}$$

and write $\xi_* = (A_1 A_2 \dots A_m)$. We can see that ξ_* is the permutation which maps A_i into A_{i+1} $(1 \leq i < m)$ and maps A_m into A_1. Finally, define $\pi_* \in T_E(X)$ by

$$x\pi_* = \begin{cases} x+n & \text{if } \ x \in A_1; \\ x & \text{otherwise} \end{cases}$$

and write $\pi_* = [A_1 A_2]$.

In [2], the author proved the following results.

Theorem 7.1 ([2], Theorem 3.7)**.** $T_E(X) = \langle \tau, \xi, \pi, \tau_*, \xi_*, \pi_* \rangle$.

Corollary 7.2 ([2], Corollary 3.8)**.** *If $m = 2$ then the rank of $T_E(X)$ is no more than* 5. *If $m \geq 3$ the rank of $T_E(X)$ is no more than* 6.

Recall that, in [7], the authors defined a subset F of $T(X,Y)$ by

$$F = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}.$$

Now, we consider the subsemigroup $\overline{F_E}$ of $T_E(X,Y)$ defined by

$$\overline{F_E} = F \cap T_E(X).$$

We see that $F_E \subseteq \overline{F_E}$. Clearly, if $X = Y$, then $\overline{F_E} = T_E(X)$ and if $E = X \times X$, then $\overline{F_E} = F$. Moreover, $\overline{F_E}$ becomes $T(X)$ if $X = Y$ and $E = X \times X$. In this paper, we study the rank of $\overline{F_E}$ in a special case when $|X \setminus Y| = 1$.

From now on, assume that the subset Y of X is $\{1,2,\ldots,mn\}$ where $m \geq 2$ and $n \geq 3$. The equivalence E_Y of Y is defined by

$$E = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \cdots \cup (A_m \times A_m)$$

where $A_i = \{(i-1)n+1, (i-1)n+2, \ldots, in\}$, $i = 1,2,\ldots,m.$

The equivalence E of X is defined by

$$E = (\overline{A_1} \times \overline{A_1}) \cup (A_2 \times A_2) \cup \cdots \cup (A_m \times A_m)$$

where $\overline{A_1} = A_1 \cup \{1'\}$. Clearly, $|X \setminus Y| = |\{1'\}| = 1$ and $T_E(Y) = \langle \tau, \xi, \pi, \tau_*, \xi_*, \pi_* \rangle$.

For each $k \leq n$, let $\phi_k : X \to Y$ be the map $1' \mapsto k$ and $x \mapsto x$ if $x \neq 1'$. Define $\overline{\tau} = \phi_1 \tau, \overline{\xi} = \phi_1 \xi, \overline{\pi} = \phi_1 \pi, \overline{\tau_*} = \phi_1 \tau_*, \overline{\xi_*} = \phi_1 \xi_*$ and $\overline{\pi_*} = \phi_1 \pi_*$. We can see that $\phi_k (k \leq n), \overline{\tau}, \overline{\xi}, \overline{\pi}, \overline{\tau_*}, \overline{\xi_*}$ and $\overline{\pi_*}$ are in $\overline{F_E}$.

By using Theorem 7.1, it is routinely to show that

$$\overline{F_E} = \langle \phi_1, \phi_2, \ldots, \phi_n, \overline{\tau}, \overline{\xi}, \overline{\pi}, \overline{\tau_*}, \overline{\xi_*}, \overline{\pi_*} \rangle.$$

Hence, by Corollary 7.2, we obtain the following result immediately.

Corollary 7.3. *If $m = 2$ then the rank of $\overline{F_E}$ is no more than $5+n$. If $m \geq 3$ the rank of $\overline{F_E}$ is no more than $6+n$.*

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RESEARCH ARTICLE

Regularity and Green's relations on semigroups of transformations with restricted range that preserve an equivalence

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Abstract

Let Y be a subset of X and $T(X, Y)$ the set of all functions from X into Y. Then, under the operation of composition, $T(X, Y)$ is a subsemigroup of the full transformation semigroup $T(X)$. Let E be an equivalence on X. Define a subsemigroup $T_E(X, Y)$ of $T(X, Y)$ by

$$T_E(X, Y) = \{\alpha \in T(X, Y) : \forall (x, y) \in E, (x\alpha, y\alpha) \in E\}.$$

Then $T_E(X, Y)$ is the semigroup of all continuous self-maps of the topological space X for which all E-classes form a basis carrying X into a subspace Y. In this paper, we give a necessary and sufficient condition for $T_E(X, Y)$ to be regular and characterize Green's relations on $T_E(X, Y)$. Our work extends previous results found in the literature.

Keywords Transformation semigroups · Restricted range · Equivalence · Regularity · Green's relations

1 Introduction

The full transformation semigroup is the collection of all functions from a set X into X with the composition which is denoted by $T(X)$. In 2008, Sanwong and Sommanee [4] studied the subsemigroup $T(X, Y)$ of $T(X)$ which is defined by

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$$T(X,Y)=\{\alpha\in T(X)\,:\,X\alpha\subseteq Y\}$$

where Y is a fixed subset of X. In [4], they discussed the regularity of elements and then determined the Green's relations on $T(X, Y)$. Moreover, they obtained a class of maximal inverse subsemigroups of $T(X, Y)$. Furthermore, a natural partial order on $T(X, Y)$ was studied in some detail in [3, 6].

Let E be an equivalence on X. Write

$$T_E(X)=\{\alpha\in T(X)\,:\,\forall(x,y)\in E,(x\alpha,y\alpha)\in E\},$$

then $T_E(X)$ is a subsemigroup of $T(X)$. We see that $T_E(X)$ is the semigroup of all continuous self-maps of the topological space X for which all E-classes form a basis. In 2005, Pei [2] studied regularity of elements and Green's relations for $T_E(X)$. In 2008, Sun, Pei and Cheng [5] investigated $T_E(X)$ with the natural partial order.

Now, we aim to generalize the results of Pei [2], Sanwong and Sommanee [4] by defining a transformation semigroup with restricted range that preserve an equivalence as follows. Let $T(X, Y)$ be the full transformation semigroup with restricted range and E an equivalence on X. Define

$$T_E(X,Y)=\{\alpha\in T(X,Y)\,:\,\forall(x,y)\in E,(x\alpha,y\alpha)\in E\}=T_E(X)\cap T(X,Y).$$

Then $T_E(X,Y)$ is a subsemigroup of $T(X)$. It is clear that if $X=Y$, then $T_E(X,Y)=T_E(X)$, which means that $T_E(X)$ is a special case of $T_E(X,Y)$. Furthermore, if E is the universal relation, $E=X\times X$, then $T_E(X,Y)$ becomes $T(X, Y)$. The latter semigroup was studied in detail in [3, 4, 6]. Moreover, it is not difficult to check that $T_E(X,Y)$ is the semigroup of all continuous self-maps of the topological space X for which all E-classes form a basis carrying X into a subspace Y and is referred to as a semigroup of continuous functions (see [1] for details).

In this paper, we give a necessary and sufficient condition for $T_E(X,Y)$ to be regular and characterize Green's relations on $T_E(X,Y)$.

Let X/E denote the quotient set of X and let Y be a subset of X. The restriction of the equivalence E on Y, denoted by E_Y, is defined by

$$E_Y=\{(x,y)\,:\,x,y\in Y,(x,y)\in E\}=E\cap(Y\times Y).$$

For each $\alpha\in T_E(X,Y)$, let

$$\pi(\alpha)=\{y\alpha^{-1}\,:\,y\in X\alpha\}$$

be the partition of X induced by α. As in [2], for each $A\subseteq X$, we write

$$\pi_A(\alpha)=\{M\in\pi(\alpha)\,:\,M\cap A\neq\emptyset\}.$$

We also define

$$\tilde{\pi}_A(\alpha)=\{M\in\pi(\alpha)\,:\,M\cap A\cap Y\neq\emptyset\}.$$

It is clear that $\tilde{\pi}_A(\alpha)$ is an appropriate extension of $\pi_A(\alpha)$ in the sense that if $Y=X$, then $\tilde{\pi}_A(\alpha)=\pi_A(\alpha)$. Obviously, $\tilde{\pi}_A(\alpha)\subseteq\pi_A(\alpha)$. For each $\alpha\in T_E(X,Y)$, define a function $\alpha_*\,:\,\pi(\alpha)\to X\alpha$ by

$P\alpha_* = x\alpha$ for each $P \in \pi(\alpha)$ and each $x \in P$.

