รายงานวิจัยฉบับสมบูรณ์

The iterations for nonexpansive semigroup mapping in Banach spaces

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มิถุนายน 2560

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สนับสนุนโดยสำนักงานกองทุนสนับสนุนการวิจัยและ
ม.เทคโนโลยีราชมงคลล้านนา ตาก

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บทคัดย่อ: งานวิจัยนี้ทำขึ้นเพื่อสร้างขั้นตอนวิธีการทำซ้ำ สำหรับการประมาณค่าจุดตรึงของการส่งกึ่งกลุ่มไม่ขยาย ในปริภูมิบานาค ด้วยการพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มภายใต้เงื่อนไขที่เหมาะสม นอกจากนี้ผู้ทำวิจัยมีแผนที่จะสร้างขั้นตอนวิธีการทำซ้ำเพื่อประมาณค่าคำตอบของปัญหาระบบเชิงดุลยภาพผสมทั่วไป หรือปัญหาอสมการการแปรผัน อีกทั้ง ผู้วิจัยยังมีแผนที่จะศึกษาทฤษฎีบทจุดตรึงที่มีความสัมพันธ์กับปัญหาดังกล่าวข้างต้นและหัวข้อที่น่าสนใจอื่น ๆ

คำหลัก: จุดตรึง ปริภูมิบานาค กึ่งกลุ่มไม่ขยาย กระบวนการทำซ้ำ

Abstract

Project Code : TRG5880203

Project Title : The iterations for nonexpansive semigroup mapping in Banach spaces

Investigator : Phayap Katchang

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Project Period : Two years

Abstract : The purpose of this project is to introduce new iterative algorithms for approximating a fixed point of nonexpansive semigroup mapping in Banach spaces which is proved the strong convergence theorems under some suitable conditions. All in all, we plan to construct the algorithms to approximate the set of solutions of the system of generalized mixed equilibrium problems or the set of solutions of the system of variational inequality problems. Furthermore, we also plan to study the relationships between the above problem and an interesting topic, as fixed point theory.

Keywords : Fixed point, Banach space, Nonexpansive semigroup, Iteration.

วัตถุประสงค์ : สร้างทฤษฎีบทระเบียบวิธีการทำซ้ำขึ้นใหม่ โดยพิสูจน์การลู่เข้าแบบเข้ม (Strong convergence) เพื่อหาจุดตรึงของการส่งกึ่งกลุ่มไม่ขยายในปริภูมิบานาค เพื่อการตีพิมพ์ในวารสารระดับนานาชาติที่อยู่ในฐานข้อมูล Science Citation Index (SCI) ของ ISI Web of Science Q1-Q3 จำนวน 2 บทความ
หรือ Q1 จำนวน 1 บทความ ในเวลา 2 ปี

ระเบียบวิธีวิจัย :

1. ค้นคว้าหาเอกสารที่เกี่ยวข้องกับเงื่อนไขที่จำเป็นของปัญหาจุดตรึงสำหรับการส่งกึ่งกลุ่มไม่ขยาย
2. สร้างและพิสูจน์ทฤษฎีใหม่ของปัญหาจุดตรึงสำหรับการส่งกึ่งกลุ่มไม่ขยาย
3. พิมพ์และส่งผลงานวิจัยตีพิมพ์ในวารสารนานาชาติ
4. สรุปผลโครงการ

ผลการวิจัย :

Paper 1

In this research, we focus on two main problems, the first one is a fixed point problem of a nonexpansive semigroup and the other is a variational inequality problem for an inverse strongly accretive mapping. Passing through the modified Mann iterative method, we propose the new iterative scheme to find the common elements solving our mentioned problems. Furthermore, we aim to obtain some strong convergence theorems under certain appropriate conditions in the q-uniformly smooth Banach spaces. Our results improve and extend resulting outcomes in the literature.

Paper 2

In this research, we focus on a common fixed point problem of a nonexpansive semigroup with the generalized viscosity methods for implicit iterative algorithms. Our main objective is to construct the new strong convergence theorems under certain appropriate conditions in uniformly convex and uniformly smooth Banach spaces. Specifically, the main results make a contribution to the implicit midpoint theorems. The findings for theorems in Hilbert spaces and the other forms of a nonexpansive semigroup can be used in several practical purposes. Finally, a numerical example in 3 dimensions is provided to support our main results.

สรุปผลการวิจัย :

Paper 1

***Theorem* 1.** *Let C be a sunny nonexpansive retract and nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let Q_C be a sunny nonexpansive retraction from E onto C, $A : C \to E$ be an β-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from C into itself, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a κ-Lipschitzian and η-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$, $\{\mu_n\} \subset (0,\infty)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ where c_q is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$ and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S(\mu_n)y_n, \end{cases} \tag{1}$$

which satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty;$ *and* $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$*;*

(C2) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0, \liminf_{n\to\infty}\lambda_n > 0$*;*

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$*;*

(C4) $\lim_{n\to\infty}\mu_n = 0$*;*

(C5) $\lim_{n\to\infty}\sup_{x\in\tilde{C}}\|S(\mu_{n+1})x - S(\mu_n)x\| = 0$, *$\tilde{C}$ bounded subset of C;*

(C6) $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0, 0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1.$

Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F. \tag{2}$$

Paper 2

***Theorem* 2.** *Let C be a nonempty closed convex subset of uniformly convex and uniformly smooth Banach space E. Let f be a generalized contraction mapping from C into itself and $\mathcal{S} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup from C into itself such that $F(\mathcal{S}) \neq 0$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = \lambda_n x_n + (1-\lambda_n)x_{n+1}, \\ y_n = \delta_n f(x_n) + (1-\delta_n)z_n, \\ x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(\mu_n)y_n, \end{cases} \tag{3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\lambda_n\}$ are the sequences in $(0,1)$. The following conditions are satisfied:

(i) $\alpha_n + \beta_n + \gamma_n = 1$*;*

(ii) $\lim\limits_{n\to\infty}\beta_n = \lim\limits_{n\to\infty}\delta_n = \lim\limits_{n\to\infty}\mu_n = \lim\limits_{n\to\infty}|\alpha_{n+1} - \alpha_n| = \lim\limits_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$ *and* $\lim\limits_{n\to\infty}\sup\limits_{x\in\tilde{C}}\|T(\mu_{n+1})x - T(\mu_n)x\| = 0$, *$\tilde{C}$ bounded subset of C;*

(iii) $0 < \liminf\limits_{n\to\infty} \alpha_n \le \limsup\limits_{n\to\infty} \alpha_n < 1$;

(iv) $\sum_{n=0}^{\infty} \beta_n = \infty$ *for all* $n \ge 1$.

Then $\{x_n\}$ *converges strongly to* $x^* \in F(\mathcal{S})$ *which also solves the following variational inequality:*

$$\langle (I - f)x^*, J(z - x^*)\rangle \le 0, \forall z \in F(\mathcal{S}). \quad (4)$$

Output : Paper 1

ON SOLVING THE VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS IN q-UNIFORMLY SMOOTH BANACH SPACES

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Abstract. In this research, we focus on two main problems, the first one is a fixed point problem of a nonexpansive semigroup and the other is a variational inequality problem for an inverse strongly accretive mapping. Passing through the modified Mann iterative method, we propose the new iterative scheme to find the common elements solving our mentioned problems. Furthermore, we aim to obtain some strong convergence theorems under certain appropriate conditions in the q-uniformly smooth Banach spaces. Our results improve and extend resulting outcomes in the literature.

Keywords: Banach space, Fixed point, Inverse-strongly accretive mapping, Nonexpansive semigroup, q-uniformly smooth, Variational inequality
2000 Mathematics Subject Classification: 46B80, 47H06, 47H09, 47H10, 47H20, 47J20.

1. Introduction

According to our framework throughout this research, we first preview some definitions involving a Banach space E as follows. Let $U = \{x \in E : \|x\| = 1\}$.

- E is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$.
  It is known that a uniformly convex Banach space is reflexive and strictly convex.
- E is said to be *smooth* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$.
  It is also said to be *uniformly smooth* if the limit is attained uniformly for all $x, y \in U$. The *modulus of smoothness* of E is defined by
  $$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\right\},$$
  where $\rho : [0, \infty) \to [0, \infty)$ is a function.
  It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$.
- E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$ where q is a fixed real number with $1 < q \leq 2$.

Let E be a real Banach space and E^* be the dual space of E with norm $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ pairing between E and E^*. For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by
$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$
for all $x \in E$. In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and written by $J_2 = J$ as usual. Further, we have the following properties of the generalized duality mapping J_q:

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(i) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$;
(ii) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$;
(iii) $J_q(-x) = -J_q(x)$ for all $x \in E$.
Certainly, if E is smooth, then J_q is single-valued and can be written by j_q (see also [1, 2]).

Let C be a nonempty closed convex subset of a real Banach space E. Recall that a mapping $A : C \to C$ is said to be
(i) *Lipschitzian* with Lipschitz constant $L > 0$ if $\|Ax - Ay\| \le L\|x - y\|, \quad \forall x, y \in C$;
(ii) *nonexpansive* if $\|Ax - Ay\| \le \|x - y\|, \quad \forall x, y \in C$.

An operator $A : C \to E$ is said to be
(i) *accretive* if there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y)\rangle \ge 0, \quad \forall x, y \in C;$$

(ii) *β-strongly accretive* if for any $\beta > 0$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Ax - Ay, j_q(x - y)\rangle \ge \beta\|x - y\|^q, \quad \forall x, y \in C;$$

(iii) *β-inverse strongly accretive* if, for any $\beta > 0$ there exists $j_q(x - y) \in J_q(x - y)$,

$$\langle Ax - Ay, j_q(x - y)\rangle \ge \beta\|Ax - Ay\|^q, \quad \forall x, y \in C.$$

Let D be a subset of C and $Q : C \to D$. Then Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$, whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \ge 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D (see [3, 4, 5]). A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z are in the range of Q.

A family $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;
(ii) $S(s + t) = S(s)S(t)$ for all $s, t \ge 0$;
(iii) $\|S(s)x - S(s)y\| \le \|x - y\|$ for all $x, y \in C$ and $s \ge 0$;
(iv) for each $x \in C$, the mapping $S(\cdot)x$ from $[0, \infty)$ into C is continuous.

Let $F(\mathcal{S})$ stands for the common fixed point set of the semigroup $\mathcal{S}$, i.e., $F(\mathcal{S}) = \{x \in C : S(s)x = x, \forall s > 0\}$. It is easy to see that $F(\mathcal{S})$ is closed and convex (see also [6, 7, 8, 9]).

In 1969, Takahashi [10] proved the first fixed point theorem for a noncommutative semigroup of nonexpansive mappings which generalizes De Marr's [11] fixed point theorem. For works related to semigroups of nonexpansive, asymptotically nonexpansive, and asymptotically nonexpansive type related to reversibility of a semigroup, we refer the reader to [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. In 2007, Lau et al. [18] introduced the following Mann's explicit iteration process;

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \quad \forall n \ge 1,$$

for a semigroup $\mathcal{S} = \{T(s) : s \in S\}$ of nonexpansive mappings on a compact convex subset C of a smooth and strictly convex Banach space. In 2012, Wangkeeree and Preechasilp [24] introduced the iterative scheme:

$$\begin{cases} x_1 \in C, \\ z_n = \gamma_n x_n + (1 - \gamma_n)T(t_n)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)T(t_n)z_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, n \ge 0. \end{cases}$$

They proved the strong convergence theorems by using a nonexpansive semigroup in Banach spaces.

In 2006, Aoyama et al. [25] proved a weak convergence theorem in Banach spaces by using the iterative algorithm as the following

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n), \end{cases}$$

for all $n \ge 1$. They solved the *generalized variational inequality problem* for finding a point $x \in C$ such that

$$\langle Ax, J(y - x)\rangle \ge 0 \tag{1.1}$$

for all $y \in C$. The solution set of (1.1) is denoted by $VI(C, A)$. Variational inequality has become a rich of inspiration in pure and applied mathematics. Recently, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences and have witnessed an explosive growth in theoretical advances, algorithmic development, etc; see e.g. [26, 27, 28].

In 2013, Song and Ceng [29] proved a strong convergence theorem in a q-uniformly smooth Banach space as the following:

$$\begin{cases} x_1 \in C, \\ z_n = Q_C(x_n - \sigma B x_n), \\ k_n = Q_C(z_n - \lambda A z_n), \\ y_n = \beta_n k_n + (1-\alpha_n)x_n, \\ x_{n+1} = Q_C[\alpha_n \gamma f x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu V)T_n y_n], n \geq 0. \end{cases} \tag{1.2}$$

They introduced a general iterative algorithm for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and the solution set of systems of variational inequalities.

Motivated and inspired by Wangkeeree and Preechasilp [24] and Song and Ceng [29]. In this paper, we introduce a new iterative scheme for finding common solutions of a variational inequality for an inverse-strongly accretive mapping and the solutions of a fixed point problem for a nonexpansive semigroup by using the modified Mann iterative method. We shall prove the strong convergence theorem in a q-uniformly smooth Banach spaces under some parameters controlling conditions. Our results extend and improve the recent results of Aoyama et al. [25], Wangkeeree and Preechasilp [24], Song and Ceng [29] and other authors.

2. Preliminaries

A Banach space E is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x (n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \forall y \in E \quad \text{with} \quad x \neq y.$$

By [30, Theorem 1], it is well known that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition, and E is smooth.

We need the following lemmas for proving our main results.

Proposition 2.1. ([3]) *Let E be a smooth banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;
(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.
If J_q is the generalized duality mapping on E then $\langle x - Qx, J_q(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$ is equivalent to this Proposition (see [29]).

Proposition 2.2. ([4, 5, 31]) *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Lemma 2.3. ([25]) *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$VI(C, A) = F(Q(I - \lambda A)),$$

where $VI(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \forall x \in C\}$.

Lemma 2.4. ([32])*Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ then x is a fixed point of T.*

Lemma 2.5. ([33]) *Let $r > 0$ and let E be a uniformly convex Banach space. Then, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|)$$

for all $x, y \in B_r := \{z \in E : \|z\| \leq r\}$ and $0 \leq \lambda \leq 1$.

Lemma 2.6. ([34]) *Let E be a real smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \leq \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \forall x, y \in B_r,$$

where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.7. ([33]) *Let E be a real q-uniformly smooth Banach space, then there exists a constant $c_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q, \forall x, y \in E.$$

In particular, if E is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2K\|y\|^2, \forall x, y \in E.$$

Lemma 2.8. ([35]) *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$ with $x \neq y$.