We obtain the following lemma which will prove useful.

Lemma 1.1 *Let $\alpha \in T_E(X, Y)$. Then for each $B \in X/E$, there exists some $B' \in X/E$ such that $B\alpha \subseteq B' \cap Y \subseteq B'$. Consequently, for each $A \in X/E$, $A\alpha^{-1}$ is either the empty set or a union of some E-classes.*

For each $\alpha \in T_E(X, Y)$, let

$$E(\alpha) = \{A\alpha^{-1} : A \in X/E \text{ and } A\alpha^{-1} \neq \emptyset\}.$$

We can see that $E(\alpha)$ is also a partition of X and x, y are contained in the same $U \in E(\alpha)$ if and only if $(x\alpha, y\alpha) \in E$.

Let E be an equivalence relation on a set X and U, V subsets of X. Let $\alpha : U \to V$. If $(u, u') \in E$ implies $(u\alpha, u'\alpha) \in E$ for each $u, u' \in U$, then α is said to be *E-preserving*. In addition, for each $u, u' \in U$, if $(u, u') \in E$ if and only if $(u\alpha, u'\alpha) \in E$, then α is called *E^*-preserving*. We remark that if α is an E^*-preserving bijection, then so is α^{-1}.

2 Regularity

In this section, we characterize regular elements in $T_E(X, Y)$ and then give a necessary and sufficient condition for $T_E(X, Y)$ to be regular.

Theorem 2.1 *Let $\alpha \in T_E(X, Y)$. Then α is regular if and only if for all $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha \subseteq (B \cap Y)\alpha$.*

Proof Fix $y_0 \in Y$. For each $A \in X/E$, define a function β as follows. If $A \cap X\alpha = \emptyset$, define $x\beta = y_0$ for all $x \in A$. If $A \cap X\alpha \neq \emptyset$, by assumption, there is a class $B \in X/E$ such that $A \cap X\alpha \subseteq (B \cap Y)\alpha$. Fix $b_0 \in B \cap Y$. For each $x \in A \cap X\alpha$, there is an element $b_x \in B \cap Y$ such that $x = b_x\alpha$. Define

$$x\beta = \begin{cases} b_x & \text{if } x \in A \cap X\alpha \\ b_0 & \text{if } x \in A \setminus X\alpha. \end{cases}$$

It is not hard to see that β preserves E and $X\beta \subseteq Y$. Hence, $\beta \in T_E(X, Y)$. We claim that $\alpha = \alpha\beta\alpha$. Let $x \in X$. Then $x\alpha \in A \cap X\alpha$ for some E-class A and, by assumption, $A \cap X\alpha \subseteq (B \cap Y)\alpha$. By construction, $x\alpha = b_{x\alpha}\alpha$, where $b_{x\alpha} \in B \cap Y$. Note that

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = b_{x\alpha}\alpha = x\alpha.$$

Therefore, $\alpha = \alpha\beta\alpha$ since x is arbitrary, as claimed.

Conversely, suppose that α is regular. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_E(X, Y)$. Let $A \in X/E$. Then $A\beta \subseteq B \cap Y$ for some $B \in X/E$. We claim that $A \cap X\alpha \subseteq (B \cap Y)\alpha$. Let $y \in A \cap X\alpha$. Then $y \in A$ and $y = x\alpha$ for some $x \in X$.

We obtain that $y\beta \in A\beta \subseteq B \cap Y$. Hence $x\alpha\beta \in B \cap Y$ which implies that $y = x\alpha = x\alpha\beta\alpha \in (B \cap Y)\alpha$ and the proof completes. □

By the above theorem, if $X = Y$, then α is regular if and only if for all $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha \subseteq B\alpha$. Hence Theorem 2.1 is a generalization of Corollary 2.3 in [2].

In [4], the authors defined a subset F of $T(X, Y)$ by

$$F = \{\alpha \in T(X, Y) : X\alpha \subseteq Y\alpha\}$$

and proved that F is the set of all regular elements in $T(X, Y)$. Moreover, they also proved that F is the largest regular subsemigroup of $T(X, Y)$.

Now, we define a subset F_E of $T_E(X, Y)$ by $\alpha \in F_E$ if $\alpha \in T_E(X, Y)$ and for each $A \in X/E$, there exists $B \in X/E$ such that $A\alpha \subseteq (B \cap Y)\alpha$. It is easy to see that $F = F_E$ if $E = X \times X$ and $F_E = T_E(X)$ if $X = Y$. In general, F_E is a proper subset of $F \cap T_E(X, Y)$. To see this, consider the following example.

Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 2, 4\}$. Define $X/E = \{A, B\}$ by $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

Clearly, $\alpha \in F \cap T_E(X, Y)$. We see that $A\alpha = \{1, 2\}, (A \cap Y)\alpha = \{1\}$ and $(B \cap Y)\alpha = \{2\}$. Thus $\alpha \notin F_E$.

The subset F_E plays an essential role in the characterization of Green's relations, as shown in Sect. 3.

Lemma 2.2 *F_E is a right ideal of $T_E(X, Y)$. Consequently, it is a subsemigroup of $T_E(X, Y)$.*

Proof Let $\alpha \in F_E$ and $\beta \in T_E(X, Y)$. Then for each $A \in X/E$, there is $B \in X/E$ such that

$$A\alpha\beta = (A\alpha)\beta \subseteq ((B \cap Y)\alpha)\beta = (B \cap Y)\alpha\beta.$$

Thus $\alpha\beta \in F_E$. □

Remark 2.3 F_E contains the set of all regular elements in $T_E(X, Y)$.

Proof Let $\alpha \in T_E(X, Y)$ be a regular element and $A \in X/E$. Then $A\alpha \subseteq B \cap X\alpha$ for some $B \in X/E$ and so there exists $C \in X/E$ such that $A\alpha \subseteq B \cap X\alpha \subseteq (C \cap Y)\alpha$ since α is regular. Hence $\alpha \in F_E$. □

In general, the set F_E is not a regular subsemigroup of $T_E(X, Y)$. For example, let E be an equivalence on $X = \{1, 2, 3, 4\}$ where $X/E = \{\{1, 2\}, \{3\}, \{4\}\}$ and $Y = \{1, 2, 3\}$. Define $\alpha \in T_E(X, Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

It is easy to check that $\alpha \in F_E$ but not regular. Consequently, the set of all regular elements in $T_E(X,Y)$ is a proper subset of F_E.

Next, we give a necessary and sufficient condition for $T_E(X,Y)$ to be regular. Note that if $|Y| = 1$, then $T_E(X,Y)$ contains exactly one element and so $T_E(X,Y)$ is regular. Here, $\Delta(Y)$ stands for the diagonal relation on Y, that is, $\Delta(Y) = \{(y,y) : y \in Y\}$.

Theorem 2.4 *Let $Y \subseteq X$ such that $|Y| > 1$. Then $T_E(X,Y)$ is regular if and only if the following statements hold.*

(1) *For each $G \in X/E$, $G \cap Y$ is non-empty.*
(2) *Either $E_Y = \Delta(Y)$ or $E = X \times X$ and $X = Y$.*

Proof ($\Rightarrow$) We prove by contrapositive. Suppose that there exists a class G with $G \cap Y$ is empty. Since $|Y| > 1$, there are $a, b \in Y$ such that $a \neq b$. Define a function $\alpha : X \to Y$ by $x\alpha = a$ if $x \in G$ and $x\alpha = b$ if $x \notin G$. We can see that $\alpha \in T_E(X,Y)$. Let A be the class containing a. We obtain $a \in A \cap X\alpha$. By the definition of α, for each $B \in X/E$ such that $B \cap Y \neq \emptyset$, we have $(B \cap Y)\alpha = \{b\}$ since $(B \cap Y) \cap G = \emptyset$. Hence $a \notin (B \cap Y)\alpha$ which implies that $A \cap X\alpha \nsubseteq (B \cap Y)\alpha$. Therefore, α is not regular by Theorem 2.1.