Lemma 2.9. ([36]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.10. ([37]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(2) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.11. ([38, 39]) *Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space E, $L_2 : C \to E$ be a κ-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$ and let $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$, $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$, then for $t \in (0, \min\{1, \frac{1}{\tau}\})$, the mapping $S : C \to E$ defined by $S := (I - t\mu L_2)$ is a contraction with a constant $1 - t\tau$.*

Lemma 2.12. ([29]) *Let C be a nonempty, closed and convex subset of a real reflexive and q-uniformly smooth Banach space E which admits a weakly sequentially continuous generalized duality mapping J_q from E into E^*. Let Q_C be a sunny nonexpansive retraction from E onto C, $V : C \to E$ a k-Lipschitzian and η-strongly accretive operator with constants k, $\eta > 0$. Suppose $f : C \to E$ is a L-Lipschitzian mapping with constant $L > 0$ and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ and $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$. Then $\{x_t\}$ defined by $x_t = Q_C[t\gamma f x_t + (I - t\mu V)Tx_t]$ converges strongly to some point $x^* \in F(T)$ as $t \to 0$, which is the unique solution of the variational inequality:*

$$\langle \gamma f x^* - \mu V x^*, J_q(p - x^*)\rangle \leq 0, \forall p \in F(T).$$

Lemma 2.13. ([29]) *Let C be a closed convex subset of a smooth Banach space E. Let $\tilde{C}$ be a nonempty subset of C. Let $Q : C \to \tilde{C}$ be a retraction and let J, J_q be the normalized duality mapping and generalized duality mapping on E, respectively. Then the following are equivalent:*
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;
(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in C, y \in \tilde{C}$;
(iv) $\langle x - Qx, J_q(y - Qx)\rangle \leq 0, \forall x \in C, y \in \tilde{C}$.

Lemma 2.14. ([41]) *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

3. Main results

Theorem 3.1. *Let C be a sunny nonexpansive retract and nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let Q_C be a sunny nonexpansive retraction from E onto C, $A : C \to E$ be an β-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from C into itself, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a κ-Lipschitzian and η-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$, $\{\mu_n\} \subset (0,\infty)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ where c_q is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$ and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S(\mu_n)y_n, \end{cases} \tag{3.1}$$

which satisfy the following conditions:

- (C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$; *and* $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$;
- (C2) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0, \liminf_{n\to\infty}\lambda_n > 0$;
- (C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;
- (C4) $\lim_{n\to\infty}\mu_n = 0$;
- (C5) $\lim_{n\to\infty}\sup_{x\in\tilde{C}}\|S(\mu_{n+1})x - S(\mu_n)x\| = 0$, *$\tilde{C}$ bounded subset of C;*
- (C6) $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0, 0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$.

Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F. \tag{3.2}$$

Proof. First of all, we prove that $\{x_n\}$ is bounded. Let $p \in F$ and $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, we have

$$\begin{aligned} \|z_n - p\|^q &= \|Q_C(x_n - \lambda_n A x_n) - Q_C(p - \lambda_n A p)\|^q \\ &\leq \|(I - \lambda_n A)x_n - (I - \lambda_n A)p\|^q \\ &= \|(x_n - p) - \lambda_n(Ax_n - Ap)\|^q \\ &\leq \|x_n - p\|^q - q\lambda_n\langle Ax_n - Ap, j_q(x_n - p)\rangle + c_q\lambda_n^q\|Ax_n - Ap\|^q \\ &\leq \|x_n - p\|^q - q\beta\lambda_n\|Ax_n - Ap\|^2 + c_q\lambda_n^q\|Ax_n - Ap\|^q \\ &= \|x_n - p\|^q - \lambda_n(q\beta - c_q\lambda_n^{q-1})\|Ax_n - Ap\|^q \\ &\leq \|x_n - p\|^q. \end{aligned} \tag{3.3}$$

Therefore $\|z_n - p\| \leq \|x_n - p\|$ and $I - \lambda_n A$ is a nonexpansive where I is an identity mapping. By condition $(C1)$, we may assume, without loss of generality, that $\alpha_n < \min\{\alpha, \frac{\alpha}{\tau}\}$ where $0 < \alpha < \liminf_{n\to\infty}(1-\gamma_n)$. From Lemma 2.11, we conclude that $\|(1-\gamma_n)I - \alpha_n\mu L_2\| \leq (1-\gamma_n) - \alpha_n\tau$. Since $0 \leq \gamma L < \tau$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(S(\mu_n)y_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\ &= \beta_n\|x_n - p\| + (1-\beta_n)\big\|Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n \\ &\quad +((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - p\big\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|[(1-\gamma_n)I - \alpha_n\mu L_2][S(\mu_n)z_n - p] \\ &\quad +\alpha_n(\gamma L_1 x_n - \mu L_2 p) + \gamma_n(x_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)(1-\gamma_n-\alpha_n\tau)\|S(\mu_n)z_n - p\| \\ &\quad +(1-\beta_n)\alpha_n\|\gamma L_1 x_n - \mu L_2 p\| + (1-\beta_n)\gamma_n\|x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)(1-\gamma_n-\alpha_n\tau)\|x_n - p\| \\ &\quad +(1-\beta_n)\alpha_n\gamma\|L_1 x_n - L_1 p\| + (1-\beta_n)\alpha_n\|\gamma L_1 p - \mu L_2 p\| \\ &\quad +(1-\beta_n)\gamma_n\|x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| - (1-\beta_n)\gamma_n\|x_n - p\| \end{aligned}$$

$$
\begin{aligned}
&-(1-\beta_n)\alpha_n\tau\|x_n-p\|+(1-\beta_n)\alpha_n\gamma L\|x_n-p\|\\
&+(1-\beta_n)\alpha_n\|\gamma L_1p-\mu L_2p\|+(1-\beta_n)\gamma_n\|x_n-p\|\\
= &\ \|x_n-p\|-(1-\beta_n)\alpha_n\tau\|x_n-p\|\\
&+(1-\beta_n)\alpha_n\gamma L\|x_n-p\|+(1-\beta_n)\alpha_n\|\gamma L_1p-\mu L_2p\|\\
= &\ \|x_n-p\|-(1-\beta_n)\alpha_n(\tau-\gamma L)\|x_n-p\|\\
&+(1-\beta_n)\alpha_n(\tau-\gamma L)\frac{\|\gamma L_1p-\mu L_2p\|}{\tau-\gamma L}.
\end{aligned}
$$

By induction, we conclude that

$$\|x_n-p\|\le \max\big\{\|x_1-p\|,\frac{\|\gamma L_1p-\mu L_2p\|}{\tau-\gamma L}\big\},\forall n\ge 1.$$

This implies that $\{x_n\}$ is bounded, so are $\{Ax_n\}$, $\{y_n\}$, $\{S(\mu_n)y_n\}$, $\{z_n\}$ and $\{S(\mu_n)z_n\}$.

Next, we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and we observe that

$$
\begin{aligned}
\|z_{n+1}-z_n\| &= \|Q_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})-Q_C(x_n-\lambda_nAx_n)\|\\
&\le \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_nAx_n)\|\\
&= \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_{n+1}Ax_n)+(\lambda_n-\lambda_{n+1})Ax_n\|\\
&\le \|(I-\lambda_{n+1}A)x_{n+1}-(I-\lambda_{n+1}A)x_n\|+|\lambda_{n+1}-\lambda_n\|Ax_n\|\\
&\le \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|,
\end{aligned}
$$

$$
\begin{aligned}
\|S(\mu_{n+1})z_{n+1}-S(\mu_n)z_n\| &\le \|S(\mu_{n+1})z_{n+1}-S(\mu_{n+1})z_n\|\\
&\quad+\|S(\mu_{n+1})z_n-S(\mu_n)z_n\|\\
&\le \|z_{n+1}-z_n\|+\|S(\mu_{n+1})z_n-S(\mu_n)z_n\|\\
&\le \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|\\
&\quad+\sup_{z\in\{z_n\}}\|S(\mu_{n+1})z-S(\mu_n)z\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \big\|Q_C\big[\alpha_{n+1}\gamma L_1x_{n+1}+\gamma_{n+1}x_{n+1}\\
&\quad+((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big]\\
&\quad-Q_C\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]\big\|\\
&\le \big\|\big[\alpha_{n+1}\gamma L_1x_{n+1}+\gamma_{n+1}x_{n+1}\\
&\quad+((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big]\\
&\quad-\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]\big\|\\
&= \big\|\big[\alpha_{n+1}\gamma L_1x_{n+1}+\gamma_{n+1}x_{n+1}\\
&\quad+((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_{n+1})z_{n+1}\big]\\
&\quad-\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]\\
&\quad+\alpha_{n+1}\gamma L_1x_n-\alpha_{n+1}\gamma L_1x_n+\gamma_{n+1}x_n-\gamma_{n+1}x_n\\
&\quad+((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_n)z_n\\
&\quad-((1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2)S(\mu_n)z_n\big\|\\
&\le \alpha_{n+1}\gamma\|L_1x_{n+1}-L_1x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|\\
&\quad+\big\|\big[(1-\gamma_{n+1})I-\alpha_{n+1}\mu L_2\big]\big[S(\mu_{n+1})z_{n+1}-S(\mu_n)z_n\big]\big\|\\
&\quad+|\alpha_{n+1}-\alpha_n|\gamma\|L_1x_n\|+|\alpha_{n+1}-\alpha_n|\mu\|L_2S(\mu_n)z_n\|\\
&\quad+|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\|\\
&\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\|+\gamma_{n+1}\|x_{n+1}-x_n\|\\
&\quad+[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\|S(\mu_{n+1})z_{n+1}-S(\mu_n)z_n\|\\
&\quad+|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\|+\mu\|L_2S(\mu_n)z_n\|\big]\\
&\quad+|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n-x_n\|
\end{aligned}
$$

$$\begin{aligned}
&\le \alpha_{n+1}\gamma L\|x_{n+1}-x_n\| + \gamma_{n+1}\|x_{n+1}-x_n\| \\
&\quad +[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\big[\|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad + \sup_{z\in\{z_n\}}\|S(\mu_{n+1})z - S(\mu_n)z\|\big] \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\| + \mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n - x_n\| \\
&= [1-\alpha_{n+1}(\tau-\gamma L)]\|x_{n+1}-x_n\| \\
&\quad +[(1-\gamma_{n+1})I-\alpha_{n+1}\tau]\big[|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad + \sup_{z\in\{z_n\}}\|S(\mu_{n+1})z - S(\mu_n)z\|\big] \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\| + \mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n - x_n\| \\
&\le \|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\
&\quad + \sup_{z\in\{z_n\}}\|S(\mu_{n+1})z - S(\mu_n)z\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\big[\gamma\|L_1x_n\| + \mu\|L_2S(\mu_n)z_n\|\big] \\
&\quad +|\gamma_{n+1}-\gamma_n|\|S(\mu_n)z_n - x_n\| \\
&\le \|x_{n+1}-x_n\| + \big[|\alpha_{n+1}-\alpha_n| + |\gamma_{n+1}-\gamma_n| + |\lambda_{n+1}-\lambda_n|\big]M \\
&\quad + \sup_{z\in\{z_n\}}\|S(\mu_{n+1})z - S(\mu_n)z\|,
\end{aligned}$$

where $M = \sup_{n\ge 0}\big\{\|Ax_n\|, \gamma\|L_1x_n\| + \mu\|L_2S(\mu_n)z_n\|, \|S(\mu_n)z_n - x_n\|\big\} < \infty$. It follows that

$$\begin{aligned}
\|S(\mu_{n+1})y_{n+1} - S(\mu_n)y_n\| &\le \|S(\mu_{n+1})y_{n+1} - S(\mu_{n+1})y_n\| + \|S(\mu_{n+1})y_n - S(\mu_n)y_n\| \\
&\le \|y_{n+1}-y_n\| + \|S(\mu_{n+1})y_n - S(\mu_n)y_n\| \\
&\le \|x_{n+1}-x_n\| + \big[|\alpha_{n+1}-\alpha_n| + |\gamma_{n+1}-\gamma_n| + |\lambda_{n+1}-\lambda_n|\big]M \\
&\quad + \sup_{z\in\{z_n\}}\|S(\mu_{n+1})z - S(\mu_n)z\| \\
&\quad + \sup_{y\in\{y_n\}}\|S(\mu_{n+1})y - S(\mu_n)y\|. \qquad (3.4)
\end{aligned}$$

Form the condition (C1), (C2), (C5)-(C6) and 3.4, we have

$$\limsup_{n\to\infty}\big(\|S(\mu_{n+1})y_{n+1} - S(\mu_n)y_n\| - \|x_{n+1}-x_n\|\big) \le 0.$$

Applying Lemma 2.9, we obtain

$$\lim_{n\to\infty}\|S(\mu_n)y_n - x_n\| = 0.$$

Therefore, we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \qquad (3.5)$$

Next, we will show that $\lim_{n\to\infty}\|x_n - S(\mu_n)x_n\| = 0$, by the convexity of $\|\cdot\|^q$ for all $q>1$, Lemma 2.7 and (3.3), we have

$$\begin{aligned}
\|y_n - p\|^q &= \|Q_C\big[\alpha_n\gamma L_1x_n + \gamma_nx_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - p\|^q \\
&\le \|\gamma_n(x_n-p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big) + \alpha_n\big(\gamma L_1x_n - \mu L_2S(\mu_n)z_n\big)\|^q \\
&\le \|\gamma_n(x_n-p) + (1-\gamma_n)\big(S(\mu_n)z_n - p\big)\|^q \\
&\quad +q\big\langle \alpha_n\big(\gamma L_1x_n - \mu L_2S(\mu_n)z_n\big), J_q\big(\gamma_n(x_n-p) \\
&\quad +(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\big)\big\rangle \\
&\quad +c_q\|\alpha_n\big(\gamma L_1x_n - \mu L_2S(\mu_n)z_n\big)\|^q \\
&\le \gamma_n\|x_n-p\|^q + (1-\gamma_n)\|S(\mu_n)z_n - p\|^q \\
&\quad +q\alpha_n\|\gamma L_1x_n - \mu L_2S(\mu_n)z_n\| \\
&\quad \times\|\gamma_n(x_n-p) + (1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^{q-1} \\
&\quad +c_q\alpha_n^q\|\gamma L_1x_n - \mu L_2S(\mu_n)z_n\|^q
\end{aligned}$$

$$\begin{aligned}
&\le \gamma_n\|x_n-p\|^q+(1-\gamma_n)\|z_n-p\|^q+\alpha_nM_1\\
&\le \gamma_n\|x_n-p\|^q+(1-\gamma_n)\Big[\|x_n-p\|^q-\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax_n-Ap\|^q\Big]\\
&\quad+\alpha_nM_1\\
&= \|x_n-p\|^q-(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax_n-Ap\|^q+\alpha_nM_1,
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= \sup_{n\ge 0}\Big\{q\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|\|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^{q-1}\\
&\quad+c_q\alpha_n^{q-1}\|\gamma L_1x_n-\mu L_2S(\mu_n)z_n\|^q\Big\}<\infty.
\end{aligned}$$

By the convexity of $\|\cdot\|^q$ for all $q>1$, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^q &\le \beta_n\|x_n-p\|^q+(1-\beta_n)\|S(\mu_n)y_n-p\|^q\\
&\le \beta_n\|x_n-p\|^q+(1-\beta_n)\|y_n-p\|^q\\
&\le \beta_n\|x_n-p\|^q+(1-\beta_n)\big[\|x_n-p\|^q\\
&\quad-(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q+\alpha_nM_1\\
&= \|x_n-p\|^q-(1-\beta_n)(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q\\
&\quad+(1-\beta_n)\alpha_nM_1.
\end{aligned}$$