Assume that $E_Y \neq \Delta(Y)$ and $E \neq X \times X$. Then there is a class $A \neq X$ such that $|A \cap Y| > 1$. Let $a, b \in A \cap Y$ such that $a \neq b$. Define a function $\alpha : X \to Y$ by $x\alpha = a$, $\forall x \in A \cap Y$ and $x\alpha = b$, $\forall x \notin A \cap Y$. We can see that $\alpha \in T_E(X,Y)$. For each $B \in X/E$, $(B \cap Y)\alpha = \{a\}$ if $B = A$ and $(B \cap Y)\alpha = \{b\}$ if $B \neq A$. We obtain $A \cap X\alpha = \{a,b\} \nsubseteq (B \cap Y)\alpha$ for all class B. Hence α is not regular by Theorem 2.1.

Suppose that $E_Y \neq \Delta(Y)$ and $X \neq Y$. Then there exists $y \in X \setminus Y$. Since $E_Y \neq \Delta(Y)$, there is a class A such that $|A \cap Y| > 1$. Let $a, b \in A \cap Y$ such that $a \neq b$. Define a function $\alpha : X \to Y$ by $x\alpha = a$ if $x = y$ and $x\alpha = b$ if $x \neq y$. It is obvious that $\alpha \in T_E(X,Y)$. We see that for each $B \in X/E$, $(B \cap Y)\alpha = \{b\}$. Therefore, $A \cap X\alpha = \{a,b\} \nsubseteq \{b\} = (B \cap Y)\alpha$ which implies that α is not regular.

($\Leftarrow$) We can see that if $E = X \times X$ and $X = Y$, then $T_E(X,Y) = T(X)$ is regular. Now, we suppose that $E_Y = \Delta(Y)$. Let $\alpha \in T_E(X,Y)$ and $A \in X/E$. Then $A \cap X\alpha \subseteq A \cap Y = \{a\}$ for some a by (1). If $A \cap X\alpha \neq \emptyset$, then $A \cap X\alpha = \{a\}$ which implies that $a = x\alpha$ for some $x \in X$. Let B be a class containing x. Then $B \cap Y$ is non-empty from which it follows that $B \cap Y = \{b\}$ for some b. Further, since $(b,x) \in E$, we get $(b\alpha, a) = (b\alpha, x\alpha) \in E$ which follows that $b\alpha \in A \cap X\alpha = \{a\}$. Thus $A \cap X\alpha = \{a\} = \{b\alpha\} = (B \cap Y)\alpha$. Therefore, α is regular. □

By Theorem 2.5 of [2], we obtain some properties of regular elements in $T_E(X,Y)$ as follows.

Theorem 2.5 *Let α and β be regular elements in $T_E(X, Y)$. Then the following statements hold.*

(1) *If $\pi(\alpha) = \pi(\beta)$, then $E(\alpha) = E(\beta)$.*
(2) *If $X\alpha = X\beta$, then for each $A \in X/E$, there are $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.*

3 Green's relations

In this section, we characterize Green's relations on $T_E(X, Y)$. We start this section by recalling the definition of Green's relations.

Let S be a semigroup. The following definitions are due to Green. For any $a, b \in S$, define

$$(a, b) \in \mathscr{L} \text{ if and only if } S^1a = S^1b,$$

or equivalently; $(a, b) \in \mathscr{L}$ if and only if $a = xb, b = ya$ for some $x, y \in S^1$.

Dually,

$$(a, b) \in \mathscr{R} \text{ to mean } aS^1 = bS^1,$$

or equivalently; $(a, b) \in \mathscr{R}$ if and only if $a = bx, b = ay$ for some $x, y \in S^1$ and then define

$$(a, b) \in \mathscr{J} \text{ to mean } S^1aS^1 = S^1bS^1,$$

or equivalently; $(a, b) \in \mathscr{J}$ if and only if $a = xby, b = uav$ for some $x, y, u, v \in S^1$.

Finally,

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R} \text{ and } \mathscr{D} = \mathscr{L} \circ \mathscr{R}.$$

Note that the above relations are equivalence relations. The relation $\mathscr{D}$ is the join $\mathscr{L} \vee \mathscr{R}$, that is, $\mathscr{D}$ is the smallest equivalence relation containing $\mathscr{L} \cup \mathscr{R}$. It is well-known that $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. Moreover, $\mathscr{H} \subseteq \mathscr{L} \subseteq \mathscr{D} \subseteq \mathscr{J}$ and $\mathscr{H} \subseteq \mathscr{R} \subseteq \mathscr{D} \subseteq \mathscr{J}$. The relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ and $\mathscr{J}$ are called *Green's relations* on S. For each $a \in S$, we denote $\mathscr{L}$-class, $\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class and $\mathscr{J}$-class containing a by L_a, R_a, H_a, D_a and J_a, respectively.

In general, if $X \neq Y$, then the semigroup $T_E(X, Y)$ does not contain the identity element. Hence $T_E(X, Y)^1 \neq T_E(X, Y)$.

Now, we prove the following theorem which extends Theorem 3.1 of [2].

Theorem 3.1 *Let $\alpha, \beta \in T_E(X, Y)$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{R}$.
(2) *$\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$.*
(3) *There exists a bijective E^*-preserving $\phi : X\alpha \to X\beta$ such that $\beta = \alpha\phi$.*

Proof

(1)⇒(2). Suppose that $(\alpha, \beta) \in \mathscr{R}$. Then there are $\gamma, \mu \in T_E(X, Y)^1$ such that $\alpha = \beta\gamma$ and $\beta = \alpha\mu$. If $\alpha = \beta$, then $\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$. If $\alpha \neq \beta$, then both γ and μ belong to $T_E(X, Y)$ which implies that $\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$ by Theorem 3.1 of [2].
(2)⇒(3). The proof also follows from Theorem 3.1 of [2].
(3)⇒(1). The proof is an appropriate modification of the proof of (3)⇒(1) in Theorem 3.1 of [2]. In fact, suppose that there exists a bijective E^*-preserving $\phi : X\alpha \to X\beta$ such that $\beta = \alpha\phi$. For each $A \in X/E$, let $A' = A \cap X\alpha$. If $A' = \emptyset$, define $x\mu = y_0$ for each $x \in A$ and $y_0 \in Y$ is fixed. If $A' \neq \emptyset$, then $A'\phi = (A \cap X\alpha)\phi \subseteq B \cap X\beta$ for some class B since ϕ is E^*-preserving. In the case $A' \neq \emptyset$, we fix $b_0 \in B \cap X\beta$ and define μ by

$$x\mu = \begin{cases} x\phi & \text{if} \quad x \in A'; \\ b_0 & \text{if} \quad x \in A \setminus A'. \end{cases}$$

It is easy to verify that $\mu \in T_E(X, Y)$ and $\beta = \alpha\mu$. Similarly, we can show that $\alpha = \beta\gamma$ for some $\gamma \in T_E(X, Y)$. □

Lemma 3.2 *Let $\alpha, \beta \in T_E(X, Y)$. If $\pi(\alpha) = \pi(\beta)$, then either both α and β are in F_E, or neither is in F_E.*

Proof Assume that $\pi(\alpha) = \pi(\beta)$ and let $\alpha \in F_E$. It suffices to show $\beta \in F_E$. Let $A \in X/E$. Then $A\alpha \subseteq (B \cap Y)\alpha$ for some class B. We claim that $A\beta \subseteq (B \cap Y)\beta$. Indeed, let $a \in A$. Then there is $x \in X$ such that $(a\beta)\beta^{-1} = (x\alpha)\alpha^{-1}$ since $(a\beta)\beta^{-1} \in \pi(\beta) = \pi(\alpha)$. Obviously, $a \in (a\beta)\beta^{-1} = (x\alpha)\alpha^{-1}$ which implies that $x\alpha = a\alpha \in A\alpha \subseteq (B \cap Y)\alpha$. Thus $x\alpha = b\alpha$ for some $b \in B \cap Y$. Hence $b \in (x\alpha)\alpha^{-1} = (a\beta)\beta^{-1}$ and so $a\beta = b\beta \in (B \cap Y)\beta$. Therefore, $\beta \in F_E$. □

By Theorem 3.1 and Lemma 3.2, we have the following corollary.

Corollary 3.3 *For $\alpha \in T_E(X, Y)$, the following statements hold.*

(1) *If $\alpha \in F_E$, then* $R_\alpha = \{\beta \in F_E : \pi(\alpha) = \pi(\beta) \text{ and } E(\alpha) = E(\beta)\}$.
(2) *If $\alpha \in T_E(X, Y) \setminus F_E$, then*

$$R_\alpha = \{\beta \in T_E(X, Y) \setminus F_E : \pi(\alpha) = \pi(\beta) \text{ and } E(\alpha) = E(\beta)\}.$$

Now, we have already characterized Green's $\mathscr{R}$-relation of $T_E(X, Y)$. To study the remaining Green's relations, we introduce some definitions for using throughout this paper. Actually, we extend the notions of E-admissibility and E^*-admissibility presented in [2].

Let $\alpha, \beta \in T_E(X, Y)$ and let ϕ be a mapping from $\pi(\alpha)$ into $\pi(\beta)$. We say that ϕ is $\tilde{E}$*-admissible* if and only if for each $A \in X/E$, there exists $B \in X/E$ such that

$$\pi_A(\alpha)\phi \subseteq \tilde{\pi}_B(\beta).$$

Equivalently, $\phi : \pi(\alpha) \to \pi(\beta)$ is $\tilde{E}$-admissible if and only if for each $A \in X/E$, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $B \cap P\phi \cap Y \neq \emptyset$.

If ϕ is a bijection such that ϕ and ϕ^{-1} are $\tilde{E}$-admissible, then ϕ is called $\tilde{E}^*$*-admissible*.

We remark that if $X = Y$, then the notions of E-admissibility (resp. $\tilde{E}$-admissibility) and $\tilde{E}$-admissibility (resp. $\tilde{E}^*$-admissibility) are the same.

Now, we determine Green's $\mathscr{L}$-relation on $T_E(X, Y)$. The proof of the following lemma is straightforward and so it is omitted.

Lemma 3.4 *Let $\alpha, \beta \in T_E(X, Y)$. If for each $A \in X/E$, there are $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$, then $X\alpha = X\beta$.*

Theorem 3.5 *Let $\alpha, \beta \in F_E$. Then the following statements are equivalent.*

(1) *$(\alpha, \beta) \in \mathscr{L}$ in $T_E(X, Y)$.*
(2) *For each $A \in X/E$, there are $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.*
(3) *There is a bijective $\tilde{E}^*$-admissible $\phi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \phi\beta_*$.*

Proof

(1)⇒(2). Suppose that $(\alpha, \beta) \in \mathscr{L}$. Then $\alpha = \gamma\beta$ and $\beta = \mu\alpha$ for some $\gamma, \mu \in T_E(X, Y)^1$. If $\alpha = \beta$, then (2) holds. If $\alpha \neq \beta$, then $\gamma, \mu \in T_E(X, Y)$. The item (2) follows by [2, Theorem 3.2].

(2)⇒(3). Suppose (2) holds. Note that $X\alpha = X\beta$ by Lemma 3.4. Then for each $P \in \pi(\alpha)$, we have $P\alpha_* \in X\alpha = X\beta$. We can see that $(P\alpha_*)\beta^{-1} \in \pi(\beta)$. Define $\phi : \pi(\alpha) \to \pi(\beta)$ by $P\phi = (P\alpha_*)\beta^{-1}$. It is clear that ϕ is well-defined and $\alpha_* = \phi\beta_*$. Now, we show that ϕ is injective. Let $P_1, P_2 \in \pi(\alpha)$ be such that $P_1\phi = P_2\phi$. Let $y \in P_1$. Then $P_1\alpha_* = y\alpha$ and so $P_1\phi = (P_1\alpha_*)\beta^{-1} = (y\alpha)\beta^{-1}$. Since $P_1\phi = P_2\phi$, we have $(y\alpha)\beta^{-1} = P_2\phi = (P_2\alpha_*)\beta^{-1}$ which implies that $y\alpha = P_2\alpha_*$. Thus $y \in P_2$ and then $P_1 \subseteq P_2$. Similarly, we can show that $P_2 \subseteq P_1$. To show that ϕ is surjective, let $Q \in \pi(\beta)$. Then $Q = x\beta^{-1}$ for some $x \in X\beta = X\alpha$. Choose $P = x\alpha^{-1} \in \pi(\alpha)$. We obtain $Q = x\beta^{-1} = (P\alpha_*)\beta^{-1} = P\phi$. Therefore, ϕ is a bijection.

Next, we show that ϕ is $\tilde{E}$-admissible. Let $A \in X/E$. Then there is $B \in X/E$ such that $A\alpha \subseteq B\beta$ by the assumption. Hence there is a class D such that $B\beta \subseteq (D \cap Y)\beta$ since $\beta \in F_E$. Thus $A\alpha \subseteq (D \cap Y)\beta$. Let $P \in \pi_A(\alpha)$. Then $P \in \pi(\alpha)$ and $P \cap A \neq \emptyset$. Choose $x \in P \cap A$. Then $P\alpha_* = x\alpha \subseteq A\alpha \subseteq (D \cap Y)\beta$ from which it follows that $x\alpha = y\beta$ for some $y \in D \cap Y$. We have $y \in D \cap (y\beta)\beta^{-1} \cap Y$ and then

$$y \in D \cap (y\beta)\beta^{-1} \cap Y = D \cap (x\alpha)\beta^{-1} \cap Y = D \cap (P\alpha_*)\beta^{-1} \cap Y = D \cap P\phi \cap Y.$$

Thus ϕ is $\tilde{E}$-admissible since $D \cap P\phi \cap Y$ is non-empty. Finally, we prove that ϕ^{-1} is $\tilde{E}$-admissible. Let $P \in \pi(\alpha)$ and $Q \in \pi(\beta)$ be such that $Q = P\phi$. We obtain $Q = (P\alpha_*)\beta^{-1}$ which implies that $Q\beta_* = P\alpha_*$. Hence

$$Q\phi^{-1} = P = (P\alpha_*)\alpha^{-1} = (Q\beta_*)\alpha^{-1}.$$

By the same argument as ϕ, we obtain ϕ^{-1} is also $\tilde{E}$-admissible. Therefore, ϕ is $\tilde{E}^*$-admissible.
(3)⇒(1). Assume that (3) holds. For each $A \in X/E$, there is $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $B \cap P\phi \cap Y \neq \emptyset$. For each $x \in A$, let $P_x = (x\alpha)\alpha^{-1}$. We can see that $P_x \in \pi_A(\alpha)$. Then $B \cap P_x\phi \cap Y \neq \emptyset$. Choose $d_x \in B \cap P_x\phi \cap Y$ and define $x\gamma = d_x$. First, we show that $\gamma \in T_E(X,Y)$. Let $(a,b) \in E$. Then $a,b \in A$ for some $A \in X/E$. By the definition of γ, we obtain $a\gamma, b\gamma \in B$ for some $B \in X/E$. Thus $(a\gamma, b\gamma) \in E$. Next, we prove that $\alpha = \gamma\beta$. Let $x \in A$ for some $A \in X/E$. Then $x\gamma\beta = d_x\beta$ where $d_x \in B \cap P_x\phi \cap Y$ for some $B \in X/E$. Moreover, since $d_x \in P_x\phi$, we get $d_x\beta = (P_x\phi)\beta_*$. Hence

$$x\gamma\beta = d_x\beta = (P_x\phi)\beta_* = P_x\alpha_* = ((x\alpha)\alpha^{-1})\alpha_* = x\alpha.$$

Similarly, we can show that $\beta = \mu\alpha$ for some $\mu \in T_E(X,Y)$. Therefore, $(\alpha,\beta) \in \mathscr{L}$. □

By the above theorem, if $X = Y$, then we obtain Theorem 3.2 of [2].