By the fact that $a^r-b^r\le ra^{r-1}(a-b),\forall r\ge 1$, we get
$(1-\beta_n)(1-\gamma_n)\lambda_n(q\beta-c_q\lambda_n^{q-1})\|Ax-Ay\|^q$

$$\begin{aligned}
&\le \|x_n-p\|^q-\|x_{n+1}-p\|^q+(1-\beta_n)\alpha_nM_1\\
&\le q\|x_n-p\|^{q-1}\big(\|x_n-p\|-\|x_{n+1}-p\|\big)+(1-\beta_n)\alpha_nM_1\\
&\le q\|x_n-p\|^{q-1}\|x_n-x_{n+1}\|+(1-\beta_n)\alpha_nM_1.
\end{aligned}$$

From $0<a\le\lambda_n\le b<(\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, the conditions (C1)-(C3), (C6) and (3.5), we conclude that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.6}$$

From Proposition 2.1 (ii) and Lemma 2.6, we also have

$$\begin{aligned}
\|z_n-p\|^2 &= \|Q_C(x_n-\lambda_nAx_n)-Q_C(p-\lambda_nAp)\|^2\\
&\le \big\langle(x_n-\lambda_nAx_n)-(p-\lambda_nAp),J(z_n-p)\big\rangle\\
&= \big\langle(x_n-p)-\lambda_n(Ax_n-Ap),J(z_n-p)\big\rangle\\
&= \big\langle x_n-p,J(z_n-p)\big\rangle-\lambda_n\big\langle Ax_n-Ap,J(z_n-p)\big\rangle\\
&\le \frac{1}{2}\big[\|x_n-p\|^2+\|z_n-p\|^2-g\|x_n-z_n\|\big]+\lambda_n\|Ax_n-Ap\|\|z_n-p\|.
\end{aligned}$$

So, we get

$$\|z_n-p\|^2 \le \|x_n-p\|^2-g\|x_n-z_n\|+2\lambda_n\|Ax_n-Ap\|\|z_n-p\|.$$

By Lemma 2.8, it follows that

$$\begin{aligned}
\|y_n-p\|^2 &= \|Q_C\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]-p\|^2\\
&\le \|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)+\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\|^2\\
&\le \|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^2\\
&\quad+2\alpha_n\big\langle\gamma L_1x_n-\mu L_2S(\mu_n)z_n,J\big(\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\\
&\quad+\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\big)\big\rangle\\
&\le \gamma_n\|x_n-p\|^2+(1-\gamma_n)\|z_n-p\|^2+\alpha_nM_2\\
&\le \gamma_n\|x_n-p\|^2+(1-\gamma_n)\Big[\|x_n-p\|^2-g\|x_n-z_n\|\\
&\quad+2\lambda_n\|Ax_n-Ap\|\|z_n-p\|\Big]+\alpha_nM_2\\
&= \|x_n-p\|^2-(1-\gamma_n)g\|x_n-z_n\|+2(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|\\
&\quad+\alpha_nM_2,
\end{aligned}$$

where

$$\begin{aligned} M_2 &= \sup_{n\geq 0}\Big\{2\big\langle \gamma L_1 x_n - \mu L_2 S(\mu_n)z_n, J\big(\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\\ &\quad +\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\big)\big\rangle\Big\} < \infty. \end{aligned}$$

We obtain

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|S(\mu_n)y_n-p\|^2\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\big[\|x_n-p\|^2-(1-\gamma_n)g\|x_n-z_n\|\\ &\quad +2(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+\alpha_nM_2\big]\\ &= \|x_n-p\|^2-(1-\beta_n)(1-\gamma_n)g\|x_n-z_n\|\\ &\quad +2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+(1-\beta_n)\alpha_nM_2. \end{aligned}$$

Then we get

$$\begin{aligned} (1-\gamma_n)g\|x_n-z_n\| &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+(1-\beta_n)\alpha_nM_2\\ &\leq \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\\ &\quad +2(1-\beta_n)(1-\gamma_n)\lambda_n\|Ax_n-Ap\|\|z_n-p\|+(1-\beta_n)\alpha_nM_2. \end{aligned}$$

By the conditions (C1)-(C3), (C6), (3.5) and (3.6), we have

$$\lim_{n\to\infty} g(\|x_n-z_n\|)=0.$$

It follows from the property of g that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.7}$$

Similar to the proof of (3.7), we start by using Lemma 2.5 and Lemma 2.8

$$\begin{aligned} \|y_n-p\|^2 &= \|Q_C\big[\alpha_n\gamma L_1x_n+\gamma_nx_n+((1-\gamma_n)I-\alpha_n\mu L_2)S(\mu_n)z_n\big]-p\|^2\\ &\leq \|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)+\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\|^2\\ &\leq \|\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\|^2\\ &\quad +2\alpha_n\big\langle\gamma L_1x_n-\mu L_2S(\mu_n)z_n, J\big(\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\\ &\quad +\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\big)\big\rangle\\ &\leq \gamma_n\|x_n-p\|^2+(1-\gamma_n)\|S(\mu_n)z_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2\\ &\leq \gamma_n\|x_n-p\|^2+(1-\gamma_n)\|z_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2\\ &\leq \gamma_n\|x_n-p\|^2+(1-\gamma_n)\|x_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2\\ &= \|x_n-p\|^2-\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2, \end{aligned}$$

where

$$\begin{aligned} M_2 &= \sup_{n\geq 0}\Big\{2\big\langle \gamma L_1 x_n - \mu L_2 S(\mu_n)z_n, J\big(\gamma_n(x_n-p)+(1-\gamma_n)\big(S(\mu_n)z_n-p\big)\\ &\quad +\alpha_n\big(\gamma L_1x_n-\mu L_2S(\mu_n)z_n\big)\big)\big\rangle\Big\} < \infty. \end{aligned}$$

We obtain

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|S(\mu_n)y_n-p\|^2\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2\\ &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\big[\|x_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)g(\|x_n-S(\mu_n)z_n\|)+\alpha_nM_2\big] \end{aligned}$$

$$= \|x_n - p\|^2 - (1-\beta_n)\gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) + (1-\beta_n)\alpha_n M_2.$$

Then we get

$$\begin{aligned}(1-\beta_n)\gamma_n(1-\gamma_n)g(\|x_n - S(\mu_n)z_n\|) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\beta_n)\alpha_n M_2 \\ &\leq \|x_n - x_{n+1}\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big) \\ &\quad +(1-\beta_n)\alpha_n M_2.\end{aligned}$$

By the conditions (C1),(C3),(C6) and (3.5), we have

$$\lim_{n\to\infty} g(\|x_n - S(\mu_n)z_n\|) = 0.$$

It follows from the property of g that

$$\lim_{n\to\infty} \|x_n - S(\mu_n)z_n\| = 0. \tag{3.8}$$

Since $S(\mu_n)$ is a nonexpansive and from the proof of Lemma 2.12, we get $Q_C S(\mu_n)z_n = S(\mu_n)z_n$ and observe that

$\|y_n - S(\mu_n)z_n\|$

$$\begin{aligned} &= \big\|Q_C\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - S(\mu_n)z_n\big\| \\ &\leq \big\|\big[\alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n\big] - S(\mu_n)z_n\big\| \\ &= \big\|\alpha_n\big(\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\big) + \gamma_n(x_n - S(\mu_n)z_n)\big\| \\ &\leq \alpha_n\|\gamma L_1 x_n - \mu L_2 S(\mu_n)z_n\| + \gamma_n\|x_n - S(\mu_n)z_n\|.\end{aligned}$$

It follows from the conditions (C1), (C6) and (3.8), we get

$$\lim_{n\to\infty} \|y_n - S(\mu_n)z_n\| = 0. \tag{3.9}$$

Since

$$\begin{aligned}\|x_n - S(\mu_n)x_n\| &\leq \|x_n - S(\mu_n)z_n\| + \|S(\mu_n)z_n - S(\mu_n)x_n\| \\ &\leq \|x_n - S(\mu_n)z_n\| + \|z_n - x_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - S(\mu_n)x_n\| = 0.$$

Now, we show that $z \in F := F(\mathcal{S}) \cap VI(C,A)$. We can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is bounded and there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup z$.

(I) First, we show that $z \in F(\mathcal{S})$. Let $\mu_{n_k} \geq 0$ such that $\mu_{n_k} \to 0$ and $\frac{\|S(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} \to 0, k \to \infty$. Fix $s > 0$, we can notice that

$\|x_{n_k} - S(s)z\|$

$$\begin{aligned} &\leq \sum_{i=0}^{[s/\mu_{n_k}]-1} \big\|S\big((i+1)\mu_{n_k}\big)x_{n_k} - S(i\mu_{n_k})x_{n_k}\big\| \\ &\quad +\big\|S\big([s/\mu_{n_k}]\mu_k\big)x_{n_k} - S\big([s/\mu_{n_k}]\mu_{n_k}\big)z\big\| \\ &\quad +\big\|S\big([s/\mu_{n_k}]\mu_{n_k}\big)z - S(s)z\big\| \\ &\leq [s/\mu_{n_k}]\|S(\mu_{n_k})x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| + \big\|S\big(s - [s/\mu_{n_k}]\mu_{n_k}\big)z - z\big\| \\ &\leq s\frac{\|S(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - z\| + \big\|S\big(s - [s/\mu_{n_k}]\mu_{n_k}\big)z - z\big\| \\ &\leq s\frac{\|S(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - z\| + \max\{\|S(\mu)z - z\| : 0 \leq \mu \leq \mu_{n_k}\}.\end{aligned}$$

For all $k \in \mathbb{N}$, we have

$$\limsup_{k\to\infty} \|x_{n_k} - S(s)z\| \leq \limsup_{k\to\infty} \|x_{n_k} - z\|.$$

Since a Banach space E with a weakly sequentially continuous duality mapping satisfies the Opial's condition, this implies $S(s)z = z$.

(II) Next, we show that $z \in VI(C,A)$. From the assumption, we see that the control sequence $\{\lambda_{n_k}\}$

is bounded. So, there exists a subsequence $\{\lambda_{n_{k_j}}\}$ converges to λ_0. We may assume, without loss of generality, that $\lambda_{n_k} \rightharpoonup \lambda_0$. Observe that

$$\begin{aligned}\|Q_C(x_{n_k}-\lambda_0 Ax_{n_k})-x_{n_k}\| &\leq \|Q_C(x_{n_k}-\lambda_0 Ax_{n_k})-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|\\ &\leq \|(x_{n_k}-\lambda_0 Ax_{n_k})-(x_{n_k}-\lambda_{n_k}Ax_{n_k})\|\\ &\quad +\|x_{n_k}-S(\mu_{n_k})z_{n_k}\|+\|S(\mu_{n_k})z_{n_k}-y_{n_k}\|\\ &\leq M\|\lambda_{n_k}-\lambda_0\|+\|x_{n_k}-S(\mu_{n_k})z_{n_k}\|\\ &\quad +\|S(\mu_{n_k})z_{n_k}-y_{n_k}\|,\end{aligned}$$

where M is as appropriate constant such that $M \geq \sup_{n\geq 1}\{\|Ax_n\|\}$. It follows from (3.8), (3.9) and $\lambda_{n_k} \rightharpoonup \lambda_0$ that

$$\lim_{k\to\infty}\|Q_C(x_{n_k}-\lambda_0 Ax_{n_k})-x_{n_k}\|=0.$$

We know that $Q_C(I-\lambda_0 A)$ is nonexpansive and it follows from Lemma 2.4 that $z \in F(Q_C(I-\lambda_0 A))$. By using Lemma 2.3, we obtain that $z \in F(Q_C(I-\lambda_0 A)) = VI(C,A)$.
Therefore, from (I) and (II), we conclude that $z \in F := F(\mathcal{S}) \cap VI(C,A)$.

Next, we show that $\limsup_{n\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle \leq 0$, where x^* is the solution of the variational inequality (1.1). Since the Banach space E has a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$ and $y_{n_k} \rightharpoonup z$, we obtain that
$\limsup_{n\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle$

$$\begin{aligned} &= \lim_{k\to\infty}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_{n_k} - x^*)\rangle\\ &= \langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0. \end{aligned} \tag{3.10}$$

Finally, we show that $\{x_n\}$ converges strongly to x^*. Setting $u_n = \alpha_n\gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu L_2)S(\mu_n)z_n, \forall n \geq 0$, it follows from Lemma 2.11, 2.13 and 2.14 that

$$\begin{aligned}\|y_n-x^*\|^q &= \langle Q_C u_n - u_n, J_q(y_n-x^*)\rangle + \langle u_n - x^*, J_q(y_n-x^*)\rangle\\ &\leq \langle u_n - x^*, J_q(y_n-x^*)\rangle\\ &= \langle [(1-\gamma_n)I-\alpha_n\mu L_2][S(\mu_n)z_n - x^*], J_q(y_n-x^*)\rangle\\ &\quad +\alpha_n\langle \gamma L_1 x_n - \mu L_2 x^*, J_q(y_n-x^*)\rangle + \gamma_n\langle x_n - x^*, J_q(y_n-x^*)\rangle\\ &\leq (1-\gamma_n-\alpha_n\tau)\|S(\mu_n)z_n - x^*\|\|y_n-x^*\|^{q-1}\\ &\quad +\gamma_n\|x_n-x^*\|\|y_n-x^*\|^{q-1} + \alpha_n\langle \gamma L_1 x_n - \gamma L_1 x^*, J_q(y_n-x^*)\rangle\\ &\quad +\alpha_n\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n-x^*)\rangle\\ &\leq (1-\gamma_n-\alpha_n\tau)\|x_n-x^*\|\|y_n-x^*\|^{q-1}\\ &\quad +\gamma_n\|x_n-x^*\|\|y_n-x^*\|^{q-1} + \alpha_n\gamma L\|x_n-x^*\|\|y_n-x^*\|^{q-1}\\ &\quad +\alpha_n\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n-x^*)\rangle\\ &\leq [1-\alpha_n(\tau-\gamma L)]\|x_n-x^*\|\|y_n-x^*\|^{q-1}\\ &\quad +\alpha_n\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n-x^*)\rangle\\ &\leq [1-\alpha_n(\tau-\gamma L)]\Big[\frac{1}{q}\|x_n-x^*\|^q + \frac{q-1}{q}\|y_n-x^*\|^q\Big]\\ &\quad +\alpha_n\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n-x^*)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|y_n-x^*\|^q &\leq \frac{1-\alpha_n(\tau-\gamma L)}{1+(q-1)\alpha_n(\tau-\gamma L)}\|x_n-x^*\|^q\\ &\quad +\frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n-x^*)\rangle\\ &\leq [1-\alpha_n(\tau-\gamma L)]\|x_n-x^*\|^q\\ &\quad +\frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n-x^*)\rangle.\end{aligned}$$

Therefore,
$\|x_{n+1}-x^*\|^q$

$$\leq \beta_n\|x_n-x^*\|^q + (1-\beta_n)\|S(\mu_n)y_n - x^*\|^q$$

$$
\begin{aligned}
&\leq \beta_n\|x_n - x^*\|^q + (1-\beta_n)\|y_n - x^*\|^q \\
&\leq \beta_n\|x_n - x^*\|^q + (1-\beta_n)\Big[[1-\alpha_n(\tau-\gamma L)]\|x_n - x^*\|^q \\
&\quad + \frac{q\alpha_n}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle\Big] \\
&= [1-\alpha_n(\tau-\gamma L)(1-\beta_n)]\|x_n - p\|^q \\
&\quad + \frac{q\alpha_n(1-\beta_n)}{1+(q-1)\alpha_n(\tau-\gamma L)}\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(y_n - x^*)\rangle. \qquad (3.11)
\end{aligned}
$$