Theorem 3.6 *For $\alpha \in T_E(X,Y)$, the following statements hold.*

(1) *If $\alpha \in T_E(X,Y) \setminus F_E$, then $L_\alpha = \{\alpha\}$.*
(2) *If $\alpha \in F_E$, then*

$$L_\alpha = \{\beta \in F_E : (\forall A \in X/E)(\exists B, C \in X/E)\, A\alpha \subseteq B\beta \text{ and } A\beta \subseteq C\alpha\}.$$

Proof Let $\alpha \in T_E(X,Y)$ and let $\beta \in L_\alpha$. Then $(\alpha,\beta) \in \mathscr{L}$ which implies that $\alpha = \gamma\beta$ and $\beta = \mu\alpha$ for some $\gamma, \mu \in T_E(X,Y)^1$. If $\gamma, \mu \in T_E(X,Y)$, then for each $A \in X/E$, there are $B, C \in X/E$ such that $A\alpha = A\gamma\beta = A\gamma\mu\alpha \subseteq (B \cap Y)\alpha$ and $A\beta = A\mu\alpha = A\mu\gamma\beta \subseteq (C \cap Y)\beta$. Thus $\alpha, \beta \in F_E$.

(1) If $\alpha \in T_E(X,Y) \setminus F_E$, then $\gamma = 1$ or $\mu = 1$ and hence $\alpha = \beta$.
(2) If $\alpha \in F_E$, then there are two cases to consider. The case $\alpha = \beta$ is clear. If $\alpha \neq \beta$, then $\gamma, \mu \in T_E(X,Y)$ and hence $\beta \in F_E$. In addition, for each $A \in X/E$, there are $B, C \in X/E$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$ by Theorem 3.5. The other containment is obvious. □

As a direct consequence of Corollary 3.3, Theorems 3.1, 3.5 and 3.6, we have the following theorems.

Theorem 3.7 *Let $\alpha, \beta \in F_E$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{H}$ *in* $T_E(X, Y)$.
(2) $\pi(\alpha) = \pi(\beta), E(\alpha) = E(\beta)$ *and for each* $A \in X/E$, *there are* $B, C \in X/E$ *such that* $A\alpha \subseteq B\beta$ *and* $A\beta \subseteq C\alpha$.
(3) *There exist a bijective* $\tilde{E}^*$*-admissible* $\phi : \pi(\alpha) \to \pi(\beta)$ *such that* $\alpha_* = \phi\beta_*$ *and a bijective* E^**-preserving* $\psi : X\alpha \to X\beta$ *such that* $\beta = \alpha\psi$.

Theorem 3.8 *For* $\alpha \in T_E(X, Y)$, *the following statements hold.*

(1) *If* $\alpha \in T_E(X, Y) \setminus F_E$, *then* $H_\alpha = \{\alpha\}$.
(2) *If* $\alpha \in F_E$, *then*

$$H_\alpha = \{\beta \in F_E : \pi(\alpha) = \pi(\beta), E(\alpha) = E(\beta) \text{ and } (\forall A \in X/E)(\exists B, C \in X/E)\ A\alpha \subseteq B\beta \text{ and } A\beta \subseteq C\alpha\}.$$

Now, we characterize Green's $\mathscr{D}$ relation.

Theorem 3.9 *Let $\alpha, \beta \in F_E$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{D}$ *in* $T_E(X, Y)$.
(2) *There is an* E^**-preserving bijection* $\Phi : X\alpha \to X\beta$ *such that for each* $A \in X/E$, *there exist* $B, C \in X/E$ *with*

$$A\beta \subseteq (B \cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C \cap Y)\beta\Phi^{-1}.$$

(3) *There exist a bijective* $\tilde{E}^*$*-admissible* $\phi : \pi(\alpha) \to \pi(\beta)$ *and a bijective* E^**-preserving* $\psi : X\alpha \to X\beta$ *such that* $\alpha_*\psi = \phi\beta_*$.

Proof

(1)⇒(3). Suppose that $(\alpha, \beta) \in \mathscr{D}$ in $T_E(X, Y)$. Then there exists $\gamma \in T_E(X, Y)$ such that $(\alpha, \gamma) \in \mathscr{R}$ and $(\gamma, \beta) \in \mathscr{L}$. By Theorems 3.1 and 3.5, there exist a bijective E^*-preserving $\psi : X\alpha \to X\gamma$ such that $\gamma = \alpha\psi$ and a bijective $\tilde{E}^*$-admissible $\phi : \pi(\gamma) \to \pi(\beta)$ such that $\gamma_* = \phi\beta_*$. In addition, we obtain $\pi(\alpha) = \pi(\gamma)$ and $X\gamma = X\beta$. Hence $\phi : \pi(\alpha) \to \pi(\beta)$ and $\psi : X\alpha \to X\beta$. Now, we show that $\alpha_*\psi = \phi\beta_*$. We claim that $\gamma_* = \alpha_*\psi$. Indeed, let $x\gamma^{-1} \in \pi(\gamma)$. Then $(x\gamma^{-1})\gamma_* = x$. From $\pi(\alpha) = \pi(\gamma)$, there is $y \in X\alpha$ such that $x\gamma^{-1} = y\alpha^{-1}$ which implies that $y = (y\alpha^{-1})\alpha_* = (x\gamma^{-1})\alpha_*$. Since $y \in X\alpha$, we get $y = z\alpha$ for some $z \in X$. Thus $z \in y\alpha^{-1} = x\gamma^{-1}$ from which it follows that $z\gamma = x = (x\gamma^{-1})\gamma_*$. We obtain

$$(x\gamma^{-1})\alpha_*\psi = y\psi = z\alpha\psi = z\gamma = (x\gamma^{-1})\gamma_*.$$

Therefore, $\alpha_*\psi = \gamma_* = \phi\beta_*$.
(3)⇒(2). Assume that there exist a bijective $\tilde{E}^*$-admissible $\phi : \pi(\alpha) \to \pi(\beta)$ and a bijective E^*-preserving $\psi : X\alpha \to X\beta$ such that $\alpha_*\psi = \phi\beta_*$. Define an E^*-preserving bijection $\Phi : X\alpha \to X\beta$ by $x\Phi = x\psi$ for all $x \in X\alpha$. Let $A \in X/E$. We have $\pi_A(\beta)\phi^{-1} \subseteq \tilde{\pi}_B(\alpha)$ for some $B \in X/E$ since $\phi^{-1} : \pi(\beta) \to \pi(\alpha)$ is $\tilde{E}^*$-admissible. We claim that $A\beta \subseteq (B \cap Y)\alpha\Phi$. Indeed, let $a \in A$. Then $(a\beta)\beta^{-1} \in \pi_A(\beta)$ which implies that $(a\beta)\beta^{-1}\phi^{-1} \in (\pi_A(\beta))\phi^{-1} \subseteq \tilde{\pi}_B(\alpha)$. Hence $(a\beta)\beta^{-1}\phi^{-1} = (b\alpha)\alpha^{-1}$ for some $b \in X$ and $(b\alpha)\alpha^{-1} \cap B \cap Y \neq \emptyset$. There exists $y \in (b\alpha)\alpha^{-1} \cap B \cap Y$ and then $b\alpha = y\alpha$. We see that

$$a\beta = (a\beta)\beta^{-1}\beta_* = (a\beta)\beta^{-1}\phi^{-1}\alpha_*\psi = (b\alpha)\alpha^{-1}\alpha_*\psi = b\alpha\psi$$

and so $a\beta = b\alpha\psi = y\alpha\psi \in (B \cap Y)\alpha\psi = (B \cap Y)\alpha\Phi$. Therefore, $A\beta \subseteq (B \cap Y)\alpha\Phi$. Similarly, we can show that $A\alpha \subseteq (C \cap Y)\beta\Phi^{-1}$ for some class C.
(2)⇒(3). Assume that (2) holds. Define a function $\phi : \pi(\alpha) \to \pi(\beta)$ by $(x\alpha)\alpha^{-1}\phi = (x\alpha)\Phi\beta^{-1}$ for all $x \in X$. Then ϕ is well-defined. We first show that ϕ is injective. Let $(x\alpha)\alpha^{-1}, (y\alpha)\alpha^{-1}$ be such that $(x\alpha)\alpha^{-1}\phi = (y\alpha)\alpha^{-1}\phi$. Then $(x\alpha)\Phi\beta^{-1} = (y\alpha)\Phi\beta^{-1}$ and so $x\alpha = y\alpha$ since Φ is injective. Hence $(x\alpha)\alpha^{-1} = (y\alpha)\alpha^{-1}$. Next, we prove the surjectivity of ϕ. Let $(x\beta)\beta^{-1} \in \pi(\beta)$. Then $x\beta \in A\beta \subseteq (B \cap Y)\alpha\Phi$ for some classes A and B. Then there is $y \in B \cap Y$ such that $y\alpha\Phi = x\beta$. Thus $(y\alpha)\alpha^{-1}\phi = (y\alpha)\Phi\beta^{-1} = (x\beta)\beta^{-1}$. Therefore, ϕ is bijective.
Finally, we show that ϕ is $\tilde{E}^*$-admissible. Let $A \in X/E$. Then there exists $C \in X/E$ such that $A\alpha \subseteq (C \cap Y)\beta\Phi^{-1}$. Let $(x\alpha)\alpha^{-1} \in \pi_A(\alpha)$. Then $(x\alpha)\alpha^{-1} \cap A \neq \emptyset$ which implies that there is $a \in A$ such that $a\alpha = x\alpha$. Hence $x\alpha = a\alpha \in A\alpha \subseteq (C \cap Y)\beta\Phi^{-1}$ and so $x\alpha = c\beta\Phi^{-1}$ for some $c \in C \cap Y$. Thus $c \in x\alpha\Phi\beta^{-1} = (x\alpha)\alpha^{-1}\phi$ which implies that $c \in (x\alpha)\alpha^{-1}\phi \cap C \cap Y \neq \emptyset$ and so ϕ is $\tilde{E}$-admissible. On the other hand, let $A \in X/E$. Then there exists $B \in X/E$ such that $A\beta \subseteq (B \cap Y)\alpha\Phi$. Let $(x\beta)\beta^{-1} \in \pi_A(\beta)$. Then $(x\beta)\beta^{-1} \cap A \neq \emptyset$ which implies that there is $a \in A$ such that $a\beta = x\beta$. Hence $x\beta = a\beta \in A\beta \subseteq (B \cap Y)\alpha\Phi$ and so $x\beta = b\alpha\Phi$ for some $b \in B \cap Y$. Thus

$$b \in (b\alpha)\alpha^{-1} = (b\alpha)\alpha^{-1}\phi\phi^{-1} = (b\alpha)\Phi\beta^{-1}\phi^{-1} = (x\beta)\beta^{-1}\phi^{-1}$$

which implies that $b \in (x\beta)\beta^{-1}\phi^{-1} \cap B \cap Y \neq \emptyset$. Therefore, ϕ^{-1} is $\tilde{E}$-admissible. Moreover, we define a function $\psi : X\alpha \to X\beta$ by $x\psi = x\Phi$ for each $x \in X\alpha$. It remains to show that $\alpha_*\psi = \phi\beta_*$. Indeed, let $(x\alpha)\alpha^{-1} \in \pi(\alpha)$. Then

$$(x\alpha)\alpha^{-1}\alpha_*\psi = x\alpha\psi = x\alpha\Phi = (x\alpha)\Phi\beta^{-1}\beta_* = (x\alpha)\alpha^{-1}\phi\beta_*.$$

(3) ⇒(1). Assume that (3) holds. Define a function $\gamma : X \to Y$ by $x\gamma = x\alpha\psi$. Since ψ is E^*-preserving, we have $\gamma \in T_E(X, Y)$. We first show that $\pi(\gamma) = \pi(\alpha)$. Let $A = x\gamma^{-1} \in \pi(\gamma)$. Then $\{x\} = A\gamma = A\alpha\psi$ which implies that $A\alpha = x\psi^{-1}$ since ψ is a bijection. We obtain $A\alpha = \{y\}$ where $\{y\} = x\psi^{-1}$ and then $y\psi = x$.

Hence $A \subseteq y\alpha^{-1}$. Let $z \in y\alpha^{-1}$. Then $z\alpha = y$ from which it follows that $z\gamma = z\alpha\psi = y\psi = x$. Thus $z \in x\gamma^{-1} = A$ implies $y\alpha^{-1} \subseteq A$. So $A = y\alpha^{-1} \in \pi(\alpha)$. We conclude that $\pi(\gamma) \subseteq \pi(\alpha)$. On the other hand, let $B = a\alpha^{-1} \in \pi(\alpha)$. Then $B\alpha = \{a\}$ which implies that $B\gamma = B\alpha\psi = \{a\}\psi = \{b\}$ for some $b \in Y$. Hence $B \subseteq b\gamma^{-1}$. Let $c \in b\gamma^{-1}$. Then $c\alpha\psi = c\gamma = b$ from which it follows that $\{c\alpha\} = b\psi^{-1} = \{a\}$ since ψ is a bijection. Then $c\alpha = a$ implies $c \in a\alpha^{-1} = B$. So $b\gamma^{-1} \subseteq B$. We conclude that $B = b\gamma^{-1} \in \pi(\gamma)$. Therefore, $\pi(\alpha) \subseteq \pi(\gamma)$ and then $\pi(\gamma) = \pi(\alpha)$. Moreover, since $\phi : \pi(\gamma) \to \pi(\beta)$ is bijective $\tilde{E}^*$-admissible and $\gamma_* = \alpha_*\psi = \phi\beta_*$, we get $(\gamma, \beta) \in \mathscr{L}$ and $X\gamma = X\beta$ by Theorem 3.5. Since $\psi : X\alpha \to X\gamma$ is bijective E^*-preserving and $\gamma = \alpha\psi$, we obtain $(\alpha, \gamma) \in \mathscr{R}$ by Theorem 3.1. Therefore, $(\alpha, \beta) \in \mathscr{D}$. □

The above result extends Theorem 3.4 of [2]. In fact, we obtain an additional characterization of the $\mathscr{D}$-relation in the case when $X = Y$, as shown in the following corollary.

Corollary 3.10 *Let $\alpha, \beta \in T_E(X)$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{D}$.
(2) *There is an E^*-preserving bijection $\Phi : X\alpha \to X\beta$ such that for each $A \in X/E$, there exist $B, C \in X/E$ with*

$$A\beta \subseteq B\alpha\Phi \text{ and } A\alpha \subseteq C\beta\Phi^{-1}.$$

We remark that for condition (2) of Theorem 3.9 to be true, it suffices to check that Φ is injective since the surjectivity of Φ follows from the fact that for each $A \in X/E$, there exists $B \in X/E$ with

$$A\beta \subseteq (B \cap Y)\alpha\Phi.$$

Theorem 3.11 *For $\alpha \in T_E(X, Y)$, the following statements hold.*

(1) *If $\alpha \in T_E(X, Y) \setminus F_E$, then*

$$D_\alpha = \{\beta \in T_E(X, Y) \setminus F_E : \pi(\alpha) = \pi(\beta) \text{ and } E(\alpha) = E(\beta)\} = R_\alpha.$$

(2) *If $\alpha \in F_E$, then*

$$D_\alpha = \{\beta \in F_E : \beta \text{ satisfies (2) of Theorem 3.9}\}.$$

Proof

(1) Let $\alpha \in T_E(X,Y) \setminus F_E$ and let $\beta \in D_\alpha$. Then there exists $\gamma \in T_E(X,Y)$ such that $(\alpha,\gamma) \in \mathscr{R}$ and $(\gamma,\beta) \in \mathscr{L}$. Hence $\gamma \in T_E(X,Y) \setminus F_E$ such that $\pi(\alpha) = \pi(\gamma)$ and $E(\alpha) = E(\gamma)$. Moreover, we obtain $\beta = \gamma$ since $L_\gamma = \{\gamma\}$. Thus $\beta \in T_E(X,Y) \setminus F_E$ such that $\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$. The other containment is clear since $\mathscr{R} \subseteq \mathscr{D}$.

(2) Let $\alpha \in F_E$ and let $\beta \in D_\alpha$. Then there exists $\gamma \in T_E(X,Y)$ such that $(\alpha,\gamma) \in \mathscr{R}$ and $(\gamma,\beta) \in \mathscr{L}$. By Corollary 3.3 and Theorem 3.6, we obtain both γ and β belong to F_E. It is easy to see that β satisfies (2) of Theorem 3.9. The other containment is clear.