Now, from (C1), (3.10) and applying Lemma 2.10 to (3.11), we get $\|x_n - x^*\| \to 0$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ converges strongly to $x^* \in F$. The proof is complete. □

Corollary 3.2. *Let C be a sunny nonexpansive retract and nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous generalized duality mapping $J : E \to E^*$ with the best smooth constant K. Let Q_C be a sunny nonexpansive retraction from E onto C, $A : C \to E$ be an β-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from C into itself, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a κ-Lipschitzian and η-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$, $\{\mu_n\} \subset (0,\infty)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < \frac{\eta}{K^2\kappa^2}$, $0 < a \leq \lambda_n \leq b < \frac{\beta}{K^2}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - K^2\mu\kappa^2)$ and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$
\begin{cases}
z_n = Q_C(x_n - \lambda_n A x_n) \\
y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) S(\mu_n) z_n\big], \\
x_{n+1} = \beta_n x_n + (1-\beta_n) S(\mu_n) y_n,
\end{cases}
$$

which satisfy the conditions (C1)-(C6) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J(z - x^*)\rangle \leq 0, \forall z \in F.$$

Corollary 3.3. *Let C be a sunny nonexpansive retract and nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let Q_C be a sunny nonexpansive retraction from E onto C, $A : C \to E$ be an β-inverse-strongly accretive operator, $\mathcal{S} = \{S(s) : s \geq 0\}$ be a nonexpansive semigroup from C into itself, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a κ-Lipschitzian and η-strongly accretive operator with constant $\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$, $\{\mu_n\} \subset (0,\infty)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ where c_q is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$ and $F := F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$
\begin{cases}
z_n = Q_C(x_n - \lambda_n A x_n) \\
y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) \frac{1}{t_n}\int_0^{t_n} S(s) z_n ds\big], \\
x_{n+1} = \beta_n x_n + (1-\beta_n) \frac{1}{t_n}\int_0^{t_n} S(s) y_n ds,
\end{cases}
$$

which satisfy the conditions (C1)-(C3) and (C6) in Theorem 3.1 and assume that

$$\lim_{n\to\infty} \sup_{x \in \tilde{C}} \left\| \frac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)x ds - \frac{1}{t_n}\int_0^{t_n} S(s)x ds \right\| = 0,$$

$\tilde{C}$ bounded subset of C, $\lim_{n\to\infty} \mu_n = \infty$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} = 1$. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F.$$

Corollary 3.4. *Let C be a sunny nonexpansive retract and nonempty closed convex subset of a q-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous generalized duality mapping $J_q : E \to E^*$. Let Q_C be a sunny nonexpansive retraction from E onto C, $A : C \to E$ be an β-inverse-strongly accretive operator, $L_1 : C \to E$ be a L-Lipschitzian mapping with constant $L \geq 0$ and $L_2 : C \to E$ be a κ-Lipschitzian and η-strongly accretive operator with constant*

$\kappa, \eta > 0$. Assume $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\} \subset (0,1)$ such that $\{\lambda_n\} \subset [a,b] \subset (0,1)$, $0 < \mu < (\frac{q\eta}{c_q\kappa^q})^{\frac{1}{q-1}}$ where c_q is a positive real number, $0 < a \leq \lambda_n \leq b < (\frac{q\beta}{c_q})^{\frac{1}{q-1}}$, $0 \leq \gamma L < \tau$ where $\tau = \mu(\eta - \frac{c_q\mu^{q-1}\kappa^q}{q})$ and $F := VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and

$$\begin{cases} z_n = Q_C(x_n - \lambda_n A x_n) \\ y_n = Q_C\big[\alpha_n \gamma L_1 x_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu L_2) z_n\big], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) y_n, \end{cases}$$

which satisfy the conditions (C1)-(C3) and (C6) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $x^* \in F$ which also solves the following variational inequality:

$$\langle \gamma L_1 x^* - \mu L_2 x^*, J_q(z - x^*)\rangle \leq 0, \forall z \in F.$$

Proof. Taking $\mu_n = 0$ in Theorem 3.1, we can conclude the desired conclusion easily. The proof is complete. □

4. Numerical Example

In this section, we illustrate a real numerical example by using main theorem.

Example 4.1. Let $E = \mathbb{R}$, $C = [0,1]$, $q = 2$, $j_q = I$, $\gamma = \mu = \dfrac{1}{2}$, $\alpha_n = \mu_n = \dfrac{1}{3n}$, $\beta_n = \dfrac{n+1}{2n}$, $\gamma_n = \dfrac{2n-1}{7n}$, $\lambda_n = \dfrac{6n-1}{14n}$ and $x_1 = 1$ which satisfy the conditions $(C1)-(C6)$ in Theorem 3.1. We define the mappings as follows: $Q_C x = \begin{cases} \dfrac{x}{|x|}, x \in \mathbb{R} - C \\ x, x \in C \end{cases}$, $(S(s))(x) = xe^{-s}$, $Ax = \dfrac{x}{2}$, $L_1 x = x^2$ and $L_2 x = \dfrac{1}{3}(x^2 + 2x)$, where A is $\dfrac{1}{2}$-inverse strongly accretive, L_1 is 1-Lipschitzian and L_2 is 1-Lipschitzian and $\dfrac{2}{3}$-strongly accretive. Then the sequence

$$\begin{cases} z_n = x_n\left(\dfrac{22n+1}{28n}\right), \\ y_n = \dfrac{x_n}{n}\left(\dfrac{x_n}{6} + \dfrac{2n-1}{7}\right) + \dfrac{z_n e^{-1/3n}}{n}\left(\dfrac{5n+1}{7n} - \dfrac{z_n e^{-1/3n} + 2}{18}\right), \\ x_{n+1} = \dfrac{n+1}{2n} x_n + \dfrac{n-1}{2n} y_n e^{-1/3n} \end{cases}$$

converges strongly to 0 shown in Figure 1 and Table 1.

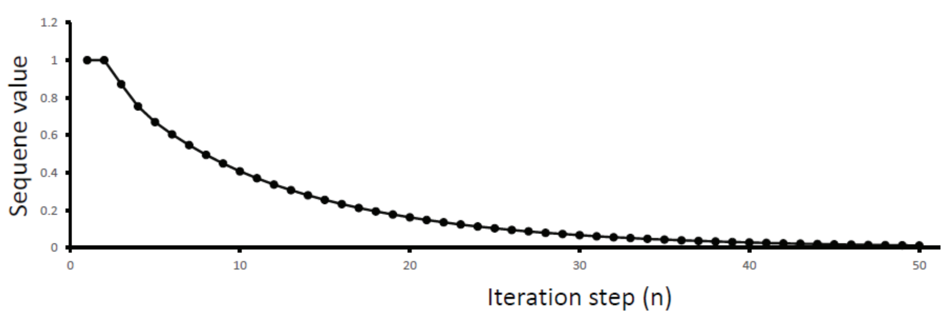

Figure 1. The iteration process.

TABLE 1. The value of sequence $\{x_n\}$

Iteration step (n)	Sequence value (x_n)	Iteration step (n)	Sequence value (x_n)
1	1	10	0.456821
2	1	20	0.182023
3	0.915366	50	0.013817
4	0.828039	100	0.000222
5	0.747896	168	0.0000009

Example 4.2. Let $E = \mathbb{R}^3$ and an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x^1 \cdot y^1 + x^2 \cdot y^2 + x^3 \cdot y^3, \forall \mathbf{x} = (x^1, x^2, x^3), \mathbf{y} = (y^1, y^2, y^3)$$

and the usual norm $\| \cdot \| : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\|\mathbf{x}\| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}.$$

Let $C = \{\mathbf{x} \in \mathbb{R}^3 \big| \|\mathbf{x}\| \leq 1\}$, $\gamma = \mu = \frac{1}{2}$, $\alpha_n = \mu_n = \frac{1}{3n}$, $\beta_n = \frac{n+1}{2n}$, $\gamma_n = \frac{2n-1}{7n}$, $\lambda_n = \frac{6n-1}{14n}$ and $\mathbf{x}_1 = (1, 2, 3)$ which satisfy the conditions $(C1) - (C6)$ in Theorem 3.1. We define the mappings as follows: $Q_C \mathbf{x} = \begin{cases} \frac{\mathbf{x}}{\|\mathbf{x}\|}, \mathbf{x} \notin C \\ \mathbf{x}, \mathbf{x} \in C \end{cases}$, $(S(s))(\mathbf{x}) = \mathbf{x} e^{-s}$, $A\mathbf{x} = \frac{\mathbf{x}}{2}$, $L_1 \mathbf{x} = \mathbf{x}^2$ and $L_2 \mathbf{x} = \frac{1}{3}(\mathbf{x}^2 + 2\mathbf{x})$, For $n = 1, \ldots, 6$, we have the sequence

$$\begin{cases} \mathbf{z}_n = \dfrac{\mathbf{Z}_n}{\|\mathbf{Z}_n\|}, \\ \mathbf{y}_n = \dfrac{\mathbf{Y}_n}{\|\mathbf{Y}_n\|}, \\ \mathbf{x}_{n+1} = \dfrac{n+1}{2n}\mathbf{x}_n + \dfrac{n-1}{2n}\mathbf{y}_n e^{-1/3n}, \end{cases}$$

where

$$\mathbf{Z}_n = \mathbf{x}_n \left(\frac{22n+1}{28n} \right),$$

$$\mathbf{Y}_n = \frac{\mathbf{x}_n}{n} \left(\frac{\mathbf{x}_n}{6} + \frac{2n-1}{7} \right) + \frac{\mathbf{z}_n e^{-1/3n}}{n} \left(\frac{5n+1}{7n} - \frac{\mathbf{z}_n e^{-1/3n} + 2}{18} \right).$$

For $n = 7$, we have the sequence

$$\begin{cases} \mathbf{z}_n = \mathbf{x}_n \left(\dfrac{22n+1}{28n} \right), \\ \mathbf{y}_n = \dfrac{\mathbf{Y}_n}{\|\mathbf{Y}_n\|}, \\ \mathbf{x}_{n+1} = \dfrac{n+1}{2n}\mathbf{x}_n + \dfrac{n-1}{2n}\mathbf{y}_n e^{-1/3n}, \end{cases}$$

where

$$\mathbf{Y}_n = \frac{\mathbf{x}_n}{n} \left(\frac{\mathbf{x}_n}{6} + \frac{2n-1}{7} \right) + \frac{\mathbf{z}_n e^{-1/3n}}{n} \left(\frac{5n+1}{7n} - \frac{\mathbf{z}_n e^{-1/3n} + 2}{18} \right).$$

For $n = 8, 9, 10, \ldots$, we have the sequence

$$\begin{cases} \mathbf{z}_n = \mathbf{x}_n \left(\dfrac{22n+1}{28n} \right), \\ \mathbf{y}_n = \dfrac{\mathbf{x}_n}{n} \left(\dfrac{\mathbf{x}_n}{6} + \dfrac{2n-1}{7} \right) + \dfrac{\mathbf{z}_n e^{-1/3n}}{n} \left(\dfrac{5n+1}{7n} - \dfrac{\mathbf{z}_n e^{-1/3n} + 2}{18} \right), \\ \mathbf{x}_{n+1} = \dfrac{n+1}{2n}\mathbf{x}_n + \dfrac{n-1}{2n}\mathbf{y}_n e^{-1/3n}. \end{cases}$$

Then the sequence converges strongly to $\mathbf{0} = (0, 0, 0)$, shown in Figure 2 and Table 2.

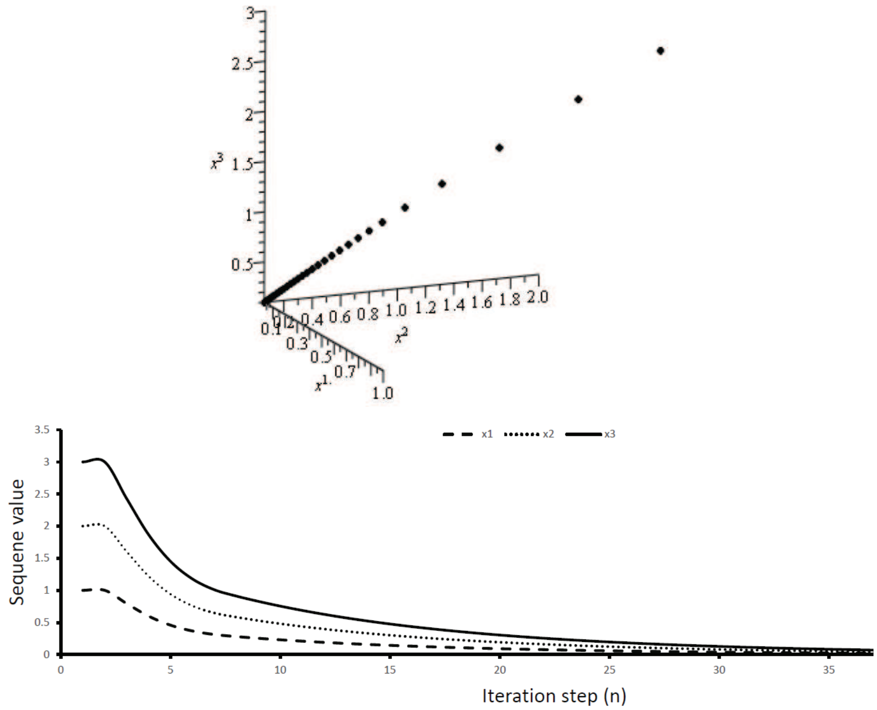

Figure 2. The iteration process.

Table 2. The value of sequence $\{\mathbf{x}_n\}$

n	x_n^1	x_n^2	x_n^3	n	x_n^1	x_n^2	x_n^3
1	1	2	3	10	0.229555	0.480061	0.753976
2	1	2	3	20	0.090760	0.191436	0.303513
3	0.794057	1.603757	2.429100	50	0.006872	0.014536	0.023119
4	0.598144	1.220303	1.867191	100	0.000110	0.000233	0.000371
5	0.457286	0.940376	1.450720	174	0.0000003	0.0000006	0.0000009

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The generalized viscosity implicit rule of nonexpansive semigroup in Banach spaces

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Abstract

In this research, we focus on a common fixed point problem of a nonexpansive semigroup with the generalized viscosity methods for implicit iterative algorithms. Our main objective is to construct the new strong convergence theorems under certain appropriate conditions in uniformly convex and uniformly smooth Banach spaces. Specifically, the main results make a contribution to the implicit midpoint theorems. The findings for theorems in Hilbert spaces and the other forms of a nonexpansive semigroup can be used in several practical purposes. Finally, a numerical example in 3 dimensions is provided to support our main results. ©2017 All rights reserved.