□

For each $x \in X$, denote the equivalence class containing x by $[x]$. Now, we characterize Green's $\mathscr{J}$-relation. First, we prove the following lemmas.

Lemma 3.12 *Let $\alpha,\beta \in T_E(X,Y)$. Then the following statements are equivalent.*

(1) *There is an E-preserving surjection $\Phi : X\alpha \to X\beta$ such that for each $A \in X/E$ there exists $B \in X/E$ with $A\beta \subseteq (B \cap Y)\alpha\Phi$.*

(2) *$\beta = \gamma\alpha\mu$ for some $\gamma,\mu \in T_E(X,Y)$.*

Proof

(1)⇒(2). Suppose that (1) holds. Note that for each $A \in X/E$, $(A \cap X\alpha)\Phi \subseteq B$ for some $B \in X/E$ since Φ is E-preserving. Furthermore, $(A \cap X\alpha)\Phi \subseteq X\beta \subseteq Y$ so that $(A \cap X\alpha)\Phi \subseteq B \cap Y$. Now, we define a function μ as follows. For each $A \in X/E$ such that $A \cap X\alpha \neq \emptyset$, $(A \cap X\alpha)\Phi \subseteq B \cap Y$ for some $B \in X/E$. Choose $b \in B \cap Y$ and define

$$x\mu = \begin{cases} b & \text{if } \ x \in A \setminus X\alpha; \\ x\Phi & \text{if } \ x \in A \cap X\alpha. \end{cases}$$

Let $x\mu = x\beta$ if $x \in \bigcup\{A \in X/E : A \cap X\alpha = \emptyset\}$. Clearly, $\mu \in T_E(X,Y)$. Next, we define a function γ as follows. Let $A \in X/E$. By the condition of Φ, there exists $B \in X/E$ such that $A\beta \subseteq (B \cap Y)\alpha\Phi$. Let $x \in A$. Then $x\beta = y\alpha\Phi$ for some $y \in B \cap Y$. Define $x\gamma = y$. It then follows that $\gamma \in T_E(X,Y)$ and $\beta = \gamma\alpha\mu$.

(2)⇒(1). Let $\beta = \gamma\alpha\mu$ for some $\gamma,\mu \in T_E(X,Y)$. Fix $y_0 \in X\beta$. If $[x] \cap X\gamma\alpha \neq \emptyset$, choose $x_0 \in [x] \cap X\gamma\alpha$ for each $x \in X\alpha$. Define a function $\Phi : X\alpha \to X\beta$ by

$$x\Phi = \begin{cases} x\mu & \text{if } \ x \in X\gamma\alpha; \\ x_0\mu & \text{if } \ x \notin X\gamma\alpha \text{ and } [x] \cap X\gamma\alpha \neq \emptyset; \\ y_0 & \text{if } \ [x] \cap X\gamma\alpha = \emptyset. \end{cases}$$

It is easy to verify that Φ is E-preserving. Let $A \in X/E$. There exists $B \in X/E$ such that $A\gamma \subseteq B \cap Y$. Therefore,

$$A\beta = A\gamma\alpha\mu = A\gamma\alpha\Phi \subseteq (B \cap Y)\alpha\Phi.$$

In addition, we obtain that Φ is surjective. □

As a direct consequence of Theorem 3.9, we obtain the following lemma.

Lemma 3.13 *Let $\alpha, \beta \in F_E$. If $(\alpha, \beta) \in \mathscr{D}$ in $T_E(X, Y)$, then there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with*

$$A\beta \subseteq (B \cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C \cap Y)\beta\Psi.$$

Theorem 3.14 *Let $\alpha, \beta \in T_E(X, Y)$. Then $(\alpha, \beta) \in \mathscr{J}$ if and only if either*

(1) *$\pi(\alpha) = \pi(\beta)$ and $E(\alpha) = E(\beta)$; or*
(2) *there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with*

$$A\beta \subseteq (B \cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C \cap Y)\beta\Psi.$$

Proof Assume that $(\alpha, \beta) \in \mathscr{J}$. Then there exist $\gamma, \delta, \lambda, \mu \in T_E(X, Y)^1$ such that $\alpha = \gamma\beta\delta$ and $\beta = \lambda\alpha\mu$. If $\gamma = 1 = \lambda$, then $\alpha = \beta\delta$ and $\beta = \alpha\mu$ which implies $(\alpha, \beta) \in \mathscr{R} \subseteq \mathscr{D}$. If $\delta = 1 = \mu$, then $\alpha = \gamma\beta$ and $\beta = \lambda\alpha$ which implies $(\alpha, \beta) \in \mathscr{L} \subseteq \mathscr{D}$. Hence (1) or (2) holds by Theorem 3.11 and Lemma 3.13.

If $\{\gamma, \lambda\} \neq \{1\}$ and $\{\delta, \mu\} \neq \{1\}$, then we have $\alpha = \eta\beta\zeta$ and $\beta = \rho\alpha\sigma$ for some $\eta, \zeta, \rho, \sigma \in T_E(X, Y)$. For example, if $\gamma = 1 = \mu$ and $\lambda, \delta \in T_E(X, Y)$, then

$$\alpha = \beta\delta = \lambda\alpha\delta \text{ and } \beta = \lambda\alpha = \lambda\beta\delta.$$

We have (2) holds by Lemma 3.12. The converse is clear by Theorem 3.1 and Lemma 3.12. □

By using Lemma 3.13 with the same proof as given in the above theorem, we obtain the following result.

Corollary 3.15 *Let $\alpha, \beta \in F_E$. Then $(\alpha, \beta) \in \mathscr{J}$ in $T_E(X, Y)$ if and only if there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with*

$$A\beta \subseteq (B \cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C \cap Y)\beta\Psi.$$

The above result leads to the following corollary which extends the result in [2].

Corollary 3.16 *Let $\alpha, \beta \in T_E(X)$. Then $(\alpha, \beta) \in \mathscr{J}$ if and only if there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with*

$$A\beta \subseteq B\alpha\Phi \text{ and } A\alpha \subseteq C\beta\Psi.$$

Next, we aim to characterize the $\mathscr{J}$-relation on the finite case.

Lemma 3.17 *Let $\alpha, \beta \in T_E(X, Y)$ be such that $\pi(\alpha)$ is finite. If $\theta : \pi(\alpha) \to \pi(\beta)$ and $\varphi : \pi(\beta) \to \pi(\alpha)$ are bijective $\tilde{E}$-admissible, then θ is $\tilde{E}^*$-admissible.*

Proof The proof of this lemma follows the same steps as in Lemma 3.7 of [2]. For the sake of completeness, we give the proof as follows. Note that $\theta\varphi$ is a bijection from $\pi(\alpha)$ onto $\pi(\alpha)$ which implies that $\theta\varphi$ is a permutation on $\pi(\alpha)$. Moreover, since $\pi(\alpha)$ is finite, we obtain $(\theta\varphi)^m$ is the identity permutation for some natural number m. We claim that $(\theta\varphi)^n$ is $\tilde{E}$-admissible for all natural number n. Indeed, let $A \in X/E$. There are $B, C \in X/E$ such that $\pi_A(\alpha)\theta \subseteq \tilde{\pi}_B(\beta)$ and $\pi_B(\beta)\varphi \subseteq \tilde{\pi}_C(\alpha)$ since $\theta : \pi(\alpha) \to \pi(\beta)$ and $\varphi : \pi(\beta) \to \pi(\alpha)$ are $\tilde{E}$-admissible. We have

$$\pi_A(\alpha)\theta\varphi \subseteq \tilde{\pi}_B(\beta)\varphi \subseteq \pi_B(\beta)\varphi \subseteq \tilde{\pi}_C(\alpha).$$

Thus $\theta\varphi$ is $\tilde{E}$-admissible. By induction, we conclude that $(\theta\varphi)^n$ is $\tilde{E}$-admissible for all natural number n.