Keywords: Nonexpansive semigroup, Fixed point, Generalized viscosity, Implicit, Banach space.
2010 MSC: MSC 47H10, 47H20, 47J20.

1. Introduction

Let E be a real Banach space, C be a nonempty closed convex subset of E and E^* be the dual space of E with norm $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ pairing between E and E^*.

- The *duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2, \|x^*\| = \|x\|\right\}$$

for all $x \in E$. It is well known that if E is a Hilbert space then J is the identity mapping, and if E is smooth then J is single-valued.

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- A mapping $T : C \to C$ is called a *nonexpansive*, if

$$\|Tx - Ty\| \leqslant \|x - y\|, \ \forall x, y \in C$$

  and $F(T) = \{x \in C : Tx = x\}$ is the set of fixed points of T.

- A mapping $f : C \to C$ is called a *contraction*, if there exists $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leqslant \alpha\|x - y\|, \ \forall x, y \in C.$$

- A family $\mathcal{S} = \{S(s) : 0 \leqslant s < \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

  (i) $S(0)x = x$ for all $x \in C$;
  (ii) $S(s + t) = S(s)S(t)$ for all $s, t \geqslant 0$;
  (iii) $\|S(s)x - S(s)y\| \leqslant \|x - y\|$ for all $x, y \in C$ and $s \geqslant 0$;
  (iv) for each $x \in C$, the mapping $S(\cdot)x$ from $[0, \infty)$ into C is continuous.

  $\mathcal{S}$, i.e., $F(\mathcal{S}) = \{x \in C : S(s)x = x, \forall s > 0\}$ is a common fixed point set of a nonexpansive semigroup. It is easy to see that $F(\mathcal{S})$ is closed and convex (see also [1, 2, 3, 4, 5, 6, 7, 8]).

Definition 1.1. A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an *L-function* if $\psi(0) = 0, \psi(t) > 0, \forall t > 0$ and for every $s > 0$ there exists $u > s$ such that

$$\psi(t) \leqslant s, \text{for } t \in [s, u].$$

Note that every L-function ψ satisfies $\psi(t) < t, \forall t > 0$.

Definition 1.2. Let (X, d) be a matric space. A mapping $f : X \to X$ is said to be:
(i) (ψ, L)-*contraction* if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an L-function and $d(f(x), f(y)) < \psi(d(x, y))$ for all $x, y \in X$ with $x \neq y$;
(ii) *Meir-Keeler type mapping* if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in X$ with $d(x, y) < \epsilon + \delta$ we have $d(f(x), f(y)) < \epsilon$.

Theorem 1.3. ([9]) *Let (X, d) be a metric space and $f : X \to X$ a mapping. Then the following assertions are equivalent:*
(i) f is a Meir-Keeler type mapping;
(ii) there exists an L-function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is a $(\psi, L)-$contraction.

Proposition 1.4. ([10]) *Let C be a convex subset of a Banach space E. Let $f : C \to C$ be a Meir-Keeler type mapping. Then for each $\epsilon > 0$ there exists $r \in (0, 1)$ such that for each $x, y \in C$ with $\|x - y\| \geqslant \epsilon$, we have*

$$\|f(x) - f(y)\| \leqslant r\|x - y\|.$$

Now, a Meir-Keeler type mapping or $(\psi, L)-$contraction is called generalized contraction mapping. We suppose that the function ψ based on the definition of the $(\psi, L)-$contraction is continuous and strictly increasing and $\lim_{t\to\infty} \eta(t) = \infty$, where $\eta(t) = t - \psi(t), t \in \mathbb{R}^+$. In consequence, we have that η is a bijection on $\mathbb{R}^+$.

In 2015, the viscosity implicit midpoint theorem in Hilbert spaces was introduced by Xu et al. [11]:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), n \geqslant 0. \tag{1.1}$$

In the same year, Ke et al. [12] introduced the two viscosity implicit midpoint theorem in Hilbert spaces:

$$x_{n+1} = \alpha_n Q(x_n) + (1 - \alpha_n)T(s_n x_n + (1 - s_n)x_{n+1}), \tag{1.2}$$

and

$$x_{n+1} = \alpha_n x_n + \beta_n Q(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}). \tag{1.3}$$

One year later, Yan et al. [13] introduced an implicit iteration for a generalized contraction mapping in Banach space:

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}). \tag{1.4}$$

They proved the strong convergence theorems.

Motivated and inspired by the idea of Yan et al. [13]. We introduce the new implicit iterative scheme and the new implicit midpoint rule with viscosity approximation method based on a generalized contraction mapping for finding solutions of fixed point problems for nonexpansive semigroup. We shall prove the strong convergence theorems in uniformly convex and uniformly smooth Banach space under some parameters controlling conditions. Our results extend and improve the recent results of Yan et al. [13] and other authors.

2. Preliminaries

According to our framework throughout this research, we first preview some definitions involving a Banach space E as follows. Let $U = \{x \in E : \|x\| = 1\}$.

- E is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geqslant \epsilon$ implies $\left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta$.
  It is known that a uniformly convex Banach space is reflexive and strictly convex.

- E is said to be *smooth* if $\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$ exists for all $x, y \in U$.
  It is also said to be *uniformly smooth* if the limit is attained uniformly for all $x, y \in U$. The *modulus of smoothness* of E is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\right\},$$

  where $\rho : [0, \infty) \to [0, \infty)$ is a function.
  It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$.

A Banach space E is said to satisfy *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x (n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \forall y \in E \text{ with } x \neq y.$$

By [14, Theorem 1], it is well known that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition, and E is smooth.

The following lemmas are very useful for proving our main results.

Lemma 2.1. ([15]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geqslant 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2. ([16]) *Let C be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : C \to C$ be a generalized contraction mapping. Then $\{x_t\}$ defined by $x_t = tf(x_t) + (1 - t)Tx_t$ for $t \in (0, 1)$, converges strongly to $x^* \in F(T)$ as $t \to 0$, which solves the variational inequality:*

$$\langle f(x^*) - x^*, J(z - x^*)\rangle \leqslant 0, \forall z \in F(T).$$

Lemma 2.3. ([16]) *Let C be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : C \to C$ be a generalized contraction mapping. Assume that $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$, converges strongly to $x^* \in F(T)$ as $t \to 0$. Suppose that $\{x_n\}$ is bounded sequence such that $x_n - Tx_n \to 0$ as $n \to \infty$. Then*

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, J(x_n - x^*) \rangle \leqslant 0.$$

Lemma 2.4. ([17]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1-\alpha_n)a_n + \delta_n, \ n \geqslant 0$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of uniformly convex and uniformly smooth Banach space E. Let f be a generalized contraction mapping from C into itself and $\mathcal{S} = \{T(t) : t \geqslant 0\}$ be a nonexpansive semigroup from C into itself such that $F(\mathcal{S}) \neq 0$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = \lambda_n x_n + (1-\lambda_n)x_{n+1}, \\ y_n = \delta_n f(x_n) + (1-\delta_n)z_n, \\ x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(\mu_n)y_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\lambda_n\}$ are the sequences in $(0,1)$. The following conditions are satisfied:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim\limits_{n\to\infty} \beta_n = \lim\limits_{n\to\infty} \delta_n = \lim\limits_{n\to\infty} \mu_n = \lim\limits_{n\to\infty} |\alpha_{n+1} - \alpha_n| = \lim\limits_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$ *and* $\lim\limits_{n\to\infty} \sup\limits_{x\in\tilde{C}} \|T(\mu_{n+1})x - T(\mu_n)x\| = 0$, *$\tilde{C}$ bounded subset of C;*

(iii) $0 < \liminf\limits_{n\to\infty} \alpha_n \leqslant \limsup\limits_{n\to\infty} \alpha_n < 1$;

(iv) $\sum\limits_{n=0}^{\infty} \beta_n = \infty$ *for all $n \geqslant 1$.*

Then $\{x_n\}$ converges strongly to $x^* \in F(\mathcal{S})$ which also solves the following variational inequality:

$$\langle (I-f)x^*, J(z-x^*) \rangle \leqslant 0, \forall z \in F(\mathcal{S}). \tag{3.2}$$

Proof. First of all, we prove that $\{x_n\}$ is bounded. Let $p \in F(\mathcal{S})$, we have

$$\begin{aligned} \|y_n - p\| &\leqslant \delta_n\|f(x_n) - p\| + (1-\delta_n)\|z_n - p\| \\ &\leqslant \delta_n\|f(x_n) - f(p)\| + \delta_n\|f(p) - p\| + (1-\delta_n)\|z_n - p\| \\ &\leqslant \delta_n\psi\|x_n - p\| + \delta_n\|f(p) - p\| + (1-\delta_n)\left[\lambda_n\|x_n - p\| + (1-\lambda_n)\|x_{n+1} - p\|\right] \\ &= \left[\delta_n\psi + (1-\delta_n)\lambda_n\right]\|x_n - p\| + \delta_n\|f(p) - p\| + (1-\delta_n)(1-\lambda_n)\|x_{n+1} - p\| \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant \alpha_n\|x_n - p\| + \beta_n\|f(x_n) - p\| + \gamma_n\|T(\mu_n)y_n - p\| \\ &\leqslant \alpha_n\|x_n - p\| + \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + \gamma_n\|y_n - p\| \\ &\leqslant \alpha_n\|x_n - p\| + \beta_n\psi\|x_n - p\| + \beta_n\|f(p) - p\| \\ &\quad + \gamma_n\left\{\left[\delta_n\psi + (1-\delta_n)\lambda_n\right]\|x_n - p\| + \delta_n\|f(p) - p\| + (1-\delta_n)(1-\lambda_n)\|x_{n+1} - p\|\right\} \end{aligned}$$

$$\begin{aligned}
&= [\alpha_n+\beta_n\psi+\gamma_n\delta_n\psi+\gamma_n(1-\delta_n)\lambda_n]\,\|x_n-p\|+(\beta_n+\gamma_n\delta_n)\|f(p)-p\|\\
&\quad +\gamma_n(1-\delta_n)(1-\lambda_n)\|x_{n+1}-p\|\\
&= [1-\gamma_n(1-\delta_n)(1-\lambda_n)-(\beta_n+\gamma_n\delta_n)(1-\psi)]\,\|x_n-p\|+(\beta_n+\gamma_n\delta_n)\|f(p)-p\|\\
&\quad +\gamma_n(1-\delta_n)(1-\lambda_n)\|x_{n+1}-p\|\\
&= [1-\gamma_n(1-\delta_n)(1-\lambda_n)-(\beta_n+\gamma_n\delta_n)\eta]\,\|x_n-p\|+(\beta_n+\gamma_n\delta_n)\eta\eta^{-1}\|f(p)-p\|\\
&\quad +\gamma_n(1-\delta_n)(1-\lambda_n)\|x_{n+1}-p\|.
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-p\| &\leqslant \alpha_n\|x_n-p\|+\beta_n\|f(x_n)-p\|+\gamma_n\|T(\mu_n)y_n-p\|\\
&\leqslant \alpha_n\|x_n-p\|+\beta_n\|f(x_n)-f(p)\|+\beta_n\|f(p)-p\|+\gamma_n\|y_n-p\|\\
&\leqslant \alpha_n\|x_n-p\|+\beta_n\psi\|x_n-p\|+\beta_n\|f(p)-p\|\\
&\quad +\gamma_n\{[\delta_n\psi+(1-\delta_n)\lambda_n]\,\|x_n-p\|+\delta_n\|f(p)-p\|+(1-\delta_n)(1-\lambda_n)\|x_{n+1}-p\|\}\\
&= [\alpha_n+\beta_n\psi+\gamma_n\delta_n\psi+\gamma_n(1-\delta_n)\lambda_n]\,\|x_n-p\|+(\beta_n+\gamma_n\delta_n)\|f(p)-p\|\\
&\quad +\gamma_n(1-\delta_n)(1-\lambda_n)\|x_{n+1}-p\|\\
&= [1-\gamma_n(1-\delta_n)(1-\lambda_n)-(\beta_n+\gamma_n\delta_n)(1-\psi)]\,\|x_n-p\|+(\beta_n+\gamma_n\delta_n)\|f(p)-p\|\\
&\quad +\gamma_n(1-\delta_n)(1-\lambda_n)\|x_{n+1}-p\|\\
&= [1-\gamma_n(1-\delta_n)(1-\lambda_n)-(\beta_n+\gamma_n\delta_n)\eta]\,\|x_n-p\|+(\beta_n+\gamma_n\delta_n)\eta\eta^{-1}\|f(p)-p\|\\
&\quad +\gamma_n(1-\delta_n)(1-\lambda_n)\|x_{n+1}-p\|.
\end{aligned}$$

It follows that

$$\|x_{n+1}-p\| \leqslant \left[1-\frac{(\beta_n+\gamma_n\delta_n)\eta}{1-\gamma_n(1-\delta_n)(1-\lambda_n)}\right]\|x_n-p\|+\frac{(\beta_n+\gamma_n\delta_n)\eta}{1-\gamma_n(1-\delta_n)(1-\lambda_n)}\eta^{-1}\|f(p)-p\|.$$

By induction, we conclude that

$$\|x_n-p\|\leqslant \max\left\{\|x_1-p\|,\eta^{-1}\|f(p)-p\|\right\},\forall n\geqslant 0.$$

This implies that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{y_n\}$, $\{T(\mu_n)y_n\}$ and $\{z_n\}$.

Next, we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and we observe that

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|(\lambda_{n+1}x_{n+1}+(1-\lambda_{n+1})x_{n+2})-(\lambda_n x_n+(1-\lambda_n)x_{n+1})\|\\
&= \|\lambda_{n+1}x_{n+1}-\lambda_{n+1}x_n+\lambda_{n+1}x_n+(1-\lambda_{n+1})x_{n+2}\\
&\quad -(1-\lambda_{n+1})x_{n+1}+(1-\lambda_{n+1})x_{n+1}-\lambda_n x_n-(1-\lambda_n)x_{n+1}\|\\
&= \|\lambda_{n+1}(x_{n+1}-x_n)+(\lambda_{n+1}-\lambda_n)x_n+(1-\lambda_{n+1})(x_{n+2}-x_{n+1})+(\lambda_n-\lambda_{n+1})x_{n+1}\|\\
&\leqslant \lambda_{n+1}\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|)+(1-\lambda_{n+1})\|x_{n+2}-x_{n+1}\|,
\end{aligned}$$

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|\,(\delta_{n+1}f(x_{n+1})+(1-\delta_{n+1})z_{n+1})-(\delta_n f(x_n)+(1-\delta_n)z_n)\,\|\\
&= \|\delta_{n+1}f(x_{n+1})-\delta_{n+1}f(x_n)+\delta_{n+1}f(x_n)+(1-\delta_{n+1})z_{n+1}\\
&\quad -(1-\delta_{n+1})z_n+(1-\delta_{n+1})z_n-\delta_n f(x_n)-(1-\delta_n)z_n\|\\
&= \|\delta_{n+1}(f(x_{n+1})-f(x_n))+(\delta_{n+1}-\delta_n)f(x_n)+(1-\delta_{n+1})(z_{n+1}-z_n)\\
&\quad +(\delta_n-\delta_{n+1})z_n\|\\
&\leqslant \delta_{n+1}\|f(x_{n+1})-f(x_n)\|+|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|)+(1-\delta_{n+1})\|z_{n+1}-z_n\|\\
&\leqslant \delta_{n+1}\psi\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|)+(1-\delta_{n+1})[\lambda_{n+1}\|x_{n+1}-x_n\|\\
&\quad +|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|)+(1-\lambda_{n+1})\|x_{n+2}-x_{n+1}\|]\\
&= (\delta_{n+1}\psi+(1-\delta_{n+1})\lambda_{n+1})\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|)\\
&\quad +(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|)+(1-\delta_{n+1})(1-\lambda_{n+1})\|x_{n+2}-x_{n+1}\|,
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|(\alpha_{n+1}x_{n+1}+\beta_{n+1}f(x_{n+1})+\gamma_{n+1}T(\mu_{n+1})y_{n+1}) \\
&\quad -(\alpha_n x_n+\beta_n f(x_n)+\gamma_n T(\mu_n)y_n)\| \\
&= \|\alpha_{n+1}x_{n+1}-\alpha_{n+1}x_n+\alpha_{n+1}x_n+\beta_{n+1}f(x_{n+1})-\beta_{n+1}f(x_n)+\beta_{n+1}f(x_n) \\
&\quad +\gamma_{n+1}T(\mu_{n+1})y_{n+1}-\gamma_{n+1}T(\mu_n)y_n+\gamma_{n+1}T(\mu_n)y_n-\alpha_n x_n-\beta_n f(x_n) \\
&\quad -\gamma_n T(\mu_n)y_n\| \\
&= \|\alpha_{n+1}(x_{n+1}-x_n)+\beta_{n+1}(f(x_{n+1})-f(x_n))+\gamma_{n+1}(T(\mu_{n+1})y_{n+1}-T(\mu_n)y_n) \\
&\quad +(\alpha_{n+1}-\alpha_n)x_n+(\beta_{n+1}-\beta_n)f(x_n)+(\gamma_{n+1}-\gamma_n)T(\mu_n)y_n\| \\
&\leqslant \alpha_{n+1}\|x_{n+1}-x_n\|+\beta_{n+1}\|f(x_{n+1})-f(x_n)\|+\gamma_{n+1}\|T(\mu_{n+1})y_{n+1}-T(\mu_n)y_n\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\|x_n\|+|\beta_{n+1}-\beta_n|\|f(x_n)\|+|\gamma_{n+1}-\gamma_n|\|T(\mu_n)y_n\| \\
&\leqslant \alpha_{n+1}\|x_{n+1}-x_n\|+\beta_{n+1}\psi\|x_{n+1}-x_n\| \\
&\quad +\gamma_{n+1}\left(\|T(\mu_{n+1})y_{n+1}-T(\mu_{n+1})y_n\|+\|T(\mu_{n+1})y_n-T(\mu_n)y_n\|\right) \\
&\quad +|\alpha_{n+1}-\alpha_n|\|x_n\|+|\beta_{n+1}-\beta_n|\|f(x_n)\|+|\gamma_{n+1}-\gamma_n|\|T(\mu_n)y_n\| \\
&= (\alpha_{n+1}+\beta_{n+1}\psi)\|x_{n+1}-x_n\|+\gamma_{n+1}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\|x_n\|+|\beta_{n+1}-\beta_n|\|f(x_n)\|+|\gamma_{n+1}-\gamma_n|\|T(\mu_n)y_n\| \\
&\quad +\gamma_{n+1}\|y_{n+1}-y_n\| \\
&\leqslant (\alpha_{n+1}+\beta_{n+1}\psi)\|x_{n+1}-x_n\|+\gamma_{n+1}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\|x_n\|+|\beta_{n+1}-\beta_n|\|f(x_n)\|+|\gamma_{n+1}-\gamma_n|\|T(\mu_n)y_n\| \\
&\quad +\gamma_{n+1}[(\delta_{n+1}\psi+(1-\delta_{n+1})\lambda_{n+1})\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|) \\
&\quad +(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|)+(1-\delta_{n+1})(1-\lambda_{n+1})\|x_{n+2}-x_{n+1}\|] \\
&= (\alpha_{n+1}+\beta_{n+1}\psi+\gamma_{n+1}\delta_{n+1}\psi+\gamma_{n+1}(1-\delta_{n+1})\lambda_{n+1})\|x_{n+1}-x_n\| \\
&\quad +\gamma_{n+1}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|+|\alpha_{n+1}-\alpha_n|\|x_n\|+|\beta_{n+1}-\beta_n|\|f(x_n)\| \\
&\quad +|\gamma_{n+1}-\gamma_n|\|T(\mu_n)y_n\|+\gamma_{n+1}|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|) \\
&\quad +\gamma_{n+1}(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|) \\
&\quad +\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})\|x_{n+2}-x_{n+1}\| \\
&= [1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})-(\beta_{n+1}+\gamma_{n+1}\delta_{n+1})\eta]\,\|x_{n+1}-x_n\| \\
&\quad +\gamma_{n+1}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|+|\alpha_{n+1}-\alpha_n|\|x_n\|+|\beta_{n+1}-\beta_n|\|f(x_n)\| \\
&\quad +|\gamma_{n+1}-\gamma_n|\|T(\mu_n)y_n\|+\gamma_{n+1}|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|) \\
&\quad +\gamma_{n+1}(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|) \\
&\quad +\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})\|x_{n+2}-x_{n+1}\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leqslant \left[1-\frac{(\beta_{n+1}+\gamma_{n+1}\delta_{n+1})\eta}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right]\|x_{n+1}-x_n\| \\
&\quad +\frac{\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\| \\
&\quad +\frac{|\alpha_{n+1}-\alpha_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\|x_n\|+\frac{|\beta_{n+1}-\beta_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\|f(x_n)\| \\
&\quad +\frac{|\gamma_{n+1}-\gamma_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\|T(\mu_n)y_n\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\delta_{n+1}-\delta_n|\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}(\|f(x_n)\|+\|z_n\|)\\
&+\frac{|\lambda_{n+1}-\lambda_n|\gamma_{n+1}(1-\delta_{n+1})}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}(\|x_n\|+\|x_{n+1}\|)\\
\leqslant\ & \left[1-\frac{(\beta_{n+1}+\gamma_{n+1}\delta_{n+1})\eta}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right]\|x_{n+1}-x_n\|\\
&+\frac{\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|\\
&+\left[\frac{|\alpha_{n+1}-\alpha_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}+\frac{|\beta_{n+1}-\beta_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right.\\
&+\frac{|\gamma_{n+1}-\gamma_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}+\frac{|\delta_{n+1}-\delta_n|\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\\
&\left.+\frac{|\lambda_{n+1}-\lambda_n|\gamma_{n+1}(1-\delta_{n+1})}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right]M
\end{aligned}
$$

where $M=\sup_{n\geqslant 1}\{\|x_n\|,\|f(x_n)\|+\|T(\mu_n)y_n\|,\|f(x_n)\|+\|z_n\|,\|x_n\|+\|x_{n+1}\|\}<\infty$.

Setting $x_{n+1}=(1-\alpha_n)w_n+\alpha_n x_n$ for all $n\geqslant 1$, we see that $w_n=\dfrac{x_{n+1}-\alpha_n x_n}{1-\alpha_n}$, then we have

$$
\begin{aligned}
\|w_{n+1}-w_n\| &= \left\|\frac{x_{n+2}-\alpha_{n+1}x_{n+1}}{1-\alpha_{n+1}}-\frac{x_{n+1}-\alpha_n x_n}{1-\alpha_n}\right\|\\
&= \left\|\frac{\beta_{n+1}f(x_{n+1})+\gamma_{n+1}T(\mu_{n+1})y_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n f(x_n)+\gamma_n T(\mu_n)y_n}{1-\alpha_n}\right\|\\
&= \left\|\frac{\beta_{n+1}f(x_{n+1})+\gamma_{n+1}T(\mu_{n+1})y_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_{n+1}f(x_n)}{1-\alpha_{n+1}}+\frac{\beta_{n+1}f(x_n)}{1-\alpha_{n+1}}\right.\\
&\quad\left.-\frac{\gamma_{n+1}T(\mu_n)y_n}{1-\alpha_{n+1}}+\frac{\gamma_{n+1}T(\mu_n)y_n}{1-\alpha_{n+1}}-\frac{\beta_n f(x_n)+\gamma_n T(\mu_n)y_n}{1-\alpha_n}\right\|\\
&= \left\|\frac{\beta_{n+1}}{1-\alpha_{n+1}}(f(x_{n+1})-f(x_n))+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(T(\mu_{n+1})y_{n+1}-T(\mu_n)y_n)\right.\\
&\quad\left.+\left(\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right)f(x_n)+\left(\frac{\gamma_{n+1}}{1-\alpha_{n+1}}-\frac{\gamma_n}{1-\alpha_n}\right)T(\mu_n)y_n\right\|\\
&= \left\|\frac{\beta_{n+1}}{1-\alpha_{n+1}}(f(x_{n+1})-f(x_n))+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(T(\mu_{n+1})y_{n+1}-T(\mu_n)y_n)\right.\\
&\quad\left.+\left(\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right)f(x_n)+\left(\frac{\beta_n}{1-\alpha_n}-\frac{\beta_{n+1}}{1-\alpha_{n+1}}\right)T(\mu_n)y_n\right\|\\
&\leqslant \frac{\beta_{n+1}}{1-\alpha_{n+1}}\|f(x_{n+1})-f(x_n)\|+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}\|T(\mu_{n+1})y_{n+1}-T(\mu_n)y_n\|\\
&\quad+\left|\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right|(\|f(x_n)\|+\|T(\mu_n)y_n\|)\\
&\leqslant \frac{\beta_{n+1}\psi}{1-\alpha_{n+1}}\|x_{n+1}-x_n\|+\left|\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right|(\|f(x_n)\|+\|T(\mu_n)y_n\|)\\
&\quad+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(\|T(\mu_{n+1})y_{n+1}-T(\mu_{n+1})y_n\|+\|T(\mu_{n+1})y_n-T(\mu_n)y_n\|)\\
&\leqslant \frac{\beta_{n+1}\psi}{1-\alpha_{n+1}}\|x_{n+1}-x_n\|+\left|\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right|(\|f(x_n)\|+\|T(\mu_n)y_n\|)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}\|y_{n+1}-y_n\|+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\| \\
\leqslant\ & \frac{\beta_{n+1}\psi}{1-\alpha_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\| \\
&+\left|\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right|(\|f(x_n)\|+\|T(\mu_n)y_n\|) \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}[(\delta_{n+1}\psi+(1-\delta_{n+1})\lambda_{n+1})\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|) \\
&+(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|)+(1-\delta_{n+1})(1-\lambda_{n+1})\|x_{n+2}-x_{n+1}\|] \\
\leqslant\ & \frac{1}{1-\alpha_{n+1}}[\beta_{n+1}\psi+\gamma_{n+1}\delta_{n+1}\psi+\gamma_{n+1}(1-\delta_{n+1})\lambda_{n+1}]\|x_{n+1}-x_n\| \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|+\left|\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right|(\|f(x_n)\|+\|T(\mu_n)y_n\|) \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|)+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|) \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(1-\delta_{n+1})(1-\lambda_{n+1})\|x_{n+2}-x_{n+1}\| \\
\leqslant\ & \frac{1}{1-\alpha_{n+1}}[\beta_{n+1}\psi+\gamma_{n+1}\delta_{n+1}\psi+\gamma_{n+1}(1-\delta_{n+1})\lambda_{n+1}]\|x_{n+1}-x_n\| \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|+\left|\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right|(\|f(x_n)\|+\|T(\mu_n)y_n\|) \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|)+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|) \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(1-\delta_{n+1})(1-\lambda_{n+1})\Bigg\{\left[1-\frac{(\beta_{n+1}+\gamma_{n+1}\delta_{n+1})\eta}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right]\|x_{n+1}-x_n\| \\
&+\frac{\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\| \\
&+\Bigg[\frac{|\alpha_{n+1}-\alpha_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}+\frac{|\beta_{n+1}-\beta_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})} \\
&+\frac{|\gamma_{n+1}-\gamma_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}+\frac{|\delta_{n+1}-\delta_n|\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})} \\
&+\frac{|\lambda_{n+1}-\lambda_n|\gamma_{n+1}(1-\delta_{n+1})}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\Bigg]M\Bigg\} \\
\leqslant\ & \frac{1}{1-\alpha_{n+1}}[\beta_{n+1}\psi+\gamma_{n+1}\delta_{n+1}\psi+\gamma_{n+1}(1-\delta_{n+1})\lambda_{n+1} \\
&+\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})]\|x_{n+1}-x_n\| \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|+\left|\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right|(\|f(x_n)\|+\|T(\mu_n)y_n\|) \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|)+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|) \\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(1-\delta_{n+1})(1-\lambda_{n+1})\Bigg\{\frac{\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{|\alpha_{n+1}-\alpha_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}+\frac{|\beta_{n+1}-\beta_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right.\\
&+\frac{|\gamma_{n+1}-\gamma_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}+\frac{|\delta_{n+1}-\delta_n|\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\\
&\left.\left.+\frac{|\lambda_{n+1}-\lambda_n|\gamma_{n+1}(1-\delta_{n+1})}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right]M\right\}\\
= &\left[1-\frac{(\beta_{n+1}+\gamma_{n+1}\delta_{n+1})\eta}{1-\alpha_{n+1}}\right]\|x_{n+1}-x_n\|\\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|+\left|\frac{\beta_{n+1}}{1-\alpha_{n+1}}-\frac{\beta_n}{1-\alpha_n}\right|(\|f(x_n)\|+\|T(\mu_n)y_n\|)\\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}|\delta_{n+1}-\delta_n|(\|f(x_n)\|+\|z_n\|)+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(1-\delta_{n+1})|\lambda_{n+1}-\lambda_n|(\|x_n\|+\|x_{n+1}\|)\\
&+\frac{\gamma_{n+1}}{1-\alpha_{n+1}}(1-\delta_{n+1})(1-\lambda_{n+1})\left\{\frac{\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\sup_{y\in\{y_n\}}\|T(\mu_{n+1})y-T(\mu_n)y\|\right.\\
&+\left[\frac{|\alpha_{n+1}-\alpha_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}+\frac{|\beta_{n+1}-\beta_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right.\\
&+\frac{|\gamma_{n+1}-\gamma_n|}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}+\frac{|\delta_{n+1}-\delta_n|\gamma_{n+1}}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\\
&\left.\left.+\frac{|\lambda_{n+1}-\lambda_n|\gamma_{n+1}(1-\delta_{n+1})}{1-\gamma_{n+1}(1-\delta_{n+1})(1-\lambda_{n+1})}\right]M\right\}.
\end{aligned}
$$

Therefore,

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\leqslant 0.$$

Applying Lemma 2.1, we obtain $\lim_{n\to\infty}\|w_n-x_n\|=0$ and by setting $\{w_n\}$, we also have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3}$$