Since $(\theta\varphi)^m$ is the identity permutation on $\pi(\alpha)$, we obtain $(\theta\varphi)^m = \theta\varphi(\theta\varphi)^{m-1}$ which implies that $\theta^{-1} = \varphi(\theta\varphi)^{m-1}$. Moreover, since φ and $(\theta\varphi)^{m-1}$ are $\tilde{E}$-admissible, by the same argument as above, we can show that θ^{-1} is $\tilde{E}$-admissible. Therefore, θ is $\tilde{E}^*$-admissible. □

Lemma 3.18 *Let $U, V \subseteq X$ be finite. If $\alpha : U \to V$ and $\beta : V \to U$ are surjective E-preserving, then α and β are bijective E^*-preserving.*

Proof It is clear that α and β are bijective since U and V are finite sets. Let $U/E_U = \{A_1, A_2, \dots, A_m\}$ and $V/E_V = \{B_1, B_2, \dots, B_n\}$ be the sets of all distinct equivalence classes of U and V, respectively. For each class B_i, we see that $B_i\alpha^{-1}$ is non-empty since α is surjective. Note that $B_i\alpha^{-1}$ is a union of some classes in U/E_U. Let k_i be the number of classes which are contained in $B_i\alpha^{-1}$. It is not difficult to see that $k_i \geq 1$ for all i and $k_1 + k_2 + \cdots + k_n = m$. Hence $m \geq n$. Similarly, we can show that $n \geq m$ by using β and so $m = n$. Moreover, we have $k_1 + k_2 + \cdots + k_n = n$ which implies that $k_1 = k_2 = \cdots = k_n = 1$ and hence $U/E_U = \{B_1\alpha^{-1}, B_2\alpha^{-1}, \dots, B_n\alpha^{-1}\}$. Now, we show that α is E^*-preserving. Let $(x\alpha, y\alpha) \in E$. Then $x\alpha, y\alpha \in B$ for some class $B \in V/E_V$ from which it follows that $(x\alpha)\alpha^{-1}, (y\alpha)\alpha^{-1} \subseteq B\alpha^{-1}$. Thus $x, y \in B\alpha^{-1} \in U/E_U$ and so $(x, y) \in E$. Similarly, we can show that β is also E^* -preserving. □

From the above lemma, we obtain the following corollary.

Corollary 3.19 *If Z is a finite set and $\alpha : Z \to Z$ is bijective E-preserving, then α is E^*-preserving.*

Now, we have the following result which covers Theorem 3.8 of [2].

Theorem 3.20 *If $\alpha, \beta \in F_E$ such that both $X\alpha$ and $X\beta$ are finite, then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathscr{D}$ *in* $T_E(X, Y)$.
(2) $(\alpha, \beta) \in \mathscr{J}$ *in* $T_E(X, Y)$.
(3) *There are E^*-preserving bijections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with*

$$A\beta \subseteq (B \cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C \cap Y)\beta\Psi.$$

Proof The implication (1)⇒(2) is trivial.

(2)⇒(3). Assume that $(\alpha, \beta) \in \mathscr{J}$ in $T_E(X, Y)$. Then, by Corollary 3.15, there are E-preserving surjections $\Phi : X\alpha \to X\beta$ and $\Psi : X\beta \to X\alpha$ such that for each $A \in X/E$ there exist $B, C \in X/E$ with

$$A\beta \subseteq (B \cap Y)\alpha\Phi \text{ and } A\alpha \subseteq (C \cap Y)\beta\Psi.$$

By Lemma 3.18, we have Φ and Ψ are bijective E^*-preserving since $X\alpha$ and $X\beta$ are finite.

(3)⇒(1). Define $\theta : \pi(\alpha) \to \pi(\beta)$ by $(x\alpha)\alpha^{-1}\theta = (x\alpha)\Psi^{-1}\beta^{-1}$ for all $x \in X$. Then θ is a well-defined bijection such that $\theta\beta_* = \alpha_*\Psi^{-1}$. We only need to check that θ is $\tilde{E}^*$-admissible. Let $A \in X/E$. Then there is $C \in X/E$ such that $A\alpha \subseteq (C \cap Y)\beta\Psi$ by assumption. Let $P = (x\alpha)\alpha^{-1} \in \pi_A(\alpha)$. Then $P \cap A \neq \emptyset$ which implies that there exists $a \in P \cap A$. We obtain $x\alpha = a\alpha \in (P \cap A)\alpha \subseteq A\alpha \subseteq (C \cap Y)\beta\Psi$ from which it follows that $x\alpha = c\beta\Psi$ for some $c \in C \cap Y$. Hence

$$c \in (x\alpha)\Psi^{-1}\beta^{-1} \cap C \cap Y = (x\alpha)\alpha^{-1}\theta \cap C \cap Y = P\theta \cap C \cap Y \neq \emptyset.$$

Thus θ is bijective and $\tilde{E}$-admissible. On the other hand, define $\tau : \pi(\beta) \to \pi(\alpha)$ by $(x\beta)\beta^{-1}\tau = (x\beta)\Phi^{-1}\alpha^{-1}$ for all $x \in X$. Similarly, we can show that τ is also bijective and $\tilde{E}$-admissible. Hence θ is $\tilde{E}^*$-admissible by Lemma 3.17. Therefore, $(\alpha, \beta) \in \mathscr{D}$ by Theorem 3.9. □

Finally, we determine Green's relations on the regular elements in $T_E(X, Y)$.

Theorem 3.21 *Let α, β be regular elements in $T_E(X, Y)$. If there exists a bijective E^*-preserving $\psi : X\alpha \to X\beta$, then there is a bijective $\tilde{E}^*$-admissible $\phi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_*\psi = \phi\beta_*$.*

Proof The proof of this theorem is a slight modification of the proof of Theorem 3.12 of [2]. Actually, we define $\phi : \pi(\alpha) \to \pi(\beta)$ by

$$P\phi = (P\alpha_*\psi)\beta_*^{-1} \text{ for all } P \in \pi(\alpha).$$

Since α_*, ψ and β_* are bijective, we obtain ϕ is also bijective. Moreover, we can see that $\alpha_*\psi = \phi\beta_*$. Now, we show that ϕ is $\tilde{E}^*$-admissible. Let $A \in X/E$ and

$A\alpha = B'$. Then $A\alpha = B' \subseteq B$ for some $B \in X/E$ by Lemma 1.1. We see that $B'\psi = C' \subseteq C$ for some $C \in X/E$ since ψ is E^*-preserving. Moreover, since β is regular, we obtain $C' \subseteq C \cap X\beta \subseteq (D \cap Y)\beta$ for some $D \in X/E$ by Theorem 2.1. We claim that $\pi_A(\alpha)\phi \subseteq \tilde{\pi}_D(\beta)$. Indeed, let $P \in \pi_A(\alpha)$. Then $P\alpha_* \in A\alpha = B'$. Hence $P\alpha_*\psi \in C' \subseteq (D \cap Y)\beta$ since $B'\psi = C'$ and $C' \subseteq (D \cap Y)\beta$. Hence $P\phi \cap D \cap Y = P\alpha_*\psi\beta^{-1} \cap D \cap Y \neq \emptyset$ which implies that $P\phi \in \tilde{\pi}_D(\beta)$. We conclude that ϕ is $\tilde{E}$-admissible. Similarly, we have ϕ^{-1} is also $\tilde{E}$-admissible since ϕ is bijective and $\phi^{-1} = \beta_*\psi^{-1}\alpha_*^{-1}$. Therefore, ϕ is $\tilde{E}^*$-admissible. □

Recall that the set of all regular elements in $T_E(X, Y)$ is contained in F_E. Then we get the following theorem.

Theorem 3.22 *Let α, β be regular elements in $T_E(X, Y)$. Then the following statements hold.*

(1) $(\alpha, \beta) \in \mathscr{R}$ *if and only if* $\pi(\alpha) = \pi(\beta)$.
(2) $(\alpha, \beta) \in \mathscr{L}$ *if and only if* $X\alpha = X\beta$.
(3) $(\alpha, \beta) \in \mathscr{H}$ *if and only if* $\pi(\alpha) = \pi(\beta)$ *and* $X\alpha = X\beta$.
(4) $(\alpha, \beta) \in \mathscr{D}$ *if and only if there exists a bijective E^*-preserving* $\psi : X\alpha \to X\beta$.

The proof of this theorem follows from Theorems 2.5 and 3.21 so it is omitted. Moreover, if $X = Y$, then Theorem 3.13 of [2] is true.

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