Next, we will show that $\lim_{n\to\infty}\|x_n-T(\mu_n)x_n\|=0$. We observe that

$$\|z_n-x_n\| \;=\; (1-\lambda_n)\|x_{n+1}-x_n\|$$

and

$$\|y_n-z_n\| \;=\; \delta_n\|f(x_)-z_n\|,$$

which imply that

$$\lim_{n\to\infty}\|z_n-x_n\|=0$$

and

$$\lim_{n\to\infty}\|y_n-z_n\|=0.$$

Therefore, we conclude that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.4}$$

Consider that

$$
\begin{aligned}
\|x_n-T(\mu_n)x_n\| &\leqslant \|x_n-x_{n+1}\|+\|x_{n+1}-T(\mu_n)x_n\|\\
&= \|x_n-x_{n+1}\|+\|\alpha_n x_n+\beta_n f(x_n)+\gamma_n T(\mu_n)y_n-T(\mu_n)x_n\|\\
&\leqslant \|x_n-x_{n+1}\|+\alpha_n\|x_n-T(\mu_n)x_n\|+\beta_n\|f(x_n)-T(\mu_n)x_n\|+\gamma_n\|T(\mu_n)y_n-T(\mu_n)x_n\|\\
&\leqslant \|x_n-x_{n+1}\|+\alpha_n\|x_n-T(\mu_n)x_n\|+\beta_n\|f(x_n)-T(\mu_n)x_n\|+\gamma_n\|y_n-x_n\|,
\end{aligned}
$$

which implies that

$$\|x_n - T(\mu_n)x_n\| \leqslant \frac{1}{1-\alpha_n}\|x_n - x_{n+1}\| + \frac{\beta_n}{1-\alpha_n}\|f(x_n) - T(\mu_n)x_n\| + \frac{\gamma_n}{1-\alpha_n}\|y_n - x_n\|.$$

It follows from the conditions (ii), (iii) and (3.4), we get

$$\lim_{n\to\infty}\|x_n - T(\mu_n)x_n\| = 0. \tag{3.5}$$

Now, we show that $z \in F(\mathcal{S})$. We can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is bounded and there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup z$. Let $\mu_{n_k} \geqslant 0$ such that $\mu_{n_k} \to 0$ and $\frac{\|T(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} \to 0, k \to \infty$. Fix $t > 0$, we can notice that

$$\begin{aligned}
\|x_{n_k} - T(t)z\| &\leqslant \sum_{i=0}^{[t/\mu_{n_k}]-1} \big\|t\big((i+1)\mu_{n_k}\big)x_{n_k} - T(i\mu_{n_k})x_{n_k}\big\| \\
&\quad + \big\|T\big([t/\mu_{n_k}]\mu_k\big)x_{n_k} - T\big([t/\mu_{n_k}]\mu_{n_k}\big)z\big\| \\
&\quad + \big\|T\big([t/\mu_{n_k}]\mu_{n_k}\big)z - T(t)z\big\| \\
&\leqslant [t/\mu_{n_k}]\|T(\mu_{n_k})x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| + \big\|T\big(t - [t/\mu_{n_k}]\mu_{n_k}\big)z - z\big\| \\
&\leqslant t\frac{\|T(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - z\| + \big\|T\big(t - [t/\mu_{n_k}]\mu_{n_k}\big)z - z\big\| \\
&\leqslant t\frac{\|T(\mu_{n_k})x_{n_k} - x_{n_k}\|}{\mu_{n_k}} + \|x_{n_k} - z\| + \max\{\|T(\mu)z - z\| : 0 \leqslant \mu \leqslant \mu_{n_k}\}.
\end{aligned}$$

For all $k \in \mathbb{N}$, we have

$$\limsup_{k\to\infty}\|x_{n_k} - T(t)z\| \leqslant \limsup_{k\to\infty}\|x_{n_k} - z\|.$$

Since a Banach space E with a weakly sequentially continuous duality mapping satisfies the Opial's condition, this implies $T(t)z = z$. Therefore, $z \in F(\mathcal{S})$.

By Lemma 2.2, the sequence $\{x_t\}$ is defined by $x_t = tf(x_t) + (1-t)Tx_t$ such that $t \in (0,1)$ which is converses strongly to a fixed point $x^* \in F(\mathcal{S})$ and solves the variational inequality:

$$\langle (f - I)x^*, J(z - x^*)\rangle \leqslant 0, \forall z \in F(\mathcal{S}).$$

It follows from Lemma 2.3 and (3.3)–(3.5), we conclude that

$$\limsup_{n\to\infty}\langle (f - I)x^*, J(y_n - x^*)\rangle \leqslant 0 \tag{3.6}$$

and

$$\limsup_{n\to\infty}\langle (f - I)x^*, J(x_{n+1} - x^*)\rangle \leqslant 0. \tag{3.7}$$

Finally, we show that $\{x_n\}$ converges strongly to $x^* \in F(\mathcal{S})$. Assume that the sequence $\{x_n\}$ does not converse strongly to x^*, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\epsilon > 0$ such that $\|x_{n_k} - x^*\| \geqslant \epsilon, \forall k = 1, 2, \ldots$. For this ϵ there exists $r \in (0,1)$ such that

$$\|f(x_{n_k}) - f(x^*)\| \leqslant r\|x_{n_k} - x^*\|.$$

We compute that

$$\begin{aligned}
\|y_{n_k}-x^*\|^2 &= \langle y_{n_k}-x^*, J(y_{n_k}-x^*)\rangle \\
&= \langle \delta_{n_k} f(x_{n_k}) + (1-\delta_{n_k})z_{n_k} - x^*, J(y_{n_k}-x^*)\rangle \\
&= \langle \delta_{n_k}(f(x_{n_k})-x^*) + (1-\delta_{n_k})(z_{n_k}-x^*), J(y_{n_k}-x^*)\rangle \\
&= \delta_{n_k}\langle f(x_{n_k})-f(x^*), J(y_{n_k}-x^*)\rangle + \delta_{n_k}\langle f(x^*)-x^*, J(y_{n_k}-x^*)\rangle \\
&\quad +(1-\delta_{n_k})\langle z_{n_k}-x^*, J(y_{n_k}-x^*)\rangle \\
&= \delta_{n_k}\langle f(x_{n_k})-f(x^*), J(y_{n_k}-x^*)\rangle + \delta_{n_k}\langle f(x^*)-x^*, J(y_{n_k}-x^*)\rangle \\
&\quad +(1-\delta_{n_k})\langle \lambda_{n_k}x_{n_k} + (1-\lambda_{n_k})x_{n_{k+1}} - x^*, J(y_{n_k}-x^*)\rangle \\
&= \delta_{n_k}\langle f(x_{n_k})-f(x^*), J(y_{n_k}-x^*)\rangle + \delta_{n_k}\langle f(x^*)-x^*, J(y_{n_k}-x^*)\rangle \\
&\quad +(1-\delta_{n_k})\langle \lambda_{n_k}(x_{n_k}-x^*) + (1-\lambda_{n_k})(x_{n_{k+1}} - x^*), J(y_{n_k}-x^*)\rangle \\
&= \delta_{n_k}\langle f(x_{n_k})-f(x^*), J(y_{n_k}-x^*)\rangle + \delta_{n_k}\langle f(x^*)-x^*, J(y_{n_k}-x^*)\rangle \\
&\quad +(1-\delta_{n_k})\lambda_{n_k}\langle x_{n_k}-x^*, J(y_{n_k}-x^*)\rangle + (1-\delta_{n_k})(1-\lambda_{n_k})\langle x_{n_{k+1}}-x^*, J(y_{n_k}-x^*)\rangle \\
&\leqslant r\delta_{n_k}\|x_{n_k}-x^*\|\|y_{n_k}-x^*\| + \delta_{n_k}\langle f(x^*)-x^*, J(y_{n_k}-x^*)\rangle \\
&\quad +(1-\delta_{n_k})\lambda_{n_k}\|x_{n_k}-x^*\|\|y_{n_k}-x^*\| + (1-\delta_{n_k})(1-\lambda_{n_k})\|y_{n_k}-x^*\|\|x_{n_{k+1}}-x^*\| \\
&\leqslant [r\delta_{n_k} + (1-\delta_{n_k})\lambda_{n_k}]\frac{1}{2}\left(\|x_{n_k}-x^*\|^2 + \|y_{n_k}-x^*\|^2\right) + \delta_{n_k}\langle f(x^*)-x^*, J(y_{n_k}-x^*)\rangle \\
&\quad +(1-\delta_{n_k})(1-\lambda_{n_k})\frac{1}{2}\left(\|y_{n_k}-x^*\|^2 + \|x_{n_{k+1}}-x^*\|^2\right) \\
&= \frac{r\delta_{n_k} + (1-\delta_{n_k})\lambda_{n_k}}{2}\|x_{n_k}-x^*\|^2 + \frac{1-\delta_{n_k}(1-r)}{2}\|y_{n_k}-x^*\|^2 \\
&\quad +\delta_{n_k}\langle f(x^*)-x^*, J(y_{n_k}-x^*)\rangle + \frac{(1-\delta_{n_k})(1-\lambda_{n_k})}{2}\|x_{n_{k+1}}-x^*\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_{n_k}-x^*\|^2 &\leqslant \frac{r\delta_{n_k} + (1-\delta_{n_k})\lambda_{n_k}}{1+\delta_{n_k}(1-r)}\|x_{n_k}-x^*\|^2 + \frac{2\delta_{n_k}}{1+\delta_{n_k}(1-r)}\langle f(x^*)-x^*, J(y_{n_k}-x^*)\rangle \\
&\quad + \frac{(1-\delta_{n_k})(1-\lambda_{n_k})}{1+\delta_{n_k}(1-r)}\|x_{n_{k+1}}-x^*\|^2.
\end{aligned}$$

We observe that

$$\begin{aligned}
\|x_{n_{k+1}}-x^*\|^2 &= \langle x_{n_{k+1}}-x^*, J(x_{n_{k+1}}-x^*)\rangle \\
&= \langle \alpha_{n_k}x_{n_k} + \beta_{n_k}f(x_{n_k}) + \gamma_{n_k}T(\mu_{n_k})y_{n_k} - x^*, J(x_{n_{k+1}}-x^*)\rangle \\
&= \langle \alpha_{n_k}(x_{n_k}-x^*) + \beta_{n_k}(f(x_{n_k})-x^*) + \gamma_{n_k}(T(\mu_{n_k})y_{n_k}-x^*), J(x_{n_{k+1}}-x^*)\rangle \\
&= \alpha_{n_k}\langle x_{n_k}-x^*, J(x_{n_{k+1}}-x^*)\rangle + \beta_{n_k}\langle f(x_{n_k})-f(x^*), J(x_{n_{k+1}}-x^*)\rangle \\
&\quad +\beta_{n_k}\langle f(x^*)-x^*, J(x_{n_{k+1}}-x^*)\rangle + \gamma_{n_k}\langle T(\mu_{n_k})y_{n_k}-x^*, J(x_{n_{k+1}}-x^*)\rangle \\
&\leqslant \alpha_{n_k}\|x_{n_k}-x^*\|\|x_{n_{k+1}}-x^*\| + r\beta_{n_k}\|x_{n_k}-x^*\|\|x_{n_{k+1}}-x^*\| \\
&\quad +\beta_{n_k}\langle f(x^*)-x^*, J(x_{n_{k+1}}-x^*)\rangle + \gamma_{n_k}\|y_{n_k}-x^*\|\|x_{n_{k+1}}-x^*\| \\
&\leqslant (\alpha_{n_k} + r\beta_{n_k})\frac{1}{2}\left(\|x_{n_k}-x^*\|^2 + \|x_{n_{k+1}}-x^*\|^2\right) + \beta_{n_k}\langle f(x^*)-x^*, J(x_{n_{k+1}}-x^*)\rangle \\
&\quad +\gamma_{n_k}\frac{1}{2}\left(\|y_{n_k}-x^*\|^2 + \|x_{n_{k+1}}-x^*\|^2\right) \\
&= \frac{\alpha_{n_k} + r\beta_{n_k}}{2}\|x_{n_k}-x^*\|^2 + \frac{\alpha_{n_k} + r\beta_{n_k}}{2}\|x_{n_{k+1}}-x^*\|^2 \\
&\quad +\beta_{n_k}\langle f(x^*)-x^*, J(x_{n_{k+1}}-x^*)\rangle + \frac{\gamma_{n_k}}{2}\|y_{n_k}-x^*\|^2 + \frac{\gamma_{n_k}}{2}\|x_{n_{k+1}}-x^*\|^2
\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{\alpha_{n_k}+r\beta_{n_k}}{2}\|x_{n_k}-x^*\|^2+\frac{\alpha_{n_k}+r\beta_{n_k}}{2}\|x_{n_{k+1}}-x^*\|^2+\beta_{n_k}\langle f(x^*)-x^*,J(x_{n_{k+1}}-x^*)\rangle\\
&\quad+\frac{\gamma_{n_k}}{2}\|x_{n_{k+1}}-x^*\|^2+\frac{\gamma_{n_k}}{2}\left[\frac{r\delta_{n_k}+(1-\delta_{n_k})\lambda_{n_k}}{1+\delta_{n_k}(1-r)}\|x_{n_k}-x^*\|^2\right.\\
&\quad\left.+\frac{2\delta_{n_k}}{1+\delta_{n_k}(1-r)}\langle f(x^*)-x^*,J(y_{n_k}-x^*)\rangle+\frac{(1-\delta_{n_k})(1-\lambda_{n_k})}{1+\delta_{n_k}(1-r)}\|x_{n_{k+1}}-x^*\|^2\right]\\
&= \left[\frac{\alpha_{n_k}+r\beta_{n_k}}{2}+\frac{\gamma_{n_k}}{2}\left(\frac{r\delta_{n_k}+(1-\delta_{n_k})\lambda_{n_k}}{1+\delta_{n_k}(1-r)}\right)\right]\|x_{n_k}-x^*\|^2\\
&\quad+\left[\frac{\alpha_{n_k}+r\beta_{n_k}}{2}+\frac{\gamma_{n_k}}{2}+\frac{\gamma_{n_k}}{2}\left(\frac{(1-\delta_{n_k})(1-\lambda_{n_k})}{1+\delta_{n_k}(1-r)}\right)\right]\|x_{n_{k+1}}-x^*\|^2\\
&\quad+\beta_{n_k}\langle f(x^*)-x^*,J(x_{n_{k+1}}-x^*)\rangle+\frac{\gamma_{n_k}\delta_{n_k}}{1+\delta_{n_k}(1-r)}\langle f(x^*)-x^*,J(y_{n_k}-x^*)\rangle\\
&= \frac{(1-\gamma_{n_k}-\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))+\gamma_{n_k}\,(r\delta_{n_k}+(1-\delta_{n_k})\lambda_{n_k})}{2\,(1+\delta_{n_k}(1-r))}\|x_{n_k}-x^*\|^2\\
&\quad+\frac{(1-\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))+\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}{2\,(1+\delta_{n_k}(1-r))}\|x_{n_{k+1}}-x^*\|^2\\
&\quad+\beta_{n_k}\langle f(x^*)-x^*,J(x_{n_{k+1}}-x^*)\rangle+\frac{\gamma_{n_k}\delta_{n_k}}{1+\delta_{n_k}(1-r)}\langle f(x^*)-x^*,J(y_{n_k}-x^*)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n_{k+1}}-x^*\|^2 &\leqslant \frac{(1-\gamma_{n_k}-\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))+\gamma_{n_k}\,(r\delta_{n_k}+(1-\delta_{n_k})\lambda_{n_k})}{(1+\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\|x_{n_k}-x^*\|^2\\
&\quad+\frac{2\beta_{n_k}\,(1+\delta_{n_k}(1-r))}{(1+\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\langle f(x^*)-x^*,J(x_{n_{k+1}}-x^*)\rangle\\
&\quad+\frac{2\gamma_{n_k}\delta_{n_k}}{(1+\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\langle f(x^*)-x^*,J(y_{n_k}-x^*)\rangle\\
&= \left[1-\frac{(1+\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}{(1+\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\right.\\
&\quad\left.-\frac{-(1-\gamma_{n_k}-\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}\,(r\delta_{n_k}+(1-\delta_{n_k})\lambda_{n_k})}{(1+\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\right]\|x_{n_k}-x^*\|^2\\
&\quad+\frac{2\beta_{n_k}\,(1+\delta_{n_k}(1-r))}{(1+\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\langle f(x^*)-x^*,J(x_{n_{k+1}}-x^*)\rangle\\
&\quad+\frac{2\gamma_{n_k}\delta_{n_k}}{(1+\beta_{n_k}(1-r))\,(1+\delta_{n_k}(1-r))-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\langle f(x^*)-x^*,J(y_{n_k}-x^*)\rangle,
\end{aligned}$$

where

$$\begin{aligned}
\alpha'_{n_k} &= \frac{\big(1+\beta_{n_k}(1-r)\big)\big(1+\delta_{n_k}(1-r)\big)-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})-\big(1-\gamma_{n_k}-\beta_{n_k}(1-r)\big)\big(1+\delta_{n_k}(1-r)\big)-\gamma_{n_k}\big(r\delta_{n_k}+(1-\delta_{n_k})\lambda_{n_k}\big)}{\big(1+\beta_{n_k}(1-r)\big)\big(1+\delta_{n_k}(1-r)\big)-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\\
&= \frac{2\beta_{n_k}(1-r)\big(1+\delta_{n_k}(1-r)\big)+2\gamma_{n_k}\delta_{n_k}(1-r)}{\big(1+\beta_{n_k}(1-r)\big)\big(1+\delta_{n_k}(1-r)\big)-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\\
&= \frac{2(1-\alpha_{n_k}-\gamma_{n_k})(1-r)\big(1+\delta_{n_k}(1-r)\big)+2\gamma_{n_k}\delta_{n_k}(1-r)}{\big(1+\beta_{n_k}(1-r)\big)\big(1+\delta_{n_k}(1-r)\big)-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\in(0,1)
\end{aligned}$$

and

$$\begin{aligned}
\delta'_{n_k} &= \frac{2\beta_{n_k}\big(1+\delta_{n_k}(1-r)\big)}{\big(1+\beta_{n_k}(1-r)\big)\big(1+\delta_{n_k}(1-r)\big)-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\langle f(x^*)-x^*,J(x_{n_{k+1}}-x^*)\rangle\\
&+\frac{2\gamma_{n_k}\delta_{n_k}}{\big(1+\beta_{n_k}(1-r)\big)\big(1+\delta_{n_k}(1-r)\big)-\gamma_{n_k}(1-\delta_{n_k})(1-\lambda_{n_k})}\langle f(x^*)-x^*,J(y_{n_k}-x^*)\rangle.
\end{aligned}$$

Now, from the condition (iv), (3.7), (3.6) and Lemma 2.4, we have $\sum_{k=0}^{\infty} \alpha'_{n_k} = \infty$ and $\limsup_{k\to\infty} \frac{\delta'_{n_k}}{\alpha'_{n_k}} \leqslant 0$. Then we get $\|x_{n_k} - x^*\| \to 0$ as $k \to \infty$. This is a contradiction, hence the sequence $\{x_n\}$ converges strongly to $x^* \in F(\mathcal{S})$. The proof is complete. □

Corollary 3.2. *Let C be a nonempty closed convex subset of uniformly convex and uniformly smooth Banach space E. Let f be a generalized contraction mapping from C into itself and $\mathcal{S} = \{T(t) : t \geqslant 0\}$ be a nonexpansive semigroup from C into itself such that $F(\mathcal{S}) \neq 0$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = \dfrac{x_n + x_{n+1}}{2}, \\ y_n = \delta_n f(x_n) + (1-\delta_n) z_n, \\ x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(\mu_n) y_n, \end{cases} \tag{3.8}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $(0,1)$. The following conditions are satisfied:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ *and* $\lim_{n\to\infty} \sup_{x\in\tilde{C}} \|T(\mu_{n+1})x - T(\mu_n)x\| = 0$, $\tilde{C}$ *bounded subset of* C;

(iii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $\sum_{n=0}^{\infty} \beta_n = \infty$ *for all* $n \geqslant 1$.

Then $\{x_n\}$ converges strongly to $x^* \in F(\mathcal{S})$ which also solves the following variational inequality:

$$\langle (I-f)x^*, J(z-x^*) \rangle \leqslant 0, \forall z \in F(\mathcal{S}). \tag{3.9}$$

Proof. Putting $\lambda_n = \frac{1}{2}$ in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

4. Applications

(I) Application to the other forms of semigroups.

Theorem 4.1. *Let C be a nonempty closed convex subset of uniformly convex and uniformly smooth Banach space E. Let f be a generalized contraction mapping from C into itself and $\mathcal{S} = \{T(t) : t \geqslant 0\}$ be a nonexpansive semigroup from C into itself such that $F(\mathcal{S}) \neq 0$. Let $\{x_n\}$ be the sequences defined by $x_1 \in C$ and*

$$\begin{cases} z_n = \lambda_n x_n + (1-\lambda_n) x_{n+1}, \\ y_n = \delta_n f(x_n) + (1-\delta_n) z_n, \\ x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n \dfrac{1}{t_n} \displaystyle\int_0^{t_n} T(t) y_n dt, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\lambda_n\}$ are the sequences in $(0,1)$, $\{t_n\}$ is an increasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. The following conditions are satisfied:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$ *and* $\lim_{n\to\infty} \sup_{x\in\tilde{C}} \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} T(t)x ds - \frac{1}{t_n} \int_0^{t_n} T(t)x ds \right\| = 0$, $\tilde{C}$ *bounded subset of* C;

(iii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $\sum_{n=0}^{\infty} \beta_n = \infty$ *for all* $n \geqslant 1$.

Then $\{x_n\}$ *converges strongly to* $x^* \in F(\mathcal{S})$ *which also solves the following variational inequality:*

$$\langle (I - f)x^*, J(z - x^*)\rangle \leqslant 0, \forall z \in F(\mathcal{S}). \tag{4.2}$$

Proof. Define $\mu_n(f) = \frac{1}{t_n}\int_0^{t_n} f(t)dt, \forall n = 0, 1, 2, \ldots$ where f belong to the space of all real valued bounded continuous functions on positive real number with supremum norm. Then, $\{\mu_n\}$ is a regular sequence of means [18]. Further, we have $T(\mu_n)x = \frac{1}{t_n}\int_0^{t_n} T(t)xdt, \forall x \in C$ and apply theorem 3.1 to conclude the result. □

Corollary 4.2. *Let* C *be a nonempty closed convex subset of uniformly convex and uniformly smooth Banach space* E. *Let* f *be a generalized contraction mapping from* C *into itself and* $\mathcal{S} = \{T(t) : t \geqslant 0\}$ *be a nonexpansive semigroup from* C *into itself such that* $F(\mathcal{S}) \neq 0$. *Let* $\{x_n\}$ *be the sequences defined by* $x_1 \in C$ *and*

$$\begin{cases} z_n = \dfrac{x_n + x_{n+1}}{2}, \\ y_n = \delta_n f(x_n) + (1 - \delta_n)z_n, \\ x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n \dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(t)y_n dt, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ *and* $\{\lambda_n\}$ *are the sequences in* $(0, 1)$, $\{t_n\}$ *is an increasing sequence in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. *The following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$ *and* $\lim_{n\to\infty} \sup_{x\in\tilde{C}} \left\| \frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(t)xds - \frac{1}{t_n}\int_0^{t_n} T(t)xds \right\| = 0$, $\tilde{C}$ *bounded subset of* C;

(iii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $\sum_{n=0}^{\infty} \beta_n = \infty$ *for all* $n \geqslant 1$.

Then $\{x_n\}$ *converges strongly to* $x^* \in F(\mathcal{S})$ *which also solves the following variational inequality:*

$$\langle (I - f)x^*, J(z - x^*)\rangle \leqslant 0, \forall z \in F(\mathcal{S}). \tag{4.4}$$

(II) Application to Hilbert spaces.

From the *duality mapping* $J : E \to 2^{E^*}$, if $E = H$ is a real Hilbert space then $J = I$ is the identity mapping. A

Theorem 4.3. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* f *be a generalized contraction mapping from* C *into itself and* $\mathcal{S} = \{T(t) : t \geqslant 0\}$ *be a nonexpansive semigroup from* C *into itself such that* $F(\mathcal{S}) \neq 0$. *Let* $\{x_n\}$ *be the sequences defined by* $x_1 \in C$ *and*

$$\begin{cases} z_n = \lambda_n x_n + (1 - \lambda_n)x_{n+1}, \\ y_n = \delta_n f(x_n) + (1 - \delta_n)z_n, \\ x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(\mu_n)y_n, \end{cases} \tag{4.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ *and* $\{\lambda_n\}$ *are the sequences in* $(0, 1)$. *The following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$ *and* $\lim_{n\to\infty} \sup_{x\in\tilde{C}} \|T(\mu_{n+1})x - T(\mu_n)x\| = 0$, $\tilde{C}$ *bounded subset of* C;

(iii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $\sum_{n=0}^{\infty} \beta_n = \infty$ *for all* $n \geqslant 1$.

Then $\{x_n\}$ *converges strongly to* $x^* \in F(\mathcal{S})$ *which also solves the following variational inequality:*

$$\langle (I - f)x^*, z - x^*\rangle \leqslant 0, \forall z \in F(\mathcal{S}). \tag{4.6}$$

Corollary 4.4. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* f *be a generalized contraction mapping from* C *into itself and* $\mathcal{S} = \{T(t) : t \geqslant 0\}$ *be a nonexpansive semigroup from* C *into itself such that* $F(\mathcal{S}) \neq 0$. *Let* $\{x_n\}$ *be the sequences defined by* $x_1 \in C$ *and*

$$\begin{cases} z_n = \dfrac{x_n + x_{n+1}}{2}, \\ y_n = \delta_n f(x_n) + (1 - \delta_n)z_n, \\ x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(\mu_n)y_n, \end{cases} \tag{4.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *are the sequences in* $(0, 1)$. *The following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ *and* $\lim_{n\to\infty} \sup_{x\in\tilde{C}} \|T(\mu_{n+1})x - T(\mu_n)x\| = 0$, $\tilde{C}$ *bounded subset of* C;

(iii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $\sum_{n=0}^{\infty} \beta_n = \infty$ *for all* $n \geqslant 1$.

Then $\{x_n\}$ *converges strongly to* $x^* \in F(\mathcal{S})$ *which also solves the following variational inequality:*

$$\langle (I - f)x^*, z - x^*\rangle \leqslant 0, \forall z \in F(\mathcal{S}). \tag{4.8}$$

5. Numerical Example

Example 5.1. Let $E = \mathbb{R}^3$ and an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y} = x^1 \cdot y^1 + x^2 \cdot y^2 + x^3 \cdot y^3, \forall \mathbf{x} = (x^1, x^2, x^3), \mathbf{y} = (y^1, y^2, y^3)$$

and the usual norm $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\|\mathbf{x}\| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}.$$

Let $C = \{\mathbf{x} \in \mathbb{R}^3 | \|\mathbf{x}\| \leqslant 1\}$, $\alpha_n = \lambda_n = \dfrac{n+1}{3n}$, $\beta_n = \dfrac{1}{3n}$, $\delta_n = \mu_n = \dfrac{1}{n}$, $\gamma_n = \dfrac{2n-2}{3n}$ and $\mathbf{x}_1 = (1, 2, 3)$ for all $n = 1, 2, \ldots$ which satisfy the conditions $(I) - (IV)$ in Theorem 3.1. We define the mappings $T(t), f : \mathbb{R}^3 \to \mathbb{R}^3$ as follows: $(T(t))(\mathbf{x}) = \mathbf{x}e^{-t}$ and $f(\mathbf{x}) = \dfrac{\mathbf{x}}{10}$ for all $\mathbf{x} \in \mathbb{R}^3$. Then the sequence

$$\mathbf{x}_{n+1} = \frac{30n^3 + 33n^2 + (20n^3 - 14n^2 - 26n + 20)e^{-1/n}}{90n^3 - (40n^3 - 100n^2 + 80n - 20)e^{-1/n}} \mathbf{x}_n$$

converges strongly to $\mathbf{0} = (0, 0, 0)$ shown in Figure 1 and Table 1.

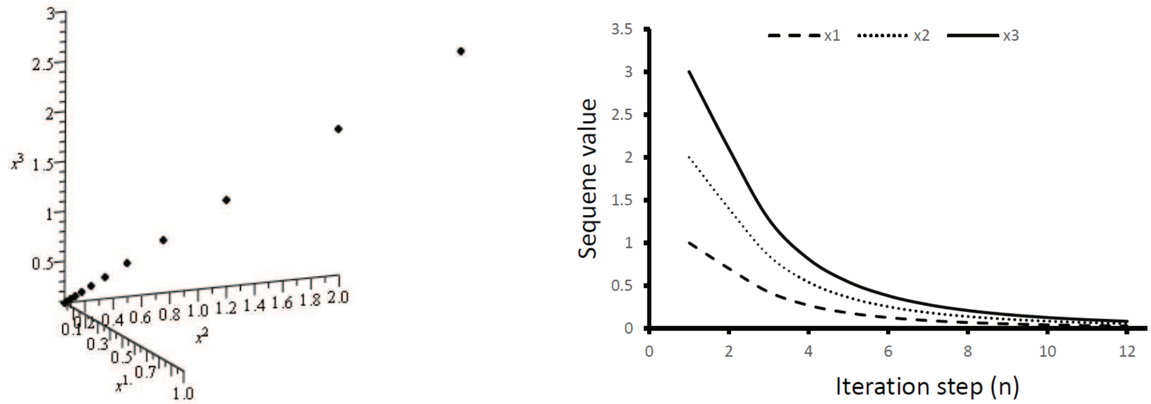

Figure 1: The iteration process.

Table 1: The value of sequence $\{\mathbf{x}_n\}$

n	x_n^1	x_n^2	x_n^3	n	x_n^1	x_n^2	x_n^3
1	1	2	3	10	0.042547	0.085093	0.127640
2	0.7	1.4	2.1	50	0.000722	0.001445	0.002167
3	0.425637	0.851275	1.276912	100	0.000109	0.000218	0.000327
4	0.270485	0.540970	0.811455	400	0.0000023	0.0000046	0.0000069
5	0.181409	0.362819	0.544228	797	0.0000003	0.0000006	0.00000099

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